29.8 Show that the one-point compactification of $\mathbb{Z}_+$ is homeomorphic with the subspace $\{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}$ of $\mathbb{R}$.

Solution Call the one-point compactification $\mathbb{Z}_+^*$ and the subspace A.

$\mathbb{Z}_+$ as a subspace of $\mathbb{R}$ has the discrete topology.

Note that the basic open sets of A are of the form $\{1/n\}$ or $0 \cup \{1/n \mid n \in \mathbb{Z}_+\} \cap (0, x)$, for any $x > 0$. I.e., the second set contains 0 and infinitely many $1/n$.

We define the homeomorphism $f\colon A \to \mathbb{Z}_+^*$ as follows:

$$f(x) = \begin{cases} \infty & \text{if } x = 0 \\ n & \text{if } x = 1/n. \end{cases}$$

Notice that the map $n \longmapsto 1/n$ is a bijection between $n \in \mathbb{Z}_+$ and $\{1/n \mid n \in \mathbb{Z}_+\}$; its inverse is itself, so f is also a bijection.

Let U be a basic open set in $\mathbb{Z}_+^*$. There are two cases:

U is an open set in $\mathbb{Z}_+$.

Then $f^{-1}(U) = U$, which is still open.

$U = \mathbb{Z}_+^* - C$, where C is compact subspace of $\mathbb{Z}_+$

If C is compact, then C is necessarily finite. Indeed, if there were infinitely many n in C, then the open cover $\{\{n\} \mid n \in C\}$ has no finite subcover, since we need every set.

If there are finitely many n, then we can simply take one open cover per n from any open cover. Hence, we can write

$$U = {}^{\mathrm{c}}\{n_1, \dots, n_N\} \implies f^{-1}(U) = {}^{\mathrm{c}}f^{-1}(\{n_1, \dots, n_N\}) = {}^{\mathrm{c}}\{1/n_1, \dots, 1/n_N\},$$

which contains infinitely many $1/n$ and 0, so it contains an open neighborhood of 0. We can simply take unions with the rest of the $1/n$ in addition to the open neighborhood of 0 to see that $f^{-1}(U)$ is open.

Lastly, we need to show that f is open. Let U be a basic open set in A.

$0 \in A$

Then U contains infinitely many $1/n$, since it's an open neighborhood of 0. This means that ${}^{\mathrm{c}}U$ contains finitely many $1/n$, so we can write ${}^{\mathrm{c}}U = \{1/n_1, \dots, 1/n_N\}$. Since f is surjective,

$${}^{\mathrm{c}}f(U) = f({}^{\mathrm{c}}U) = \{n_1, \dots, n_N\},$$

which is compact since we have finitely many n_i. So, $f(U)$ is the complement of a compact set in $\mathbb{Z}_+$, which means $f(U)$ is open, by definition.

$0 \notin A$

Then $f(U) \subseteq \mathbb{Z}_+$, which is open since $\mathbb{Z}_+$ has the discrete topology.

Taking all of the steps together, we see that f is a homeomorphism, so the two sets are homeomorphic.

30.5 a. Show that every metrizable space with a countable dense subset has a countable basis.

b. Show that every metrizable Lindelöf space has a countable basis.

Solution a. Let X be a metrizable space with a countable dense subset $Q = \{q_1, q_2, \ldots\}$. For each q_i, we take the countable neighborhood basis $\mathcal{U}_i := \{B(q_i, 1/n) \mid n \in \mathbb{Z}_+\}$.

By an analysis theorem, a countable union of countable sets is countable, so $\mathcal{B} := \bigcup_{i=1}^{\infty} \mathcal{U}_i$ is a countable collection.

All that's left is to show that it's a basis.

Let U be an open neighborhood in X, and pick $x \in U$. Then for some $n_x \in \mathbb{N}$, we can fit $B(x, 1/n_x)$ and $B(x, 1/2n_x)$ in U. Since Q is dense in X, there exists some $q(x) \in B(x, 1/2n_x)$. Then $x \in B(q(x), 1/2n_x) \subseteq B(x, 1/2n_x) \subseteq B(x, 1/n_x) \subseteq U$. Hence,

$$U = \bigcup_{x \in U} B(q(x), 1/2n_x),$$

so $\mathcal{B}$ is a countable basis.

b. Let X be a metrizable Lindelöf space.

For each $n \in \mathbb{N}$, take the collection $\mathcal{U}_n := \{B(x, 1/n) \mid x \in X\}$. This is an open cover of X, so by hypothesis, there exists $x_1^{(n)}, x_2^{(n)}, \ldots, x_{N_n}^{(n)} \in X$ such that $\mathcal{B}_n := \{B(x_i^{(n)}, 1/n) \mid 1 \leq i \leq N_n\}$ covers X.

Then $\mathcal{B} := \bigcup_{n=1}^{\infty} \mathcal{B}$ is a countable basis for X. It is countable since each $\mathcal{B}_n$ is finite, and we're taking a countable union of finitely many elements. It also covers X, since each $\mathcal{B}_n$ covers X.

Let U be open in X. Pick $x \in U$. Then we can fit $B(x, 1/2n) \subseteq B(x, 1/n) \subseteq U$ for some $n \geq 1$. Since $\mathcal{B}$ covers X, there exists some k and ℓ such that $x \in B(x_k^{(\ell)}, 1/\ell) \subseteq U$. Indeed, if $1/\ell < 1/2n$, then by the triangle inequality, if we let $y \in B(x_k^{(\ell)}, 1/\ell)$,

$$d(x, y) \leq d(x, x_k^{(\ell)}) + d(x_k^{(\ell)}, y) < \frac{1}{\ell} + \frac{1}{\ell} < \frac{1}{2n} < \frac{1}{n},$$

which means that $B(x_k^{(\ell)}, 1/\ell) \subseteq B(x, 1/n) \subseteq U$. Since we can do this for any x in U, it follows that U is the union of these sets, so $\mathcal{B}$ is a countable basis.

30.14 Show that if X is Lindelöf and Y is compact, then $X \times Y$ is Lindelöf.

Solution Let $\mathcal{B}$ be an open covering of $X \times Y$.

For each $y \in Y$, $\mathcal{B}$ covers $X \times \{y\}$. This set is homeomorphic to X Lindelöf, so $\mathcal{B}$ admits a countable covering of $X \times \{y\}$, which we call $\mathcal{B}_y$.

The projection of the collection $\bigcup_{y \in Y} \mathcal{B}_y$ onto Y covers Y compact, so there exist $y_1, \ldots y_n$ such that the projection of $\bigcup_{i=1}^{n} \mathcal{B}_{y_i}$ covers Y. Since for each y, $\mathcal{B}_y$ covers $X \times \{y\}$, it follows that $\bigcup_{i=1}^{n} \mathcal{B}_{y_i}$ covers $X \times Y$.

Thus, since a finite union of countably many elements is countable, $\bigcup_{i=1}^{n} \mathcal{B}_{y_i}$ is a countable covering of $X \times Y$, so $X \times Y$ is Lindelöf.

31.6 Let $p\colon X \to Y$ be a closed continuous surjective map. Show that if X is normal, then so is Y.

[*Hint:* If U is an open set containing $p^{-1}(\{y\})$, show there is a neighborhood W of y such that $p^{-1}(W) \subseteq U$.]

Solution Let $y \in Y$, and suppose U is an open set containing $p^{-1}(\{y\})$. Then

$$p^{-1}(\{y\}) \subseteq U \implies {}^{\mathrm{c}}U \subseteq p^{-1}({}^{\mathrm{c}}\{y\}).$$

Since p is closed and ${}^{\mathrm{c}}U$ is closed, $p({}^{\mathrm{c}}U)$ is closed in Y, so ${}^{\mathrm{c}}p({}^{\mathrm{c}}U)$ is open in Y. Thus,

$$p({}^{\mathrm{c}}U) \subseteq {}^{\mathrm{c}}\{y\} \implies y \in {}^{\mathrm{c}}p({}^{\mathrm{c}}U),$$

so there exists an open neighborhood $W \ni y$ such that $W \subseteq {}^{\mathrm{c}}p({}^{\mathrm{c}}U)$. Then

$$p^{-1}(W) \subseteq p^{-1}({}^{\mathrm{c}}p({}^{\mathrm{c}}U)) = {}^{\mathrm{c}}p^{-1}(p({}^{\mathrm{c}}U)) \overset{\text{surjectivity}}{=} {}^{\mathrm{c}}p^{-1}({}^{\mathrm{c}}p(U)) = {}^{\mathrm{c}}({}^{\mathrm{c}}U) = U,$$

so the hint is proved.

Let A and B be disjoint, closed subsets of Y.

Since f is continuous, $p^{-1}(A)$ and $p^{-1}(B)$ are closed and disjoint in X, also. Since X is normal, there exist open sets U and V with $U \cap V = \emptyset$, $p^{-1}(A) \subseteq U$, and $p^{-1}(B) \subseteq V$.

By the hint, for every $y \in A$, there exists an open neighborhood U_y of y in Y such that $p^{-1}(U_y) \subseteq U$. Moreover, $p^{-1}(U_y) \cap B \subseteq U \cap V = \emptyset$.

We can perform the same argument to get V_y for every $y \in B$ which is contained entirely in V. Then we can take our separation to be

$$\bigcup_{y \in A} U_y \supseteq A \quad \text{and} \quad \bigcup_{y \in B} V_y \supseteq B,$$

so Y is normal.

31.7 Let $p\colon X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact for each $y \in Y$. (Such a map is called a perfect map.)

a. Show that if X is Hausdorff, then so is Y.

b. Show that if X is regular, then so is Y.

c. Show that if X is locally compact, then so is Y.

d. Show that if X is second-countable, then so is Y. [*Hint:* Let $\mathscr{B}$ be a countable basis for X. For each finite subset J of $\mathcal{B}$, let U_J be the union of all sets of the form $p^{-1}(W)$, for W open in Y, that are contained in the union of the elements of J.]

Solution a. Let $x \neq y \in Y$. Then $p^{-1}(\{x\})$ and $p^{-1}(\{y\})$ are compact and disjoint. Since X is Hausdorff and $p^{-1}(\{x\})$ is compact, for each $z \in p^{-1}(\{y\})$, we can find disjoint open neighborhoods U_z and V_z such that $p^{-1}(\{x\}) \subseteq U_z$ and $z \in V_z$.

Then $\{V_z \mid z \in p^{-1}(\{y\})\}$ covers $p^{-1}(\{y\})$ compact, so there exist $z_1, \ldots, z_n$ so that $V_{z_1}, \ldots, V_{z_n}$ cover $p^{-1}(\{y\})$. Since open sets are closed under finite intersections, $\bigcap_{i=1}^n U_{z_i}$ is open, and we have

$$p^{-1}(\{x\}) \subseteq \bigcap_{i=1}^n U_{z_i} \coloneqq U \quad \text{and} \quad p^{-1}(\{y\}) \subseteq \bigcup_{i=1}^n V_{z_i} \coloneqq V.$$

The two coverings are also disjoint, since if $z \in U$, then $z \notin V_{z_i}$ for all i, by construction.

By the hint in (31.6), there exist open neighborhoods W and Z of x and y, respectively, such that $p^{-1}(W) \subseteq U$ and $p^{-1}(Z) \subseteq V$ and

$$p^{-1}(W) \cap p^{-1}(Z) \subseteq U \cap V = \emptyset \implies W \cap Z = \emptyset,$$

so Y is Hausdorff.

b. Let $x \in Y$ and $A \subseteq Y$ be closed with $x \notin A$. Then $p^{-1}(\{x\})$ is compact, and $p^{-1}(A)$ is closed in X. Since X is regular, for every $y \in p^{-1}(\{x\})$, there exist U_y and V_y open and disjoint with $y \in U_y$ and $p^{-1}(A) \subseteq V_y$. Then $\{U_y \mid y \in p^{-1}(\{x\})\}$ forms an open cover of $p^{-1}(\{x\})$ compact, so there exist $y_1, \dots, y_n$ such that

$$p^{-1}(\{x\}) \subseteq \bigcup_{i=1}^{n} U_{y_i}.$$

Moreover, $A \subseteq \bigcap_{i=1}^{n} V_{y_i}$ is open and disjoint from $\bigcup_{i=1}^{n} U_{y_i}$ by construction.

Similar to (a), we take an open neighborhood around each point in A whose preimage is contained in $\bigcap_{i=1}^{n} V_{y_i}$ and take their union. We also take one around x whose preimage is contained in $\bigcup_{i=1}^{n} U_{y_i}$, and conclude that this neighborhood and the union are disjoint, so Y is regular.

c. Let $y \in Y$. Then $p^{-1}(\{y\})$ is compact in X, so since X is locally compact, for each $x \in p^{-1}(\{x\})$, there exists a compact subspace C_x in X which contains an open neighborhood U of x. Then $\{U_x \mid x \in p^{-1}(\{y\})\}$ is an open cover of $p^{-1}(\{y\})$ compact, so there exist $x_1, \dots, x_n$ such that

$$p^{-1}(\{y\}) \subseteq \bigcup_{i=1}^{n} U_{x_i} \subseteq \bigcup_{i=1}^{n} C_{x_i}.$$

A finite union of compact sets is still compact. Indeed, given an open cover $\mathcal{U}$ of the union, for each C_{x_i} we can find a finite subcover $\mathcal{U}_i \subseteq \mathcal{U}$ of C_{x_i}. Then the union $\bigcup_{i=1}^{n} \mathcal{U}_i$ is a finite union of finitely many elements, which is finite.

By the hint in (31.6), we can find $W \ni y$ open in Y such that

$$p^{-1}(W) \subseteq \bigcup_{i=1}^{n} U_{x_i} \subseteq \bigcup_{i=1}^{n} C_{x_i} \implies W \subseteq \bigcup_{i=1}^{n} C_{x_i}.$$

Since p is continuous, compactness is preserved under p, so

$$y \in \underbrace{W}_{\text{open}} \subseteq f\left(\bigcup_{i=1}^{n} C_{x_i}\right) = \underbrace{\bigcup_{i=1}^{n} f(C_{x_i})}_{\text{compact}},$$

so Y is locally compact.

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1 Show that a space in which each compact set is closed has the property that each convergent sequence has at most one limit, and the points of a space with the latter property are closed.

Solution Let X be a space such that every compact set is closed.

Let $(x_n)_{n \geq 1}$ be a convergent sequence which converges to some $x \in X$. Then the set $K = \{x_n \mid n \geq 1\} \cup \{x\}$ is compact. Indeed, take an open cover $\{U_i\}_{i \in I}$ of K. Then there exists i_0 such that $x \in U_{i_0}$.

By definition of a convergent sequence, there exists $N \in \mathbb{N}$ such that $x_n \in U_{i_0}$ for all $n \geq N$. Then take $U_{i_1}, \dots, U_{i_n}$ such that $x_j \in U_{i_j}$ for $1 \leq j \leq N$. This gives us the finite subcover $\bigcup_{j=0}^{n} U_{i_j}$, so the set is compact.

By hypothesis, K is closed also, so it contains all of its limit points. Hence, x is the only limit of $(x_n)_{n \geq 1}$.

Let X be a space such that every convergent sequence has at most one limit.

Let $x \in X$. Then the sequence $(x)_{n \geq 1}$ converges to x only, by hypothesis. Hence, $\{x\}$ has no limit points, so it's closed, since $\overline{A} = A \cup A' = A$.

2 Let X be a normal space, and let A be a closed subset and U_1 and U_2 be open subsets of X such that $A \subseteq U_1 \cup U_2$. Show that $A = B_1 \cup B_2$ for some closed subsets B_i of X with $B_i \subseteq U_i$ for $i \in \{1, 2\}$.

Solution We can take $F_1 = A \cap {}^{\mathrm{c}}U_1$ and $F_2 = A \cap {}^{\mathrm{c}}U_2$. Then

$$F_1 \cap F_2 = A \cap {}^{\mathrm{c}}U_1 \cap {}^{\mathrm{c}}U_2 = A \cap {}^{\mathrm{c}}(U_1 \cup U_2) = \emptyset,$$

since $A \subseteq U_1 \cup U_2$. By normality of X, there exist V_1 and V_2 open disjoint subsets of X such that $F_1 \subseteq V_1$ and $F_2 \subseteq V_2$.

We can then take $B_1 = A \cap {}^{\mathrm{c}}V_1$ and $B_2 = A \cap {}^{\mathrm{c}}V_2$. We claim that these are the sets that we want.

$$B_1 \cup B_2 = (A \cap {}^{\mathrm{c}}V_1) \cup (A \cap {}^{\mathrm{c}}V_2) = A \cap {}^{\mathrm{c}}(V_1 \cap V_2) = A \cap {}^{\mathrm{c}}\emptyset = A.$$

Also, for each $i \in \{1, 2\}$,

$$B_i \subseteq {}^{\mathrm{c}}V_i \subseteq {}^{\mathrm{c}}F_i = {}^{\mathrm{c}}A \cup U_i,$$

so since $B_i \subseteq A$, we must have that

$$B_i = B_i \cap A \subseteq {}^{\mathrm{c}}A \cup U_i \cap A = U_i,$$

as desired.

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3 Let X be the space of all bounded sequences in $\mathbb{R}$ with the metric $d(x, y) = \sup\{|x_n - y_n| \mid n \geq 1\}$ for $x = (x_n)_{n\geq 1}$ and $y = (y_n)_{n\geq 1}$ in X. Show that X does not contain a countable dense subset (i.e., X is not "separable").

Solution Suppose otherwise, and that there exists a countable dense subset $Q = \{(q_n^{(1)})_{n\geq 1}, (q_n^{(2)})_{n\geq 1}, \ldots\}$ of X.

Pick $x_1 = q_1^{(1)} + 1$. Then we define

$$x_n = \begin{cases} x_1 & \text{if } |x_1 - q_n^{(n)}| \geq 1 \\ q_n^{(n)} - 1 & \text{if } x_1 \leq q_n^{(n)} < x_1 + 1 \\ q_n^{(n)} + 1 & \text{if } x_1 - 1 < q_n^{(n)} \leq x_1 \end{cases}$$

By construction, we have $|x_n - q_n^{(n)}| \geq 1$. Moreover, in the second case, we have

$$x_1 \leq q_n^{(n)} < x_1 + 1 \implies x_1 - q_n^{(n)} + 1 \leq 1,$$

and a similar inequality for the third case. So, we have $|x_1 - x_n| \leq 1$ for all n, so $|x_n| \leq 1 + |x_1|$ for all $n \geq 1$. Thus, $d((x_n)_{n\geq 1}, (q_n^{(m)})_{n\geq 1}) \geq |x_m - q_m^{(m)}| \geq 1$ for all $m \geq 1$, so $(x_n)_{n\geq 1}$ does not lie in any ball of radius $1/2$ around any $q_n^{(m)}$.

But we assumed Q was dense, which is a contradiction. Hence, X is not separable.