4.3.14 $y'' + 2y' + 3y = 0$

Solution $\lambda^2 + 2\lambda + 3 = 0$

$$\begin{aligned}\lambda_{1/2} &= \frac{-2 \pm \sqrt{4-12}}{2} \\ &= -1 \pm \sqrt{2}i\end{aligned}$$

$$\boxed{y(t) = e^{-t}\left(c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t\right) \quad c_1, c_2 \in \mathbb{R}}$$

4.3.22 $y'' + 4y' + 4y = 0$

Solution $\lambda^2 + 4\lambda + 4 = 0$

$(\lambda + 2)(\lambda + 2) = 0$

$\lambda = -2$

$$\boxed{y(t) = c_1 e^{-2t} + c_2 t e^{-2t} \quad c_1, c_2 \in \mathbb{R}}$$

4.3.30 $y'' - 2y' - 3y = 0, \quad y(0) = 2, \quad y'(0) = -3$

Solution $\lambda^2 - 2\lambda - 3 = 0$

$$\begin{aligned}\lambda_{1/2} &= \frac{2 \pm \sqrt{4+12}}{2} \\ &= 1 \pm 2\end{aligned}$$

$$\begin{aligned}y(t) &= c_1 e^{-t} + c_2 e^{3t} \\ y(0) &= 2 \Rightarrow c_1 + c_2 = 2 \\ y'(0) &= -3 \Rightarrow -c_1 + 3c_2 = -3 \\ 4c_2 &= -1 \Rightarrow c_2 = -\frac{1}{4} \\ c_1 &= 2 - c_2 = 2 + \frac{1}{4} = \frac{9}{4}\end{aligned}$$

$$\boxed{y(t) = \frac{9}{4}e^{-t} - \frac{1}{4}e^{3t}}$$

4.3.37 Suppose that λ_1 and λ_2 are characteristic roots of the characteristic equation $\lambda^2 + p\lambda + q = 0$, where p and q are real constants.

a. Prove that $\lambda_1\lambda_2 = q$.

b. Prove that $\lambda_1 + \lambda_2 = -p$.

Solution $\lambda_1 = \dfrac{-p + \sqrt{p^2 - 4q}}{2}, \quad \lambda_2 = \dfrac{-p - \sqrt{p^2 - 4q}}{2}$

a. $\lambda_1\lambda_2 = \dfrac{p^2 - p^2 + 4q}{4} = q \ \square$

b. $\lambda_1 + \lambda_2 = \dfrac{-2p}{2} = -p \ \square$

4.5.18 $y'' + 3y' + 2y = 3e^{-4t}, \quad y(0) = 1, \quad y'(0) = 0$

Solution We first consider the associated homogeneous ODE:

$$y'' + 3y' + 2y = 0$$
$$\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda_1 = -1,\ \lambda_2 = -2$$
$$y_h(t) = c_1 e^{-t} + c_2 e^{-2t} \quad c_1, c_2 \in \mathbb{R}$$

We now look for a particular solution of the form $y = ae^{-4t}$:

$$16ae^{-4t} - 12ae^{-4t} + 2ae^{-4t} = 3e^{-4t}$$
$$6a = 3 \Rightarrow a = \frac{1}{2}$$

Thus, the general solution is given by

$$y(t) = \frac{1}{2}e^{-4t} + c_1 e^{-t} + c_2 e^{-2t}$$
$$y(0) = 1 \Rightarrow \frac{1}{2} + c_1 + c_2 = 1$$
$$y'(0) = 0 \Rightarrow -2 - c_1 - 2c_2 = 0$$
$$c_1 = 3, c_2 = -\frac{5}{2}$$
$$\boxed{y(t) = \frac{1}{2}e^{-4t} + 3e^{-t} - \frac{5}{2}e^{-2t}}$$

4.5.26 $y'' + 4y = 4\cos 2t$

Solution We look for a particular solution in the form $y = t(a\sin 2t + b\cos 2t)$:

$$y' = a\sin 2t + b\cos 2t + 2t(a\cos 2t - b\sin 2t)$$
$$y'' = 2a\cos 2t - 2b\sin 2t + 2(a\cos 2t - b\sin 2t) - 4t(a\sin 2t + b\cos 2t)$$
$$= 4a\cos 2t - 4b\sin 2t - 4t(a\sin 2t + b\cos 2t)$$

$$y'' + 4y = 4\cos 2t$$
$$4a\cos 2t - 4b\sin 2t - 4t(a\sin 2t + b\cos 2t) + 4t(a\sin 2t + b\cos 2t) = 4\cos 2t$$
$$4a\cos 2t - 4b\sin 2t = 4\cos 2t \Rightarrow a = 1,\ b = 0$$
$$\boxed{y_p(t) = t\sin 2t}$$