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Symmetric group $S_n = \{\text{bijections } X \to X\}$

$\uparrow$ permutations

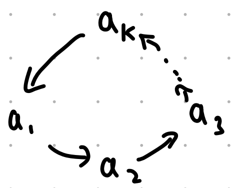
$X = \{1, 2, 3, \dots, n\}$

<u>Array notation:</u> $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{bmatrix}$

<u>Cyclic notation:</u> $\sigma = (1\ 2)(3\ 4)\cancel{(5)} = (12)(34)$

disjoint

Recall <u>k-cycle</u> is $(a_1, \dots, a_k)$



the rest not moved

Every $\sigma \in S_n$ is a product of disjoint cycles:

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 2 & 4 & 8 & 5 & 7 & 6 \end{bmatrix} = (1)(2\ 3)(4)(5\ 8\ 6)(7)$$
$$= (2\ 3)(5\ 8\ 6)$$

Note: disjoint cycles commute: $\alpha\beta = \beta\alpha$, so $(\alpha\beta)^k = \alpha^k \beta^k$

$(\alpha, \beta \text{ disjoint})$

not true for not disjoint, e.g. $(1\ 2)(2\ 3) \neq (2\ 3)(1\ 2)$

$$(1\ 2)(2\ 3) = (1\ 2\ 3) \neq (1\ 3\ 2) = (2\ 3)(1\ 2)$$

Recall, if $\sigma \in S_n$, <u>order</u> $O(\sigma) = k$ where k is min s.t. $\alpha^k = 1$. Thus $O(k \text{ cycle}) = k$.

<u>Definition 4.12:</u> A <u>transposition</u> is a 2-cycle.

Example: $(2\ 3)$, $(6, 8)$, $(\dots)$

<u>Proposition 4.13:</u> Any k-cycle is a product of transposition.

Proof: $(1, 2, 3, \dots, k) = (1\ k)(1\ k{-}1)(1\ k{-}2) \cdots (1\ 3)(1\ 2)$ $(= (1\ 2\ 3 \dots))$

(check!)

Corollary 4.14: Any permutation is a product of transposition.

Proof: apply 4.13 to cyclic decomposition.

Proposition 4.15: Every permutation can be written as a product of either even # of transposition or odd # of transposition, but not both

Proof: Idea: suppose $\underbrace{(..)(..)..}_{\text{odd}} = \sigma = \underbrace{(..)(..)...(ab)}_{\text{even}}$

then $...(ab) = \sigma(ab) = ...\underbrace{(ab)^2}_{\text{identity}}$

so $\underbrace{(..)...(..)}_{\text{odd}} = 1_x$

Idea 2: $(a\,b) = (..)...(..)$ where $(..) = (a\ a+1)$

Full proof in MA3131 Groups and Symmetry.

Definition 4.16: If $\sigma \in S_n$ can be written as the product of an even (resp. odd) number of transposition, then we say that σ is even (resp. odd) and its sign $sgn(\sigma) = +1$ (resp. -1).

Note: Have well-defined function $sgn: S_n \longrightarrow \{+1, -1\}$

Lemma 4.17: $sgn(\sigma\mu) = sgn(\sigma) \cdot sgn(\mu)$

Proof: clear. (check for σ, μ odd / even)

(Note: by 4.17, $sgn: S_n \longrightarrow \{\pm 1\}$ is a group homomorphism.)

Lemma 4.18: (1) If $\sigma = 1_x$ (identity) then $sgn(1_x) = 1$ $(1_x = (1\,2)(1\,2))$

(2) If σ is a product of m transposition, then $sgn(\sigma) = (-1)^m$

(3) If σ is k-cycle then $sgn(\sigma) = (-1)^{k-1}$.

Proof: (1) $1_X = (1\ 2)(1\ 2)$ so $sgn(1_X) = 1$.

(2) by 4.17, $sgn(\sigma) = sgn((..)(..)...(..)) = sgn(..)...\ sgn(..) \Rightarrow m$ transposition $= (-1)^m$

(3) use 4.13 (k-cycle = product of k-1 transposition)

Example: Easy to find $sgn(\sigma)$ for any σ. e.g. if $\sigma = (1\ 4\ 2)(3\ 8\ 7\ 9)(5\ 7)$ then

$$sgn(\sigma) = sgn(1\ 4\ 2) \cdot sgn(3\ 8\ 7\ 9) \cdot sgn(5\ 7) = (-1)^2 \cdot (-1)^3 \cdot (-1)^1$$
$$= (-1)^6$$
$$= 1$$

so σ is even.