STAT 638: Solution to Homework 2

3.1 (a) We know $Y_1, \ldots, Y_{100} \mid \theta$ are iid Benoulli(θ). So by definition, $Y = \sum_{i=1}^{100} Y_i \mid \theta \sim$ Binomial$(100, \theta)$.

(b) $P(Y = 57 \mid \theta) = \binom{100}{57}\theta^{57}(1-\theta)^{100-57}$. Now we can simply put $\theta = 0, 0.1, \ldots, 1$ in the equation to get the probabilities.

(c) $P(\theta \mid Y = 57) = \frac{P(\theta)\ P(Y=57|\theta)}{\sum_\theta\ P(\theta)\ P(Y=57|\theta)}$. Since we know $p(\theta) = 1/11$, we can use the $P(Y = 57 \mid \theta)$ values calculated in 3.2, we can obtain $P(\theta \mid Y = 57)$ via the formula above (Bayes rule).

(d) Since $\theta \sim$ Uniform$(0, 1)$, $P(\theta \mid Y) \propto P(Y \mid \theta)P(\theta)$

(e) The prior of part (c) is discrete uniform and that of part (d) is uniform(0,1). The posterior of part (c) is discrete but well approximated by the Beta(58,44), which is the posterior in (d).

3.3 (a) Since $P(\theta \mid Y) \propto P(Y \mid \theta)P(\theta)$, $\theta_A(237, 20)$. So, $E(\theta_A|Y_A) = \alpha/\beta = 11.85$, $Var(\theta_A|Y_A) = \alpha/\beta^2 = 0.5925$, 95% quantile based Confidence Interval : (10.39,13.41).

Also, $\theta_B(125, 14)$. So, $E(\theta_A|Y_A) = \alpha/\beta = 8.92$, $Var(\theta_A|Y_A) = \alpha/\beta^2 = 0.63$, 95% quantile based Confidence Interval : (7.43,10.56).

(b) Since $\theta_B \sim$ Gamma$(12 \times n_0, n_0)$, then the posterior density is Gamma$(12 \times n_0 + 113, n_0 + 13)$. In order for the posterior expectation of θ_B to be close to the expectation of θ_A, we need $\frac{12\times n_0+113}{n_0+13} = 11.85 \implies n_0 = 274$. That means in order for the posterior expectation of θ_B to be close to that of θ_A, the variance of the prior of θ_B should be small. In other words, strong beliefs about θ_B are necessary.

3.4 (a) Since $\theta \sim$ Beta(α, β), $y \mid \theta$ follows Binomial(n, θ), then $\theta \mid y \sim \beta(\alpha+y, \beta+n-y)$. Hence, the posterior distribution is Beta$(17, 36)$. Hence $E(\theta|y) = \frac{\alpha}{\alpha+\beta} = 0.32$, Mode$(\theta \mid y) = \frac{\alpha-1}{\alpha+\beta-2} = 0.32$, SD $= \sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta-1)}} = 0.06$, 95% quantile-based confidence interval = (0.2033, 0.4510).

(b) Since $\theta \sim$ Beta$(8,2)$, the posterior distribution is Beta$(23,30)$, $E(\theta|Y) = 0.43$, mode$(\theta|Y) = 0.43$, sd$(\theta|Y) = 0.07$, 95% quantile-based confidence interval = (0.3047, 0.5680).

(c) Based on historical or outside information, we believe there is a 75% chance that θ is near 0.2 and a 25% chance that it is near 0.8.

(d) $P(\theta \mid Y) \propto P(\theta)P(Y \mid \theta)$ is the mixture distribution 0.75 Beta(17,36) + 0.25 Beta(23,30). The posterior mode in part (c) is 0.314, which is close to posteior mode in part (a). The prior of part (c) put more weight on Beta(2,8), therefore, the posterior mode in part (c) will be close to that in part (a).

(e) The marginal distribution $P(y) = \int P(Y \mid \theta)P(\theta)d\theta = \frac{1}{4}\frac{1}{B(2,8)}\frac{1}{B(16,29)}(3B(17,36) + B(23,30))$.

Hence, the weights are respectively given by $\frac{3B(17,36)}{3B(17,36)+B(23,30)}$ and $\frac{B(23,30)}{3B(17,36)+B(23,30)}$.

3.7(a) Since $P(\theta) = 1, y \mid \theta \sim Bin(n,\theta)$, we have $P(\theta \mid y) \sim \beta(y+1, n-y+1)$ i.e Beta(3, 14). Now we can proceed as in (3.4)(a) to compute the summary statistics.

(b) Since Y_1, Y_2 are conditionally independent given θ,

$P(Y_2 = y_2 \mid Y_1 = y_1) = \frac{\int P(Y_2=y_2|\theta)P(\theta|Y_1=y_1)d\theta}{P(Y_1=y_1)}$.

Now putting $y_1 = 2$, we have $P(Y_2 = y_2 \mid Y_1 = 2) = \binom{278}{y_2}\frac{\Gamma(17)}{\Gamma(3)\Gamma(14)}\frac{\Gamma(y_2+3)\Gamma(292-y_2)}{\Gamma(295)}$.

(c) $E(Y_2|Y_1 = 2) = 49.06$, Var$(Y_2|Y_1 = 2) = 662.13$.

(d) Here $Y_2 \mid \hat{\theta}$ is Binomial$(n, 2/15)$, so we have $E(Y_2 \mid \hat{\theta}) = 37.06$, Var$(Y_2 \mid \hat{\theta}) = 32.12$. Note that the variance is smaller than part (c).

3.12(a) $-E\big[\frac{\partial^2 \log P(Y|\theta)}{\partial\theta^2}\big| \, \theta\big]^{\frac{1}{2}} = -E\big[-\frac{Y}{\theta^2} - \frac{n-Y}{(1-\theta)^2}\big| \, \theta\big]^{\frac{1}{2}} = \sqrt{\frac{n}{\theta(1-\theta)}}$. Hence, Jeffreys' prior is Beta(1/2,1/2).

(b) $\log p(y|\psi) = c(y) + \psi y - n\log(1+e^{\psi})$.
$\frac{\partial}{\partial\psi}\log p(y|\psi) = y - ne^{\psi}/(1+e^{\psi})$.
$\frac{\partial^2}{\partial\psi^2}\log p(y|\psi) = -\frac{(1+e^{\psi})ne^{\psi}-ne^{\psi}e^{\psi}}{(1+e^{\psi})^2} = -ne^{\psi}\frac{1+e^{\psi}-e^{\psi}}{(1+e^{\psi})^2}$.
Hence, the Fisher info is $\mathcal{I}(\psi) = -E(\frac{\partial^2}{\partial\psi^2}\log p(y|\psi)|\psi) = \frac{ne^{\psi}}{(1+e^{\psi})^2}$, and the Jeffreys prior is $p_{\psi}(\psi)\propto(\mathcal{I}(\psi))^{1/2}\propto\frac{e^{\psi/2}}{1+e^{\psi}}$.

(c) We have $\phi = \log\frac{\theta}{1=\theta}$; $\theta = h(\phi) = \frac{e^{\phi}}{1+e^{\phi}}$. Then the jacobian is $J = \frac{e^{\phi}}{(1+e^{\phi})^2}$. Then $P_J(\phi) =\mid J \mid P_{\theta}(h(\phi))$, yields the result.