

STAT 638: Solution for Homework #1

2.1 a)

$$P(Y_1 = \text{Farm}) = 0.11$$

$$P(Y_1 = \text{Operatives}) = 0.279$$

$$P(Y_1 = \text{Craftsmen}) = 0.277$$

$$P(Y_1 = \text{Sales}) = 0.099$$

$$P(Y_1 = \text{Professional}) = 0.235$$

b)

$$P(Y_2 = \text{Farm}) = 0.023$$

$$P(Y_2 = \text{Operatives}) = 0.260$$

$$P(Y_2 = \text{Craftsmen}) = 0.240$$

$$P(Y_2 = \text{Sales}) = 0.125$$

$$P(Y_2 = \text{Professional}) = 0.352$$

c)

$$P(Y_2 = \text{Farm} | Y_1 = \text{Farm}) = 0.164$$

$$P(Y_2 = \text{Operatives} | Y_1 = \text{Farm}) = 0.318$$

$$P(Y_2 = \text{Craftsmen} | Y_1 = \text{Farm}) = 0.282$$

$$P(Y_2 = \text{Sales} | Y_1 = \text{Farm}) = 0.073$$

$$P(Y_2 = \text{Professional} | Y_1 = \text{Farm}) = 0.164$$

d)

$$P(Y_1 = \text{Farm} | Y_2 = \text{Farm}) = 0.783$$

$$P(Y_1 = \text{Operatives} | Y_2 = \text{Farm}) = 0.087$$

$$P(Y_1 = \text{Craftsmen} | Y_2 = \text{Farm}) = 0.043$$

$$P(Y_1 = \text{Sales} | Y_2 = \text{Farm}) = 0.043$$

$$P(Y_1 = \text{Professional} | Y_2 = \text{Farm}) = 0.043$$

2.2 a)

$$E(a_1 Y_1 + a_2 Y_2) = a_1 \mu_1 + a_2 \mu_2$$

$$\text{Var}(a_1 Y_1 + a_2 Y_2) = a_1^2 \text{Var}(Y_1) + a_2^2 \text{Var}(Y_2) + 2a_1 a_2 \text{Cov}(Y_1, Y_2) = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2, \text{ since independence of } Y_1 \text{ and } Y_2 \text{ implies that } \text{Cov}(Y_1, Y_2) = 0.$$

b)

$$E(a_1 Y_1 - a_2 Y_2) = a_1 \mu_1 - a_2 \mu_2$$

$$\text{Var}(a_1 Y_1 - a_2 Y_2) = a_1^2 \text{Var}(Y_1) + a_2^2 \text{Var}(Y_2) - 2a_1 a_2 \text{Cov}(Y_1, Y_2) = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2$$

2.3 a)

$$p(x|y, z) = \frac{p(x, y, z)}{p(y, z)} \propto f(x, z)g(y, z)h(z) \propto f(x, z)$$

b)

$$p(y|x, z) = \frac{p(x, y, z)}{p(x, z)} \propto f(x, z)g(y, z)h(z) \propto g(y, z)$$

c)

$$p(x, y|z) = \frac{p(x, y, z)}{p(z)} \propto f(x, z)g(y, z)h(z) \propto f(x, z)g(y, z)$$

As a function of x and y , the last expression is a product of a function of x alone and a function of y alone, and therefore X and Y are conditionally independent given Z .

2.5 a)

$$P(X = 0, Y = 0) = P(Y = 0|X = 0)P(X = 0) = 0.4(0.5) = 0.2$$

$$P(X = 0, Y = 1) = P(Y = 1|X = 0)P(X = 0) = 0.6(0.5) = 0.3$$

$$P(X = 1, Y = 0) = P(Y = 0|X = 1)P(X = 1) = 0.6(0.5) = 0.3$$

$$P(X = 1, Y = 1) = P(Y = 1|X = 1)P(X = 1) = 0.4(0.5) = 0.2$$

b)

$$E(Y) = 0 \cdot P(Y = 0) + 1 \cdot P(Y = 1) = 0.5$$

c)

The distribution of Y given $X = 0$ is Bernoulli with success probability 0.60, and hence $\text{Var}(Y|X = 0) = 0.6(0.4) = 0.24$. The distribution of Y given $X = 1$ is Bernoulli with success probability 0.40, and hence $\text{Var}(Y|X = 1) = 0.4(0.6) = 0.24$. We have

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = 1^2 \cdot P(Y = 1) - (0.5)^2 = 0.5 - 0.25 = 0.25.$$

It makes sense that $\text{Var}(Y)$ is larger than the two conditional variances since there is less uncertainty about the value of Y when we know what the value of X is.

d)

$$P(X = 0|Y = 1) = \frac{P(X = 0, Y = 1)}{P(Y = 1)} = \frac{0.3}{0.5} = 0.6$$

2.6

$$P(A^c \cap B|C) + P(A \cap B|C) = P(B|C) \implies$$

$$P(A^c \cap B|C) = P(B|C) - P(A \cap B|C) \implies$$

$$P(A^c \cap B|C) = P(B|C) - P(A|C)P(B|C) \implies$$

$$P(A^c \cap B|C) = P(B|C)(1 - P(A|C)) \implies$$

$$P(A^c \cap B|C) = P(B|C)P(A^c|C)$$

The other two proofs are done in a similar way.

Define the following probabilities:

$$P(A) = 0.5 \quad P(B) = 0.6 \quad P(C) = 0.6$$

$$P(A \cap C) = 0.2 \quad P(B \cap C) = 0.3 \quad P(A \cap B) = 0.3$$

$$P(A \cap B \cap C) = 0.1$$

We have

$$P(A \cap B|C) = \frac{0.1}{0.6} = \frac{1}{6},$$

$$P(A|C) = \frac{0.2}{0.6} = \frac{1}{3} \quad \text{and} \quad P(B|C) = \frac{0.3}{0.6} = \frac{1}{2},$$

and so A and B are conditionally independent given C . Now,

$$P(A|C^c) = \frac{P(A \cap C^c)}{P(C^c)} = \frac{0.5 - 0.2}{1 - 0.6} = \frac{3}{4},$$

$$P(B|C^c) = \frac{P(B \cap C^c)}{P(C^c)} = \frac{0.6 - 0.3}{1 - 0.6} = \frac{3}{4},$$

and

$$P(A \cap B|C^c) = \frac{P(A \cap B \cap C^c)}{P(C^c)} = \frac{0.3 - 0.1}{1 - 0.6} = \frac{1}{2}.$$

Since $(3/4)^2 = 9/16 \neq 1/2$, A and B are not conditionally independent given C^c .