1 Let $F \subseteq \mathbb{R}$ be a closed set bounded above. Prove that the least upper bound of F belongs to F.

Solution If F is empty, then there is nothing to prove. Assume that F is non-empty from now on. As $\mathbb{R}$ has the least upper bound property and F is non-empty and bounded above, $L = \sup F$ exists in $\mathbb{R}$.

Consider the ball $B_r(L)$, where $r > 0$. $B_r(L) \cap F$ is non-empty for all r. Otherwise, there exists $R > 0$ such that $a \leq L - R < L$ for all $a \in F$, which cannot happen since L is the least upper bound of F. Hence, since F is closed, $L \in \bar{F} = F$.

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2 Let (X, d) be a metric space. Prove that a set $F \subseteq X$ is closed if and only if every convergent sequence in F has the property that its limit belongs to F.

Solution "$\implies$"

Suppose $F \subseteq X$ is closed. Let $\{x_n\}_{n \geq 1} \subseteq F$ be a convergent sequence, and let x be its limit. Then by definition, for all $r > 0$, there $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d(x, x_n) < r$$

Hence, $B_r(x) \cap F$ is non-empty for all r. Thus, by definition, $x \in \bar{F} = F$. So, every convergent sequence in F converges in F.

"$\impliedby$"

Suppose every convergent sequence $\{x_n\}_{n \geq 1} \subseteq F$ converges in F. Let $x \in \bar{F}$.

Let $r_1 = 1$. As $x \in \bar{F}$, $B_{r_1}(x) \cap F \neq \emptyset$. Define $x_1 \in B_{r_1}(x) \cap F$.

Let $0 < r_2 < \min\{\frac{1}{2}, d(x, x_1)\}$. As $x \in \bar{F}$, $B_{r_2} \cap F \neq \emptyset$. Define $x_2 \in B_{r_2} \cap F$. We proceed inductively.

Suppose we have defined $r_1 > r_2 > \cdots > r_n > 0$ and $x_1, x_2, \ldots, x_n$. Let $0 < r_{n+1} < \min\{\frac{1}{n+1}, d(x, x_n)\}$. Then as $x \in \bar{F}$, $B_{r_{n+1}} \cap F \neq \emptyset$. Let $x_{n+1} \in B_{r_{n+1}} \cap F$.

Thus, we have defined a sequence $\{x_n\}_{n \geq 1} \subseteq F$ with $d(x, x_n) < \frac{1}{n}$ for all $n \geq 1$. The sequence converges trivially to x by the Archimedean principle.

Since every convergent sequence in F converges in F, $x \in F$. Hence, $F = \bar{F}$.

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3 Consider the metric space $(X, d) = (\mathbb{R}, |\cdot|)$. For each of the following subsets of $\mathbb{R}$ decide if they are open, closed, or not open and not closed, connected or not connected. Also, in each case write down the set of accumulation points. Justify your answer.

a. $A = \mathbb{Q}$.

b. $A = \mathbb{Q} \cap [0, 1]$.

c. $A = \{(-1)^n(1 + \frac{1}{n})\}$.

d. $A = \bigcup_{n \in \mathbb{N}} [n, n + \frac{1}{n}]$.

e. $A = \bigcup_{n \in \mathbb{N}} [\frac{1}{2^{n+1}}, \frac{1}{2^n}]$.

Solution a. $\mathbb{Q}$ is neither open or closed. As $\mathbb{Q}$ is dense in $\mathbb{R}$, $\bar{\mathbb{Q}} = \mathbb{R} \neq \mathbb{Q}$, so $\mathbb{Q}$ is not closed. Every open ball of $\mathbb{Q}$ includes elements of $\mathbb{R} \setminus \mathbb{Q}$ since the irrationals are dense in $\mathbb{R}$, so open balls are not subsets of $\mathbb{Q}$. Hence, $\mathbb{Q}$ is not open.

$\mathbb{Q}$ is disconnected. Consider the sets $F_1 = \{q \in \mathbb{Q} \mid q \leq \sqrt{2}\}$ and $F_2 = \{q \in \mathbb{Q} \mid q \geq \sqrt{2}\}$. The two sets are clearly closed and satisfy the following: $\mathbb{Q} \subseteq F_1 \cup F_2$, $\mathbb{Q} \cap F_1 \neq \emptyset$, $\mathbb{Q} \cap F_2 \neq \emptyset$, and $\mathbb{Q} \cap F_1 \cap F_2 = \mathbb{Q} \cap \{\sqrt{2}\} = \emptyset$. Thus, by a theorem, $\mathbb{Q}$ is disconnected.

$\mathbb{Q}' = \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, every ball in $\mathbb{R}$ contains infinitely many elements of $\mathbb{Q}$. So, every element of $\mathbb{R}$ is an accumulation point of $\mathbb{Q}$.

b. A is neither closed or open for the same reasoning as above, except applied on the interval $[0, 1]$ instead of all of $\mathbb{R}$.

A is also disconnected if we take the closed sets $\{q \in A \mid q \leq \frac{1}{\sqrt{2}}\}$ and $\{q \in A \mid q \geq \frac{1}{\sqrt{2}}\}$ and apply the same reasoning as above.

$A' = [0, 1]$ since none of the adherent points are isolated.

c. A is not open since each element is an isolated point, so its interior is the empty set. It is not closed because it does not contain 1, which is in its closure.

A is disconnected since we can write it as $\{-2\} \cup \{(-1)^n(1+\frac{1}{n}) \mid n \geq 2\}$, where both of these sets are clearly separated.

$A' = \{-1, 1\}$. Every point in A is isolated, so $A \not\subseteq A'$. 1 and -1 have infinitely many elements in A close to it, since if we take $k_n = 2n$, the subsequence converges to 1 and if we take $k_n = 2n+1$, the subsequence converges to -1.

d. A is closed and isn't open.

The set is disconnected since we can write it as $[1, 2+\frac{1}{2}] \cup \bigcup_{n\geq 2}[n, n+\frac{1}{n}]$ which are clearly separated since they are closed and disjoint sets.

Since $\mathbb{R}$ is dense in each interval of A, we can always find elements arbitrarily close to any element in A. Any element outside of A is not an accumulation point. Hence, $A = A'$.

e. A is not closed because its closure includes 0, which is not in the set A. It is not open either because $\frac{1}{2}$ is in A but not in the

A is connected since it is equal to $\left(0, \frac{1}{2}\right]$, which is clearly connected.

The accumulation points are in the interval $[0, \frac{1}{2}]$. Every point inside is clearly an accumulation point, but any point outside is not since a ball can be made small enough so that it does not intersect A.

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4 Assume that the sets A and B are separated and let $A_1 \subseteq A$ and $B_1 \subseteq B$. Prove that A_1 and B_1 are separated.

Solution $A_1 \cap \bar{B}_1 \subseteq A \cap \bar{B} = \emptyset$. Similarly, $\bar{A}_1 \cap B_1 = \emptyset$. Thus, by definition, A_1 and B_1 are separated.

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5 Assume that the sets A and B are separated and that the sets A and C are separated. Prove that the sets A and $B \cup C$ are separated.

Solution $A \cap \overline{B \cup C} \subseteq A \cap (\bar{B} \cup \bar{C}) = (A \cap \bar{B}) \cup (A \cap \bar{C}) = \emptyset \cup \emptyset = \emptyset$.

$\bar{A} \cap (B \cup C) = (\bar{A} \cap B) \cup (\bar{A} \cap C) = \emptyset \cup \emptyset = \emptyset$.

By definition, A and $B \cup C$ are separated.

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6 If A and B are closed sets, prove that $A \setminus B$ and $B \setminus A$ are separated.

Solution Note that $A \setminus B = A \cap B^{\mathrm{C}}$. Then

$$\overline{A \setminus B} \cap (B \setminus A) = \overline{A \cap B^{\mathrm{C}}} \cap (B \cap A^{\mathrm{C}}) \subseteq \bar{A} \cap \overline{B^{\mathrm{C}}} \cap B \cap A^{\mathrm{C}} = A \cap A^{\mathrm{C}} \cap \overline{B^{\mathrm{C}}} \cap B = \emptyset.$$

Similarly, by the same argument, we can show that $\overline{B \setminus A} \cap (A \setminus B) = \emptyset$. Thus, $A \setminus B$ and $B \setminus A$ are separated.

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7 Let (X, d) be a connected metric space and let A be a connected subset of X. Assume that the complement of A is the union of two separated sets B and C. Prove that $A \cup B$ and $A \cup C$ are connected. Prove also that if A is closed, then so are $A \cup B$ and $A \cup C$.

Solution Suppose $A \cup B$ is disconnected. Then there exist two separated sets D and E such that $D \cup E = A \cup B$. Then $A = A \cap (D \cup E) = (A \cap D) \cup (A \cap E)$.

Suppose $A \cap D = \emptyset$. Then since $A \cup B = D \cup E$, we must have that $A = E$ and $B = D$. I.e., A and B are separated. Then since $A \cup (B \cup C) = X$, we have

$$\bar{A} \cap (B \cup C) = (\bar{A} \cap B) \cup (\bar{A} \cap C) = \emptyset$$
$$A \cap \overline{B \cup C} = (A \cap \bar{B}) \cup (A \cap \bar{C}) = \emptyset$$

which implies that X is disconnected, which is a contradiction. The same argument holds for $A \cap C = \emptyset$, but with C and D switched initially. So, $A \cap D$ and $A \cap E$ must both be non-empty.

Assume from now on that $A \cap D$ and $A \cap E$ are both non-empty. We claim that $A \cap D$ and $A \cap E$ are separated.

$$\overline{A \cap D} \cap A \cap E \subseteq \bar{A} \cap \bar{D} \cap A \cap E = A \cap (\bar{D} \cap E) = \emptyset$$
$$A \cap D \cap \overline{A \cap E} \subseteq A \cap D \cap \bar{A} \cap \bar{E} = A \cap (D \cap \bar{E}) = \emptyset$$

This implies that A is disconnected, which is a contradiction. Hence, $A \cup B$ must be connected. We can apply the same argument to $A \cup C$ with B and C switched and reach the same contradiction. Thus, $A \cup B$ and $A \cup C$ are both connected.

If A is closed, then A^{C} is open. By a theorem, since B and C are separated, B and C are both open, so B^{C} and C^{C} are both closed.

Note that $A \cup B \cup C = X$ and A, B, and C are pairwise disjoint. Then clearly, $A \cup B = C^{\mathrm{C}}$ and $A \cup C = B^{\mathrm{C}}$. Thus, $A \cup B$ and $A \cup C$ are closed.

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8 Let (X, d) be a metric space and let A, B be two closed subsets of X such that $A \cup B$ and $A \cap B$ are connected. Prove that A is connected.

Solution Suppose A is disconnected. Then there exist two separated sets C snd D such that $A = C \cup D$. Then

$$A \cap B = (C \cup D) \cap B = (C \cap B) \cup (D \cap B)$$

Suppose $C \cap B = \emptyset$. Then we must have that $A = C$ and $B = D$. So, $A \cup B = C \cup D \implies A \cup B$ is disconnected, which cannot happen. We can apply the same argument for $D \cap B = \emptyset$ with D and C switched. Thus, $D \cap B$ and $C \cap B$ must both be non-empty.

From now on, assume those two sets are non-empty. Then we claim that they are separated:

$$\overline{C \cap B} \cap D \cap B \subseteq \bar{C} \cap \bar{B} \cap D \cap B = B \cap (\bar{C} \cap D) = \emptyset$$
$$C \cap B \cap \overline{D \cap B} \subseteq C \cap B \cap \bar{D} \cap \bar{B} = B \cap (C \cap \bar{D}) = \emptyset$$

Then by definition, $A \cap B$ is separated, which is a contradiction. Hence, A must be connected.

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9 Let $\{A_i\}_{i \in I}$ be a family of connected sets such that one set of the family intersects all the others. Prove that $\bigcup_{i \in I} A_i$ is connected.

Solution Let $A = \bigcup_{i \in I} A_i$, $n \in I$ be so that A_n is the set that intersects all the others, and assume there exists $\emptyset \neq B \subset A$ such that it is both closed and open in A. Then $A \setminus B$ is also closed and open in A. Note that $A_i \in B \cup (A \setminus B)$ for all $i \in I$ since $A_i \in A$, and $A_i \cap B \cap (A \setminus B) = \emptyset$ since B and $A \setminus B$ are disjoint.

Define $I_B = \{i \in I \mid B \cap A_i \neq \emptyset\}$ and $I_{A \setminus B} = \{i \in I \mid (A \setminus B) \cap A_i \neq \emptyset\}$. Note that $I_B \cup I_{A \setminus B} = I$.

Suppose $I_B \cap I_{A \setminus B} = \emptyset$. Then $A_a \subseteq B$ or $A_a \subseteq A \setminus B$. In the first case, A_a intersects all the other sets, so B must also intersect all the other sets. $B \neq A$, so there exists $i \in I$ such that $A_i \cap B \neq \emptyset$ and $A_i \cap (A \setminus B) \neq \emptyset$. Then by a theorem, A_i must be disconnected since B and $A \setminus B$ are open. If $A_a \subseteq A \setminus B$, then we can apply the same argument, but with B and $A \setminus B$ switched. This is a contradiction, so $I_B \cap I_{A \setminus B} \neq \emptyset$.

Let $i \in I_B \cap I_{A \setminus B}$. Then by definition, $A_i \cap (A \setminus B) \neq \emptyset$ and $A_i \cap B \neq \emptyset$. Then A_i must be disconnected, which is a contradiction. So, $I_B \cap I_{A \setminus B} = \emptyset$.

$I_B \cap I_{A \setminus B}$ can't be empty and non-empty at the same time, which contradicts our initial assumption that B exists. Hence, the only sets that are open and closed in A are $\emptyset$ and A, so by a theorem, A is connected.

10 Let (X, d) be a metric space and let A, B be two subsets of X such that $A \cap B \neq \emptyset$ and $B \setminus A \neq \emptyset$. Assume also that B is connected. Prove that $\mathrm{Fr}(A) \cap B \neq \emptyset$. Deduce that if X is connected, then every subset of X, other than $\emptyset$ or X, has at least one frontier point.

Solution Note that $B = (B \cap A) \cup (B \cap A^{\mathrm{C}})$.

Suppose that $\mathrm{Fr}(A) \cap B = \bar{A} \cap \overline{A^{\mathrm{C}}} \cap B = \emptyset$. We claim that $(B \cap A)$ and $(B \cap A^{\mathrm{C}})$ are separated.

$$\overline{(B \cap A)} \cap B \cap A^{\mathrm{C}} \subseteq \bar{B} \cap \bar{A} \cap B \cap A^{\mathrm{C}} \subseteq (\bar{A} \cap \overline{A^{\mathrm{C}}} \cap B) \cap \bar{B} = \emptyset$$
$$(B \cap A) \cap \overline{B \cap A^{\mathrm{C}}} \subseteq B \cap A \cap \bar{B} \cap \overline{A^{\mathrm{C}}} \subseteq (\bar{A} \cap \overline{A^{\mathrm{C}}} \cap B) \cap \bar{B} = \emptyset$$

By definition, B is disconnected, which is a contradiction. Hence, $\mathrm{Fr}(A) \cap B \neq \emptyset$.

If X is connected, then let $\emptyset \neq A \subset X$. $A \cap X = A \neq \emptyset$ and $X \setminus A \neq \emptyset$ since $A \neq X$. Since X is connected, we have that $\mathrm{Fr}(A) \cap X = \mathrm{Fr}(A) \neq \emptyset$ for all $\emptyset \neq A \subset X$.