18/1/23

Classification of DE

① $\frac{dy}{dt} = F(y(t), y) \Rightarrow$ non-autonomous ODE

② $\frac{dy}{dt} = F(y(t)) \Rightarrow$ autonomous ODE

I) Ordinary and Partial DE $(y = f(x))$

$\frac{dy}{dx} \quad \frac{d^2y}{dx^2} \quad \frac{d^3y}{dx^3} \cdots \frac{d^ny}{dx^n}$

$F(t, u(t), u'(t), \dots, u^{(n)}(t))$

II) PDE: $y = f(t, x_1, x_2, \dots, x_n)$

$F = (t, y, y'_t, y''_{x_1 x_n}, \dots, y'''_{x_1 x_3 x_n})$

Example 1: Charge $Q(t)$ on a capacitor $L \frac{d^2Q(t)}{dt} + R\frac{dQ(t)}{dt} + \frac{1}{C} Q(t) = E$

Example PDE: The Heat Condution Equation $\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$

III) Order: <u>Highest</u> derivative order in DE

$F(t, u(t), u'(t), \dots, u^{(n)}(t)) = 0 \rightarrow$ ODE of n-th order

V) Linear and Nonlinear DE:

$a_0(t) \cdot y^{(n)} + a_1(t) \cdot y^{(n-1)} + \dots + a_n(t) \cdot y = g(t)$

Example PDE: The Heat Condution Equation $\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$

$\frac{\partial^3 u(x,t)}{\partial x^2 \, dt}$

$y(t), x(t), z(t)$

$$\begin{cases} \frac{dx}{dt} = ax - axy \\ \frac{dy}{dt} = -cy + \gamma x \cdot y \end{cases}$$

$+1 \; \frac{dz}{dt} = F(x, y, z, t)$

Important Questions:

① **Solutions:** $F(t, \phi(t), \phi'(t), \dots, \phi^{(n)}(t)) = 0$

↓ is a real-valued
with real-valued solution $y = \phi(t)$

$y_1 = \phi(t)$

② **Question of Existence**

- Certain Restrictions on F

③ **Question of Uniqueness**

④ **Can we actually determine a solution?**

⇒ Not always solution expressible in terms of Elementary functions

Example: 10^{th} slide

1.

First Order DE

- Linear ⊛
- Seperable
- Exact

} Method for solving

$$\boxed{\frac{dy}{dt} + p(t)\cdot y = g(t)}$$ standard form of General FODE

$p(t)$ and $g(t)$

⁎ immediate integration

Example: $(4+t^2)\frac{dy}{dt} + 2t\cdot y = 4t$

$$\frac{dy}{dt}\left[(4+t^2)\cdot y\right] = \frac{dy}{dt}\cdot(4+t^2) + 2t\cdot y$$

$$\int \frac{d}{dt}\left[(4+t^2)\cdot y\right] = \int 4t \; dt$$

$$(4+t^2)y + C_1 = 2t^2 + C_2$$

$$\boxed{y = \frac{2t^2}{4+t^2} + \frac{C}{4+t^2}}$$ is the general solution

Integrating Factor:

- Left sides are not Derivative of Product.
- If the DE is multiplied by a $\mu(t)$, called the Integrating Factor, then our DE is converted into one.

$$\frac{dy}{dt} + p(t)\cdot y = g(t) \qquad (\times \mu(t))$$

$$\underbrace{\mu(t)\cancel{\frac{dy}{dt}} + p(t)\cdot\mu(t)\cdot\cancel{y}} = g(t)\cdot\mu(t$$

is Derivative of the product

$$[\mu(t)\cdot y]' = \cancel{\frac{dy}{dt}}\cdot\cancel{\mu(t)} + \frac{d\mu(t)}{dt}\cdot\cancel{y}$$

$$\frac{d\mu(t)}{dt} = p(t)\cdot\underbrace{\mu(t)}$$

is positive

$$\int \frac{d\mu(t)}{\mu(t)} = \int p(t)\cdot dt \qquad \boxed{c_1 = 0}$$

$$\ln[\mu(t)] = \int p(t)\cdot dt + C_1$$

$$\boxed{\mu(t) = e^{\int p(t)\,dt}}$$

$$\int \frac{d[\mu(t)\cdot y]}{dt} = \int \mu(t)\cdot g(t)$$

$$\mu(t)\cdot y = \int \mu(t)\cdot g(t)\,dt + C \rightarrow \boxed{y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)\cdot g(s) + C\right]} \Rightarrow \text{general solution}$$

Example: $t\cdot y' + 2y = 4t^2 \qquad y(1) = 2$

standard Form $\Rightarrow y' + \underbrace{\frac{2}{t}}_{p(t)}\cdot y = 4t = g(t)$

$$\mu(t) = e^{\int p(t)dt} = e^{\int 2/t\,dt} = e^{2\cdot \ln|t|} \overset{t>0}{=} e^{\ln|t^2|} = t^2$$

$$t^2\cdot y' + 2t^2\cdot y = (t^2\cdot y)' = 4t^3$$

$$\int (t^2\cdot y)' = \int 4t^3$$

$$t^2\cdot y = t^4 + C$$

$\boxed{y = t^2 + \frac{c}{t^2}}$ is General Solution

IVP $y(1) = 2 \Rightarrow 2 = 1^2 + \frac{c}{1^2} \Rightarrow \boxed{c = 1}$

$\boxed{y_{IVP} = t^2 + \frac{1}{t^2}, \quad t > 0}$ particular solution

$t^2 \neq 0$

$\boxed{t \neq 0}$

$\lim_{t\to 0^+} t^2 + \frac{c}{t^2}$

$c = 0 \quad y = t^2$

Separable Equations:

$$\frac{dy}{dt} = f(x,y)$$

$$\underbrace{M(x,y)}_{-f(x,y)} + \underbrace{N(x,y)}_{1}\frac{dy}{dx} = 0$$

$$\boxed{M(x) + N(y)\frac{dy}{dx} = 0}$$

$$M(x)\,dx + N(y)\,dy \Rightarrow \int M(x)\,dx = \int -N(y)\,dy$$

Example: Show that the equation

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

is separable, and then find an equation for it's integral curves.

$$-x^2 + (1-y^2)\cdot\frac{dy}{dx} = 0$$

$$\frac{d}{dx} f(y) = \frac{d}{dy} f(y)\cdot\frac{dy}{dx} = f'(y)\cdot\frac{dy}{dx}$$

$$\boxed{f(y(x))}$$

$$\frac{d}{dx}\left(y - \frac{1}{3}y^3\right) = (1-y^2)\frac{dy}{dx}$$

$$\frac{d}{dx}\left[-\frac{x^3}{3}\right] + \frac{d}{dx}\left(y - \tfrac{1}{3}y^3\right) = 0$$

$$\frac{d}{dx}\left[-\frac{x^3}{3} + y - \tfrac{1}{3}y^3\right] = 0$$

by integrating, $-x^3 + 3y - y^3 = C$