Homework 1

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Homework Description

Read Chapters 1 and 2 in the Hoff book. Then, do the following exercises in Hoff (p. 225-226): 2.1, 2.2, 2.3, 2.5 You must turn in your solutions as a pdf file here on Canvas. Use the Submit Assignment button on the top right. If your solutions are on paper, please scan them to pdf using a scanner or a scanner app on your phone. Please do not take a regular photo, as this can result in very large file sizes. Make sure that everything is legible. Please note that late homework will not be accepted and will result in a score of zero. To avoid late submissions due to technical issues, we recommend turning in your homework the night before the due date.

- Deadline: Sep. 8 by 12:01pm

Problem 2.1

Marginal and conditional probability: The social mobility data from Section 2.5 gives a joint probability distribution on (Y_1, Y_2) = (fathers's occupation, son's occupation)

Table 1: The social mobility data (Hoff 2009, 580:24)

			son's occupation		
father's occupation	farm	operatives	craftsmen	sales	professional
farm	0.018	0.035	0.031	0.008	0.018
operatives	0.002	0.112	0.064	0.032	0.069
craftsman	0.001	0.066	0.094	0.032	0.084
sales	0.001	0.018	0.019	0.010	0.051
professional	0.001	0.029	0.032	0.043	0.130

(a) The marginal probability distribution of a father's occupation

According to Table 1, let $\mathbb{Y}_{\mathbb{1}}$ and $\mathbb{Y}_2$ be sets of father's and son's occupations:

$$\mathbb{Y}_{\mathbb{1}} = \mathbb{Y}_2 = \{\text{farm}, \text{operatives}, \text{craftsmen}, \text{sales}, \text{professional}\}$$

$$\begin{aligned}
Pr(Y_1 = \text{farm}) &= \sum_{y_2 \in \mathbb{Y}_2} Pr(Y_1 = \text{farm} \cap Y_2 = y_2) \\
&= Pr(Y_1 = \text{farm} \cap Y_2 = \text{farm}) + Pr(Y_1 = \text{farm} \cap Y_2 = \text{operatives}) \\
&+ Pr(Y_1 = \text{farm} \cap Y_2 = \text{craftsmen}) + Pr(Y_1 = \text{farm} \cap Y_2 = \text{sales}) \\
&+ Pr(Y_1 = \text{farm} \cap Y_2 = \text{professional}) \\
&= 0.018 + 0.035 + 0.031 + 0.008 + 0.018 \\
&= 0.11
\end{aligned}$$

$$\begin{aligned}
Pr(Y_1 = \text{operatives}) &= \sum_{y_2 \in \mathbb{Y}_2} Pr(Y_1 = \text{operatives} \cap Y_2 = y_2) \\
&= Pr(Y_1 = \text{operatives} \cap Y_2 = \text{farm}) + Pr(Y_1 = \text{operatives} \cap Y_2 = \text{operatives}) \\
&+ Pr(Y_1 = \text{operatives} \cap Y_2 = \text{craftsmen}) + Pr(Y_1 = \text{operatives} \cap Y_2 = \text{sales}) \\
&+ Pr(Y_1 = \text{operatives} \cap Y_2 = \text{professional}) \\
&= 0.002 + 0.112 + 0.064 + 0.032 + 0.069 \\
&= 0.279
\end{aligned}$$

$$\begin{aligned}
Pr(Y_1 = \text{craftsman}) &= \sum_{y_2 \in \mathbb{Y}_2} Pr(Y_1 = \text{craftsman} \cap Y_2 = y_2) \\
&= Pr(Y_1 = \text{craftsman} \cap Y_2 = \text{farm}) + Pr(Y_1 = \text{craftsman} \cap Y_2 = \text{operatives}) \\
&+ Pr(Y_1 = \text{craftsman} \cap Y_2 = \text{craftsmen}) + Pr(Y_1 = \text{craftsman} \cap Y_2 = \text{sales}) \\
&+ Pr(Y_1 = \text{craftsman} \cap Y_2 = \text{professional}) \\
&= 0.001 + 0.066 + 0.094 + 0.032 + 0.084 \\
&= 0.277
\end{aligned}$$

$$\begin{aligned}
Pr(Y_1 = \text{sales}) &= \sum_{y_2 \in \mathbb{Y}_2} Pr(Y_1 = \text{sales} \cap Y_2 = y_2) \\
&= Pr(Y_1 = \text{sales} \cap Y_2 = \text{farm}) + Pr(Y_1 = \text{sales} \cap Y_2 = \text{operatives}) \\
&+ Pr(Y_1 = \text{sales} \cap Y_2 = \text{craftsmen}) + Pr(Y_1 = \text{sales} \cap Y_2 = \text{sales}) \\
&+ Pr(Y_1 = \text{sales} \cap Y_2 = \text{professional}) \\
&= 0.001 + 0.018 + 0.019 + 0.010 + 0.051 \\
&= 0.099
\end{aligned}$$

$$
\begin{aligned}
Pr(Y_1 = \text{professional}) &= \sum_{y_2 \in \mathbb{Y}_2} Pr(Y_1 = \text{professional} \cap Y_2 = y_2) \\
&= Pr(Y_1 = \text{professional} \cap Y_2 = \text{farm}) + Pr(Y_1 = \text{professional} \cap Y_2 = \text{operatives}) \\
&+ Pr(Y_1 = \text{professional} \cap Y_2 = \text{craftsmen}) + Pr(Y_1 = \text{professional} \cap Y_2 = \text{sales}) \\
&+ Pr(Y_1 = \text{professional} \cap Y_2 = \text{professional}) \\
&= 0.001 + 0.029 + 0.032 + 0.043 + 0.130 \\
&= 0.235
\end{aligned}
$$

Table 2: Marginal probability of father's occupation

marginal probability	value
$p(Y_1 = \text{farm})$	0.11
$p(Y_1 = \text{operatives})$	0.279
$p(Y_1 = \text{craftsmen})$	0.277
$p(Y_1 = \text{sales})$	0.099
$p(Y_1 = \text{professional})$	0.235
SUM	1.0

Table 2 shows that the sum of marginal probability is 1.

(b) The marginal probability distribution of a son's occupation

$$
\begin{aligned}
Pr(Y_2 = \text{farm}) &= \sum_{y_1 \in \mathbb{Y}_1} Pr(Y_2 = \text{farm} \cap Y_1 = y_1) \\
&= Pr(Y_2 = \text{farm} \cap Y_1 = \text{farm}) + Pr(Y_2 = \text{farm} \cap Y_1 = \text{operatives}) \\
&+ Pr(Y_2 = \text{farm} \cap Y_1 = \text{craftsmen}) + Pr(Y_2 = \text{farm} \cap Y_1 = \text{sales}) \\
&+ Pr(Y_2 = \text{farm} \cap Y_1 = \text{professional}) \\
&= 0.018 + 0.002 + 0.001 + 0.001 + 0.001 \\
&= 0.023
\end{aligned}
$$

$$
\begin{aligned}
Pr(Y_2 = \text{operatives}) = & \sum_{y_1 \in \mathbb{Y}_1} Pr(Y_2 = \text{operatives} \cap Y_1 = y_1) \\
& = Pr(Y_2 = \text{operatives} \cap Y_1 = \text{farm}) + Pr(Y_2 = \text{operatives} \cap Y_1 = \text{operatives}) \\
& + Pr(Y_2 = \text{operatives} \cap Y_1 = \text{craftsmen}) + Pr(Y_2 = \text{operatives} \cap Y_1 = \text{sales}) \\
& + Pr(Y_2 = \text{operatives} \cap Y_1 = \text{professional}) \\
& = 0.035 + 0.112 + 0.066 + 0.018 + 0.029 \\
& = 0.26
\end{aligned}
$$

$$
\begin{aligned}
Pr(Y_2 = \text{craftsmen}) = & \sum_{y_1 \in \mathbb{Y}_1} Pr(Y_2 = \text{craftsmen} \cap Y_1 = y_1) \\
& = Pr(Y_2 = \text{craftsmen} \cap Y_1 = \text{farm}) + Pr(Y_2 = \text{craftsmen} \cap Y_1 = \text{operatives}) \\
& + Pr(Y_2 = \text{craftsmen} \cap Y_1 = \text{craftsmen}) + Pr(Y_2 = \text{craftsmen} \cap Y_1 = \text{sales}) \\
& + Pr(Y_2 = \text{craftsmen} \cap Y_1 = \text{professional}) \\
& = 0.031 + 0.064 + 0.094 + 0.019 + 0.032 \\
& = 0.24
\end{aligned}
$$

$$
\begin{aligned}
Pr(Y_2 = \text{sales}) = & \sum_{y_1 \in \mathbb{Y}_1} Pr(Y_2 = \text{sales} \cap Y_1 = y_1) \\
& = Pr(Y_2 = \text{sales} \cap Y_1 = \text{farm}) + Pr(Y_2 = \text{sales} \cap Y_1 = \text{operatives}) \\
& + Pr(Y_2 = \text{sales} \cap Y_1 = \text{craftsmen}) + Pr(Y_2 = \text{sales} \cap Y_1 = \text{sales}) \\
& + Pr(Y_2 = \text{sales} \cap Y_1 = \text{professional}) \\
& = 0.008 + 0.032 + 0.032 + 0.010 + 0.043 \\
& = 0.125
\end{aligned}
$$

$$
\begin{aligned}
Pr(Y_2 = \text{professional}) = & \sum_{y_1 \in \mathbb{Y}_1} Pr(Y_2 = \text{professional} \cap Y_1 = y_1) \\
& = Pr(Y_2 = \text{professional} \cap Y_1 = \text{farm}) + Pr(Y_2 = \text{professional} \cap Y_1 = \text{operatives}) \\
& + Pr(Y_2 = \text{professional} \cap Y_1 = \text{craftsmen}) + Pr(Y_2 = \text{professional} \cap Y_1 = \text{sales}) \\
& + Pr(Y_2 = \text{professional} \cap Y_1 = \text{professional}) \\
& = 0.018 + 0.069 + 0.084 + 0.051 + 0.130 \\
& = 0.352
\end{aligned}
$$

Table 3: Marginal probability of son's occupation

marginal probability	value
$p(Y_2 = \text{farm})$	0.023
$p(Y_2 = \text{operatives})$	0.26
$p(Y_2 = \text{craftsmen})$	0.24
$p(Y_2 = \text{sales})$	0.125
$p(Y_2 = \text{professional})$	0.352
SUM	1.0

Table 3 shows that the sum of marginal probability is 1.

(c) The conditional distribution of a son's occupation, given that the father is a farmer;

The conditional distribution of a son's occupation can be expressed as $p(y_2|y_1 = \text{farmer})$.

$$p(y_2 = *|y_1 = \text{farmer}) = \frac{p(y_1 = \text{farm} \cap y_2 = *)}{p(y_1 = \text{farm})}$$

where $* \in \mathbb{Y}_2$. As described in Table 2, $p(y_1 = \text{farm}) = 0.11$. Use Table 1 to calculate the distribution:

$$\begin{aligned}
p(y_2 = \text{farm}|y_1 = \text{farm}) &= \frac{0.018}{0.11} \approx 0.16 \\
p(y_2 = \text{operatives}|y_1 = \text{farm}) &= \frac{0.035}{0.11} \approx 0.32 \\
p(y_2 = \text{craftsman}|y_1 = \text{farm}) &= \frac{0.031}{0.11} \approx 0.28 \\
p(y_2 = \text{sales}|y_1 = \text{farm}) &= \frac{0.008}{0.11} \approx 0.072 \\
p(y_2 = \text{professional}|y_1 = \text{farm}) &= \frac{0.018}{0.11} \approx 0.16
\end{aligned}$$

(d) The conditional distribution of a father's occupation, given that the son is a farmer.

$$p(y_1 = *|y_2 = \text{farm}) = \frac{p(y_1 = * \cap y_2 = \text{farm})}{p(y_2 = \text{farm})}$$

According to Table 3, $p(y_2 = \text{farm}) = 0.023$. Use Table 1 to calculate the distribution:

$$
\begin{aligned}
p(y_1 = \text{farm}|y_2 = \text{farm}) &= \frac{0.018}{0.023} \approx 0.78 \\
p(y_1 = \text{operatives}|y_2 = \text{farm}) &= \frac{0.002}{0.023} \approx 0.09 \\
p(y_1 = \text{craftsman}|y_2 = \text{farm}) &= \frac{0.001}{0.023} \approx 0.04 \\
p(y_1 = \text{sales}|y_2 = \text{farm}) &= \frac{0.001}{0.023} \approx 0.04 \\
p(y_1 = \text{professional}|y_2 = \text{farm}) &= \frac{0.001}{0.023} \approx 0.04
\end{aligned}
$$

Problem 2.2

Expectations and variances: Let Y_1 and Y_2 be two independent random variables, such that $E[Y_i] = \mu_i$ and $Var[Y_i] = \sigma_i^2$. Using the definition of expectation and variance, computing the following quantities, where a_1 and a_2 are given constants.

(a) $E[a_1Y_1 + a_2Y_2]$, $Var[a_1Y_1 + a_2Y_2]$

Because Y_1 and Y_2 are independent:

$$E[Y_1Y_2] = E[Y_1]E[Y_2]$$

Thus

$$
\begin{aligned}
E[a_1Y_1 + a_2Y_2] &= E[a_1Y_1] + E[a_2Y_2] \\
&= a_1E[Y_1] + a_2E[Y_2] \\
&= \underline{a_1\mu_1 + a_2\mu_2}
\end{aligned}
$$

$$
\begin{aligned}
Var[a_1Y_1 + a_2Y_2] &= E[[(a_1Y_1 + a_2Y_2) - E[a_1Y_1 + a_2Y_2]]^2] \\
&= E[(a_1Y_1 + a_2Y_2)^2] - E[a_1Y_1 + a_2Y_2]^2 \\
&= E[a_1^2Y_1^2 + 2a_1a_2Y_1Y_2 + a_2^2Y_2^2] - (a_1\mu_1 + a_2\mu_2)^2 \\
&= a_1^2\sigma_1^2 + 2a_1a_2 \underbrace{\mu_1\mu_2}_{E[Y_1Y_2]=E[Y_1]E[Y_2]} + a_2^2\sigma_2^2 - (a_1\mu_1 + a_2\mu_2)^2 \\
&= a_1^2\sigma_1^2 + 2a_1a_2\mu_1\mu_2 + a_2^2\sigma_2^2 - (a_1^2\mu_1^2 + 2a_1a_2\mu_1\mu_2 + a_2^2\mu_2^2) \\
&= \underline{a_1^2(\sigma_1^2 - \mu_1^2) + a_2^2(\sigma_2^2 - \mu_2^2)}
\end{aligned}
$$

(b) $E[a_1Y_1 - a_2Y_2]$, $Var[a_1Y_1 - a_2Y_2]$

$$\begin{aligned} E[a_1Y_1 - a_2Y_2] &= E[a_1Y_1] - E[a_2Y_2] \\ &= a_1E[Y_1] - a_2E[Y_2] \\ &= \underline{a_1\mu_1 - a_2\mu_2} \end{aligned}$$

$$\begin{aligned} Var[a_1Y_1 - a_2Y_2] &= E[[(a_1Y_1 - a_2Y_2) - E[a_1Y_1 - a_2Y_2]]^2] \\ &= E[(a_1Y_1 - a_2Y_2)^2] - E[a_1Y_1 - a_2Y_2]^2 \\ &= E[a_1^2Y_1^2 - 2a_1a_2Y_1Y_2 + a_2^2Y_2^2] - (a_1\mu_1 - a_2\mu_2)^2 \\ &= a_1^2\sigma_1^2 - 2a_1a_2 \underbrace{\mu_1\mu_2}_{E[Y_1Y_2]=E[Y_1]E[Y_2]} + a_2^2\sigma_2^2 - (a_1\mu_1 - a_2\mu_2)^2 \\ &= a_1^2\sigma_1^2 + 2a_1a_2\mu_1\mu_2 + a_2^2\sigma_2^2 - (a_1^2\mu_1^2 - 2a_1a_2\mu_1\mu_2 + a_2^2\mu_2^2) \\ &= \underline{a_1^2(\sigma_1^2 - \mu_1^2) + a_2^2(\sigma_2^2 - \mu_2^2) + 4a_1a_2\mu_1\mu_2} \end{aligned}$$

Problem 2.3

Full conditionals: Let X, Y, Z be random variables with joint density (discrete or continuous) $p(x, y, z) \propto f(x, z)g(y, z)h(z)$. Show that

(a) $p(x|y, z) \propto f(x, z)$ **i.e.** $p(x|y, z)$ **is a function of** x **and** z**;**

Let $c, d \in \mathbb{R}$ constants

$$\begin{aligned} p(x|y, z) &= \frac{p(x, y, z)}{p(y, z)} \\ &= \frac{c \cdot f(x, z)g(y, z)h(z)}{p(y, z)} \qquad = \frac{c \cdot f(x, z)g(y, z)h(z)}{\int_{x \in \mathbb{X}} p(x, y, z)dx} \\ &= \frac{c \cdot f(x, z)g(y, z)h(z)}{d \cdot \int_{x \in \mathbb{X}} f(x, z)g(y, z)h(z)dx} \\ &= \frac{c \cdot f(x, z)g(y, z)h(z)}{d \cdot g(y, z)h(z) \int_{x \in \mathbb{X}} f(x, z)dx} \\ &= \frac{c \cdot f(x, z)}{d \cdot \int_{x \in \mathbb{X}} f(x, z)dx} \\ &\propto \underline{\frac{f(x, z)}{\int_{x \in \mathbb{X}} f(x, z)dx}} \end{aligned}$$

Thus, $p(x|y, z)$ is a function of $f(x, z)$.

(b) $p(y|x,z) \propto g(y,z)$ **i.e.** $p(y|x,z)$ **is a function of** y **and** z**;**

Let $c, d \in \mathbb{R}$ constants

$$\begin{aligned}
p(y|x,z) &= \frac{p(x,y,z)}{p(x,z)} \\
&= \frac{p(x,y,z)}{\int_{y\in\mathbb{Y}} p(x,y,z)dy} \\
&= \frac{c \cdot f(x,z)g(y,z)h(z)}{d \cdot \int_{y\in\mathbb{Y}} f(x,z)g(y,z)h(z)dy} \\
&= \frac{c \cdot f(x,z)g(y,z)h(z)}{d \cdot f(x,z)h(z) \int_{y\in\mathbb{Y}} g(y,z)dy} \\
&\propto \frac{f(y,z)}{\int_{y\in\mathbb{Y}} g(y,z)dy}
\end{aligned}$$

(c) X **and** Y **are conditionally independent given** Z**.**

Let $a_1, a_2 \in \mathbb{R}$ constant,

$$\begin{aligned}
p(x|z) &= \frac{p(x,z)}{p(z)} \\
&= \frac{\int_{y\in\mathbb{Y}} p(x,y,z)dy}{\int_{x\in\mathbb{X}}\int_{y\in\mathbb{Y}} p(x,y,z)dydx} \\
&= \frac{\int_{y\in\mathbb{Y}} f(x,z)g(y,z)h(z)dy}{\int_{x\in\mathbb{X}}\int_{y\in\mathbb{Y}} p(x,y,z)dydx} \\
&= \frac{f(x,z)h(z)\int_{y\in\mathbb{Y}} g(y,z)dy}{\int_{x\in\mathbb{X}}\int_{y\in\mathbb{Y}} p(x,y,z)dydx} \\
&= \frac{f(x,z)\int_{y\in\mathbb{Y}} g(y,z)dy}{\int_{x\in\mathbb{X}}\int_{y\in\mathbb{Y}} f(x,z)g(y,z)dydx}
\end{aligned}$$

$$
\begin{aligned}
p(y|z) &= \frac{p(y,z)}{p(z)} \\
&= \frac{\int_{x\in\mathbb{X}} p(x,y,z)dx}{\int_{x\in\mathbb{X}}\int_{y\in\mathbb{Y}} p(x,y,z)dydx} \\
&= \frac{\int_{x\in\mathbb{X}} f(x,z)g(y,z)h(z)dx}{\int_{x\in\mathbb{X}}\int_{y\in\mathbb{Y}} p(x,y,z)dydx} \\
&= \frac{g(y,z)h(z)\int_{x\in\mathbb{X}} f(x,z)dx}{\int_{x\in\mathbb{X}}\int_{y\in\mathbb{Y}} p(x,y,z)dydx} \\
&= \frac{g(y,z)\int_{x\in\mathbb{X}} f(x,z)dx}{\int_{x\in\mathbb{X}}\int_{y\in\mathbb{Y}} f(x,z)g(y,z)dydx}
\end{aligned}
$$

$$
\begin{aligned}
p(x|z)p(y|z) &= \frac{f(x,z)\int_{y\in\mathbb{Y}} g(y,z)dy}{\int_{x\in\mathbb{X}}\int_{y\in\mathbb{Y}} f(x,z)g(y,z)dydx} \cdot \frac{g(y,z)\int_{x\in\mathbb{X}} f(x,z)dx}{\int_{x\in\mathbb{X}}\int_{y\in\mathbb{Y}} f(x,z)g(y,z)dydx} \\
&= \frac{f(x,z)g(y,z)}{\int_{x\in\mathbb{X}}\int_{y\in\mathbb{Y}} f(x,z)g(y,z)dydx}
\end{aligned}
$$

$$
\begin{aligned}
p(x,y|z) &= \frac{p(x,y,z)}{p(z)} \\
&= \frac{f(x,z)g(y,z)h(z)}{\int_{x\in\mathbb{X}}\int_{y\in\mathbb{Y}} p(x,y,z)dydx} \\
&= \frac{f(x,z)g(y,z)h(z)}{\int_{x\in\mathbb{X}}\int_{y\in\mathbb{Y}} f(x,z)g(y,z)h(z)dydx} \\
&= \frac{f(x,z)g(y,z)h(z)}{\int_{x\in\mathbb{X}} f(x,z)h(z)\left[\int_{y\in\mathbb{Y}} g(y,z)dy\right]dx} \\
&= \frac{f(x,z)g(y,z)h(z)}{\int_{x\in\mathbb{X}} f(x,z)h(z)dx \cdot \int_{y\in\mathbb{Y}} g(y,z)dy} \\
&= \frac{f(x,z)}{\int_{x\in\mathbb{X}} f(x,z)dx}\frac{g(y,z)}{\int_{y\in\mathbb{Y}} g(y,z)dy} \\
&= \frac{f(x,z)g(y,z)}{\int_{x\in\mathbb{X}}\int_{y\in\mathbb{Y}} f(x,z)g(y,z)dydx} \\
&= \underline{p(x|z)p(y|z)}
\end{aligned}
$$

Thus, $p(x,y|z) = p(x|z)p(y|z)$ that means $p(x,y|z)$ is conditionally independent given z.

Problem 2.5

Urns: Suppose urn H is filled with 40% green balls and 60% red balls, and urn T is filled with 60% green balls and 40% red balls. Someone will flip a coin and then select a ball from urn H or urn T depending on whether the coin lands heads or tails, respectively. Let X be 1 or 0 if the coin lands heads or tails, and let Y be 1 or 0 if the ball is green or red.

Table 4: Probability of choosing a certain ball in a given urn.

	Green	Red
H (chosen if head[1])	0.4	0.6
T (chosen if tail[0])	0.6	0.4

Table 5: Event coding

	1	0
X	Head	Tail
Y	Green	Red

(a) Write out the joint distribution of X and Y in a table.

Suppose the coin is fair,

$$
\begin{aligned}
p(X = 0 \cap Y = 0) = p(X = 0)p(Y = 0|X = 0) = 0.5 \cdot 0.4 = 0.2 \\
p(X = 0 \cap Y = 1) = p(X = 0)p(Y = 1|X = 0) = 0.5 \cdot 0.6 = 0.3 \\
p(X = 1 \cap Y = 0) = p(X = 1)p(Y = 0|X = 1) = 0.5 \cdot 0.6 = 0.3 \\
p(X = 1 \cap Y = 1) = p(X = 1)p(Y = 1|X = 1) = 0.5 \cdot 0.4 = 0.2
\end{aligned}
$$

(b) Find $E[Y]$. What is the probability that the ball is green?

$$
\begin{aligned}
E[Y] &= \sum_{y\in\{0,1\}} p(Y=y)y \\
&= p(Y=1)\cdot 1 \\
&= \sum_{x\in\{0,1\}} p(Y=1|X=x)p(X=x) \\
&= p(Y=1|X=0)p(X=0) + p(Y=1|X=1)p(X=1) \\
&= 0.6\cdot 0.5 + 0.4\cdot 0.5 \\
&= 0.3 + 0.2 \\
&= 0.5
\end{aligned}
$$

(c) Find $Var[Y|X=0]$, $Var[Y|X=1]$ and $Var[Y]$. Thinking of variance as measuring uncertainty, explain intuitively why one of these variances is larger than others.

$$
\begin{aligned}
E[Y|X=0] &= \sum_{y\in\{0,1\}} P_{Y|X=0}(Y=y|X=0)y \\
&= P_{Y|X=0}(Y=1|X=0)\cdot 1 \\
&= 0.6
\end{aligned}
$$

$$
\begin{aligned}
E[(Y|X=0)^2] &= \sum_{y\in\{0,1\}} P_{Y|X=0}(Y=y|X=0)y^2 \\
&= P_{Y|X=0}(Y=1|X=0)\cdot 1 \\
&= 0.6
\end{aligned}
$$

$$
\begin{aligned}
E[Y|X=1] &= \sum_{y\in\{0,1\}} P_{Y|X=1}(Y=y|X=1)y \\
&= P_{Y|X=1}(Y=1|X=1)\cdot 1 \\
&= 0.4
\end{aligned}
$$

$$
\begin{aligned}
E[(Y|X=1)^2] &= \sum_{y\in\{0,1\}} P_{Y|X=0}(Y=y|X=1)y^2 \\
&= P_{Y|X=1}(Y=1|X=1)\cdot 1 \\
&= 0.4
\end{aligned}
$$

$$
\begin{aligned}
E[Y^2] &= \sum_{y \in \{0,1\}} P(Y=y)y^2 \\
&= P(Y=1) \cdot 1^2 \\
&= P(Y=1 \cap X=1) + P(Y=1 \cap X=0) \\
&= 0.2 + 0.3 = 0.5
\end{aligned}
$$

Thus,

$$Var[Y|X=0] = E[(Y|X=0)^2] - E[Y|X=0]^2 = 0.6 - 0.6^2 = \underline{0.24}$$

$$Var[Y|X=1] = E[(Y|X=1)^2] - E[Y|X=1]^2 = 0.4 - 0.4^2 = \underline{0.24}$$

$$Var[Y] = E[Y^2] - E[Y]^2 = 0.5 - 0.5^2 = \underline{0.25}$$

Explaination

$Var[Y]$ is larger than $Var[Y|X=0]$ and $Var[Y|X=1]$ because Y can be more determined by the information of X. With known X, the distribution of Y is set, and less uncertain with single confirmed distribution.

(d) Suppose you see that the ball is green. What is the probability that the coin turned up tails?

$$
\begin{aligned}
p(X=0|Y=1) &= \frac{p(X=0 \cap Y=1)}{p(Y=1)} \\
&= \frac{p(X=0 \cap Y=1)}{p(Y=1|X=0)p(X=0) + p(Y=1|X=1)p(X=1)} \\
&= \frac{0.5 \cdot 0.6}{0.6 \cdot 0.5 + 0.4 \cdot 0.5} \\
&= \frac{0.3}{0.3 + 0.2} \\
&= \frac{0.3}{0.5} \\
&= \underline{0.6}
\end{aligned}
$$

References

Hoff, Peter D. 2009. *A First Course in Bayesian Statistical Methods*. Vol. 580. Springer.