Classical + Quantum Algorithms for Local Max-Cut

Conjectures

Happiness

Let $G = (V, E)$ be a graph of order $n \geq 3$ with degree 2 (a ring). A *cut* is a bitstring $c = (c_0 \dots c_{2^n-1})$. A $v \in V$ is *happy for cut c* when

$$|\{u \in V \setminus \{v\} : c_u \neq c_v\}| \geq |\{u \in V \setminus \{v\} : c_u = c_v\}| \tag{1}$$

ie when there are at least a many of v's neighbors on the opposite side of the cut as on the same side.

Definition 1. *For $d = 2$, define the happiness function $h : \mathbb{Z}_5 \times \{0,1\}^5$ by*

$$h(v, \mathbf{c}) = !(c_{v-1} = c_v = c_{v+1}) \tag{2}$$

Should be full n but need to determine how to get a vertex's (second) neighborhood from the bitstring $\mathbf{c}$

Hamiltonian

The goal is to construct a Hermitian operator $H : \{0,1\}^n \to \mathbb{N}$ that represents how many happy vertices are contained in G for each possible cut. That is, $\langle i| H |i\rangle$ is equal to the amount of happy vertices there are in G for cut i.

Definition 2. *For $\mathbf{c} \in \{0,1\}^n$,*

$$H(\mathbf{c}) = \sum_{i \in V} h(i, \mathbf{c}) \tag{3}$$

Full Graph Solution

Given access to the full graph $G = (V, E)$, then we can naively construct H by iterating over the possible cuts and counting how many $v \in V$ are happy.

Second Neighborhood Approach

We construct H by decomposing into H_i for all $i \in V$ where H_i counts the number of happy vertices in the neighborhood of i. To determine if $i-1$ and $i+1$ are happy, we only need to know the second neighborhood of i. Therefore we are only interested in the 5 slots of the cut that correspond to $v's$ second neighbors.

We define $H_i : \{0,1\}^5 \to \mathbb{N}$ as

$$H_i(\mathbf{c}) = h(i-1, \mathbf{c}) + h(i, \mathbf{c}) + h(i+1, \mathbf{c}) \tag{4}$$

Theorem 1. $H = \frac{1}{3} \sum_{i \in V} H_i$.

Proof: For any cut $c \in \{0,1\}^n$, we have

$$\begin{aligned}
\langle c| \frac{1}{3} \sum_{i \in V} H_i |c\rangle &= \frac{1}{3} \sum_{i \in V} \langle c| H_i |c\rangle && \text{Linearity of expectation} \\
&= \frac{1}{3} \sum_{i \in V} [h(i-1, \mathbf{c}) + h(i, \mathbf{c}) + h(i+1, \mathbf{c})] && (4) \\
&= \sum_{i \in V} h(i, \mathbf{c}) && \text{Triple count each } h(j, \mathbf{c}) \text{ above} \\
&= H && (3)
\end{aligned}$$

□

Permutations

Let $\sigma_n : \{0,1\}^n \to \{0,1\}^n$ be the finite left shift operator:

$$\sigma_n(c_0, c_1, \ldots, c_{n-1}) = (c_1, c_2, \ldots, c_{n-1}, c_0) \tag{5}$$

Lemma 1. *σ_n is a permutation on $\{0,1\}^n$.*

Proof: For any $y = (y_0, \ldots, y_{n-1}) \in \{0,1\}^n$, we have $\sigma_n(y_{n-1}, y_0 \ldots, y_{n-2}) = y$. Now let $x = (x_0, \ldots, x_{n-1}), y = (y_0, \ldots, y_{n-1}) \in \{0,1\}^n$ such that $\sigma_n(x) = \sigma_n(y)$. Then $(x_1, \ldots, x_{n-1}, x_0) = (y_1, \ldots, y_{n-1}, y_0)$ which is true iff $x_i = y_i$ for all i and thus $x = y$. Therefore σ_n is a bijection and since its domain and codomain are equal, it is a permutation. □

Lemma 2. *All cycles of σ_n are of order n (except for the all 0s and all 1s strings in which σ_n acts as the identity).*

Proof: Let $x = (x_0, \ldots, x_{n-1}) \in \{0,1\}^n$. Then $\sigma(x) = (x_1, \ldots, x_{n-1}, x_0)$, $\sigma^2(x) = (x_2, \ldots, x_0, x_1)$, etc, to $\sigma^{n-1}(x) = (x_{n-1}, x_0, \ldots, x_{n-2}) \neq x$. Then

$$\begin{aligned}
\sigma^n(x) &= \sigma(\sigma^{n-1})(x) \\
&= \sigma(x_{n-1}, x_0, \ldots, x_{n-2}) \\
&= (x_0, \ldots, x_{n-1}) \\
&= x
\end{aligned}$$

Lastly consider the string $(x, \dots, x)$ where $x \in \{0, 1\}$. Then $\sigma(x, \dots, x) = (x, \dots, x)$. $\square$

Next define the following function on $\mathbb{Z}_{2^n}$:

$$\pi_n(i) = \begin{cases} 2i & \text{if } 0 \leq i < 2^{n-1} \\ 2i+1 & \text{if } 2^{n-1} \leq i < 2^n \end{cases} \tag{6}$$

Remember that $i \in \mathbb{Z}_{2^n}$ so if $2i + 1 > 2^n$ then we need to subtract 2^n it.

Lemma 3. *π_n is a permutation of $\mathbb{Z}_{2^n}$.*

Proof: Let $y \in \mathbb{Z}_{2^n}$. If y is even, then it can be written as $y = 2m$ for some $m \in \mathbb{Z}_{2^{n-1}}$. Then $\pi_n(m) = y$. If y is odd, then it can be written as $y = 2m + 1$ for some $m \in \mathbb{Z}_{2^n}$. If $2^{n-1} \leq m < 2^n$, then $\pi_n(m) = y$. Otherwise, consider what happens if we replace m with $m' = m + 2^{n-1}$:

$$2m' + 1 = 2(m + 2^{n-1}) + 1 = 2m + 2^n + 1 \equiv 2m + 1 \mod 2^n$$

And so $\pi_n(m') = y$ where $2^{n-1} \leq m' < 2^n$. Therefore π_n is a bijection and since its domain and codomain are equal, it is a permutation. $\square$

Now consider the function $f : \{0, 1\}^n \to \mathbb{Z}_{2^n}$ given by

$$f_n((x_0, \dots, x_{n-1})) = \sum_{i=0}^{n-1} x_i \cdot 2^i \tag{7}$$

Lemma 4. *$\{0, 1\}^n \cong \mathbb{Z}_{2^n}$ as vector spaces via f_n*

(proof omitted, is it necessary?)

Lemma 5. *$\pi_n \circ f_n = f_n \circ \sigma_n$*

(proof omitted, is it necessary? Can do if needed)

This gives us the following commuting diagram

$$\begin{tikzcd}
\{0,1\}^n \arrow[r, "f_n"] \arrow[d, "\sigma_n"'] \arrow[dr, dashed, "\pi_n \circ f_n"] & \mathbb{Z}_{2^n} \arrow[d, "\pi_n"] \\
\{0,1\}^n \arrow[r, "f_n"] & \mathbb{Z}_{2^n}
\end{tikzcd}$$

Corollary 1. *All cycles of π_n are of order n (except for the all 0s and all 1s strings in which π_n acts as the identity).*

Proof: Apply lemmas 2 and 5. $\square$

(Attempted proof of conjecture 2): Consider a cut $\mathbf{c} = (c_0, c_1, c_2, c_3, c_4) \in \{0,1\}^5$. For any $i \in V$ denote its second neighborhood as $\boldsymbol{c_i} = (c_{i-2}, c_{i-1}, c_i, c_{i+1}, c_{i+2})$ with respect to $\mathbf{c}$ where the indices are in $\mathbb{Z}_5$. Let $j \in \mathbb{Z}_5$. Then

$$H_j \boldsymbol{c_i} = h(j-1, \boldsymbol{c_i}) + h(j, \boldsymbol{c_i}) + h(j+1, \boldsymbol{c_i})$$

$$\begin{aligned} P_{\pi^{-j}} H_0 P_{\pi^j} \boldsymbol{c_i} &= P_{\pi^{-j}} H_0 \boldsymbol{c_{\pi^{-j}(i)}} \\ &= P_{\pi^{-j}} (h(-1, \boldsymbol{c_{\pi^{-j}(i)}}) + h(0, \boldsymbol{c_{\pi^{-j}(i)}}) + h(1, \boldsymbol{c_{\pi^{-j}(i)}})) \\ &= h(\pi^j(-1), \boldsymbol{c_{\pi^j \pi^{-j}(i)}}) + h(\pi^j(0), \boldsymbol{c_{\pi^j \pi^{-j}(i)}}) + h(\pi^j(1), \boldsymbol{c_{\pi^j \pi^{-j}(i)}}) \end{aligned}$$

$$\begin{aligned} P_{\pi^{-j}} H_0 P_{\pi^j} \boldsymbol{c_i} &= P_{\pi^{-j}} H_0 \boldsymbol{c_{\pi^{-j}(i)}} \\ &= P_{\pi^{-j}} (h(-1, \boldsymbol{c_{\pi^{-j}(i)}}) + h(0, \boldsymbol{c_{\pi^{-j}(i)}}) + h(1, \boldsymbol{c_{\pi^{-j}(i)}})) \\ &= h(-1, \boldsymbol{c_i}) + h(0, \boldsymbol{c_i}) + h(1, \boldsymbol{c_i}) \\ &= h(\pi^j(-1), \boldsymbol{c_i}) + h(0, \boldsymbol{c_i}) + h(1, \boldsymbol{c_i}) \qquad \text{NOT CLEAR ENOUGH} \end{aligned}$$

TODO FINISH. Need to clean up definitions of h and H. □

Lemma 6. *For all $v \in V$, H_v is a real diagonal matrix so they are Hermitian. This also true for their sum H.*

(Proof omitted, necessary?)

Unitaries

Given a Hermition operator H on $\mathbb{C}^{2^n}$ and an angle $\gamma \in [0, 2\pi)$, we have the following definition

$$U_{H,\gamma} = e^{-i\gamma H} \tag{8}$$

Consider the operator $X : \mathbb{C} \to \mathbb{C}$ given by

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{9}$$

(the quantum NOT gate). We can extend this to an operator on an n-dimensional Hilbert space: for each $j \in [n]$, define

$$X_j = \left(\bigotimes_{k=1}^{j-1} I \right) \bigotimes X \bigotimes \left(\bigotimes_{k=j+1}^{N} I \right) \tag{10}$$

where I is the 2×2 identity operator. Then X_j is the one qubit operator that acts as a NOT gate on the j^{th} qubit and as the identity on everything else.

Letting $d \equiv 2^n$, X_j is a $d \times d$ matrix. Lastly, we consider the sum of all the X_j's:

$$\overline{X_n} = \sum_{j=1}^{n} X_i \tag{11}$$

Given an angle $\beta \in [0, \pi)$, we also have the following definition

$$U_\beta = e^{-i\beta\overline{X_n}} \tag{12}$$

Recursive definition of $\overline{X_n}$? Need to figure out how to use it for something useful. Preferably, the spectrum.
Both $U_{H,\gamma}$ and U_β are unitary since they are the matrix exponential of Hermitian matrices.

Conjectures

Conjecture 1. *For all $j \in \mathbb{Z}_n$, $H_j = P_{\pi^{-j}} H_0 P_{\pi^j}$.*

1 is proven for $n = 5$. In order to prove higher dimensions, I think it is sufficient to prove the following conjecture.

Conjecture 2. *Let $H_0 \in M_{2^5}$ be the diagonal matrix whose i^{th} entry is the amount of happy vertices in the second neighborhood of 0 and let $H_0^n \in M_{2^n}$ be defined similarly for $n \geq 5$. Then*

$$H_0^n = H_0 \otimes \overbrace{I_2 \otimes \cdots \otimes I_2}^{n-1}$$

Conjecture 3. *$\langle +| \, U(H, -\gamma) U(-\beta) H U(\beta) U(H, \gamma) \, |+\rangle$ can be decomposed into a sum of $\langle +| \, U(H_0, -\gamma) U(-\beta) H_0 U(\beta) U(H_0, \gamma) \, |+\rangle$ where $U(H_0, \gamma)$ needs information on the second neighborhood of H_0.*

Conjecture 4. *There exists an $\alpha \in [0, 2\pi)$ and a $\beta \in [0, \pi)$ such that*

$$\langle +| \, U(H, -\gamma) U(-\beta) H U(\beta) U(H, \gamma) \, |+\rangle > 0.95$$