3.3.5 Find the most general linear transformation of the circle $|z| = R$ into itself.

Solution Without loss of generality, $R = 1$. We can just scale our numbers to meet an arbitrary R.

Take three distinct points on the circle: $e^{i\alpha}, e^{i\beta}, e^{i\gamma}$.

Take three more distinct points on the circle (they may be the same as the first three points): $e^{i\alpha'}, e^{i\beta'}, e^{i\gamma'}$.

Then for any z on the circle different from the first three points, z must be mapped to the same circle, since Möbius transformations send circles to circles. Thus, if T sends e^{ix} to $e^{ix'}$, we can use the invariance of cross ratios to get

$$(z, e^{i\alpha}, e^{i\beta}, e^{i\gamma}) = (T(z), e^{i\alpha'}, e^{i\beta'}, e^{i\gamma'})$$
$$\implies \frac{z - e^{i\beta}}{z - e^{i\gamma}} \cdot \frac{e^{i\alpha} - e^{i\gamma}}{e^{i\alpha} - e^{i\gamma}} = \frac{T(z) - e^{i\beta'}}{T(z) - e^{i\gamma'}} \cdot \frac{e^{i\alpha'} - e^{i\gamma'}}{e^{i\alpha'} - e^{i\gamma'}}.$$

We can solve $T(z)$ to get our transformation.

3.3.6 Suppose that a linear transformation carries one pair of concentric circles into another pair of concentric circles. Prove that the ratios of the radii must be the same.

Solution We can assume without loss of generality that all the circles are centered at the origin.

We can consider the two concentric circles as an annulus centered at the origin.

It is well known that a Möbius transformation is conformal, so it preserves angles and distances. Thus, take a transformation that sends the real axis to the real axis, and maps the circles to concentric circles at the origin. So, we can consider the transformation as one from an annulus to another annulus centered at the origin. Since there is no translation, the transformation is the composition of a dilation and rotation, so the ratios are preserved: For $c \in \mathbb{C}$ such that $|c| = 1$,

$$\frac{cr_1}{cr_2} = \frac{r_1}{r_2}.$$

3.3.7 Find a linear transformation which carries $|z| = 1$ and $|z - 1/4| = 1/4$ into concentric circles. What is the ratio of the radii?

Solution Assume without loss of generality that the circles are sent to the center.

Notice that $1/2$ is symmetric to 2, with respect to the large circle, and $5/16$ is symmetric to 1, with respect to the small circle.

We want a transformation which does the following:

$$\begin{aligned} \frac{1}{2} &\mapsto r_1 \\ 1 &\mapsto r_2 \\ 2 &\mapsto r_2^2/r_1 \\ \frac{5}{16} &\mapsto r_1^2/r_2 \end{aligned}$$

This is because r_1 and r_2^2/r_1 are symmetric, and r_2 and r_1^2/r_2 are symmetric, and we want symmetric points to map to symmetric points.

By the symmetry principle, we know that such a transformation will map the circles to concentric circles, because for each of the circles, we do the following: map a point on the circle to another, map a pair of symmetric points to symmetric points, with respect to the circle.

By invariance of cross products,

$$\left(\frac{1}{2}, 1, 2, \frac{5}{16}\right) = \left(r_1, r_2, \frac{r_2^2}{r_1}, \frac{r_1^2}{r_2}\right).$$

We can solve for r_1, r_2, which gives us the linear transformation by replacing $1/2$ with z and r_1 with $T(z)$. After finding r_1, r_2, we can easily find their ratio.

1.3.3 Compute

$$\int_{|z|=2} \frac{\mathrm{d}z}{z^2-1}$$

for the positive sense of the circle.

Solution By partial fractions, we get

$$\frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right).$$

Thus, we get

$$\int_{|z|=2} \frac{\mathrm{d}z}{z^2-1} = \frac{1}{2}\left[\int_{|z|=2} \frac{\mathrm{d}z}{z-1} - \int_{|z|=2} \frac{\mathrm{d}z}{z+1}\right] = \frac{2\pi i}{2}\Big[n(\{|z|=2\},1) - n(\{|z|=2\},-1)\Big] = i\pi - i\pi = 0.$$

1.3.4 Compute

$$\int_{|z|=1} |z-1| \cdot |\mathrm{d}z|.$$

Solution Consider the change of variables $z \mapsto e^{i\theta}$, $\mathrm{d}z = ie^{i\theta}\,\mathrm{d}\theta$. Then

$$\begin{aligned}
\int_{|z|=1} |z-1| \cdot |\mathrm{d}z| &= \int_0^{2\pi} \sqrt{(\cos^2\theta - 1) + \sin^2\theta}|ie^{i\theta}|\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \sqrt{\cos^2\theta + \sin^2\theta - 2\sin\theta + 1}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} 2\sqrt{\frac{1-\cos\theta}{2}}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right|\mathrm{d}\theta \\
&= 2\cdot 2\int_0^{\pi} \sin\theta\,\mathrm{d}\theta \\
&= 8.
\end{aligned}$$

1.3.6 Assume that $f(z)$ is analytic and satisfies the inequality $|f(z)-1| < 1$ in a region Ω. Show that

$$\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = 0.$$

Solution By the given inequality, we know that $f(\gamma)$ lies in $\{z \mid |z-1| < 1\}$, i.e., $0 \notin f(\gamma)$. Taking the substitution $f(z) \mapsto u$, we get

$$\int_{f(\gamma)} \frac{\mathrm{d}u}{u} = 2\pi i n(f(\gamma), 0) = 0.$$

1.3.7 If $P(z)$ is a polynomial and C denotes the circle $|z-a|=R$, what is the value of $\int_C P(z)\,\mathrm{d}\bar{z}$? *Answer*: $-2\pi i R^2 P'(a)$.

Solution By linear algebra, we can write

$$P(z)=\sum_{n=0}^{N} a_n (z-a)^n,$$

for some $N\in\mathbb{N}$ and coefficients a_n. Taking the substitution $z\mapsto a+Re^{i\theta}$, we get

$$\bar{z}=\bar{a}+Re^{-i\theta} \implies \mathrm{d}\bar{z}=-iRe^{-i\theta}\,\mathrm{d}\theta.$$

Thus,

$$\begin{aligned}\int_C P(z)\,\mathrm{d}\bar{z} &= \int_0^{2\pi}\sum_{n=0}^{N} a_n R^n e^{in\theta}\cdot(iRe^{-i\theta})\,\mathrm{d}\theta\\ &= \sum_{n=0}^{N} a_n \int_0^{2\pi} -iR^{n+1}e^{-i(n-1)\theta}\,\mathrm{d}\theta.\end{aligned}$$

Notice that for $n\in\mathbb{Z}-\{0\}$,

$$\int_0^{2\pi} e^{in\theta}\,\mathrm{d}\theta=0,$$

by simple calculation. Thus,

$$\int_C P(z)\,\mathrm{d}\bar{z}=a_1\int_0^{2\pi} -iR^2\,\mathrm{d}\theta=-2\pi i a_1 R^2.$$

Lastly,

$$P'(z)=\sum_{n=1}^{N} n a_n (z-a)^{n-1} \implies P'(a)=a_1,$$

so

$$\int_C P(z)\,\mathrm{d}\bar{z}=-2\pi i R^2 P'(a).$$

2.2.2 Compute

$$\int_{|z|=2} \frac{\mathrm{d}z}{z^2+1}$$

by decomposition of the integrand in partial fractions.

Solution By partial fractions,

$$\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right).$$

By the same argument as in problem 1.3.3,

$$\int_{|z|=2} \frac{\mathrm{d}z}{z^2+1} = \frac{2\pi i}{2}\Big[n(\{|z|=2\}, i) - n(\{|z|=2\}, -i)\Big] = 0.$$

2.2.3 Compute

$$\int_{|z|=\rho} \frac{|\mathrm{d}z|}{|z-a|^2}$$

under the condition $|a| \neq p$. *Hint*: Make use of the equations $z\bar{z} = \rho^2$ and

$$|\mathrm{d}z| = -i\rho\frac{\mathrm{d}z}{z}.$$

Solution We first prove the hint: Take $z \mapsto \rho e^{i\theta}$. Then $\mathrm{d}z = i\rho e^{i\theta}\,\mathrm{d}\theta$, so

$$|\mathrm{d}z| = \rho\,\mathrm{d}\theta = \rho\frac{i\rho e^{i\theta}}{i\rho e^{i\theta}}\,\mathrm{d}\theta = \frac{\rho}{iz}\,\mathrm{d}z = -i\rho\frac{\mathrm{d}z}{z}.$$

Substituting into the integral,

$$\begin{aligned}
\int_{|z|=\rho} \frac{|\mathrm{d}z|}{|z-a|^2} &= \int_0^{2\pi} \frac{-i\rho}{z(z-a)(\bar{z}-\bar{a})}\,\mathrm{d}z \\
&= -i\rho\int_0^{2\pi} \frac{1}{(z-a)(\rho^2-\bar{a}z)}\,\mathrm{d}z \\
&= -\frac{i\rho}{\rho^2-|a|^2}\int_0^{2\pi} \frac{1}{z-a} + \frac{\bar{a}}{\rho^2-\bar{a}z}\,\mathrm{d}z \\
&= -\frac{i\rho}{\rho^2-|a|^2}\int_0^{2\pi} \frac{1}{z-a} - \frac{1}{z-\frac{\rho^2}{\bar{a}}}\,\mathrm{d}z
\end{aligned}$$

Now there are two cases: $|a| < \rho$ and $|a| > \rho$.

$|a| < \rho$:

In this case,

$$\left|\frac{\rho^2}{\bar{a}}\right| = \rho\frac{\rho}{|a|} > \rho,$$

so we get

$$\int_{|z|=\rho} \frac{|\mathrm{d}z|}{|z-a|^2} = -\frac{2\pi i^2\rho}{\rho^2-|a|^2}\left[n(\{|z|=\rho\}, a) - n\left(\{|z|=\rho\}, \frac{\rho^2}{\bar{a}}\right)\right] = \frac{2\pi\rho}{\rho^2-|a|^2}.$$

$|a| > \rho$:

In this case,

$$\left|\frac{\rho^2}{\bar{a}}\right| = \rho\frac{\rho}{|a|} < \rho,$$

so by the same argument, we get

$$\int_{|z|=\rho} \frac{|\mathrm{d}z|}{|z-a|^2} = -\frac{2\pi\rho}{\rho^2 - |a|^2} = \frac{2\pi\rho}{|a|^2 - \rho^2}.$$

In either case, we get

$$\int_{|z|=\rho} \frac{|\mathrm{d}z|}{|z-a|^2} = \left|\frac{2\pi\rho}{\rho^2 - |a|^2}\right|.$$

2.3.2 Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$ reduces to a polynomial.

Solution Since f is analytic,

$$f(z) = \sum_{k=0}^{\infty} a_k z^{k+1},$$

where

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^k}\,\mathrm{d}z,$$

for some circle C containing the origin with radius R large enough so that $|f(z)| < |z|^n$.

By Cauchy's estimate, for $k \geq n$, we have

$$|a_k| \leq \frac{k!}{2\pi} \cdot 2\pi R \cdot \frac{|z|^k}{R^{k+1}} = \frac{k!|z|^k}{R^k} \xrightarrow{R\to\infty} 0.$$

Thus, f truncates to

$$f(z) = \sum_{k=0}^{n} a_k z^k,$$

which is a polynomial.

2.3.3 If $f(z)$ is analytic and $|f(z)| \leq M$ for $|z| \leq R$, find an upper bound for $|f^{(n)}(z)|$ in $|z| \leq \rho < R$.

Solution By Cauchy's estimate,

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \cdot \sup_{|\zeta|=\rho}\left|\frac{f(z)}{(\zeta - z)^{n+1}}\right| \cdot 2\pi\rho \leq \frac{M\rho}{(R-\rho)^{n+1}}n!,$$

since $|\zeta - z|$ is smallest when they lie on the same line passing through the origin.

2.3.4 If $f(z)$ is analytic for $|z| < 1$ and $|f(z)| \leq 1/(1-|z|)$, find the best estimate of $|f^{(n)}(0)|$ that Cauchy's inequality will yield.

Solution Taking $0 < r < 1$, Cauchy's estimate yields

$$|f^{(n)}(0)| \leq \frac{n!}{1-r} \cdot \frac{1}{r^n}.$$

We can maximize $(1-r)r^n$ with calculus, which gives us a number $m(r)$, so the best estimate is

$$|f^{(n)}(0)| \leq m(r) \cdot n!$$

2.3.6 Let the function $\varphi(z,t)$ be continuous as a function of both variables with z lies in a region Ω and $\alpha \le t \le \beta$. Suppose further that $\varphi(z,t)$ is analytic as a function of $z \in \Omega$ for any fixed t. Then

$$F(z) = \int_\alpha^\beta \varphi(z,t)\,\mathrm{d}t$$

is analytic in z and

$$F'(z) = \int_\alpha^\beta \frac{\partial \varphi(z,t)}{\partial z}\,\mathrm{d}t.$$

Solution By Fubini's theorem and the fact that φ is holomorphic for any t, we see that

$$F(z) = \frac{1}{2\pi i}\int_\alpha^\beta \int_C \frac{\varphi(\zeta,t)}{\zeta - z}\,\mathrm{d}\zeta\,\mathrm{d}t = \frac{1}{2\pi i}\int_C \int_\alpha^\beta \varphi(\zeta,t)\,\mathrm{d}t\,\frac{\mathrm{d}\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_C \frac{F(\zeta)}{\zeta - z}\,\mathrm{d}\zeta,$$

for some circle C containing z.

By uniform convergence of the geometric series, if we pick z_0 in the bounded component of the circle C,

$$F(z) = \frac{1}{2\pi i}\int_C \sum_{n=0}^\infty \frac{F(\zeta)}{(\zeta - z_0)^{n+1}}(z - z_0)^n\,\mathrm{d}\zeta = \sum_{n=0}^\infty \left(\frac{1}{2\pi i}\int_C \frac{F(\zeta)}{(\zeta - z_0)^{n+1}}\,\mathrm{d}\zeta\right)(z - z_0)^n,$$

so F is analytic in a neighborhood of $z \in \Omega$. z was arbitrary, so F is analytic in Ω.

Since φ is analytic as a function of z, we can write

$$\begin{aligned}
f'(z) &= \frac{1}{2\pi i}\int_C \frac{F(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\int_C \int_\alpha^\beta \frac{\varphi(\zeta,t)}{(\zeta - z)^2}\,\mathrm{d}t\,\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\int_C \int_\alpha^\beta \frac{1}{(\zeta - z)^2}\sum_{n=0}^\infty b_n(z)(\zeta - z)^n\,\mathrm{d}t\,\mathrm{d}\zeta \\
&= \sum_{n=0}^\infty \frac{1}{2\pi i}\int_\alpha^\beta b_n(z)\int_C (\zeta - z)^{n-2}\,\mathrm{d}\zeta\,\mathrm{d}t.
\end{aligned}$$

Notice that $-(\zeta - z)^{-1}$ is the primitive of $(\zeta - z)^{-2}$ so the integral of $(\zeta - z)^{-2}$ over C is 0, and for $n \ge 2$, we get analytic functions (i.e., polynomials), so the integrals for those are also 0. Thus, the sum reduces to

$$f'(z) = \frac{1}{2\pi i}\int_\alpha^\beta \int_C \frac{b_1(z)}{\zeta - z}\,\mathrm{d}\zeta\,\mathrm{d}t = \frac{1}{2\pi i}\int_\alpha^\beta 2\pi i b_1(z)\,\mathrm{d}t = \int_\alpha^\beta b_1(z)\,\mathrm{d}t = \int_\alpha^\beta \frac{\partial \varphi(z,t)}{\partial z}\,\mathrm{d}t.$$

3.2.2 Show that a function which is analytic in the whole plane and has a nonessential singularity at ∞ reduces to a polynomial.

Solution Suppose f is entire and has a nonessential singularity at ∞.

If the singularity is removable, then

$$\lim_{z\to 0}\left|zf\left(\frac{1}{z}\right)\right| = 0,$$

so $f(1/z)$ extends to be analytic at 0, which means that f extends to be analytic on $\mathbb{C}^*$. But $\mathbb{C}^*$ is compact, which means that f is bounded and attains its maximum. Thus, by the maximum principle, f is constant.

Otherwise, there exists $h \in \mathbb{Z}$ such that

$$\lim_{z\to 0}|z|^n\left|f\left(\frac{1}{z}\right)\right| = \begin{cases} 0 & \text{if } n > h \\ \infty & \text{if } n < h \end{cases}$$

Since f is analytic but not a polynomial, there exists $N > h$ such that $a_N \neq 0$. But

$$z^N f\left(\frac{1}{z}\right) = a_N + \sum_{n=0, n\neq N}^{\infty} \frac{a_n}{z^{n-N}} \overset{z\to 0}{\not\longrightarrow} 0,$$

since for $n < N$, $a_n z^n \xrightarrow{z\to 0} 0$, and for $n > N$, the terms diverge, and $a_N \neq 0$.

This is a contradiction. Hence, eventually, $a_n = 0$, so f truncates to a polynomial.

3.2.3 Show that the functions e^z, $\sin z$ and $\cos z$ have essential singularities at ∞.

Solution Recall that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Then

$$z^N e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^{n-N}} \xrightarrow{z\to 0} \infty,$$

for any N, since there are infinitely many non-zero coefficients.

Similarly, since

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cos z = \sum_{n=0}^{\infty} (-1)^0 \frac{x^{2n}}{(2n)!}$$

have infinitely many non-zero coefficients also, $z^N \cos 1/z \xrightarrow{z\to 0} \infty$ and $z^N \sin 1/z \xrightarrow{z\to 0} \infty$ for any N, so they have essential singularities at ∞.

3.2.5 Prove that an isolated singularity of $f(z)$ is removable as soon as either $\operatorname{Re} f(z)$ or $\operatorname{Im} f(z)$ is bounded above or below. *Hint*: Apply a fractional linear transformation.

Solution We can just consider the case where $\operatorname{Re} f(z)$ is bounded. In the case that $\operatorname{Im} f(z)$ is bounded, we can rotate f so that $\operatorname{Re} e^{i\alpha} f(z)$ is bounded. Similarly, we can just consider the case where $\operatorname{Re} f(z)$ is bounded above by some M, since we can perform the following argument on $-f$.

Let z_0 be a singularity of f.

Suppose that there exists M so that $\operatorname{Re} f(z) \leq M$. We can take a Möbius transformation which takes the upper-half plane to the unit disk, e.g.,

$$T(z) = \frac{z-i}{z+i}.$$

Then $T(f)$ is bounded on the unit disk close to the singularity, since $T(f)$ is continuous and $T(f(z_0)) = 1$. Thus, $T(f)$ extends to be holomorphic on a neighorhood of the singularity.

Since $T(f)$ and T are analytic on a neighborhood of the singularity, $f = T^{-1} \circ T \circ f$ is analytic in a neighborhood of the singularity.

3.2.6 Show that an isolated singularity of $f(z)$ cannot be a pole of $\exp f(z)$. *Hint*: f and e^f cannot have a common pole.

Solution Let z_0 be a pole of f. Then $1/f$ extends to be analytic at z_0, since $1/f(z_0) = 1/\infty = 0$.

By the open mapping theorem, for some $\delta > 0$, we have $(1/f)(\{z \mid |z - z_0| < \delta\})$ is an open set containing 0, which we'll call U. Thus, there exists a sequence $\{z_n\}_{n\geq 1}$ in $B(z_0, \delta)$ such that $\mathbb{R} \ni f(z_n) < 0$ and $f(z_n) \xrightarrow{n\to\infty} -\infty$, and $f(z_n) \in U$ for all z_n.

But then

$$\liminf_{z\to z_0} |\exp f(z)| = \lim_{n\to\infty} |\exp f(z_n)| = 0,$$

so it cannot be the case that $\exp f$ has a pole at z_0.