

Numerical solution of a first-order Hamilton–Jacobi equation for dynamic equilibrium of pathogen cohorts

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1 Introduction

In this report, we study a zero-sum feedback game for a system of nonlinear ordinary differential equations describing one-seasonal dynamics of two biotrophic fungal cohorts within a common host plant. The cohorts can be understood as resident and mutant populations. The invasion functional takes the form of the difference between the two marginal fitness criteria and represents the cost in the definition of the value of the differential game. A general dynamic game approach of [1] is exploited. The Cauchy problem for the resulting first-order Hamilton–Jacobi partial differential equation (HJ PDE) is numerically solved via a finite difference method using the ROC-HJ software [2]. Interpretations of the obtained numerical simulation results are also provided.

2 Statement of the model

Consider one-seasonal dynamics of two biotrophic fungal cohorts within one host plant. Let us refer to them as to cohorts 1 and 2. For cohort i , let n_i be the lesion density, which is the number of mycelia per unit area of the host. For simplicity, n_1 and n_2 are assumed to be constant during the entire infection period within the season, which means that new infectious agents do not penetrate into the host during the analyzed period. Even though this is typically a strong restriction for infections in nature, it still allows for drawing significant conclusions with relevant biological interpretations and also applies to experimental infection protocols.

Denote the average size of a mycelium in cohort i by M_i . Let S_i be the average quantity of spores produced by a mycelium in cohort i . The infection age t within the season is interpreted as a time variable. Note that the mycelial sizes M_1 and M_2

can be measured in terms of equivalent amounts of infecting spores (for instance, if a mycelium appears from one spore at the beginning of the infection period, the initial mycelial size can be set as one equivalent of an infecting spore or simply as one spore).

The nutrient flux uptaken by cohort i is determined by a function $f_i = f_i(M_1, M_2)$ and allocated between two different pathogen activities such as within-host multiplication (mycelial growth) and production of asexual spores. For brevity, we do not explicitly show the dependence on the constant parameters n_1, n_2 when writing $f_i = f_i(M_1, M_2)$, $i = 1, 2$. Let $u_i = u_i(t)$ be a related resource allocation (control) function taking values between zero and one. Suppose that the whole flux goes to mycelial growth when $u_i(t) = 0$ and to spore production when $u_i(t) = 1$, while, for $0 < u_i(t) < 1$, there is an intermediate configuration.

Let $g = g(M)$ be the rate of mycelial decay for both cohorts. Assume also that spores are produced with a constant yield $\delta > 0$ as compared to mycelial growth.

The time span of the infections within the season is a fixed interval $[0, T]$.

The following dynamic model can thus be formulated [1]:

$$\left\{ \begin{array}{l} \frac{dM_1(t)}{dt} = (1 - u_1(t)) f_1(M_1(t), M_2(t)) - g(M_1(t)), \\ \frac{dM_2(t)}{dt} = (1 - u_2(t)) f_2(M_1(t), M_2(t)) - g(M_2(t)), \\ \frac{dS_1(t)}{dt} = \delta u_1(t) f_1(M_1(t), M_2(t)), \\ \frac{dS_2(t)}{dt} = \delta u_2(t) f_2(M_1(t), M_2(t)), \\ M_1(0) = M_1^0, \quad M_2(0) = M_2^0, \quad S_1(0) = S_1^0, \quad S_2(0) = S_2^0, \\ 0 \leq u_1(t) \leq 1, \quad 0 \leq u_2(t) \leq 1, \quad t \in [0, T]. \end{array} \right. \quad (1)$$

Since the right-hand sides of the third and fourth equations in (1) do not contain S_1 and S_2 , one does not need to treat S_1 and S_2 explicitly, so that M_1 and M_2 can be considered as the only state variables.

The reproductive success of cohort i is determined by

$$\int_0^T \frac{dS_i(t)}{dt} e^{-\mu t} dt = \delta \int_0^T u_i(t) f_i(M_1(t), M_2(t)) e^{-\mu t} dt, \quad (2)$$

where $e^{-\mu t}$ describes exponential extinction of the infections, and μ is a positive constant. Since δ is a positive constant, we can divide (2) by δ and arrive at

$$J_i(u_1(\cdot), u_2(\cdot)) = \int_0^T u_i(t) f_i(M_1(t), M_2(t)) e^{-\mu t} dt.$$

In line with [1,3], we study one-seasonal competition between the two pathogen cohorts in terms of their average reproductive outputs by seeking the saddle control strategies in the following zero-sum two-player differential game:

$$\begin{cases} \frac{dM_1(t)}{dt} = (1 - u_1(t)) f_1(M_1(t), M_2(t)) - g(M_1(t)), \\ \frac{dM_2(t)}{dt} = (1 - u_2(t)) f_2(M_1(t), M_2(t)) - g(M_2(t)), \\ M_1(0) = M_1^0, \quad M_2(0) = M_2^0, \\ 0 \leq u_1(t) \leq 1, \quad 0 \leq u_2(t) \leq 1, \quad t \in [0, T], \end{cases} \quad (3)$$

$$J(u_1(\cdot), u_2(\cdot)) \longrightarrow \inf_{u_1(\cdot)} \sup_{u_2(\cdot)} \text{ or } \sup_{u_2(\cdot)} \inf_{u_1(\cdot)}, \quad (4)$$

$$\begin{aligned} J(u_1(\cdot), u_2(\cdot)) &= J_2(u_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \\ &= \int_0^T (u_2(t) f_2(M_1(t), M_2(t)) - u_1(t) f_1(M_1(t), M_2(t))) e^{-\mu t} dt. \end{aligned} \quad (5)$$

According to [1], we represent the nutrient fluxes and decay rates as

$$f_i(M_1, M_2) = \nu(n_1 M_1 + n_2 M_2) \cdot \rho(M_i) \quad \forall M_1 \geq 0 \quad \forall M_2 \geq 0, \quad i = 1, 2, \quad (6)$$

$$\rho(M) = \alpha \frac{M}{M + k} \quad \forall M \geq 0, \quad (7)$$

$$\nu(n_1 M_1 + n_2 M_2) = \frac{1}{1 + \beta(n_1 M_1 + n_2 M_2)} \quad \forall M_1 \geq 0 \quad \forall M_2 \geq 0, \quad (8)$$

$$g(M) = \gamma M \quad \forall M \geq 0. \quad (9)$$

Moreover, the following parameter values are chosen for numerical simulations:

$$\begin{aligned} \alpha &= 0.2 \cdot 10^4 \text{ spores/day}, \quad k = (1/6) \cdot 10^4 \text{ spores}, \\ \beta &= 10^{-5} \text{ cm}^2/\text{spores}^2, \quad \gamma = 0.06 \text{ 1/day}, \quad \mu = 0.03 \text{ 1/day}, \\ n_1 &= 9 \text{ spores/cm}^2, \quad n_2 = 1 \text{ spore/cm}^2, \\ T &= 60 \text{ days}. \end{aligned} \quad (10)$$

As was shown in [1], the bounded domain

$$G = \{(M_1, M_2) \in \mathbb{R}^2 : 0 < M_1 < \bar{M}_1, 0 < M_2 < \bar{M}_2\}$$

with

$$\bar{M}_1 = \bar{M}_2 > \alpha/\gamma - k$$

is an invariant set in the state space (that is, if a state trajectory starts in G , it cannot leave G). The parameters \bar{M}_1, \bar{M}_2 can be understood as carrying capacity estimates.

3 Uninvadable and evolutionary stable strategies

In this section, we describe a motivation for the game-theoretic statement (3)–(5).

Based on the terminology of Adaptive Dynamics [4], let us interpret cohort 1 as a resident and cohort 2 as a mutant. Denote the corresponding classes of considered strategies as \mathcal{U}_1 and \mathcal{U}_2 . For a pair of strategies $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$, we agree that the resident is not invaded by the mutant if and only if

$$J(u_1, u_2) = J_2(u_1, u_2) - J_1(u_1, u_2) \leq 0.$$

A strategy $\hat{u}_1 \in \mathcal{U}_1$ is hence called uninvadable if and only if

$$J(\hat{u}_1, u_2) \leq 0 \quad \forall u_2 \in \mathcal{U}_2,$$

which is equivalent to

$$\sup_{u_2 \in \mathcal{U}_2} J(\hat{u}_1, u_2) \leq 0.$$

Such a \hat{u}_1 exists if

$$\inf_{u_1 \in \mathcal{U}_1} \sup_{u_2 \in \mathcal{U}_2} J(u_1, u_2) = \min_{u_1 \in \mathcal{U}_1} \sup_{u_2 \in \mathcal{U}_2} J(u_1, u_2) \leq 0 \quad (11)$$

or

$$\inf_{u_1 \in \mathcal{U}_1} \sup_{u_2 \in \mathcal{U}_2} J(u_1, u_2) < 0$$

(the latter inequality is mentioned in order to include the case when the infimum with respect to u_1 is not reached). This motivates our game-theoretic statement (3)–(5), where the first player tries to maximize its resistance to an invasion by the second one, and vice versa.

If $M_1^0 = M_2^0$ and $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{U}$, then $J(u, u) = 0$ for all $u \in \mathcal{U}$, and (11) is simplified to

$$\min_{u_1 \in \mathcal{U}} \sup_{u_2 \in \mathcal{U}} J(u_1, u_2) = 0.$$

In this case, a strategy

$$\hat{u}_1 \in \operatorname{Arg} \min_{u_1 \in \mathcal{U}} \left(\sup_{u_2 \in \mathcal{U}} J(u_1, u_2) \right)$$

is called evolutionary stable if the related maximum with respect to u_2 is unique:

$$\operatorname{Arg} \max_{u_2 \in \mathcal{U}} J(\hat{u}_1, u_2) = \{\hat{u}_1\}.$$

This approach to define evolutionary stable strategies was initially proposed in [3] and further developed in [1].

4 Hamilton–Jacobi–Isaacs equation

Introduce the following function known as Hamiltonian:

$$\begin{aligned} H(t, M_1, M_2, u_1, u_2, p_1, p_2) &= p_1 ((1 - u_1) f_1(M_1, M_2) - g(M_1)) \\ &\quad + p_2 ((1 - u_2) f_2(M_1, M_2) - g(M_2)) \\ &\quad + e^{-\mu t} (u_2 f_2(M_1, M_2) - u_1 f_1(M_1, M_2)) \end{aligned}$$

$\forall t \in [0, T] \quad \forall (M_1, M_2) \in G \quad \forall (u_1, u_2) \in [0, 1]^2 \quad \forall (p_1, p_2) \in \mathbb{R}^2$

(this is the sum of the dot product of $p = (p_1, p_2)$ with $(dM_1/dt, dM_2/dt)$ and the integrand in $J(u_1(\cdot), u_2(\cdot))$, see (3) and (5)). The Hamiltonian satisfies the saddle point condition with respect to the control variables u_1 and u_2 :

$$\begin{aligned} \min_{u_1 \in [0, 1]} \max_{u_2 \in [0, 1]} H(t, M_1, M_2, u_1, u_2, p_1, p_2) \\ = \max_{u_2 \in [0, 1]} \min_{u_1 \in [0, 1]} H(t, M_1, M_2, u_1, u_2, p_1, p_2) = \mathcal{H}(t, M_1, M_2, p_1, p_2) \end{aligned}$$

$\forall t \in [0, T] \quad \forall (M_1, M_2) \in G \quad \forall (p_1, p_2) \in \mathbb{R}^2$

(here minimax and maximin give the same result). Due to the theoretical results of [5, §XI.6], the value function $V: [0, T] \times G \rightarrow \mathbb{R}$ in the formulated feedback differential game is a unique solution of the following Cauchy problem for the Hamilton–Jacobi–Isaacs partial differential equation (HJI PDE):

$$\begin{cases} \frac{\partial V(t, M_1, M_2)}{\partial t} + \mathcal{H}\left(t, M_1, M_2, \frac{\partial V(t, M_1, M_2)}{\partial M_1}, \frac{\partial V(t, M_1, M_2)}{\partial M_2}\right) = 0, \\ V(T, M_1, M_2) = 0, \\ t \in [0, T], \quad (M_1, M_2) \in G. \end{cases} \quad (12)$$

The solution may in general be nonsmooth and should be understood in a generalized (viscosity or minimax) sense [5, 6].

Due to the general properties of value functions of zero-sum feedback differential games [7], the value function V in our problem is Lipschitz continuous in $[0, T] \times G$. By Rademacher's theorem, it is differentiable almost everywhere in $(0, T) \times G$, except for possibly a subset of Lebesgue measure zero.

For points (t, M_1, M_2) at which the value function is differentiable, the saddle feedback control strategies can be obtained from the relations

$$\begin{aligned} u_1(t, M_1, M_2) &\in \operatorname{Arg} \min_{w_1 \in [0, 1]} \left\{ \max_{w_2 \in [0, 1]} H \left(t, M_1, M_2, w_1, w_2, \frac{\partial V(t, M_1, M_2)}{\partial M_1}, \frac{\partial V(t, M_1, M_2)}{\partial M_2} \right) \right\}, \\ u_2(t, M_1, M_2) &\in \operatorname{Arg} \max_{w_2 \in [0, 1]} \left\{ \min_{w_1 \in [0, 1]} H \left(t, M_1, M_2, w_1, w_2, \frac{\partial V(t, M_1, M_2)}{\partial M_1}, \frac{\partial V(t, M_1, M_2)}{\partial M_2} \right) \right\}, \end{aligned}$$

which are reduced to

$$u_1(t, M_1, M_2) = \begin{cases} 0, & e^{-\mu t} + \frac{\partial V(t, M_1, M_2)}{\partial M_1} < 0, \\ 1, & e^{-\mu t} + \frac{\partial V(t, M_1, M_2)}{\partial M_1} > 0, \\ \text{arbitrary from } [0, 1], & e^{-\mu t} + \frac{\partial V(t, M_1, M_2)}{\partial M_1} = 0, \end{cases} \quad (13)$$

$$u_2(t, M_1, M_2) = \begin{cases} 0, & e^{-\mu t} - \frac{\partial V(t, M_1, M_2)}{\partial M_2} < 0, \\ 1, & e^{-\mu t} - \frac{\partial V(t, M_1, M_2)}{\partial M_2} > 0, \\ \text{arbitrary from } [0, 1], & e^{-\mu t} - \frac{\partial V(t, M_1, M_2)}{\partial M_2} = 0. \end{cases}$$

These are resource allocation strategies that enable for equilibrium coexistence of the two pathogen cohorts.

5 Numerical solution

The Cauchy problem (12) for the HJI PDE is considered with the nutrient fluxes (6)–(8), decay rate (9), and parameter values (10). The numerical solution has been obtained via a finite difference method [5, 7–9] using the ROC-HJ software [2].

For practical purposes, it is reasonable to rewrite (12) in reverse time $\tau = T - t$ as

$$\left\{ \begin{array}{l} \frac{\partial V(T - \tau, M_1, M_2)}{\partial \tau} + \max_{u_1 \in [0, 1]} \min_{u_2 \in [0, 1]} \left(-H \left(T - \tau, M_1, M_2, u_1, u_2, \right. \right. \\ \left. \left. \frac{\partial V(T - \tau, M_1, M_2)}{\partial M_1}, \frac{\partial V(T - \tau, M_1, M_2)}{\partial M_2} \right) \right) = 0, \\ V(T - \tau, M_1, M_2) |_{\tau=0} = 0, \\ \tau \in [0, T], \quad (M_1, M_2) \in G, \end{array} \right. \quad (14)$$

and then to rewrite (14) in the new normalized state variables

$$m_i = M_i \cdot 10^{-4}, \quad i = 1, 2$$

(such changes of the time and state variables lead to an equivalent problem).

We specify the carrying capacity estimates as $\bar{M}_1 = \bar{M}_2 = \bar{M} = 3.5 \cdot 10^4$, $\bar{m} = \bar{M} \cdot 10^{-4} = 3.5$, and discretize the computational region

$$\{(m_1, m_2, \tau) : 0 \leq m_1 \leq \bar{m}, 0 \leq m_2 \leq \bar{m}, 0 \leq \tau \leq T\}$$

by the spatial step $\Delta m_1 = \Delta m_2 = \Delta m = 2 \cdot 10^{-3}$ and time step $\Delta \tau = 5 \cdot 10^{-3}$.

Monotone finite difference methods (including the classical Lax–Friedrichs scheme) ensure theoretical convergence properties with error estimates, but they are mostly of first order and may generate so-called numerical diffusion problems on long time

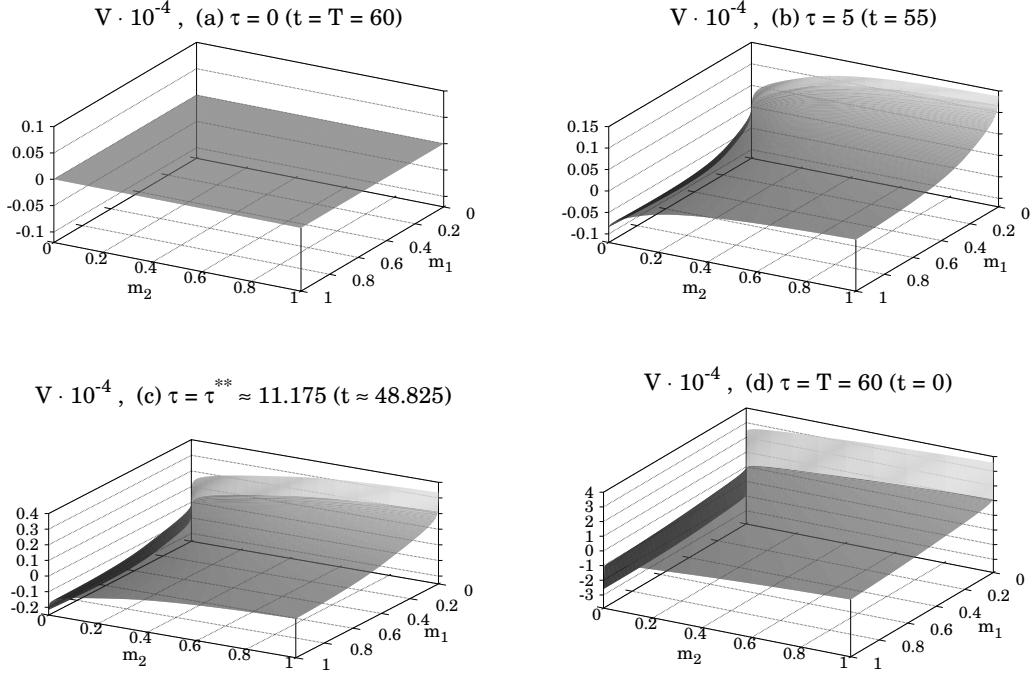


Figure 1: Finite-difference approximation of the value function V at several time instants.

intervals. One way to attenuate such practical issues is to use higher-order ENO (essentially non-oscillatory) schemes such as those originally proposed in [8]. They are not necessarily monotone, and there is lack of associated convergence theorems. However, their relevance has been justified by various numerical experiments [8, 9].

We have used the second-order ENO scheme (to approximate the partial derivatives with respect to the state variables) which is coupled with the second-order Runge–Kutta time discretization scheme and implemented in [2]. In order to evaluate the partial derivatives $\partial V / \partial M_i$, $i = 1, 2$, in the saddle feedback control representations (13), we have applied the standard second-order symmetrized approximation [10, §5.7].

Figs. 1 and 2 show the reductions of the approximate value function and corresponding saddle feedback control strategies to the coordinate plane (m_1, m_2) for the time instants $\tau = 0$ ($t = T = 60$), $\tau = 5$ ($t = T - 5 = 55$), $\tau = \tau^{**} \approx 11.175$ ($t = T - \tau^{**} \approx 48.825$), and $\tau = T = 60$ ($t = 0$). Here τ^{**} is a reverse time instant that triggers the appearance of singular control regimes and is obtained together with the equilibrium steady state (M^{**}, M^{**}) from the analysis based on Pontryagin’s principle [1]. Note that Pontryagin’s principle for zero-sum two-player differential games describes necessary conditions for saddle open-loop strategies [11].

Fig. 2 illustrates the appearance and time evolution of four approximate switch-

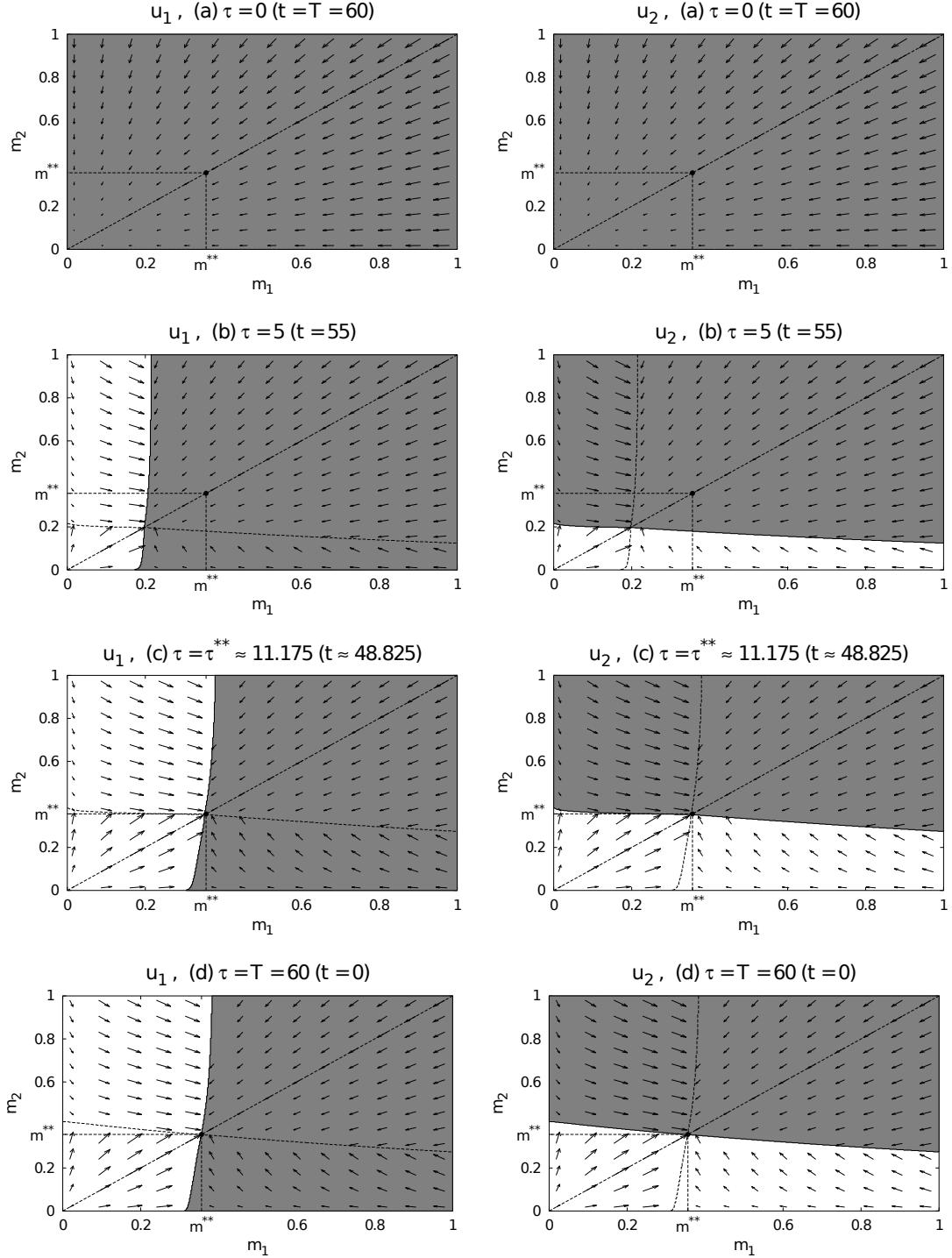


Figure 2: Finite-difference approximations of the saddle feedback control strategies at several time instants. White and gray colors show the regions of the control values 0 and 1, respectively. The fields of the corresponding forward-time velocities are also illustrated.

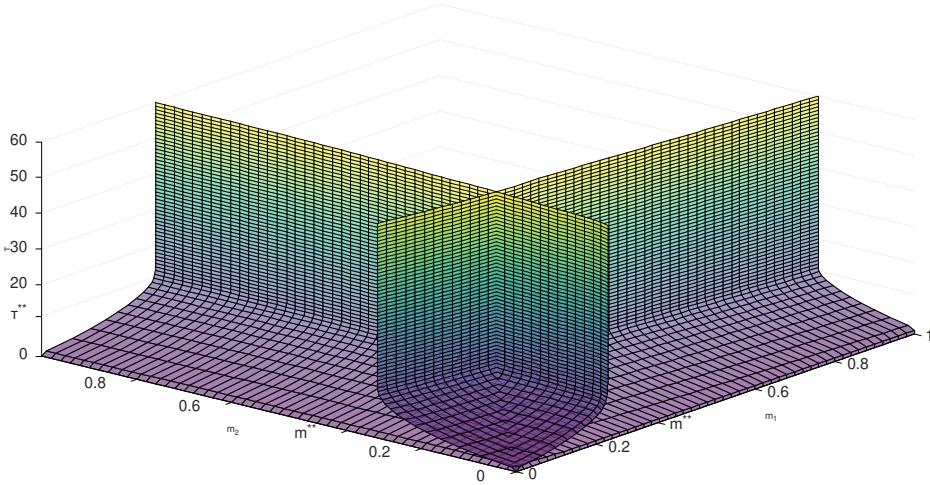


Figure 3: The four control switching surfaces in the three-dimensional space (m_1, m_2, τ) .

ing curves. For $\tau \geq \tau^{**}$, they intersect at the point $(m^{**}, m^{**}) = 10^{-4} \cdot (M^{**}, M^{**})$. With the further increase of τ , the feedback control portrait approaches a stationary form. A three-dimensional picture of the four control switching surfaces is given in Fig. 3. For $\tau \geq \tau^{**}$, the point (m^{**}, m^{**}) attracts forward-time trajectories and determines an equilibrium steady-state regime for both cohorts, while the four surfaces can be interpreted as turnpikes.

From Fig. 1d, one can see that, in the considered domain of initial states, the value $V(0, M_1^0, M_2^0)$ of the zero-sum feedback differential game is negative for $M_1^0 > M_2^0$, zero for $M_1^0 = M_2^0$, and positive for $M_1^0 < M_2^0$. The saddle feedback resource allocation strategy of cohort 1 is uninvadable if $M_1^0 \geq M_2^0$. Similarly, the saddle feedback resource allocation strategy of cohort 2 is uninvadable if $M_1^0 \leq M_2^0$.

6 Conclusion

The current report discusses a mathematical development that allows for the construction of benchmark pathogenic resource allocation strategies against which actual infection mechanisms may be compared.

We have studied resource allocation equilibrium for one-seasonal dynamics of two biotrophic fungal cohorts within a common host plant. It is also relevant to investigate long-seasonal dynamics and associated evolutionary equilibria of competing pathogen cohorts. One may exploit certain discrete rules so as the transition from one season to another [12, 13]. Another important research direction is to characterize not only such equilibria themselves but also whether they represent evolutionary

attractors or not, and how they can be approached through evolution.

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