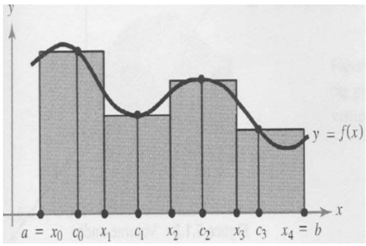
Recall the notion of integration in one variable ...

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$



We call this the Reimann integral of f.

Fundamental Theorem of Calculus

Theorem:

If there exists differentiable function F such that $F' = f$, for an (integrable) function f, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Second form of the FTC:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Differentiation and integration are inverse operations!

Integrals:

Let f, g (integrable) functions. Then

- $\int f+g\,dx = \int f\,dx + \int g\,dx$
- $\int h(y) f(x)\,dx = h(y)\int f(x)\,dy$ for h (integrable) function.
- $\int f(x) g'(x)\,dx = f(x)g(x) - \int_a^b f'(x) g(x)\,dx$

or

$$\int_a^b f(x) g'(x)\,dx = \left[f(x)g(x)\right]_a^b - \int_a^b f'(x) g(x)\,dx$$

Integration by parts! ... maybe the most important formula in applied analysis.

Double Integrals

Let $f: \mathbb{R}^2 \to \mathbb{R}$ function. We define the double integral of f by

$$\iint_\Omega f(x,y)\,dx\,dy = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_i)(x_i - x_{i-1})(y_i - y_{i-1})$$

Double Integrals over Rectangles:

Let $f: [a,b]\times[c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ (integrable) function. The double integral of f is then

$$\iint_\Omega f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$$

The double integrals of a continuous function over a rectangular domain is equal to the iterated integrals.

Remark: depending on which is more convenient, we can use either red or blue forms.

Example 1: Let $f(x,y) = \cos x \sin y$. Compute the double integrals of f over the rectangle

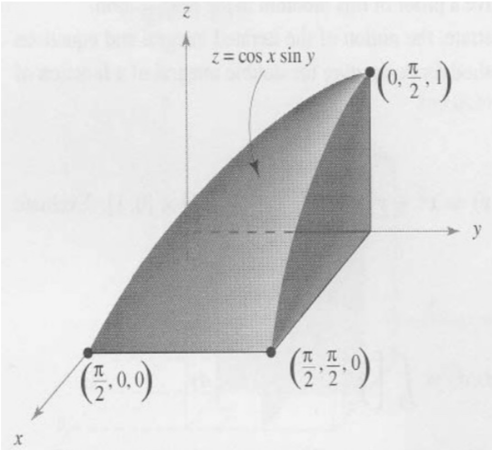
$$\Omega = [0, \pi/2] \times [0, \pi/2].$$

Solution: we have

$$\int_0^{\pi/2} \left(\int_0^{\pi/2} \cos x \sin y \,dx\right) dy = \cdots = 1.$$

similarly

$$\int_0^{\pi/2} \left(\int_0^{\pi/2} \cos x \sin y \,dy\right) dx = \cdots = 1.$$



Example 2: Let $f(x,y) = x \sin y$. Find the double integral of f over the rectangle $\Omega = [0,1]\times[0,\pi]$.

Solution: we have

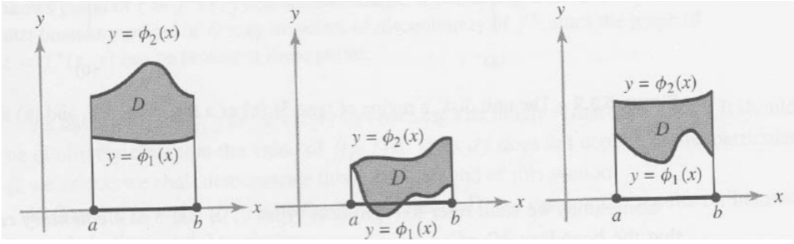
$$\iint_\Omega x \sin y \,dx\,dy = \int_0^\pi \left(\int_0^1 x \sin y \,dx\right) dy = \cdots = 1$$

similarly

$$\iint_\Omega x \sin y \,dx\,dy = \int_0^1 \left(\int_0^\pi x \sin y \,dy\right) dx = \cdots = 1$$

Double Integrals over Simple Domains:

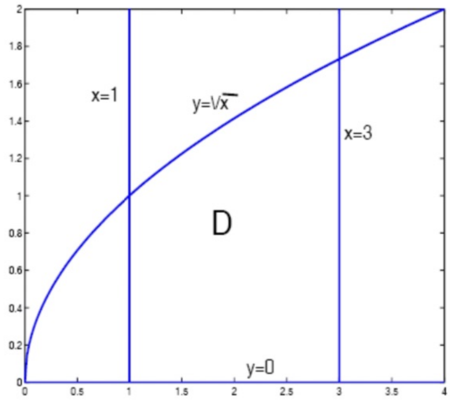
Problem: Let $f: D \subset \mathbb{R}^2 \to \mathbb{R}$. What is the double integral f over D, when D is of the following form?



Solution: $\iint_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right) dx$

Domains of the form $D = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b,\ \varphi_1(x) \leq y \leq \varphi_2(x)\}$ are called x-simple domains.

Example: Let $f(x,y) = xy = xy$. Compute the double integral of f over the domain bounded by $y = \sqrt{x}$, the x-axis, $x = 1$, and $x = 3$.



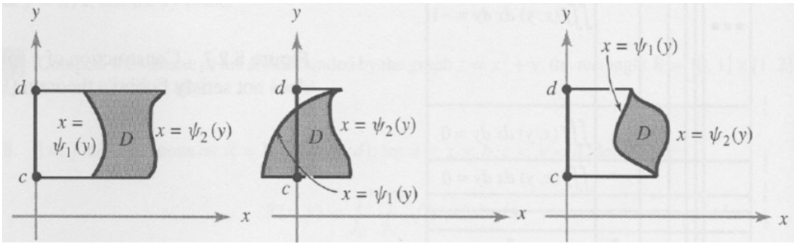
Solution: D is x-simple. indeed,

$$D = \{(x,y) : 1 \leq x \leq 3,\ 0 \leq y \leq \sqrt{x}\}.$$

we have: $\int_1^3 \int_0^{\sqrt{x}} xy\,dy\,dx = \cdots = 13/3$

Double Integrals over Simple Domains:

Problem: Let $f: D \subset \mathbb{R}^2 \to \mathbb{R}$. What is the double integral f over D, when D is of the following form?



Solution: $\iint_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)\,dx \right) dy$

Domains of the form $D = \{(x,y) \in \mathbb{R}^2 : c \leq y \leq d,\ \varphi_1(y) \leq x \leq \varphi_2(y)\}$ are called y-simple domains.

Example 1: $I = \int_{-2}^{2} dx \int_{x^2}^{4} f(x,y)\,dy$

change the order of integration.

$$I = \int_{-2}^{2} dx \int_{x^2}^{4} f(x,y)\,dy = \int_0^4 dy \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dy$$