Many of the laws, underlying the behaviours of the Natural world are:

- Relation $\Rightarrow$ are in math terms, Equations
- Rates $\Rightarrow$ Derivatives, at which things happen

Equation 1: $F(t, y(t), y'(t), y''(t), \dots, y^{(n)}(t)) = 0$ is the differential equation

$y(t) \longrightarrow$ is the general solution, containing constant $C$.

IVP: $y(t=0) = y_0 \longrightarrow$ uniquely determine $C$ and Particular Solution DE for IVP.

Example: Falling Object Problem. Suppose that an object is falling in the atmosphere near sea level.

force due to air resistance (drag)

$\gamma v$ drag coefficient

$mg$ $9.8\ m/s^2$ acceleration due to G

gravity force

Newtons second law $F = m \cdot a$ $\quad a = \frac{dv}{dt}$

$F \to$ net force on the object

$$F = m \cdot \frac{dv}{dt}$$

$$m \cdot \frac{dv}{dt} = mg - \gamma \cdot v$$

(eq) $\frac{dv}{dt} = g - \frac{\gamma \cdot v}{m}$ $\Rightarrow$ we need to find a function $v = v(t)$ that satisfies the (eq)

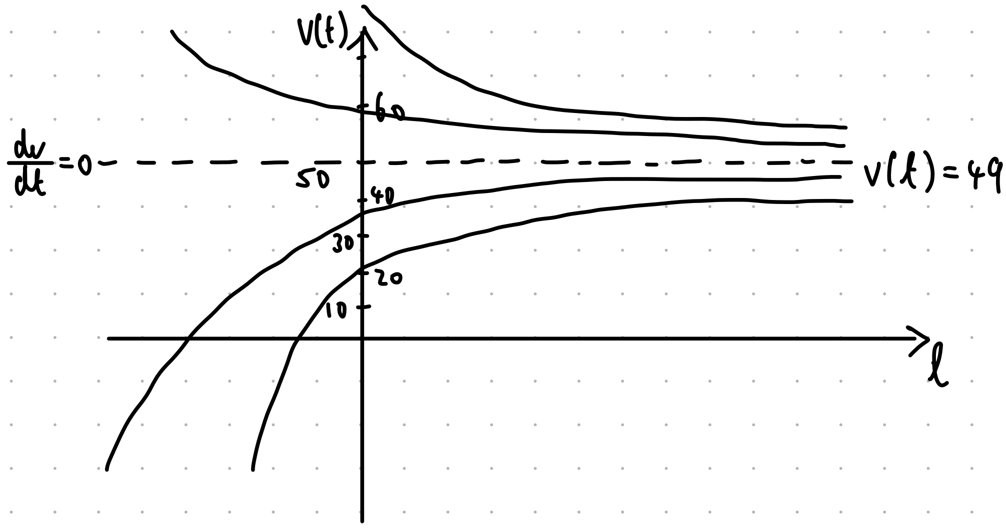
$m = 10\,kg \quad \gamma = 9\ kg/s \Rightarrow \boxed{\frac{dv}{dt} = 9.8 - \frac{v}{5}}$ (eq)

**What can we learn about solutions without actually finding any of them?**

what information can be obtained directly from $\frac{dv}{dt} = 9.8 - \frac{v}{5}$?

① Suppose $v$ has a certain given value: $v = 40 \Rightarrow \frac{dv}{dt} = 9.8 - \frac{40}{5} = 1.8 > 0 \quad v(t) \nearrow$

$v = 50 \Rightarrow \frac{dv}{dt} = 9.8 - \frac{50}{5} = -0.2 < 0 \quad v(t) \searrow$

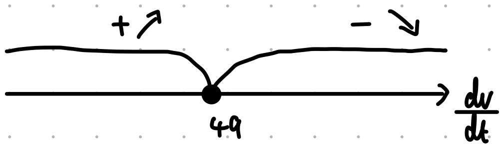



$$9.8 - \frac{v}{5} = 0 \Rightarrow 9.8 = \frac{v}{5} \Rightarrow v = 9.8 \cdot 5 = 49$$

$\boxed{V = 49}$ Equilibrium Solution (ES)

$$\frac{d49}{dt} = 9.8 - \frac{49}{5} \Rightarrow 0 = 0 \text{ it is a solution!}$$

$\Longrightarrow$ Determining ES: $9.8 - \frac{v}{5} = 0$ $\quad V = 49$



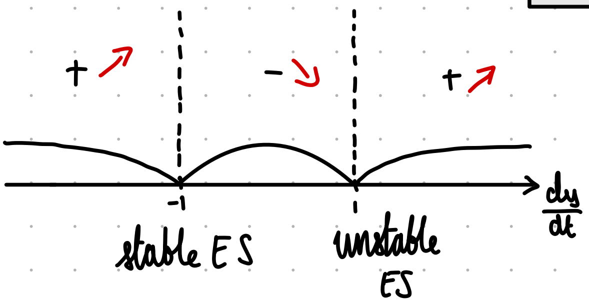

Example: a) $\frac{dy}{dt} = t^2 + y^2 - 1 \Rightarrow$ is non-autonomous DE

$\frac{dy}{dt} = F(y(t), t)$

- forward
- pull back attractor
- global

b) $\frac{dy}{dt} = 2(y^2 - 1) = 0$

$y^2 - 1 = 0 \Rightarrow (y-1)(y+1) = 0 \Rightarrow \boxed{y = 1 \vee y = -1}$




$\Rightarrow$ is autonomous DE $\qquad \frac{dy}{dt} = F(y(t))$

**Qualitative Conclusions:** $\frac{dv}{dt} = 9.8 - \frac{v}{5}$

$$\Rightarrow \frac{dv}{dt} = \frac{49 - v}{5}$$

$$\Rightarrow \frac{1}{49 - v}\, dv = \frac{1}{5}\, dt$$

$$\Rightarrow \frac{1}{v - 49}\, dv = -\frac{1}{5}\, dt$$

$$\Rightarrow \int \frac{1}{v-49}\, dv = \int -\frac{1}{5}\, dt$$

$$\Rightarrow \ln|v - 49| = -\tfrac{1}{5} t + c$$

$$\Rightarrow v - 49 = e^{[-\frac{1}{5} t + c]}$$

$$\Rightarrow v = e^{[-\frac{1}{5} t + c]} + 49$$

$$\Rightarrow v = 49 + C_1 e^{-t/5} \quad \Rightarrow \text{General Solution}$$

$C_1$ ↓ is arbitary

IVP $\Rightarrow v(t=0) = 0$

$$0 = 49 + C_1 e^{-0/5} \Rightarrow 0 = 49 + C_1 \cdot 1$$

$$C_1 = -49$$

$$v(t) = 49 - 49 e^{-t/5} \Rightarrow v(t) = 49(1 - e^{-t/5}) \text{ Particular Solution}$$

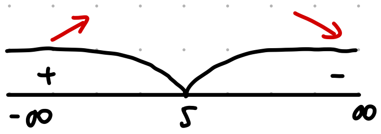
**Example:** $\frac{dy}{dt} = -2y + 10;\ y(0) = y_0$

$$-2y + 10 = 0$$

$$-2y = -10$$

$$2y = 10$$

$$y = 5$$




stable ES

$$\frac{dy}{10 - 2y} = dt \Rightarrow -\tfrac{1}{2}\ln|10 - 2y| = t + c$$

$$\ln|10 - 2y| = -2t + C_1$$

$$10 - 2y = e^{[-2t + C_1]}$$

$$-2y = -10 + e^{-2t} \cdot e^{c_1}$$

$$y = 5 + e^{-2t} \cdot \left[-\frac{e^{c_1}}{2}\right]$$

$$\boxed{y_g = 5 + e^{-2t} \cdot C_2}$$

$$y_0 = 5 + C_2 \cdot e^{0}$$

$$C_2 = y_0 - 5$$

$$\boxed{y_{IVP}^{(t)} = 5 + (y_0 - 5)\, e^{-2t}}$$