17.2.24 Compute $\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, where $\mathbf{F}(x,y) = \left\langle \frac{-y}{(x^2+y^2)^2}, \frac{x}{(x^2+y^2)^2} \right\rangle$, and $\mathcal{C}$ is the circle of radius R with the center at the origin oriented counterclockwise.

Solution A parameterization of $\mathcal{C}$ is $\mathbf{r}(t) = R\langle \cos t, \sin t\rangle$, with $t \in [0, 2\pi]$. Our line integral then becomes

$$\begin{aligned}
&\int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} \left\langle \frac{-R\sin t}{((R\cos t)^2 + (R\sin t)^2)^2}, \frac{R\cos t}{((R\cos t)^2 + (R\sin t)^2)^2} \right\rangle \cdot \langle -R\sin t, R\cos t\rangle\ \mathrm{d}t \\
&= \int_0^{2\pi} \frac{1}{R^2}\langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\ \mathrm{d}t \\
&= \frac{1}{R^2}\int_0^{2\pi} \sin^2 t + \cos^2 t\,\mathrm{d}t \\
&= \frac{1}{R^2}\int_0^{2\pi} \mathrm{d}t \\
&= \frac{1}{R^2}\left[t\right]_0^{2\pi} \\
&\boxed{= \frac{2\pi}{R^2}}
\end{aligned}$$

17.2.44 Calculate the total mass of a metal tube in the helical shape $\mathbf{r}(t) = \langle \cos t, \sin t, t^2\rangle$ (distance in centimeters) for $0 \le t \le 2\pi$ if the mass density is $\delta(x,y,z) = \sqrt{z}$ g/cm.

Solution With $\mathcal{C}$ as the curve of the metal tube, the mass of the tube is given by

$$\begin{aligned}
&\int_{\mathcal{C}} \delta(x,y,z)\,\mathrm{d}s \\
&= \int_0^{2\pi} \delta(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|\ \mathrm{d}t \\
&= \int_0^{2\pi} \sqrt{t^2}\,\|\langle -\sin t, \cos t, 2t\rangle\|\ \mathrm{d}t \\
&= \int_0^{2\pi} |t|\,\sqrt{\sin^2 t + \cos^2 t + 4t^2}\,\mathrm{d}t \\
&= \int_0^{2\pi} |t|\,\sqrt{1+4t^2}\,\mathrm{d}t
\end{aligned}$$

On the interval $[0, 2\pi]$, $t \ge 0$, so $|t| = t$. Thus,

$$\begin{aligned}
&= \int_0^{2\pi} t\sqrt{1+4t^2}\,\mathrm{d}t \\
&= \left[\frac{1}{12}(1+4t^2)^{3/2}\right]_0^{2\pi} \\
&\boxed{= \left(\frac{1}{12}(1+16\pi^2)^{3/2} - \frac{1}{12}\right)\mathrm{g}}
\end{aligned}$$

17.2.67 Calculate the flux of $\mathbf{F}(x,y) = \langle e^y, 2x-1 \rangle$ on the parabola $y = x^2$ for $0 \le x \le 1$, oriented left to right.

Solution A parameterization of the curve is $\mathbf{r}(t) = \left\langle t, t^2 \right\rangle$, for $t \in [0,1]$. With this parameterization, $\mathbf{r}'(t) = \langle 1, 2t \rangle$ and the normal pointing towards the right is $\mathbf{n}(t) = \langle 2t, -1 \rangle$. Thus, the flux passing through the curve is

$$\begin{aligned}
&\int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{n}(t)\,\mathrm{d}t \\
&= \int_0^1 \left\langle e^{t^2}, 2t-1 \right\rangle \cdot \langle 2t, -1 \rangle\,\mathrm{d}t \\
&= \int_0^1 2te^{t^2} - 2t + 1\,\mathrm{d}t \\
&= \left[e^{t^2} - t^2 + t \right]_0^1 \\
&= e - 1 + 1 - 1 \\
&\boxed{= e - 1}
\end{aligned}$$