1 Let $\{a_n\}_{n\geq 1}$ be a Cauchy sequence of real numbers. Show that $\{a_n^2\}_{n\geq 1}$ is also a Cauchy sequence.

Solution Since $\{a_n\}n \geq 1$ is Cauchy, it is bounded, meaning there exists a positive M such that $|a_n| < M$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$. As $\{a_n\}_{n\geq 1}$ is Cauchy, $\exists N \in \mathbb{N}$ such that for all $n, m \geq N$, we have $|a_n - a_m| < \frac{\epsilon}{2M}$ and $a_n, a_m \leq M$.

$$|a_n^2 - a_m^2| = |a_n - a_m||a_n + a_m| \leq |a_n - a_m|(|a_n| + |a_m|) < \frac{\epsilon}{2M}(2M) = \epsilon$$

Thus, by definition, $\{a_n^2\}_{n\geq 1}$ is Cauchy.

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2 Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence defined by the following rule:

$$a_1 = 3 \quad \text{and} \quad a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \quad \text{for all} \quad n \geq 1.$$

a. Show that the sequence is bounded below.

b. Show that this is a sequence of rational numbers.

c. Prove that the sequence is monotonically decreasing.

d. Deduce that $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

Solution a. $(x - \sqrt{2})^2 \geq 0 \implies x^2 - 2\sqrt{2}x + 2 \geq 0 \implies \frac{x^2+2}{2x} \geq \sqrt{2}\ \forall x > 0$.

Since $a_n > 0$, $a_{n+1} = \frac{a_n^2+2}{2a_n} \geq \sqrt{2}$. Thus, for all $n \geq 2$, $a_n \geq \sqrt{2}$. If $n = 1$, we have $a_1 = 3 = \sqrt{9} > \sqrt{2}$, so $a_n \geq \sqrt{2}$ for all $n \geq 1$. Thus $\sqrt{2}$ is a lower bound.

b. We will prove this by induction.

Base step:

$a_1 = 3 \in \mathbb{Q}$

Inductive step:

Suppose $a_n \in \mathbb{Q} \implies \frac{1}{a_n} \in \mathbb{Q}$. Then since $\mathbb{Q}$ is a field, it is closed under addition and scalar multiplication. It follows that $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \in \mathbb{Q}$.

By the principle of induction, the sequence is one of rational numbers.

c. Since a_n is rational, it can never be equal to $\sqrt{2}$. Thus, we now have a strict inequality: $a_n > \sqrt{2}\ \forall n \geq 1$. We wish to show that $\frac{1}{a_n} < \frac{a_n}{2}$. Then the result follows immediately. $a_n > \sqrt{2} > 0 \implies a_n^2 > a_n\sqrt{2} > 2$. Also, $\frac{1}{a_n} > 0$, so dividing through by $2a_n$ yields $\frac{a_n}{2} > \frac{1}{a_n}$. Thus, $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} < \frac{a_n}{2} + \frac{a_n}{2} = a_n$. The inequality holds for all $n \geq 1$, so the sequence is monotonically decreasing.

d. The sequence is bounded below and monotonically decreasing, so by a theorem, the sequence converges to $\inf\{a_n \mid n \geq 1\}$. We will now find its limit, a. $a \geq \sqrt{2} > 0$ since $\sqrt{2}$ is a lower bound for a_n.

First, we show that $\lim_{n\to\infty} \frac{a_n^2+2}{2a_n} = \frac{a^2+2}{2a}$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|a_n - a| < \frac{\epsilon}{\sqrt{2}}$$

Then

$$\begin{aligned}
\left|\frac{a_n^2+2}{2a_n} - \frac{a^2+2}{2a}\right| &= \left|\frac{a_n^2a + 2a - a^2a_n - 2a_n}{2a_na}\right| \\
&= \left|\frac{a_na(a_n - a) + 2(a - a_n)}{2a_na}\right| \\
&< \left|\frac{a_n - a}{2}\right| + \left|\frac{a_n - a}{\sqrt{2}a}\right| < \frac{\epsilon}{2\sqrt{2}} + \frac{\epsilon}{\sqrt{2}a} < \frac{\epsilon}{2\sqrt{2}} + \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

Using this result, we have

$$\begin{aligned}\lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \frac{a_n^2+2}{2a_n}\\ a &= \frac{a^2+2}{2a}\\ a^2-2 &= 0\\ a &= \sqrt{2}\end{aligned}$$

a cannot be $-\sqrt{2}$ since we determined that $a > 0$. Thus, the limit of $\{a_n\}_{n\geq 1}$ is $\sqrt{2}$.

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3 Consider the following sequence:

$$a_1 = \sqrt{2} \quad \text{and} \quad a_{n+1} = \sqrt{2+a_n} \quad \text{for all} \quad n \geq 1.$$

a. Show that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded above.

b. Prove that the sequence is monotonically increasing.

c. Deduce that $\{a_n\}_{n\in\mathbb{N}}$ and find its limit.

Solution a. We will prove by induction that the sequence is bounded above by 2.

Base step:

$a_1 = \sqrt{2} < \sqrt{4} = 2$

Inductive step:

Suppose $a_n < 2$. We wish to show that $a_{n+1} < 2$.

$$a_{n+1} = \sqrt{2+a_n} < \sqrt{2+2} = 2$$

Thus, a_{n+1} is bounded above by 2.

By the principle of induction, the sequence is bounded above by 2.

b. We will now prove by induction that the sequence is increasing monotonically.

Base step:

$a_1 = \sqrt{2} < \sqrt{2+\sqrt{2}} = a_2$

Inductive step:

Suppose $a_n < a_{n+1}$. We wish to show that $a_{n+1} < a_{n+2}$.

$$a_{n+2} = \sqrt{2+a_{n+1}} > \sqrt{2+a_n} = a_{n+1}$$

Thus, the inductive step holds.

If we take both steps, then by the principle of induction, the inequality holds for all $n \geq 1$. So, the sequence is increasing monotonically.

c. The sequence is monotonically increasing and it is bounded above, so by a theorem, the sequence must converge. We will now find the limit, a, of the sequence.

We will first show that $\lim_{n\to\infty} \sqrt{2+a_n} = \sqrt{2+a}$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - a| < \epsilon\sqrt{2+a}$. Then

$$\left|\sqrt{2+a_n} - \sqrt{2+a}\right| = \left|\frac{a_n - a}{\sqrt{2+a_n}+\sqrt{2+a}}\right| < \frac{|a_n - a|}{\sqrt{2+a}} < \frac{\epsilon\sqrt{2+a}}{\sqrt{2+a}} = \epsilon$$

Thus, $\lim_{n\to\infty} \sqrt{2+a_n} = \sqrt{2+a}$. Then

$$\begin{aligned}\lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \sqrt{2+a_n}\\ a &= \sqrt{2+a}\\ a^2 &= 2+a\\ (a-2)(a+1) &= 0\end{aligned}$$

Every term of the sequence is greater than 0 since it is a sequence of real square roots, $a \neq -1$, so $a = 2$. Thus, $\lim_{n\to\infty} a_n = 2$.

4 Let a_1, b_1 be two real numbers such that $0 < a_1 < b_1$. For $n \geq 1$, we define

$$a_{n+1} = \sqrt{a_n b_n} \quad \text{and} \quad b_{n+1} = \frac{a_n + b_n}{2}.$$

a. Prove that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is monotonically increasing and that the sequence $\{b_n\}_{n\in\mathbb{N}}$ is monotonically decreasing.

b. Show that the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are bounded.

c. Deduce that the two sequences converge and prove that they converge to the same limit.

Solution a. We will prove these by induction. We first prove that $b_{n+1} \geq a_{n+1}$ for all $n \geq 1$.

$$\begin{aligned}
(\sqrt{a_n} - \sqrt{b_n})^2 &\geq 0 \\
a_n - 2\sqrt{a_n b_n} + b_n &\geq 0 \\
\frac{a_n + b_n}{2} &\geq \sqrt{a_n b_n} \\
b_{n+1} &\geq a_{n+1}
\end{aligned}$$

We only have equality when $a_n = b_n$.
We start with $\{a_n\}_{n\in\mathbb{N}}$.
Base step:
$a_2 = \sqrt{a_1 b_1} > \sqrt{a_1 a_1} = a_1$
Inductive step:
Suppose $a_{n+1} > a_n$. Then $\sqrt{a_n b_n} > a_n \implies b_n > a_n \implies b_{n+1} > a_{n+1}$. We wish to show that $a_{n+2} > a_{n+1}$.

$$a_{n+2} = \sqrt{a_{n+1} b_{n+1}} > \sqrt{a_{n+1} a_{n+1}} = a_{n+1}$$

Thus the inductive step holds.
Both steps hold, so $\{a_n\}_{n\in\mathbb{N}}$ increases monotonically for $n \geq 1$.
We now prove that $\{b_n\}_{n\in\mathbb{N}}$ decreases monotonically through induction.
Base step:
$b_2 = \frac{a_1+b_1}{2} = b_1 - \frac{b_1-a_1}{2} < b_1$ since $b_1 > a_1$.
Inductive step:
Suppose $b_{n+1} < b_n \implies a_n + b_n < 2b_n \implies b_n > a_n \implies b_{n+1} > a_{n+1}$. We wish to show that $b_{n+2} < b_{n+1}$. Since $\{a_n\}_{n\in\mathbb{N}}$ increases monotonically, $a_{n+1} > a_n$.

$$b_{n+2} = \frac{a_{n+1} + b_{n+1}}{2} < \frac{b_{n+1} + b_{n+1}}{2} = b_{n+1}$$

Thus, the inductive step holds.
By induction, $\{b_n\}_{n\in\mathbb{N}}$ decreases monotonically.

b. Since $\{a_n\}_{n\in\mathbb{N}}$ is monotonically increasing, we have that $a_1 < a_n$ for all $n \geq 1$. Similarly, since $\{b_n\}_{n\in\mathbb{N}}$ is monotonically decreasing, $b_1 > b_n$ for all $n \geq 1$. By the inequality shown in the beginning of part (a), we have

$$b_1 > b_{n+1} \geq a_{n+1} > a_1$$

for all $n \geq 1$. Thus, $\{a_n\}_{n\in\mathbb{N}}$ is bounded above by b_1 and $\{b_n\}_{n\in\mathbb{N}}$ is bounded below by a_1. By a theorem, since $\{a_n\}_{n\in\mathbb{N}}$ increases monotonically and is bounded above, it must converge. Similarly, since $\{b_n\}_{n\in\mathbb{N}}$ decreases monotonically and is bounded below by a_1, so it must also converge.

c. Suppose $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. We wish to show that $a = b$.
Let $\epsilon > 0$. As both limits exist, there exists $N \in \mathbb{N}$ such that for all $n + 1 > n \geq N$, we have both

$$|a_n - a| < \epsilon \quad \text{and} \quad |b_n - b| < \epsilon.$$

Then

$$\left|b_{n+1} - \frac{a+b}{2}\right| = \left|\frac{a_n - a}{2} + \frac{b_n - b}{2}\right| \leq \frac{|a_n - a|}{2} + \frac{|b_n - b|}{2} < \epsilon$$

Thus b_n converges to $\frac{a+b}{2}$. But b_n converges to b also, and since the limit must be unique, we have $b = \frac{a+b}{2} \implies 2b = a + b \implies b = a$. Thus, the two limits are equal, meaning $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

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5 Let $\alpha > 1$ and define the sequence $\{x_n\}_{n\geq 1}$ of real numbers as follows:

$$x_1 > \sqrt{\alpha} \quad \text{and} \quad x_{n+1} = \frac{x_n + \alpha}{x_n + 1} \quad \text{for all} \quad n \geq 1.$$

a. Show that $\{x_{2n-1}\}_{n\geq 1}$ is decreasing and bounded below by $\sqrt{\alpha}$.

b. Show that $\{x_{2n}\}_{n\geq 1}$ is increasing and bounded above by $\sqrt{\alpha}$.

c. Show that the sequence $\{x_n\}_{n\geq 1}$ converges to $\sqrt{\alpha}$.

Solution a. First note that

$$x_{n+2} = \frac{\frac{x_n+\alpha}{x_n+1} + \alpha}{\frac{x_n+\alpha}{x_n+1} + 1} = \frac{x_n + 2\alpha + x_n\alpha}{2x_n + \alpha + 1}$$

We will now prove by induction that $\{x_{2n-1}\}_{n\geq 1}$ is decreasing.

Base step:

Since $x_1 > \sqrt{\alpha} > 1$, $x_1^2 > \alpha$, so we have

$$x_3 - x_1 = \frac{x_1 + 2\alpha + x_1\alpha}{2x_1 + \alpha + 1} - x_1 = \frac{2\alpha - 2x_1^2}{2x_1 + \alpha + 1} < 0$$

Thus $x_1 > x_3$.

Inductive step:

Suppose $x_{2n-1} > x_{2n+1}$. We wish to show that $x_{2n+3} > x_{2n+1}$.

$$x_{2n+3} - x_{2n+1} = \frac{x_{2n+1} + 2\alpha + x_{2n+1}}{2x_{2n+1} + \alpha + 1} - \frac{x_{2n-1} + 2\alpha + x_{2n-1}}{2x_{2n-1} + \alpha + 1} = \frac{(\alpha - 1)^2(x_{2n+1} - x_{2n-1})}{(2x_{2n} + \alpha + 1)(2x_{2n-2} + \alpha + 1)} < 0$$

Thus, $x_{2n+3} > x_{2n+1}$, so the inductive step holds.

We now wish to show that x_{2n-1} is bounded below by $\sqrt{\alpha}$. We will also prove this by induction.

Base step:

$x_1 > \sqrt{\alpha}$

Inductive step:

Suppose $x_{2n-1} > \sqrt{\alpha}$. We wish to show that $x_{2n+1} > \sqrt{\alpha}$.

$$\begin{aligned} x_{2n+1} - \sqrt{\alpha} = \frac{x_{2n-1} + 2\alpha + x_{2n-1}\alpha}{2x_{2n-1} + \alpha + 1} - \sqrt{\alpha} &= \frac{x_{2n-1} + 2\alpha + x_{2n-1}\alpha - 2x_{2n-1}\sqrt{\alpha} - \alpha\sqrt{\alpha} - \sqrt{\alpha}}{2x_{2n-1} + \alpha + 1} \\ &= \frac{x_{2n-1}(\alpha - 2\sqrt{\alpha} + 1) - \sqrt{\alpha}(\alpha - 2\sqrt{\alpha} + 1)}{2x_{2n-1} + \alpha + 1} \\ &= \frac{(x_{2n-1} - \sqrt{\alpha})(\sqrt{\alpha} - 1)^2}{2x_{2n-1} + \alpha + 1} > 0 \end{aligned}$$

Thus, we have $x_{2n+1} > \sqrt{\alpha}$, as desired.

By induction, $\{x_{2n-1}\}_{n\geq 1}$ is bounded below by by $\sqrt{\alpha}$.

b. We will prove this part similarly to the part above. We start by showing that the sequence is increasing.

Base step:

$x_2 = \frac{x_1+\alpha}{x_1+1}$. Then

$$\begin{aligned}
x_4 - x_2 = \frac{x_2 + 2\alpha + x_2\alpha}{2x_2+\alpha+1} - x_2 &= \frac{x_2 + 2\alpha + x_2\alpha - 2x_2^2 - x_2\alpha - x_2}{2x_2+\alpha+1} \\
&= \frac{2\alpha - 2x_2^2}{2x_2+\alpha+1} \\
&= \frac{2(\alpha - \frac{x_1^2+2x_1\alpha+\alpha^2}{x_1^2+2x_1+1})}{2x_2+\alpha+1} \\
&= \frac{2(\alpha x_1^2 + 2x_1\alpha + \alpha - x_1^2 - 2x_1\alpha - \alpha^2)}{(2x_2+\alpha+1)(x_1+1)^2} \\
&= 2\frac{x_1^2(\alpha-1) - \alpha(\alpha-1)}{(2x_2+\alpha+1)(x_1+1)^2} \\
&= 2\frac{(\alpha-1)(x_1-\sqrt{\alpha})(x_1+\sqrt{\alpha})}{(2x_2+\alpha+1)(x_1+1)^2}
\end{aligned}$$

Since $x_1 > \sqrt{\alpha}$ and $\alpha > 1$, the entire term must be greater than 0. Thus, we have $x_4 - x_2 > 0 \implies x_4 > x_2$.

Inductive step:

Suppose $x_{2n} > x_{2n-2}$. We wish to show that $x_{2n+2} > x_{2n}$.

$$x_{2n+2} - x_{2n} = \frac{x_{2n} + 2\alpha + x_{2n}\alpha}{2x_{2n}+\alpha+1} - \frac{x_{2n-2} + 2\alpha + x_{2n-2}\alpha}{2x_{2n-2}+\alpha+1} = \frac{(\alpha-1)^2(x_{2n}-x_{2n-2})}{(2x_{2n}+\alpha+1)(2x_{2n-2}+\alpha+1)} > 0$$

Thus, the inductive step holds.

By induction, we have $x_{2n+2} > x_{2n}$ for all $n \geq 1$, so the sequence $\{x_{2n}\}_{x\geq 1}$ is increasing. We will now prove by induction that it is bounded above by $\sqrt{2}$.

Base case:

$x_2 - \sqrt{\alpha} = \dfrac{x_1+\alpha}{x_1+1} - \sqrt{\alpha} = \dfrac{x_1 + \alpha - \sqrt{\alpha}x_1 - \sqrt{\alpha}}{x_1+1} = \dfrac{\sqrt{\alpha}(\sqrt{\alpha}-x_1) - (\sqrt{\alpha}-x_1)}{x_1+1} = \dfrac{(\sqrt{\alpha}-1)(\sqrt{\alpha}-x_1)}{x_1+1}$

$\sqrt{\alpha} > 1$ and $x_1 > \sqrt{\alpha}$, so we have $x_2 - \sqrt{\alpha} < 0 \implies x_2 < \sqrt{\alpha}$.

Inductive step:

Suppose $x_{2n} < \sqrt{\alpha}$. We wish to show that $x_{2n+2} < \sqrt{\alpha}$ also.

$$\begin{aligned}
x_{2n+2} - \sqrt{\alpha} = \frac{x_{2n} + 2\alpha + x_{2n}\alpha}{2x_{2n}+\alpha+1} - \sqrt{\alpha} &= \frac{x_{2n} + 2\alpha + x_{2n}\alpha - 2x_{2n}\sqrt{\alpha} - \alpha\sqrt{\alpha} - \sqrt{\alpha}}{2x_{2n}+\alpha+1} \\
&= \frac{-\sqrt{\alpha}(\alpha - 2\sqrt{\alpha}+1) + x_{2n}(\alpha - 2\sqrt{\alpha}+1)}{2x_{2n}+\alpha+1} \\
&= \frac{(\sqrt{\alpha}-1)^2(x_{2n}-\alpha)}{2x_{2n}+\alpha+1}
\end{aligned}$$

Since $x_{2n} < \sqrt{\alpha}$, we have $x_{2n} - \sqrt{\alpha} < 0$, so the entire term above is less than 0. Hence, $x_{2n+2} - \sqrt{\alpha} < 0 \implies x_{2n+2} < \sqrt{\alpha}$ as desired.

By the principle of mathematical induction, the sequence $\{x_{2n}\}_{n\geq 1}$ is bounded above by $\sqrt{\alpha}$.

c. Consider the set $\{x_n \mid n \geq N\}$. If N is odd, then $x_N \in \{x_{2n-1} \mid n \geq 1\}$, so it is an upper bound for the set, since $\{x_{2n-1} \mid n \geq 1\}$ is decreasing and $x_{2n} < \sqrt{\alpha} < x_N$ for all n. Since it is in the set, is must be the supremum because any smaller number will be smaller than x_N, meaning that number cannot be an upper bound. Similarly, since $N+1$ is even, $x_{N+1} \in \{x_{2n} \mid n \geq 1\}$ will be a lower bound since $x_{N+1} < \sqrt{\alpha} < x_{2n-1}$ for all n. It is also the infimum for the same reasoning for when N is odd.

If N is even, then $x_{N+1} \in \{a_{2n-1} \mid n \geq 1\}$, and it will be the supremum of the set $\{x_n \mid n \geq N\}$, and $x_N \in \{x_{2n} \mid n \geq 1\}$ will be the infimum of that same set using the same reasoning as above. Thus, $\{x_{2n-1}\}_{n\geq 1}$ is the sequence of supremums, and $\{x_{2n}\}_{n\geq 1}$ is the sequence of infimums of $\{x_n \mid n \geq N\}$. So, we have $\lim_{n\to\infty} x_{2n-1} = \limsup_{n\to\infty} x_n$ and $\lim_{n\to\infty} x_{2n} = \liminf_{n\to\infty} x_n$

We now wish to prove that that $\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} x_{2n} = \sqrt{\alpha}$. We will first show that if $\lim_{n\to\infty} x_{n+2} = x$, then $\lim_{n\to\infty} \frac{x_n+2\alpha+x_n\alpha}{2x_n+\alpha+1} = \frac{x+2\alpha+x\alpha}{2x+\alpha+1}$.

$\lim_{n\to\infty} x_n + 2\alpha + x_n\alpha$ is the limit of a sum of convergent sequence. So, by a theorem proved in class, its limit is $x + 2\alpha + x\alpha$. Similarly, $\lim_{n\to\infty} 2x_n + \alpha + 1 = 2x + \alpha + 1$. Thus, by a theorem, we have

$$\lim_{n\to\infty} \frac{x_n + 2\alpha + x_n\alpha}{2x_n + \alpha + 1} = \frac{\lim_{n\to\infty} x_n + 2\alpha + x_n\alpha}{\lim_{n\to\infty} 2x_n + \alpha + 1} = \frac{x + 2\alpha + x\alpha}{2x + \alpha + 1}$$

Thus, we have

$$\begin{aligned} \lim_{n\to\infty} x_{n+2} &= \lim_{n\to\infty} \frac{x_n + 2\alpha + x_n\alpha}{2x_n + \alpha + 1} \\ x &= \frac{x + 2\alpha + x\alpha}{2x + \alpha + 1} \\ 2x^2 - 2\alpha &= 0 \\ x &= \sqrt{\alpha} \end{aligned}$$

$x \neq -\sqrt{\alpha} < 0$ as $x_{2n+1} > \sqrt{2} > 0$ and $x_{2n} > x_2 > 0$. This applies to both sequences since we did not make any assumptions about x_{n+2} when computing the limit. Thus, we have

$$\lim_{n\to\infty} x_{2n-1} = \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n} = \sqrt{\alpha}$$

Hence, by a theorem proved in class, $\{x_n\}_{n\geq 1}$ converges and

$$\lim_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \sqrt{\alpha}.$$

---

6 Let

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \left[1 - \frac{1}{(n+1)^2}\right] a_n \quad \text{for all} \quad n \geq 1.$$

a. Show that the sequence $\{a_n\}_{n\geq 1}$ converges.

b. Find its limit.

Solution a. Since $(n+1)^2 > 0$, $1 - \frac{1}{(n+1)^2} < 1$. So, $a_{n+1} = \left[1 - \frac{1}{(n+1)^2}\right] a_n < a_n$. Thus, the sequence decreases monotonically. We will prove by induction that it is bounded below by 0.

Base step:

$a_1 = 1 > 0$

Inductive step:

Suppose $a_n > 0$. Then $a_{n+1} = \left[1 - \frac{1}{(n+1)^2}\right] a_n > 0$ since $1 - \frac{1}{(n+1)^2} > 0$ also. So, the inductive step holds.

Thus, by induction, the sequence is bounded below by 0. Since the sequence bounded below and is monotonically decreasing, then by a theorem, the sequence must converge.

b. Note that $a_{n+1} = a_n \frac{n(n+2)}{(n+1)^2}$.

$$\begin{aligned} a_2 &= \frac{1\cdot 3}{2\cdot 2} \\ a_3 &= \frac{1\cdot 3}{2\cdot 2} \cdot \frac{2\cdot 4}{3\cdot 3} = \frac{1}{2} \cdot \frac{3+1}{3} \\ a_4 &= \frac{1\cdot 3}{2\cdot 2} \cdot \frac{2\cdot 4}{3\cdot 3} \cdot \frac{3\cdot 5}{4\cdot 4} = \frac{1}{2} \cdot \frac{4+1}{4} \end{aligned}$$

We proceed by induction to show that $a_n = \frac{1}{2} \cdot \frac{n+1}{n}$. The base case is taken care above, so we only need to show the inductive step.

Inductive step:

Suppose $a_n = \frac{1}{2} \cdot \frac{n+1}{n}$. We wish to show that $a_{n+1} = \frac{1}{2} \cdot \frac{n+2}{n+1}$.

$$\begin{aligned} a_{n+1} &= a_n \frac{n(n+2)}{(n+1)^2} \\ &= \frac{1}{2} \cdot \frac{n+1}{n} \cdot \frac{n(n+2)}{(n+1)^2} \\ &= \frac{1}{2} \cdot \frac{n+2}{n+1} \end{aligned}$$

Thus, the inductive step holds.

By the principle of mathematical induction, $a_n = \frac{1}{2} \cdot \frac{n+1}{n}$ for all $n \geq 1$. Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{n}\right) = \frac{1}{2}(1+0) = \frac{1}{2}.$$

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7 Let A be a non-empty bounded subset of $\mathbb{R}$ and suppose $\sup A \notin A$. Show that there exists an increasing sequence of points $\{a_n\}_{n\geq 1}$ in A such that $\lim_{n\to\infty} a_n = \sup A$.

Solution We will first construct a sequence of elements of A that is increasing.

Let $a_1 \in A$. By definition, $a_1 \leq \sup A$. But $\sup A \notin A$, so $a_1 < \sup A$. Thus, a_1 cannot be an upper bound of A, so there exists $a_2 \in A$ such that $a_1 < a_2 < \sup A$. We proceed by induction.

Inductive step:

Suppose $a_n \in A$. Then by definition, $a_n \leq \sup A$. But $\sup A \notin A$, so $a_n < \sup A$ and a_n cannot be an upper bound of A. Hence, there exists $a_{n+1} \in A$ such that $a_n < a_{n+1} < \sup A$.

The inductive step holds.

By induction, we have shown that we can indeed construct a sequence of elements of A that is increasing. In fact, the sequence we constructed, which depends on our choice of a_1, is monotonically increasing and is bounded above by $\sup A$, it must converge to its supremum. We will now show that we can find a_1 so that $\sup\{a_n \mid n \geq 1\} = \sup A$.

$\sup A$ is clearly an upper bound of the sequences by construction, so we need to show that it is the least upper bound for at least one sequence. Then that sequence is increasing and converges to $\sup A$.

Suppose otherwise. Then there exists $M < \sup A$ that is an upper bound for all the sequences. M cannot be an upper bound for A, so there exists $a \in A$ such that $M < a < \sup A$. If we take $a_1 = a$, then M is not an upper bound of that sequence, which is a contradiction. Thus, there exists at least one increasing and bounded sequence $\{a_n \mid n \geq 1\}$ whose supremum is $\sup A$, as desired.

8 Let $\mathcal{C}$ be the set of Cauchy sequences of rational numbers. Define the relation $\sim$ as follows: if $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1} \in \mathcal{C}$, we write $\{a_n\}_{n\geq 1} \sim \{b_n\}_{n\geq 1}$ if the sequence $\{a_n - b_n\}_{n\geq 1}$ converges to zero.

a. Prove that $\sim$ is an equivalence relation on $\mathcal{C}$.

b. For $\{a_n\}_{n\geq 1} \in \mathcal{C}$, we denote the equivalence class by $[a_n]$. Let R denote the set of equivalence classes in $\mathcal{C}$. We define addition and multiplication on R as follows:

$$[a_n] + [b_n] = [a_n + b_n] \quad \text{and} \quad [a_n] \cdot [b_n] = [a_n b_n].$$

Show that these three internal laws of composition are well defined and that R together with these operations is a field.

c. We define a relation on R as follows: we write $[a_n] < [b_n]$ if $[a_n] \neq [b_n]$ and there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n < b_n$. Prove that this relation is well defined. Show that the set of positive elements in R, that is,

$$P = \{[a_n] \in R \mid [a_n] > 0\}$$

satisfies the following properties:

(01') For every $[a_n] \in R$, exactly one of the following holds: either $[a_n] = [0]$ or $[a_n] \in P$ or $-[a_n] \in P$, where $[0]$ denotes the equivalence class of the sequence identically equal to zero.

(02') For every $[a_n]$, $[b_n] \in P$, we have $[a_n] + [b_n] \in P$ and $[a_n] \cdot [b_n] \in P$. Conclude that R is an ordered field.

Solution a. Reflexivity: $\{a_n - a_n\}_{n\geq 1} = \{0\}_{n\geq 1}$ which obviously converges to 0. Thus, $\{a_n\}_{n\geq 1} \sim \{a_n\}_{n\geq 1}$.

Symmetry: $\{a_n\}_{n\geq 1} \sim \{b_n\}_{n\geq 1} \iff \{a_n - b_n\}_{n\geq 1}$ converges to 0. By a theorem proved in class, the sequence $\{-1 \cdot (a_n - b_n)\}_{n\geq 1} = \{b_n - a_n\}_{n\geq 1}$ also converges to 0 $\iff \{b_n\}_{n\geq 1} \sim \{a_n\}_{n\geq 1}$.

Transitivity: Let $\{a_n\}_{n\geq 1} \sim \{b_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1} \sim \{c_n\}_{n\geq 1}$. Then by a theorem proved in class, $\lim_{n\to\infty} a_n - c_n = \lim_{n\to\infty}(a_n - b_n) + (b_n - c_n) = \lim_{n\to\infty}(a_n - b_n) + \lim_{n\to\infty}(b_n - c_n) = 0$. Thus, $\{a_n - c_n\}_{n\geq 1}$ converges to 0 $\iff \{a_n\}_{n\geq 1} \sim \{c_n\}_{n\geq 1}$.

b. We first show that the operations are well defined. Let $\{a_n\}_{n\geq 1} \sim \{a'_n\}_{n\geq 1} \iff \lim_{n\to\infty} a_n - a'_n = 0$ and $\{b_n\}_{n\geq 1} \sim \{b'_n\}_{n\geq 1} \iff \lim_{n\to\infty} b_n - b'_n = 0$.
We wish to show that $\{a_n + b_n\}_{n\geq 1} \sim \{a'_n + b'_n\}_{n\geq 1}$.

$$\lim_{n\to\infty}(a_n + b_n) - (a'_n + b'_n) = \lim_{n\to\infty}(a_n - a'_n) + (b_n - b'_n) = 0 \iff \{a_n + b_n\}_{n\geq 1} \sim \{a'_n + b'_n\}_{n\geq 1}.$$

Thus addition is well defined.
We now do the same for multiplication. That is, we wish to show that $\{a_n b_n\}_{n\geq 1} \sim \{a'_n b'_n\}_{n\geq 1}$. b_n and a'_n are Cauchy, so they are convergent. Let b be the limit of b_n, and a' be the limit of a'_n. So, by a theorem proved in class,

$$\lim_{n\to\infty} a_n b_n - a'_n b'_n = \lim_{n\to\infty} b_n(a_n - a'_n) + a'_n(b_n - b'_n) = \lim_{n\to\infty} b \cdot 0 + a' \cdot 0 = 0 \iff \{a_n b_n\}_{n\geq 1} \sim \{a'_n b'_n\}_{n\geq 1}.$$

So multiplication is also well defined.
To show that $\mathcal{C}$ is a field, we will make heavy use of the fact that $\mathbb{Q}$ is also a field since the terms of the sequences of $\mathcal{C}$ are rational.
Let $[a_n]$, $[b_n]$, and $[c_n]$ be in $\mathcal{C}$.

(A1) $[a_n] + [b_n] = [a_n + b_n]$. $\{a_n + b_n\}_{n\geq 1}$ is also Cauchy, since the sum of two convergent sequences is also convergent by a theorem proved in class, so $\{a_n + b_n\}_{n\geq 1} \in \mathcal{C} \implies [a_n + b_n] \in R$

(A2) $[a_n] + [b_n] = [a_n + b_n] = [b_n + a_n] = [b_n] + [a_n]$

(A3) $[a_n] + ([b_n] + [c_n]) = [a_n] + [b_n + c_n] = [a_n + (b_n + c_n)] = [(a_n + b_n) + c_n] = [a_n + b_n] + [c_n] = ([a_n] + [b_n]) + [c_n]$

(A4) The equivalence class $[0]$ (where 0 is the sequence of 0's) is additive identity. $[a_n] + [0] = [a_n]$.

(A5) The equivalence class $-[a_n] = [-a_n]$ is the additive inverse. $[a_n] - [a_n] = [a_n - a_n] = [0]$.

(M1) $[a_n]\cdot[b_n] = [a_nb_n]$. $\{a_nb_n\}_{n\geq 1}$ is also Cauchy as the product of two convergent sequences is convergent (by a theorem proved in class), so $\{a_nb_n\}_{n\geq 1} \in \mathcal{C} \implies [a_nb_n] \in R$.

(M2) $[a_n]\cdot[b_n] = [a_nb_n] = [b_na_n] = [b_n]\cdot[a_n]$

(M3) $[a_n]\cdot([b_n]\cdot[c_n]) = [a_n]\cdot[b_nc_n] = [a_n(b_nc_n)] = [(a_nb_n)c_n] = [a_nb_n]\cdot[c_n] = ([a_n]\cdot[b_n])\cdot[c_n]$.

(M4) The equivalence class $[1]$ (where 1 is the sequence of 1's) is the multiplicative identity. $[a_n]\cdot[1] = [a_n1] = [a_n]$.

(M5) The equivalence class $[\frac{1}{a_n}]$ is the multiplicative inverse, except for the class $[0]$, which doesn't have one. $[a_n]\cdot[\frac{1}{a_n}] = [a_n\cdot\frac{1}{a_n}] = [1]$.

(D) $[c_n]\cdot([a_n]+[b_n]) = [c_n]\cdot[a_n+b_n] = [c_n(a_n+b_n)] = [c_na_n+c_nb_n] = [c_na_n]+[c_nb_n] = [c_n]\cdot[a_n]+[c_n]\cdot[b_n]$.

c. We first show that the order is well defined. Let $\{a_n\}_{n\geq 1} \sim \{a'_n\}_{n\geq 1} \iff \lim_{n\to\infty} a_n - a'_n = 0$ and $\{b_n\}_{n\geq 1} \sim \{b'_n\}_{n\geq 1} \iff \lim_{n\to\infty} b_n - b'_n = 0$. We wish to show that if there exists $N \in \mathbb{N}$ such that if $a_n > b_n$ for all $n \geq N$, then $a'_n > b'_n$ for the same interval. Note that if $a_n > b_n$, then $a = \lim_{n\to\infty} a_n \geq \lim_{n\to\infty} b_n = b$. Since the order relation is defined when the two limits aren't equal, we have the strict inequality $a > b$.

Let $\epsilon = \frac{b-a}{2}$. As $\lim_{n\to\infty} a'_n - a_n = 0 \implies \lim_{n\to\infty} a'_n - \lim_{n\to\infty} a_n = 0 \implies \lim_{n\to\infty} a'_n = a$, there exists $N_a \in \mathbb{N}$ such that for all $n \geq N_a$ we have

$$|a'_n - a| < \frac{a-b}{2} \implies \frac{a+b}{2} < a'_n$$

Similarly, as $\lim_{n\to\infty} b'_n - b_n = 0$, we have $\lim_{n\to\infty} b'_n = b$. So, there exists $N_b \in \mathbb{N}$ such that for all $n \geq N_b$, we have

$$|b'_n - b| < \frac{a-b}{2} \implies b'_n < \frac{a+b}{2}$$

Let $N = max N_a, N_b$. Then both inequalities hold, so $a'_n > \frac{a+b}{2} > b'_n$ for all $n \geq N$. Thus, by definition, $[a'_n] > [b'_n]$, so the order relation is well defined.

(O1') Consider $\{a_n\}_{n\geq 1} \in \mathcal{C}$. Since it is Cauchy, the sequence is also convergent. Suppose $\lim_{n\to\infty} a_n = a$.

Let $\epsilon > 0$. Then we can find $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - a| < \epsilon \implies a - \epsilon < a_n < a + \epsilon$. there are three cases:

$a > 0$:

If we choose $\epsilon = a$ then $0 < a_n$ for all $n \geq N$. Thus, in this case, the equivalence class $[a_n] > [0] \iff [a_n] \in P$.

$a < 0$:

In this case, $-a > 0$. If we choose $\epsilon = -a$ then $a_n < 0$ for all $n \geq N$. Thus, in this case, the equivalence class $[a_n] < [0] \implies -[a_n] > [0] \implies -[a_n] \in P$.

$a = 0$:

Then $\lim_{n\to\infty} a_n - 0 = 0 \iff \{a_n\}_{n\geq 1} \sim \{0\}_{n\geq 1} \implies [a_n] = [0]$.

Thus (O1') holds.

(02') Let $[a_n], [b_n] \in P$. Then there exists $N \in \mathbb{N}$ such that $a_n > 0$ and $b_n > 0$ for all $n \geq N$. Then $a_n + b_n > 0$ for all $n \geq N$, so $[a_n + b_n] > [0] \implies [a_n + b_n] \in P$. Similarly, $a_nb_n > 0$ for all $n \geq N$, so $[a_nb_n] > [0] \implies [a_nb_n] \in P$.

Thus, R is an ordered field with this order relation.