The logistic operator equation, Bose gas, and market fluctuations

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Abstract

I will discuss a mathematical link between the Quantum Statistical Mechanics and the logistic growth and decay processes. It is based on an observation that a certain nonlinear operator evolution equation, which we refer to as the Logistic Operator Equation (LOE), provides an extension of the standard model of noninteracting bosons.

The classical logistic equation is frequently used in economics, e.g. as a model for exploitation of a renewable resource, a model for the demand curve, etc. LOE also seems to lend itself to macroeconomic interpretations. For instance, it may be viewed as a model for the value fluctuations in a basket of physical commodities or tradable goods. In this interpretation the items in the basket are viewed as possibly interdependent, affecting one another via a complex pattern of interactions. The primary parameter of the LOE model is the reciprocal of "temperature" (rather than the more typically used time variable). However, as analysis shows, the LOE-governed change is not determined by the initial conditions. In fact, the actual effects of logistic heating or cooling, as determined from the LOE, may depend on a plethora of extraneous parameters.

Significant insight into the nature of this model can be gained by analyzing the exact solutions of LOE. I will discuss a fairly general set of solutions obtained for a special calibration of the model, which sets it in the number-theoretic framework. This trick, in the tradition of Julia and Bost-Connes, makes it possible for us to tap into the vast resources of classical mathematics and, in particular, to construct explicit solutions of LOE via the Dirichlet series. The theoretical results and numerical simulations obtained in this way shed light at the unique complexities of this rich and multifaceted model.

1 Introduction

The demand function is frequently modelled by the saturating logistic curve. e.g. [1].

The number-theoretic version of the QSM has a number of advantages. In particular, it allows one to draw the intuitions, insight, as well as the techniques from two sources: physics and number theory.

2 The operator logistic equation and the partition function

New theories in physics often begin with a mathematical opportunity. To give an example, I have always had this feeling about the Bardeen-Cooper theory of superconductivity, which takes its root in Cooper's observation that a certain energy is minimized by bound pairs of particles – a purely mathematical observation in the first instance of its existence. From a certain point of view, perhaps quite a haughty one, physics is but mathematics plus a story to go with it. Therefore, to my mind, it makes sense to explore the mathematical reality via a fairly abstract approach, which starts with an application problem but focuses on the mathematical structures. Before a developed version of the "story" can be provided one needs to probe the mathematical language capable of capturing the essence of the phenomenology encountered in a real life problem. This is the route I take int his article. Let us see how far one can go with the logistic equation, generalizing it to a model as complex as it is possible within the mathematical reality. At the same time, let us study the developing interpretation — the natural suggestion is to interpret the structure within the poetic of Economics. We note that the logistic operator equation we consider here features a special form of quadratic nonlinearity. Even so, it ought to be emphasized, the type of nonlinearity is distinct from that in the well-known Lotka-Volterra equations, and the resulting dynamics displays different features. At the same time, as is typically the case, a mathematical idea is portable, and similar interpretations can be found within the poetic of bona fide Physics or Environmental modelling, etc.

2.1 The scalar logistic equation and the Dirichlet series

The logistic equation (LE) describes a stabilizing growth or decay process. In biology, it may be taken to describe the dynamics of a population of some species when there is a limited supply of resources favorable to its expansion. In economics, LE supplies the basic model for harvesting a natural resource, expansion of industrial sectors, etc. Logistic *decay* curve provides a basic (overidealized) model for gradual depreciation of overpriced stock-price, for diminishing demand for a new product in a saturating market, etc. LE is typically formulated in the form

$$\lambda f' = f\left(1 - \frac{f}{L}\right), \quad (\lambda > 0,\, L > 0). \tag{1}$$

When the initial value, say $f(1)$ is small, $f(s)$ at first grows nearly exponentially, but as it increases, the rate of growth is retarded, and the solution asymptotically reaches a plateau at $f(\infty) = L$. A solution that starts at a high value $f(1) > L$, on the other hand, will decrease at an abating rate and, again, stabilize at $f(\infty) = L$.

Since the equation is separable, it is easy to find the general solution, which is given in a closed form by an elementary formula. Namely, since

$$\int \left(\frac{1}{f} + \frac{1}{L-f}\right) df = \int \frac{1}{\lambda}\, ds,$$

it follows that the solution satisfying the initial condition $f(1) = f_1$ is

$$f(s) = L\left(1 + \frac{L - f_1}{f_1}\exp(-\frac{s-1}{\lambda})\right)^{-1}.$$

From now on we set $L = 1$ for simplicity. Note that f may be written in the form

$$f(s) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1-f_1}{f_1}\right)^n \exp\frac{-n(s-1)}{\lambda}, \tag{2}$$

which is a *generalized* Dirichlet series of the form

$$f(s) = \sum a^n\, e^{-\mu_n s},$$

where

$$a = e^{1/\lambda}(1 - 1/f_1), \qquad \mu_n = n/\lambda. \tag{3}$$

Next, we wish to exploit the relation of LE to the *classical* Dirichlet series (which will be referred to simply as the Dirichlet series), i.e. expressions of the form $D(s) = \sum d_n/n^s$. This prepares the ground for a radical generalization of LE, which is one of the main themes of this article. As an immediate consequence of (3) we obtain the following fact

Proposition 2.1 *Logistic equation (1) admits a Dirichlet series solution if and only if*

$$\lambda = \lambda_m = \frac{1}{\log m} \qquad \text{for an integer } m > 1.$$

To every eigenvalue λ_m *there corresponds a one parameter family of eigenfunctions*

$$f(s) = E_m(a,s) = \frac{1}{1 - am^{-s}} = 1 + \frac{a}{m^s} + \frac{a^2}{m^{2s}} + \frac{a^3}{m^{3s}} + \ldots \tag{4}$$

Proof. It follows from (3) that f is a classical Dirichlet series iff for every n there exists an integer $k(n)$ such that $n/\lambda = \log k(n)$. Setting $n = 1$ we se that necessarily $\lambda = 1/\log m$, where $m = k(1)$, and subsequently $\mu_n = n/\lambda = \log m^n$. Substituting of this value of μ_n into (2) one obtains (4). ⋄

Remark 1. It is interesting to observe the values the logistic curves of the Dirichlet type assume at $s = 1$:

$$E_m(a,s)|_{s=1} = m/(m-a),$$

i.e. as m increases the solutions become flatter on the semi-axis $[1, \infty)$. In other words, the higher the index m, the closer the curve is to its long-range equilibrium already at $s = 1$.

For reasons that will become clear as we proceed, we typically set $a = 1$. This means that the logistic curves explode to infinity as s approaches 0 from the right. That is why we prefer $s = 1$, rather than $s = 0$, to be the "initial moment" for the logistic decay.

Remark 2. It is interesting to consider Proposition 2.1 in light of the following general fact: *If a Dirichlet series $D(s) = \sum d_n/n^s$ satisfies a differential-algebraic equation, then the set of prime factors of those numbers n for which $d_n \neq 0$ is finite.* (This seems to have been observed long ago, see e.g. [4]. A modern proof may be found in [2], Theorem 4.) Note that the set of prime factors corresponding to each E_m is limited to the prime factors of a single number m.

2.2 The logistic operator equation

We wish to consider an operator version of (1), which we will refer to as the logistic operator equation (LOE). Namely,

$$\Lambda \frac{d}{ds} F = F - F^2, \tag{5}$$

where Λ is an infinite diagonal matrix

$$\Lambda = \mathrm{diag}\left(\frac{1}{\log 2}, \frac{1}{\log 3}, \frac{1}{\log 5}, \frac{1}{\log 7}, \ldots, \frac{1}{\log p}, \ldots\right). \tag{6}$$

(We chose this specific Λ to conform with the patterns observed in the preceding section, and take advantage of the Dirichlet series computational machinery. Other calibrations of the eigenvalues may be necessary depending on the intended application of the model. However, one is inclined to believe that at least the generic properties of models with different Λ's should be reflected in this specially calibrated one.)

Note that only prime numbers p are included in the sequence $1/\log p$. Note that Proposition 2.1 immediately gives one special solution of (5), namely

$$F(s) = \mathrm{diag}(\frac{1}{1-2^{-s}}, \frac{1}{1-3^{-s}}, \frac{1}{1-5^{-s}}, \frac{1}{1-7^{-s}}, \ldots, \frac{1}{1-p^{-s}}, \ldots). \tag{7}$$

We now invoke Euler's majestic formula

$$\prod_{p \text{ prime}} \frac{1}{1-p^{-s}} = \sum_{n=1}^{\infty} n^{-s}. \tag{8}$$

As is well known, the identity above is satisfied whenever $\Re s > 1$. Furthermore, the product extends via analytic continuation to the Riemann's zeta function $\zeta(s)$, which is holomorphic in the entire complex plane, except for $s = 1$ where it has a simple pole.

It seems quite natural to regard the product of the entries of the infinite diagonal matrix $F(s)$ as its determinant. Therefore, we write

$$\det F(s) = \zeta(s).$$

In this article we demonstrate that the diagonal F is not the only solution of (5). In fact, the logistic operator equation is a model for a much richer evolutionary phenomenon. However, before we can get to studying this equation, we need to outline its natural conceptual context. That is found in the Quantum Statistical Mechanics.

2.3 Relation to the partition function for the boson gas

The quantity

$$Z = \det F(s)$$

may be interpreted as a *partition function*. In order to motivate this terminology we will refer to the quantum boson gas. In what follows we recollect some known facts, e.g. [16], and introduce additional assumptions suitable for our aims.

At the departure, we point out that the logistic decay curves (4) are encountered in the Quantum Statistical Mechanics (QSM) in the bosonic partition function. We will make this analogy complete in what follows. Let us define the single-particle Hilbert space by

$$\mathrm{H} = \mathrm{span}\left\{e_p : p \text{ prime }\right\}. \tag{9}$$

Recall that the bosonic Fock space is defined as

$$\mathrm{H}^{\odot} = C \oplus \mathrm{H} \oplus \bigodot_{k=1}^{2} \mathrm{H} \oplus \bigodot_{k=1}^{3} \mathrm{H} \oplus \bigodot_{k=1}^{4} \mathrm{H} \oplus \ldots$$

where C is the 1-dimensional Hilbert space, and $\bigodot_{k=1}^{n} \mathrm{H}$ denotes the symmetric tensor product of n copies of the single-particle Hilbert space. We observe that $\mathrm{H}^{\odot}$ has a natural basis

$$\mathrm{H}^{\odot} = \mathrm{span}\left\{e_n : n = 1, 2, 3, \ldots\right\},$$

where $e_1 = 1 \in C$, and if the prime factorization of $n > 1$ is $n = p_1 p_2 \ldots p_r$, then $e_n := e_{p_1} \odot e_{p_2} \odot \ldots \odot e_{p_r}$. Note the natural embedding

$$\mathrm{H} \subset \mathrm{H}^{\odot}.$$

Also, note that every operator T on H has a natural extension to $\mathrm{H}^{\odot}$, denoted $T^{\odot}$, which is defined by setting

$$T^{\odot}\, e_n := Te_{p_1} \odot Te_{p_2} \odot \ldots \odot Te_{p_r}, \qquad \text{and } T^{\odot}\, e_1 = e_1.$$

We have

$$T^{\odot}|_{\mathrm{H}} = T.$$

Once the bosonic Fock space is introduced, we return to the discussion of operator $F(s)$ constructed above, (7). Since

$$I - F(s)^{-1} = \sum_{p \text{ prime}} p^{-s}\, |e_p\rangle\langle e_p|, \tag{10}$$

therefore, clearly,

$$[I - F(s)^{-1}]^{\odot} = \sum_{n=1}^{\infty} n^{-s}\, |e_n\rangle\langle e_n|. \tag{11}$$

Thus, by virtue of Euler's formula (8), we have

$$\det F(s) = \mathrm{Tr}\ [I - F(s)^{-1}]^{\odot}. \tag{12}$$

This should be taken as the rigorous definition of the determinant on the left. In other words, we define the determinant of an operator T acting on H as follows

$$\det T = \mathrm{Tr}\ [I - T^{-1}]^{\odot}. \tag{13}$$

We now observe that in view of (12) and (11) the partition function defined above may be expressed as

$$Z = \det F(s) = \mathrm{Tr}\ \exp[-sH],\ \text{where } H = \sum_{n=1}^{\infty} \log n\, |e_n\rangle\langle e_n|. \tag{14}$$

This, of course, is the standard partition function for the operator H (with the chemical potential set to zero). Note that H has the lowest energy 0, and is a non-negative operator.

This type of setting for a number-theoretic QSM, save the operator F, was discussed already in [16]. At this point, however, we introduce a new point of view, and a new object. Namely, we will focus on the logistic operator equation (5), and its solutions. We will describe and interpret these objects within the frame of QSM.

First, we wish to interpret operator $F(s)$, by tying it to the quantum dynamics. In view of (14), the single particle Hamiltonian is

$$H_{SP} = \sum_{p \text{ prime}} \log p\, |e_p\rangle\langle e_p|. \tag{15}$$

Subsequently, (10) translates into

$$I - F(s)^{-1} = \exp(-sH_{SP}),$$

or, equivalently,

$$H_{SP} = -\frac{1}{s} \log\,[I - F(s)^{-1}]. \tag{16}$$

However, we also have

$$H_{SP} = -\frac{1}{2\Re s} \log\,[I - F(s)^{-1}][I - F(s)^{-1}]^{*}, \tag{17}$$

or

$$H_{SP} = -\frac{1}{2\Re s} \log\,[I - F(s)^{-1}]^{*}[I - F(s)^{-1}]. \tag{18}$$

The three formulas above are equivalent in the special case of F considered above, but we wish to use them for more general solutions of the LOE. Formulas (17) and (18) are more useful in this discussion, because they guarantee a selfadjoint Hamiltonian. In comparison, (16) may generally lead to a non-selfadjoint (diffusive) Hamiltonian. Moreover, we can use the spectral theorem to interpret the latter two formulas, which is not always possible in case of (16). In general, the effective single-particle Hamiltonian will depend on s. It is only for the special $F(s)$ given in (7) that cancelations occur, which yield a time-independent Hamiltonian (15). Also, let us observe that in case of, say, (17) the single-particle dynamics is given by the unitary semi-group

$$U(t) = \exp(itH_{SP}) = \Big([I - F(s)^{-1}][I - F(s)^{-1}]^*\Big)^{-it/2\Re s}.$$

An analogous formula, with a reverse order of the two operators in the base of the exponential, holds in the complementary case (18).

3 A Dirichlet series approach to the LOE

In this section we will demonstrate that there are very many solutions of (5) other than the diagonal (7). Moreover, such general solutions depend on plethora of exogenous variables. In order to carry out the analysis we introduce operator-valued Dirichlet series.

3.1 Setting up the solution space

An operator equation needs to be *interpreted* by prescribing, *a priori*, the space in which one is to look for its solution. This is hardly surprising, since one is forced to do the same even in the case of the better known differential equations. It seems natural to look for solutions of (5) that are bounded on H, and holomorphic in the variable s. However, in this particular case the structure seems to be better revealed by the following Ansatz. Namely, we will look for solutions of (5) in the form of operator-valued Dirichlet series:

$$F(s) = F_1 + 2^{-s}F_2 + 3^{-s}F_3 + 4^{-s}F_4 + \dots \tag{19}$$

We will consider F_k and F as operators on a Hilbert space H, i.e.

$$F_k, F \in \mathrm{B} = \{B : \mathrm{H} \to \mathrm{H},\ ||B|| < \infty\}.$$

where H is as in (9). It is convenient to set the standard resolution of the identity in H:

$$I = \sum_{p \text{ prime}} |e_p\rangle\langle e_p|. \tag{20}$$

We also use the basis of H to clarify definition (6). Namely, Λ is given by

$$\Lambda = \sum_{p \text{ prime}} \frac{1}{\log p} |e_p\rangle\langle e_p| \in \mathrm{B}. \tag{21}$$

It is interesting to observe that Λ is a compact (i.e. absolutely continuous) operator.

All calculations with operator-valued Dirichlet series are first carried out at the formal, algebraic level. After the calculation is done, in order to verify that a given series represents an operator, one needs to verify its convergence. We will only consider convergence in operator norm. Calculations with operator-valued Dirichlet series are similar to calculations with regular series, [13], e.g.

$$\frac{d}{ds}F(s) = -\log 2 \cdot 2^{-s}F_2 - \log 3 \cdot 3^{-s}F_3 - \log 4 \cdot 4^{-s}F_4 + \dots$$

We will always interpret the derivative via convergence in norm. Multiplication follows the usual pattern. If

$$G(s) = G_1 + 2^{-s}G_2 + 3^{-s}G_3 + 4^{-s}G_4 + \dots,$$

then

$$F(s)G(s) = \sum_{n=1}^{\infty} n^{-s}H_n, \text{ where } H_n = \sum_{d|n} F_d G_{n/d}$$

Of course, multiplication is generally noncommutative, i.e. $[F, G]$ is typically nonzero. It follows from the above that

$$F(s)^{-1} = G_1 + 2^{-s}G_2 + 3^{-s}G_3 + 4^{-s}G_4 + \dots,$$

where $G_1 = F_1^{-1}$ and G_n satisfy the recurrence

$$G_n = -F_1^{-1} \sum_{d|n, d>1} F_d G_{n/d}.$$

In particular F is (formally) invertible if and only if F_1 is invertible. Of course, F is rigorously invertible iff the Dirichlet series representing the formal inverse converges. In this article, we are only interested in invertible solutions, i.e. we assume a priori that F_1^{-1} exists. This is because we wish to examine "full rank" solution, as such is the one given by (7).

3.2 Solving LOE via the Dirichlet series

Next, using the Ansatz (19) we will reduce the logistic operator equation (5) to a set of discrete conditions of F_n. This is similar in flavor to the standard use of the Fourier transform to solve a (linear) differential equation. First, substituting (19) into (5) we immediately obtain

$$F_1 - F_1^2 = 0, \quad \text{i.e., since the operator is invertible, } F_1 = I.$$

Next, observe

$$F - \Lambda \frac{d}{ds} F = I + 2^{-s}[I + \log 2 \cdot \Lambda]F_2 + 3^{-s}[I + \log 3 \cdot \Lambda]F_3 + 4^{-s}[I + \log 4 \cdot \Lambda]F_4 + \dots$$

and

$$F(s)^2 = I + \sum_{n=2}^{\infty} n^{-s} \sum_{d|n} F_d F_{n/d}.$$

Therefore, (5) is equivalent to the infinite sequence of conditions

$$\sum_{d|n} F_d F_{n/d} = [I + \log n \cdot \Lambda]F_n, \quad n = 2, 3, \dots$$

Next, by grouping the F_n terms (there are exactly two F_n terms on the left, which correspond to $d = 1$ and $d = n$), one obtains

$$\sum_{d|n, 1<d<n} F_d F_{n/d} = -[I - \log n \cdot \Lambda]F_n. \tag{22}$$

We may take a note of the fact that $[I - \log n \cdot \Lambda]$ is a Fredholm operator. The crucial observation is that $[I - \log n \cdot \Lambda]$ is invertible iff n *is not* a prime. Indeed, in view of (21)

$$I - \log n \cdot \Lambda = \sum_{p \text{ prime}} \left(1 - \frac{\log n}{\log p}\right) |e_p\rangle\langle e_p|.$$

In particular,

$$-[I - \log n \cdot \Lambda]^{-1} = \sum_{p \text{ prime}} \frac{\log p}{\log n/p} |e_p\rangle\langle e_p|, \qquad \text{when } n \text{ is not a prime.} \tag{23}$$

Note that the above is a bounded operator. We will use this formula later on. First, however, let us consider the case of $n = q$ prime. In such a case (22) is simply

$$[I - \log q \cdot \Lambda]F_q = 0.$$

It follows that for all $x \in \mathrm{H}$ $\langle e_p|F_q x\rangle = 0$ as long as $p \neq q$. Therefore, F_q is a rank one operator, whose range is the e_q-line. We can summarize this fact as follows

Proposition 3.1 *If q is prime, then*

$$F_q = |e_q\rangle\langle x_q|, \qquad \textit{for a certain } x_q \in H. \tag{24}$$

Automatically,

$$\|F_q\| = \|x_q\|,$$

i.e., the operator norm of F_q is equal to the length of vector x_q.

Remark. Note that vectors x_q are completely arbitrary — they comprise a set of exogenous variables of the model.

In what follows we will need to examine compositions of prime-indexed F_p.

Proposition 3.2 *If $p_1, p_2, \dots p_k$ are prime numbers, then*

$$F_{p_1} F_{p_2} \dots F_{p_k} = \left(\prod_{j=1}^{k-1} \langle x_{p_j} | e_{p_{j+1}} \rangle \right) |e_{p_1}\rangle\langle x_{p_k}|.$$

Proof. The formula is a direct consequence of (24).$\diamond$

Next, we will demonstrate that the sequence of vectors $x_2, x_3, x_5, x_7, \dots$ determines the solution $F(s)$ of the form (19) uniquely, i.e. all the coefficients F_n are determined via those whose indexes are prime. Indeed, if n is not prime (22) may be written in the form

$$F_n = -[I - \log n \cdot \Lambda]^{-1} \sum_{d|n, 1<d<n} F_d F_{n/d}. \tag{25}$$

Note that the factors F_d and $F_{n/d}$ on the right can be further reduced via this formula. This, when carried out, will yield a representation of F_n via a polynomial in variables F_p where p are the prime factors of n. Before we formulate the general result, we need the following

Definition 3.1 *For a natural n let $n = p_1 p_2 \dots p_r$ be its prime decomposition. We denote the total number of primes by $r = r(n)$. Note that the list of primes may contain repetitions. We let $W(n)$ denote the set of words that consist of all the letters $p_1, p_2, \dots, p_r$, so that each prime occurs as many times as its multiplicity. If $\sigma \in W(n)$, we denote $p_{\sigma(k)}$ the k'th letter in the word σ.*

Remark. Note that the number of words is less that $r!$ whenever n has at least one prime factor of multiplicity greater than 1.

Proposition 3.3 *Let $n = p_1 p_2 \dots p_r$ be the prime decomposition of n. Then,*

$$F_n = \sum_{\sigma \in W(n)} C[p_{\sigma(1)}, p_{\sigma(2)}, \dots, p_{\sigma(r)}]\, F_{p_{\sigma(1)}} F_{p_{\sigma(2)}} \dots F_{p_{\sigma(r)}}, \tag{26}$$

where $r = r(n)$. The scalar coefficients $C[\dots]$ are prescribed by the recurrence: $C[p] = 1$ for a prime p, and

$$\begin{aligned} C[p_1, p_2, \dots, p_r] = \tfrac{\log p_1}{\log p_2 \dots p_r} \{ \quad & C[p_1] \cdot C[p_2, \dots, p_r] + C[p_1, p_2] \cdot C[p_3, \dots, p_r] + \\ & C[p_1, p_2, p_3] \cdot C[p_4, \dots, p_r] + \dots + C[p_1, \dots, p_{r-1}] \cdot C[p_r] \} . \end{aligned} \tag{27}$$

Remark. Before we tackle the details, let us consider the simple case $n = q^2$ with q prime. In view of (23), (24), and (25), we obtain

$$F_{q^2} = \left(\sum_{p \text{ prime}} \frac{\log p}{\log q^2/p} \, |e_p\rangle\langle e_p| \right) \langle x_q | e_q \rangle \, |e_q\rangle\langle x_q| = \langle x_q | e_q \rangle \, F_q,$$

or, equivalently,

$$F_{q^2} = F_q F_q.$$

Since (27) implies $C[q, q] = 1$, this conforms with (26). Let us compare this to the case of $n = pq$ for $p \neq q$ (both p and q prime). One obtains immediately $C[p, q] = \log p / \log q$. Hence, (26) implies

$$F_{pq} = \frac{\log p}{\log q} \, F_p F_q + \frac{\log q}{\log p} \, F_q F_p.$$

The same formula may be obtained by a direct application of (23) and (25).

Proof of Proposition 3.3: We use induction with respect to the number of prime factors of n. Formula (26) is obviously true when $n = p$ is a prime. Next, assume that the formula is true for all n whose prime decomposition consists of less than r primes (not necessarily all different), where $r > 1$. Now, let $n = p_1 p_2 \dots p_r$. Since n is not prime, (25) applies. Consider the sum

$$\sum_{d|n, 1<d<n} F_d F_{n/d}.$$

Indices d and n/d have less than r prime factors. Therefore by assumption each F_d and $F_{n/d}$ may be written in the form (26). It follows that F_n is a linear combination of compositions of the form $F_{p_{\sigma(1)}} F_{p_{\sigma(2)}} \dots F_{p_{\sigma(r)}}$ where at least one factor in front of the composition is contributed by F_d. Fixing σ we see that the only values of d that contributed to this particular composition are: $d = p_{\sigma(1)}$, or $d = p_{\sigma(1)} p_{\sigma(2)}$, etc...., until finally $d = p_{\sigma(1)} \cdots p_{\sigma(r-1)}$. Of course, the tailing factors are in every case contributed by the corresponding $F_{n/d}$. It suffices to find the scalar coefficient at this composition of prime-index operators, which comes from grouping of the like terms, which are the terms we have just named. We assume, without loss of generality, $p_{\sigma(k)} = p_k$, $k = 1, 2, \dots r$. Obviously, the coefficient at $F_{p_1} F_{p_2} \dots F_{p_r}$ is given by formula (27). Indeed, the expression in the curly brackets is nothing but a recipe for grouping the like terms. The coefficient in front comes from the composition of term (23) with $F_{p_1} F_{p_2} \dots F_{p_r}$, and follows directly from Proposition 3.2. $\diamond$

Remark. Note that coefficients $C[\ldots]$ are universal, i.e. they do not depend on the exogenous variables x_q. However, the compositions $F_{p_{\sigma(1)}}F_{p_{\sigma(2)}}\ldots F_{p_{\sigma(r)}}$ do depend on x_q as summarized in Proposition 3.2.

To summarize our findings we collect Propositions 3.2 and 3.3. The result is a closed-form formula for the solutions of LOE:

Theorem 3.1 *Solutions of (5) that conform with Ansatz (19) are of the form*

$$F(s) = I + \sum_{n=2}^{\infty} n^{-s} \sum_{\sigma \in W(n)} C[p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(r(n))}] \left(\prod_{j=1}^{r(n)-1} \langle x_{p_{\sigma(j)}} | e_{p_{\sigma(j+1)}} \rangle \right) |e_{p_{\sigma(1)}}\rangle\langle x_{p_{\sigma(r(n))}}|. \quad (28)$$

Vectors $x_p \in H$ for $p = 2, 3, 5, 7, \ldots$ are arbitrary and constitute the set of exogenous parameters. Coefficients $C[\ldots]$ are universal — they are determined by the set of all primes and by the natural logarithm function log *via recurrence (27).*

Remark 1. Note that every coefficient F_n is a finite-rank operator: if $n = p_1 p_2 \ldots p_r$, then

$$\text{Range}\{F_n\} \subset \text{span}\ \{e_{p_1}, e_{p_2}, \ldots e_{p_r}\}.$$

Remark 2. Note that the Theorem gives a necessary condition for F to be a solution, but not a sufficient one. Indeed, at this point we do not know if the series converges for arbitrary vectors x_q.

When we choose $x_q = e_q$ for $q = 2, 3, 5, \ldots$ we obtain the special solution (7). Note that in this case, and only in this case, all $F_p = |e_p\rangle\langle e_p|$ are self-adjoint. Of course, $F(s)$ is also self-adjoint for all s real (where defined). In fact, this is the only case when $F(s)$ is self-adjoint. Namely, we have

Proposition 3.4 *A solution (28) of (5) is self-adjoint for all s real where it is well defined iff it is the special solution (7).*

In order to prove this fact, we need a simple

Lemma 3.1 *For any prime p,*

$$C[p, p, \ldots, p] = 1.$$

Proof. The claim follows from (27) by induction.$\diamond$

Proof of Proposition 3.4. The special solution (7) is clearly self-adjoint. To see that this is the only solution of the form (28) with this property we proceed as follows. Let s be a fixed real number, and assume that that the series (28) converges. We claim that $F(s) = F(s)^*$ implies $F_n = F_n^*$ for all n. Indeed, the matrix elements satisfy $\langle e_k|F(s) - I|e_l\rangle = \langle e_k|F(s)^* - I|e_l\rangle$. At the same time, two (scalar) Dirichlet series are equal iff all the coefficients are equal. This implies the identity $\langle e_k|F_n|e_l\rangle = \langle e_k|F_n^*|e_l\rangle$ for all n, k, l. Therefore, $F_n = F_n^*$, *a fortiori* $F_p = F_p^*$ for p prime. Thus, by virtue of Proposition 3.1, $x_p = e_p$ for all p, and by Propositions 3.2 and 3.3, $F_n = 0$ iff n has two distinct prime factors. Therefore, by virtue of Lemma 3.1, the only nonzero coefficients are those of the form

$$F_{p^r} = |e_p\rangle\langle e_p|.$$

Combining this with the resolution of identity (20) we see that

$$F(s) = \sum_{p \text{ prime}} (1 + p^{-s} + p^{-2s} + p^{-3s} + \ldots)\, |e_p\rangle\langle e_p|,$$

which coincides with the special solution (7).$\diamond$

3.3 More about the universal coefficients

Theorem 3.1 reveals the form of the solutions of LOE. The dependence on the exogenous variables x_p is rather clear. However, the universal coefficients $C[...]$ remain somewhat mysterious. In this subsection, we will shed some light at their properties. First, let us introduce the lexicographic order in every set of words $W(n)$. Recall that for $n = p_1 p_2 \ldots p_r$ (prime decomposition) the set consists of all words of length r formed with the alphabet $p_1, p_2, \ldots, p_r$ (which may contain repetitions of one or more of the primes). We fix a lexicographic order $\prec$ of primes $p_1, \ldots p_r$, which coincides with their order on the number line, and assume $p_1 < p_2 < p_3 < \ldots < p_r$. This induces a lexicographic order of word in $W(n)$, e.g.

$$[p_1, p_2, p_3, \ldots, p_r] \prec [p_2, p_1, p_3, \ldots, p_r].$$

In fact, the word on the left precedes any other word in $W(n)$. At a first glance it may appear that the coefficients $C[...]$ should increase as we go down the list of words in this order. However, examples shown in Figure 1 quickly disprove this hypothesis. Even for n as small as $n = 100 = 2 \cdot 2 \cdot 5 \cdot 5$ the consecutive words are:

$$[2, 2, 5, 5] \prec [2, 5, 2, 5] \prec [2, 5, 5, 2] \prec [5, 2, 2, 5] \prec [5, 2, 5, 2] \prec [5, 5, 2, 2],$$

but

$$C[5, 5, 2, 2] < C[5, 2, 5, 2].$$

The coefficients behave even more mysteriously for numbers that consist of more primes.

4 Market interpretation of LOE

We postulate that there is an analogy between the boson gas and the market phenomena. On the one hand we have the well-known boson calculus, and on the other LOE. We have seen the relation between these two formalisms in subsection 2.3. The boson calculus is the standard way of describing the quantum boson gas. The novel approach to it discussed here is based on LOE. Having considered the "mathematical reality" of the logistic equation we venture to suggest a macroeconomic interpretation. Such an interpretation is based on a hypothesis that LOE supplies a fundamental model for the heating and cooling of the market. Of course, heating/cooling are terms only made precise by the model itself.

First, we define a market as a collection of goods G_p (where p runs over all primes) with the corresponding values $E_p(1, s)$. Our aim is to exploit LOE as a model for the fluctuation of the values. The phenomenon is dependent on a number of parameters, including the reciprocal temperature s. The interpretation of s as 'market temperature' is a consequence of its meaning within the underlying concept of boson gas. We suggest that the diagonal solution (7) of (5) describes a simple (decoupled) cooling of a market. As the inverse temperature s increases, all the values $E_p(1, s)$ decay to 1. This, of course, is due to the special calibration of Λ. However, different goods have different sensitivity to temperature, some of the goods cool off faster and some slower.

The diagonal solution is realistic only if there is no interaction between the different goods G_p, and their histories are completely independent. Such a constraint is artificial and rather self-imposed, as the equation does not require F to be diagonal. As we have seen, there exist non-diagonal solutions of LOE as well. We suggest that an introduction of the off-diagonal terms models interaction between the market histories of different goods. The values of the actual goods (at temperature s) are in this picture represented as the eigenvalues of the matrix

$$R(s) = \frac{1}{2}(F(s) + F(s)^*).$$

The symmetrization is necessary to ensure that the eigenvalues are real numbers. In addition, it is convenient to represent the values in the logarithmic scale, see top graph in Fig. 3.

The history of the system is partly determined by a path in the complex plane. If the path is taken to be rectangular, see Fig 4, then the intervals parallel to the real axis represent heating or cooling of the market, while the other edges represent evolution at constant temperature. The market undergoes a "correction" during the heating/cooling periods.

As we have seen, Theorem 3.1, off-diagonal solutions depend on the parameters x_p. Variables x_p, which are vectors in the underlying Hilbert space, play the role of exogenous variables. In particular, the history of the market in this model is not self-determined in a deterministic way.

The usual invariant, trace and determinant, play the role of market indices, see lower two graphs in Fig. 3. Recall that the determinant of F, made precise via definition (13), has the meaning of the partition function for the underlying boson gas.

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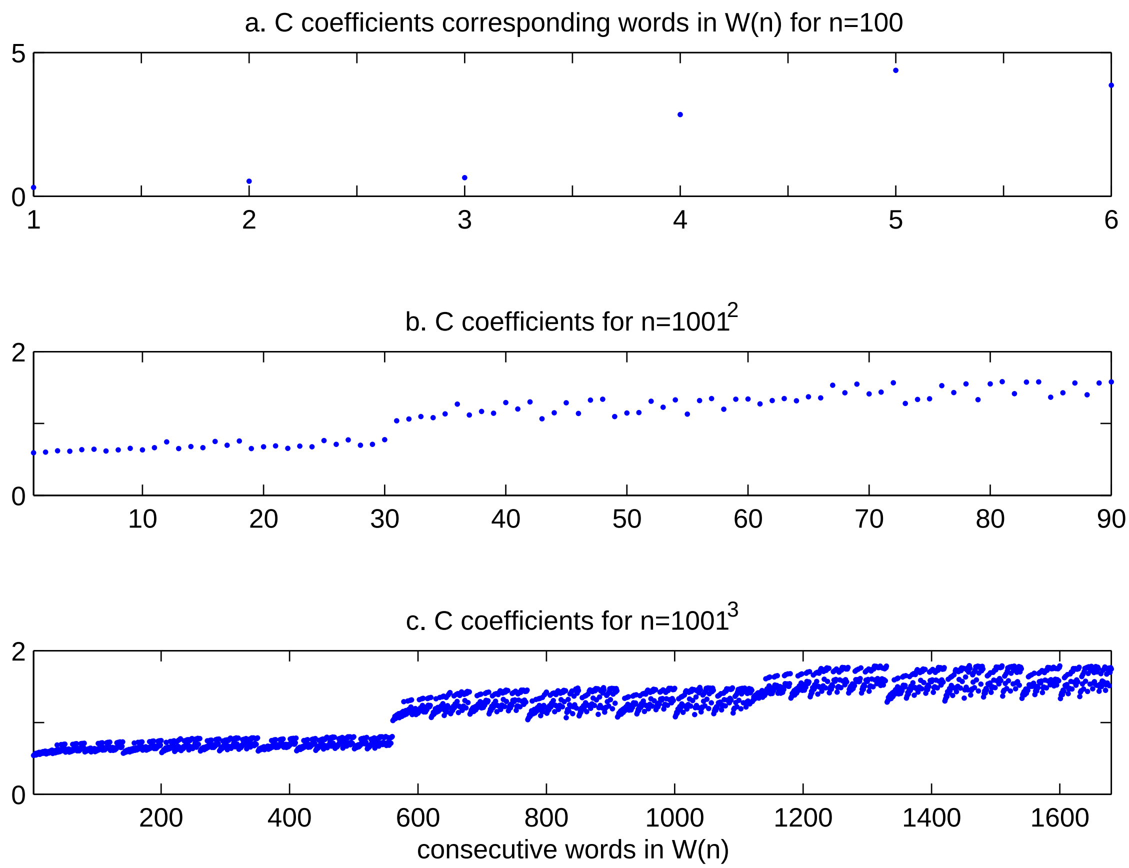

Figure 1. Coefficients C obtained for three different values of n. Integer points on the x-axis number the consecutive words obtained from the prime decomposition of n in the lexicographic order. Since $100 = 2 \cdot 2 \cdot 5 \cdot 5$ there are six words. Since $1001 = 7 \cdot 11 \cdot 13$, one can compute that there are 90 words for $n = 1001^2$, and 1680 words for $n = 1001^3$.

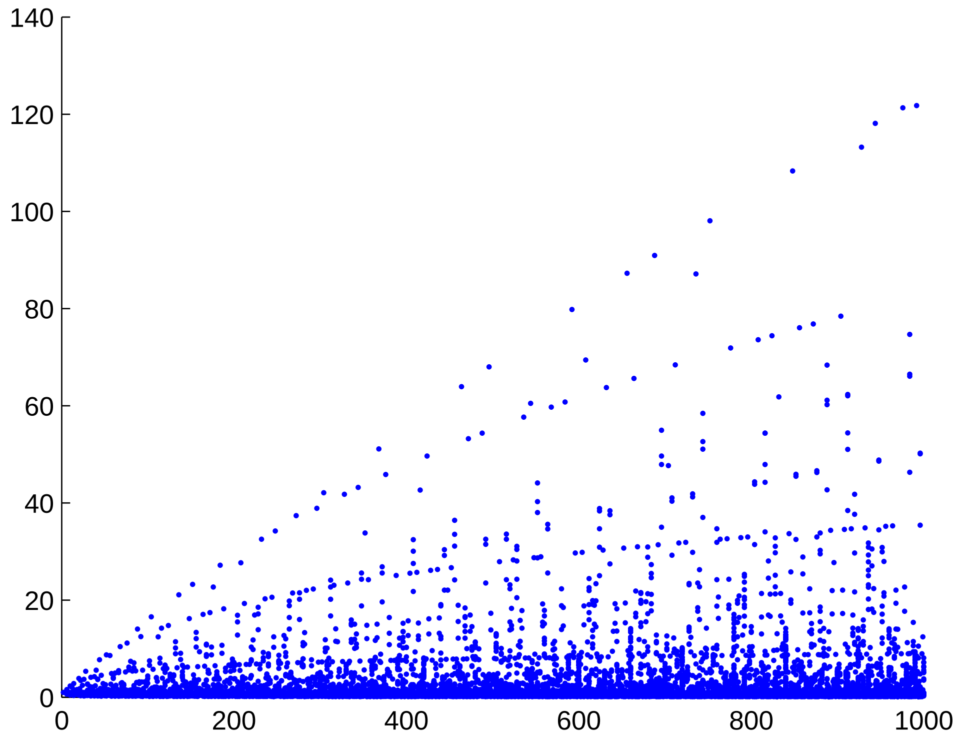

Figure 2. Coefficients C are here stacked above their corresponding n for $n = 2, 3, \ldots, 1000$.

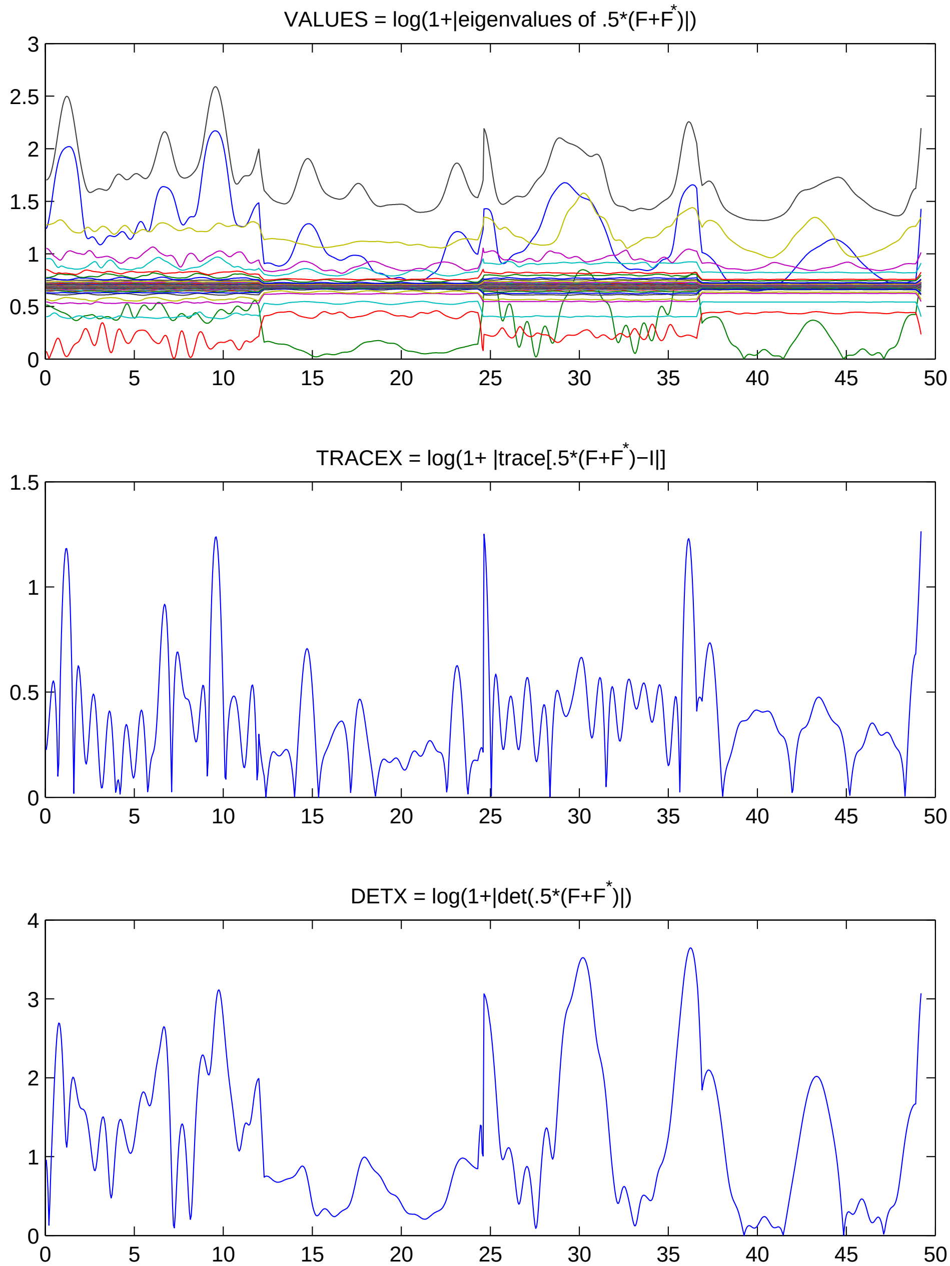

Figure 3. Here, we depict two cycles of the "logistic market". The simulation is based on the approximation of the Dirichlet series with the first $n = 1000$ terms. Since there are 168 primes between 2 and 1000, we select 168 vectors x_p at random. (Each vector x_p has 168 coordinates, which are normally distributed random variables with mean 0 and variance 1).

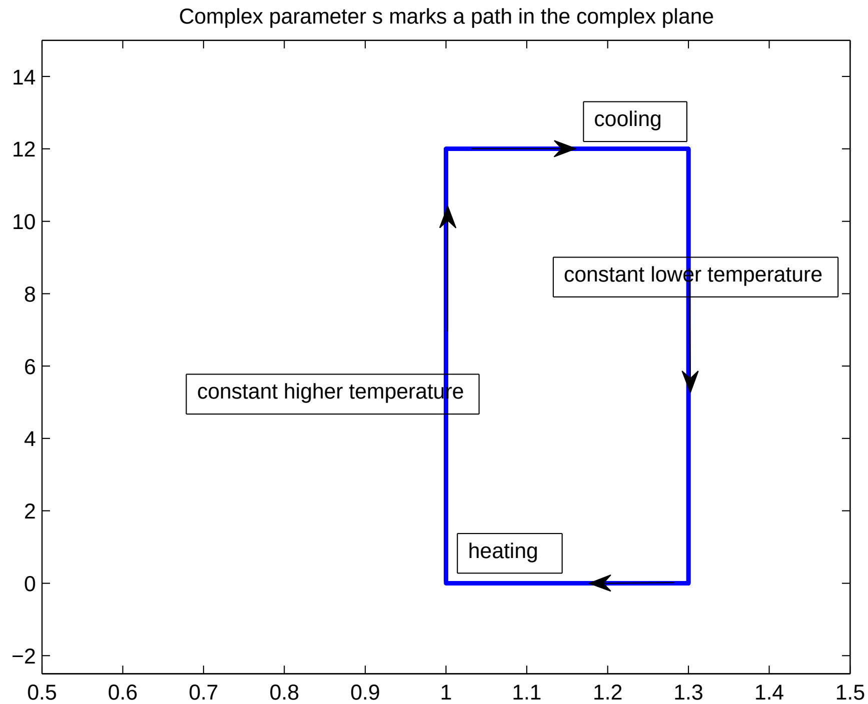

Figure 4. The path traced by parameter s corresponding to graphs in Fig. 3. The horizontal axis consists of two sets of four segments. The segments correspond to consecutive sides of the rectangle shown in Fig. 4. First, the reciprocal temperature is set at $s = 1$, and the market propagates at constant temperature along the line joining $s = 1$ with $s = 1 + 12i$. This evolution is depicted in the first segment $[0, 12]$. Then the temperature is lowered as the market propagates from $s = 1 + 12i$ to $s = 1.5 + 12i$. This is captured in the very short interval $[12, 12.5]$. Subsequently, the market propagates along the right edge of the rectangle toward $s = 1.5$. (The corresponding segment on the x-axis is $[12.5, 24.5]$.) Finally, the rectangular trajectory is closed by "heating" along the lower edge (another .5 segment on the x-axis). The cycle is repeated with a different set of x_p selected at random.

Since the periods of cooling and heating are very short, the graph of 'VALUES' appears to consist of just four equal-length segments interspersed by brief moments of rapid market "correction".

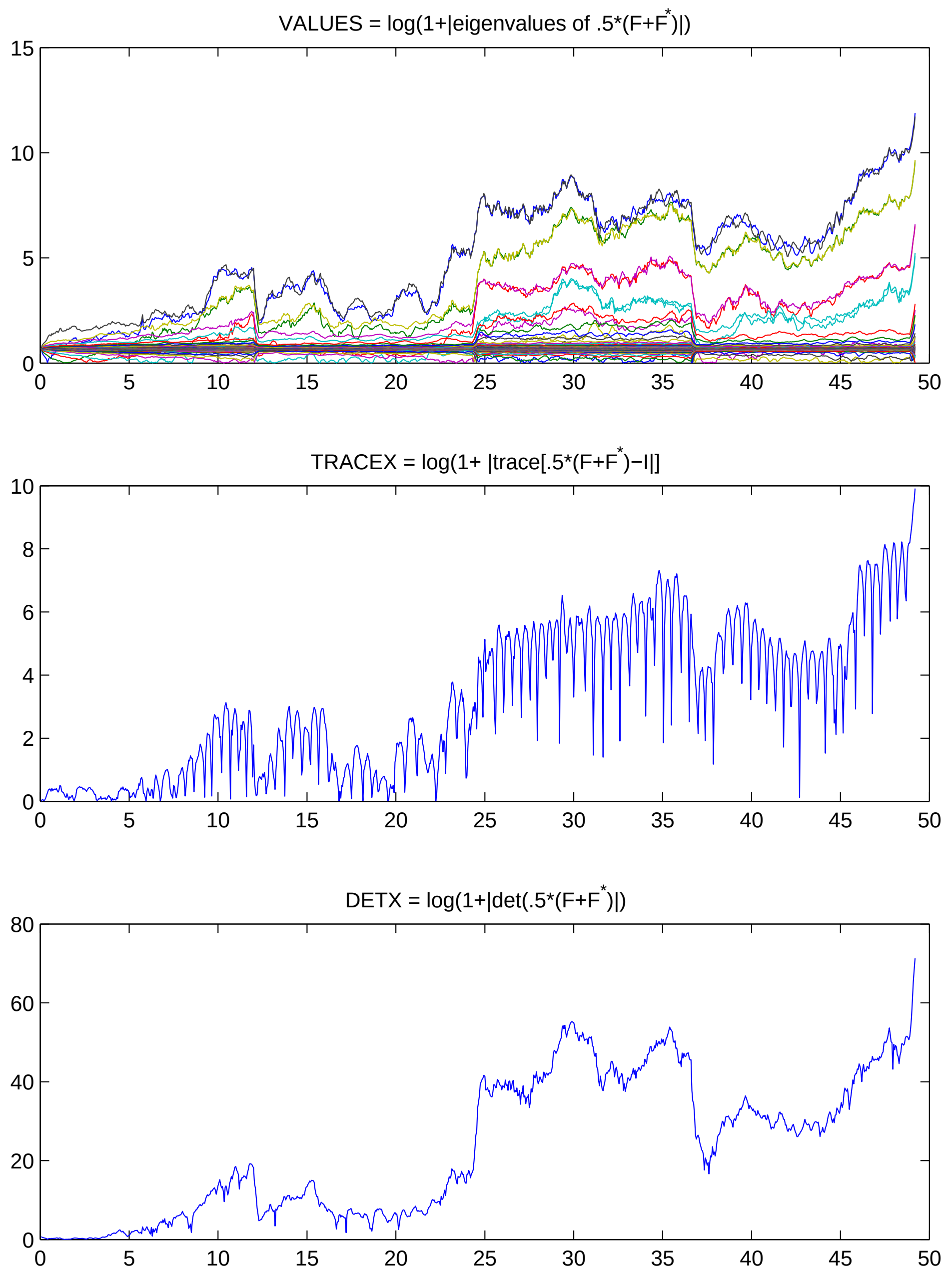

Figure 6. In this run of the simulation, the exogenous parameters are all zero at the initial moment, but then they are allowed to drift in a Brownian motion.

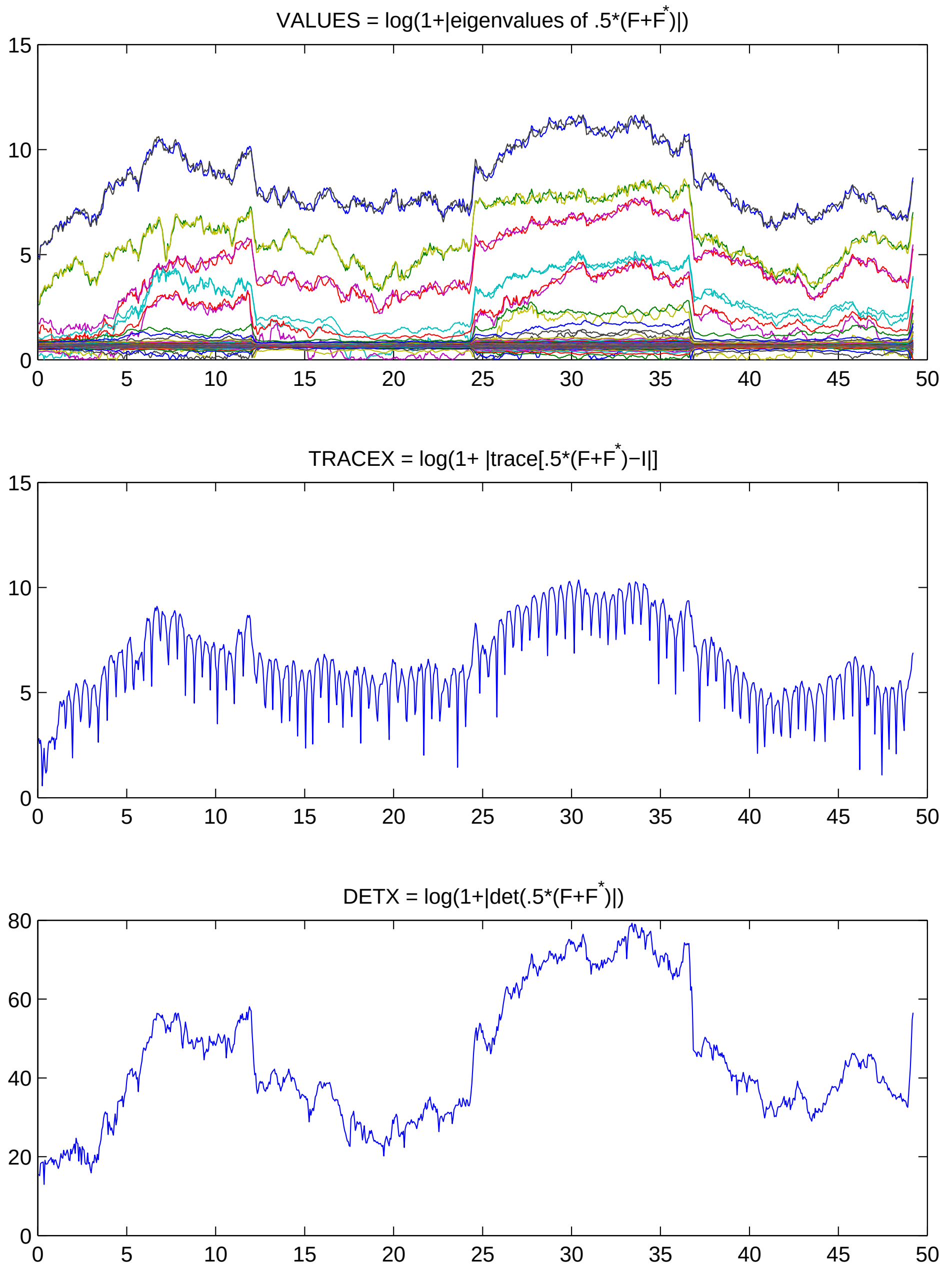

Figure 7. Here, the exogenous parameters start as random (the coordinates of these vectors have mean zero variance one), but then they are allowed to drift in a Brownian motion.