Homework 6

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Description

- Course: STAT638, 2022 Fall

  Read Hoff (2009, ch. 6). Then, do Hoff (2009, Exercise 6.1). You may assume that θ and are a priori independent, and that Y_A and Y_B are conditionally independent given θ and γ.

Computational Environmnent Setup

Third-party libraries

```
%matplotlib inline
import sys # system information
import matplotlib # plotting
import scipy # scientific computing
import random
import pandas as pd # data managing
from scipy.special import comb
from scipy import stats as st
from scipy.special import gamma
import numpy as np
import matplotlib.pyplot as plt
from itertools import permutations
# Matplotlib setting
plt.rcParams['text.usetex'] = True
matplotlib.rcParams['figure.dpi']= 300
np.random.seed(20220928) # Consistent random effect
```

Version

```
print(sys.version)
print(matplotlib.__version__)
print(scipy.__version__)
print(np.__version__)
print(pd.__version__)
```

```
3.8.14 (default, Sep  6 2022, 23:26:50)
[Clang 13.1.6 (clang-1316.0.21.2.5)]
3.3.1
1.5.2
1.19.1
1.1.1
```

Problem 6.1

Poisson population comparisons: Let's reconsider the number of children data of Exercise 4.8. We'll assume Poisson sampling models for the two groups as before, but now we'll parameterize θ_A and θ_B as $\theta_A = \theta, \theta_B = \theta \times \gamma$. In the parameterization, γ represents the relative rate $\frac{\theta_B}{\theta_A}$. Let $\theta \sim gamma(a_\theta, b_\theta)$ and let $\gamma \sim gamma(a_\gamma, b_\gamma)$.

(a)

Are θ_A and θ_B independent or dependent under this prior distribution? In what situations is such a joint prior distribution justified?

$$
\begin{aligned}
Cov(\theta_A, \theta_B) &= E[\theta_A \theta_B] - E[\theta_A]E[\theta_B] && (1)\\
&= E[\theta\theta\gamma] - E[\theta]E[\theta\gamma] && (2)\\
&= E[\theta^2\gamma] - E[\theta]E[\theta\gamma] && (3)\\
&= E[\theta^2]E[\gamma] - E[\theta]^2 E[\gamma] && (4)\\
&= E[\gamma](E[\theta^2] - E[\theta]^2) && (5)\\
&= E[\gamma]Var[\theta] && (6)\\
&= \frac{a_\gamma}{b_\gamma}\frac{a_\theta}{b_\theta^2} && (7)\\
&\neq 0 && (8)
\end{aligned}
$$

Because $Cov(\theta_A, \theta_B) > 0$, θ_A and θ_B are dependent.

(b)

Obtain the form of the full conditional distribution of θ given $\mathbb{y}_A$, $\mathbb{y}_B$ and γ.

$$
\begin{aligned}
p(\theta|y_A, y_B, \gamma) &= \frac{p(y_A, y_B|\theta, \gamma)p(\theta|\gamma)}{p(y_A, y_B|\gamma)} && (9)\\
&= \frac{p(y_A|\theta, \gamma)p(y_B|\theta, \gamma)p(\theta|\gamma)}{p(y_A|\gamma)p(y_B|\gamma)} && (10)\\
&= \frac{p(y_A|\theta)p(y_B|\theta, \gamma)p(\theta)}{p(y_A)p(y_B|\gamma)} && (11)\\
&\propto p(y_A|\theta)p(y_B|\theta, \gamma)p(\theta) && (12)\\
&= \prod_{i=1}^{n_A} poisson(y_{Ai};\theta) \times \prod_{i=1}^{n_B} poisson(y_{Bi};\theta\gamma) \times gamma(\theta; a_\theta, b_\theta) && (13)\\
&= \theta^{\sum_{i=1}^{n_A} y_{Ai}} e^{-n_A\theta} \times (\theta\gamma)^{\sum_{i=1}^{n_B}} e^{-n_B\theta\gamma} \times \theta^{a_\theta - 1} e^{-b_\theta\theta} && (14)\\
&= \theta^{n_A\bar{y}_A} e^{-n_A\theta} \times (\theta\gamma)^{n_B\bar{y}_B} e^{-n_B\theta\gamma} \times \theta^{a_\theta - 1} e^{-b_\theta\theta} && (15)\\
&= \theta^{n_A\bar{y}_A} e^{-n_A\theta} \times (\theta)^{n_B\bar{y}_B} e^{-n_B\theta\gamma} \times \theta^{a_\theta - 1} e^{-b_\theta\theta} && (16)\\
&\propto \theta^{n_A\bar{y}_A + n_B\bar{y}_B + a_\theta - 1} e^{-(n_A + n_B\gamma + b_\theta)\theta} && (17)
\end{aligned}
$$

Therefore, $p(\theta|y_A, y_B, \gamma) \sim gamma(\theta|n_A\bar{y}_A + n_B\bar{y}_B + a_\theta, n_A + n_B\gamma + b_\theta)$

(c)

Obtain the form of the full conditional distribution of γ given $\mathtt{y}_A$, $\mathtt{y}_B$ and θ.

$$
\begin{aligned}
p(\gamma|y_A, y_B, \theta) &= \frac{p(y_A, y_B|\gamma, \theta)p(\gamma|\theta)}{p(y_A, y_B|\theta)} && (18)\\
&\propto p(y_A, y_B|\gamma, \theta)p(\gamma|\theta) && (19)\\
&= p(y_A|\gamma, \theta)p(y_B|\gamma, \theta)p(\gamma) && (20)\\
&= p(y_A|\theta)p(y_B|\gamma, \theta)p(\gamma) && (21)\\
&= \prod_{i=1}^{n_A} p(y_{Ai}|\theta) \prod_{i=1}^{n_B} p(y_{Bi}|\gamma, \theta)p(\gamma) && (22)\\
&= \theta^{s\bar{y}_A} e^{-n_A\theta} (\gamma\theta)^{n_B\bar{y}_B} e^{-n_B\gamma\theta} \gamma^{a_\gamma - 1} e^{-b_\gamma\gamma} && (23)\\
&\propto \gamma^{n_B\bar{y}_B + a_\gamma - 1} e^{-n_A\theta - n_B\gamma\theta - b_\gamma\gamma} && (24)\\
&\propto \gamma^{n_B\bar{y}_B + a_\gamma - 1} e^{-(n_B\theta + b_\gamma)\gamma} && (25)
\end{aligned}
$$

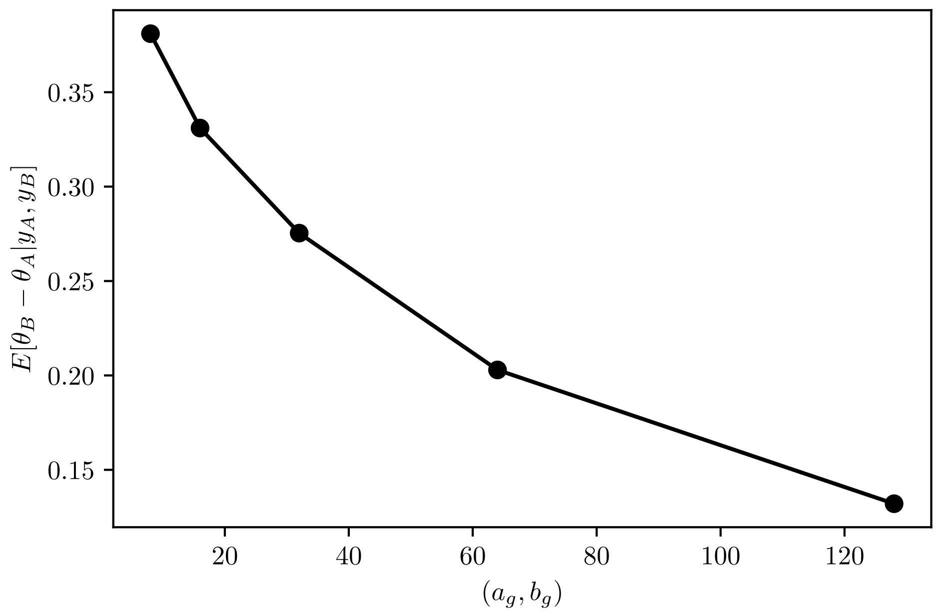
Therefore, $p(\gamma|y_A, y_B, \theta) \sim Gamma(n_B\bar{y}_B + a_\gamma, n_B\theta + b_\gamma)$

(d)

Set $a_\theta = 2$ and $b_\theta = 1$. Let $a_\gamma = b_\gamma \in \{8, 16, 32, 64, 128\}$. For each of these five values, run a Gibbs sampler of at least 5000 iterations and obtain $E[\theta_B - \theta_A | \mathbb{y}_A, \mathbb{y}_B]$. Describe the effects of the prior distribution for γ on the results.

```
class GibbSampler:
    def __init__(self, a_t, b_t, a_g, b_g):
        # Theta
        self.a_t = 2
        self.b_t = 1
        # Gamma
        self.a_g = a_g
        self.b_g = b_g
        self.theta = st.gamma(a_t, scale=1/b_t)
        self.gamma = st.gamma(a_g, scale=1/b_g)

    def expection(self, obs_func, dataA, dataB, n_sampling):
        # Sample number
        nA=len(dataA)
        nB= len(dataB)
        # Sample mean
        mA = np.mean(dataA)
        mB = np.mean(dataB)


        thetaAs = np.zeros(n_sampling)
        thetaBs = np.zeros(n_sampling)

        # initial
        theta_pre = st.gamma.rvs(self.a_t, scale=(self.b_t)**-1)
        gamma_pre = st.gamma.rvs(self.a_g, scale=(self.b_g)**-1)

        # Gibbs iteration
        thetaAs[0] = theta_pre
        thetaBs[0] = theta_pre*gamma_pre

        for i in range(1,n_sampling):
            # posterior
            theta = st.gamma.rvs(nA*mA + nB*mB + self.a_t, scale= (nA + nB*gamma_pre + sel
            theta_pre = theta
            gamma = st.gamma.rvs(nB*mB + self.a_g, scale= (nB*theta_pre + self.b_g)**-1)
```

```
            gamma_pre = gamma

            # transformation
            thetaAs[i] = theta
            thetaBs[i] = theta*gamma

        # Apply obs
        obs = [obs_func(thetaAs[i], thetaBs[i]) for i in range(0,n_sampling)]

        return np.mean(obs)


def obs_func(theta_A, theta_B):
    return theta_B - theta_A


vals = np.array([8,16,32,64,128], dtype=float)

priors = [{\
    "a_t": 2.0,
    "b_t": 1.0,
    "a_g": a,
    "b_g": a
} for a in vals]

obs = {
    "obs_func": obs_func,
    "dataA": np.loadtxt("data/menchild30bach.dat"),
    "dataB": np.loadtxt("data/menchild30nobach.dat"),
    "n_sampling": 5000
}

exps = np.zeros(len(vals))

for i in range(0, len(vals)):
    g = GibbSampler(**priors[i])
    exps[i] = g.expection(**obs)


plt.plot(vals, exps, "-o",color="black");
plt.xlabel("$(a_{g}, b_{g})$");
plt.ylabel("$E[\\theta_{B}-\\theta_{A} | y_A, y_{B}]$");
```



Problem External

Also complete the following problem: We would like to study the survival times after patients receive a new cancer treatment. We observe the following survival times (in years) for 6 patients: 3, 5, x, 4, x, x. Here, x denotes a censored observation, meaning that the respective patient survived for more than 5 years after the treatment (which is when the study ended). We consider the following model:

$$Y_i = \begin{cases} Z_i, & Z_i \le c \\ \times, & Z_i > c \end{cases}, i = 1, \dots, n \tag{26}$$

$$Z_1, \dots, Z_n | \theta \sim^{iid} Exponential(\theta)$$

$$\theta \sim Gamma(a, b)$$

We have $a = 1$, $b = 4$, $c = 5$, and $n = 6$.

(a)

Find the full-conditional distribution (FCD) of θ

$$
\begin{aligned}
p(\theta|y,z) &\propto \underbrace{p(y|\theta,z)}_{=1} p(\theta|z) && (27)\\
&= p(\theta|z) && (28)\\
&= \frac{p(z|\theta)p(\theta)}{p(z)} && (29)\\
&\propto p(z|\theta)p(\theta) && (30)\\
&= \prod_{i=1}^{n=6} p(z_i|\theta)p(\theta) && (31)
\end{aligned}
$$

- $p(z_i|\theta) \propto \theta e^{-\theta z_i}$
- $p(\theta) \propto \theta^{a-1} e^{-b\theta}$

$$
\begin{aligned}
p(\theta|y,z) &= \left(\prod_{i=1}^{n=6} \theta e^{-\theta z_i}\right) \theta^{a-1} e^{-b\theta} && (32)\\
&= \theta^{n} e^{-\theta n \bar{z}} \theta^{a-1} e^{-b\theta} && (33)\\
&= \theta^{n+a-1} e^{-b\theta-\theta n \bar{z}} && (34)\\
&= \theta^{(n+a)-1} e^{-(b+n\bar{z})\theta} && (35)
\end{aligned}
$$

Therefore, $\theta \sim gamma(n+a, b+n\bar{z})$ - where $\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$

(b)

Find the FCD of each Z_i.

(Hint: For uncensored i, this distribution will be a degenerate point mass; for censored i, the resulting distribution will be a so-called truncated exponential distribution, which is proportional to a exponential density but constrained to lie in an interval. Each FCD does not depend on other Z's)

$$
p(z_i|\theta, y_i = x) = \theta e^{-\theta(z_i - c)} \tag{36}
$$

- $p(z_i|\theta, y_i \text{ is uncensored }) = 1$

$$
\begin{aligned}
p(z_i|\theta, y_i = x) &= p(z_i = c + t|\theta, z_i > c) &(37)\\
&= \frac{p(z_i = c + t \cap z_i > c|\theta)}{p(z_i > c|\theta)} &(38)\\
&= \frac{p(z_i = c + t|\theta)}{p(z_i > c|\theta)} &(39)\\
&= \frac{\theta e^{-\theta(c+t)}}{\int_c^\infty \theta e^{-\theta x} dx} &(40)\\
&= \theta e^{-\theta t} &(41)\\
&= \theta e^{-\theta(z_i - c)} &(42)
\end{aligned}
$$

This is the memoryless property of Exponential distribution[1].

In summary,

$$
p(z_i|\theta, y_i) = \begin{cases} 1, & y = z_i \\ \theta e^{-\theta(z_i - c)} & y = x \end{cases} \tag{43}
$$

(c)

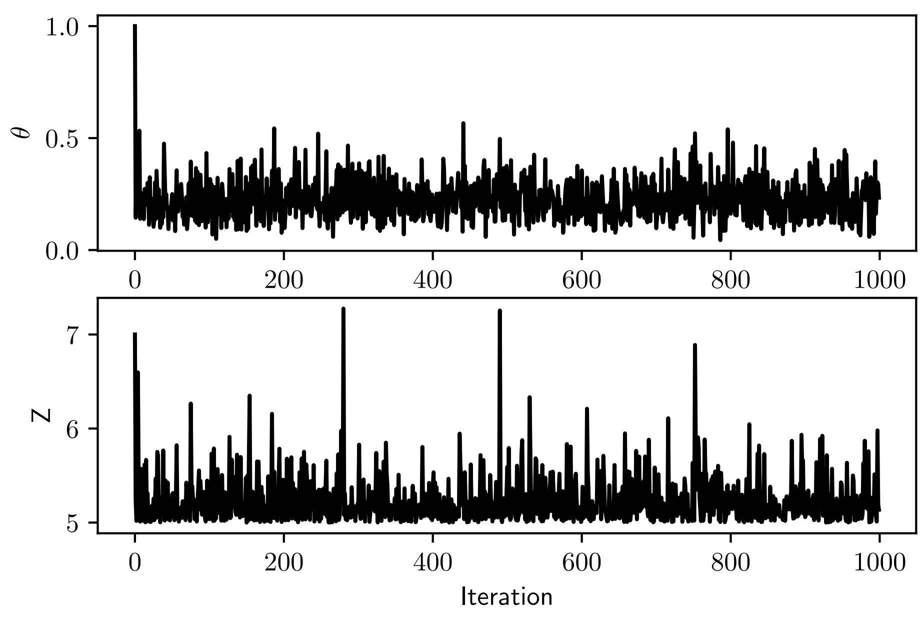
> Implement a Gibbs sampler that approximate the joint posterior of θ and $Z_1, \dots, Z_n$. (For example, you can use `truncdist::rtrunc(3, spec=exp, a=c, rate=theta)` (Use `stats.truncexpon`) to sample from a truncated exponential in R.) Run the sampler for enough iterations such that each of the effective sample sizes for θ and for the three censored Z_i are all greater than 1000. Provide the corresponding trace plots and discuss the mixing of the Markov chain.

- Both sampling converges to a range of stochastic state, that means the sampling is not stuck in certain region. The dependence between two subsequence steps is low.

```
data = np.array([3,5,4], dtype=float)
n = 6
n_cs = n - len(data)
a = 1
b = 4
c = 5
nSamp = 1000
```

[1] https://byjus.com/maths/exponential-distribution/

```python
thetas = np.zeros(nSamp)
zs = np.zeros((nSamp, n_cs))

thetas[0] = 1
zs[0] = [7,10,8]

for i in range(1, nSamp):
    df = np.concatenate([data, zs[i-1]])
    thetas[i] = st.gamma.rvs(n+a, scale= (b+np.sum(df))**-1)
    zs[i] = st.expon.rvs(loc = c, scale= thetas[i], size =3)

# Plotting
fig, ax = plt.subplots(2,1)

ax[0].plot(thetas, "-", color="k")
ax[1].plot(zs[:,0], "-", color="k")

ax[1].set_xlabel("Iteration")
ax[0].set_ylabel("$\\theta$")
ax[1].set_ylabel("Z");
```



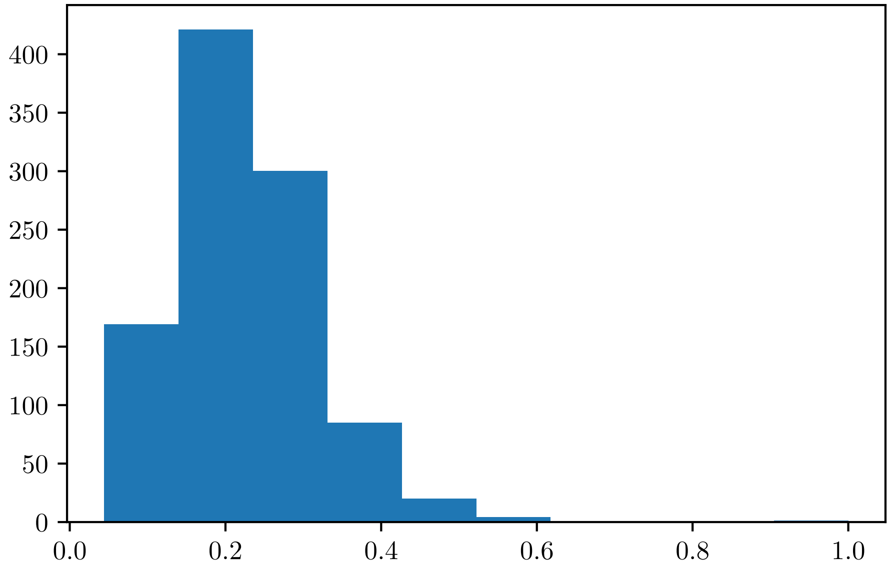
- Truncated exponential distribution is sampled by `scipy.stats.truncexpon` [2]

[2] $f(x, b) = \frac{\exp(-x)}{1-\exp(-b)}$. $y = (x - loc)/scale$

(d)

> Obtain an approximate 96% posterior credible interval for θ based on the samples from (c).

```
plt.hist(thetas);
```




```
invs = st.mstats.mquantiles(thetas, prob=[0.02, 0.98])

pd.DataFrame({"96% Credible Interval of theta": ["Left bound", "Right bound"], "Value": in
```

	96% Credible Interval of theta	Value
0	Left bound	0.085378
1	Right bound	0.444496

Hoff, Peter D. 2009. *A First Course in Bayesian Statistical Methods*. Vol. 580. Springer.