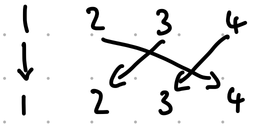
27/1/23 L8

Recall: Symmetric group $S_n = \{$ all bijections $X \to X \}$ where $X = \{1, 2, 3, \dots, n\}$

Let $\sigma \in S_n$, say $\sigma =$  $(n=4)$ permutation

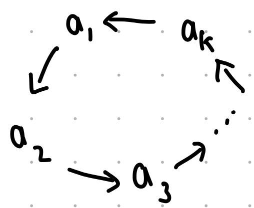
then in <u>array notation</u> $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{bmatrix}$

then inverse $\sigma^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & \dots & \end{bmatrix}$

Multiplication:

$$\sigma\mu = \sigma \circ \mu = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{bmatrix}$$

Recall: Def 4.4: k-cycle $(a_1, \dots, a_k)$



$a_i \in X$

Example: $(1\,2) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{bmatrix}$ (2-cycle)

$(2\,4\,3) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{bmatrix}$ (3-cycle)

cyclic notation array notation

<u>Definition 4.5:</u> Cycles are <u>disjoint</u> if no common elements.

Example. $(2\,6\,5)$ and $(1\,4\,3\,7)$ are disjoint.

$(1\,2)$ and $(2\,5\,6)$ are <u>not</u> disjoint.

Note: $\underline{(2\,6\,5)} = (6\,5\,2) = (5\,2\,6)$

↑ smallest number first.

Lemma 4.7: Any permutation σ can be written as a unique product of disjoint cycles (up to ordering). This product is called <u>cyclic decomposition</u> of σ.

Proof: see ex

Example: $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 7 & 2 & 6 & 5 & 3 & 8 \end{bmatrix}$

then $\sigma = (1\ 4\ 2)(3\ 7)(5\ 6)\cancel{(8)}$ Note: $(8) = 1_X$

Remark: We do not write cycles of length 1 as they $= 1 = 1_X$ (id. map)

Multiplication in cyclic notation:

Example: (1) $(1\ 2\ 3)\cdot(1\ 2) = (1\ 3)(2) = (1\ 3)$

(2) $(1\ 2\ 4)\cdot(2\ 3\ 5)\cdot(1\ 2\ 3\ 4) = (1\ 3)(2\ 5\ 4)$ disjoint

Cyclic notation more compact: $|S_n| = n!$

$S_1 = \{1\}$ $1! = 1$

$S_2 = \{1, (1\ 2)\}$ $(X = \{1, 2\})$ $2! = 4$

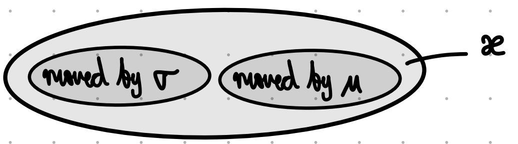
$S_3 = \{1, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ $3! = 6$

$S_4 = \{\ldots, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ $4! = 24$

Lemma 4.9: If σ and μ are disjoint then $\sigma\mu = \mu\sigma$, so $(\sigma\mu)^k = \sigma^k\mu^k$.

Note: not true if not disjoint, e.g. $(1\ 2)(2\ 3) = (2\ 3)(1\ 2)$

Proof: $\sigma\mu = \mu\sigma$ is clear:



now

$$(\sigma\mu)^k = \sigma\underbrace{\mu\sigma}_{=\sigma\mu}\mu\ldots\sigma\mu = \sigma\cdots\sigma\mu\cdots\mu = \sigma^k\mu^k$$

Example: Let $\sigma = (1\ 4\ 2)(3\ 7)(5\ 6)$ (cyclic decomp.)

(disjoint cycles) By 4.9, also $\sigma = (3\ 7)(1\ 4\ 2)(5\ 6) = \cdots$

Also by 4.9, $\sigma^k = (1\ 4\ 2)^k (3\ 7)^k (5\ 6)^k$

e.g. $\sigma^2 = (1\ 4\ 2)^2 (3\ 7)^2 (5\ 6)^2$

$= (1\ 2\ 4)\ \cancel{(3)}\cancel{(7)}\ \cancel{(5)}\cancel{(6)}$

$= (1\ 2\ 4).$

<u>Lemma 4.10:</u> If σ is a k-cycle then $O(\sigma) = k$ (recall order of σ ($O(\sigma)$) is min m s.t. $\sigma^m = 1$)

Proof: Let $\sigma = (a_1, \dots, a_k)$. Then

$\sigma^1(a_1) = a_2$
$\sigma^2(a_1) = a_3$
$\vdots$
$\sigma^{k-1}(a_1) = a_k$
$\sigma^k(a_1) = a_1$

But similarly, $\sigma^k(a_i) = a_i \quad \forall i$ so $\sigma^k = 1_x$.

<u>Lemma 4.11:</u> Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_m$ be the cyclic decomposition of σ (so σ_i's are disjoint). Then if σ_i is a k_i-cycle,

$O(\sigma) = \text{lcm}(k, k_2, \dots k)$.

Proof: by 4.9, $\sigma^s = \sigma_1^s \sigma_2^s \dots \sigma_m^s$

now $\sigma^s = 1 \iff \sigma_i^s = 1 \quad \forall i \iff k_i \mid s$

so $O(\sigma) = $ smallest such $s = \text{lcm}(k_1, \dots, k_m)$.

Example: $\sigma = (1\ 4\ 2)(3\ 7)(5\ 6)$, $O(\sigma) = \text{lcm}(3, 2, 2)$

$= 6$