The Standard Probability Space for Playing Cards

Let Ω be the collection of all permutations of the standard set of playing cards i.e.

$$\Omega := S(\{1\heartsuit, \dots A\heartsuit, 1\diamondsuit, \dots A\diamondsuit, 1\clubsuit, \dots, A\clubsuit, 1\spadesuit, \dots, A\spadesuit\})$$

Define $p : \Omega \mapsto \mathbb{R}$ by $p(\omega) = 1/(52!)$ and a probability $P : \mathcal{P}(\Omega) \mapsto \mathbb{R}$ by

$$P(A) = \sum_{\omega \in A} p(\omega)$$

The triple $(\Omega, \mathcal{P}(\Omega), P)$ is a probability space.

Let $A_m(n)$ be the event that a four-of-a-kind of rank m is observed in the first n draws. Let $X : \Omega \mapsto \mathbb{N}$ be a random variable defined by

$$X(\omega) = \min\{n : \omega \in A_m(n), m \in \{1, \ldots, 13\}\}.$$

Then the average number of draws required to observe any four-of-a-kind is given by the expectation of X:

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{n=1}^{52} nP(X = n) \\
&= \sum_{n=1}^{52} n(P(X \leq n) - P(X \leq n-1)) \\
&= 52 \underbrace{P(X \leq 52)}_{=1} - \sum_{n=1}^{51} P(X \leq n) \\
&= 52 - \sum_{n=1}^{51} P(\bigcup_{m=1}^{13} A_m(n)) \\
&= 52 - \sum_{n=1}^{51} \underbrace{\sum_{\emptyset \neq M \subseteq \{1,\ldots,13\}} (-1)^{|M|+1} P(\bigcap_{m \in M} A_m(n))}_{\text{inclusion-exclusion}} \\
&= 52 - \sum_{n=1}^{51} \sum_{m=1}^{13} (-1)^{m+1} \underbrace{\binom{13}{m}}_{\text{symmetry ①}} \underbrace{\binom{52-4m}{n-4m} \Big/ \binom{52}{n}}_{\text{hypergeometric ②}}
\end{aligned}$$

Symmetry ①

All the possible choices of $M \subseteq \{1, \ldots, 13\}$ have a cardinality (number of elements) in the range $1, 2, \ldots, 13$. If we choose M, M' such that $|M| = |M'|$, then

$$(-1)^{|M|+1} P(\bigcap_{m \in M} A_m(n)) = (-1)^{|M'|+1} P(\bigcap_{m \in M'} A_m(n))$$

by symmetry. For example, for n draws, the probability of getting four aces, four queens and four kings is the same as getting four ones, four twos and four threes.
For each $m \in \{1, \ldots, 13\}$, the number of sets M with $|M| = m$ is $\binom{13}{m}$.

Hypergeometric ②

We have chosen m four-of-a-kinds (e.g. $m = 3$ and we are looking for four aces, four kings and four queens) and decided on the number n of draws to make. What is the probability of that particular selection of cards being chosen?

Let us "paint" the cards we want to turn up in the draw red and the rest green. Then the problem collapses into a familiar problem of "choosing coloured balls from an urn without replacement". Google "hypergeometric distribution" for more info.