MA1114

Linear Algebra

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Questions: 1

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Question 1

a) V vector space, $S = \{v_1, v_2, \ldots, v_k\}$, S vectors in V

S is linearly independant if

$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k = 0$

with $\lambda_i \in \mathbb{R}$, implies that $\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_k = 0$

span of S = span(S) = span $(v_1, v_2, \ldots, v_k)$

$= \langle v_1, v_2, \ldots, v_k \rangle$

is the set of all linear combinations of the vectors in S.

b) ~~suppose~~ S linearly independent, $v \in V$ s.t. $v \notin$ span(S)

prove $S \cup \{v\} = \{v_1, v_2, \ldots, v_k, v\}$ is linearly independant

suppose $\left(\sum_{i=1}^{k} a_i v_i\right) + av = 0$

and $(a_1, \ldots, a_k, a) \neq (0, 0, \ldots, 0)$

$\sum_{i=1}^{k} a_i v_i = -av$

$(-1) \sum_{i=1}^{k} a_i v_i = av$

when $a = 0$

$(-1) \sum_{i=1}^{k} a_i v_i = 0 \times v$

$(-1) \sum_{i=1}^{k} a_i v_i = 0$

$\sum_{i=1}^{k} a_i v_i = 0$

$\Rightarrow$ linearly independant.

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Question 1 ~~ctd~~

b) ~~ctd~~ --

when $a \neq 0$

~~then $(-1)\sum_{i=1}^{k} a_i v_i = av \neq 0$~~

$\Rightarrow (-1)\sum_{i=1}^{k} a_i v_i = av \neq 0$

~~so if S is linearly independant~~

~~$\Rightarrow \sum_{i=1}^{k} a_i v_i + av = 0$~~

$av \neq 0 \Rightarrow v \in span(S)$

$\Rightarrow$ linearly independant (by COGOF)

Question 1 ctd...

c) $V = \mathbb{R}^3 \quad S = \{v_1, v_2\}$

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \notin \operatorname{span}(S)$$

$B = \{v_1, v_2, v\}$ is a basis of V.

$\lambda_1 v_1 + \lambda_2 v_2 = v$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{array}\right]$$

$x = -1$

$y = 1$

$0x + 0y = 2$ ✗. $0 \neq 2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3 pivots $\Rightarrow$ it's a basis

Question 1 ctd...

d) $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in V$, determine the coordinate vector $[x]_B$ of x.

$$Ax = b \qquad [x]_B \qquad A[x]_B = x$$

$$\lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V = x$$

$$\lambda_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 \\ 0 \\ \lambda_1 \end{bmatrix} + \begin{bmatrix} \lambda_2 \\ \lambda_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \lambda_3 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\cancel{\lambda_1 + \lambda_2 = x_1} \qquad \cancel{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}$$
$$\cancel{\lambda_2 + \lambda_3 = x_2}$$
$$\cancel{\lambda_1 + \lambda_3 = x_3}$$

$$\cancel{[x]_B =}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & x_1 \\ 0 & 1 & 1 & x_2 \\ 1 & 0 & 1 & x_3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & x_1 \\ 0 & 1 & 1 & x_2 \\ 0 & -1 & 1 & x_3 + x_1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & x_1 \\ 0 & 1 & 1 & x_2 \\ 0 & 0 & 2 & x_3 - x_1 + x_2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & x_1 \\ 0 & 1 & 0 & \frac{x_3 - x_1 + x_2}{2} \\ 0 & 0 & 1 & \frac{x_3 - x_1 + x_2}{2} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3x_1 + x_2 + x_3}{2} \\ 0 & 1 & 0 & \frac{x_3 - x_1 + x_2}{2} \\ 0 & 0 & 1 & \frac{x_3 - x_1 + x_2}{2} \end{array}\right]$$

$$[x]_B = \begin{bmatrix} \frac{3x_1 + x_2 + x_3}{2} \\ \frac{x_3 - x_1 + x_2}{2} \\ \frac{x_3 - x_1 + x_2}{2} \end{bmatrix}$$

Question 1 ctd...

e) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$\det(tI - A) = \det\left(\begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix} - \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} t-1 & -1 & 0 \\ 0 & t-1 & -1 \\ -1 & 0 & t-1 \end{pmatrix}$$

$= (t-1)^3 - 1 = 0$

$= (t-1)^3 = 1$

$= t-1 = 1$

$\boxed{\Rightarrow t = 2}$

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Question 2

a) $A \in M_{mn}(\mathbb{R})$

~~A is an inverse when m=n and B is~~

An inverse of A when $m = n$ is a matrix B of the same size such that $BA = AB = I$, where I is the identity matrix of the same size.

b) $\neg$ invertible $\Rightarrow$ matrix with a zero column.

let $A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow & & \uparrow \\ v_1 & v_2 & \cdots & v_k & \cdots & v_n \\ \downarrow & \downarrow & & \downarrow & & \downarrow \end{pmatrix}$ be a square matrix

where $v_k = 0$, a zero column.

Then any matrix B of the ~~matrix~~ same size, the product.

$$BA = \begin{pmatrix} \uparrow & & \uparrow & & \uparrow \\ Bv_1 & \cdots & Bv_k & \cdots & Bv_n \\ \downarrow & & \downarrow & & \downarrow \end{pmatrix} = \begin{pmatrix} \uparrow & & \uparrow & & \uparrow \\ Bv_1 & \cdots & 0 & \cdots & Bv_n \\ \downarrow & & \downarrow & & \downarrow \end{pmatrix}$$

also has a zero column, so it can never be a identity matrix.

c) i) $\begin{bmatrix} 22/7 & 1 \\ \pi & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 7/22 \\ \pi & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 7/22 \\ 0 & 1-\frac{7\pi}{22} \end{bmatrix} \to \cdots$

$\to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\Rightarrow$ is invertible.

ii) $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 5 & 8 & 13 & 21 \\ 34 & 55 & 89 & 144 \end{bmatrix}$ $\Rightarrow \neg$ invertible since determinant is 0 by ~~sw~~ cofactor expansion after swapping R_1 and R_2

iii) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $\Rightarrow \neg$ invertible since not a square matrix

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Question 2 ctd...

c) iv) det $D = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{bmatrix}$

$$\det(D) = (-1)^2 d_{11} D_{11} + (-1)^3 d_{12} D_{12} + (-1)^4 d_{13} D_{13} + (-1)^5 d_{14} D_{14} + (-1)^6 d_{15} D_{15}$$

$$= 0 + 0 + 0 + 0 + 0$$

$\det(D) = 0$ $\Rightarrow \neg$ invertible since $\det(D) = 0$

d) $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & +1 & 1 & -1 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 2 \end{array}\right]$$

so $A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

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Question 2 ctd...

e) i) $T: \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 5x_1 - x_3 \\ 5x_2 - x_3 \\ 2x_2 + 2x_3 \end{pmatrix}$

$U, V, W = \mathbb{R}^3, \mathbb{R}^3$ $\quad E = \left\{ \underset{e_1''}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{e_2''}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{e_3''}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$

$\left[T(e_1)\right]_E = T(e_1) = T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$

$\left[T(e_2)\right]_E = T(e_2) = T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$

$\left[T(e_3)\right]_E = T(e_3) = T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

$_E[T]_E = \begin{bmatrix} 5 & 0 & -1 \\ 0 & 5 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

ii) $B = \left\{ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\left[T(v_1)\right]_B = T\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$

$\left[T(v_2)\right]_B = T\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix}$

$\left[T(v_3)\right]_B = T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$

$_B[T]_B = \begin{bmatrix} 4 & 5 & 5 \\ 4 & 5 & 0 \\ 4 & 2 & 0 \end{bmatrix}$

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Question 2 ctd...

e) iii) $[T]_E = \begin{bmatrix} 5 & 0 & -5 \\ 0 & 5 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

$$\det([T]_E) = 5 \times (10 - 2)$$
$$= 5 \times 8$$
$$= 40$$

$\Rightarrow$ invertible $\quad \det \neq 0$

$[T]_B = \begin{bmatrix} 9 & 5 & 5 \\ 4 & 5 & 0 \\ 4 & 2 & 0 \end{bmatrix}$

$$\det([T]_B) = 5 \times (8 - 20)$$
$$= 5 \times -12$$
$$= -60$$

$\Rightarrow$ invertible $\quad \det \neq 0$

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Question 8

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

i) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \mapsto R_3 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 \mapsto R_3 + R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

ii) rank $= 2$

nullity $= 1$

iii) $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \\ x_3 = 0 \end{cases} \qquad \begin{aligned} &\Rightarrow x_1 = -x_3 \\ &\Rightarrow x_2 = -x_3 \\ &\Rightarrow x_3 = 0 \end{aligned}$$

so basis $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

b) i) does not exist

since the dimensions is less than the amount of vectors

ii) $\{x^4, x^3, x^2, x, 5n\}$

iii) $\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

iv) $\{A \in M_{2,2}(\mathbb{R}) \mid A^T = A\}$

$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

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Question 3 ctd...

c) $T: \mathbb{R}^2 \to \mathbb{R}$
$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 \\ 3x_1 - x_2 \end{pmatrix}$

i) $E = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$[T(e_1)]_E = T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$[T(e_2)]_E = T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$T = [T]_E = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

ii) $\det(tI - T) = 0$, $\cancel{\chi_T(t)} = 0$ $\chi_T(t) = 0$

$\det\left(\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} - \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\right)$

$= \det\begin{bmatrix} t-1 & -1 \\ -3 & t+1 \end{bmatrix}$

$= ((t-1)(t+1)) - 3$

$= t^2 - 1 - 3 \quad = t^2 - 4$

$\Rightarrow t^2 - 4 = 0$

$t = \sqrt{4}$

$t = \pm 2$

eigenvalues: $2, -2$

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Question 3 ctd...

c) iii) T is diagonalisable

there some invertible P (a base change matrix) such that $T = P^{-1}DP$ since ~~eigenvalues exists~~ eigenvectors exists.

$t = 2$

$$\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} = B$$

$Bx = 0$

$$= \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_2 + 3R_1$

$x_1 - x_2 = 0 \implies x_2 = x_1$ suppose $x_1 = a$

eigenspace: $E_{t=2} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $t = 2$.

$t = -2$

$$\left(\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} - \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\right) = \begin{bmatrix} -3 & -1 \\ -3 & -1 \end{bmatrix} = B$$

$Bx = 0$

$$= \left[\begin{array}{cc|c} -3 & -1 & 0 \\ -3 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_1 + R_2$

$-3x_1 = x_2$
$x_1 = -\frac{1}{3}x_2$

suppose $x_2 = a$
eigenspace: $E_{t=-2} = a\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$x = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ for $t = -2$

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Question 3 ctd...

c) iii) ctd...

$$\lambda_1 x_1 + \lambda_2 x_2 = \cancel{\begin{pmatrix} x \\ x y \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & x \\ 1 & -3 & y \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & x \\ 0 & -4 & y-x \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & x \\ 0 & 4 & x-y \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 1 & x \\ 0 & 1 & \frac{x-y}{4} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{3x-y}{4} \\ 0 & 1 & \frac{x-y}{4} \end{array}\right]$$

$$\lambda_1 = \frac{3x-y}{4}$$

$$\lambda_2 = \frac{x-y}{4}$$

$$\text{basis} = \cancel{\begin{bmatrix} \frac{3x-y}{4} \\ \frac{x-y}{4} \end{bmatrix}} \begin{bmatrix} \frac{3x-y}{4} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{x-y}{4} \end{bmatrix}$$

iv) $P^{-1} [T]_E P = [T]_B$

$$P^{-1} \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ since ~~too~~ $[T]_B$ can be reduced.

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Question 4

a) The standard inner ~~product~~ product on $\mathbb{R}^n$ is the function : $\langle -, - \rangle : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\langle v, w \rangle = v^T w$$

suppose w is similar to v

$$= (v_1, \dots, v_n) \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

$$= v_1 w_1 + v_2 w_2 + \dots + v_n w_n$$

$$= \sum_{i=1}^{n} v_i w_i$$

b) $V^\perp = \{ w \in V \mid \langle v, w \rangle = 0 \}$ is subspace of V and determine $0^\perp$

① let $0 \in \mathbb{R}^n$ and $w_{in} = 0$

$$\langle v, w \rangle = \sum_{i=1}^{n} v_i w_i = \sum_{i=1}^{n} v_i 0 = \sum_{i=1}^{n} v_i \cdot \sum_{i=1}^{n} 0$$

$$= \left(\sum_{i=1}^{n} v_i\right) \times 0 = 0 \in V^\perp$$

② let $v, w \in \mathbb{R}^n$, $u+w \in S$

$$\langle v, u+w \rangle = \sum_{i=1}^{n} v_i (u+w)_i = \sum_{i=1}^{n} (v_i u_i + v_i w_i)$$

$$= \sum_{i=1}^{n} (v_i u_i) + \sum_{i=1}^{n} (v_i w_i)$$

$$= \langle v, u \rangle + \langle v, w \rangle \text{ so } 0 + 0 = 0$$

③ let $w \in V^\perp$ $\lambda w \in S$ $\lambda \in \mathbb{R}$

$$\langle v, \lambda w \rangle = \sum_{i=1}^{n} v_i (\lambda w)_i = \sum_{i=1}^{n} \lambda v_i w_i$$

$$= \lambda \sum_{i=1}^{n} v_i w_i$$

$$= \lambda \langle v, w \rangle \text{ so } 0 \times \lambda = 0$$

①, ②, ③ hold so it is a subspace.

$$0^\perp = \langle 0, 0 \rangle \Rightarrow \lambda 0 \times 0 = 0$$

$$\Rightarrow 0 + 0 = 0 \ \& \ 0 \in 0^\perp$$

so $0^\perp$ is orthogonal.

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Question 4 ctd

c)

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Question 4 ctd...

d) $B = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right.$

$$\hat{u}_1 = v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\hat{u}_2 = v_2 - \frac{\langle v_1 \hat{u}_1 \rangle}{\|\hat{u}_1\|^2} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{-1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}$$

$$\hat{u}_2 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\tilde{u}_3 = v_3 - \frac{\langle v_3, \hat{u}_1 \rangle}{\|\hat{u}_1\|^2} - \hat{u}_2 \frac{\langle v_3, \hat{u}_2 \rangle}{\|u_3\|^2}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{0}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{0}{4} \times \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\langle u_1 \; v_2 \rangle = 0$

$\langle u_1, u_3 \rangle = 0$

$\langle u_2 \; u_3 \rangle = 0$

$$u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$u_2 = \frac{1}{\|\hat{u}_2\|} \hat{u}_2 = -\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$u_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$