1 Consider the linear program

$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 \\ \text{subject to} \quad & 0 \le x_1 \le 5 \\ & 0 \le x_2 \le 7 \\ & x_1 + x_2 \le 9. \end{aligned}$$

Convert the problem to standard form and solve it using the simplex method.

Solution The problem can be written in standard form as

$$\begin{aligned} \text{minimize} \quad & \begin{pmatrix} -2 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix} \\ \text{subject to} \quad & x_1 + t_1 = 5 \\ & x_2 + t_2 = 7 \\ & x_1 + x_2 + t_3 = 9 \\ & \begin{pmatrix} x_1 & x_2 & t_1 & t_2 & t_3 \end{pmatrix}^\top \ge 0 \end{aligned}$$

which is in standard form. Then the canonical augmented matrix is

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 9 \\ 0 & 1 & 0 & 1 & 0 & 7 \\ 1 & 0 & 1 & 0 & 0 & 5 \end{array}\right) \iff \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & -1 & 3 \end{array}\right)$$

if we choose B to be the first three columns. Then the reduced cost vector is given by

$$\mathbf{r}_D = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

which has a negative entry. Thus, we will pull the 4-th column into our basis B.
Also,

$$p := \arg\min \left\{ \frac{\mathbf{y}_{i0}}{\mathbf{y}_{iq}}, \mathbf{y}_{i0} > 0 \right\} = 3$$

so we pivot about $(3, 4)$. This gives us

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 & -1 & 3 \end{array}\right)$$

and the reduced cost coefficient vector is then

$$\mathbf{r}_D = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ge 0.$$

Thus, the basic solution

$$\mathbf{x}^* = \begin{pmatrix} 5 & 4 & 0 & 3 & 0 \end{pmatrix}^\top$$

is our optimal basic feasible solution.

2 Consider the problem

$$\begin{aligned} \text{maximize} \quad & f(x) \\ \text{subject to} \quad & h(x) = 0, \end{aligned}$$

where $f, h\colon \mathbb{R}^2 \to \mathbb{R}$, and $\nabla f(\mathbf{x}) = \begin{pmatrix} x_1 & x_1 + 4 \end{pmatrix}^\top$. Suppose that $\mathbf{x}^*$ is an optimal solution and $\nabla h(\mathbf{x}^*) = \begin{pmatrix} 1 & 4 \end{pmatrix}^\top$. Find $\nabla f(\mathbf{x}^*)$.

Solution By Lagrange's theorem, there exists $\lambda^* \in \mathbb{R}$ such that

$$\nabla f(\mathbf{x}^*) + \nabla h(\mathbf{x}^*)\lambda = \mathbf{0} \implies x_1 = \frac{4}{3},$$

so

$$\nabla f(\mathbf{x}^*) = \begin{pmatrix} \frac{4}{3} & \frac{16}{3} \end{pmatrix}^\top.$$

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3 Consider the problem

$$\begin{aligned} \text{maximize} \quad & \|\mathbf{x} - \mathbf{x}_0\|^2 \\ \text{subject to} \quad & \|\mathbf{x}\|^2 = 9, \end{aligned}$$

where $\mathbf{x_0} = \begin{pmatrix} 1 & \sqrt{3} \end{pmatrix}^\top$. Find all points satisfying the Lagrange condition for the problem.

Solution We can write the constraint as $h(\mathbf{x}) := \|\mathbf{x}\|^2 - 9 = 0$. This is a quadratic form, and its gradient is $2\mathbf{x}$. Also, the objective function is a quadratic form with $\nabla f(\mathbf{x}) = 2(\mathbf{x} - \mathbf{x}_0)$. Thus, we want to find $\lambda \in \mathbb{R}$ such that

$$2\mathbf{x} - 2\mathbf{x}_0 + 2\mathbf{x}\lambda = \mathbf{0} \implies \mathbf{x} = \frac{\mathbf{x}_0}{1+\lambda}.$$

Substituting it into our constraint, we get

$$\begin{aligned} \frac{\|\mathbf{x}_0\|^2}{(1+\lambda)^2} &= 9 \\ 9\lambda^2 + 18\lambda + 5 &= 0 \\ \implies \lambda_{1/2} &= \frac{-18 \pm \sqrt{324 - 180}}{18} \\ &= \frac{-18 \pm 12}{18} \\ &= -1 \pm \frac{2}{3}. \end{aligned}$$

So $\lambda \in \left\{-\frac{5}{3}, -\frac{1}{3}\right\}$, which gives

$$\mathbf{x} \in \left\{-\frac{3\mathbf{x}_0}{2}, \frac{3\mathbf{x}_0}{2}\right\},$$

where $x_0 = \begin{pmatrix} 1 & \sqrt{3} \end{pmatrix}^\top$.

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4 Let $A \in \mathbb{R}^{m \times n}$, $m < n$, rank $A = m$, and $\mathbf{b} \in \mathbb{R}^m$. Define $\Omega = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{b}\}$ and let $\mathbf{x}_0 \in \Omega$. Show that for any $\mathbf{y} \in \mathbb{R}^n$,

$$\Pi(\mathbf{x}_0 + \mathbf{y}) = \mathbf{x}_0 + P\mathbf{y},$$

where $P = I - A^\top (AA^\top)^{-1} A$.

Solution By definition,

$$\Pi(\mathbf{x}_0 + \mathbf{y}) = \arg\min_{\mathbf{z}\in\Omega} \|\mathbf{x}_0 + \mathbf{y} - \mathbf{z}\|^2 \quad \text{subject to} \quad A\mathbf{z} = \mathbf{b}.$$

Let $\mathbf{w} \coloneqq \mathbf{x}_0 + \mathbf{y} - \mathbf{z} \implies A\mathbf{w} = A\mathbf{x}_0 + A\mathbf{y} - A\mathbf{z} = \mathbf{b} + A\mathbf{y} - \mathbf{b} = A\mathbf{y}$. This is a least squares problem, and the solution is given by

$$\begin{aligned}
\mathbf{w}^* &= A^\top (AA^\top)^{-1} A\mathbf{y} \\
\mathbf{x}_0 + \mathbf{y} - \mathbf{z}^* &= A^\top (AA^\top)^{-1} A\mathbf{y} \\
\mathbf{z}^* &= \mathbf{x}_0 + (I_n - A^\top (AA^\top)^{-1} A)\mathbf{y} \\
\Pi(\mathbf{x}_0 + \mathbf{y}) &= \mathbf{x}_0 + P\mathbf{y}
\end{aligned}$$

as desired.