1 For $n \geq 1$ let $f_n\colon [1,2] \to \mathbb{R}$ be defined as follows: for any $x \in [1,2]$,

$$f_1(x) = 0, \quad f_{n+1}(x) = \sqrt{x + f_n(x)}, \quad \forall n \geq 1.$$

Prove that $\{f_n\}_{n\geq 1}$ converges uniformly to $f(x) = \frac{1+\sqrt{1+4x}}{2}$.

Solution Fix $x \in [1,2]$. Notice that 2 is an upper bound for $f_n(x)$. We will show this by induction.

Base step:

$2 - f_1(x) = 2 > 0 \implies 2 \geq f_1(x)$

Inductive step:

Suppose we have $2 \geq f_n(x)$. We wish to show that $2 \geq f_{n+1}(x)$.

$$2 - f_{n+1}(x) = 2 - \sqrt{x + f_n(x)} = \frac{4 - x - f_n(x)}{2 + \sqrt{x + f_n(x)}} \geq \frac{2 - f_n(x)}{2 + \sqrt{x + f_n(x)}} \geq 0 \implies 2 \geq f_{n+1}(x)$$

Hence the inductive step holds.

Next, we will show that $f_n(x)$ is monotonically increasing.

$$\begin{aligned}
f_{n+1}(x) - f_n(x) &= \sqrt{x + f_n(x)} - f_n(x) \\
&= \frac{x + f_n(x) - f_n^2(x)}{\sqrt{x + f_n(x)} + f_n(x)} \\
&\geq \frac{2 + f_n(x) - f_n^2(x)}{\sqrt{x + f_n(x)} + f_n(x)} \\
&= \frac{(2 - f_n(x))(1 + f_n(x))}{\sqrt{x + f_n(x)} + f_n(x)} \geq 0
\end{aligned}$$

Since $f_n(x)$ is monotonically increasing and bounded above by 2, it converges pointwise to $f(x)$. Moreover, since f_n is continuous for all $n \geq 1$ (each f_n where $n \geq 2$ a composition of continuous functions) $[1,2]$ is compact, by Dini's theorem, f_n converges uniformly to f as $n \to \infty$.

As $\lim_{n\to\infty} f_n(x)$ exists for all $x \in [1,2]$, $\sqrt{x + f_n(x)}$ converges to $\sqrt{x + f(x)}$ since it is a composition of continuous functions. Thus,

$$f_{n+1}(x) = \sqrt{x + f_n(x)} \implies f(x) = \sqrt{x + f(x)} \implies f(x) = \frac{1 + \sqrt{1 + 4x}}{2}$$

2 For $n \geq 1$ let

$$f_n \colon [0, \infty) \to \mathbb{R}, \quad f_n(x) = \frac{nx^2 + 1}{nx + 1}.$$

Study the pointwise and uniform convergence of f_n on each of the intervals $[0, \infty)$, $(0, \infty)$, $[1, \infty)$.

Solution $[0, \infty)$:

When $x = 0$, we have $f_n(0) = 1$, so $f_n(0) \xrightarrow{n \to \infty} f(0) = 1$.

For $x > 0$, we have

$$f_n(x) = \frac{x^2 + \frac{1}{n}}{x + \frac{1}{n}} \xrightarrow{n \to \infty} f(x) = \frac{x^2}{x} = x$$

Thus, $f(x)$ is not continuous, so $f_n(x)$ cannot converge uniformly to $f(x)$ on this interval.

$(0, 1)$:

From the above calculation, we have that $f_n(x) \xrightarrow{n \to \infty} f(x) = x$. Moreover,

$$d(f_n, f) = \sup_{x \in [0, \infty)} |f_n(x) - f(x)| = \sup_{x \in [0, \infty)} \left| \frac{1 - x}{nx + 1} \right| \geq 1$$

Thus, f converges pointwise on $x \in (0, \infty)$, but it does not converge uniformly.

$[1, \infty)$:

The calculation from the first case still holds, so we have that $f(x) = x$. Note that

$$(f_n(x) - x)' = -\frac{n + 1}{(nx + 1)^2} \leq 0$$

so $f_n(x) - x$ is decreasing for all x. So, since $f_n(x) - x \leq 0$, $|f_n(x) - x|$ is increasing for all $x \geq 1$. Thus,

$$d(f_n, f) = \sup_{x \in [1, \infty)} |f_n(x) - x| = \lim_{x \to \infty} |f_n(x) - x| = \lim_{x \to \infty} \left| \frac{\frac{1}{x} - 1}{n + \frac{1}{x}} \right| = \frac{1}{n}$$

Then by the Archimedean principle, it is clear that f_n converges uniformly to f on $[1, \infty)$.

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3 Given a metric space (X, d), let $C(X)$ denote the set of bounded and continuous functions $f \colon X \to \mathbb{R}$. For $f, g \in C(X)$, we define

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

a. Prove that $(C(X), d)$ is a metric space.

b. Show that $C(X)$ is complete, connected, but not compact.

Solution a. $C(X)$ is clearly non-empty. One such function in $C(X)$ is $f(x) \equiv 0$. Then we need to show that d is a metric.

$d(f, g) \geq 0$ clearly since it is the supremum of an absolute value.

$d(f, g) = 0 \iff \sup_{x \in X} |f(x) - g(x)| = 0 \iff |f(x) - g(x)| = 0 \iff f(x) = g(x) \; \forall x \in X \iff f = g$

$d(f, g) = \sup_{x \in X} |f(x) - g(x)| = \sup_{x \in X} |g(x) - f(x)| = d(g, f)$

$$\begin{aligned} d(f, g) &= \sup_{x \in X} |f(x) - g(x)| \\ &= \sup_{x \in X} |(f(x) - h(x)) + (h(x) - g(x))| \\ &\leq \sup_{x \in X} |f(x) - h(x)| + \sup_{x \in X} |h(x) - g(x)| = d(f, h) + d(h, g) \end{aligned}$$

b. Let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence of functions in $C(X)$. Fix $x \in X$. Then $\{f_n(x)\}_{n\geq 1}$ is Cauchy in $\mathbb{R}$. As $\mathbb{R}$ is complete, $f_n(x) \xrightarrow{n\to\infty} f(x) \in \mathbb{R}$. This applies for all x, so hence, f_n converges pointwise to f. We now show that f belongs to $C(X)$.

To show that f is continuous, since all the f_n are continuous, it suffices to show that they converge uniformly to f.

Fix $\varepsilon > 0$. Then as $\{f_n\}_{n\geq 1}$ is Cauchy, there exists $N_1 \in \mathbb{N}$ such that for all $n, m \geq N$, we have

$$d(f_n, f_m) = \sup_{x\in X} |f_n(x) - f_m(x)| < \frac{\varepsilon}{100}.$$

for all $x \in X$. Taking $m \to \infty$ yields

$$d(f_n, f) = \sup_{x\in X} |f_n(x) - f(x)| \leq \frac{\varepsilon}{100} < \varepsilon$$

for all $x \in X$. Thus, $f_n \xrightarrow{n\to\infty} f$ uniformly, so f is continuous on X.

Since $f_n \xrightarrow{n\to\infty} f$ uniformly, there exists $N \in \mathbb{N}$ such for all $n \geq N$, we have that

$$d(f_n, f) = \sup_{x\in X} |f_n(x) - f(x)| < 1 \implies 1 - f_n(x) \leq f(x) \leq 1 + f_n(x)$$

for all $x \in X$. But $f_n(x) \in C(X)$, so there exists M such that $|f_n(x)| \leq M$. Thus,

$$1 - M \leq f(x) \leq 1 + M$$

so f is bounded. Hence, $f \in C(X)$, so $(C(X), d)$ is complete.

We wish to show that $C(X)$ is connected. Let $f, g \in C(X)$, and define

$$\gamma\colon [0,1] \to C(X),\ \gamma(t) = (1-t)f + tg$$

Note that $\gamma(0) = f$ and $\gamma(1) = g$. It suffices to show that γ is continuous on $[0,1]$.

Note that for all $t \in [0,1]$, $\gamma(t) \in C(X)$. Indeed, it is continuous because it is a sum and product of continuous functions, and it is bounded because f and g are both bounded. We now wish to prove that γ is continuous.

As g and f are bounded, there exists $M \in \mathbb{R}$ such that $|f(x)| \leq \frac{M}{2}$ and $|g(x)| \leq \frac{M}{2}$ for all $x \in X$.

Fix $\varepsilon > 0$, $t_0 \in [0,1]$, and choose $\delta = \frac{\varepsilon}{2M}$. Then if $|t - t_0| < \delta$, we have

$$\begin{aligned}
d(\gamma(t), \gamma(t_0)) &= \sup_{x\in X} |\gamma(t)(x) - \gamma(t_0)(x)| \\
&= \sup_{x\in X} |(1-t)f(x) + tg(x) - (1-t_0)f(x) - t_0 g(x)| \\
&= \sup_{x\in X} |f(x)(t_0 - t) + g(x)(t - t_0)| \\
&= |t - t_0| \sup_{x\in X} |g(x) - f(x)| \\
&\leq |t - t_0| \left(\sup_{x\in X} |g(x)| + \sup_{x\in X} |f(x)| \right) \\
&\leq \frac{\varepsilon}{2M} \cdot M < \varepsilon
\end{aligned}$$

Thus, $C(X)$ is path connected $\implies$ $C(X)$ is connected.

Lastly, $C(X)$ is not compact. Consider the sequence $\{f_n\}_{n\geq 1}$ with $f_n \equiv n$. The sequence clearly does not admit a convergent subsequence, so $C(X)$ is not sequentially compact $\implies$ $C(X)$ is not compact.

4 Consider the subset of $C([0,1])$ defined as follows:

$$X = \{f\colon [0,1] \to \mathbb{R} \mid f(0) = 0 \text{ and } |f(x) - f(y)| \leq |x - y|\}.$$

Prove that X is compact.

Solution We first show that X is uniformly bounded. Let $f \in X$. Then $|f(x) - f(0)| = |f(x)| \leq |x - 1| \leq 1$ for all $x \in [0,1]$. Thus, X is uniformly bounded.

We now show that X is equicontinuous. Fix $\varepsilon > 0$ and choose $\delta = \frac{\varepsilon}{2}$. Then for all $f \in X$ and all $x, y \in [0,1]$ with $|x - y| < \delta$,

$$|f(x) - f(y)| \leq |x - y| \leq \frac{\varepsilon}{2} < \varepsilon$$

Thus, X is equicontinuous.

Hence, by Arzelà–Ascoli, X is sequentially compact $\implies$ X is compact.

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5 Let X and Y be two metric spaces and let $f\colon X \to Y$ be a function. Assume A and B are open subsets of X such that f is continuous on A and f is continuous on B.

a. Show that f is continuous on $A \cup B$.

b. Is this result still true if A and B were both closed subsets of X?

c. Is the result true for the union of an infinite number of open sets?

d. Is the result true for the union of an infinite number of closed sets?

Solution a. Let $x_0 \in A \cup B$. Assume without loss of generality that $x_0 \in A$.

Fix $\varepsilon > 0$

Then as A is open, there exists δ_1 such that $B^X_{\delta_1}(x_0) \subseteq A$. As f is continuous on A, there exists δ_2 such that for all $x \in B^X_{\delta_2}(x_0) \cap A$, we have $f(x) \in B^Y_\varepsilon(f(x_0))$. Choose $\delta = \min\{\delta_1, \delta_2\}$. The $B^X_\delta(x_0) \subseteq A$, so for all $x \in A \cup B$ such that $x \in B^X_\delta(x_0)$, we have $f(x) \in B^Y_\varepsilon(f(x_0))$.

We can apply the same argument, but with A and B switched. Hence, f is continuous on $A \cup B$.

b. Yes. Let $x_0 \in A \cup B$. Assume without loss of generality that $x \in A$.

Let $\{x_n\}_{n \geq 1} \subseteq A \cup B$ be such that $x_n \xrightarrow{n \to \infty} x_0$. Let $\{a_n\}_{n \geq 1}$, where a_n is the subsequence containing all x_n such that $x_n \in A$. Similarly, let $\{b_n\}_{n \geq 1}$ be the subsequence containing all x_n such that $x_n \in B$.

If $\{b_n\}_{n \geq 1}$ has finitely many terms, then there exists $N \in \mathbb{N}$ such that if $n \geq N$, $x_n \in A$. Then $f(x_n) \xrightarrow{n \to \infty} f(x_0)$, since f is continuous on A.

If $\{a_n\}_{n \geq 1}$ has finitely many terms, then we can apply the same argument as above with a_n, b_n and A, B switched.

If both subsequences have infinitely many terms, then we have that $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = x_0$ by uniqueness of limits, so $x_0 \in A \cap B$. Then as f is continuous on both A and B, we have that

$$\left.\begin{array}{l} a_n \xrightarrow{n \to \infty} x_0 \\ b_n \xrightarrow{n \to \infty} x_0 \end{array}\right\} \implies x_n \xrightarrow{n \to \infty} x_0$$

Hence f is continuous $A \cup B$.

c. Yes. We can apply the same argument: x_0 must belong to one open set, say A, so we can fit an open ball in A. As f is continuous on A, we can make δ small enough so that $B^X_\delta(x_0) \subseteq A$ and $f(B^X_\delta(x_0)) \subseteq B^Y_\varepsilon(f(x_0))$.

d. No. Consider $f\colon \mathbb{R} \to \mathbb{R}$, with $f(0) = 0$ and $f(x) = 1$ otherwise. Then f is continuous on $\{0\}$ and $\left[\frac{1}{n}, \infty\right)$ for all $n \geq 1$.

But $\{0\} \cup \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, \infty\right) = [0, \infty)$, and f is clearly not continuous at $x = 0$.

6 Let $f\colon [0,1] \to \mathbb{R}$ be a function with Darboux's property such that for any $y \in \mathbb{R}$, the set $f^{-1}(\{y\})$ is closed. Prove that f is continuous.

Solution Suppose f is discontinuous at $x_0 \in [0,1]$. Then there exists $\varepsilon_0 > 0$ such that for all $\delta > 0$, there exists x_δ such that $|x_0 - x_\delta| < \delta$ but $|f(x_0) - f(x_\delta)| \geq \varepsilon_0$. Assume without loss of generality that this implies

$$f(x_\delta) \geq \varepsilon_0 + f(x_0)$$

If we parametrize δ as $\frac{1}{n}$, then we get a sequence $\{x_n\}_{n\geq 1}$ with $x_n \xrightarrow{n\to\infty} x_0$ and $f(x_n) \geq \varepsilon_0 + f(x_0)$.

As $f(x_0) < \varepsilon_0 + f(x_0) \leq f(x_n)$, then by the Darboux property, there exists $y_n \in (x_n, x_0)$ (or $y_n \in (x_0, x_n)$ if $x_n > x_0$) such that $f(y_n) = \varepsilon_0 + f(x_0)$. Note that $y_n \xrightarrow{n\to\infty} x_0$ also.

Since $\{y_n\}_{n\geq 1} \subseteq f^{-1}(\{\varepsilon_0 + f(x_0)\})$ closed, we have that $y_n \xrightarrow{n\to\infty} x_0 \in f^{-1}(\{\varepsilon_0 + f(x_0)\})$ by completeness and uniqueness of limits. But this implies that $f(x_0) = \varepsilon_0 + f(x_0) = f(x_0)$, which is a contradiction. Hence, f is continuous.

In the case that $f(x_\delta) \leq -\varepsilon_0 + f(x_0)$, the same argument applies, but with inequalities switched around.

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7 Let $f, g\colon [a,b] \to [a,b]$ be two continuous functions such that $f \circ g = g \circ f$. Show that there exists $x_0 \in [a,b]$ such that $f(x_0) = g(x_0)$.

Solution Assume $f(x) \neq g(x)$ for all $x \in [a,b]$. Assume without loss of generality that $f(x) < g(x)$.

By Brouwer's fixed point theorem, there exists $x_0 \in [0,1]$ such that $f(x_0) = x_0$. Then define

$$\begin{aligned} x_1 &= g(x_0) = g(f(x_0)) = f(g(x_0)) = f(x_1) \\ x_2 &= g(x_1) = g(f(x_1)) = f(g(x_1)) = f(x_2) \end{aligned}$$

Repeating this process inductively yields a sequence $\{x_n\}_{n\geq 1}$ such that

$$\begin{aligned} x_{n+1} &= g(x_n) \\ x_n &= f(x_n) \end{aligned}$$

for all $n \geq 1$. Also, as $g(x) > f(x)$, we have that $x_{n+1} = g(x_n) > f(x_n) = x_n$. So x_n is monotonically increasing. Moreover, as each $x_n \in [a,b]$, x_n is bounded above by a. Thus, the sequence must converge. Call its limit x. Since $[a,b]$ is compact, we have by uniqueness of limits that $x \in [a,b]$. Since f and g are continuous on $[a,b]$, we have

$$\lim_{n\to\infty} g(x_n) = g(x) = f(x) = \lim_{n\to\infty} f(x_n)$$

as desired.

If $g(x) < f(x)$, then the argument would be the same, but x_n would be decreasing instead. However, it would be bounded below by a and we would get the same result.