**32.12.2** Let $R$ be a domain that is not a field. Show that $R[t]$ is not a PID.

**Solution** Since $R$ is not a field, there exists $a \in R$ which is not a unit.

Now suppose that $R[t]$ is a PID, and consider $(a, t+1)$. By assumption, there exists $d(t) \in R[t]$ so that $(d) = (a, t+1)$, so there exist $f(t), g(t) \in R[t]$ such that $d(t)f(t) = a$ and $d(t)g(t) = t+1$.

From the first equation, we know that $d$ must be a constant, so we shall now consider $d(t)$ as the constant $d$.

From the second equation, $t+1$ is irreducible, so one of $d$ and $g(t)$ must be 1. $g(t)$ cannot be constant or else $dg(t)$ is constant, so $d$ must be 1, which implies that $(a, t+1) = (1) = R[t]$.

Thus, there exist $x(t), y(t) \in R[t]$ so that $x(t)a + y(t)(t+1) = 1$. This tells us that $y(t) \equiv 0$, since there is no linear term, so $x(t)a = 1$. But this means that $a$ must be a unit in $R[t]$, which is impossible, since it is not a unit in $R$, a contradiction. Thus, $R[t]$ is not a PID.

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**32.12.7** Let $y = t + a$. Show that the map $R[t] \mapsto R[y]$ given by $\sum a_i t^i \mapsto \sum a_i y^i$ is a ring isomorphism.

**Solution** We denote the map by $\varphi$.

We first show that it's a homomorphism.

It's clear that $\varphi(1) = 1$ and $\varphi(0) = 0$.

Let $f(t) = \sum a_i t^i, g(t) = \sum b_i t^i \in R[t]$. Then

$$\begin{aligned}\varphi(f(t) + g(t)) &= \sum (a_i + b_i)(t+a)^i \\ &= \sum a_i (t+a)^i + \sum b_i (t+a)^i \\ &= \varphi(f(t)) + \varphi(g(t)).\end{aligned}$$

Lastly,

$$\begin{aligned}\varphi(f(t)g(t)) &= \sum_i \sum_j a_{i-j} b_j (t+a)^j \\ &= \left(\sum a_i (t+a)^i\right)\left(\sum b_i (t+a)^i\right) \\ &= \varphi(f(t))\varphi(g(t)),\end{aligned}$$

so $\varphi$ is a homomorphism.

The map $\sum a_i t^i \mapsto \sum a_i (t-a)^i$ is an inverse for $\varphi$, and by the same argument, it is also a homomorphism, so $\varphi$ is an isomorphism.

---

**35.18.4** Let $M$ be a simple $R$-module, i.e., $M$ has no proper nonzero submodules. Prove that $\mathrm{End}_R(M)$ is a division ring.

**Solution** Let $\varphi \in \mathrm{End}_R(M)$ be non-zero. Notice that $N \coloneqq \ker \varphi$ is an $R$-submodule of $M$:

Let $r \in R$ and $a, b \in N$. Then $\varphi(ra) = r\varphi(a) = 0$, and $\varphi(a+b) = \varphi(a) + \varphi(b) = 0$, so $ra \in \ker \varphi$ and $a + b \in \ker \varphi$, so $\ker \varphi$ is an $R$-submodule of $M$.

Since $M$ is simple, $N = M$ or $N = \{0\}$. Since we assumed that $\varphi$ is non-zero, $N \neq M$, so $N = \{0\}$, i.e., it's an injection. Thus, it has a left-inverse $\varphi^{-1}\colon \operatorname{im} \varphi \to M$, so that $\varphi^{-1} \circ \varphi = \mathrm{id}_M \in \mathrm{End}_R(M)$, so $\mathrm{End}_R(M)$ is a division ring.

**35.18.7** Let $m$ be a positive integer. Determine the abelian groups ($\mathbb{Z}$-modules) $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ and $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$ up to isomorphism.

**Solution** Recall that a $\mathbb{Z}$-homomorphism $\varphi$ is $\mathbb{Z}$-linear: If $n \in \mathbb{Z}$ and $x, y \in M$, then $\varphi(rx + y) = r\varphi(x) + \varphi(y)$.

$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$:

Notice that for any $k \in \mathbb{Z}$, a homomorphism $\varphi \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ satisfies $\varphi(k) = k\varphi(1)$, so $\varphi$ is completely determined by the image of $\varphi(1)$. We have $m$ choices, so our possible homomorphisms are given by $\varphi_k$, where $\varphi_k(1) = k$, for $0 \leq k \leq m - 1$.

Hence, given $0 \leq k \leq \ell \leq m - 1$, $\varphi_k(1) + \varphi_\ell(1) = k + \ell = \varphi_{k+\ell}(1)$, so addition of the functions is the same as addition in $\mathbb{Z}/m\mathbb{Z}$. There is an obvious bijection from our homomorphisms to $\mathbb{Z}/m\mathbb{Z}$. Thus, $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \simeq \mathbb{Z}/m\mathbb{Z}$.

$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$:

As in the above, a homomorphism $\varphi$ satisfies $\varphi(k) = k\varphi(1)$ for $0 \leq k \leq m - 1$. However, we also need that $0 = \varphi(0) = \varphi(m) = m\varphi(1)$, which means that $\varphi(1) = 0$. Thus, the only homomorphism in this set is the 0 function, i.e., $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \simeq \{e\}$, the trivial group.

---

**35.18.8** Let $m$ and $n$ be positive integers with greatest common divisor $d$. Show $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/d\mathbb{Z}$.

**Solution** We need to determine what values $\varphi(1)$ can be, if $\varphi$ is a non-trivial $\mathbb{Z}$-module homomorphism from $\mathbb{Z}/m\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$.

1 has order $m$ in $\mathbb{Z}/m\mathbb{Z}$, so we need $m\varphi(1) = \varphi(m) = 0$. Thus, there exists $k \in \mathbb{Z}$ so that $nk = m\varphi(1)$. By assumption, there exist $x, y \in \mathbb{Z}$ so that $n = xd$ and $m = yd$, so we can write $kxd = yd\varphi(1)$. Notice that $\gcd(x, y) = 1$, or else we can absorb this common factor into $d$ and create a larger divisor, which can't happen.

Since $\mathbb{Z}$ is a domain, $kx = y\varphi(1)$. Because $y \nmid x$, we need $\varphi(1) \mid x$. There are $d$ choices: $x, 2x, \ldots, dx = n$.

If we set $\varphi_\ell(1) = \ell x$, then it is easy to see that $\varphi_\ell(1) + \varphi_k(1) = (\ell + k)x = \varphi_{\ell+k}(1)$, so pointwise addition is analogous to addition in $\mathbb{Z}/d\mathbb{Z}$, i.e., $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/d\mathbb{Z}$.

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**35.18.10** Let $M_i$, $i \in I$, be $R$-modules. Show that the (external) direct product $\prod_I M_i$ satisfies the following universal property relative to the $R$-homomorphisms $\pi_j \colon \prod_I M_i \to M_j$ defined by $\{m_i\}_I \mapsto m_j$ for all $j \in I$: Given an $R$-module $M$ and $R$-homomorphism $g_j \colon M \to M_j$ for all $j \in I$, there exists a unique $R$-homomorphism $h \colon M \to \prod_I M_i$ satisfying $g_j = \pi_j \circ h$ for all $j \in I$.

**Solution** Let $h$ be such a homomorphism, and let $m \in M$. Then

$$g_j(m) = (\pi_j \circ h)(m) = (h(m))_j \implies h(m) = \prod_I g_j(m),$$

and this shows uniqueness. To show existence, we just need to show that this is a homomorphism, but this is clear, since each $g_j$ is an $R$-homomorphism.

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**35.18.11** Write the Third Isomorphism Theorem using exact sequences.

**Solution** Let $\overline{\cdot}$ denote the the usual homomorphisms.

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\overline{\cdot}"] & M \arrow[r, "\overline{\cdot}"] \arrow[d, "\overline{\cdot}"] & M/N \arrow[r] \arrow[d, dashed, "\simeq"] & 0 \\
0 \arrow[r] & N/A \arrow[r] & M/A \arrow[r, "\overline{\cdot}"] & (M/A)/(N/A) \arrow[r] & 0
\end{tikzcd}$$

**35.18.12** Prove the Five Lemma.

**Solution** We have proved the full version of the Five Lemma below.

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**35.18.13** Let

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' & \end{tikzcd}$$

be a commutative diagram of $R$-modules and $R$-homomorphisms with exact rows. Prove that

$$\partial\colon \ker\gamma \to \operatorname{coker}\alpha \quad \text{defined by} \quad a \mapsto f'^{-1}\beta g^{-1}a + \operatorname{im}\alpha$$

is a well-defined $R$-homomorphism and that the following sequence is exact:

$$\ker\alpha \xrightarrow{f|_{\ker\alpha}} \ker\beta \xrightarrow{g|_{\ker\beta}} \ker\gamma \xrightarrow{\partial} \operatorname{coker}\alpha \xrightarrow{\overline{f}} \operatorname{coker}\beta \xrightarrow{\overline{g}} \operatorname{coker}\gamma.$$

Moreoever, in this diagram show that $f|_{\ker\alpha}$ is a monomorphism if $f$ is a monomorphism and $\overline{g}$ is an epimorphism if $g'$ is an epimorphism.

**Solution** Well-definedness:

First, $f'^{-1}\beta g^{-1}a$ is non-empty. Since $g$ is an epimorphism, there exists $b \in B$ so that $g(b) = a$. By commutativity, $g'\beta(b) = \gamma g(b) = \gamma(a) = 0$, so by exactness, there exists $a' \in A'$ with $f'(a') = \beta b$. $f'$ is a monomorphism, by exactness, since such an $a'$ is unique.

To finish, we just need to show that $x, y \in f'^{-1}\beta g^{-1}a$, then $x - y \in \operatorname{im}\alpha$. By assumption, there exist $n, m \in g^{-1}a$ so that $\beta n = f'(x)$ and $\beta m = f'(y)$. Then $g(n - m) = g(n) - g(m) = a - a = 0$, so $n - m \in \ker g = \operatorname{im} f$. Thus, there exists $\ell \in A$ so that $f(\ell) = n - m$. Then by commutativity of the diagram and the fact that $f'$ is mono,

$$f'\alpha(\ell) = \beta f(\ell) = \beta(n - m) = f'(x - y) \implies \alpha(\ell) = x - y \implies x - y \in \operatorname{im}\alpha$$

as required.

Exactness:

We will now show that the given sequence is exact.

$\ker\beta$:

This part is well-defined by commutativity of the diagram: $\beta f(a) = f'\alpha(a) = f'(0) = 0$.

Then $b \in \operatorname{im} f|_{\ker\alpha} \iff \exists\, a \in \ker\alpha$ s.t. $f(a) = b \iff gf(a) = g(b) = 0 \iff b \in \ker g|_{\ker\beta}$, since the first row is exact.

$\ker\gamma$:

Let $c \in \operatorname{im} g|_{\ker\beta}$. Then there exists $b \in \ker\beta$ so that $g(b) = c$. Then $\partial c = f'^{-1}\beta g^{-1}gb = f'^{-1}\beta b = 0$, so $c \in \ker\partial$.

Conversely, let $c \in \ker\partial$, so that $f'^{-1}\beta g^{-1}c \in \operatorname{im}\alpha$. Thus, there exists $a \in A$ so that $\alpha(a) = f'^{-1}\beta g^{-1}c$. By commutativity, $\beta f(a) = f'\alpha(a) = f'f'^{-1}\beta g^{-1}c = \beta g^{-1}c$, so $\beta(f(a) - g^{-1}(c)) = 0$, which gives $g^{-1}(c) - f(a) := b \in \ker\beta$. Then notice that if we apply $g$, we get $g(b) = c$, so $c \in \operatorname{im} g|_{\ker\beta}$.

coker $\alpha$:

Let $a' \in \operatorname{im} \partial$, so that there exists $a \in \ker \gamma$ with $a' = f'^{-1}\beta g^{-1} a + \operatorname{im} \alpha$. Then

$$\overline{f}(a') = f' f'^{-1} \beta g^{-1} a + \ker \beta = \beta g^{-1} a + \operatorname{im} \beta = \operatorname{im} \beta,$$

so $a' \in \ker \overline{f}$.

Now let $a' \in \ker \overline{f}$. Then $\overline{f}(a') \in \operatorname{im} \beta$, so there exists $b \in B$ such that $f'(a') = \beta b$. Then $a' = f'^{-1}\beta b$. By commutativity, exactness, and the fact that $f'$ is a monomorphism

$$g'\beta(b) = \gamma g(b) \implies 0 = f'\gamma g(b) \implies \gamma g(b) = 0 \implies g(b) \in \ker \gamma.$$

Thus, there exists $a \in \ker \gamma$ with $g(b) = a$. Hence,

$$a' = f^{-1}\beta b = f^{-1}\beta g^{-1} a \implies a' \in \operatorname{im} \partial.$$

coker $\beta$:

Let $b' + \operatorname{im} \beta \in \operatorname{im} \overline{f}$. Then there exists $a' + \operatorname{im} \alpha \in \operatorname{coker} \alpha$ so that $\overline{f}(a' + \operatorname{im} \alpha) = b' + \operatorname{im} \beta$. Then

$$\overline{g}(b' + \operatorname{im} \beta) = g'(\overline{f}(a' + \operatorname{im} \alpha)) + \operatorname{im} \gamma = g' f'(a') + \operatorname{im} \gamma = \operatorname{im} \gamma \implies b' + \operatorname{im} \beta \in \ker \overline{g},$$

by exactness.

On the other hand, suppose $b' + \operatorname{im} \beta \in \ker \overline{g}$. Then

$$\overline{g}(b' + \operatorname{im} \beta) = g'(b') + \operatorname{im} \gamma = \operatorname{im} \gamma \implies g'(b') \in \operatorname{im} \gamma,$$

so there exists $c \in C$ such that $\gamma(c) = g'(b')$. Since $g$ is epi (by exactness) there exists $b \in B$ so that $g(b) = c$. By commutativity, exactness in the bottom row, and the fact that $f'$ is mono,

$$\gamma g(b) = g'\beta(b) \implies f'\gamma g(b) = 0 \implies \gamma g(b) = 0 \implies \gamma(c) = 0 \implies c \in \ker \gamma.$$

Thus, $g'(b') = 0$, so $b' \in \ker g' = \operatorname{im} f'$, which means that there exists $a' \in A'$ such that $f'(a') = b'$. Hence,

$$\overline{f}(a' + \operatorname{im} \alpha) = f'(a') + \operatorname{im} \beta = b' + \operatorname{im} \beta \implies b' + \operatorname{im} \beta \in \operatorname{im} \overline{f},$$

as required.

$f|_{\ker \alpha}$ monomorphism $\implies$ $f$ is a monomorphism:

Let $a, a'' \in A$ so that $f(a) = f(a'')$. By commutativity,

$$\beta f(a) = f'\alpha(a) = f'\alpha(a'') = \beta f(a'').$$

Since $f'$ was a monomorphism, $\alpha(a) = \alpha(a'') \implies a - a'' \in \ker \alpha$. Since $f|_{\ker \alpha}$ is a monomorphism,

$$f(a - a'') = f|_{\ker \alpha}(a - a'') = 0 \implies a = a'',$$

so $f$ is mono.

$g'$ is an epimorphism $\implies$ $\overline{g}$ is an epimorphism:

Let $c' + \operatorname{im} \gamma \in \operatorname{coker} \gamma$. Since $g'$ is epi, there exists $b' \in B'$ so that $g'(b') = c'$. Then

$$\overline{g}(b' + \operatorname{im} \beta) = g'(b') + \operatorname{im} \gamma = c' + \operatorname{im} \gamma,$$

so $\overline{g}$ is epi.

**35.18.14** Prove the full version of the Five Lemma: Suppose that

$$\begin{array}{ccccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & D & \xrightarrow{\delta} & E \\ {\scriptstyle h_A}\downarrow & & {\scriptstyle h_B}\downarrow & & {\scriptstyle h_C}\downarrow & & {\scriptstyle h_D}\downarrow & & {\scriptstyle h_E}\downarrow \\ A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \xrightarrow{\gamma'} & D' & \xrightarrow{\delta'} & E' \end{array}$$

is a commutative diagram of $R$-modules and $R$-homomorphism with exact rows.

a. if $h_A$ is an epimorphism and $h_B$ and $h_D$ are monomorphisms, then $h_C$ is a monomorphism.

b. If $h_E$ is a monomorphism and $h_B$ and $h_D$ are epimorphisms, then $h_C$ is an epimorphism.

**Solution** a. Let $c_1, c_2 \in C$ such that $h_C(c_1) = h_C(c_2)$. Then $h_C(c_1 - c_2) = 0$, so $\gamma'(h_C(c_1 - c_2)) = 0$. Thus, by commutativity, $0 = h_D(\gamma(c_1 - c_2))$. Since $h_D$ was a monomorphism, this tells us that $\gamma(c_1 - c_2) = 0$, so $c_1 - c_2 \in \ker\gamma = \operatorname{im}\beta$, so there exists $b \in B$ such that $\beta(b) = c_1 - c_2$.

Then by commutativity, $\beta'(h_B(b)) = h_C(\beta(b)) = h_C(c_1 - c_2) = 0$, so $h_B(b) \in \ker\beta' = \operatorname{im}\alpha'$, so there exists $a' \in A'$ such that $\alpha'(a') = h_B(b)$. Since $h_A$ was an epimorphism, there exists $a \in A$ so that $h_A(a) = a'$. By commutativity,

$$h_B(\alpha(a)) = \alpha'(h_A(a)) = \alpha'(a') = h_B(b),$$

so because $h_B$ was mono, $b = \alpha(a)$, so $b \in \operatorname{im}\alpha = \ker\beta$. Hence, $0 = \beta(b) = c_1 - c_2 \implies c_1 = c_2$, so $h_C$ is mono.

b. Let $c' \in C'$, and let $d' = \gamma'(c')$. Since $h_D$ is epi, there exists $d \in D$ so that $h_D(d) = d'$. By commutativity and exactness,

$$h_E(\delta(d)) = \delta'(h_D(d)) = \delta'(\gamma'(c')) = 0.$$

Because $h_E$ is mono, this means that $\delta(d) = 0$, so $d \in \ker\delta = \operatorname{im}\gamma$, which means there exists $c \in C$ so that $\gamma(c) = d$. By commutativity,

$$\gamma'(h_C(c)) = h_D(\gamma(c)) = h_D(d) = d' = \gamma'(c') \implies \gamma'(h_C(c) - c') \in \ker\gamma' = \operatorname{im}\beta'.$$

So there exists $b' \in B'$ with $\beta'(b') = h_C(c) - c'$. Since $h_B$ was epi, there exists $b \in B$ so that $h_B(b) = b'$. By commutativity,

$$h_C(c) - c' = \beta'(b') = \beta'(h_B(b)) = h_C(\beta(b)) \implies c' = h_C(c - \beta(b)),$$

so $h_C$ is epi.

**35.18.16** Let $R$ be a commutative ring and $M, N$ $R$-modules. Recall that $\mathrm{Hom}_R(M,N)$ is an $R$-module. If $h\colon A \to B$ is an $R$-homomorphism of $R$-modules, define

$$\begin{aligned} h_*\colon\ &\mathrm{Hom}_R(N,A) \to \mathrm{Hom}_R(N,B) \text{ by } f \mapsto h \circ f \text{and} \\ h^*\colon\ &\mathrm{Hom}_R(B,N) \to \mathrm{Hom}_R(A,N) \text{ by } f \mapsto f \circ h. \end{aligned}$$

Show that these are $R$-homomorphisms and if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence, then

$$0 \longrightarrow \mathrm{Hom}_R(N,A) \xrightarrow{f_*} \mathrm{Hom}_R(N,B) \xrightarrow{g_*} \mathrm{Hom}_R(N,C) \qquad \text{and}$$

$$0 \longrightarrow \mathrm{Hom}_R(C,N) \xrightarrow{g^*} \mathrm{Hom}_R(B,N) \xrightarrow{f^*} \mathrm{Hom}_R(A,N)$$

are exact.

**Solution** Exactness at $\mathrm{Hom}_R(N,A)$:

First, notice that $\ker f_* = \{h \in \mathrm{Hom}_R(N,A) \mid f \circ h = 0\}$. Since the given sequence is exact, $\ker f = \{0\}$, the only element in the kernel must be the 0 homomprhism, so the sequence is exact at $\mathrm{Hom}_R(N,A)$.

Exactness at $\mathrm{Hom}_R(N,B)$:

Now let $h \in \operatorname{im} f_*$, so that there exists $h' \in \mathrm{Hom}_R(N,A)$ with $f \circ h' = f_*(h') = h$. Then

$$g_*(h) = g \circ f \circ h' = 0,$$

since $\operatorname{im} f = \ker g$, so $\operatorname{im} f_* \subseteq \ker g_*$. On the other hand, let $h \notin \ker g_*$, so that there exists $a \in A$ so that $(g \circ h)(a) \neq 0$. Then $h(a) \notin \ker g$, so $h(a) \notin \operatorname{im} f$, so there cannot be $h'$ with $f \circ h' = h$. Thus, $\operatorname{im} f_* \supseteq \ker g_*$, so the diagram is exact here.

Exactness at $\mathrm{Hom}_R(C,N)$:

Note that $\ker g^* = \{h \in \mathrm{Hom}_R(C,N) \mid h \circ g = 0\}$. By exactness, $g(B) = C$, so an element in the kernel must satisfy $(h \circ g)(B) = h(C) = \{0\}$, so the only element in the kernel is the 0 homomorphism. Thus, we have exactness here.

Exactness at $\mathrm{Hom}_R(B,N)$:

Let $h \in \operatorname{im} g^*$. Then there exists $h' \in \mathrm{Hom}_R(C,N)$ so that $h = h' \circ g$. Then

$$f^*(h) = h \circ f = h' \circ g \circ f = h' \circ 0 = 0,$$

by exactness, so $\operatorname{im} g^* \subseteq \ker f^*$.

Now let $h \notin \ker f^*$, so that there exists $a \in A$ so that $(h \circ f)(a) \neq 0$. So, there cannot exist $h' \in \mathrm{Hom}_R(C,N)$ so that $h = h' \circ g$. Otherwise,

$$0 \neq (h \circ f)(a) = (h' \circ f \circ g)(a) = 0,$$

which is impossible, so $\operatorname{im} g^* = \ker f^*$, which shows exactness here.

**35.18.17** Let $Q$ be an $R$-module. Then $Q$ is called *$R$-injective* if given any $R$ monomorphism, $f\colon A \to B$ and $R$-homomorphism $g\colon A \to Q$, there exists an $R$-homomorphism $h\colon B \to Q$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, "h"] \\ Q & \end{tikzcd}$$

commutes. Show that $Q$ is an $R$-injective if and only if, whenever

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence of $R$-modules and $R$-homomorphisms, then

$$0 \longrightarrow \mathrm{Hom}_R(C, Q) \xrightarrow{g*} \mathrm{Hom}_R(B, Q) \xrightarrow{f*} \mathrm{Hom}_R(A, Q) \longrightarrow 0$$

is exact.

**Solution** "$\Longrightarrow$"

Let $Q$ be $R$-injective.

By a previous problem, we already know that the sequence is exact at the first two modules, so we just need to show the last one is exact. It suffices to show that $f^*$ is epi.

Let $h \in \mathrm{Hom}_R(A, Q)$. Since $Q$ is $R$-injective and $f$ is a monomorphism by exactness, there exists an $R$-homomorphism $h' \in \mathrm{Hom}_R(B, Q)$ so that $h = h' \circ f = f^*(h')$, by commutativity of the diagram, so $f^*$ is epi. Thus, the sequence is exact.

"$\Longleftarrow$"

Suppose that the two sequences are exact. Then by assumption, $f^*$ is an epimorphism, so for any $R$-homomorphism $g \in \mathrm{Hom}_R(A, Q)$, there exists $h \in \mathrm{Hom}_R(B, Q)$ so that $f^*(h) = h \circ f = g$, so the diagram commutes.

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**35.18.18** Let $Q$ be an $R$-module. Show that $Q$ is an $R$-injective if and only if given any ideal $\mathfrak{A}$ in $R$ and an $R$-homomorphism $g\colon \mathfrak{A} \to Q$, there exists an $R$-homomorphism $h\colon R \to Q$ such that the diagram

$$\begin{tikzcd} \mathfrak{A} \arrow[r, hook, "\mathrm{inc}"] \arrow[d, "g"'] & R \arrow[dl, "h"] \\ Q & \end{tikzcd}$$

commutes.

**Solution** "$\Longrightarrow$"

Let $Q$ be an $R$-injective. The inclusion map is mono and $g$ is an $R$-homomorphism, so by the previous problem, there exists an $R$-homomorphism $h$ so that the given diagram commutes.

"$\Longleftarrow$"

Let $A$ be an $R$-submodule and let $g$ be a homomorphism as given in the problem.

Consider the poset $S$ with elements of the form $\{B, f\}$, where $B \subseteq A$ is an $R$-module containing $A$ and $f$ is a homomorphism so that the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, hook, "\mathrm{inc}"] \arrow[d, "g"'] & R \arrow[dl, "f"] \\ Q & \end{tikzcd}$$

$S \neq \emptyset$, since it contains $((0), 0)$, where 0 is the 0 function.

Define the partial order of $S$ so that

$$(B, f) \preceq (C, \tilde{f}) \iff B \subseteq C \quad \text{and} \quad f = \tilde{f}\Big|_B .$$

It's easy that this is a partial ordering.

Now let $\mathcal{C} = \{(B_\alpha, f_\alpha)\}_\alpha \subseteq S$ be a chain. Set $B = \bigcup_{\alpha \in A} B_\alpha$ and define $f(a)$ if $a \in B_\alpha$. This is well-defined: Suppose there exists $a$ and $\alpha, \beta \in A$ with $a \in B_\alpha \cap B_\beta$. Since $\mathcal{C}$ is totally ordered, $B_\alpha \subseteq B_\beta$ or $B_\beta \subseteq B_\alpha$. In either case, $f_\alpha(a) = f_\beta(a)$, so there is no issue with the definition of $f$.

$f$ is a homomorphism: if $a, b \in B$ and $r \in R$, then there exist $\alpha, \beta \in A$ such that $a \in B_\alpha$ and $b \in B_\beta$. Again, because $\mathcal{C}$ is totally ordered, $B_\alpha \subseteq B_\beta$ or $B_\beta \subseteq B_\alpha$. So, without loss of generality, assume that $a, b \in B_\alpha$. Since $f_\alpha$ is a homomorphism, we have that $f(a+b) = f_\alpha(a+b) = f_\alpha(a) + f_\alpha(b) = f(a) + f(b)$. Similarly, $f(ra) = f_\alpha(ra) = rf_\alpha(a) = rf(a)$, so $f$ is a homomorphism.

Lastly, for any $a \in A$, there exists $\alpha \in A$ so that $a \in A_\alpha$, so because the diagram commutes for $f_\alpha$, we have $f(a) = f_\alpha(a) = g(a)$, so the diagram for $f$ commutes as well.

This shows that $\{B, f\}$ is an upper bound for the chain.

By Zorn's lemma, there exists a maximal element $\{B, h\}$, with respect to $\subseteq$. We claim that $B = A$.

Suppose otherwise, and assume that there exists $x \notin B$. Then consider $(B \colon x) \coloneqq \{r \in R \mid rx \in B\}$. This is an ideal: If $r, s \in (B : x)$, then $(r+s)x = rx + rs \in B$, since $B$ is an $R$-submodule. Similarly, for any $t \in R$, $(tr)x \in B$ also.

Define $g \colon (B : x) \to Q$ via $g(r) = h(rx)$. This is a homomorphism, so by assumption, there exists an $R$-homomorphism $\tilde{g}$ so that

$$\begin{tikzcd}
(B \colon x) \arrow[r, hook, "\text{inc}"] \arrow[d, "g"'] & R \arrow[dl, "\tilde{g}"] \\
Q &
\end{tikzcd}$$

commutes.

Now define $\tilde{h}$ on $B + Rx$ via $\tilde{h}(b + rx) = h(b) + \tilde{g}(r)$. If $b = rx \in B \cap Rx$ then $h(b) = h(rx)$ and $\tilde{g}(r) = h(rx)$ by commutativity of diagrams, so $h(b) = \tilde{g}(rx)$. Thus, $\tilde{h}$ is a well-defined homomorphism. But this means that $(B, h) \preceq (B + Rx, \tilde{h})$, without equality, which contradicts the maximality of $(B, h)$. Hence, $B = A$, and the proposition holds.

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**35.18.19** Show that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

**Solution** Let $\mathfrak{A}$ be an ideal in $\mathbb{Z}$. Since $\mathbb{Z}$ is a Euclidean domain, it is a PID, so all ideals are of the form $(n) = n\mathbb{Z}$, for some $n \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$ and $g \colon n\mathbb{Z} \to \mathbb{Q}$ be a $\mathbb{Z}$-homomorphism. We wish to find a $\mathbb{Z}$-homomorphism $h \colon \mathbb{Z} \to \mathbb{Q}$ so that the following diagram commutes:

$$\begin{tikzcd}
n\mathbb{Z} \arrow[r, hook, "\text{inc}"] \arrow[d, "g"'] & \mathbb{Z} \arrow[dl, dashed, "h"] \\
\mathbb{Q} &
\end{tikzcd}$$

For $m \in \mathbb{Z}$, set $h(m) = g(nm)/n$. This is a homomorphism:

Let $a, b \in \mathbb{Z}$. Then $h(a+b) = g(n(a+b))/n = g(na+nb)/n = g(na)/n + g(nb)/n = h(a) + h(b)$, since $g$ is a $\mathbb{Z}$-homomorphism. Similarly, for $r \in \mathbb{Z}$, $h(ra) = g(r(na))/n = rg(ra)/n = rh(a)$, so $h$ is a $\mathbb{Z}$-homomorphism.

Now, if $na \in n\mathbb{Z}$, then

$$h(na) = g(n(na))/n = ng(na)/n = g(na),$$

so the diagram commutes. By the previous problem, it follows that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.