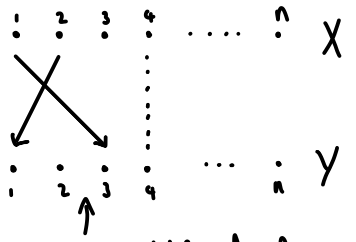
26/1/23 L7

<u>Notation:</u> $|X|$ = # elements in set X.

<u>Lemma 3.2:</u> If $|X| = |Y| = n$. Then the number of all bijections $X \to Y$ is $n!$

Proof:



n possibilities for $f(1)$

$n-1$ possibilities for $f(2)$ (must $\neq f(1)$)

$\vdots$

2 — // — $f(n-1)$

1 — // — $f(n)$

In total $n(n-1)(n-2)\cdots 2\cdot 1 = n!$ possibilities.

<u>4 Permutations and Symmetric Groups</u>

<u>Definition 4.1:</u> Let $X = \{1, 2, 3, \ldots, n\}$. Then any bijection $X \longrightarrow X$ is called a <u>permutation</u> of X The <u>symmetric group</u> S_n is the set of all permutations of $\{1, 2, 3, \ldots, n\}$:

$$S_n = \{\text{all bijections } \{1,2,3,\ldots,n\} \longrightarrow \{1,2,3,\ldots,n\}\}$$

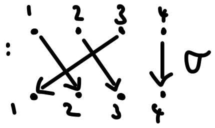
with operation compositions of maps

Note: (1) $|S_n| = n!$ by 3.12

(2) I will use greek letters for elements of S_n: $\alpha, \beta, \gamma, \delta, \ldots, \sigma$

How to represent elements of S_n?

Let $\sigma \in S_4$ s.t. $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$, $\sigma(4) = 4$

We have 3 main ways to describe σ: (1) as a map $X \longrightarrow X = \{1,2,3,4\}$: 

(2) (array notation): $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{bmatrix} \begin{matrix} \leftarrow x \\ \leftarrow \text{images} \end{matrix}$

(3) (cycle notation): $\sigma = (1\ 2\ 3)$ (see later)

Consider array notation: note $\sigma^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{bmatrix}$ (check $\sigma\sigma^{-1} = \sigma^{-1}\sigma = 1$) suppose $\mu = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{bmatrix}$.

$\sigma\mu = \sigma \circ \mu = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{bmatrix}$ $((\sigma \circ \mu)(k) = \sigma(\mu(k))\ \forall k \in X)$

Note: To find $\sigma\mu$, we do μ first and then σ.

Note: S_n has a distinguished element, the identity map $1 = 1_X = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$

<u>Powers</u> $\alpha \in S_n$ then $\alpha^2 = \alpha \circ \alpha = \alpha \cdot \alpha$. Define $\alpha^0 = 1\ (= 1_X)$, $\alpha^k = \underbrace{\alpha \cdots \alpha}_{k}$ $(k \geq 1)$

$\alpha^{-k} = \underbrace{\alpha^{-1} \cdots \alpha^{-1}}_{k}$ $(k \geqslant 1)$

Then α^n is defined $\forall n \in \mathbb{Z}$, $\alpha^{-k} = (\alpha^{-1})^k = (\alpha^k)^{-1}$, $\alpha^m \cdot \alpha^n = \alpha^{m+n}$
$(\alpha^m)^n = \alpha^{mn}$ $\forall m,n \in \mathbb{Z}$

Note: $(\alpha\beta)^m \neq \alpha^m \beta^m$ in general
$\quad\ \ \|\qquad\qquad\ \ ''$
$\alpha\beta\ \alpha\beta \cdots \alpha\beta \neq \alpha \cdots \alpha\ \beta \cdots \beta$

as $\alpha\beta \neq \beta\alpha$ in general (as $\alpha \circ \beta \neq \beta \circ \alpha$ composition of maps not commutative)

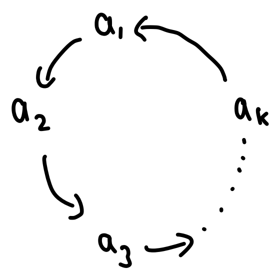
However note: $(\alpha\beta)^{-1} = \alpha^{-1}\beta^{-1}$ (as $\alpha\beta \cdot \beta^{-1}\alpha^{-1} = 1$)
$(\alpha\beta\gamma)^{-1} = \gamma^{-1}\beta^{-1}\alpha^{-1}$

<u>Definition 4.3:</u> Let $\alpha \in S_n$. The <u>order</u> of α is the smallest integer $m \geq 1$ s.t. $\alpha^m = 1$

Example: Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{bmatrix}$. Then $\alpha^2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{bmatrix} \neq 1$, $\alpha^3 = 1$. So $O(\alpha) = 3$ (order)

We will see $O(\alpha)$ must divide the order (size) of the group. Here $|S_n| = n!$

Definition 4.4: Let $k \leq n$. For k different integers $a_1, \dots a_k$ with $1 \leq a_i \leq n$ we denote by $(a_1, \dots a_k)$ the permulation which maps $a_1 \to a_2$, $a_2 \to a_3, \dots, a_k \to a_1$ and doesn't move other numbers. Such permutations are k-cycle (= cycles of length k).



Example: $(1\ 3\ 4)$ is a 3-cycle in S_5 is $(1\ 3\ 4) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{bmatrix}$.

Definition 4.5: Cycles are <u>disjoint</u> if they have no common elements.

Example: (1) $(2 5 6)$ and $(1 4 3 7)$ are disjoint.

(2) $(2 3)$ and $(1 4 3 7)$ not disjoint.

Remark 4.6: note $(1\ 4\ 3\ 7) = (4\ 3\ 7\ 1) = (3\ 7\ 1\ 4) = (7\ 1\ 4\ 3)$ the same permutation. Use the cycle with smallest number first $(1 4 3 7)$.