Bass Exercises

18.14 Let A be the set of real-valued continuous functions on $[0,1]$ such that

$$\int_0^{1/2} f(x)\,\mathrm{d}x - \int_{1/2}^1 f(x)\,\mathrm{d}x = 1.$$

Prove that A is a closed convex subset of $C([0,1])$, but there does not exist $f \in A$ such that

$$\|f\| = \inf_{g\in A}\|g\|.$$

Solution Let $\{f_n\} \subseteq A$ such that $f_n \to f$ in $C([0,1])$, i.e., $f_n \to f$ uniformly. Then

$$1 = \int_0^{1/2} f_n(x)\,\mathrm{d}x - \int_{1/2}^1 f_n(x)\,\mathrm{d}x \xrightarrow{n\to\infty} \int_0^{1/2} f(x)\,\mathrm{d}x - \int_{1/2}^1 f(x)\,\mathrm{d}x,$$

so $f \in A$, hence A is closed.

Now let $f, g \in A$ and $t \in [0,1]$. Then by linearity of integration,

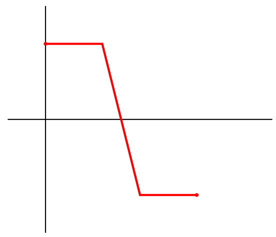
$$\begin{aligned}
&\int_0^{1/2} tf(x) + (1-t)g(x)\,\mathrm{d}x - \int_{1/2}^1 tf(x) + (1-t)g(x)\,\mathrm{d}x \\
&= t\left(\int_0^{1/2} f(x)\,\mathrm{d}x - \int_{1/2}^1 f(x)\,\mathrm{d}x\right) + (1-t)\left(\int_0^{1/2} g(x)\,\mathrm{d}x - \int_{1/2}^1 g(x)\,\mathrm{d}x\right) \\
&= t + (1-t) \\
&= 1,
\end{aligned}$$

so $tf + (1-t)g \in A$, so A is convex.

For $f \in A$, notice that

$$\begin{aligned}
1 &\le \int_0^{1/2} |f(x)|\,\mathrm{d}x + \int_{1/2}^0 |f(x)|\,\mathrm{d}x \\
&\le \frac{1}{2}\|f\| + \frac{1}{2}\|f\| \\
&= \|f\|.
\end{aligned}$$

We claim that 1 is the infimum. Indeed, we can use a function that looks like the following:



If the horizontal segments sit at $1+1/n$ and $1-1/n$, respectively, then we can adjust the slope of the middle segment until the function is in A.

Now let $f \in A$ with $\|f\| = 1$. We consider its behavior at $1/2$:

Suppose $f(1/2) \leq 1/2$. By continuity, there exists $\delta > 0$ so that $f(x) \leq 3/4$ on $[1/2 - \delta, 1/2]$. Then

$$\begin{aligned}\int_0^{1/2} f(x)\,\mathrm{d}x - \int_{1/2}^1 f(x)\,\mathrm{d}x &= \int_0^{1/2-\delta} f(x)\,\mathrm{d}x + \int_{1/2-\delta}^{1/2} f(x)\,\mathrm{d}x - \int_{1/2}^1 f(x)\,\mathrm{d}x \\ &\leq \left(\frac{1}{2} - \delta\right)\|f\| + \frac{3}{4}\delta + \frac{1}{2}\|f\| \\ &= 1 - \frac{\delta}{4} \\ &< 1.\end{aligned}$$

If $f(1/2) \geq -1/2$, then we can consider $-f$, which satisfies $-f(1/2) \leq -1/2 \leq 1/2$, and apply the same argument.

At least one of these two scenarios must occur, which shows that if $\|f\| = 1$, then $f \notin A$. Thus, no function with $\|f\| = 1$ is in A.

Folland Exercises

3.18 Prove Proposition 3.13c.

Solution We wish to show that $L^1(\nu) = L^1(|\nu|)$ and that if $f \in L^1(\nu)$, then $\left|\int f\,\mathrm{d}\nu\right| \leq \int |f|\,\mathrm{d}|\nu|$.

We have

$$f \in L^1(\nu) \iff \infty > \int |f|\,\mathrm{d}\nu = \int |f| \frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}\,\mathrm{d}|\nu| = \int |f|\,\mathrm{d}|\nu| \iff f \in L^1(|\nu|),$$

since $\mathrm{d}\nu/\mathrm{d}|\nu| = 1$ $|\nu|$-a.e, so the two sets are the same.

Now let $f \in L^1(\nu)$. Then

$$\left|\int f\,\mathrm{d}\nu\right| = \left|\int f \frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}\,\mathrm{d}|\nu|\right| = \left|\int f\,\mathrm{d}|\nu|\right| \leq \int |f|\,\mathrm{d}|\nu|.$$

3.20 If ν is a complex measure on $(X, \mathcal{M})$ and $\nu(X) = |\nu|(X)$, then $\nu = |\nu|$.

Solution Let $E \in \mathcal{M}$ and write $\mathrm{d}\nu = f\,\mathrm{d}\mu$, for some $f \in L^1(\mu)$ and positive measure μ, so that $\mathrm{d}|\nu| = |f|\,\mathrm{d}\mu$. If we write $f = f_+ - f_-$, our assumption tells us

$$\nu(X) = |\nu|(X) \implies \int f_+\,\mathrm{d}\mu - \int f_-\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu = \int |f|\,\mathrm{d}\mu = \int f_+\,\mathrm{d}\mu + \int f_-\,\mathrm{d}\mu.$$

Thus,

$$\int f_-\,\mathrm{d}\mu = 0,$$

so $f_- = 0$ μ-a.e, which tells us that f is non-negative μ-a.e, i.e., $f = |f|$ μ-a.e. Hence, $\mathrm{d}\nu = f\,\mathrm{d}\mu = |f|\,\mathrm{d}\mu = \mathrm{d}|\nu|$, so $\nu = |\nu|$.

3.21 Let ν be a complex measure on $(X, \mathcal{M})$. If $E \in \mathcal{M}$, define

$$\mu_1(E) = \sup\left\{\sum_1^n |\nu(E_j)| \mid n \in \mathbb{N}, E_1, \ldots, E_n \text{ disjoint, } E = \bigcup_1^n E_j\right\},$$
$$\mu_3(E) = \sup\left\{\left|\int_E f\,\mathrm{d}\nu\right| \mid |f| \le 1\right\}.$$

Then $\mu_1 = \mu_3 = |\nu|$.

Solution Let $E \in \mathcal{M}$.

If we take $f = 1$, then $|f| \le 1$, so

$$|\nu(E)| = \left|\int_E \mathrm{d}\nu\right|,$$

which shows that $\mu_1 \le \mu_3$.

Let $\varphi = \sum a_j \chi_{E_j}$ be simple, so that the $E_j \subseteq E$ are disjoint. We also allow one of the a_j to be zero so that we can assume $E = \bigcup E_j$. Then

$$\left|\int_E \varphi\,\mathrm{d}\nu\right| = \left|\sum_{j=1}^n a_j \nu(E_j)\right| \le \sum_{j=1}^n |a_j||\nu|(E_j) \le \mu_1(E).$$

By approximating any $f \in L^1(\nu)$ with $|f| \le 1$ from below with simple functions φ_n, monotone convergence gives us

$$\left|\int_E f\,\mathrm{d}\nu\right| = \lim_{n\to\infty}\left|\int_E \varphi_n\,\mathrm{d}\nu\right| \le \mu_1(E).$$

Since this holds for any f, we have $\mu_3(E) \le \mu_1(E)$, so $\mu_1 = \mu_e$.

Now let $f = \mathrm{d}\nu/\mathrm{d}|\nu|$, which is 1 $|\nu|$-a.e., so by adjusting it on a set of $|\nu|$-measure zero, we may assume that $|f| \le 1$ everywhere. Then

$$\mu_3(E) = \left|\int_E \frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}\,\mathrm{d}|\nu|\right| = \left|\int_E \mathrm{d}\nu\right| = |\nu(E)| = |\nu|(E),$$

so $\mu_1 = \mu_3 = |\nu|$, as required.

5.48 Suppose that $\mathcal{X}$ is a Banach space.

a. The norm-closed unit ball $B = \{x \in \mathcal{X} \mid \|x\| \le 1\}$ is also weakly closed.

b. If $E \subseteq \mathcal{X}$ is bounded (with respect to the norm), so is its weak closure.

c. If $F \subseteq \mathcal{X}^*$ is bounded (with respect to the norm), so is its weak* closure.

d. Every weak*-Cauchy sequence in $\mathcal{X}^*$ converges.

Solution a. Notice that

$$B = \bigcap_{f \in \mathcal{X}^*} \{x \mid |f(x)| \le \|f\|\}.$$

If $x \in B$, then $\|x\| \le 1 \implies |f(x)| \le \|f\|\|x\| \le \|f\|$, for any $f \in \mathcal{X}^*$. On the other hand, suppose that x is in the right-hand side, but $\|x\| > 1$. By Hahn-Banach, there exists a linear functional f so that $f(x/\|x\|) = 1$ and $\|f\| = 1$. But

$$|f(x)| = \|x\|\left|f\left(\frac{x}{\|x\|}\right)\right| = \|x\|\|f\| > \|f\|,$$

which is impossible, since x was in the right-hand side. Hence, the two sets are equal, and each set in the intersection is the preimage of a closed set under weakly continuous function, so it is weakly closed. Intersections of weakly closed sets are weakly closed, so B is weakly closed.

b. By scaling, $B_n \coloneqq \{x \in \mathcal{X} \mid \|x\| \leq n\}$ is also weakly closed.

Since E is bounded, there exists $n \geq 1$ so that $E \subseteq B$. Then because B is weakly closed, the weak closure of E is also contained in B, so the weak closure of E is bounded.

c. By Alaoglu's theorem and the fact that scaling is a homemomorphism, closed balls are weak* compact in $\mathcal{X}^*$, hence closed, since $\mathcal{X}$ is Hausdorff. Thus, if F is bounded, it is contained in some ball B which is weak* closed, so the weak* closure of F is also contained in B. Hence, F's weak* closure is bounded.

d. Let $\{f_n\} \subseteq \mathcal{X}^*$ be Cauchy in the weak* topology.

From the definition, for all $x \in \mathcal{X}$, we have $|f_n(x) - f_m(x)| \xrightarrow{n\to\infty} 0$. I.e., $\{f_n(x)\}$ is Cauchy in K, which is complete, so f_n converges pointwise to some function f. Since $\mathcal{X}^*$ is a Banach space, it follows that $f \in \mathcal{X}^*$. It's then clear by definition of f that $|f_n(x) - f(x)| \xrightarrow{n\to\infty} 0$ for all x, so $f_n \to f$ in the weak* topology.

7.21 Let $\{f_\alpha\}_{\alpha\in A}$ be a subset of $C(X)$, where X is a compact metric space, and let $\{c_\alpha\}_{\alpha\in A}$ a family of complex numbers. If for each finite set $B \subseteq A$ there exists $\mu_B \in M(X)$ such that $\|\mu_B\| \leq 1$ and $\int f_\alpha \mathrm{d}\mu_B = c_\alpha$ for $\alpha \in B$, then there exists $\mu \in M(X)$ such that $\|\mu\| \leq 1$ and $\int f_\alpha \mathrm{d}\mu = c_\alpha$ for all $\alpha \in A$.

Solution If A is finite, then there is nothing to say, so assume from now on that A is infinite.

By a corollary, because X is compact Hausdorff, we know that $C(X)^* = (X)$.

Since $\|\mu_B\| \leq 1$ for each finite B, it follows that $\|\mu_B\| \leq 1$ in $C(X)^*$ also, so each μ_B is an element of the unit ball in $C(X)^*$. By Banach-Alaoglu, the unit ball is compact.

Consider $\{\mu_B\}_B$. By compactness, these measures admit a subsequence that converges vaguely to a measure μ. Since they converge vaguely, by definition, we have

$$c_\alpha = \int f_\alpha \,\mathrm{d}\mu_B \to \int f_\alpha \,\mathrm{d}\mu.$$

Also by compactness, $\|\mu\| \leq 1$.