A Mathematical Gem: The Hyperbolic Boundary as a Relativistic Shadow

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August 29, 2025

Abstract

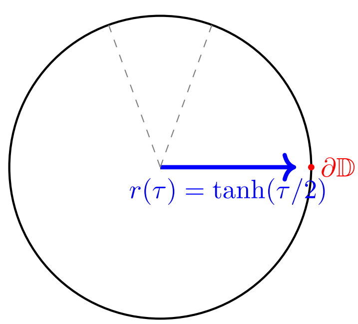
Students of hyperbolic geometry learn that geodesics reach the "ideal boundary" in finite time despite infinite distances. This paradox dissolves when embedding $\mathbb{H}^2$ into Minkowski space. The boundary is not spatial infinity but the projection of light-like geodesics. Special relativity explains clock freezing and shape flattening via time dilation and length contraction. Though implicit in the hyperboloid model, this physical intuition is rarely taught. We bridge geometry and physics, transforming paradox into clarity.

1 Introduction: The Frozen Edge of the Disk

Hyperbolic geometry's Poincaré disk reveals puzzling behavior:

- Radial geodesics reach $|z| = 1$ in finite proper time.
- Hyperbolic distance to the boundary remains infinite.
- Dynamics (clocks, rotations) appear to freeze near the boundary.

The standard explanation — "the metric behaves this way" — lacks physical intuition. Embedding $\mathbb{H}^2$ in Minkowski space resolves this.



Radial geodesic flow toward boundary

Figure 1: Proper time τ drives $r \to 1$. Dynamics freeze due to relativistic projection, not failure.

2 The Hidden Stage: Minkowski Space

The hyperboloid model $X^2 + Y^2 - T^2 = -1$ $(T > 0)$ in $(2+1)$-dimensional Minkowski space $ds^2 = dX^2 + dY^2 - dT^2$ is foundational. A radial geodesic:

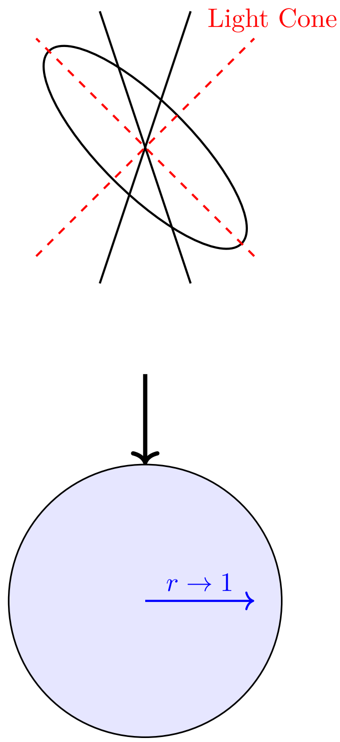
$$\Gamma(\tau) = (\sinh\tau, 0, \cosh\tau)$$

has proper time τ. As $\tau \to \infty$, it approaches the null ray $(s, 0, s)$ without intersecting it.

Projecting to the Poincaré disk via $z = \frac{X+iY}{T+1}$ gives:

$$z(\tau) = i\tanh(\tau/2) \implies |z| \to 1 \text{ as } \tau \to \infty.$$

The Poincaré metric $ds_P = \frac{2|dz|}{1-|z|^2}$ blows up near $|z| = 1$ due to stretching of null asymptotics.



Poincaré Disk Shadow

Projection flattens light-like limits into apparent boundary

Figure 2: Hyperboloid (top) projects to Poincaré disk (bottom). Light cone (red) maps to $|z| = 1$.

3 Relativity in the Geometry Classroom

As geodesics approach light-like behavior:

- **Time dilation**: Proper time slows relative to lab frame.
- **Length contraction**: Objects contract along motion direction.

These effects manifest in projections as frozen clocks and flattened shapes.

4 Boundary Point multiplicity

Each boundary point corresponds to a null ray equivalence class. Geodesics Γ_1 and Γ_2:

$$\Gamma_1(\tau) = (\sinh\tau, 0, \cosh\tau), \quad \Gamma_2(\tau) = (0, \sinh\tau, \cosh\tau)$$

project to $z = 1$ and $z = i$, but perturbations show smooth endpoint transitions. The boundary catalogs light-like futures.

5 Why This Isn't Taught

Possible reasons:

- Geometers avoid physics metaphors.
- Physicists rarely teach pure hyperbolic geometry.
- Textbooks prioritize formalism over intuition.

6 Conclusion: A Bridge Between Worlds

This framework uses existing math but shifts perspective. The boundary is a relativistic shadow, not infinity. Teaching this turns confusion into clarity.

Acknowledgments

Thanks to coffee, curiosity, and students wondering why time stops.

Conflict of Interest

The author declares none.

Funding

No external funding.

A Metric Degeneracy at the Light Cone

On the light cone $X^2 + Y^2 = T^2$:

$$ds^2_{\text{induced}} = dX^2 + dY^2 - dT^2 = 0 \quad \text{(along generators)}.$$