5.3.1 Find the poles and residues of the following functions:

a. $\dfrac{1}{z^2+5z+6}$

b. $\dfrac{1}{(z^2-1)^2}$

c. $\dfrac{1}{\sin z}$

d. $\cot z$

e. $\dfrac{1}{\sin^2 z}$

f. $\dfrac{1}{z^m(1-z)^n}$, (m, n positive integers)

Solution a. Notice that $z^2+5z+6=(z+2)(z+3)$. Thus, by partial fractions

$$\frac{1}{(z+2)(z+3)} = \frac{1}{z+2} - \frac{1}{z+3}.$$

It has poles at -2 and -3.

By definition, the residue at -2 is 1 and the residue at -3 is -1, since if we subtract off a fraction removes the associated singularity.

b. Using partial fractions again,

$$\frac{1}{(z^2-1)^2} = \frac{1}{4}\left(\frac{1}{z+1} + \frac{1}{(z+1)^2} - \frac{1}{z-1} - \frac{1}{(z-1)^2}\right).$$

The poles are -1 and 1.

So, the residue at -1 is $1/4$ and the residue at 1 is $-1/4$.

c. Notice that

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}.$$

Thus, the function has a pole whenever $e^{iz}=e^{-iz} \implies 2iz = 2\pi in \implies z=\pi n$, for some $n \in \mathbb{Z}$.

We'll show that the poles are simple. Notice that if n is integer, then $e^{n\pi}=(-1)^n$. Then

$$\lim_{z\to\pi n}\frac{2i(z-\pi n)}{e^{iz}-e^{-iz}} = \lim_{z\to 0}\frac{2iz}{e^{i(z+\pi n)}-e^{-i(z+\pi n)}} = \lim_{z\to 0}(-1)^n\frac{2iz}{e^{iz}-e^{-iz}} = (-1)^n\frac{1}{(e^z)'}\bigg|_{z=0} = (-1)^n,$$

so the poles are simple.

Moreover, if we write the function as a Laurent series, it is easy to see that the residue is simply the coefficient of $1/(z-z_0)$, which is a_{-1}, if we use uniform convergence, Cauchy's integral formula, and Goursat's theorem. Thus, if f has an isolated singularity at z_0,

$$\lim_{z\to z_0}(z-z_0)f(z) = a_{-1}.$$

Thus, if we apply this to our calculation, we see that $\operatorname{Res}(1/\sin z, \pi n) = (-1)^n$.

d. Notice that the poles of this function are exactly the same as the poles of $1/\sin z$, since there are no z such that $\cos z = \sin z = 0$.

By the same argument as the above,

$$\operatorname{Res}(\cot z, \pi n) = \lim_{z\to\pi n}\frac{z-\pi n}{\sin z}\cos z = (-1)^n\cos \pi n = (-1)^n(-1)^n = 1.$$

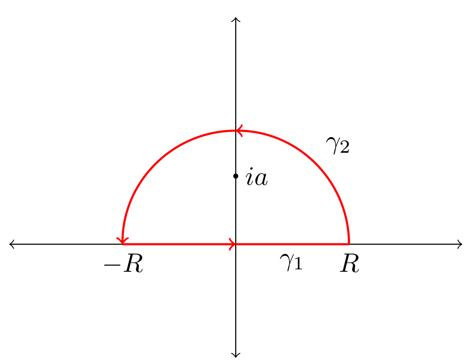
So the residue at every pole πn is 1.

e. Like before, the poles are at πn, for $n \in \mathbb{N}$.
Notice
$$\frac{\mathrm{d}}{\mathrm{d}z} - \cot z = \frac{1}{\sin^2 z}.$$
Thus,
$$\operatorname{Res}\left(\frac{1}{\sin^2 z}, \pi n\right) = \frac{1}{2\pi i}\int_C \frac{1}{\sin^2 z}\,\mathrm{d}z = \frac{1}{2\pi i}\int_C (-\cot z)'\,\mathrm{d}z = 0,$$
for every pole.

f. Notice that if we write f in its Laurent series, if z_0 is a pole of order n, we can find the residue via
$$\lim_{z\to z_0} \frac{1}{(n-1)!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}(z-z_0)^n f(z).$$
Indeed, the residue is the coefficient a_{-1}, and after multiplying by $(z-z_0)^n$, f becomes analytic in a neighborhood of z_0, and a_{-1} is the coefficient on $(z-z_0)^{n-1}$.
Thus,
$$\operatorname{Res}\left(\frac{1}{z^m(1-z)^n}, 0\right) = \lim_{z\to 0}\frac{1}{(n-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\frac{z^m}{z^m(1-z)^n} = (-1)^{m-1}\frac{n(n-1)\cdots(n-m)}{(m-1)!}.$$
Similarly,
$$\operatorname{Res}\left(\frac{1}{z^m(1-z)^n}, 1\right) = (-1)^{n-1}\frac{m(m-1)\cdots(m-n)}{(n-1)!}$$

5.3.3d
$$\int_0^\infty \frac{x^2}{(x^2+a^2)^3}\,\mathrm{d}x,\ a \text{ real}$$

Solution Notice that the integrand is even, so the following integral gives us the same result:
$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)^3}\,\mathrm{d}x.$$
Consider the curve γ, which is a semicircle of radius $R > a$ centered at the origin in the upper-half plane.



γ_1 is the horizontal segment and γ_2 is the curve.
Over this contour, we have
$$\int_{\gamma_1+\gamma_2}\frac{z^2}{(z^2+a^2)^3}\,\mathrm{d}z = 2\pi i \operatorname{Res}\left(\frac{z^2}{(z^2+a^2)^3}, ai\right).$$

By Cauchy's integral formula, we see

$$
\begin{aligned}
\frac{1}{2}\int_{\gamma_1+\gamma_2} \frac{z^2}{(z^2+a^2)^3}\,\mathrm{d}z &= \frac{1}{2}\int_{\gamma_1+\gamma_2} \frac{1}{(z-ai)^3}\left(\frac{z^2}{(z+ai)^3}\right)\mathrm{d}z \\
&= \frac{2\pi i}{2}\left(\frac{z^2}{(z+ai)^3}\right)''\Bigg|_{z=ai} \\
&= \pi i\left(\frac{12(ai)^2}{(ai+ai)^5} - \frac{12ai}{(ai+ai)^4} + \frac{2}{(ai+ai)^3}\right) \\
&= \frac{\pi}{16a^3}.
\end{aligned}
$$

Notice that

$$\left|\frac{1}{2}\int_{\gamma_2} \frac{z^2}{(z^2+a^2)^3}\,\mathrm{d}z\right| \le \frac{\pi R^3}{2(R^2-a^2)^3} \xrightarrow{R\to\infty} 0.$$

Indeed, we can write the denominator as $z^2-(ai)^2$, and the closest point on γ_2 to ai is Ri.

So,

$$\frac{1}{2}\int_{\gamma_1} \frac{x^2}{(x^2+a^2)^3}\,\mathrm{d}x + \frac{1}{2}\int_{\gamma_2} \frac{z^2}{(z^2+a^2)^3}\,\mathrm{d}z = \frac{\pi}{16a^3} \xrightarrow{R\to\infty} \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3}\,\mathrm{d}x = \frac{\pi}{16a^3}.$$

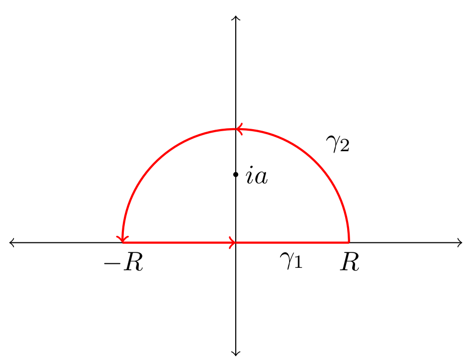
5.3.3e

$$\int_0^\infty \frac{\cos x}{x^2+a^2}\,\mathrm{d}x,\ a \text{ real}$$

Solution Consider the integral

$$\int_{\gamma_1+\gamma_2} \frac{e^{iz}}{z^2+a^2}\,\mathrm{d}z,$$

where γ_1 and γ_2 are the same contours as the previous problem.



By the residue theorem and Cauchy's integral formula, since e^{iz} is entire and ai is a simple pole,

$$\int_{\gamma_1+\gamma_2} \frac{e^{iz}}{z^2+a^2}\,\mathrm{d}z = 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{z^2+a^2}, ai\right) = 2\pi i \cdot \frac{e^{-a}}{2ai} = \frac{\pi e^{-a}}{a}.$$

Then

$$
\begin{aligned}
\int_{\gamma_1} \frac{e^{iz}}{z^2+a^2} = \int_{\gamma_1} \frac{e^{iz}}{z^2+a^2}\,\mathrm{d}z &\xrightarrow{R\to\infty} \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,\mathrm{d}x \\
\left|\int_{\gamma_2} \frac{e^{iz}}{z^2+a^2}\,\mathrm{d}z\right| \le \frac{\pi R}{R^2-a^2} \sup_{x+iy\in\gamma_2} |e^{ix}e^{-y}| &\xrightarrow{R\to\infty} 0,
\end{aligned}
$$

since y is positive, which means e^{-y} is bounded.

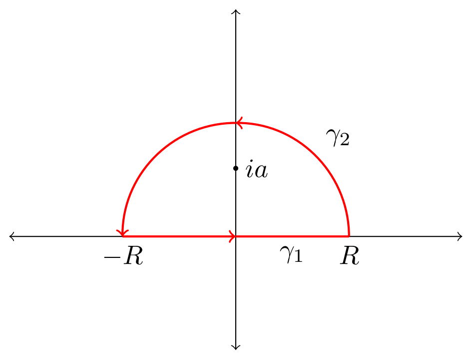
Thus, since the original integrand was even,

$$\operatorname{Re}\int_{\gamma_1+\gamma_2}\frac{e^{iz}}{z^2+a^2}\,\mathrm{d}z = 2\int_0^\infty \frac{\cos x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi e^{-a}}{a} \implies \int_0^\infty \frac{\cos x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi e^{-a}}{2a}.$$

5.3.3f

$$\int_0^\infty \frac{x\sin x}{x^2+a^2}\,\mathrm{d}x,\ a \text{ real}$$

Solution We can use the same contour as the previous problems.



By the residue theorem,

$$\int_{\gamma_1+\gamma_2}\frac{ze^{iz}}{z^2+a^2}\,\mathrm{d}z = 2\pi i\,\operatorname{Res}\left(\frac{ze^{iz}}{z^2+a^2}, ai\right).$$

The residue here is given by

$$\frac{aie^{-a}}{2ai} = \frac{e^{-a}}{2},$$

by a simple application of Cauchy's integral formula. Thus,

$$\int_{\gamma_1+\gamma_2}\frac{ze^{iz}}{z^2+a^2}\,\mathrm{d}z = 2\pi i\cdot\frac{e^{-a}}{2} = i\pi e^{-a}.$$

We also have

$$\int_{\gamma_1}\frac{ze^{iz}}{z^2+a^2}\,\mathrm{d}z = \int_{\gamma_1}\frac{ze^{iz}}{z^2+a^2}\,\mathrm{d}z \xrightarrow{R\to\infty} \int_{-\infty}^\infty \frac{x\sin x}{x^2+a^2}$$
$$\left|\int_{\gamma_2}\frac{ze^{iz}}{z^2+a^2}\,\mathrm{d}z\right| \le \frac{\pi R^2}{R^2-a^2}\sup_{x+iy\in\gamma_2}|e^{ix}e^{-y}| \xrightarrow{R\to\infty} 0,$$

by the same argument as the previous problem.

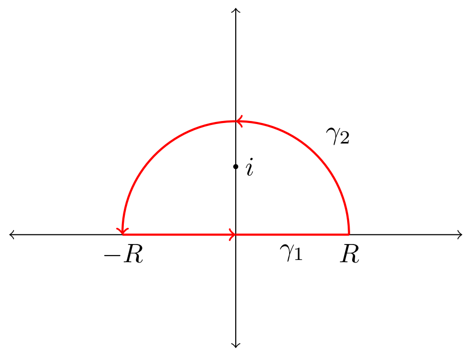
Thus, if we send $R\to\infty$ and use the fact that our original integrand is even and the imaginary part of the integrand we used,

$$\int_0^\infty \frac{x\sin x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi e^{-a}}{2}.$$

5.3.3g

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,\mathrm{d}x$$

Solution We'll use the same contour as before.



Consider

$$\int_\gamma \frac{z^{1/3}}{1+z^2}\,\mathrm{d}z.$$

While $z^{1/3}$ has a branch singularity at 0, we can simply take the contour to have a small arc around the origin and shrink it to get the same result.

The residue of the function at i is

$$\frac{i^{1/3}}{2i} = \frac{e^{i\pi/6}}{2i},$$

so

$$\int_\gamma \frac{z^{1/3}}{1+z^2}\,\mathrm{d}z = 2\pi i\frac{e^{i\pi/6}}{2i} = \pi e^{i\pi/6}$$

Notice that

$$\left|\int_{\gamma_2} \frac{z^{1/3}}{1+z^2}\,\mathrm{d}z\right| \le \frac{\pi R^{4/3}}{R^2-1} \xrightarrow{R\to\infty} 0.$$

Also notice that

$$\begin{aligned}
\int_{-\infty}^\infty \frac{z^{1/3}}{1+z^2}\,\mathrm{d}z &= \int_{-\infty}^0 \frac{z^{1/3}}{1+z^2}\,\mathrm{d}z + \int_0^\infty \frac{z^{1/3}}{1+z^2}\,\mathrm{d}z \\
&= \int_0^\infty \frac{(-z)^{1/3}}{1+(-z)^2}\,\mathrm{d}z + \int_0^\infty \frac{z^{1/3}}{1+z^2}\,\mathrm{d}z \\
&= (1+e^{i\pi/3})\int_0^\infty \frac{z^{1/3}}{1+z^2}\,\mathrm{d}z,
\end{aligned}$$

since $\arg z \in [0,\pi)$. Thus, taking $R\to\infty$,

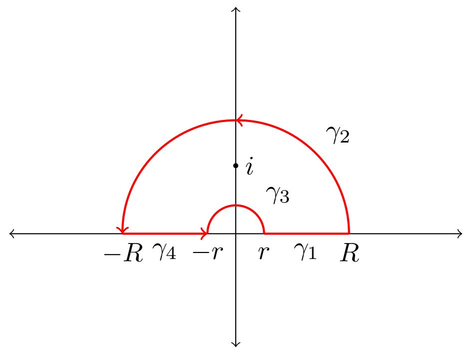
$$\int_\gamma \frac{z^{1/3}}{1+z^2}\,\mathrm{d}z = (1+e^{i\pi/3})\int_0^\infty \frac{z^{1/3}}{1+z^2}\,\mathrm{d}z = \pi e^{i\pi/6} \implies \int_0^\infty \frac{x^{1/3}}{1+x^2}\,\mathrm{d}x = \frac{\pi e^{i\pi/6}}{1+e^{i\pi/3}} = \frac{\pi}{\sqrt{3}}.$$

The last equality is gotten by expanding the exponentials and simplifying.

5.3.3h

$$\int_0^\infty (1+x^2)^{-1} \log x \, \mathrm{d}x$$

Solution We'll use the same contour, but with a bump.



Then $\log z$ is analytic and single-valued on the region bounded by the contour.

We'll also take the branch of the logarithm with angle between 0 and 2π, since we don't make a full loop around the origin.

By the residue theorem

$$\int_{\gamma_1+\gamma_2+\gamma_3} \frac{\log z}{1+z^2} \, \mathrm{d}z = 2\pi i \frac{\log i}{2i} = \pi \frac{\pi i}{2} = \frac{\pi^2}{2} i.$$

Next, notice that

$$\left| \int_{\gamma_2} \frac{\log z}{1+z^2} \, \mathrm{d}z \right| \leq \frac{\pi R((\log R)^2 + \pi^2)^{1/2}}{R^2 - 1} \xrightarrow{R\to\infty} 0$$
$$\left| \int_{\gamma_3} \frac{\log z}{1+z^2} \, \mathrm{d}z \right| \leq \frac{\pi r((\log r)^2 + \pi^2)^{1/2}}{1 - r^2} \xrightarrow{r\to 0} 0,$$

since $x \log x \xrightarrow{x\to 0} 0$, which is easy to see with L'Hôpital's.

Thus, if we take $r \to 0$ and $R \to \infty$,

$$\int_{\gamma_1+\gamma_2+\gamma_3} \frac{\log z}{1+z^2} \, \mathrm{d}z = \frac{\pi^2}{2} i = \int_{\gamma_1} \frac{\log z}{1+z^2} \, \mathrm{d}z = \int_{-\infty}^{\infty} \frac{\log z}{1+z^2} \, \mathrm{d}z.$$

If we split the integral, we see

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\log z}{1+z^2} \, \mathrm{d}z &= \int_{-\infty}^{0} \frac{\log z}{1+z^2} \, \mathrm{d}z + \int_0^\infty \frac{\log z}{1+z^2} \, \mathrm{d}z \\
&= \int_0^\infty \frac{\log z}{1+z^2} + \frac{\pi i}{1+z^2} \, \mathrm{d}z + \int_{-\infty}^{\infty} \frac{\log z}{1+z^2} \, \mathrm{d}z \\
&= 2 \int_0^\infty \frac{\log z}{1+z^2} \, \mathrm{d}z + \pi i \arctan x \Big|_0^\infty \\
&= 2 \int_0^\infty \frac{\log z}{1+z^2} \, \mathrm{d}z + \frac{\pi^2}{2} i.
\end{aligned}$$

Thus,

$$2 \int_0^\infty \frac{\log z}{1+z^2} \, \mathrm{d}z + \frac{\pi^2}{2} i = \frac{\pi^2}{2} i \implies \int_0^\infty \frac{\log z}{1+z^2} \, \mathrm{d}z = 0.$$

5.3.3i

$$\int_0^\infty \log(1+x^2)\frac{\mathrm{d}x}{x^{1+\alpha}},\ (0<\alpha<2)$$

Solution We first integrate by parts to see that

$$\int_0^\infty \log(1+x^2)\frac{\mathrm{d}x}{x^{1+\alpha}} = -\frac{x^{-\alpha}}{\alpha}\log(1+x^2)\Big|_0^\infty + \frac{1}{\alpha}\int_0^\infty \frac{2x}{x^\alpha(1+x^2)}\,\mathrm{d}x.$$

By L'Hôpital's,

$$\begin{aligned}\lim_{x\to 0} -\frac{1}{\alpha}\frac{\log(1+x^2)}{x^\alpha} &= \lim_{x\to 0} -\frac{1}{\alpha^2}\frac{2x}{(1+x^2)x^{\alpha-1}}\\ &= \lim_{x\to 0} -\frac{1}{\alpha^2}\frac{2}{2x^\alpha+(\alpha-1)(1+x^2)x^{\alpha-2}}\\ &= \lim_{x\to 0} -\frac{1}{\alpha^2}\frac{2x^{2-\alpha}}{2x^2+(\alpha-1)(1+x^2)}\\ &= 0.\end{aligned}$$

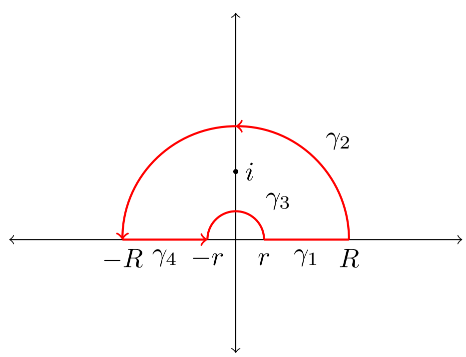
Similarly,

$$\begin{aligned}\lim_{x\to \infty} -\frac{1}{\alpha}\frac{\log(1+x^2)}{x^\alpha} &= \lim_{x\to \infty} -\frac{1}{\alpha^2}\frac{2x}{(1+x^2)x^{\alpha-1}}\\ &= \lim_{x\to \infty} -\frac{1}{\alpha^2}\frac{2}{2x^\alpha+(\alpha-1)(1+x^2)x^{\alpha-2}}\\ &= \lim_{x\to \infty} -\frac{1}{\alpha^2}\frac{2x^{2-\alpha}}{2x^2+(\alpha-1)(1+x^2)}\\ &= 0.\end{aligned}$$

This is because $2>2-\alpha>0$, by assumption. So, the integral reduces to

$$\int_0^\infty \log(1+x^2)\frac{\mathrm{d}x}{x^{1+\alpha}} = \frac{1}{\alpha}\int_0^\infty \frac{2x}{x^\alpha(1+x^2)}\,\mathrm{d}x.$$

We'll use the same contour, but with a bump.



Thus, by the residue theorem,

$$\frac{1}{\alpha}\int_0^\infty \frac{2z}{z^\alpha(1+z^2)}\,\mathrm{d}z = 2\pi i\frac{2i}{i^\alpha 2i} = \frac{2\pi}{\alpha}e^{i(1-\alpha)\pi/2}.$$

Notice that because $\alpha>0$,

$$\left|\int_{\gamma_2}\frac{2z}{z^\alpha(1+z^2)}\,\mathrm{d}z\right| \le \frac{2\pi R^2}{R^\alpha(R^2-1)} \xrightarrow{R\to\infty} 0$$

$$\left|\int_{\gamma_3}\frac{2z}{z^\alpha(1+z^2)}\,\mathrm{d}z\right| \le \frac{2\pi r^2}{r^\alpha(r^2-1)} = \frac{2\pi r^{2-\alpha}}{r^2-1} \xrightarrow{r\to 0} 0.$$

Also,

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\log(1+z^2)}{z^{1+\alpha}}\,\mathrm{d}z &= \int_{-\infty}^{0}\frac{\log(1+z^2)}{z^{1+\alpha}}\,\mathrm{d}z + \int_{0}^{\infty}\frac{\log(1+z^2)}{z^{1+\alpha}}\,\mathrm{d}z \\ &= \int_{0}^{\infty}\frac{\log(1+(-z)^2)}{(-z)^{1+\alpha}}\,\mathrm{d}z + \int_{0}^{\infty}\frac{\log(1+z^2)}{z^{1+\alpha}}\,\mathrm{d}z \\ &= \left(\frac{1}{e^{i\pi(1+\alpha)}}+1\right)\int_{0}^{\infty}\frac{\log(1+z^2)}{z^{1+\alpha}}\,\mathrm{d}z.\end{aligned}$$

Thus, when we take $R\to\infty$ and $r\to 0$, we get

$$\frac{2\pi}{\alpha}e^{i(1-\alpha)\pi/2} = \int_{\gamma_1+\gamma_2+\gamma_3}\frac{\log(1+z^2)}{z^{1+\alpha}}\,\mathrm{d}z = \left(\frac{1}{e^{i\pi(1+\alpha)}}+1\right)\int_{0}^{\infty}\frac{\log(1+z^2)}{z^{1+\alpha}}\,\mathrm{d}z$$
$$\int_{0}^{\infty}\frac{\log(1+z^2)}{z^{1+\alpha}}\,\mathrm{d}z = \frac{1}{\alpha}\frac{2\pi e^{i(1-\alpha)\pi/2}}{1+e^{-i\pi(1+\alpha)}} = \frac{\pi\csc\frac{\pi\alpha}{2}}{\alpha}.$$

The last equality comes from expanding the exponential and simplifying via rationalization.

5.3.5 Show that if $f(z)$ is analytic and bounded for $|z|<1$ and if $|\zeta|<1$, then

$$f(\zeta) = \frac{1}{\pi}\iint_{|z|<1}\frac{f(z)\,\mathrm{d}x\,\mathrm{d}y}{(1-\overline{z}\zeta)^2}$$

Solution By the Cauchy integral formula,

$$f(\zeta) = \frac{1}{2\pi i}\int_{|z|=1}\frac{f(z)}{z-\zeta}\,\mathrm{d}z.$$

Green's theorem for complex variables says that if we write $F(z)$ for the integrand,

$$\int_{|z|=1}F(z)\,\mathrm{d}z = 2i\iint_{|z|<1}\frac{\partial F}{\partial\overline{z}}\,\mathrm{d}x\,\mathrm{d}y,$$

Next, since $z=1/\overline{z}$ for z on the unit circle,

$$F(z) = \frac{f(z)}{z-\zeta} = \frac{\overline{z}f(z)}{1-\overline{z}\zeta},$$

so because $\partial f/\partial\overline{z}=0$ if f is analytic, we get

$$\frac{\partial F}{\partial\overline{z}} = \frac{f(z)(1-\overline{z}\zeta)+\overline{z}f(x)\zeta}{(1-\overline{z}\zeta)^2} = \frac{f(z)}{(1-\overline{z}\zeta)^2}.$$

Thus, we get

$$f(\zeta) = \frac{1}{2\pi i}\iint_{|z|<1}F_y - F_x\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2\pi i}\iint_{|z|<1}\frac{f(z)}{(1-\overline{z}\zeta)^2}\,\mathrm{d}x\,\mathrm{d}y$$

as desired.