ECN 101: Intermediate Macroeconomic Theory, Fall 2014

Handout on Cobb-Douglas Production Function

TA Jae-Wook Jung *(jwjung@ucdavis.edu)*

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1 Why Is the Cobb-Douglas Production Function Important?

- The Cobb-Douglas function is widely used to represent production functions (and also utility functions) in economics.

- It is based on the statistical observation that "the division of national income between capital and labor have been roughly constant over time."
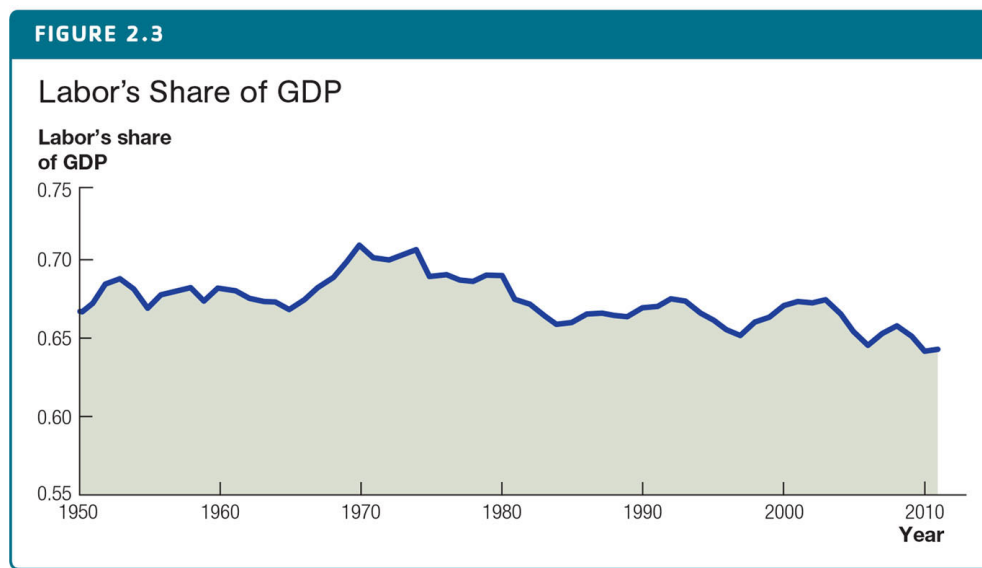

Figure 1: The income share of labor in the US has stayed around 2/3. [p.25, Jones, 3rd ed.]

- It was developed by Charles Cobb (mathematician) and Paul Douglas (economist, becoming US senator from Illinois later) in 1927. But the functional form was actually first proposed and used by Knut Wicksell (economist, 1851-1926) in the 19th century.

- It has very useful mathematical properties such as constant returns to scale.

2   Functional Form

- The most standard form of the Cobb-Douglas production function for a single output with two factors is

$$Y = AK^\alpha L^{1-\alpha}$$

  where

  Y: total production of output (e.g., the number of output produced)
  K: capital input (e.g., the total amount of machinery)
  L: labor input (e.g., the total hour of work or the measure of employment)
  A: total factor productivity (It will be shown in Chapter 6, so here just assume $A = 1$)
  α: the output elasticity of capital, constant between 0 and 1 ($0 < \alpha < 1$)

3   Mathematical Property

3.1   Constant Returns to Scale

- It exhibits constant returns to scale.

    - *Definition:* A production function exhibits *constant returns to scale* if changing all input factors by a positive proportion has changing output by the same proportion.

- Notice that the sum of two exponents for K and L is $\alpha + (1 - \alpha) = 1$. This is closely related with its constant returns to scale property.

- Show that it exhibits constant returns to scale:
  If K and L are increased by the same positive proportion $z > 0$, then Y increases by the same proportion, too.

$$
\begin{aligned}
F(zK, zL) =& A(zK)^\alpha (zL)^{1-\alpha} \quad \text{for any } z > 0 \\
=& A \times z^\alpha \times K^\alpha \times z^{1-\alpha} \times L^{1-\alpha} \\
=& z^{\alpha + 1 - \alpha} AK^\alpha L^{1-\alpha} \\
=& zY
\end{aligned}
$$

- If the sum of two exponents is less than 1, the function exhibits decreasing returns to scale: $F(zK, zL) < zY$ for any $z > 1$.

- If the sum of two exponents is greater than 1, the function exhibits increasing returns to scale: $F(zK, zL) > zY$ for any $z > 1$.

3.2   Constant Income Share of Factors

- What does α (the exponent for K) mean?

    - α represents how much capital contributes on the production of output.

- It is the same as the portion of output distributed to capital (capital income share).

$$\text{(capital income share)} = \frac{R \times K}{P \times Y} = \frac{MPK \times K}{Y}$$
$$= \frac{\partial Y}{\partial K} \times K \div Y$$
$$= \frac{\alpha A K^{\alpha-1} L^{1-\alpha} \times K}{Y}$$
$$= \frac{\alpha A K^{\alpha} L^{1-\alpha}}{Y} = \alpha$$

where R is the (nominal) rental rate of capital, and P is the price of output Y, which we can usually assume as unity.

- Labor income share $= (1 - \alpha)$ in a similar way.

3.3   Increasing Function

- It is increasing in K or L (See Figure 2 and 3).

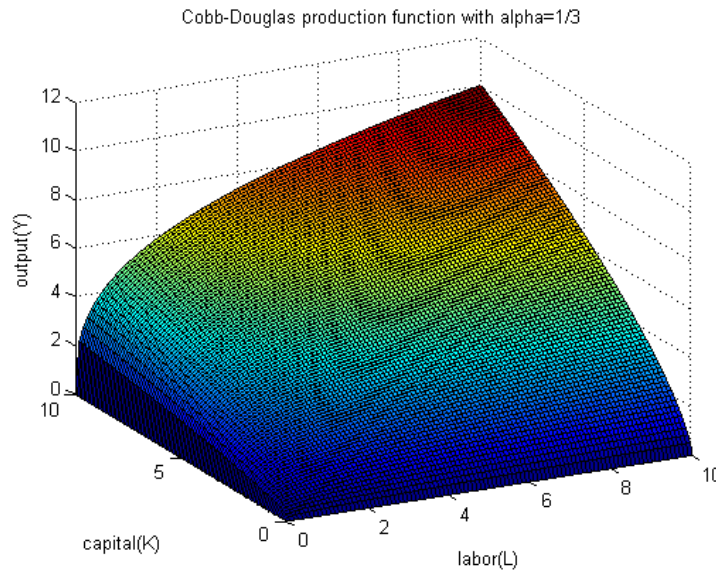

Figure 2: Cobb-Douglas Production Function

3.4   Diminishing Marginal Products of Each Input Factor

- What is the marginal product?

    - The marginal product of a factor (K or L) is the extra amount of output if one unit of the factor is added, holding all the other inputs fixed.

    - Marginal products can be calculated by the partial derivatives of Y with respect to each factor such as

$$MPL = \frac{\partial Y}{\partial L} = (1 - \alpha) A K^{\alpha} L^{-\alpha} = A(1 - \alpha) \left(\frac{K}{L} \right)^{\alpha} \tag{1}$$

$$MPK = \frac{\partial Y}{\partial K} = \alpha A K^{\alpha-1} L^{1-\alpha} = A\alpha \left(\frac{K}{L} \right)^{\alpha-1} \tag{2}$$
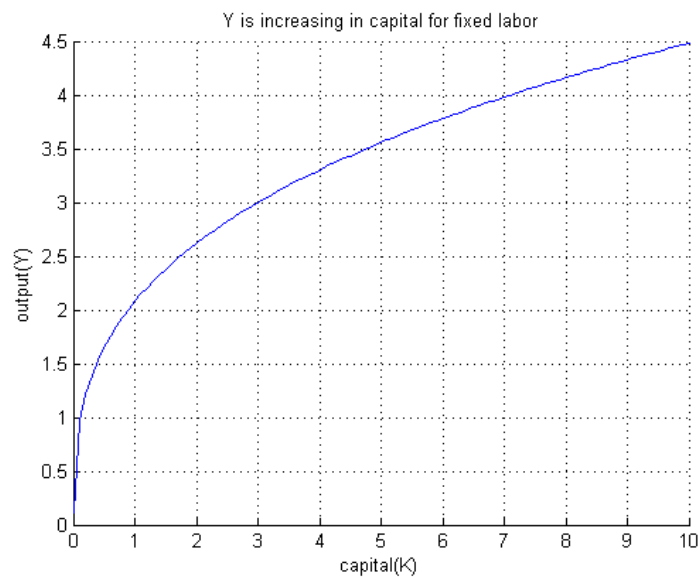
Figure 3: Y is increasing in K with fixed L.

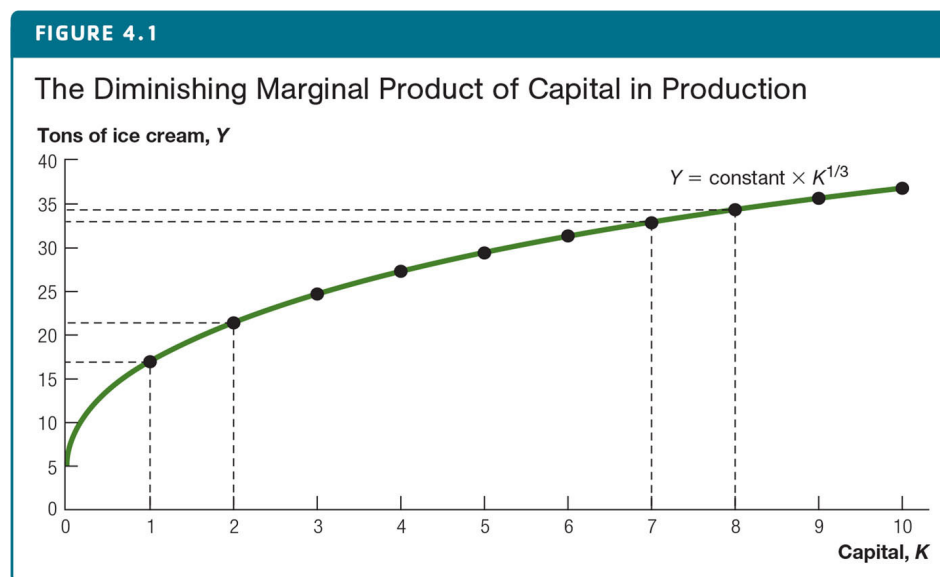- The marginal product is diminishing in its factor (See Figure 4).

Figure 4: MPK is declining as K increases. [p.72, Jones, 3th ed.]

- From (1), $MPL = (1 - \alpha)A\left(\frac{K}{L}\right)^{\alpha}$ is decreasing in L, where $0 < \alpha < 1$.

- Also $MPK = \alpha A\left(\frac{K}{L}\right)^{\alpha - 1} = \alpha A\left(\frac{L}{K}\right)^{1 - \alpha}$ is decreasing in K, where $0 < \alpha < 1$ in (2).

- Notice that MPK (MPL) increases as L (K, respectively) increases with fixed K (L, respectively).

3.5   Proportionality between Marginal Products and Average Products

- The marginal products is proportional to the output per unit of its factor.

- From (1) and (2),

$$MPL = (1-\alpha)A\left(\frac{K}{L}\right)^{\alpha} = (1-\alpha)\frac{Y}{L} \tag{3}$$

$$MPK = \alpha A\left(\frac{K}{L}\right)^{\alpha-1} = \alpha\frac{Y}{K} \tag{4}$$

- Marginal products are proportional to average products.

3.6   The Euler's Theorem

- *Definition*: If the production function has constant returns to scale, then

$$F(K,L) = (MPK \times K) + (MPL \times L)$$

- It is identical with the property of the Cobb-Douglas production function that the division of national income between capital and labor had been roughly constant over time.

- From (3), Total Payment to Labor $= MPL \times L = (1-\alpha)Y$

- From (4), Total Payment to Capital $= MPK \times K = \alpha Y$ where α is a constant, $0 < \alpha < 1$

$$F(K,L) = MPK \times K + MPL \times L = \alpha Y + (1-\alpha)Y = Y$$

4   Per Capita (Per Person) Version

- "*Per Capita*" means "per person."

- The per capita version of Cobb-Douglas production function is

$$\frac{Y}{L} \equiv y = \frac{AK^{\alpha}L^{1-\alpha}}{L}$$
$$= AK^{\alpha}\frac{L^{1-\alpha}}{L}$$
$$= AK^{\alpha}L^{-\alpha}$$
$$= A\left(\frac{K}{L}\right)^{\alpha}$$
$$= Ak^{\alpha}$$

where
y: output per worker (person)
k: capital per worker (person), $k \equiv \frac{K}{L}$.

- e.g., For $Y = K^{1/3}L^{2/3}$,

$$y = \frac{Y}{L} = \frac{K^{1/3}L^{2/3}}{L} = K^{1/3}L^{2/3-1} = \left(\frac{K}{L}\right)^{1/3} = k^{1/3}$$

- Output per capita (y) also increases in capital per capita (k). See Figure 5.

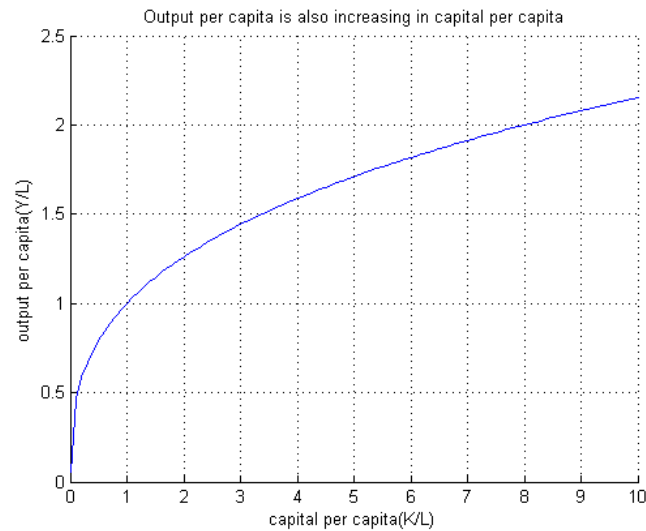- The per capita version is used in Chapter 4, 5 and 6.

Figure 5: Output Per Capita and Capital Per Capita

5   Criticism

- It is based on only statistical observations but there is no economic fundamental for the Cobb-Douglas production function.

- Labor and capital income shares are relatively constant but they can change over time. See an article from Economist for a recent debate on decreasing labor income share around the world. `http://www.economist.com/news/finance-and-economics/21588900-all-around-world-labour-losing-out-capital-labo`