Total Derivative:

Another notion of derivative for a vector field is the total derivative.

Let $\underline{F} = (F_1, F_2, F_3)$ be a (differential) vector field with variables x, y and z. The total derivative is then defined as the matrix

$$\frac{\partial(F_1, F_2, F_3)}{\partial(x,y,z)} = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{pmatrix}$$

the total derivative contains a complete information concerning the rate of change of the vector field $\underline{F}$

Remainder:

matrix and vector product (matrix times vector = vector)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{matrix} A \\ B \end{matrix} = \begin{pmatrix} aA & bB \\ cA & dB \end{pmatrix}$$

Total differential of a function:

$$df(x_0, y_0) = \frac{\partial f(x_0, y_0)}{dx} dx + \frac{\partial f(x_0, y_0)}{dy} dy$$

Total derivative of a vector function:

$$d\vec{F} = \begin{pmatrix} dF_1 \\ dF_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy \\ \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Jacobian:

We also define the the Jacobian of the vector field $\underline{F}$ as the determinant of the total derivative.

$$J = \left| \frac{\partial(F_1, F_2, F_3)}{\partial(x,y,z)} \right| = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix}$$

similarly for 2 dimensions

Jacobian - Polar Coordinates:

we want to make the following change of variables:

$$(x,y) \longrightarrow (r,\theta)$$

what is the Jacobian of this change of variables?

The Jacobian reads

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Jacobian - Spherical Coordinates:

for the following change of variables

$$x, y, z \rightarrow r, \theta, \varphi$$

the Jacobian reads:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} x_r & x_\theta & x_\varphi \\ y_r & y_\theta & y_\varphi \\ z_r & z_\theta & z_\varphi \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\cos\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$\boxed{= r^2\sin\theta}$$

Change of Variables in the Laplacian

Example (polar coordinates)

$x = r\cos\varphi \quad y = r\sin\varphi \qquad u(x,y) \leftrightarrow g(r,\theta)$

find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ functions of $\frac{\partial g}{\partial r}$ and $\frac{\partial g}{\partial \varphi}$

the solution is

$$\frac{\partial u}{\partial x} = \frac{\partial g}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial g}{\partial \varphi}\frac{\partial \varphi}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial g}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial g}{\partial \varphi}\frac{\partial \varphi}{\partial y}$$

$r = \sqrt{x^2+y^2}$; $\varphi = \arctan\left(\frac{y}{x}\right)$

the derivatives are:

$$\frac{\partial r}{\partial x} = \cos\theta \quad \frac{\partial \varphi}{\partial y} = -\sin\varphi/r \qquad \frac{\partial r}{\partial y} = \sin\varphi \quad \frac{\partial \varphi}{\partial y} = \cos\varphi/r$$

thus $\boxed{\frac{\partial u}{\partial x} = \frac{\partial g}{\partial r}\cos\varphi + \frac{\partial g}{\partial \varphi}(-\sin\varphi/r) \qquad \frac{\partial u}{\partial y} = \frac{\partial g}{\partial r}\sin\varphi + \frac{\partial g}{\partial r}\cos\varphi/r}$

The Laplacian in Polar Coordinates:

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 g}{\partial r^2} + \frac{\partial g}{\partial r}\frac{1}{r} + \frac{\partial^2 g}{\partial \varphi^2}\frac{1}{r^2}}$$

The Laplacian in Cylindrical Coordinates:

$$\boxed{= \frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \varphi^2} + \frac{\partial^2 g}{\partial z^2}}$$