**6.5.41** Suppose $1 < p \le \infty$ and $p^{-1} + q^{-1} = 1$. For the case $p = \infty$, assume that the measure is semifinite. If $T$ is a bounded operator on $L^p$ such that $\int (Tf)g = \int f(Tg)$ for all $f, g \in L^p \cap L^q$, then $T$ extends uniquely to a bounded operator on $L^r$ for all $r$ in $[p, q]$ (if $p < q$) or $[q, p]$ (if $q < p$).

**Solution** First notice that $L^p \cap L^q$ is dense in $L^q$. Indeed, $L^q$-integrable simple functions are dense in $L^q$, and $L^q$-integrable simple functions are also $L^p$-integrable, since they are supported on a set with finite measure.

We will use Theorem 6.13 from Folland: Let $f, g \in L^p \cap L^q$, where $f, g$ are simple with $\|g\|_p = 1$. Then by Hölder's inequality,

$$\left| \int (Tf)g \right| \le \int |f(Tg)| \le \|f\|_q \|Tg\|_p \le \|f\|_q \|T\| \|g\|_p = \|T\| \|f\|_q$$

where $\|T\|$ is the operator norm of $T$. Because $T$ is bounded, $\|T\| < \infty$, so $(Tf)g \in L^1$. By density of $L^p \cap L^q$ in $L^q$, the inequality holds for all $g \in L^q$, so by the theorem,

$$\|Tf\|_q \le \|T\| \|f\|_q,$$

and again by density of $L^p \cap L^q$ in $L^q$, this inequality extends to all $f \in L^q$.

Next, we will show that $T$ is linear: let $f_1, f_2, g \in L^p \cap L^q$. Then

$$\int [T(f_1 + f_2)]g = \int (f_1 + f_2)Tg = \int f_1(Tg) + \int f_2(Tg) = \int (Tf_1)g + \int (Tf_2)g = \int (Tf_1 + Tf_2)g.$$

Hence,

$$\int [T(f_1 + f_2) - Tf_1 - Tf_2]g = 0$$

for all $g \in L^q$, by the same density argument as before. Hence, by Theorem 6.13 again,

$$\|T(f_1 + f_2) - Tf_1 - Tf_2\|_q = 0 \implies T(f_1 + f_2) = Tf_1 + Tf_2.$$

We may now apply Riesz-Thorin to $T$. We have that $T$ is a bounded operator on $L^p$ and on $L^q$, and if $p < r < q$, there exists $t \in (0, 1)$ so that

$$\frac{1}{r} = \frac{1-t}{p} + \frac{t}{q},$$

so $T$ extends uniquely to a bounded operator on $L^r$.

**6.5.45** If $0 < \alpha < n$, define an operator $T_\alpha$ on functions on $\mathbb{R}^n$ by

$$T_\alpha f(x) = \int |x-y|^{-\alpha} f(y)\,\mathrm{d}y.$$

Then $T_\alpha$ is weak type $(1, n\alpha^{-1})$ and strong type $(p,r)$ with respect to Lebesgue measure on $\mathbb{R}^n$, where $1 < p < n(n-\alpha)^{-1}$ and $r^{-1} = p^{-1} - (n-\alpha)n^{-1}$.

**Solution** Notice that by Corollary 2.51,

$$\beta^p m\big(\{|x|^{-\alpha} > \beta\}\big) = \beta^p m\Big(\Big\{|x| < \beta^{-1/\alpha}\Big\}\Big) = \beta^p m\Big(B(0, \beta^{-1/\alpha})\Big) = C\beta^p \beta^{-n/\alpha}.$$

So, if we choose $p = n\alpha^{-1}$, we see that $|x|^{-\alpha} \in \text{weak}\, L^{n/\alpha}$, and so by translation invariance of the Lebesgue measure,

$$[K(x,\cdot)]_{n/\alpha} = [K(\cdot,y)]_{n/\alpha} = \big[|x|^{-\alpha}\big]_{n/\alpha} \leq C < \infty.$$

Thus, by Theorem 6.36, $T_\alpha$ is weak type $(1, n\alpha^{-1})$. By the same theorem, $T_\alpha$ is strong type $(p,r)$, where $1 < p < r < \infty$ and satisfy

$$\frac{1}{p} + \frac{1}{n/\alpha} = \frac{1}{r} + 1.$$

If we pick $r$ so that $r^{-1} = p^{-1} - (n-\alpha)n^{-1}$, we have

$$\frac{1}{p} - \frac{n-\alpha}{n} + 1 = \frac{1}{p} + \frac{\alpha}{n} - 1 + 1 = \frac{1}{p} + \frac{\alpha}{n},$$

so that value of $r$ satisfies the equation. Lastly, from the choice of $r$,

$$\frac{1}{r} = \frac{1}{p} - \frac{n-\alpha}{n} < \frac{1}{p} \implies 1 < p < r,$$

and because $p < n(n-\alpha)^{-1}$,

$$\frac{1}{r} = \frac{1}{p} - \frac{n-\alpha}{n} > 0 \implies r < \infty$$

so the inequality $1 < p < r < \infty$ is satisfied. Hence, $T_\alpha$ is strong type $(p,r)$ for the given conditions on $p$ and $r$, as required.

**8.1.4** If $f \in L^\infty$ and $\|\tau_y f - f\|_\infty \to 0$ as $y \to 0$, then $f$ agrees a.e. with a uniformly continuous function.

**Solution** We follow the hint and consider

$$A_r f(x) = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)\,\mathrm{d}y.$$

We claim that the function defined by $\lim_{r\to 0} A_r f(x)$ is well-defined and uniformly continuous.

By Lemma 3.16, we know that $A_r f(x)$ is continuous in both $r$ and $x$. Indeed, since $f \in L^\infty$, for any bounded set $K \subseteq \mathbb{R}^n$, we have

$$\int_K |f(x)|\,\mathrm{d}x \le \int_K \|f\|_\infty\,\mathrm{d}x = \|f\|_\infty m(K) < \infty \implies f \in L^1_{\mathrm{loc}},$$

so we may apply the results of the lemma. Then we have

$$\begin{aligned}
|\tau_y A_r f(x) - A_r f(x)| &= \left| \frac{1}{m(B(r,x-y))} \int_{B(r,x-y)} f(z)\,\mathrm{d}z - \frac{1}{m(B(r,x))} \int_{B(r,x)} f(z)\,\mathrm{d}z \right| \\
&= \left| \frac{1}{m(B(r,x))} \int_{B(r,x)} f(z-y)\,\mathrm{d}z - \frac{1}{m(B(r,x))} \int_{B(r,x)} f(z)\,\mathrm{d}z \right| \qquad (z \mapsto z-y) \\
&\le \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(z-y) - f(z)|\,\mathrm{d}z \\
&\le \frac{1}{m(B(r,x))} \int_{B(r,x)} \|\tau_z f - f\|_\infty\,\mathrm{d}z \\
&\le \|\tau_y f - f\|_\infty \xrightarrow{y\to 0} 0.
\end{aligned}$$

Hence, $\|\tau_y A_r f - A_r f\|_u \xrightarrow{y \to 0} 0$, so $A_r f$ is uniformly continuous.

We will now show that $A_r f$ is uniformly Cauchy in $r$: Let $r, s > 0$. Then

$$\begin{aligned}
|A_r f(x) - A_s f(x)| &= \left| A_r f(x) - \frac{1}{m(B(r,x))} \int_{B(r,x)} A_s f(y)\,\mathrm{d}y + \frac{1}{m(B(r,x))} \int_{B(r,x)} A_s f(y)\,\mathrm{d}y - A_s f(x) \right| \\
&\le \left| \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y) - A_s f(y)\,\mathrm{d}y \right| + \frac{1}{m(B(r,x))} \int_{B(r,x)} |A_s f(y) - A_s f(x)|\,\mathrm{d}y \\
&\le \frac{1}{m(B(r,x)) m(B(s,x))} \int_{B(r,x)} \int_{B(s,x)} |f(y) - f(z)|\,\mathrm{d}z\,\mathrm{d}y + \|\tau_{y-x} A_r f - A_r f\|_u \\
&\le \|\tau_{y-z} f - f\|_\infty + \|\tau_{y-x} A_r f - A_r f\|_u,
\end{aligned}$$

where $z \in B(s,x)$ and $y \in B(r,x)$. Thus, $|y - z| \le r + s$ and $|y - x| \le r$, so as $r, s \to 0$, the above expression must go to 0. Indeed, the first term tends to 0 by assumption, and the second term vanishes because of uniform continuity of $A_r f(x)$ in $x$. This shows that $A_r f$ is uniformly Cauchy in $r$.

Hence, $Af(x) \coloneqq \lim_{r\to 0} A_r f(x)$ exists for all $x$, and $\{A_r f\}_r$ is a sequence of uniformly continuous functions which converges uniformly to $Af$, so $Af$ must be uniformly continuous.

By Theorem 3.18 in Folland, $Af(x) = f(x)$ for a.e. $x \in \mathbb{R}^n$, which concludes the proof.