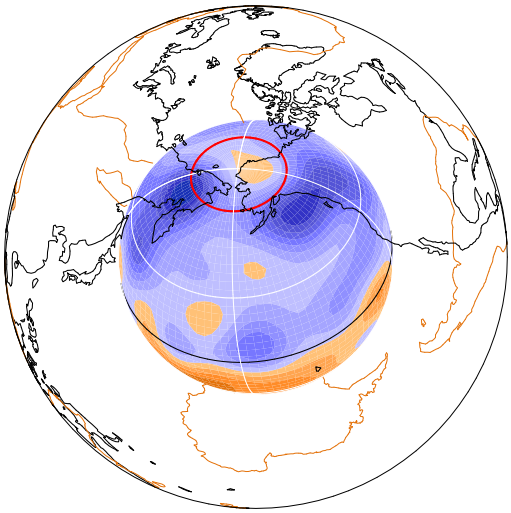
LEOPACK



kriepgrf

Kumar **R**oberts velocity **I**nstability **E**igenvalue **P**roblem
Growth **R**ate **F**ind

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1 kriepgrf

Kumar **R**oberts velocity **I**nstability **E**igenvalue **P**roblem **G**rowth **R**ate **F**ind

The majority of these notes are directly plagerised from [Sar94].

The equation describing the evolution of a magnetic field, $\boldsymbol{B}$, in a conducting fluid with velocity $\boldsymbol{u}$ is derived from the pre-Maxwell equations

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \ , \quad \nabla \times \boldsymbol{B} = \mu \boldsymbol{J} \quad \text{and} \quad \nabla . \boldsymbol{B} = 0 \ , \tag{1}$$

and Ohm's law

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}), \tag{2}$$

where $\boldsymbol{E}$ is the electric field and σ the electrical conductivity. The pre-Maxwell forms are used since the displacement current, $\partial \boldsymbol{E}/\partial t$, will be negligible for the relatively slow variations appropriate for the Earth. Assuming the electrical conductivity to be a constant, taking the curl of Equation (2) and applying the relations of (1) together with the vector identity

$$\nabla \times (\nabla \times \boldsymbol{V}) = \nabla(\nabla . \boldsymbol{V}) - \nabla^2 \boldsymbol{V},$$

gives the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \boldsymbol{B}. \tag{3}$$

The generalised form of the induction equation, as used by the programs, is

$$c_k \frac{\partial \boldsymbol{B}}{\partial t} = c_m \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + c_l \nabla^2 \boldsymbol{B}. \tag{4}$$

We can write Equation (3) in the form

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}, \tag{5}$$

where $\eta = 1/(\mu_0 \sigma)$ is the magnetic diffusivity. If L, U, L^2/η and η^2/L^3 are respectively scales for length, velocity, time and the magnetic field strength (we are employing the diffusive time-scale) then we can write Equation (5) in the non-dimensional form

$$\frac{\partial \boldsymbol{B}}{\partial t} = R_m \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \nabla^2 \boldsymbol{B}. \tag{6}$$

The magnetic Reynolds number, $R_m = (UL/\eta) = \mu_0 UL\sigma$, gives the ratio of the magnitude of the advective term, $\nabla \times (\boldsymbol{u} \times \boldsymbol{B})$, to the diffusive term, $\nabla^2 \boldsymbol{B}$.

The program is specific to the Kumar Roberts flow ([KR75]). $\boldsymbol{u}_{\rm KR}$ is defined as

$$\boldsymbol{u}_{\rm KR} = \nabla \times \nabla \times \left[\, {}^{P}_{\rm KR}v(r,t,\theta,\phi)\, \boldsymbol{r} \, \right] \; + \; \nabla \times \left[\, {}^{T}_{\rm KR}v(r,t,\theta,\phi)\, \boldsymbol{r} \, \right], \tag{7}$$

where the only non-zero radial functions for the velocity are

$${}^{T}_{\rm KR}v_1^{0c}(r) = \varepsilon_0 \left[r(1-r^2) + \Lambda r \right], \tag{8}$$

$${}^{P}_{\rm KR}v_2^{0c}(r) = \varepsilon_1 r^5 (1-r^2)^3, \tag{9}$$

$${}^{P}_{\rm KR}v_2^{2s}(r) = \varepsilon_2 r^3 (1-r^2)^2 \cos(pr) \tag{10}$$

and

$${}^{P}_{\rm KR}v_2^{2c}(r) = \varepsilon_3 r^3 (1-r^2)^2 \sin(pr). \tag{11}$$

The geometry is a sphere with outer boundary at $r = 1$: this means $r_{\rm i} = 0$ and $r_{\rm o} = 1$.

The relative strengths of different components of the velocity are given by the parameters ε_0, ε_1, ε_2 and ε_3.

The stand-alone source code version of the program is is compiled by typing

```
make kriepgrf
```

within this directory. Once the executable is created, begin execution by typing

```
kriepgrf < inputfile
```

The inputs file must have the following format.

```
* input file for kriepgrf
*
example_aOUTPUT                    : ROOT
0.0    1.0     10.0   4   20      : RI, RO, DRSV, NEV, NCV
*---------------------------------------------------------------
* NR LH ITRI ISF IOF ISP NBNO E0    E1    E2    E3  PPAR  RM
*---------------------------------------------------------------
  50  8  0    1   0   1   4  1.00  0.03  0.04  0.04  3   3860.0
  50  8  0    1   0   1   4  1.00  0.03  0.04  0.04  3   3890.0
  50 10  0    1   0   1   4  1.00  0.03  0.04  0.04  3   3860.0
  50 10  0    1   0   1   4  1.00  0.03  0.04  0.04  3   3890.0
  50 12  0    1   0   1   4  1.00  0.03  0.04  0.04  3   3860.0
  50 12  0    1   0   1   4  1.00  0.03  0.04  0.04  3   3890.0
```

Any line in the input file beginning with an asterisk, `*`, is ignored by the program and can thus be used to enter comments and notes.

The following arguments are common to all of the runs carried out by the execution:

- `ROOT`: First characters in output files to be generated by current run.
- `RI`. Inner boundary radius: essentially always set to zero.
- `RO`. Outer boundary radius: essentially always set to 1.0.
- `DRSV`. The real shift - this is $\lambda_{\mathrm{r}}^{\mathrm{s}}$.
- `NEV`. The number of eigenvalues requested.
- `NCV`. The length of the Arnoldi factorisation. See [LSY98]) for details.

All of the (uncommented) lines which follow describe a single run and contain the following variables:

- `NR`. Number of radial grid nodes.
- `LH`. Highest spherical harmonic degree, l requested.
- `ITRI`. Determines whether or not triangular truncation is used. The normal truncation of spherical harmonics is to include only terms with l up to including `LH`. This option is selected by `ITRI = 0`. If `ITRI = 1` then only radial functions with $(l + m)$ up to including `LH` are selected.
- `ISF`. Symmetry selection flag. This essential selects the seed field. Due to the symmetry of the flow (equations 8 to 11) there are four distinct symmetries of magnetic field which decouple. Each is determined by a poloidal magnetic seed field with a single spherical harmonic with degree l_{Seed} and order m_{Seed}. The options are:
  `ISF = 1`: $\rightarrow$ axial dipole. $l_{\mathrm{Seed}} = 1,\ m_{\mathrm{Seed}} = 0$.
  `ISF = 2`: $\rightarrow$ equatorial quadrupole. $l_{\mathrm{Seed}} = 2,\ m_{\mathrm{Seed}} = 0$.
  `ISF = 3`: $\rightarrow$ axial dipole. $l_{\mathrm{Seed}} = 1,\ m_{\mathrm{Seed}} = 1$.
  `ISF = 4`: $\rightarrow$ equatorial quadrupole. $l_{\mathrm{Seed}} = 2,\ m_{\mathrm{Seed}} = 1$.
  See ([Sar94]) and ([GZ93]) for details on symmetry.
- `IOF`. Output file flag. Options are:
  `IOF = 0`: $\rightarrow$ no output of eigenvectors.
  `IOF = 1`: $\rightarrow$ only output eigenfunctions corresponding to the eigenvalue with the largest real part.
  `IOF = 2`: $\rightarrow$ output all eigenfunctions.

Note that for `kriepgrf`, options `IOF = 1` and `IOF = 2` only ever produce a single eigenvector.

- `ISP`. Radial grid node spacings flag.
  `isp = 1` forces evenly spaced grid nodes from `ESNAAS` and `isp = 2` forces Chebyshev zero spaced nodes from `ZCPAAS`.

- `NBNO`. Number of bounding nodes for derivatives of velocity radial functions. Limited by the integer parameter `NBNOMX` in the source code. Whereas, for example, David Gubbins' code uses analytic expressions for the velocity radial functions and their derivatives, this code differentiates the functions numerically. The main reason for this approach was so that the subroutines used could handle general flows, e.g. steady flows locked by thermal boundary heating. However, it also has the advantage that (once thoroughly tested) any velocity may be applied without fear of making algebraic errors in the differentiation. The clear disadvantage is that it is not exact, although `NBNO` may be increased arbitrarily with little detriment to the program, other than the time required to compute these derivatives. Experiment to see what effect this has. `NBNO` must be atleast 3.

- `E0`. Parameter ε_0 in Equation (8).

- `E1`. Parameter ε_1 in Equation (9).

- `E2`. Parameter ε_2 in Equation (10).

- `E3`. Parameter ε_3 in Equation (11).

- `PPAR`. p/π where p is defined in Equations (10) and (11). `PPAR = 3.0` $\rightarrow$ $p = 3\pi$.

- `RM1`. A value for the magnetic Reynolds number which is known to be below critical.

- `RM2`. A value for the magnetic Reynolds number which is known to be below critical.

1.1 Subprograms required for kriepgrf

SUBS subroutines

```
fopen.f krvhmf.f esnaas.f zcpaas.f svkrvf.f kdthsr.f
svfdcf.f ontppf.f gauwts.f schnla.f matop.f vcpcc.f
rv0mfa.f amlp.f  mfseps.f hmfwt.f xarrwt.f evecex.f
svfwt.f fclose.f fnamer.f radvlf.f cntric.f ldgnmf.f
```

gfdcfd.f shvect.f vfcp.f vf2qst.f invmft.f vicexr.f
innlca.f amdlt.f amlica.f amta.f amsdea.f vecop.f
asvta.f asvcpl.f cubeop.f dvecz.f fftrlv.f asvdr.f
matind.f bmrcop.f powtwo.f

SUBS double precision function

emmult.f pmm.f pmm1.f plm.f dpmm.f dpmm1.f
dplm.f sqrll1.f

SUBS integer function

indfun.f indshc.f

BLAS double precision function

dnrm2.f ddot.f dasum.f

BLAS integer function

idamax.f

BLAS subroutines

daxpy.f dcopy.f dgemm.f dger.f dscal.f dswap.f
dtrsm.f dgemv.f dtbsv.f dtrmm.f drot.f dtrmv.f

ARPACK subroutines

dnaupd.f dneupd.f dnaup2.f dvout.f ivout.f second.f
dstatn.f dmout.f dgetv0.f dnaitr.f dnconv.f dneigh.f
dngets.f dnapps.f dlaqrb.f dsortc.f

LAPACK subroutines

dgbtrf.f dgbtrs.f dgetrf.f dgetri.f dgbtf2.f dlaswp.f
xerbla.f dlahqr.f dgeqr2.f dlacpy.f dlaset.f dorm2r.f
dtrevc.f dtrsen.f dgetf2.f dtrtri.f dlabad.f dlanv2.f
dlarfg.f dlarf.f dlaln2.f dlacon.f dtrexc.f dtrsyl.f
dtrti2.f dlarnv.f dlascl.f dlartg.f dlassq.f dladiv.f
dlaexc.f dlasy2.f dlaruv.f dlarfx.f

LAPACK double precision function

dlapy2.f dlamch.f dlanhs.f dlange.f

LAPACK integer function

ilaenv.f

LAPACK logical function

lsame.f

1.2 Run-time limitations

Several parameters are set at the outset which limit the physical size of the problem.

```
 INTEGER NRMAX, LHMAX, NHMAX, NPHMAX, NTHMAX, KLMAX,
1        NBNDMX, NDCS, NDRVM, ISVMAX, NPMAX, LHLH2M, NCFM,
2        NBN, NBNOMX, NRUNM, NCFMOM, NVMAX, IVELMX, NDCS0,
3        MAXNVI, NCVM
 PARAMETER ( NRMAX = 100, LHMAX = 20, MAXNVI = 30000, NRUNM = 50,
1            LHLH2M = LHMAX*(LHMAX+2), NBN = 2, NCVM = 25,
2            NHMAX = LHLH2M/2, NPHMAX = 64, NTHMAX = LHMAX + 2,
3            KLMAX = (NBN+1)*NHMAX-1, NBNDMX = 3*KLMAX+1 )
 PARAMETER ( NBNOMX = 5, NDCS = LHMAX + 1, NDRVM = 2,
1            NCFM = 2*NBN + 1, NCFMOM = 2*NBNOMX + 1,
2            ISVMAX = NRMAX*NHMAX, NVMAX = 4,
3            IVELMX = NVMAX*NRMAX, NDCS0 = 1,
4            NPMAX = (LHMAX+1)*(LHMAX+2)/2 )
```

If the values are insufficient, then change them and recompile. (Note that `NDRVM` and `NDCS0` are not size dependent and should not be changed.)

- `NRMAX` is the maximum permitted number of radial grid nodes.
- `LHMAX` is the highest permitted spherical harmonic degree, l.
- `MAXNVI` is the maximum number of spherical harmonic interactions. These are pre-calculated and stored in an array. There is no real way of knowing how many of these will be required without simply working them out. This number is therefore rather trial and error based: it clearly increases with `LH`. (This is probably quite an inefficient way of calculating non-linear interactions. Anybody is welcome to think up new ways. However, since the majority of time in this program is at the linear algebra stage, this aspect of the code never seemed worth examining.)
- `NRUNM` is the maximum number of independent runs permitted.
- `NBN` is the number of bounding nodes on either side of the finite difference stencil. `NBN = 2` is recommended value. Since the matrix describing the advective term couples distinct spherical harmonic radial functions (i.e. we are using the **IFORMF = 3** option, the bandwidth of the matrix soon becomes very large as the number of radial functions increases. `NBN = 3` therefore is a far higher computational cost than `NBN = 2`. (`NBN = 3` is necessary for the vorticity equation, for example in `linons1`, because of the fourth derivatives for the poloidal velocity. However, for the induction equation, `NBN = 2` gives fourth order accuracy.)

- `NPHMAX` is the maximum permitted number of grid nodes in ϕ for the Fast Fourier Transforms.
- `NBNOMX` is the upper limit for `NBNO`.
- `NVMAX` is the maximum number of velocity spherical harmonic radial functions. `NVMAX = 4` is quite adequate for the Kumar and Roberts flow (which has exactly 4 spherical harmonics). This will need to be changed if this program is to be modified to deal with other flows.

1.3 Outputs from KRIEPGRF

If the filename stem "root" was specified in the input file, the file `root.log` will be generated, along with any output solution vectors.

For each run demanded from the input file, the following information is given in the file `root.log`:

```
ri:    0.0000000D+00 ro:    1.0000000D+00 nr:    50 isp:  1 nbn0:  4
lh:   8 itri:  0 Field sym: Axial_dipole
e0:    1.0000000D+00 e1:    3.0000000D-02 e2:    4.0000000D-02
e3:    4.0000000D-02 pp:    3.0000000D+00 rm:    3.8600000D+03
   4 eigenvalues converged.
eval:   1 (  -3.5027366D-02,   0.0000000D+00) res:    6.3415803D-14
eval:   2 (  -3.5978165D+01,   0.0000000D+00) res:    5.7301550D-11
eval:   3 (  -3.7009492D+01,  -4.2301365D+01) res:    1.6574127D-09
eval:   4 (  -3.7009492D+01,   4.2301365D+01) res:    1.6574127D-09
```

This simply echos the specified parameters together with the growth rate eigenvalues and their direct residuals.

1.4 Sample runs of kriepgrf

The directory

`$LEOPACK_DIR/SAMPLERUNS/KRIEPGRF`

contains example input files and model output. Do not under any circumstances edit these files, as these examples should serve as a control for the correct working of the code. After compiling the program, copy the `.input` files to another directory, run the code and confirm that the output agrees with that in the directory.

1.4.1 Example a

With the file `example_a.input`, we simply calculate the same growth rate eigenvalues as we do in the input file of the same name for the program `krcmrnif`. Compare the output between the two files

`$LEOPACK_DIR/SAMPLERUNS/KRIEPGRF/example_aOUTPUT.log`

and

`$LEOPACK_DIR/SAMPLERUNS/KRCMRNIF/example_aOUTPUT.log`

References

[GZ93] D. Gubbins and K. Zhang. Symmetry properties of the dynamo equations for paleomagnetism and geomagnetism. *Phys. Earth Planet. Inter.*, 75:225–241, 1993.

[KR75] S. Kumar and P. H. Roberts. A three-dimensional kinematic dynamo. *Proc. Roy. Soc. Lond. A*, 344:235–258, 1975.

[LSY98] R. B. Lehoucq, D. C. Sorensen, and C. Yang. *Arpack users guide: Solution of large scale eigenvalue problems by implicitly restarted Arnoldi methods*. SIAM, 1998.

[Sar94] G. R. Sarson. *Kinematic Dynamo Calculations for Geomagnetism*. PhD thesis, University of Leeds, 1994.