1 Let $f\colon [0,1] \to \mathbb{R}$. We say that f is Hölder continuous of order $\alpha \in (0,1)$ and write $f \in C^\alpha([0,1])$ if

$$\|f\|_{C^\alpha} = \sup\{|f(x)| \mid x \in [0,1]\} + \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} \mid x, y \in [0,1] \text{ with } x \neq y\right\} < \infty.$$

Let $d_\alpha\colon C^\alpha([0,1]) \times C^\alpha([0,1]) \to \mathbb{R}$ be given by

$$d_\alpha(f,g) = \|f-g\|_{C^\alpha}.$$

a. Show that $(C^\alpha([0,1]), d_\alpha)$ is a complete metric space.

b. Show that any bounded sequence in $(C^{1/2}([0,1], d_{1/2}))$ admits a subsequence that converges in $(C^{1/3}([0,1]), d_{1/3})$.

Solution a. We'll show that d_α is a metric.

$d_\alpha(f,g) \geq 0$ since it is the sum of two supremums of non-negative functions.

$d_\alpha(g,f) = d_\alpha(f,g)$ since we can switch the order of subtraction in an absolute value.

$d_\alpha(f,g) = 0 \iff \sup\{|f(x)|\} + \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha}\right\} = 0 \iff f(x) \equiv 0.$

d_α also satisfies the triangle inequality since the $|\cdot|$ metric satisfies the triangle inequality, and since $\sup A + B \leq \sup A + \sup B$.

Hence, d_α is a metric.

Let $\{f_n\}_{n\geq 1} \subseteq C^\alpha([0,1])$ be a Cauchy sequence with respect to the d_α metric.

Let $\varepsilon > 0$. Then as $\{f_n\}_{n\geq 1}$ is Cauchy, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\begin{aligned}\|f_n - f_m\|_{C^\alpha} < \frac{\varepsilon}{2} &\iff \sup\{|f_n(x) - f_m(x)|\} + \sup\left\{\frac{|f_n(x) - f_n(y) - (f_m(y) - f_m(y))|}{|x-y|^\alpha}\right\} < \frac{\varepsilon}{2} \\ &\implies \sup\{|f_n(x) - f_m(x)|\} < \frac{\varepsilon}{2} \\ &\implies |f_n(x) - f_m(x)| < \frac{\varepsilon}{2}\end{aligned}$$

Thus, $\{f_n(x)\}_{n\geq 1}$ is Cauchy in $\mathbb{R}$, which is complete, so it converges to some $f(x)$. Thus $f_n \xrightarrow{n\to\infty} f$. Indeed, we can take $m \to \infty$ in the above inequality (for the same N) to get

$$\sup\{|f_n(x) - f(x)|\} + \sup\left\{\frac{|f_n(x) - f_n(y) - (f(y) - f(y))|}{|x-y|^\alpha}\right\} \leq \frac{\varepsilon}{2} < \varepsilon \iff \|f - f_n\|_{C^\alpha} < \varepsilon.$$

We now show that $f \in C^\alpha([0,1])$. For $n \geq N$,

$$\begin{aligned}\big|\|f\|_{C^\alpha} - \|f_n\|_{C^\alpha}\big| &\leq \|f - f_n\|_{C^\alpha} < \varepsilon \\ \implies \|f\|_{C^\alpha} &< \varepsilon + \|f_n\|_{C^\alpha} < \infty\end{aligned}$$

so $f \in C^\alpha([0,1])$. Hence, $(C^\alpha([0,1]), d_\alpha)$ is a complete metric space.

b. Let $\{f_n\}_{n\geq 1}$ be a bounded sequence in $(C^{1/2}([0,1]), d_{1/2})$

If $x, y \in [0,1]$, then $|x-y| \leq 1 \implies |x-y| \leq |x-y|^{1/2} \leq |x-y|^{1/3}$. Thus,

$$\frac{|f(x)-f(y)|}{|x-y|^{1/3}} \leq \frac{|f(x)-f(y)|}{|x-y|^{1/2}} \leq \frac{|f(x)-f(y)|}{|x-y|} \implies \|f\|_{C^{1/3}} \leq \|f\|_{C^{1/2}} \tag{1}$$

So, if $\{f_n\}_{n\geq 1}$ is bounded in $(C^{1/2}([0,1]), d_{1/2})$, it is also bounded in $(C^{1/3}([0,1]), d_{1/3})$.

Note that if $f \in C^\alpha([0,1])$, $f \in C([0,1])$. Indeed, $f \in C^{1/2}([0,1])$ is bounded by some $M > 0$. Then for $\delta > 0$, if $0 < |x-y| < \delta$, we have

$$|f(x) - f(y)| \leq M|x-y|^{1/3} < M\delta^{1/3}$$

which we can make sufficiently small by shrinking $\delta > 0$, so f is continuous on $[0,1]$. Thus, it is sufficient to show that $\{f_n\}_{n\geq 1}$ is equicontinuous and uniformly bounded.

$\{f_n\}_{n\geq 1}$ is equicontinuous by using the above argument, which is independent of what n is. The sequence is also uniformly bounded by assumption. Hence, by Arzelà-Ascoli, $\{f_n\}_{n\geq 1}$ admits a uniformly convergent subsequence $\{f_{k_n}\}_{n\geq 1}$ in the uniform metric.

We'll show that $f_{k_n} \xrightarrow{n\to\infty} f$ in the metric $d_{1/3}$ also by showing that it is Cauchy in that metric.

Let $\varepsilon > 0$. Let $M_n = \sup\left\{\frac{|f_{k_n}(x)-f_{k_n}(y)|}{|x-y|^{1/3}}\right\}$. Then as $\{f_{k_n}(x)\}_{n\geq 1}$ is Cauchy in $\mathbb{R}$, so is $\{M_n\}_{n\geq 1}$. Hence, there exists $N \in \mathbb{N}$ such that for all $N \geq N$, we have both

$$\begin{aligned}|f_{k_n}(x) - f_{k_m}(x)| &< \frac{\varepsilon}{2}\\ |M_n - M_m| &< \frac{\varepsilon}{2}.\end{aligned}$$

Hence,

$$\begin{aligned}\|f_{k_n} - f\|_{C^{1/3}} &\leq \sup\{|f_{k_n}(x) - f_{k_m}(x)|\} + \sup\left\{\frac{|f_{k_n}(x) - f_{k_m}(x) - (f_{k_n}(y) - f_{k_m}(y))|}{|x-y|^{1/3}}\right\}\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\\ &= \varepsilon,\end{aligned}$$

so f_{k_n} converges. Since $(C^{1/3}([0,1]), d_{1/3})$ is complete, f_{k_n} converges in $C^{1/3}([0,1])$, by uniqueness of limits, as desired.

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2 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, that is,

$$\sup\left\{\frac{|f(x)-f(y)|}{|x-y|} \mid x,y \in \mathbb{R} \text{ with } x \neq y\right\} < \infty.$$

Suppose that for every $x \in \mathbb{R}$,

$$\lim_{n\to\infty} n\left[f\left(x + \frac{1}{n}\right) - f(x)\right] = \lim_{n\to\infty} n\left[f\left(x - \frac{1}{n}\right) - f(x)\right] = 0.$$

Prove that f is differentiable on $\mathbb{R}$.

Solution Fix $x \in \mathbb{R}$.

Notice that we can rewrite the limits as

$$\lim_{n\to\infty} \frac{f\left(x+\frac{1}{n}\right) - f(x)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{f\left(x-\frac{1}{n}\right) - f(x)}{\frac{1}{n}} = 0.$$

We claim that that $f'(x) \equiv 0$.

Without loss of generality, consider the interval (x,∞).

On that interval, the function $g(y) := \frac{f(y)-f(x)}{y-x}$ is continuous.

Consider

$$v_n = \sup\left\{\frac{|f(x)-f(y)|}{|x-y|} \mid y \in \left(x, x+\frac{1}{n}\right)\right\}.$$

$\{v_n\}_{n\geq 1}$ is bounded below by 0 and is monotonically decreasing since v_{n+1} is the supremum of a subset of the interval associated with v_n. Hence, it converges to some $L \geq 0$. Thus, there exists $\{x_n\}_{n\geq 1} \subseteq (x,\infty)$ with $x_n \xrightarrow{n\to\infty} x$ and $g(x_n) \xrightarrow{n\to\infty} L$.

Suppose $L > 0$. Then there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$|g(x_n) - L| < \frac{L}{4}$$

As the above limits exist, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have

$$\left| g\left(x + \frac{1}{n} \right) - L \right| > \frac{L}{2}$$

Let $N = \max\{N_1, N_2\}$ and consider the interval $\left[x + \frac{1}{N}, x_N \right]$ (or the other way around). On this interval, g is uniformly continuous. Then

$$\begin{aligned}
\left| g(x_N) - g\left(x + \frac{1}{N} \right) \right| &= \left| g(x_N) - L - \left(g\left(x + \frac{1}{N} \right) - L \right) \right| \\
&\geq \left| |g(x_N) - L| - \left| \left(g\left(x + \frac{1}{N} \right) - L \right) \right| \right| \\
&> \frac{L}{2}
\end{aligned}$$

which is a contradiction, because no $\delta > 0$ can make the difference between these two values less than $\frac{L}{2}$.

Hence, we must have $L = 0$. We can repeat the same argument on the interval $(-\infty, x)$ to get that

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = 0$$

for all x.

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3 Let $f \colon \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that

$$f(x) \geq 0 \quad \text{and} \quad f''(x) \leq 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

Show that f is constant.

Solution Suppose that f is not constant.

Assume without loss of generality that $f'(y) < 0$ for some $y \in \mathbb{R}$. Then consider $L(x) = f'(y)(x - y) + f(y)$. Then consider $g \colon [y, \infty) \to \mathbb{R}$, $g(x) = L(x) - f(x)$. Notice that $g(y) = 0$.

Note that since $f''(x) \leq 0$, we have that $f'(x) \leq f'(y) < 0$ for all $x \geq y$. Then

$$g'(x) = L'(x) - f'(x) = f'(y) - f'(x) \geq 0$$

for all $x \in [y, \infty)$, so g is monotonically increasing on the same interval, meaning $g(x) \geq 0$ for all $x \in [y, \infty)$.

Since $f'(y) < 0$, there exists $z \geq y$ such that $L(z) < 0$. But since $f(x) \geq 0$, this implies that

$$0 \leq g(z) = L(z) - f(x) < 0$$

which is a contradiction. Hence, there is no $y \in \mathbb{R}$ such that $f'(y) < 0$.

We can repeat the same argument for $f'(y) > 0$ by replace $f(x)$ with $h \colon \mathbb{R} \to \mathbb{R}$, $h(x) \coloneqq f(-x)$ and y with $-y$ to show that no such y exists. Indeed, for all $x \in \mathbb{R}$,

$$\begin{aligned}
h(x) &= f(-x) \geq 0 \\
h''(x) &= f''(-x) \leq 0 \\
h'(x) &= -f'(-x) \implies h'(-y) = -f'(y) < 0,
\end{aligned}$$

so h satisfies our hypotheses for the above argument.

Hence, $0 \leq f'(x) \leq 0$ for all $x \in \mathbb{R}$, so $f'(x) \equiv 0 \implies f$ is constant.

4 Assume $f\colon (a,b) \to \mathbb{R}$ be a twice differentiable function. Show that for any $x \in (a,b)$, the limit

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$$

exists and equals $f''(x)$.

Solution Let $x \in (a,b)$. As (a,b) is open, there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq (a,b)$. On this interval, for $|h| \leq \delta$, $f(x+h)$ and $f(x-h)$ are both functions on $(x-\delta, x+\delta)$ and are both differentiable on that interval. By the chain rule, their derivatives with respect to h are $f'(x+h)$ and $-f'(x-h)$, respectively.

Also, both the numerator and denominator approach 0, and both are differentiable twice, so by applying L'Hôpital's rule twice, we get

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} &= \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h} \\
&= \lim_{h\to 0} \frac{f''(x+h)+f''(x-h)}{2} \\
&= f''(x) \quad \text{(by the last problem on our midterm)}
\end{aligned}$$

as desired.

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5 We say that a function $f\colon [a,b] \to \mathbb{R}$ is a convex function if $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$ for all $x, y \in [a,b]$ and for all $t \in [0,1]$. Show that for any $x \in (a,b)$, the limits

$$\lim_{y \searrow x} \frac{f(y)-f(x)}{y-x} \quad \text{and} \quad \lim_{y \nearrow x} \frac{f(y)-f(x)}{y-x}$$

exist and are finite.

Hint: Show that for all $a \leq x < y < z \leq b$ we have

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x} \leq \frac{f(z)-f(y)}{z-y}.$$

Solution Let $a \leq x < y < z \leq b$. We will show

$$\frac{f(z)-f(x)}{z-x} \leq \frac{f(z)-f(y)}{z-y}.$$

Let $L_{xz}(t) = tf(z)+(1-t)f(x)$ and $L_{yz} = tf(z)+(1-t)f(y)$.

Since f is convex,

$$\begin{aligned}
f(tz+(1-t)y) &\leq L_{yz}(t) \\
f(tz+(1-t)x) &\leq L_{xz}(t)
\end{aligned}$$

Note that since $x < y < z$, there exists $t_y \in [0,1]$ such that $L_{xz}(t_y) = y$. Then from the second inequality, we have

$$L_{yz}(0) = f(y) \leq L_{xz}(t_y)$$

Also note that $L_{yz}(1) = L_{xz}(1)$. Thus

$$\frac{f(z)-f(x)}{z-x} = \frac{L_{xz}(1)-L_{xz}(t_y)}{1-t_y} \leq \frac{L_{yz}(1)-L_{yz}(0)}{1-0} = \frac{f(z)-f(y)}{z-y}$$

as desired.

We can use the same argument to prove the other inequality, so we now have

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x} \leq \frac{f(z)-f(y)}{z-y}.$$

Look at the left inequality, we see that for fixed x,

$$\frac{f(y)-f(x)}{y-x}$$

is monotonically increasing as y increases. Moreover, it is bounded above, so

$$\lim_{y \nearrow x} \frac{f(y)-f(x)}{y-x}$$

exists and is finite.

We can use the same argument, but with the right inequality to find that for a fixed z,

$$\frac{f(z)-f(x)}{z-x}$$

is monotonically decreasing as x decreases. Thus,

$$\lim_{z \searrow x} \frac{f(z)-f(x)}{z-x}$$

exists and is finite also.