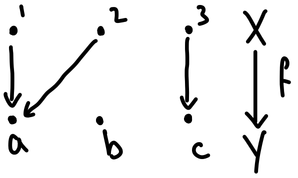
Maps $f: X \to Y$ (X, Y sets)
$x \mapsto f(x)$ ← <u>image</u> of x

<u>im f</u> $= f(x) = \{ f(x) \mid x \in X \} \subseteq Y$

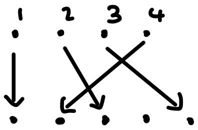
<u>preimage</u> of y is $f^{-1}(y) = \{ x \in X \mid f(x) = y \}$ ($y \in Y$)

<u>Definition 3.3:</u> $f: X \longrightarrow Y$ is <u>injective</u> if distinct elements $\longrightarrow$ distinct elements (equiv. if $f(x_1) = f(x_2)$ then $x_1 = x_2$)

Example: f in Ex 3.2 

is not injective

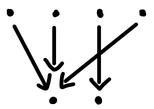
Note: f is injective $\Leftrightarrow \forall y \in Y$ the preimage $f^{-1}(y)$ is empty or singleton.

Example: 

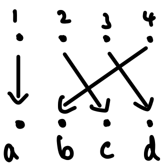
is injective

<u>Definition 3.4:</u> $f: X \to Y$ is <u>surjective</u> (<u>onto</u>) if $f(X) = Y$, i.e $\forall y \in Y\ \exists x \in X$ s.t. $f(x) = y$.

Note: f is surjective $\Leftrightarrow f^{-1}(y) \neq \phi \ \ \forall y \in Y$.

Example:  is surjective

<u>Definition 3.5:</u> $f: X \longrightarrow Y$ is <u>bijective</u> (or 1-1 correspondence) if f is injective & surjective (i.e. $f^{-1}(y)$ is a singleton $\forall y \in Y$).

Example: 

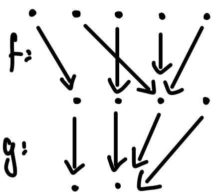
<u>Definition 3.6:</u> Assume $X \xrightarrow{f} Y \xrightarrow{g} Z$. Then the <u>composite</u> map $g \circ f$ (or simply gf) is the map $X \to Z$ given by $(gf)(x) = g(f(x))\ \forall x \in X$. ($(f \circ g)(y) = \underline{f(g(y))}$??)

Example: Let $X = Y = Z = \mathbb{R}$, $f(x) = x^2$, $g(x) = x+1$. Then $(gf)(x) = g(f(x)) = g(x^2) = x^2+1$

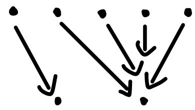
$$(fg)(x) = f(g(x)) = f(x+1) = (x+1)^2 = x^2+2x+1.$$

so $fg \neq gf$

Example:



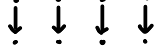
$g \circ f$:



<u>**Definition 3.7:**</u> Let X be a set. The <u>identity map</u> 1_X or Id_X is $1_X : X \longrightarrow X$ given by

$$1_X(x) = x \ \forall x \in X$$

Example: 1_X:



<u>**Lemma 3.8:**</u> (1) Let $f : X \longrightarrow Y$. Then $f \circ 1_X = f$ and $1_Y \circ f = f$

(2) Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$. Then $h \circ (g \circ f) = (h \circ g) \circ f$ (so "$\circ$" is <u>associative</u>)

Proof: (1) clear

(2) $(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$

$((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$ $\quad \forall x \in X$

<u>**Definition 3.9:**</u> Let $f : X \longrightarrow Y$. Then $g : Y \longrightarrow X$ is <u>inverse</u> of f (we write $f = g^{-1}$) if $fg = 1_Y$ and $gf = 1_X$. If such g exists then f is <u>invertible</u>

<u>**Proposition 3.10:**</u> $f : X \longrightarrow Y$ is invertible $\Leftrightarrow$ f is a bijection.

Proof: ($\Rightarrow$) suppose f is invert., so g as in 3 g exists. need to show f is inj + surj

surj: Let $y \in Y$. Take $x = g(y)$ $(g : Y \to X)$.

then $f(x) = f(g(y)) = (f \circ g)(y)$

by 3.9, $1_Y(y) = y$

inj: suppose $f(x_1) = f(x_2)$.

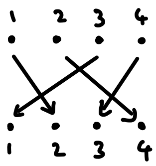
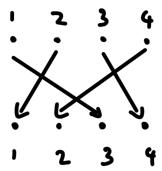
then $g(f(x_1)) = g(f(x_2))$

$(g \circ f)(x_1) = (g \circ f)(x_2)$

$1_X(x_1) = 1_X(x_2)$

$x_1 = x_2$

$(\Leftarrow)$ suppose f is bij. = inj. + surj. Then for every $y \in Y$ the preimage $f^{-1}(y)$ is a singleton, so $f^{-1}(y) = x$ for some $x \in X$. Define $g: Y \longrightarrow X$ as $g(y) = f^{-1}(y)$ $\forall y \in Y$. then $fg = 1_Y$ and $gf = 1_X$ (check!), so g is the inverse of f.

Example: f:  f^{-1}: 

Proposition 3.11: Let $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ be bijective. Then $gf: X \longrightarrow Z$ is a bijection and $(gf)^{-1} = f^{-1}g^{-1}$

Proof: Part 2 only: $(f^{-1}g^{-1})(gf) = f^{-1}(g^{-1}g)f = f^{-1} \circ 1_Y \circ f = f^{-1} \circ f = 1_X$

$(gf)(f^{-1}g^{-1}) = g(ff^{-1})g^{-1} = g \circ 1_X \circ g^{-1} = g \circ g^{-1} = 1_Z$