**13** The following deals with $G_\delta$ and $F_\sigma$ sets.

a. Show that a closed set is a $G_\delta$ and an open set an $F_\sigma$.
[Hint: If $F$ is closed, consider $\mathcal{O}_n = \{x \mid d(x, F) < 1/n\}$.]

b. Give an example of an $F_\sigma$ set which is not a $G_\delta$.
[Hint: This is more difficult; let $F$ be a denumerable set that is dense.]

c. Give an example of a Borel set which is not a $G_\delta$ nor an $F_\sigma$.

**Solution** a. Let $F$ be a closed set. Then $\mathcal{O}_n = \{x \mid d(x, F) < 1/n\}$, which is open.
Indeed, let $y \in \mathcal{O}_n \implies d(y, F) < 1/n$, and let $r = d(y, F)/2$. Then $B_r(y) \subseteq \mathcal{O}_n$. Otherwise, there exists $z \in B_r(y) \cap F$, but since $d(x, F) = \inf\{d(x, z) \mid z \in F\}$, this means

$$d(y, F) \le d(y, z) < r = \frac{d(y, F)}{2},$$

which is a contradiction. Hence, each $\mathcal{O}_n$ is open.
We claim that $F = \bigcap_{n\ge1} \mathcal{O}_n$.
If $d(x, F) = 0$, then $x \in F$, since $F$ is closed. If that were not the case, then we can construct a sequence $\{x_k\}_{k\ge1} \subseteq F$ such that $d(x, x_k) < 1/k$ for all $k \ge 1$. This means that $x_k \xrightarrow{k\to\infty} x \notin F$, which is a contradiction, since closed subsets of $\mathbb{R}^d$ are complete. Hence,

$$x \in F \iff d(x, F) = 0 \iff d(x, F) < \frac{1}{n}\ \forall n \ge 1 \iff x \in \bigcap_{n\ge1} \mathcal{O}_n.$$

Hence, $F$ is $G_\delta$.
Let $U$ be an open set. For all $x \in U$, there exists $r_x > 0$ such that $B_{r_x}(x) \subseteq U$. Then the closed ball $K_{r_x/2}(x) \subseteq U$. Hence, since $x \in B_{r_x}(x)$,

$$U = \bigcup_{x\in U} K_{r_x/2}(x),$$

so $U$ is $F_\sigma$.

b. Consider $\mathbb{Q}$, which is $F_\sigma$. Indeed, since $\mathbb{Q}$ is countable, we can enumerate it via $\mathbb{Q} = \{r_1, r_2, \ldots\}$. Then since singleton sets $\{x\}$ are closed in $\mathbb{R}$,

$$\mathbb{Q} = \bigcup_{n\ge1} \{r_n\},$$

so $\mathbb{Q}$ is $F_\sigma$.
$\mathbb{Q}$ is not $G_\delta$. Suppose otherwise, and that there exists open sets $\{U_n\}_{n\ge1}$ such that

$$\mathbb{Q} = \bigcap_{n\ge1} U_n.$$

Then each $U_n$ must be dense in $\mathbb{R}$, since $\mathbb{Q}$ is dense in $\mathbb{R}$. If not, then there exists $U_i$ such that $\overline{U_i} \subsetneq \mathbb{R}$, but this implies that $\overline{\mathbb{Q}} \subseteq \overline{U_i} \subsetneq \mathbb{R}$, which is a contradiction.
Note that we can write $\mathbb{R} \setminus \mathbb{Q} = \bigcap_{n\ge1} {}^{\mathrm{c}}\{r_n\}$. Indeed, $x$ is irrational if and only if it is not any rational number. Since the irrationals are dense in $\mathbb{R}$ and each ${}^{\mathrm{c}}\{r_n\}$ is open, it follows that ${}^{\mathrm{c}}\{r_n\}$ is dense.
By the Baire Category Theorem, since $\mathbb{R}$ is a complete metric space, a countable intersection of open, dense sets in $\mathbb{R}$ is also dense. Hence,

$$\mathbb{R} = \overline{\bigcap_{n\ge1} U_n \cap \bigcap_{n\ge1} {}^{\mathrm{c}}\{r_n\}} = \overline{\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q})} = \emptyset,$$

which is a contradiction. So, $\mathbb{Q}$ is an $F_\sigma$ set which is not $G_\delta$.

c. Note that since $\mathbb{Q}$ is not $G_\delta$, then $\mathbb{R} \setminus \mathbb{Q}$ is not $F_\sigma$. Otherwise, there exist closed sets $\{F_n\}_{n\geq 1}$ such that

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n\geq 1} F_n \implies \mathbb{Q} = {}^{\mathrm{c}}\left(\bigcup_{n\geq 1} F_n\right) = \bigcap_{n\geq 1} {}^{\mathrm{c}}F_n,$$

which implies that $\mathbb{Q}$ is $G_\delta$, since ${}^{\mathrm{c}}F_n$ is open. Using complements again, it follows that $\mathbb{R} \setminus \mathbb{Q}$ is $G_\delta$. Then consider the following sets:

$$\begin{aligned} E_1 &= (\mathbb{R} \setminus \mathbb{Q}) \cap (-\infty, 0] \\ E_2 &= \mathbb{Q} \cap [0, \infty). \end{aligned}$$

$E_1$ is $G_\delta$ since $\mathbb{R} \setminus \mathbb{Q}$ and $(-\infty, 0)$ are both $G_\delta$. Moreover, it's not $F_\sigma$ since $\mathbb{R} \setminus \mathbb{Q}$ is not $F_\sigma$.
Similarly, $E_2$ is $F_\sigma$ and not $G_\delta$. Indeed, $E_2$ is dense in $[0, \infty)$, which is a closed subset of $\mathbb{R}$, so it is complete. Using the same argument as in part (b), it follows that $E_2$ is $F_\sigma$ and not $G_\delta$.
Note that using the same arguments, we can switch between $(-\infty, 0]$ and $[0, \infty)$ in the definitions of $E_1$ and $E_2$ to get similar sets, i.e., $F_\sigma$ but not $G_\delta$ and vice versa.
Hence $E \coloneqq E_1 \cup E_2$ is Borel. Indeed, every open set is a Borel set, by definition, and since the Borel sets are a $\sigma$-algebra, every closed set is Borel. Hence, by the definition of a $\sigma$-algebra, $F_\sigma$ and $G_\delta$ sets are Borel, so $E_1$ and $E_2$ are Borel. Since $\sigma$-algebras are closed under unions, $E_1 \cup E_2$ is Borel.
Moreover, $E$ is neither $F_\sigma$ nor $G_\delta$, since $E_1$ is not $F_\sigma$ and $E_2$ is not $G_\delta$.
If $E$ were $F_\sigma$, then there exist closed sets $\{F_n\}_{n\geq 1}$ such that $E = \bigcup_{n\geq 1} F_n$. But

$$\mathbb{R} \setminus \mathbb{Q} = \underbrace{E_1}_{F_\sigma} \cup \underbrace{(\mathbb{R} \setminus \mathbb{Q}) \cap [0, \infty)}_{F_\sigma} \implies \mathbb{R} \setminus \mathbb{Q} \text{ is } F_\sigma,$$

which is a contradiction.
Similarly, we can take complements to see that $E$ is not $G_\delta$.
Thus, $E$ is a Borel set but it's neither $G_\delta$ nor $F_\sigma$.

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**16** Suppose $\{E_k\}_{k\geq 1}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$\begin{aligned} E &= \{x \in \mathbb{R}^d \mid x \in E_k, \text{ for infinitely many } k\} \\ &= \limsup_{k\to\infty} E_k. \end{aligned}$$

a. Show that $E$ is measurable.
b. Prove that $m(E) = 0$.
[Hint: Write $E = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k$.]

**Solution** We'll show that $m_*(E) = 0$. Then this shows that $E$ is measurable and $m(E) = 0$.
Note that $E = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k \implies E \subseteq \bigcup_{k\geq n} E_k$ for all $n \geq 1$. Then by countable subadditivity,

$$m_*(E) \leq \sum_{k=n}^{\infty} m_*(E_k) = \sum_{k=1}^{\infty} m_*(E_k) - \sum_{k=1}^{n-1} m_*(E_k) \ \forall n \geq 1.$$

Since

$$\sum_{k=1}^{\infty} m_*(E_k) = \sum_{k=1}^{\infty} m(E_k) < \infty,$$

we can take $n \to \infty$ in the inequality to get

$$m_*(E) \leq \sum_{k=1}^{\infty} m_*(E_k) - \sum_{k=1}^{\infty} m_*(E_k) = 0.$$

**18** Prove the following assertion: Every measurable function is the limit a.e. of a sequence of continuous functions.

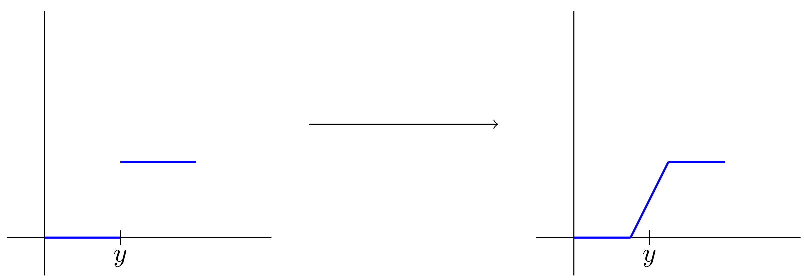
**Solution** Let $f$ be a measurable function. Then there exists a sequence of step functions which converge to $f$ almost everywhere.

It then suffices to show that step functions can be approximated by a sequence of continuous function which converge to it almost everywhere.

Let $\varphi = \sum_{k=1}^{N} = a_k \chi_{R_k}$ be a step function in canonical form, i.e., each rectangle $R_k$ is disjoint or almost disjoint from the others. Then consider the endpoints of each rectangle, $\{a_k, b_k\}$, with possibly $a_k = b_k$. These endpoints "separate" the steps. More precisely, for an endpoint $x$, there exists $\delta_x > 0$ such that $\varphi$ is constant on $(x - \delta_x, x)$ and $(x, x + \delta_x)$, but it attains different constants on each interval. Then we can define $l_x = \varphi(y)$, $y \in (x - \delta_x, x)$ and $r_x = \varphi(y)$, $y \in (x, x + \delta_x)$.

For each endpoint $y$, we "connect" the steps with a line as follows: Let $d_n = \min\{1/n, \min_y\{\delta_y\}\}$. Then

$$\psi_n(x) = \begin{cases} \frac{1}{2}\left(y - \frac{d_n}{2} + x\right) l_y + \frac{1}{2}\left(y - \frac{d_n}{2} - x\right) r_y & \text{if } y \text{ is an endpoint and } x \in \left[y - \frac{d_n}{2}, y + \frac{d_n}{2}\right] \\ \varphi(x) & \text{otherwise} \end{cases}.$$




This $\psi_m$ is continuous since we "fixed" all the discontinuous of our step function $\varphi_n$.

Moreover, $\psi_n \xrightarrow{n\to\infty} \psi$ almost everywhere. The only areas where we have issues of convergence is at each endpoint $y$ of the rectangles, since the connecting line gets more vertical as $n$ gets large, so $\psi_n$ still converges to $\varphi$ at points arbitrarily close, but not equal to $y$. However, we have a finite number of these endpoints, since we have finitely many rectangles.

Hence, $\psi_n(x)$ converges to $\varphi(x)$ at all but finitely many $x$, so it converges almost everywhere.

**22** Let $\chi_{[0,1]}$ be the characteristic function of $[0,1]$. Show that there is no everywhere continuous function $f$ on $\mathbb{R}$ such that

$$f(x) = \chi_{[0,1]}(x) \quad \text{almost everywhere.}$$

**Solution** Suppose $f = \chi_{br(0,1)}$ almost everywhere, but $f$ is continuous. By definition,

$$m(\{f \neq \chi_{[0,1]}\}) \coloneqq m(M) = 0.$$

Since $f$ is continuous, $D \coloneqq f^{-1}\big(\mathbb{R} \setminus \{0,1\}\big)$ is open. Moreover, $D$ is non-empty. Otherwise, we have three cases to consider:

$f \equiv 0$:

Then $f$ differs from $\chi_{[0,1]}$ on the set $[0,1]$ which has Lebesgue measure 1, so this is a contradiction.

$f \equiv 1$:

Then $f$ differs from $\chi_{[0,1]}$ on $\mathbb{R} \setminus [0,1]$ which has infinite Lebesgue measure, another contradiction.

$f(x) \in \{0,1\}$ $\forall x \in \mathbb{R}$:

Since $f$ is continuous and $\mathbb{R}$ is connected, $f(\mathbb{R})$ must be connected, so it must be an interval. However, $f(\mathbb{R}) = \{0,1\}$, which is not an interval. Hence, we have a contradiction.

Thus, $D$ is open, so it has positive Lebesgue measure. It is measurable since it is open, and it has positive measure since there exists a ball of radius $r > 0$ contained in $D$, so by the subset property of Lebesgue measure, $0 < 2r < m(D)$.

But $D \subseteq M$, since $f(x) \notin \{0,1\} \ni \chi_{[0,1]}(x)$ for all $x \in D$, so by the subset property of Lebesgue measure, $0 < m(D) \leq m(M) = 0$, which is a contradiction.

Hence, no such $f$ exists.

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**29** Suppose $E$ is a measurable subset of $\mathbb{R}$ with $m(E) > 0$. Prove that the difference set of $E$, which is defined by

$$\{z \in \mathbb{R} \mid z = x - y \text{ for some } x, y \in E\},$$

contains an open interval centered at the origin.

[Hint: Indeed, by Exercise 28, there exists an open interval $I$ so that $m(E \cap I) \geq (9/10)m(I)$. If we denote $E \cap I$ by $E_0$, and suppose that the difference set of $E_0$ does not contain an open interval around the origin, then for arbitrary small $a$, the sets $E_0$, and $E_0 + a$ are disjoint. From the fact that $(E_0 \cup (E_0 + a)) \subseteq (I \cup (I + a))$ we get a contradiction, since the left-hand side has measure $2m(E_0)$, while the right-hand side has measure only slightly larger than $m(I)$.]

**Solution** Note that 0 is in the difference set, since we could take $x = y$.

By Exercise 28, since $m(E) = m_*(E) > 0$, then for all $0 < \alpha < 1$, there exists an open interval $I$ such that $m_*(E \cap I) \geq \alpha m_*(I)$.

In particular, there exists an open interval $I$ so that $m(E \cap I) \geq (9/10)m(I)$. Define $E_0 \coloneqq E \cap I$.

Suppose the difference set of $E_0$ did not contain an open interval about the origin. Then for an arbitrary small $a$, $E_0$ and $E_0 + a$ are disjoint. Indeed, if for some $M > 0$, $E_0 \cap (E_0 + M) \neq \emptyset$ for all $a < M$, then $\{0, a\} \subseteq E_0$ for all $a < M$. This implies that $(0, M) \subseteq E_0$, which is a contradiction.

Then note that $E_0 \cup (E_0 + a) \subseteq I \cup (I + a)$, for an arbitrarily small $a$. $E_0$ is measurable since $E$ and $I$ are measurable. $E_0 + a$ and $I + a$ have the same measure as $E_0$ and $I$, respectively, since Lebesgue measure is translation invariant. Hence, by the subset and disjoint properties of Lebesgue measure,

$$m(E_0 \cup (E_0 + a)) = 2m(E_0) \leq m(I) + a \implies m(E_0) \leq \frac{1}{2}m(I) + a.$$

Since $a$ was arbitrary, we get that $m(E_0) \leq (1/2)m(I)$. But $m(E_0) \geq (9/10)m(I)$, which is a contradiction. Hence, the difference set of $E_0$ must contain an open interval at the origin. Since $E_0 \subseteq E$, $E$ must contain an open interval as well.

**5** Suppose $E$ is measurable with $m(E) < \infty$, and

$$E = E_1 \cup E_2, \qquad E_1 \cap E_2 = \emptyset.$$

If $m(E) = m_*(E_1) + m_*(E_2)$, then $E_1$ and $E_2$ are measurable.

In particular, if $E \subseteq Q$, where $Q$ is a finite cube, then $E$ is measurable if and only if $m(Q) = m_*(E) + m_*(Q - E)$.

**Solution** Consider the inner measure of $E$, which is

$$M_*(E) = \sup m(F), \text{ where } F \subseteq E \text{ closed.}$$

where $F$ is a closed subset of $E$.

Our first claim is that $E$ is measurable if and only if $M_*(E) = m_*(E)$.

"$\implies$"

This is clear since by Theorem 3.4(ii), for all $\varepsilon > 0$, there exists a closed set $F \subseteq E$ with $m(E - F) \leq \varepsilon$. Then

$$M_*(E) \geq m(F) = m(E) - m(E - F) \geq m(E) - \varepsilon.$$

Since $\varepsilon$ was arbitrary, $m(E) = m_*(E) \leq M_*(E)$.

Note that $M_*(E) \leq m_*(E)$, since any set that covers $E$ covers a closed subset of $E$, so $M_*(E) = m_*(E)$ if $E$ is measurable.

"$\impliedby$"

Let $M_*(E) = m_*(E)$. Let $U$ be an open set containing $E$.

Then for all $n \geq 1$, there exists a closed subset of $F_n$ and a collection of open cubes $Q_n$ covering $E$ such that

$$m_*(E) - \frac{1}{n} = M_*(E) - \varepsilon \leq m(F_n) \leq \sum |Q_n| \leq m_*(E) + \frac{1}{n}.$$

Consider $F \coloneqq \bigcup_n F_n$. $m(F) = m_*(E)$ by the above inequality, and $F$ is $F_\sigma$. Hence, $E$ differs from an $F_\sigma$ set by a set of measure 0, so $E$ is measurable.

Our next claim is that if $A \subseteq E$, then $m(E) = m_*(A) + M_*(E - A)$.

Since $E$ is measurable, $m_*(E) = M_*(E)$. Hence, it suffices to show two inequalities:

$$M_*(E) - m_*(A) \leq M_*(E - A)$$
$$m_*(E) - M_*(E - A) \geq m_*(A).$$

Let $\varepsilon > 0$, $F$ be a closed subset of $E$ with $m(F) \geq m(E) - \varepsilon$, and let $\{Q_n\}$ be a covering of $A$ by open cubes with $\sum |Q_n| \leq m_*(A) + \varepsilon$. Then

$$F - \left(\bigcup Q_n\right) \coloneqq F - Q = F \cap {}^{c}Q$$

is closed, since $Q$ is open. Moreover, this is a closed subset of $M_*(E - A)$. Hence, $F - Q \subseteq E - A$ implies

$$M_*(E - A) \geq M_*(F - Q) = m(F - Q) = m(F) - m(Q) \geq m(E) - m_*(A) - 2\varepsilon = M_*(E) - m_*(A) - 2\varepsilon.$$

Since $\varepsilon$ was arbitrary, we have $M_*(E) - m_*(A) \leq M_*(E - A)$ as desired. The second inequality follows from a similar argument.

Thus, by the second claim and by hypothesis,

$$m(E) = m_*(E_2) + M_*(E - E_2) = m_*(E_2) + M_*(E_1) = m_*(E_2) + m_*(E_1) \implies m_*(E_1) = M_*(E_1).$$

Hence $E_1$ is measurable by the first claim. Moreover, $E_2 = E - E_1 = E \cap {}^{c}E_1$ is measurable, since complements of a measurable set are measurable and since measurable sets are closed under intersections.