12/10/22 Finite Sets

A <u>set</u> is a collection of things called <u>elements</u>

<u>Example:</u> {odd numbers}

$= 2\mathbb{N} - 1$

A <u>set</u> is finite if it is not infinite.

Part One:

Q1) What is counting?

Counting is the act of assigning numbers to a set of things

Q2) Explain what it means for a set to have n elements.

For a set to have n elements and if it is possible to label the elements in the set with the numbers 1, 2, 3, ..., n such that every element has no same label and is labeled with no missing numbers from 1, 2, 3, ... n

Q3) Recall the notion of Injectivity, Surjectivity and Bijectivity of a function.

A function $f: X \to Y$ is called injective if for any $a, b \in X$. $f(a) = f(b)$ implies $a = b$.

f is called surjective if $\text{Im}(f) = Y$

A bijective function, $f: X \to Y$ where it is a one-to-one (injective) and onto (surjective).

Q4) How can we define the cardinality of a finite set?

The cardinality of a finite set if the set can be assigned one number without it repeating. A bijective function $f: S \to \underline{n}$ between S (a finite <u>set</u>) and $\underline{n} := \{1, 2, 3, \ldots, n\}$

What is the cardinality of the empty set $\emptyset$?

empty set $= \emptyset = \{\}$ so $|\emptyset| = 0$

Q5) How would you define an infinite set i.e. a set with infinitely many elements?

an infinite set is when we can count on and on with seemingly no end.

symbol $\Rightarrow \infty$. They cannot be counted.

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<u>Goal</u>: understand <u>cardinality</u> for all sets

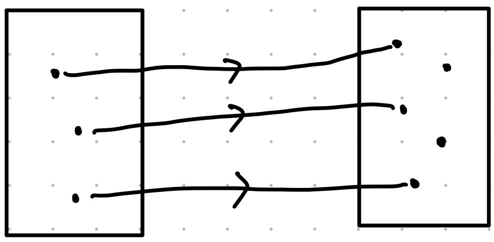
<u>Seen</u>: finite set S $\qquad |S| = \#$ elements of S.

<u>really</u> $|S| = n \Leftrightarrow \exists$ bijective function, $\quad S \longrightarrow \{1, 2, 3, \dots, n\}$

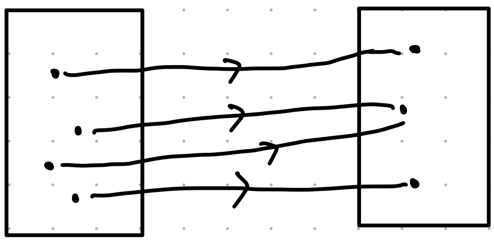
S and T are <u>equinumerous</u> (same size) if there is a bijective function $f: S \to T$.

i.e. f is injective $\Leftrightarrow f(a) = f(b) \Rightarrow a = b \ \forall a, b \in S$

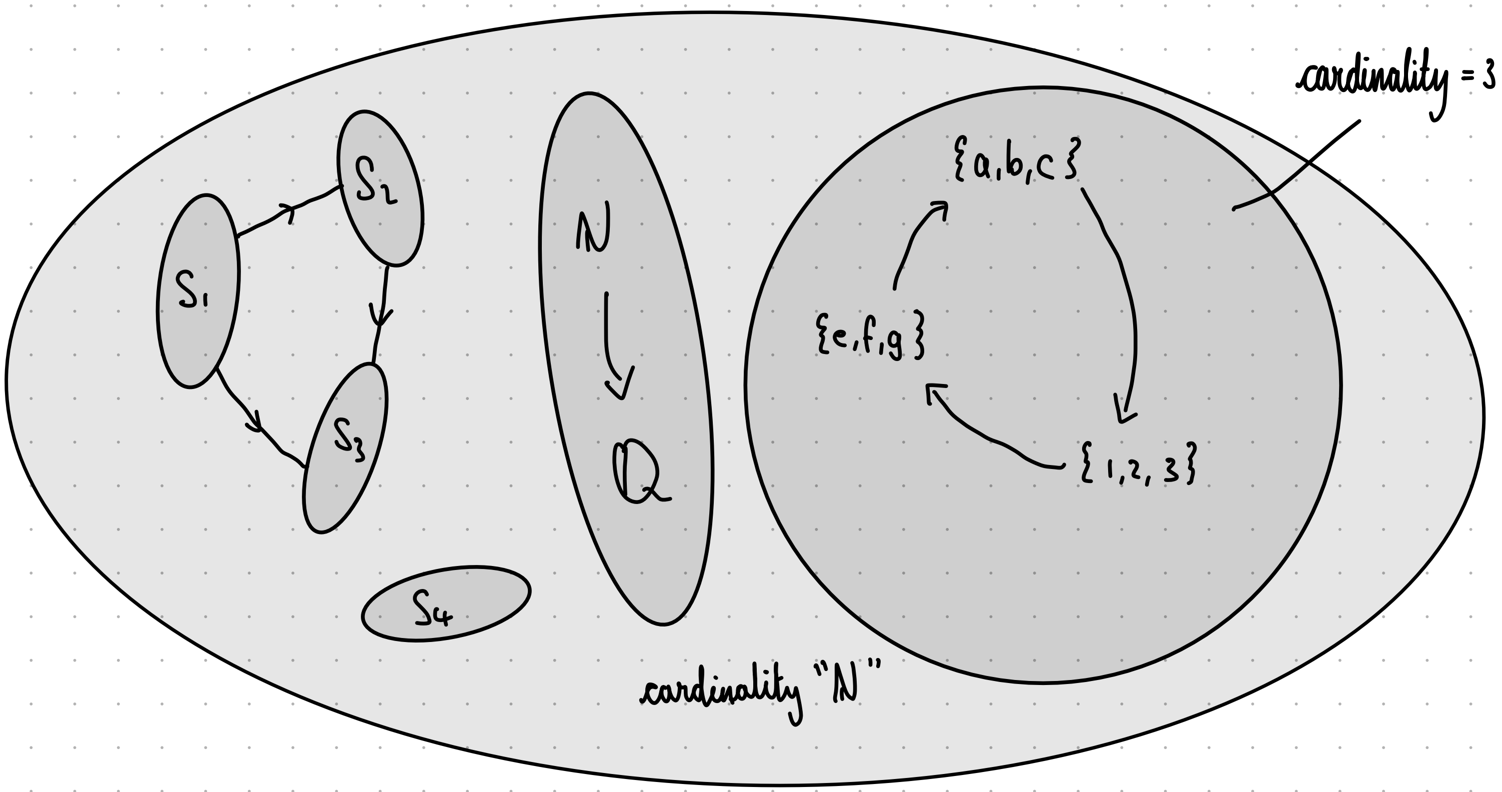
f is surjective $\Leftrightarrow f(S) = \{f(a) : a \in S\} = T$



injective, not surjective



surjective, not injective



To understand "equivalence" in maths we use <u>equivalence relations</u>.

<u>Definition:</u> If X is a set, a <u>equivalence relation</u> on X is a subset $R \subseteq X \times X$

relation "$\{(x,y) \mid x,y \in X\}$

collection of pairs

(reads as x is equivalent to y). Satisfying (a) $(x,x) \in R \;\; \forall x \in X$ reflectivity

(b) $(x,y) \in R \Rightarrow (y,x) \in R$ symmetry

(c) $(x,y) \in R, (y,z) \in R \Rightarrow (x,z) \in R$ transivity

<u>Example (odd + even numbers)</u>

$X = \mathbb{Z}$ (integers)

$R = \{(x,y) \mid x-y \text{ is even}\}$ check there is an equivalence relation

(a) $(x,x) \in R$ since $x - x = 0$

(b) $(x,y) \in R \Rightarrow x-y$ is even

$\Rightarrow y - x = -(x-y)$ is even

(c) $(x,y), (y,z) \in R \Rightarrow x-y, \; y-z$ are even

$\Rightarrow (x-y) + (y-z) = x - z$ is even

so R <u>is</u> an equivalence relation

<u>Lemma:</u> Suppose R is an <u>equivalence relation</u> on X $\forall x \in X$, set $[X]_R = \{y \in X \mid (y,x) \in R\}$ then

(a) $\bigcup_{x \in X} [x]_R = X$

(b) $[x]_R \neq [y]_R \Leftrightarrow [x]_R \cap [y] = \emptyset$

<u>e.g with odd & even numbers:</u>

$[3]_R = \{y \in \mathbb{Z} \mid y-3 \text{ is even}\}$ is odd numbers

$[6]_R = \{y \in \mathbb{Z} \mid y-3 \text{ is even}\}$ is even numbers

$\mathbb{Z} = [3]_R \cup [6]_R$

$= [1]_R \cup [2]_R$

Part One:

Q6) Show that "equinumerousity" is an equivalence relation on the class C of all sets. That is, Show that the subset: $R = \{(S, T) \mid S \text{ and } T \text{ are equinumerous}\} \subseteq C \times C$ is an equivalence on C.

pf: by showing the properties of reflexivity, symmetry and transitivity we can show that equinumerosity is an equivalence relation.

reflexive: the identity map $Id_S : S \to S$ where $Id_S(s) = s \ \forall s \in S$

$\Rightarrow$ bijection

symmetry: suppose S is equinumerous to T

$\Rightarrow$ bijection $f : S \to T$

since f is bijective, $\exists f^{-1}$

$\Rightarrow$ bijection $f : T \to S$

transitivity: suppose S equinumerous to U and U to T

$\Rightarrow$ bijection for $f : S \to U$ & $g : U \to T$

then $g \circ f : S \to T$

$\Rightarrow$ bijective so S and T are equinumerous

we have proved that equinumerosity is an equivalence relation $\square$

Q7) Describe the equivalence class $[S]_R$ of a set $S \in C$. The cardinality $|S|$ of a set S is defined to be the equivalence class $[S]_R$. We choose a representative set in each equivalence class, which allows us to extend counting from finite to infinite sets.

pf: suppose $[S]_R = \{1, \dots, n\} \ \forall n \in \mathbb{N}$

we can choose any representative set, r, in $[S]_R$

so $r = \{i\}, \{j\}$ where $1 \leq i, j \leq n$ and know $|[S]_R| = n$

class $[S]_R$ is finite but we want $[S]_R$ to be infinite

$\longrightarrow$

$\rightarrow$ so we can take multiples of r, s.t. $i \neq j$ and $1 \leq i, j \leq |[S]_R|$

therefore we can extend counting to be infinite by doing the following:

(1) $\{i\} + \{j\}$

(2) $\{i\} \times \{j\}$

(3) $\{i\} - \{j\}$

(4) $\frac{\{i\}}{\{j\}}$

so (1), (2), (3), (4) $\cup [S]_R$

$\square$

Q8) Let $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ be the set of positive integers or natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$ be the set of non-negative integers, $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ be the set of integers and $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$ be the set of rational numbers. Show the following are equivalent

a) $|\mathbb{N}| = |S|$; b) $[S]_R = [\mathbb{N}]_R$; c) S is equinumerous to $\mathbb{N}$;

d) the elements of S can be written as a sequence $s_1, \dots, s_n, s_{n+1}, \dots$

pf: (c) if S is equinumerous to $\mathbb{N}$

it holds that S is reflexive, symmetric, and transitive

$\Rightarrow S$ is a bijection we can assign

therefore we can have elements in S as $\{s_1, \dots, s_n, s_{n+1}, \dots\}$ where it can be $n \in \mathbb{N}$

$\Rightarrow$ (d) then (c) = (d)

$|S| = |s_1, \dots, s_n, s_{n+1}, \dots|$

$= |n + t| \ \forall t \in \mathbb{N}$ (assumed)

$= |\mathbb{N}|$

$\Rightarrow$ (a) is equal to (c) & (d) $\Rightarrow$ (a) = (c) = (d)

finally, then (a) is equal to (b) by question 7

$\therefore$ (c) = (d) = (a) = (b) $\square$

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<u>Definition:</u> A set is countably infinite $\Leftrightarrow \exists$ a bijection $f: \mathbb{N} \to S$

$$\Leftrightarrow [S]_R = [\mathbb{N}]_R$$

$$\shortparallel$$

$$\{T \in C \mid \exists \text{ bijection } f: S \to T\}$$

(Relation R: $\{(S,T) \mid S$ and T are equinumerous$\}$)

Recall: $[S]_R = [S']_R \Leftrightarrow (S, S') \in R$

<u>Example:</u> $S = \{1, 2, 3, \ldots\}$

S is countably infinite since there is a bijection $S \to \mathbb{N}$

Part One:

Q9) Answer the following:

a) Are the sets $2\mathbb{N} = \{0, 2, 4, 6, 8, \ldots\}$ of even numbers and $2\mathbb{N} - 1 = \{1, 3, 5, 7, 9, \ldots\}$ of odd numbers countaby infinite? Justify.

pf: Let $f: \mathbb{N} \to S$ where $S = 2\mathbb{N}$ $\forall x \in \mathbb{N}$ s.t $f(x) = \frac{x}{2}$

f is well-defined (mapped) as x is even $\frac{x}{2} \in \mathbb{Z}$, integer

let $x, y \in \mathbb{N}$ s.t. $f(x) = f(y)$

then: $f(x) = f(y)$

$x/2 = y/2$ (definition of f)

$x = y \Rightarrow f$ is injective

considering f^{-1}, $\forall x \in \mathbb{Z}$ s.t. $f^{-1}(x) = 2x$

f^{-1} is well defined and $2x$ is even $\Rightarrow f^{-1}: S \to \mathbb{N}$

then: $f^{-1}(x) = f^{-1}(y)$

$\Rightarrow 2x = 2y \Rightarrow x = y \Rightarrow f^{-1}$ is injective

$\Rightarrow$ if f, f^{-1} is both injective $\exists$ bijection (by Cantor-Bernstein-Schröder theorem) $\square$

$2\mathbb{N}-1 = \{1,3,5,7,\dots\}$

Let $f: \mathbb{N} \to S$ where $S = 2\mathbb{N}-1$ $\forall x \in \mathbb{N}$

f is well-defined as $x+1$ is even and so $\frac{x+1}{2} \in \mathbb{Z}$

let $x, y \in \mathbb{N}$ s.t. $f(x) = f(y)$

then: $f(x) = f(y)$

$\Rightarrow \frac{x+1}{2} = \frac{y+1}{2}$

$\Rightarrow x+1 = y+1$

$\Rightarrow x = y \Rightarrow f$ is injective

considering f^{-1}, $\forall x \in \mathbb{Z}$ s.t. $f^{-1}(x) = 2x-1$

f^{-1} is well-defined and $2x-1$ is odd $\Rightarrow f^{-1}: S \to \mathbb{N}$

then $f^{-1}(x) = f^{-1}(y)$

$\Rightarrow 2x-1 = 2y-1$

$\Rightarrow 2x = 2y$

$\Rightarrow x = y \Rightarrow f^{-1}$ is injective

$\Rightarrow f, f^{-1}$ is injective $\exists$ bijective (by Cantor-Bernstein-Schröder theorem) $\square$

Theorem (Cantor-Bernstein-Schröder theorem): (shortened)

$\exists f: S \to T$ and $\exists g: T \to S$ are both injective then $\exists$ bijection between S and T

Q9) b) Is the set $\mathbb{N}_0$ countably infinite? Justify

$\mathbb{N}_0 = \{0,1,2,3,4,\dots\}$

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

we know the sets individually $\{0\}$ and $\mathbb{N}$ are countably infinite

$f: \mathbb{N} \to \mathbb{N} \Rightarrow \exists$ bijection

for $g: \mathbb{N}_0 \to \mathbb{N}$ it means $0 \to 1$, $1 \to 2$, $2 \to 3$

then $\forall x \in \mathbb{N}$ s.t. $f(x) = x+1$

let $x, y \in \mathbb{N}$ s.t. $f(x) = f(y)$

then: $f(x) = f(y)$

$\Rightarrow x+1 = y+1$

$\Rightarrow x = y \Rightarrow$ injective

consider f^{-1}, let $g = f^{-1}$, $g: \mathbb{N} \to \mathbb{N}_0$ then $x \in \mathbb{N}_0$ gives $f(x) = x-1$

let $x, y \in \mathbb{N}_0$ s.t. $f(x) = f(y)$

then: $f(x) = f(y)$

$\Rightarrow x-1 = y-1$

$\Rightarrow x = y \Rightarrow$ injective

$\Rightarrow f, g$ both injective then $\exists$ a bijection between $\mathbb{N}$ and $\mathbb{N}_0$

hence $\mathbb{N}_0$ is countably infinite. $\square$

Q9) c) Is the set $\mathbb{Z}$ countably infinite? Justify.

well $\mathbb{Z}$ composes of $\mathbb{N}$, $\{0\}$ and $-\mathbb{N}$ $\Rightarrow \mathbb{Z} = \mathbb{N} \cup \{0\} \cup -\mathbb{N}$

pf: define $f: \mathbb{N} \to \mathbb{Z}$

by $\begin{cases} f(1) = 0 \\ f(n) = n/2 \text{ if } n \text{ is even} \\ f(n) = -\left(\frac{n-1}{2}\right) \text{ if } n \text{ is odd}, n > 1 \end{cases}$

we now show that f maps $\mathbb{N}$ onto $\mathbb{Z}$. Let $w \in \mathbb{Z}$

if $w = 0$, then note that $f(1) = 0$

suppose $w > 0$, then $f(2w) = \frac{2w}{2} = w$

suppose $w < 0$, solving $w = -\left(\frac{n-1}{2}\right) \Rightarrow n = -2w+1 \Rightarrow$ odd and positive

so $f(-2w+1) = -\left[\frac{-2w+1-1}{2}\right] = w$. Hence f maps $\mathbb{N}$ onto $\mathbb{Z} \Rightarrow$ injective

and $f^{-1}: \mathbb{Z} \to \mathbb{N}$ s.t. $n = f^{-1}(w) = 2w$ if $w \in \mathbb{Z}^{>0}$ and $n = f^{-1}(\mathbb{Z}) = -(2w+1)$

if $w \in \mathbb{Z}^{<0}$ $\Rightarrow \exists$ bijection since f and f^{-1} are both injective $\square$

Q10) Let S and T be sets. In terms of the existence (or not) of injective, surjective or bijective functions between S and T, what should be meant by:

a) $|S| < |T|$;

"the size of set s is less than the size of set s"

means $\exists\ f: S \to T$, injective

it is easy to see that now each element in T can only come from one certain input from S, but not every element in T is reached from this function.

b) $|S| = |T|$;

"the size of S is the same as the size of T"

means $\exists\ f: S \to T$, bijective

every element in S can be labelled with a unique element in T

c) $|S| > |T|$

"the size of s is greater than the size of T"

means $\exists f: S \to T$, surjective

every element in T is reached from an input from S.

Q11) We denote the cardinality of the set $\mathbb{N}$ of natural numbers by $|\mathbb{N}| = \aleph_0$. (say: "aleph zero". $\aleph$ is the first letter in the Hebrew alphabet). Discuss the following definition:

<u>Definition:</u> A set S is countable if $|S| \leq \aleph_0$.

A set S is countably infinite if $|S| = \aleph_0$.

A set S is uncountably infinite $|S| > \aleph_0$.

Using your definition in Question 10, show that a set S is finite if $|S| < \aleph_0$

pf: "$\Rightarrow$" Let S be finite

by definition of finite sets $\Rightarrow \exists n \in \mathbb{N} : S \sim \mathbb{N}_n$

where $\mathbb{N}_n = \{1, 2, 3, \dots, n\} \Rightarrow \mathbb{N}_n = n$

then by definition of cardinality $|S| = n$

$\forall i \in \mathbb{N} : i \subseteq \mathbb{N}$ then $n+1 \subseteq \mathbb{N}$

$\Rightarrow n+1 = |n+1| \leq |\mathbb{N}| = \aleph_0$ also $n < n+1$

thus $|S| < \aleph_0$

"$\Leftarrow$" Let $|S| < \aleph_0$

by definition of aleph mapping: $\aleph_0 = \omega \Rightarrow \mathbb{N} = \omega$

we can see that $|S| \in \mathbb{N}$

and by definition of cardinal $\exists n \in \mathbb{N} : S \sim n$

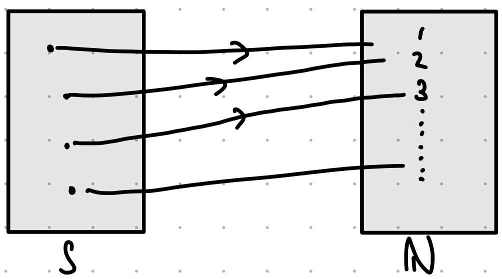
$\exists n \in \mathbb{N} : S \sim \mathbb{N}_n$

thus by definition: S is finite. $\square$

for Q11) alternative answer:

$f : S \to \mathbb{N}$, injective and not bijective

if S were infinite, then could construct an inverse to f ⚡ $\Rightarrow S$ is finite $\square$



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Interested in cardinality of sets where $|S|$ is $[S]$ (equivalent class of S)

S is countable $\Leftrightarrow [S]=[\mathbb{N}]$

Examples: - $\mathbb{N}$

- $\mathbb{N}_0$
- $2\mathbb{N}$
- $2\mathbb{N}-1$
- $\mathbb{Q}$
- $\mathbb{Z}$

$\mathbb{N}\times\mathbb{N} \Rightarrow \{1,2\}\times\{a,b,c\} = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}$

$\mathbb{N}^2 = \{(x,y) \mid x,y\in\mathbb{N}\}$

(later: we want sets $|S|>|\mathbb{N}|$) $\rightarrow \exists$ surjective $S\rightarrow\mathbb{N}$, not injection

uncountable gives a new type of infinity

Part 2:

Q1) Prove the following theorem:

Theorem 2.1 (Cantor, 1875). The set $\mathbb{Q}$ of all rational numbers is countably infinite i.e. $|\mathbb{Q}|=\aleph_0$.

pf: the rational numbers is defined by: $\mathbb{Q}=\{\frac{a}{b} : a,b\in\mathbb{Z},\ b\neq 0\}$

every rational number r can be uniquely written as $r = p/q$ where $p,q\in\mathbb{Z}, q\neq 0$

and p and q are relatively prime, that is GCD of p and q is 1

$\forall$ rational number $r\in\mathbb{Q} \Rightarrow f:\mathbb{Q}\rightarrow\mathbb{Z}\times\mathbb{Z}$

then $f(r)=(p,q) \Rightarrow f$ is injective

$\mathbb{Z}$ countable $\Rightarrow \mathbb{Z}\times\mathbb{Z}$ countable also $\therefore f(\mathbb{Q})$ is countably infinite then $\exists$ bijection

note that the function $h:\mathbb{Q}\rightarrow f(\mathbb{Q})$ defined by $h(q)=f(q)$ is a bijection therefore $g \circ h : \mathbb{Q} \to \mathbb{N}$ is a bijection

$\Rightarrow \mathbb{Q}$ is countably infinite $\square$

Q2) Let S and T be two sets. The cartesian product $S \times T$ of S and T is defined as:

$S \times T = \{(s,t) \mid s \in S,\ t \in T\}$. Prove the following assertions:

a) $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ is countably infinite

pf: let the map $f : \mathbb{N}^2 \longrightarrow \mathbb{N}^2$ where $\mathbb{N}^2 = \{(1,1),(1,2),\dots,(1,n),(2,1),\dots,(2,n), \dots (n,1),(n,2),\dots,(n,n)\}$

we know $h : \mathbb{N} \times \mathbb{N} \Rightarrow h$ injective

$\Longrightarrow$ both countable

similarly $k : T \to \mathbb{N}$

can say $\exists \forall (s_k, t_l) \in S \times T$

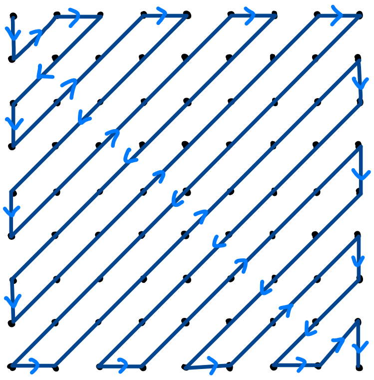
so can say $f(s_k, t_l) = \frac{(k+l-1)(k+l-2)}{2} + \frac{l+(-1)^k}{2} k + \frac{l+(-1)^{k+l-1}}{2} l$

then f gives enumeration of $S \times T$ where $S \times T = \mathbb{N} \times \mathbb{N} = \mathbb{N}^2$

bijection $F : \mathbb{N} \to \mathbb{N}^2 \Rightarrow f^{-1}$

$$\begin{tikzcd}
(s_1,t_1) \arrow[d] & (s_1,t_2) \arrow[r] & (s_1,t_3) \arrow[dl] & (s_1,t_4) \\
(s_2,t_1) \arrow[ur] & (s_2,t_2) \arrow[dl] & (s_2,t_3) \arrow[ur] & \\
(s_3,t_1) \arrow[d] & (s_3,t_2) \arrow[ur] & & \\
(s_4,t_1) \arrow[ur] & & &
\end{tikzcd}$$

gives the general shape:



$\square$

b) The Cartesian product $S \times T$ of two countable sets S and T is countable

pf: if $T \neq \{0\}$ then $\rightarrow$ T is countable

$\rightarrow$ surjective function $f: \mathbb{N} \rightarrow T$

$\rightarrow$ injective function $g: T \rightarrow \mathbb{N}$

S and T non-empty sets then $\exists$ surjective $f: \mathbb{N} \rightarrow S$ and $g: \mathbb{N} \rightarrow T$

suppose $\exists$ function $h: \mathbb{N} \times \mathbb{N} \rightarrow S \times T$ s.t. $h(n,m) = (f(n), g(m))$

choose $(a,b) \in S \times T$ be arbitrary since $a \in S$ and $b \in T$

and f, g surjective functions $\exists\, p, q \in \mathbb{N}$ s.t. $f(p) = a$ & $g(q) = b$

thus $h(p,q) = (f(p), g(q)) = (a,b)$

and h is surjective from a countable set $\mathbb{N}^2 \rightarrow S \times T$

$\Rightarrow$ $S \times T$ countable. $\square$

c) The cartesian product $S_1 \times S_2 \times \cdots \times S_k$ of finitely many countable sets $S_1, S_2, \ldots, S_k$ is countable.

pf: let $S_1, S_2, \ldots, S_k$ be the countable infinite sets

so we define $Y_n = S_1 \times \cdots \times S_k$ when $n = 1, \ldots, k$

thus $Y_k := S_1 \times S_2 \times \cdots \times S_k$

by induction

if $k=1$ then the set $Y_1 = S_1$ is countably infinite

assume Y_n $(n-k,\ 1 \leq n \leq k)$ is countably infinite

then $Y_{n+1} = (S_1 \times S_2 \times \cdots \times S_n) \times S_{n+1}$

$\Rightarrow Y_{n+1} = Y_n \times S_{n+1}$

where Y_n and S_{n+1} both countably infinite

from Q2(b) $\Rightarrow$ Y_{n+1} is countably infinite

Therefore, the cartesian product of a finite number of countable infinite sets is countable too

$\square$

Q3) Prove the following:

a) Any subset T of a countable set S is countable.

pf: suppose $T \subseteq S$

$\exists$ injection $f: S \to \mathbb{N}$

let $i: T \to S$ be the inclusion mapping

we have that i is injection and composite of injections is an injection $\Rightarrow f \circ i: T \to \mathbb{N}$ is an injection.

Hence T is countable $\square$

b) Every infinite subset T of a countably infinite set S is countably infinite.

pf: sps $\exists$ an injection $f: \mathbb{N} \to S$

let T be img of f, $f^{-1}: T \to \mathbb{N} \Rightarrow$ bijection

hence T is a countably infinite subset of s $\square$

c) The set $\mathbb{Z}$ of integers is countable.

answered from Q9(c).

d) If S_n is a countable set for each $n \in \mathbb{N}$ then the union $\bigcup_{n \in \mathbb{N}} S_n$ is a countable set.

pf: $\forall n \in \mathbb{N}$ let F_n be set of all surjection from $\mathbb{N}$ to S_n

since S_n is countable

$\Rightarrow F_n$ is not empty.

$\exists$ a sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ s.t. $f_n \in F_n \quad \forall n \in \mathbb{N}$

let $\varphi: \mathbb{N} \times \mathbb{N} \to S$

surjection defined by $\varphi(m,n) = f_m(n)$

since $\mathbb{N}^2$ is countable

$\Rightarrow \exists$ surjections $\alpha: \mathbb{N} \to \mathbb{N}^2$

composition of surjection $\varphi \cdot \alpha: \mathbb{N} \to S \Rightarrow$ surjection $\varphi \cdot \alpha$

by surjection from Natural numbers iff countable, then S is countable $\square$

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If s is any set

$\mathcal{P}(s) = \{A \mid A \subseteq s\}$ (set of all subsets)

S finite $\Rightarrow |\mathcal{P}(s)|$ finite

$S = \{\}$	$\{\}$	1
$\{1\}$	$\{\}, \{1\}$	2
$\{1,2\}$	$\{\}, \{1\}, \{2\}, \{1,2\}$	4
$\{1,2,3\}$	$\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}, \{1,2\}, \{2,3\}, \{1,2,3\}$	8

$$\boxed{|\mathcal{P}(s)| = 2^{|s|}}$$

$$\begin{array}{c} \quad 1 \quad 2 \quad 3 \\ \{(0,0,0), \\ (1,0,0), \\ (0,1,0), \\ (0,0,1), \\ (1,1,0), \\ (0,1,1), \\ (1,0,1), \\ (1,1,1)\} \end{array}$$

size $= 2^3$

$\{\} = (0,0,0)$

$\{1\} = (1,0,0)$

$\{1,2\} = (1,1,0)$

Part 2:

Q4) (Galileo's Paradox, 1638) The subset of perfect squares $S = \{n^2 : n \in \mathbb{N}\} \subset \mathbb{N}$ has the same cardinality as the set $\mathbb{N}$ of natural numbers i.e is countably infinite

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$

$S = \{1, 4, 9, 16, \dots\}$

Let $f : \mathbb{N} \to S$ defined as $n \in \mathbb{N}$

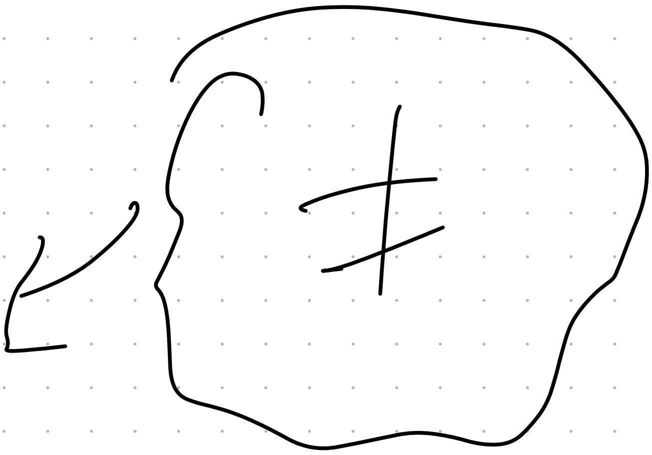
s.t. $f(n) = n^2$

f well defined and n^2 is a perfect square

let $n, m \in S$ s.t $f(n) = f(m)$

$$f(n) = f(m)$$
$$n^2 = m^2$$
$$n = m$$

$\Rightarrow f$ is injective



pre image

let us consider the inverse f^{-1}

so $f^{-1} : S \to \mathbb{N}$, $\forall n \in \mathbb{N} : f^{-1}(n) = \sqrt{n}$

f^{-1} well defined as n is square rootable and so $\sqrt{n} \in \mathbb{N}$

then: $f^{-1}(n) = f^{-1}(m)$

$$\sqrt{n} = \sqrt{m}$$
$$n = m$$

$\Rightarrow f^{-1}$ injective

bijective

if f^{-1} and f are both injective (one-to-one) then we can see that $|S| = |\mathbb{N}|$

so $S = \{n^2 : n \in \mathbb{N}\}$ is countably infinite

by Cantor - Bernstein - Schröder Theorem.

More generally n^k would still have an injection with $n \in \mathbb{N}$

since n^k can be k-rooted to find n anywhere

vice versa. from n we can do anything to it since we know where we started from.

generally perfect k-times

∎

Q5) What is the cardinality of the set $\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, \dots\}$ of all prime numbers? Justify.

we can set up a map between the integer numbers $\mathbb{Z}$ and $\mathbb{P}$ and we can show $\exists$ bijection.

① its easy to see that $\mathbb{P} \subset \mathbb{N}$

the following map f from the set of natural numbers is such a bijection:

$$f(0) = 2$$
$$f(1) = 3$$
$$f(-1) = 5$$
$$f(2) = 7$$
$$f(-2) = 11$$
$$\vdots$$

and so on, easy to see that we can enumerate primes like integers

In general: if $n = 0 \Rightarrow f(n) =$ first prime

$n > 0 \Rightarrow f(n) = (2n)$ th prime

$n < 0 \Rightarrow f(n) = (2|n|)$ th prime

$$|\mathbb{Z}| = \aleph_0 = |\mathbb{P}|$$

② $\mathbb{P} \subseteq \mathbb{N} \Rightarrow |\mathbb{P}| \leq |\mathbb{N}|$

$\Rightarrow \exists f : \mathbb{P} \to \mathbb{N}$ s.t $f(x) = x$

$\Rightarrow f$ is injection

since $\mathbb{P} \leq \mathbb{N}$ then $\exists\, g : \mathbb{P} \to \mathbb{N}$ s.t. $g(n) = n + 1^{th}$ prime

$\Rightarrow g$ is injection

$\Rightarrow |\mathbb{P}| = |\mathbb{N}|$ by Schroeder - Bernstein Thm.

■

Part 3:

Q1) a) Power set $\mathcal{P}(S)$, $S = \{1,2,3,4,5,6\}$

$\mathcal{P}(S) = \{\}, \{1\}, \{2\}, \dots, \{6\},$

$\{1,2\}, \{1,3\}, \dots, \{1,6\}, \{2,1\}, \{2,2\}, \dots, \dots, \{6,6\},$

$\{1,2,3\}, \{1,2,4\}, \dots, \{1,2,6\}, \dots, \{1,6,2\}, \dots, \dots, \{1,5,6\}, \{1,6,5\},$

$\{1,2,3,4\}, \dots,$

$\{1,2,3,4,5,6\}$

$|\mathcal{P}(S)| = 2^6$

b) $\mathcal{P}(S)$ for $S = \emptyset$

$\mathcal{P}(S) = \{\}$

$|\mathcal{P}(S)| = 1$

Q2) S is a finite set with k elements then the p.s $\mathcal{P}(S)$ has cardinality $|\mathcal{P}(S)| = 2^k$

Let S denote arbitory set s.t. $|S| = n$
then $|\mathcal{P}(S)| = 2^n$

base case : $|S| = 0$, clearly $S = \emptyset$. But the empty set is the only subset of itself, so $|\mathcal{P}(S)| = 1 = 2^0$

Suppose $|S| = n$, by induction hypothesis we know that $|\mathcal{P}(S)| = 2^n$.

let T be set with $n+1$ elements, $T = S \cup \{a\}$.

two kinds of subsets of T s.t. those that include a and those that don't

- subset of $S \to 2^n$ of them
- sets of the form $Z \cup \{a\}$ where $Z \in \mathcal{P}(S)$; since there are 2^n possible choices for Z

there must be exactly 2^n subsets of T of which a is an element

$\Rightarrow |\mathcal{P}(T)| = 2^n + 2^n = 2^{n+1}$

$\blacksquare$ $\quad |S| < |\mathcal{P}(S)|$

Q3) Show that if S is any set (not necessarily finite), there is a bijection between the power set $P(S)$ and the set 2^S of all functions from S to $\{0,1\}$, i.e $2^S = \{f : S \longrightarrow \{0,1\}$

suppose $f: S \rightarrow P(S)$, surjection

indicator of function $1_s : S \Rightarrow \{0,1\}$ defined by $1_s(x) = \begin{cases} 1 & \text{if } x \in s \\ 0 & \text{otherwise} \end{cases}$

if suffices to show that we can point to an indicator function $1_A : S \Rightarrow \{0,1\}$

not form of $1_{f(x)}$ then $s \in P(S)$ is not of the sets $f(x)$ and $\Rightarrow f$ cannot surjective

define $1_s(x) := 1 - 1_{f(x)}(x) \Rightarrow f$ not surjective

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$\mathbb{P}(S) \longleftrightarrow \{0,1\}^S$

$|\mathbb{P}(S)|$ size $2^{|S|}$

uncountable, no bijection

$|\mathbb{P}(S)| > |\mathbb{N}|$ new type of infinity (cantors theorems)

S any set $\nexists\ f: S \longrightarrow \mathbb{P}(S)$, surjective

so $|\mathbb{P}(S)| > |S|$

S finite, clear cardinalities:

$$|\mathbb{P}(S)| = 2^{|S|} > |S|$$

Part 3:

Q4) (Cantor's Theorem, 1891) Let f be a map from set A to its power set $P(A)$. Then $f: A \to P(A)$ is not surjective. As a consequence, $|A| < |P(A)|$ holds for any set A.

pf: consider the set $B = \{x \in A \mid x \notin f(x)\}$

suppose to the contrary that f is surjective. Then $\exists\, a \in A$ s.t. $f(a) = B$.

but by construction, $a \in B \iff a \notin f(a) = B$

$\searrow$ contradiction $\cdot\!\times\!\cdot$

$\Rightarrow f$ cannot be surjective

o.t.o.h, $g: \begin{matrix} A \to P(A) \\ x \mapsto \{x\} \end{matrix}$ is an injective map.

$\Longrightarrow |A| < |P(A)|$ $\square$

Q5) $P(\mathbb{N})$ of the set of natural numbers is not countable.

pf: by previous question, cantors theorem

$\nexists$ bijection from a set to its power set.

if $\mathbb{N} = \emptyset$, the empty map suffices as it's vacuously an injection

let $f: \mathbb{N} \to P(\mathbb{N})$ be the mapping defined as $\forall n \in \mathbb{N}: f(n) = \{n\}$

let $n, m \in \mathbb{N}$ s.t. $f(n) = f(m)$
then $\{n\} = \{m\}$
$\Rightarrow n = m$

$\Rightarrow f$ injective

from Cantor-Bernstein-Schröder Theorem $f^{-1} = g: P(\mathbb{N}) \to \mathbb{N}$

$\Rightarrow P(\mathbb{N})$ is not bijective
$\Rightarrow$ not countable

$\square$

Q6) (Hilbert's Paradox of the Infinite Hotel)

Rm1 ⊠ Rm2 ⊠ Rm3 ⊠ Rm4 ⊠ .... Rm ∞ ⊠

move current guest before giving new guest a room

a) Someone arrives in the night at the asking for a room.

we move Rm1 → Rm2
Rm2 → Rm3
⋮
Rm(n) → Rm(n+1) ... (carries to infinity)

for finite # of people

then we move everyone down one room, so now room 1 is vacant, for the new guest.

b) a coach with a countably infinite number of people arrives.

we can move every guest into a new room which is double their current room number so

Rm 1 → Rm 2
Rm 2 → Rm 4
Rm 3 → Rm 6
Rm 4 → Rm 8
⋮
Rm(n) → Rm(2n) since the set of even numbers is countably infinite
⋮

now every odd numbered rooms are free to vacate since we also know that the odd number is also countably infinite.

c) Countably infinite many coaches with countably infinite many people arrive. can they be accommodated?

Prime numbers, countably infinite → proven previously

- assign current guests to the prime number 2 raised to the power of their current room number.
- we use the next prime for each bus, and raise it to the power of seat number on the bus:

bus 1 ← prime number, 3

bus 2 ← prime number, 5

⋮

bus t ← $t+1$ prime number

overall, $(P_{t+1})^M$ where m is assigned seat number on the bus.

↳ all the buses passengers fan out into rooms based on prime numbers

room 6 not filled, not a power of prime

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The questions set was P4: 1, 2, 4
P4: 4 is the final question we answered and didnt get onto Part 5.

(4) $\exists\, \mathbb{N} \xrightarrow[\text{inj.}]{\text{sur.}} \mathbb{P}(\mathbb{N})$

$\Rightarrow \nexists\, \mathbb{N} \xrightarrow{\sim} \mathbb{P}(\mathbb{N})$

$\Rightarrow \mathbb{P}(\mathbb{N})$ is not countable

$\mathbb{N} = \aleph_0$

$|\mathbb{P}(\mathbb{N})| > \aleph_0$

$\mathbb{P}(\mathbb{N})$ is <u>uncountable</u>

$\mathbb{R} = \{a_1 a_2 a_3 \dots a_n \cdot b_1 \dots b_n \mid$ where $a_1 \neq 0,\ n \in \mathbb{N},\ 0 \leq a_i, b_i \leq 9$

all possible lim of sequences of rational number

$\sqrt{2} \in \mathbb{R},\ \sqrt{2} \notin \mathbb{Q}$

$\Rightarrow \mathbb{R} \supsetneq \mathbb{Q}$

$\mathbb{Q} \cup \{\sqrt{2}\}$ countable
$\mathbb{Q} \cup \{\sqrt{p}\}$: p prime

sps. can order $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$

e.g.

1	0.132487
2	0.196732
3	0.2739...
4	$\pi/4$
⋮	

1	0 1 0 1 0 1 0 0 ...
2	1 1 1 0 1 0 0 1 ...
3	0 0 1 0 0 1 0 1 ...
(new*) 4	

4 constructed by 0 1 1 ...
↓
1 0 0 ...

can keep making new number since every new entry into set is a new number, never seen before

Part 4:

Q1) Prove the following theorem:

Theorem 4.1 (Cantor, 1874). The set of real numbers $\mathbb{R}$ is uncountably infinite. i.e the cardinality of the continuum $c = |\mathbb{R}|$ is strictly greater than the card($\aleph_0$) of the natural numbers $\mathbb{N}$ and so $c > \aleph_0$.

pf: sps that $[0,1]$ is countable. Clearly $[0,1]$ is not a finite set, so we are assuming that $[0,1]$ is countably infinfinite

then $\exists$ bijection from $\mathbb{N}$ to $[0,1]$ we can create an infinite list which contains every real number

e.g.

$$\begin{array}{ll} 1 & 0.02342 \\ 2 & 0.32434 \\ 3 & 0.50000 \\ 4 & 0.20342 \\ \vdots & \vdots \end{array}$$

let N be the number obtained. obtained as follows. for each $n \in \mathbb{N}$ let n^{th} decimal spot of N be equal to the n^{th} decimal spot of n^{th} number in the list $+1$ if < 9

$= 0$ if $= 9$

by construction, N different from every number in our list

$\Rightarrow$ list is incomplete $\therefore \not\exists$ of a bijection : $\mathbb{N} \to [0,1]$

$\Rightarrow [0,1]$ uncountable

and now since $[0,1]$ is uncountable and $[0,1] \subset \mathbb{R}$ we have that $\mathbb{R}$ is uncountable $\square$

Q2) Show that the cardinality of the set T of all infinite sequences having only 0s and 1s is uncountably infinite with cardinality $|T| = 2^{\aleph_0}$ using cantor's diagonal argument.

pf: follows from Q1, instead of creating an infinite list which contains real number we use 0 and 1.

e.g.
$$\begin{array}{ll} 1 & 0\ 0\ 1\ 0\ 1\ 1 \\ 2 & 1\ 0\ 0\ 0\ 1\ 1 \\ 3 & 0\ 0\ 1\ 1\ 0\ 1 \\ 4 & 0\ 0\ 0\ 1\ 0\ 1 \\ \vdots & \vdots \end{array}$$

now instead of +1 if <9 or =0 if 9. we simple flip the 0s and 1s

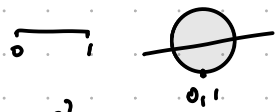
$\Rightarrow$ uncountable $\square$

Q4) Describe geometrically a bijective correspondant between the half open unit interval $(0,1]$ and the real line $\mathbb{R}$.

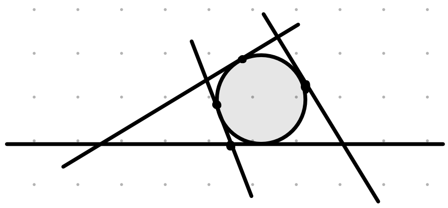
shows $(0,1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$

really want $\mathbb{R}$ is uncountable, turns out there is a bijection

$(0,1] \to \mathbb{R}$



$f: (0,1] \to \mathbb{R}^2$
$x \longmapsto (\cos(2\pi x), \sin(2\pi x))$



tangent lines

References:

all proofs are taken from: proofwiki.org