1 Show that a subset U of a topological space X is open if and only if for every $x \in U$, there exists an open neighborhood V of x that is contained in U.

Solution "$\implies$"

Let U be open in X. For every $x \in U$, U is an open neighborhood of x contained in U.

"$\impliedby$"

Let U be a subset such that for all $x \in U$, there exists an open neighborhood V_x of x contained in U. Then

$$U = \bigcup_{x \in U} V_x.$$

Since each V_x is open and topologies are closed under arbitrary unions, U is open.

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2 Let $A \subseteq B$ be subsets of a topological space X. Show that the subspace topology on A from X is the same as the subspace topology on A from B, where B has the subspace topology from X.

Solution Let U be an open set from the subspace topology on A from X. Then there exists an open set $D \subseteq X$ such that $U = D \cap A$. Since $A \subseteq B$, $A = A \cap B$, so $U = D \cap A \cap B = (D \cap B) \cap A$. Hence, since topologies are closed under finite unions, U is also in the subspace topology on A from B, where B has the subspace topology from X.

Let U be an open set from the subspace topology on A from B. Then there exists an open set $D \subseteq B$ such that $U = D \cap A$. Since D is an open set in B with the subspace topology from X, there exists an open set $E \subseteq X$ such that $D = E \cap B$. Hence, $U = E \cap B \cap A = E(\cap B \cap A) = E \cap A$. Hence U is also in the subspace topology from A on X.

Thus, a set U is in the subspace topology on A from X if and only if it is in the subspace topology on A from B, so they are the same topology.

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3 Let X be a nonempty set, and choose $b \in X$. Set $A = X - \{b\}$, and let $\mathcal{T}_A$ be a topology on A. Find the finest topology on X for which $\mathcal{T}_A$ is the subspace topology on A from X. On the other hand, show that there is not, in general, a topology on X for which $\mathcal{T}_A$ is the subspace topology on A and which is coarser than all the other topologies on X with this property.

Solution Choose $\mathcal{B} = \mathcal{T}_A \cup \{b\}$. This forms a basis for a topology $\mathcal{T}_X$ on X.

$X - \{b\} = A \in \mathcal{B}$, so $A \cup \{b\} = X$. Thus, $\bigcup_{U \in \mathcal{B}} = X$.

Let $U, V \in \mathcal{B}$ such that $U \cap V \neq \emptyset$. Then we must have either $U, V \in \mathcal{T}_A$ or $U = V = \{b\}$. Otherwise, $U \cap V = \emptyset$, since $\{b\} \notin \mathcal{B}$.

If $U, V \in \mathcal{T}_A$, then for all $x \in U \cap V$, $U \cap V$ is an open neighborhood of x contained in $U \cap V$.

If $U = V = \{b\}$, then we must have that $x = b$, so $\{b\}$ is an open neighborhood of x.

Hence, $\mathcal{B}$ is a basis for a topology on X. Moreover, the subspace topology on A from X is equal to $\mathcal{T}_A$, since if $U \in \mathcal{T}_X$, then $U \cap A = U$ or $U \cap A = U - \{b\} \in \mathcal{T}_A$, and we can get all elements of $\mathcal{T}_A$ also since $\mathcal{T}_A \subseteq \mathcal{T}_X$.

This is also the finest topology on X satisfying the properties. Indeed, if there were a finer topology $\mathcal{T}'$ on X, then there exists $U \in \mathcal{T}'$ such that $U \notin \mathcal{T}_X$. Then $U - \{b\} \notin \mathcal{T}_X$ either because otherwise, $U - \{b\} \in \mathcal{T}_A \implies U = (U - \{b\}) \cup \{b\} \in \mathcal{T}_X$. But $U \cap A = U - \{b\} \notin \mathcal{T}_A$, which is a contradiction. Hence, the topology generated by $\mathcal{B}$ is the finest topology satisfying the conditions of the problem.

Consider the trivial topology on A, $\mathcal{T}_A = \{\emptyset, A\}$ and the trivial topology on X, $\mathcal{T}_X = \{\emptyset, X\}$. Then $\emptyset \cap A = \emptyset$ and $X \cap A = A$, so the subspace topology of A from X is $\mathcal{T}_A$. But $\mathcal{T}_X$ is clearly the most coarse topology on X.

4 Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on a set X. Show that $\mathcal{T}$ is the intersection of all topologies that contain $\mathcal{B}$.

Solution Let B be the intersection of all topologies containing $\mathcal{B}$.

$B \subseteq \mathcal{T}$

This is clearly true since $\mathcal{T}$ is a topology containing $\mathcal{B}$.

$\mathcal{T} \subseteq B$

Let $U \in \mathcal{T}$. Then there exists $\{U_i\}_{i \in I} \subseteq \mathcal{B}$ such that $U = \bigcup_{i \in I} U_i$. Hence, U must be in every topology containing $\mathcal{B}$, since topologies are closed under arbitrary unions.

Thus, by double inclusion, $\mathcal{T} = B$.

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5 The lower limit topology on $\mathbb{R}$ has basis the intervals $[a, b)$ with $a < b$, and in the textbook it is shown that it is strictly finer than the Euclidean topology.

a. Show that the lower limit topology is strictly coarser than the discrete topology.

b. Find the closure of (a, b) for $a < b$ in the lower limit topology.

Solution a. Note that the discrete topology contains the lower limit topology since the discrete topology contains every subset of $\mathbb{R}$.

The lower limit topology does not contain singletons, which are open sets in the discrete topology. Any union of the intervals $[a, b)$ contains an open interval, and finite intersections also contain an open interval or are empty.

Hence, the lower limit topology is strictly coarser than the discrete topology.

b. The closed sets under the lower limit topology are of the form $(c, d]$, which we can get through intersections, where $c < d$. Hence, the closure of (a, b) must be $(a, b]$. Any smaller closed set does not contain (a, b).

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6 Let A and B be subsets of a topological space. Prove or give a counterexample to each of the following statements regarding closures:

a. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

b. $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

Solution a. Note that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Indeed, if $x \in \overline{A}$, then for any open neighborhood U of x,

$$\emptyset \neq U \cap A \subseteq U \cap (A \cup B).$$

The same argument holds for if $x \in \overline{B}$. Thus, the claim holds.

Let $x \in \overline{A \cup B}$. Suppose that $x \notin \overline{A} \cup \overline{B}$. Then

$$x \in (\overline{A} \cup \overline{B})^{\mathrm{c}} = (\overline{A})^{\mathrm{c}} \cap (\overline{B})^{\mathrm{c}}$$

which is open, since $\overline{A}$ and $\overline{B}$ are closed, and open sets are closed under finite intersections. Hence, there exists an open neighborhood U of x such that $U \cap A = U \cap B = \emptyset$. But this implies that

$$U \cap (A \cup B) = (U \cap A) \cup (U \cap B) = \emptyset \iff x \notin \overline{A \cup B}$$

which is a contradiction. Hence, the two sets are the same.

b. A counterexample is $A = (0, 1)$ and $B = (1, 2)$ under the Euclidean topology in $\mathbb{R}$.

$$\overline{A \cap B} = \emptyset \neq \{1\} = [0, 1] \cap [1, 2] = \overline{A} \cap \overline{B}$$

7 Consider $\mathbb{R}$ with the Euclidean topology.

a. Show that the subspace topology on $\mathbb{Z}$ from $\mathbb{R}$ is the discrete topology.

b. Show that $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e., that $\mathbb{R}$ is the closure of $\mathbb{Q}$ in $\mathbb{R}$.

Solution a. The subspace topology will be a subset of the discrete topology, so it suffices to show that the discrete topology is a subset of the subspace topology on $\mathbb{Z}$ from $\mathbb{R}$.

Let U be an element of the discrete topology. Then we can write $U = \{n_1, \ldots, n_k\}$, with $0 \leq k \leq \infty$, where $k = 0$ corresponds to the empty set, and $k = \infty$ corresponds to a set with infinitely many elements.

If $k = 0$, then $U = \emptyset$, which is clearly part of the subspace topology.

Assume from now on that $k > 0$. Then for each $n_i \in U$, we can cover $\{n_i\}$ with the open set $V_i = \left(n_i - \frac{1}{2}, n_i + \frac{1}{2}\right)$. Note that this interval contains no other integer, since the distance between any two different integers is at least 1. Then

$$U = \bigcup_{i=1}^{k} \{n_i\} = \bigcup_{i=1}^{k} (V_i \cap \mathbb{Z}) = \left(\bigcup_{i=1}^{k} V_i\right) \cap \mathbb{Z}.$$

Since topologies are closed under arbitrary unions, $\bigcup_{i=1}^{k} V_i$ is open, so U must be in the subspace topology. Hence, the two topologies are the same.

b. It suffices to show that for any open set U in $\mathbb{R}$ that $U \cap \mathbb{Q} \neq \emptyset$. Indeed, then we would have that for all $x \in \mathbb{R}$, any open neighborhood of x intersects $\mathbb{Q}$, which implies that $x \in \overline{\mathbb{Q}}$.

Since the open intervals are a basis for the Euclidean topology on $\mathbb{R}$, we can show that $(a, b) \cap \mathbb{Q}$ is non-empty for all $a < b$.

By the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < |b - a|$. By the same property, there exists $m \in \mathbb{Z}$ such that $m < na < m + 1 \implies \frac{m}{n} < a < \frac{m}{n} + \frac{1}{n}$. Hence,

$$a < \frac{m}{n} + \frac{1}{n} < a + \frac{1}{n} < b$$

and $\frac{m}{n} + \frac{1}{n} \in \mathbb{Q}$ since $m, n \in \mathbb{Z}$ and the rational numbers are a field under regular addition and multiplication.

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8 Show that a subset A of a topological space X is both open and closed if and only if it has empty boundary.

Solution "$\implies$"

Let A be open and closed in X. Then $A = \overline{A} = \mathring{A} \implies \partial A = \overline{A} - \mathring{A} = \emptyset$.

"$\impliedby$"

Let A be a subset of X with empty boundary. Then $\overline{A} - \mathring{A} = \emptyset \implies \overline{A} \subseteq \mathring{A}$.

Note that $\mathring{A} \subseteq A$ since $\mathring{A}$ is a union of subsets of A. Also, $A \subseteq \overline{A}$ since $\overline{A}$ is the intersection of all closed sets containing A. Hence,

$$\mathring{A} \subseteq A \subseteq \overline{A} \subseteq \mathring{A} \implies \mathring{A} = A = \overline{A},$$

so A is open and closed in X.