Homework 5

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Description

- Course: STAT638, 2022 Fall

  Read Chapter 5 in the Hoff book.

  Then, do the following exercises in Hoff: 5.1, 5.2, 5.5.

  For 5.2, _A denotes the mean exam score for students assigned to method A, and _B denotes the mean exam score for students assigned to method B.

  Be careful: Some of the notation differs between the lecture slides and the Hoff book.

- Deadline: `Oct 12 by 12:01pm`

Computational Enviromnent Setup

Third-party libraries

```python
%matplotlib inline
import sys # system information
import matplotlib # plotting
import scipy # scientific computing
import random
import pandas as pd # data managing
from scipy.special import comb
from scipy import stats as st
from scipy.special import gamma
import numpy as np
import matplotlib.pyplot as plt
from itertools import permutations
# Matplotlib setting
plt.rcParams['text.usetex'] = True
matplotlib.rcParams['figure.dpi']= 300
np.random.seed(20220928) # Consistent random effect
```

Version

```python
print(sys.version)
print(matplotlib.__version__)
print(scipy.__version__)
print(np.__version__)
print(pd.__version__)
```

```
3.8.14 (default, Sep  6 2022, 23:26:50)
[Clang 13.1.6 (clang-1316.0.21.2.5)]
3.3.1
1.5.2
1.19.1
1.1.1
```

Problem 5.1

Studying: The files school1.dat, school2.dat and school3.dat contina data on the amount of time students from 3 high schools spent on studying or homework during an exam period. Analyze data from each of these schools separately, using the normal model with a conjugate prior distribution, in which $\{\mu_0 = 5, \sigma_0^2 = 4, \kappa_0 = 1, \nu_0 = 2\}$ and compute or approximate following:

```
schs = [np.loadtxt("data/school{}.dat".format(i)) for i in range(1,4)]
```

(a)

Posterior means and 95% confidence intervals for the mean θ and standard deviation σ from each school;

```
class NormalModel:
  def __init__(self, data, mu0, var0, k0, v0):
    # Prior
    self.mu0 = mu0 # prior mean
    self.var0= var0 # prior variance
    self.k0= k0 # prior observation
    self.v0 = v0
    # Data
    self.data = data
    self.mean = np.mean(data) # sample mean
    self.n = len(data) # sample counts
    self.kn = self.n + self.k0
    self.s2 = np.sum( (self.data- self.mean)**2 ) / (self.n - 1)

    # Posterior parameters
    self.vn = self.get_vn(self.v0, self.n) # Posterior v
    self.varn = self.get_varn(self.vn, self.v0, self.var0, self.k0, self.n, self.kn, self.
    self.mun = self.get_mun(self.k0, self.mu0, self.n, self.mean, self.kn) # Posterior mea


  def get_vn(self, v0, n):
    return v0 + n
```

```
  def get_varn(self, vn, v0, var0, k0, n, kn, mean, mu0):
    varn = (1/vn)*(v0*var0 + (n-1)*self.s2 + (k0*n/kn)*(mean - mu0)**2)
    return varn

  def get_mun(self, k0, mu0, n, mean, kn):
    return (k0*mu0 + n*mean)/kn

  def rv_theta_post(self):
    mu0 = self.mu0; k0 = self.k0;
    return st.norm(loc=self.mun, scale= np.sqrt(self.varn/self.kn))

  def rv_pred(self):
    mun = self.mun
    varn = self.varn
    return st.norm(loc=self.mun, scale= np.sqrt(self.varn))

setting = {"mu0": 5, "var0":4, "k0":1, "v0":2}
nms = [NormalModel(sch, **setting) for sch in schs]
rv_thps = [nm.rv_theta_post() for nm in nms]

# Display data
pd.DataFrame({"School":[i+1 for i in range(0, len(nms))],"Mean":[rv_thp.mean() for rv_thp
```

	School	Mean	95% confidence interval (left, right)
0	1	9.292308	(7.832422909482735, 10.752192475132652)
1	2	6.948750	(5.242990325564609, 8.654509674435392)
2	3	7.812381	(6.26475553664974, 9.360006368112167)

(b)

The posterior probability that $\theta_i < \theta_j < \theta_k$ for all size permutataions $\{i, j, k\}$ of $\{1, 2, 3\}$;

```
n_mc = 10000
samps_thp = [rv_thp.rvs(size=n_mc) for rv_thp in rv_thps]
perm = list(permutations([0,1,2]))
text = [""] * len(perm)
probs = [int] * len(perm)
for (a, i) in enumerate(perm):
  text[a] = "theta_{} < theta_{} < theta_{}".format(i[0]+1, i[1]+1, i[2]+1)
```

```
  probs[a] = np.sum(np.logical_and(samps_thp[i[0]] < samps_thp[i[1]], samps_thp[i[1]] < sa

pd.DataFrame({"Permuatations": text, "Probabilities": probs})
```

	Permuatations	Probabilities
0	theta_1 < theta_2 < theta_3	0.0058
1	theta_1 < theta_3 < theta_2	0.0030
2	theta_2 < theta_1 < theta_3	0.0788
3	theta_2 < theta_3 < theta_1	0.6929
4	theta_3 < theta_1 < theta_2	0.0123
5	theta_3 < theta_2 < theta_1	0.2072

(c)

The posterior probability that $\tilde{Y}_i < \tilde{Y}_j < \tilde{Y}_k$ for all siz permuatations $\{i, j, k\}$ of $\{1, 2, 3\}$, where $\tilde{Y}_i$ is a sample from the posterior predictive distribution of school i.

```
rv_posts = [nm.rv_pred() for nm in nms]

samps_post = [rv_post.rvs(size=n_mc) for rv_post in rv_posts]
perm = list(permutations([0,1,2]))
text = [""] * len(perm)
probs = [int] * len(perm)
for (a, i) in enumerate(perm):
  text[a] = "Y_tilde_{} < Y_tilde_{} < Y_tilde_{}".format(i[0]+1, i[1]+1, i[2]+1)
  probs[a] = np.sum(np.logical_and(samps_post[i[0]] < samps_post[i[1]], samps_post[i[1]] <

pd.DataFrame({"Permuatations": text, "Probabilities": probs})
```

	Permuatations	Probabilities
0	Y_tilde_1 < Y_tilde_2 < Y_tilde_3	0.1020
1	Y_tilde_1 < Y_tilde_3 < Y_tilde_2	0.1029
2	Y_tilde_2 < Y_tilde_1 < Y_tilde_3	0.1857
3	Y_tilde_2 < Y_tilde_3 < Y_tilde_1	0.2686
4	Y_tilde_3 < Y_tilde_1 < Y_tilde_2	0.1361
5	Y_tilde_3 < Y_tilde_2 < Y_tilde_1	0.2047

(d)

Compute the posterior probability that θ_1 is bigger than both θ_2 and θ_3, and the posterior probability that $\tilde{Y}_1$ is bigger than both $\tilde{Y}_2$ and $\tilde{Y}_3$.

```
th_biggest = np.sum(np.logical_and(samps_thp[0]>samps_thp[1], samps_thp[0]>samps_thp[2]))/
post_biggest = np.sum(np.logical_and(samps_post[0]>samps_post[1], samps_post[0]>samps_post

pd.DataFrame({"Properties": ["Theta_1 is the biggest", "Tilde_1 is the biggest"], "Probabi
```

	Properties	Probabilities
0	Theta_1 is the biggest	0.9001
1	Tilde_1 is the biggest	0.4733

Problem 5.2

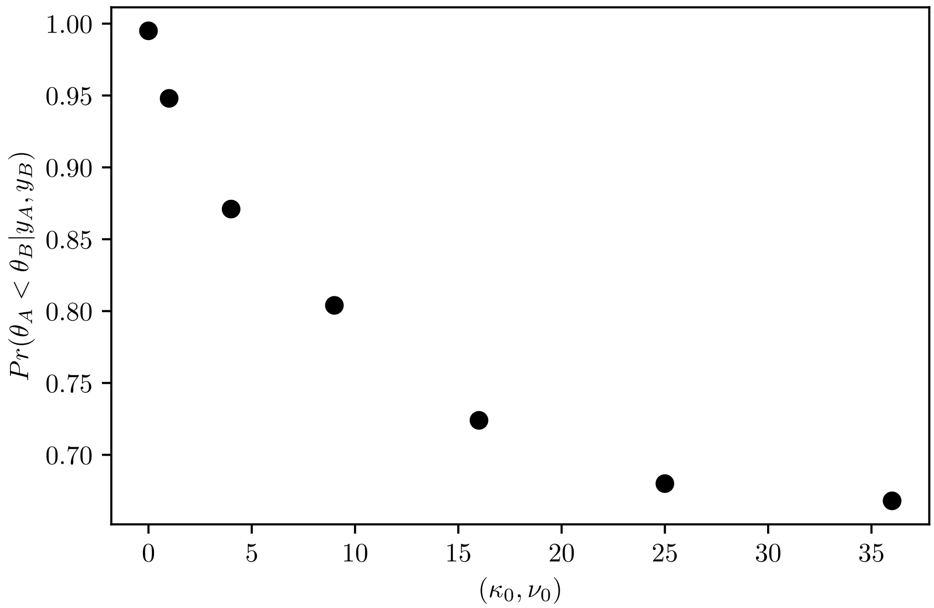
Sensitivity analysis: 32 students in a science classroom were randomly assigned to one of two study methods, A and B, so that $n_A = n_B = 16$ students were assigned to each method. After several weeks of study, students were examined on the course material with an exam designed to give an average score of 75 with a standard deviation of 10. The scores for the two groups are summarized by $\{\bar{y}_A = 75.2, s_A = 7.3\}$ and $\{\bar{y}_B = 77.5, s_b = 8.1\}$. Consider independent, conjugate normal prior distributions for each of θ_A and θ_B, with $\mu_0 = 75$ and $\sigma_0^2 = 100$ for both groups. For each $(\kappa_0, \nu_0) \in \{(1,1),(2,2),(4,4),(8,8),(16,16),(32,32)\}$ (or more values), obtain $Pr(\theta_A < \theta_B | y_A, y_B)$ via Monte Carlo Sampling. Plot this probability as a function of $\kappa_0 = \nu_0$. Describe how you might use this plot to convey the evidence that $\theta_A < \theta_B$ to people of a variety of prior opinions.

- Increase (κ_0, ν_0) can descrease the probability of B bigger that than A. But, the probability of B larger than A can hardly lower than 0.5.

```
class NormalModel:
  def __init__(self, sample_size,sample_mean, sample_var, mu0, var0, k0, v0):
    # Prior
    self.mu0 = mu0 # prior mean
    self.var0= var0 # prior variance
    self.k0= k0 # prior observation
    self.v0 = v0
    # Data
    self.mean = sample_mean # sample mean
```

```
    self.n = sample_size # sample counts
    self.kn = self.n + self.k0
    self.s2 = sample_var

    # Posterior parameters
    self.vn = self.get_vn(self.v0, self.n) # Posterior v
    self.varn = self.get_varn(self.vn, self.v0, self.var0, self.k0, self.n, self.kn, self.
    self.mun = self.get_mun(self.k0, self.mu0, self.n, self.mean, self.kn) # Posterior mea


  def get_vn(self, v0, n):
    return v0 + n

  def get_varn(self, vn, v0, var0, k0, n, kn, mean, mu0):
    varn = (1/vn)*(v0*var0 + (n-1)*self.s2 + (k0*n/kn)*(mean - mu0)**2)
    return varn

  def get_mun(self, k0, mu0, n, mean, kn):
    return (k0*mu0 + n*mean)/kn

  def rv_theta_post(self):
    mu0 = self.mu0; k0 = self.k0;
    return st.norm(loc=self.mun, scale= np.sqrt(self.varn/self.kn))

  def rv_pred(self):
    mun = self.mun
    varn = self.varn
    return st.norm(loc=self.mun, scale= np.sqrt(self.varn))

def expAB(k0,v0,size=1000):
  setting = {"sample_size":16, "mu0": 75, "var0": 100, "k0":k0, "v0":v0}
  rvtA = NormalModel(sample_mean=75.2,\
    sample_var=7.3, **setting).rv_theta_post()
  rvtB = NormalModel(sample_mean=77.5,\
    sample_var=8.1, **setting).rv_theta_post()

  spA, spB = rvtA.rvs(size=size), rvtB.rvs(size=size)
  return np.sum(spA < spB)/size

pars = [i**2 for i in range(0, 7)]
vals = [float]*len(pars)
```

```
for (i, p) in enumerate(pars):
  vals[i] = expAB(p,p)

# Plotting
figV, axV = plt.subplots()
axV.plot(pars,vals, 'o', color="k")
axV.set_xlabel("$(\\kappa_0, \\nu_0)$")
axV.set_ylabel("$Pr(\\theta_A < \\theta_B | y_A, y_B)$");
```



Problem 5.5

Unit information prior: Obtain a unit information prior for the normal model as follows:

(a)

Reparameterize the normal model as $p(y|\theta, \psi)$, where $\psi = \frac{1}{\sigma^2}$. Write out the log likelihood $l(\theta, \psi|y) = \sum \log p(y_i|\theta, \psi)$ in terms of θ and ψ.

$$p(y|\theta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{y-\theta}{\sigma})^2}$$

$$p(y|\theta,\psi) = \frac{\sqrt{\psi}}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\theta)^2\psi}$$

$$\log p(y_i|\theta,\psi) = \log\left(\sqrt{\frac{\psi}{2\pi}}\right) - \frac{1}{2}(y_i-\theta)^2\psi$$

$$\begin{align}
l(\theta,\psi|y) = \sum_{i=1}^{n} \log(y_i|\theta,\psi) &= \log\left(\frac{\psi}{2\pi}\right)^{n/2} - \frac{1}{2}\psi\sum_{i=1}^{n}(y_i-\theta)^2 \tag{1}\\
&= \frac{n}{2}\log\left(\frac{\psi}{2\pi}\right) - \frac{\psi}{2}\sum_{i=1}^{n}(y_i-\theta)^2 \tag{2}\\
&= \frac{n}{2}\log\left(\frac{\psi}{2\pi}\right) - \frac{\psi}{2}\sum_{i=1}^{n}(y_i^2-2\theta y_i+\theta^2) \tag{3}\\
&= \frac{n}{2}\log\left(\frac{\psi}{2\pi}\right) - \frac{\psi}{2}(\sum y_i^2 - 2\theta\sum y_i + n\theta^2) \tag{4}\\
&= \frac{-\psi}{2}\sum_{i=1}^{n} y_i^2 + \psi\theta\sum_{i=1}^{n} y_i - \frac{n\psi\theta^2}{2} + \frac{n}{2}\log\left(\frac{\psi}{2\pi}\right) \tag{5}
\end{align}$$

(b)

Find a probability density $P_U(\theta,\psi)$ so that $\log p_U(\theta,\psi) = \frac{l(\theta,\psi|y)}{n} + c$, where c is a constant that does not depend on θ or ψ.

Hint: Write $\sum(y_i-\theta)^2$ as $\sum(y_i-\bar{y}+\bar{y}-\theta)^2 = \sum(y_i-\bar{y})^2 + n(\theta-\bar{y})^2$, and recall that $\log p_U(\theta,\psi) = \log p_U(\theta|\psi) + \log p_U(\psi)$.

$$\begin{align}
l(\theta,\psi|y) &= \frac{n}{2}\log\left(\frac{\psi}{2\pi}\right) - \frac{\psi}{2}\sum_{i=1}^{n}(y_i-\theta)^2 \tag{6}\\
&= \frac{n}{2}\log\left(\frac{\psi}{2\pi}\right) - \frac{\psi}{2}\left[\sum(y_i-\bar{y})^2 + n(\theta-\bar{y})^2\right] \tag{7}\\
&= \frac{n}{2}\log\left(\frac{\psi}{2\pi}\right) - \frac{\psi}{2}\left[(n-1)s^2 + n(\theta-\bar{y})^2\right] \tag{8}\\
&= \frac{n}{2}\log\left(\frac{\psi}{2\pi}\right) - \frac{\psi}{2}\left((n-1)s^2 + n\theta^2 - 2n\theta\bar{y} + n\bar{y}^2\right) \tag{9}\\
&= \frac{n}{2}\log\left(\frac{\psi}{2\pi}\right) - \frac{\psi}{2}(n-1)s^2 - \frac{\psi n\theta^2}{2} + n\psi\theta\bar{y} - \frac{\psi n\bar{y}^2}{2} \tag{10}\\
&= \frac{n}{2}\log\left(\frac{\psi}{2\pi}\right) - \frac{\psi}{2}(n-1)s^2 - \frac{\psi n\bar{y}^2}{2} + n\psi\theta\bar{y} - \frac{\psi n\theta^2}{2} \tag{11}
\end{align}$$

$$\frac{l(\theta,\psi|y)}{n}+c=\underbrace{\frac{1}{2}\log\left(\frac{\psi}{2\pi}\right)-\frac{\psi}{2n}(n-1)s^2-\frac{\psi\bar{y}^2}{2}}_{=h(\psi|y)}+\underbrace{\psi\theta\bar{y}-\frac{\psi\theta^2}{2}}_{=g(\theta|\psi,y)}+c \tag{12}$$

$$\begin{aligned}
p_U(\theta,\psi) &= \exp\left(\frac{l(\theta,\psi|y)}{n}+c\right) && (13)\\
&\propto \exp\left(\frac{l(\theta,\psi|y)}{n}\right) && (14)\\
&= \exp(h(\psi|y))\exp(g(\theta|\psi,y)) && (15)
\end{aligned}$$

$$\begin{aligned}
\exp(h(\psi|y)) &= (\frac{\psi}{2\pi})^{1/2}\exp\left(-(\frac{(n-1)s^2}{2n}+\frac{\bar{y}^2}{2})\psi\right) && (16)\\
&\propto \psi^{\frac{1}{2}}\exp\left(-\frac{\psi}{(\frac{(n-1)s^2}{2n}+\frac{\bar{y}^2}{2})^{-1}}\right) && (17)\\
&\sim Gamma(\psi,\frac{3}{2},(\frac{(n-1)s^2}{2n}+\frac{\bar{y}^2}{2})^{-1}) && (18)
\end{aligned}$$

$$\begin{aligned}
\exp(g(\theta|\psi,y)) &= \exp\left(\psi\theta\bar{y}-\frac{\psi\theta^2}{2}\right) && (19)\\
&= \exp\left(-\frac{\psi}{2}\theta^2+\psi\bar{y}\theta-\frac{\psi\bar{y}^2}{2}+\frac{\psi\bar{y}^2}{2}\right) && (20)\\
&= \exp\left(-\frac{\psi}{2}(\theta^2-2\bar{y}\theta+\bar{y}^2)+\frac{\psi\bar{y}^2}{2}\right) && (21)\\
&= \exp\left(-\frac{\psi}{2}(\theta-\bar{y})^2+\frac{\psi\bar{y}^2}{2}\right) && (22)\\
&\propto \exp\left(-\frac{\psi}{2}(\theta-\bar{y})^2\right) && (23)\\
&= \exp\left(-\frac{1}{2}\left(\frac{\theta-\bar{y}}{\psi^{-1/2}}\right)^2\right) && (24)\\
&\propto Normal(\theta,\bar{y},\psi^{-1/2}) && (25)
\end{aligned}$$

Thus[1],

[1] Use the PDF formula on Wiki

$$P_U(\theta,\psi) = P_U(\theta|\psi)P_U(\psi) \propto Gamma(\psi, \frac{3}{2}, (\frac{(n-1)s^2}{2n} + \frac{\bar{y}^2}{2})^{-1}) \tag{26}$$
$$\times\, Normal(\theta, \bar{y}, \psi^{-1/2}) \tag{27}$$
$$\tag{28}$$

(c)

Find a probability density $p_U(\theta,\psi|y)$ that is proportional to $p_U(\theta,\psi) \times p(y_1,\ldots,y_n|\theta,\psi)$. It may be convenient to write this joint density as $p_U(\theta|\psi,y) \times p_U(\psi|y)$. Can this joint density be considered a posterior density?

$$P_U(\theta,\psi|y) = P_U(\psi|y) \times p_U(\theta|\psi,y) \tag{29}$$
$$= Gamma(\psi, \frac{3}{2}, (\frac{(n-1)s^2}{2n} + \frac{\bar{y}^2}{2})^{-1}) \tag{30}$$
$$\times\, Normal(\theta, \bar{y}, \psi^{-1/2}) \tag{31}$$
$$\propto \psi^{1/2} \exp\left(-(\frac{(n-1)s^2}{2n} + \frac{\bar{y}^2}{2})\psi\right) \times \exp\left(-\frac{1}{2}\left(\frac{\theta-\bar{y}}{\psi^{-1/2}}\right)^2\right) \tag{32}$$
$$\propto \psi^{1/2} \exp\left(-(\frac{(n-1)s^2}{2n} + \frac{\bar{y}^2}{2})\psi\right) \times \exp\left(-\frac{\psi}{2}\left(\theta-\bar{y}\right)^2\right) \tag{33}$$
$$\propto \psi^{1/2} \exp\left(-(\frac{(n-1)s^2}{2n} + \frac{\bar{y}^2}{2})\psi\right) \times \exp\left(-\frac{\psi}{2}\left(\theta^2-2\theta\bar{y}+\bar{y}^2\right)\right) \tag{34}$$
$$\psi^{1/2} \exp\left(-\left(\frac{(n-1)s^2}{2n} + \frac{\bar{y}^2}{2} + \frac{\bar{y}^2}{2}\right)\psi\right) \times \exp\left(-\frac{\psi}{2}(\theta^2-2\theta\bar{y})\right) \tag{35}$$

- $p_U(\theta|\psi,y) \propto Normal(\theta, \bar{y}, \psi^{-1/2})$
- $p_U(\psi|\theta,y) \sim Gamma(\frac{3}{2}, (\frac{(n-1)s^2}{2n} + \frac{\bar{y}^2}{2})^{-1})$
- Yes, it can be considered as a posterior density because the unit information prior is the product of two proper distribution.

References