4.56 Define $\Phi\colon [0,\infty] \to [0,1]$ by $\Phi(t) = t/(t+1)$ for $t \in [0,\infty)$ and $\Phi(\infty) = 1$.

a. Φ is strictly increasing and $\Phi(t+s) \le \Phi(t) + \Phi(s)$.

b. If (Y,ρ) is a metric space, then $\Phi \circ \rho$ is a bounded metric on Y that defines the same topology as ρ.

c. If X is a topological space, the function $\rho(f,g) = \Phi(\sup_{x\in X}|f(x) - g(x)|)$ is a metric on $\mathbb{C}^X$ whose associated topology is the topology of uniform convergence.

d. Let $X = \mathbb{R}^d$ and $U_n = B(0,n)$. Then the function

$$\rho(f,g) = \sum_{n=1}^{\infty} 2^{-n}\Phi\left(\sup_{x\in\overline{U_n}}|f(x) - g(x)|\right)$$

is a metric on $\mathbb{C}^X$ whose associated topology is the uniform topology of uniform convergence on compact sets.

Solution a. Notice that

$$\Phi(t) = 1 - \frac{1}{t+1}.$$

Since $t+1$ is a strictly increasing function, $1/(t+1)$ is strictly decreasing, so $-1/(t+1)$ is strictly increasing. It follows that Φ is strictly increasing on $[0,\infty)$. Lastly, $\Phi(t) < 1 = \Phi(\infty)$, so it's strictly increasing on $[0,\infty]$.

Notice that since $t,s \ge 0$,

$$\frac{1}{t+1} + \frac{1}{s+1} = \frac{t+s+2}{ts+t+s+1} \le \frac{t+s+2}{t+s+1} = 1 + \frac{1}{t+s+1}.$$

Thus, by rearranging the terms,

$$1 - \frac{1}{t+s+1} \le \left(1 - \frac{1}{t+1}\right) + \left(1 - \frac{1}{s+1}\right) \iff \Phi(t+s) \le \Phi(t) + \Phi(s).$$

b. $\Phi \circ \rho$ is bounded by 1, and it's 0 if and only if $\rho(x,y) = 0$ if and only if $x = y$, since ρ is a metric. It's also non-negative and satisfies the triangle inequality, by part (a). Lastly, it's symmetrical since ρ is symmetric, so $\Phi \circ \rho$ is a bounded metric.

Let $B_\rho(x,r)$ be an open ball in the ρ metric. Then $B_{\Phi\circ\rho}(x,r') \subseteq B_\rho(x,r)$, where $r' < r/(1+r)$: If $y \in B_{\Phi\circ\rho}(x,r')$, we have

$$(\Phi\circ\rho)(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)} < \frac{r}{1+r} = \Phi(r) \implies \rho(x,y) < r,$$

since Φ is strictly increasing.

Now let $B_{\Phi\circ\rho}(x,r)$ be an open ball in the new metric. Assume $r < 1$ (if $r \ge 1$, then the ball is the whole space). Let $r' < r/(1-r)$. Then $B_\rho(x,r') \subseteq B_{\Phi\circ\rho}(x,r)$:

$$y \in B_\rho(x,r') \implies (\Phi\circ\rho)(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)} < \frac{r'}{1+r'} < r.$$

So the two metrics define the same topology on Y.

c. We need to prove that $f_n \xrightarrow{n\to\infty} g$ uniformly if and only if f_n converges to g in the given metric.

This is easy to see as $\Phi(\sup_{x\in X}|f(x) - g(x)|)$ gives the same topology as $\sup_{x\in X}|f(x) - g(x)|$ which generates the uniform convergence topology.

d. We need to show that $f_n \to f$ uniformly on compact sets if and only if $f_n \to f$ is the given topology.

" $\implies$ "

Let $f_k \to f$ uniformly on compact sets. In particular, $f_k \to f$ uniformly on $\overline{U_n}$, which is compact. Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ be so that

$$\sum_{n=N+1}^{\infty} 2^{-n} < \frac{\varepsilon}{2}.$$

Since $f_k \to f$ uniformly on each $\overline{U_n}$, there exists k_n so that if $k \geq k_n$, $\sup_{x\in\overline{U_n}}|f_k(x) - f(x)| < \varepsilon/2$. Take k_0 to be the largest among $k_1, \dots, k_N$.

Then if $k \geq k_0$, we have

$$\begin{aligned}
\rho(f_k, f) &= \sum_{n=1}^{N} 2^{-n}\Phi\left(\sup_{x\in\overline{U_n}}|f_k(x) - f(x)|\right) + \sum_{n=N+1}^{\infty} 2^{-n}\Phi\left(\sup_{x\in\overline{U_n}}|f_k(x) - f(x)|\right) \\
&\leq \frac{\varepsilon}{2}\sum_{n=1}^{N} 2^{-n} + \sum_{n=N+1}^{\infty} 2^{-n} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon,
\end{aligned}$$

so $f_k \to f$ in the given metric.

" $\impliedby$ "

Let $\varepsilon > 0$ and let K be a compact set in $\mathbb{R}$. Since compact sets are bounded, there exists $N \geq 1$ so that $K \subseteq U_N$.

Suppose $f_k \to f$ in the topology, i.e., $\rho(f_k, f) \xrightarrow{k\to\infty} 0$. In particular, there exists $k_0 \in \mathbb{N}$ so that

$$\begin{aligned}
k \geq k_0 &\implies \rho(f_k, f) = \rho(f, g) = \sum_{n=1}^{\infty} 2^{-n}\Phi\left(\sup_{x\in\overline{U_n}}|f_k(x) - f(x)|\right) < 2^{-N}\frac{\varepsilon}{1+\varepsilon} \\
&\implies 2^{-N}\Phi\left(\sup_{x\in\overline{U_n}}|f_k(x) - f(x)|\right) < 2^{-N}\Phi(\varepsilon) \\
&\implies \sup_{x\in\overline{U_n}}|f_k(x) - f(x)| < \varepsilon \\
&\implies \sup_{x\in K}|f_k(x) - f(x)| < \varepsilon.
\end{aligned}$$

Hence, $f_k \to f$ uniformly on compact sets.

4.61 Theorem 4.43 remains valid for maps from a compact Hausdorff space X into a complete metric space Y provided the hypothesis of pointwise boundedness is replaced by pointwise total boundedness. (Make this statement precise and then prove it.)

Solution The theorem can be restated as follows:

Theorem. *Let X be a compact Hausdorff space. If $\mathcal{F}$ is an equicontinuous, pointwise totally bounded subset of $C(X, Y)$, then $\mathcal{F}$ is totally bounded in the uniform metric, and the closure of $\mathcal{F}$ in $C(X)$ is compact.*

So, we just need to show that $\mathcal{F}$ is totally bounded.

Let $\varepsilon > 0$.

By equicontinuity, for every $x \in X$, there exists an open neighborhood $U_x \ni x$ so that $\rho(f(x), f(y)) < \varepsilon/4$ whenever $y \in U_x$, for any $f \in \mathcal{F}$. Thus, $X = \bigcup_x U_x$, so by compactness, there exist $x_1, \dots, x_n$ so that $X = \bigcup_i U_{x_i}$.

Since $\{f(x) \mid f \in \mathcal{F}\}$ is totally bounded, there exists $z_1, \dots, z_m \in \{f(x) \mid f \in \mathcal{F}\} := F$ so that $\{B(z_i, \varepsilon/4)\}_{i=1}^m$ cover F.

Then the rest of the proof follows exactly as it does in Folland.

4.63 Let $K \in C([0,1] \times [0,1])$. For $f \in C([0,1])$, let $Tf(x) = \int_0^1 K(x,y)f(y)\,\mathrm{d}y$. Then $Tf \in C([0,1])$, and $\{Tf \mid \|f\|_u \leq 1\}$ is precompact in $C([0,1])$.

Solution By Arzelà-Ascoli, it's enough to prove that the family is equicontinuous and pointwise bounded.

Let $\mathcal{F} = \{Tf \mid \|f\|_u \leq 1\}$.

Equicontinuity:

Let $\varepsilon > 0$.

Since K is continuous in x, there exists $\delta > 0$ so that $|K(x,y) - K(z,y)| < \varepsilon$ whenever $|x - z| < \delta$. Then if $Tf \in \mathcal{F}$, using the fact that $\|f\|_u \leq 1$, we have

$$|Tf(x) - Tf(z)| \leq \int_0^1 |K(x,y) - K(z,y)||f(y)|\,\mathrm{d}y < \int_0^1 \varepsilon\,\mathrm{d}y = \varepsilon$$

whenever $|x - z| < \delta$, so $\mathcal{F}$ is equicontinuous.

Pointwise Boundedness:

Notice that $[0,1]^2$ is compact and $|K|$ is continuous, so it attains its maximum $M < \infty$ on $[0,1]^2$.

Let $Tf \in \mathcal{F}$. Because $\|K\|_u = M$ and $\|f\|_u \leq 1$,

$$|Tf(x)| \leq \int_0^1 |K(x,y)||f(y)|\,\mathrm{d}y \leq \int_0^1 M\mathrm{d}y = M,$$

so $\mathcal{F}$ is pointwise bounded.

Thus, $\mathcal{F}$ is precompact.

4.64 Let (X,ρ) be a metric space. A function $f \in C(X)$ is called **Hölder continuous of exponent** α ($\alpha > 0$) if the quantity

$$N_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^\alpha}$$

is finite. If X is compact, $\{f \in C(X) \mid \|f\|_u \leq 1 \text{ and } N_\alpha(f) \leq 1\}$ is compact in $C(X)$.

Solution Notice that because X is a metric space, it's Hausdorff, so we wish to apply Arzelà-Ascoli here.

Let $\mathcal{F} = \{f \in C(X) \mid \|f\|_u \leq 1 \text{ and } N_\alpha(f) \leq 1\}$.

It's clear that $\mathcal{F}$ is pointwise bounded. We just now need to show equicontinuity.

Let $\varepsilon > 0$, and choose $\delta < \varepsilon^{1/\alpha}$.

Notice that

$$N_\alpha(f) \leq 1 \implies \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^\alpha} \leq 1.$$

In particular, if $0 < \rho(x,y) < \delta$, the upper bound still holds, so

$$\frac{|f(x) - f(y)|}{\rho(x,y)^\alpha} \leq 1 \implies |f(x) - f(y)| \leq \rho(x,y)^\alpha < \delta^\alpha < \varepsilon,$$

for any $f \in \mathcal{F}$, so $\mathcal{F}$ is equicontinuous.

Thus, by Arzelà-Ascoli, $\mathcal{F}$ is precompact.

We now need to show that $\mathcal{F}$ is closed with respect to the uniform norm. Let $f_n \xrightarrow{n \to \infty} f$ uniformly, where $f_n \in \mathcal{F}$. We need to show that $f \in \mathcal{F}$.

It's clear that $f \in C(X)$ and that $\|f\|_u \leq 1$, since the uniform metric is a metric. We just need to show that it's Hölder continuous of exponent α.

Notice that for any $\varepsilon > 0$, there exist $x \neq y \in X$ so that

$$\frac{|f(x) - f(y)|}{\rho(x,y)^\alpha} > N_\alpha(f) - \varepsilon.$$

Since $f_n \xrightarrow{n\to\infty} f$ uniformly, it follows that

$$N_\alpha(f) - \varepsilon < \frac{|f(x) - f(y)|}{\rho(x,y)^\alpha} \leq 1.$$

Since ε was arbitrary, it follows that $N_\alpha(f) \leq 1$, so $f \in \mathcal{F}$. Hence $\mathcal{F}$ is compact.

4.68 Let X and Y be compact Hausdorff spaces. The algebra generated by functions of the form $f(x,y) = g(x)h(y)$, where $g \in C(X)$ and $h \in C(Y)$ is dense in $C(X \times Y)$.

Solution Let

$$\mathcal{A} := \left\{ \sum_{i=1}^{n} g_i(x)h_i(y) \mid n \in \mathbb{N},\ g_i \in C(X),\ h_i \in C(y) \right\}.$$

Now let

$$f(x,y) = \sum_{i=1}^{n} g_i(x)h_i(y) \quad \text{and} \quad \tilde{f}(x,y) = \sum_{j=1}^{m} \tilde{g}_j(x)\tilde{h}_j(y).$$

It's clear that $f + \tilde{f} \in \mathcal{A}$ since their sum is still of the correct form.

As for the product, we have

$$f\tilde{f} = \sum_{i=1}^{n}\sum_{j=1}^{m} g_i(x)h_i(y)\tilde{g}_j(x)\tilde{h}_j(y) = \sum_{i=1}^{n}\sum_{j=1}^{m} \Big[g_i(x)\tilde{g}_j(x)\Big]\Big[h_i(y)\tilde{h}_j(y)\Big],$$

and be reindexing, we can write the product as a single sum. Hence $\mathcal{A}$ is an algebra.

We now need to show that $\mathcal{A}$ separates points and vanishes nowhere.

Let $(x,y),(x_0,y_0) \in X \times Y$. Since X and Y are compact Hausdorff spaces, Urysohn's lemma gives continuous functions $g, h\colon X \to [0,1]$ so that $g(x) = h(y) = 0$ and $g(x_0) = h(y_0) = 1$. Then $g(x)h(y) \in \mathcal{A}$ and separates the two points: $g(x)h(y) = 0 \neq 1 = g(x_0)h(y_0)$.

Lastly, $\mathcal{A}$ vanishes nowhere: take the constant function 1.

Thus, $\mathcal{A}$ is dense in $C(X \times Y)$.

4.69 Let A be a nonempty set, and let $X = [0,1]^A$. The algebra generated by the coordinate maps $\pi_\alpha\colon X \to [0,1]$ ($\alpha \in A$) and the constant function 1 is dense in $C(X)$.

Solution Notice that by Tychonoff's theorem, X is compact, so X is compact Hausdorff.

We now need to show that the algebra $\mathcal{A}$ separates points and vanishes nowhere.

It clearly vanishes nowhere since $1 \in \mathcal{A}$.

Now let $\mathbf{x} := \{x_\alpha\}_{\alpha\in A} \neq \{y_\alpha\}_{\alpha\in A} := \mathbf{y} \in X$. In particular, there must exist some $\alpha \in A$ so that $x_\alpha \neq y_\alpha$. Then $\pi_\alpha(\mathbf{x}) = x_\alpha \neq y_\alpha = \pi_\alpha(\mathbf{y})$, so $\mathcal{A}$ separates points.

Thus, $\mathcal{A}$ is dense in $C(X)$.