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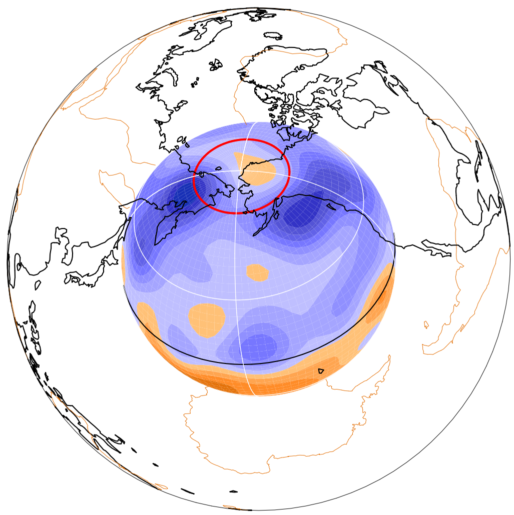

Vector spherical harmonics

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1 Vector spherical harmonics

This appendix is essentially taken straight out of my thesis, [Gib98]. However, I think it is a helpful reference in terms of understanding how the codes work, given that all of these relations are applied throughout, and therefore worth the additional space.

2 The $\mathcal{D}_l$ operator

The spherical harmonic Y_α is defined by

$$Y_\alpha(\theta,\phi) = P_{l_\alpha}^{m_\alpha}(\cos\theta)\{^{cos}_{sin}\}_\alpha m_\alpha \phi, \tag{1}$$

where the Associated Legendre Function, $P_{l_\alpha}^{m_\alpha}(\cos\theta)$, satisfies

$$\frac{d}{d\theta}\left(\sin\theta\frac{dP_{l_\alpha}^{m_\alpha}(\cos\theta)}{d\theta}\right) + \left[l_\alpha(l_\alpha+1) - \frac{(m_\alpha)^2}{\sin^2\theta}\right]P_{l_\alpha}^{m_\alpha}(\cos\theta) = 0 \tag{2}$$

and Y_α is Schmidt quasi-normalised with

$$\int_0^{2\pi}\int_0^{\pi} Y_\alpha Y_{\alpha_1}\sin\theta d\theta d\phi = \left\{\begin{array}{ll} \dfrac{4\pi}{2l_\alpha+1} & \alpha=\alpha_1 \\ 0 & \alpha\neq\alpha_1 \end{array}\right\}. \tag{3}$$

The associated Legendre functions, $P_l^m(\cos\theta)$, satisfy

$$\int_0^{\pi}\left[P_l^m(\cos\theta)\right]^2\sin\theta d\theta = \frac{2(2-\delta_{m0})}{2l+1}. \tag{4}$$

The Laplacian operator, ∇^2, which arises in the heat, momentum and induction equations, is written for a scalar, ψ, in spherical coordinates

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{5}$$

and by virtue of Equation (2),

$$\nabla^2 Y_\alpha = -\frac{l_\alpha(l_\alpha+1)}{r^2}Y_\alpha. \tag{6}$$

If the scalar function, $\Theta(r,\theta,\phi)$, is expanded in a series of spherical harmonics:

$$\Theta(r,\theta,\phi) = \sum_\alpha \Theta_\alpha(r) Y_\alpha, \tag{7}$$

then

$$\begin{array}{rl} \nabla^2\Theta = & \sum_\alpha\left[\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\Theta_\alpha(r)}{\partial r}\right) - \dfrac{l_\alpha(l_\alpha+1)}{r^2}\Theta_\alpha(r)\right]Y_\alpha \\ & \sum_\alpha \mathcal{D}_{l_\alpha}\Theta_\alpha(r)Y_\alpha \end{array} \tag{8}$$

The ordinary differential operator, $\mathcal{D}_l$, can be written in the following equivalent forms,

$$\begin{aligned} \mathcal{D}_l f &= \frac{1}{r}\frac{d^2}{dr^2}(rf) - \frac{l(l+1)}{r^2}f && (9)\\ &= \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - \frac{l(l+1)}{r^2}f && (10)\\ &= \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{l(l+1)}{r^2}f, && (11) \end{aligned}$$

and the composite operator $\mathcal{D}_l^2$, which is found in the viscosity term of the vorticity equation, is given by

$$\mathcal{D}_l^2 f = f'''' + 4\frac{f'''}{r} - 2l(l+1)\frac{f''}{r^2} + (l+2)(l+1)l(l-1)\frac{f}{r^4}, \tag{12}$$

where $'$ denotes differentiation with respect to r.

3 Poloidal and Toroidal decomposition of a solenoidal vector

In the formalism of [BG54], a vector $\boldsymbol{v}$ satisfying

$$\nabla.\boldsymbol{v} = 0, \tag{13}$$

can be decomposed into a toroidal and a poloidal component

$$\boldsymbol{v} = \sum_{\alpha} [\boldsymbol{T}_\alpha + \boldsymbol{P}_\alpha] \tag{14}$$

with

$$\begin{aligned} \boldsymbol{T}_\alpha &= \qquad\quad \nabla \times \left[\tau_\alpha(r) P_{l_\alpha}^{m_\alpha}(\cos\theta) \{{}^{cos}_{sin}\}_\alpha m\phi \;\; \boldsymbol{r}\right] \\ \boldsymbol{P}_\alpha &= \nabla\times \quad \nabla \times \left[p_\alpha(r) P_{l_\alpha}^{m_\alpha}(\cos\theta) \{{}^{cos}_{sin}\}_\alpha m\phi \;\; \boldsymbol{r}\right]. \end{aligned} \tag{15}$$

As both the magnetic field and velocity (in the Boussinesq approximation) are solenoidal vectors, the treatment given here is general to both. It can be easily verified that the vector components of $\boldsymbol{T}_\alpha$ and $\boldsymbol{P}_\alpha$ in Equation (15) are given by

$$
\begin{aligned}
[\boldsymbol{T}_\alpha]_r &= 0 \\
[\boldsymbol{T}_\alpha]_\theta &= \frac{\tau_\alpha(r)}{\sin\theta}\frac{\partial Y_\alpha}{\partial\phi} \\
[\boldsymbol{T}_\alpha]_\phi &= -\tau_\alpha(r)\frac{\partial Y_\alpha}{\partial\theta} \\
&\text{and} \\
[\boldsymbol{P}_\alpha]_r &= l_\alpha(l_\alpha+1)\frac{p_\alpha(r)}{r}Y_\alpha \\
[\boldsymbol{P}_\alpha]_\theta &= \left[\frac{p_\alpha(r)}{r}+\frac{dp_\alpha(r)}{dr}\right]\frac{\partial Y_\alpha}{\partial\theta} \\
[\boldsymbol{P}_\alpha]_\phi &= \frac{1}{\sin\theta}\left[\frac{p_\alpha(r)}{r}+\frac{dp_\alpha(r)}{dr}\right]\frac{\partial Y_\alpha}{\partial\phi}.
\end{aligned}
\tag{16}
$$

A general vector (not necessarily solenoidal) can be represented in a decomposition of scaloidal, spheroidal and toroidal vector harmonics (see [MF53])

$$\boldsymbol{v} = \sum_\alpha \left[q_\alpha(r)\boldsymbol{q}_\alpha + s_\alpha(r)\boldsymbol{s}_\alpha + t_\alpha(r)\boldsymbol{t}_\alpha\right] \tag{17}$$

with

$$
\begin{aligned}
\boldsymbol{q}_\alpha &= Y_\alpha\hat{\boldsymbol{r}} &&= [\ Y_\alpha\ ,\ 0\ ,\ 0\ ] \\
\boldsymbol{s}_\alpha &= \frac{1}{\sqrt{l_\alpha(l_\alpha+1)}}\nabla_h(rY_\alpha) &&= \frac{1}{\sqrt{l_\alpha(l_\alpha+1)}}\left[0, \frac{\partial Y_\alpha}{\partial\theta}, \frac{1}{\sin\theta}\frac{\partial Y_\alpha}{\partial\phi}\right] \\
\boldsymbol{t}_\alpha &= \frac{1}{\sqrt{l_\alpha(l_\alpha+1)}}\boldsymbol{r}\times\nabla_h(Y_\alpha) &&= \frac{1}{\sqrt{l_\alpha(l_\alpha+1)}}\left[0, -\frac{1}{\sin\theta}\frac{\partial Y_\alpha}{\partial\phi}, \frac{\partial Y_\alpha}{\partial\theta}\right].
\end{aligned}
\tag{18}
$$

This representation will be referred to here as a 'qst-decomposition' and is used for calculating the $\boldsymbol{k}\times\boldsymbol{v}$ term and non-linear terms in the momentum and induction equations. Even though $\boldsymbol{v}$ and $\boldsymbol{B}$ are solenoidal, the cross products are generally not, although because $\nabla\times$ is applied, the final result can be treated as a PT-decomposed vector.

If $\boldsymbol{v}$ is a solenoidal vector, conditions on the radial functions $q_\alpha(r)$, $s_\alpha(r)$ and $t_\alpha(r)$ can be obtained to make the expansions in (14) and (17) equivalent. A direct comparison of Equations (16) and (18) reveals

$$q_\alpha(r) = l_\alpha(l_\alpha+1)\frac{p_\alpha(r)}{r} \tag{19}$$

$$s_\alpha(r) = \sqrt{l_\alpha(l_\alpha+1)}\frac{1}{r}\frac{d}{dr}\left[rp_\alpha(r)\right] \tag{20}$$

$$t_\alpha(r) = -\sqrt{l_\alpha(l_\alpha+1)}\tau_\alpha(r) \tag{21}$$

when $\nabla.\boldsymbol{v} = 0$.

4 Curls of Vectors

Firstly we shall deal with vectors of the scaloidal, spheroidal and toroidal type.

4.1 Curl of scaloidal vectors

Let $\boldsymbol{v} = q_\alpha(r)\boldsymbol{q}_\alpha$. We use the vector identity

$$\nabla \times (\psi \boldsymbol{a}) = \psi \nabla \times \boldsymbol{a} - \boldsymbol{a} \times (\nabla \psi), \tag{22}$$

and set the general scalar ψ to be $q_\alpha(r)Y_\alpha/r$ and the general vector $\boldsymbol{a}$ to be the radial vector $\boldsymbol{r}$. Hence

$$\nabla \times (q_\alpha(r)\boldsymbol{q}_\alpha) = -\boldsymbol{r} \times \nabla_h \left(\frac{q_\alpha(r)Y_\alpha}{r}\right) \tag{23}$$

and comparing with the definition of the toroidal vector in Equation (18) shows

$$\nabla \times (q_\alpha(r)\boldsymbol{q}_\alpha) = -\sqrt{l_\alpha(l_\alpha+1)}\frac{q_\alpha(r)}{r}\boldsymbol{t}_\alpha. \tag{24}$$

4.2 Curl of spheroidal vectors

Let $\boldsymbol{v} = s_\alpha(r)\boldsymbol{s}_\alpha$. Now

$$\nabla \times (s_\alpha(r)\boldsymbol{s}_\alpha) = \frac{1}{\sqrt{l_\alpha(l_\alpha+1)}}\nabla \times [s_\alpha(r)\nabla_h(rY_\alpha)] \tag{25}$$

and

$$s_\alpha(r)\nabla_h(rY_\alpha) = \nabla [rY_\alpha s_\alpha(r)] - \hat{\boldsymbol{r}} [rs_\alpha(r)]' Y_\alpha. \tag{26}$$

Since $\nabla \times \nabla = 0$,

$$\nabla \times (s_\alpha(r)\boldsymbol{s}_\alpha) = -\frac{1}{\sqrt{l_\alpha(l_\alpha+1)}}\nabla \times \left[\frac{(rs_\alpha)'}{r}Y_\alpha \boldsymbol{r}\right] \tag{27}$$

By the definition (15), this is clearly a toroidal vector with

$$\tau_\alpha(r) = -\frac{1}{\sqrt{l_\alpha(l_\alpha+1)}}\frac{1}{r}\frac{d}{dr}[rs_\alpha(r)]$$

and applying equivalence relation (21) gives

$$\nabla \times (s_\alpha(r)\boldsymbol{s}_\alpha) = \frac{1}{r}\frac{d}{dr}[rs_\alpha(r)]\,\boldsymbol{t}_\alpha. \tag{28}$$

4.3 Curl of toroidal vectors

Let $\boldsymbol{v} = t_\alpha(r)\boldsymbol{t}_\alpha$. It follows from Equation (21) that

$$\nabla \times [t_\alpha(r)\boldsymbol{t}_\alpha] = \nabla \times \nabla \times \left[-\frac{t_\alpha(r)}{\sqrt{l_\alpha(l_\alpha+1)}} Y_\alpha \boldsymbol{r} \right] \tag{29}$$

which is a poloidal vector spherical harmonic with

$$p_\alpha(r) = -\frac{t_\alpha(r)}{\sqrt{l_\alpha(l_\alpha+1)}}.$$

Using the results (19) and (20) gives the scaloidal and spheroidal presentation of this vector:

$$\nabla \times [t_\alpha(r)\boldsymbol{t}_\alpha] = -\sqrt{l_\alpha(l_\alpha+1)}\frac{t_\alpha(r)}{r}\boldsymbol{q}_\alpha - \frac{1}{r}\frac{d}{dr}\left[r t_\alpha(r) \right] \boldsymbol{s}_\alpha \tag{30}$$

4.4 Curl of poloidal vectors

Let $\boldsymbol{v} = \nabla \times \nabla \times [p_\alpha(r) Y_\alpha \boldsymbol{r}]$.

We proceed by expressing $\boldsymbol{v}$ as a scaloidal and a spheroidal harmonic, and applying results (24) and (28).

$$\boldsymbol{v} = l_\alpha(l_\alpha+1)\frac{p_\alpha(r)}{r}\boldsymbol{q}_\alpha + \sqrt{l_\alpha(l_\alpha+1)}\frac{1}{r}\frac{d}{dr}\left[r p_\alpha(r) \right] \boldsymbol{s}_\alpha \tag{31}$$

and

$$\nabla \times \boldsymbol{v} = \sqrt{l_\alpha(l_\alpha+1)} \left[\frac{1}{r}\frac{d^2}{dr^2}\left(r p_\alpha(r) \right) - l_\alpha(l_\alpha+1)\frac{p_\alpha(r)}{r^2} \right] \boldsymbol{t}_\alpha. \tag{32}$$

Using (9) and (21) shows that this purely toroidal vector can be written

$$\nabla \times [\nabla \times \nabla \times (p_\alpha(r) Y_\alpha \boldsymbol{r})] = \nabla \times [-\mathcal{D}_{l_\alpha} p_\alpha(r) Y_\alpha \boldsymbol{r}] \tag{33}$$

5 The Laplacian

The Laplacian of a scalar function has already been dealt with in the introduction of the $\mathcal{D}_l$ operator. For a general vector $\boldsymbol{v} = (v_r, v_\theta, v_\phi)$, the Laplacian in spherical polar coordinates (see for example [AW95]) is given by

$$\left[\nabla^2 \boldsymbol{v}\right]_r = \nabla^2 v_r - \frac{2}{r^2}v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2\cos\theta}{r^2 \sin\theta}v_\theta - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{34}$$

$$\left[\nabla^2 \boldsymbol{v}\right]_\theta = \nabla^2 v_\theta - \frac{1}{r^2\sin^2\theta}v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi} \tag{35}$$

$$\left[\nabla^2 \boldsymbol{v}\right]_\phi = \nabla^2 v_\phi - \frac{1}{r^2\sin^2\theta}v_\phi + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi} \tag{36}$$

This general formula will be used in many of the derivations in this section. Firstly we shall deal with the Laplacian of vectors in the qst-decomposition.

5.1 Laplacian of scaloidal vectors

Let $\boldsymbol{v} = q_\alpha(r)\boldsymbol{q}_\alpha$. We use the preliminary result

$$\nabla^2 [\psi \boldsymbol{a}] = (\nabla^2\psi)\boldsymbol{a} + 2(\nabla\psi.\nabla)\boldsymbol{a} + \psi\nabla^2\boldsymbol{a} \tag{37}$$

with the general scalar ψ and general vector $\boldsymbol{a}$ replaced with $q_\alpha(r)$ and $\boldsymbol{q}_\alpha$ respectively. Since $q_\alpha(r)$ depends only upon r and Y_α only on θ and ϕ, the cross terms are zero and so

$$\nabla^2 [q_\alpha(r)\boldsymbol{q}_\alpha] = \nabla^2 [q_\alpha(r)]\, \boldsymbol{q}_\alpha + q_\alpha(r)\nabla^2 [\boldsymbol{q}_\alpha]\,. \tag{38}$$

Being dependent only on r,

$$\nabla^2 [q_\alpha(r)] = \frac{d^2 q_\alpha(r)}{dr^2} + \frac{2}{r}\frac{dq_\alpha(r)}{dr} \tag{39}$$

and from (34 - 36),

$$\nabla^2 [\boldsymbol{q}_\alpha] \quad = \quad \left[\nabla^2 - \frac{2}{r^2}\right] Y_\alpha \hat{\boldsymbol{r}} + \frac{2}{r}\nabla_h Y_\alpha. \tag{40}$$

Substituting (39) and (40) into (38), and applying (11) and (18), gives the relation

$$\nabla^2 [q_\alpha(r)\boldsymbol{q}_\alpha] = \left(\mathcal{D}_{l_\alpha} - \frac{2}{r^2}\right) q_\alpha(r)\boldsymbol{q}_\alpha + 2\sqrt{l_\alpha(l_\alpha+1)}\frac{q_\alpha(r)}{r^2}\boldsymbol{s}_\alpha \tag{41}$$

5.2 Laplacian of spheroidal vectors

Let $\boldsymbol{v} = s_\alpha(r)\boldsymbol{s}_\alpha$. Using the result (37), the expression

$$\nabla^2 [s_\alpha(r)\boldsymbol{s}_\alpha] = \nabla^2 [s_\alpha(r)]\, \boldsymbol{s}_\alpha + s_\alpha(r)\nabla^2 [\boldsymbol{s}_\alpha]\,. \tag{42}$$

is obtained. The Laplacian of the radial function has exactly the same form as in (39). For the $\nabla^2 [\boldsymbol{s}_\alpha]$ term, we apply the general vector identity

$$\nabla^2\boldsymbol{V} = -\nabla\times\nabla\times\boldsymbol{V} + \nabla(\nabla.\boldsymbol{V}).$$

to the vector $\boldsymbol{s}_\alpha$:

$$\nabla^2\boldsymbol{s}_\alpha = -\nabla\times\nabla\times\boldsymbol{s}_\alpha + \nabla(\nabla.\boldsymbol{s}_\alpha). \tag{43}$$

From the definition (18),

$$\nabla.\boldsymbol{s}_\alpha = -\sqrt{l_\alpha(l_\alpha+1)}\frac{Y_\alpha}{r} \tag{44}$$

and with

$$\begin{aligned} \nabla(rY_\alpha) \quad &= \quad \left(Y_\alpha, \frac{\partial Y_\alpha}{\partial\theta}, \frac{1}{\sin\theta}\frac{\partial Y_\alpha}{\partial\phi}\right) \\ &= \quad \boldsymbol{q}_\alpha + \sqrt{l_\alpha(l_\alpha+1)}\boldsymbol{s}_\alpha, \end{aligned} \tag{45}$$

we obtain

$$\nabla \times \nabla \times \boldsymbol{s}_\alpha = -\frac{\nabla \times \nabla \times \boldsymbol{q}_\alpha}{\sqrt{l_\alpha(l_\alpha+1)}}. \tag{46}$$

Substituting (44) and (46) back into Equation (43) gives

$$\nabla^2 \boldsymbol{s}_\alpha = \frac{\nabla \times \nabla \times \boldsymbol{q}_\alpha}{\sqrt{l_\alpha(l_\alpha+1)}} - \sqrt{l_\alpha(l_\alpha+1)}\nabla(\frac{Y_\alpha}{r}). \tag{47}$$

and therefore

$$\nabla^2 \boldsymbol{s}_\alpha = \frac{2\sqrt{l_\alpha(l_\alpha+1)}}{r^2}\boldsymbol{q}_\alpha - \frac{l_\alpha(l_\alpha+1)}{r^2}\boldsymbol{s}_\alpha. \tag{48}$$

Replacing this result back into (42) gives

$$\nabla^2 \left[s_\alpha(r)\boldsymbol{s}_\alpha\right] = \mathcal{D}_{l_\alpha} s_\alpha(r)\boldsymbol{s}_\alpha + 2\sqrt{l_\alpha(l_\alpha+1)}\frac{s_\alpha(r)}{r^2}\boldsymbol{q}_\alpha. \tag{49}$$

5.3 Laplacian of toroidal vectors

Let $\boldsymbol{v} = t_\alpha(r)\boldsymbol{t}_\alpha$. Then

$$\boldsymbol{v} = \nabla \times \left[-\frac{t_\alpha(r)}{\sqrt{l_\alpha(l_\alpha+1)}} Y_\alpha \boldsymbol{r}\right], \tag{50}$$

and since the toroidal vector is solenoidal,

$$\nabla^2 \boldsymbol{v} = -\nabla \times \nabla \times \boldsymbol{v} \tag{51}$$

and we proceed by taking the curl. From Equation (15) it is clear that

$$\nabla \times \boldsymbol{v} = \nabla \times \nabla \times \left[-\frac{t_\alpha(r)}{\sqrt{l_\alpha(l_\alpha+1)}} Y_\alpha \boldsymbol{r}\right],$$

and so from (33),

$$\nabla \times \nabla \times \boldsymbol{v} = \nabla \times \left[-\frac{\mathcal{D}_{l_\alpha} t_\alpha(r)}{\sqrt{l_\alpha(l_\alpha+1)}} Y_\alpha \boldsymbol{r}\right]. \tag{52}$$

Using (21), we then have the expression for the Laplacian of both types of toroidal vectors:

$$\nabla^2 t_\alpha(r)\boldsymbol{t}_\alpha = \mathcal{D}_{l_\alpha} t_\alpha(r)\boldsymbol{t}_\alpha \tag{53}$$

$$\nabla^2 \left[\nabla \times (\tau_\alpha(r) Y_\alpha \boldsymbol{r})\right] = \nabla \times (\mathcal{D}_{l_\alpha} \tau_\alpha(r) Y_\alpha \boldsymbol{r}) \tag{54}$$

The Laplacian of a toroidal vector then only involves taking derivatives of the radial function for the same vector harmonic.

5.4 Laplacian of poloidal vectors

Let $\boldsymbol{v} = \nabla \times \nabla \times [p_\alpha(r) Y_\alpha \boldsymbol{r}]$. Again, since the poloidal vector is solenoidal, the ∇^2 operator may be replaced with $-\nabla \times \nabla \times$. Applying the $\nabla \times$ operator twice to $\boldsymbol{v}$ and using Equation (33) gives the result

$$\nabla^2 \left[\nabla \times \nabla \times (p_\alpha(r) Y_\alpha \boldsymbol{r})\right] = \nabla \times \nabla \times (\mathcal{D}_{l_\alpha} p_\alpha(r) Y_\alpha \boldsymbol{r}). \tag{55}$$

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