23 As an application of the Fourier transform, show that there does not exist a function $I \in L^1(\mathbb{R}^d)$ such that

$$f * I = f \quad \text{for all} \quad f \in L^1(\mathbb{R}^d).$$

Solution Suppose otherwise, and that there exists I with those properties. Note that by translation invariance, $I(x-y)$ is also integrable on $\mathbb{R}^d$.

Consider the characteristic function χ_E, where E is a set of positive measure. Then by assumption, $\chi_E * I = \chi_E$. By commutativity of convolution, we get

$$\chi_E(x) = I * \chi_E = \int_{\mathbb{R}^d} \chi_E(y) I(x-y)\,\mathrm{d}y = \int_E I(x-y)\,\mathrm{d}y.$$

By a theorem, since $I(x-y)$ is integrable on $\mathbb{R}^d$, then for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $m(K) < \delta$, then $\int_K I(x)\,\mathrm{d}x < \varepsilon$. Take $\varepsilon = 1$, and shrink E so that $m(E) = \delta/2$. Then we get

$$\chi_E(x) = \int_E I(x-y)\,\mathrm{d}y < \varepsilon = 1 \ \forall x \in E.$$

But this implies that $\chi_E(x) \equiv 0$, which is a contradiction, since we assumed E to have positive measure.

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24 Consider the convolution

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y) g(y)\,\mathrm{d}y.$$

a. Show that $f * g$ is uniformly continuous when f is integrable and g bounded.

b. If in addition g is integrable, prove that $(f * g)(x) \to 0$ as $|x| \to \infty$.

Solution a. Let f be integrable and g be bounded. Since g is bounded, there exists $M > 0$ such that $|g(x)| \leq M$. Let $x, z \in \mathbb{R}^d$, and fix $\varepsilon > 0$. Then

$$\begin{aligned}
|(f*g)(x) - (f*g)(z)| &= \left| \int_{\mathbb{R}^d} f(x-y)g(y)\,\mathrm{d}y - \int_{\mathbb{R}^d} f(z-y)g(y)\,\mathrm{d}y \right| \\
&= \left| \int_{\mathbb{R}^d} \Big[f(x-y) - f(z-y) \Big] g(y)\,\mathrm{d}y \right| \\
&\leq \int_{\mathbb{R}^d} |f(x-y) - f(z-y)||g(y)|\,\mathrm{d}y \\
&\leq M \int_{\mathbb{R}^d} |f(x-y) - f(z-y)|\,\mathrm{d}y
\end{aligned}$$

Note that $|f(x-y) - f(z-y)|$ is integrable, since $0 \leq |f(x-y) - f(z-y)| \leq |f(x-y)| + |f(z-y)|$, and the RHS is integrable. Indeed, f is integrable, and integrals are invariant under translation, so both $f(x-y)$ and $f(z-y)$ are integrable.
Also note that by Proposition 2.5,

$$\int_{\mathbb{R}^d} |f(x-y) - f(z-y)|\,\mathrm{d}y \xrightarrow{z \to x} 0,$$

so there exists $\delta > 0$ such that if $\|x - z\| < \delta$, then

$$\int_{\mathbb{R}^d} |f(x-y) - f(z-y)|\,\mathrm{d}y < \frac{\varepsilon}{M}.$$

Thus, for this same δ, we get

$$|(f*g)(x) - (f*g)(z)| \leq M \int_{\mathbb{R}^d} |f(x-y) - f(z-y)|\,\mathrm{d}y < \varepsilon,$$

so $f * g$ is uniformly continuous.

b. Let g be integrable, in addition to being bounded.

Then by Exercise 21(d), $f(x-y)g(y)$ is integrable, so by Fubini's theorem, $f * g$ is integrable for almost every x. By part (a), $f * g$ is uniformly continuous.

If we fix every coordinate except for x_i, we can treat $(f * g)(x)$ as a function from $\mathbb{R}$ to $\mathbb{R}$, so by problem 6(b) (which was on our previous homework), we have that

$$\lim_{|x_i|\to\infty} (f * g)(x) = 0,$$

for each $1 \leq i \leq d$.

For any $\varepsilon > 0$, the definition of the limit gives us $N_i \in \mathbb{N}$ for each i such that if $|x_i| > N_i$, where x_i is the i-th coordinate of x, then $|(f * g)(x)| < \varepsilon$.

Thus, if $\|x\|$ is sufficiently large, we have at least one $|x_i| \geq N_i$, which gives us $(f * g)(x) < \varepsilon$.

Hence,

$$\lim_{\|x\|\to\infty} (f * g)(x) = 0.$$

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2 Prove the Cantor-Lebesgue theorem: if

$$\sum_{n=0}^{\infty} A_n(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges for x in a set of positive measure (or in particular for all x), then $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$. [Hint: Note that $A_n(x) \to 0$ uniformly on a set E of positive measure.]

Solution Suppose the series converges on a set U of positive measure.

Since the series converges, we must have that

$$\lim_{n\to\infty} A_n(x) = 0$$

pointwise for every $x \in U$. Hence, we can take a smaller subset with finite measure, and apply Egorov's theorem. This gives us a set $E \subseteq U$ of (finite) positive measure such that $A_n \xrightarrow{n\to\infty} 0$ uniformly on E.

Thus,

$$\int_E |A_n| \xrightarrow{n\to\infty} 0 \iff \int_E |a_n \cos nx + b_n \sin nx| \, \mathrm{d}x \xrightarrow{n\to\infty} 0.$$

Note that we can write

$$A_n^2(x) = (a_n \cos nx + b_n \sin nx)^2 = (a_n^2 + b_n^2)\cos^2\left(nx + \arctan\left(-\frac{b_n}{a_n}\right)\right).$$

Moreover, $A_n^2 \xrightarrow{n\to\infty} 0$ uniformly on E as well. Indeed, if $A_n(x) < \varepsilon < 1$ for some $n \geq N$,

$$|A_n^2(x)| < \varepsilon^2 < \varepsilon.$$

Then by Problem 1,

$$\int_E A_n^2 = \int_E (a_n^2 + b_n^2)\cos^2\left(nx + \arctan\left(-\frac{b_n}{a_n}\right)\right) \mathrm{d}x = \frac{a_n^2 + b_n^2}{2} m(E).$$

But since $A_n^2 \xrightarrow{n\to\infty} 0$ uniformly, $\int_E A_n^2 \xrightarrow{n\to\infty} 0$. Since $m(E) > 0$, this implies that $a_n^2 + b_n^2 \xrightarrow{n\to\infty} 0$, which implies that $a_n \xrightarrow{n\to\infty} 0$ and $b_n \xrightarrow{n\to\infty} 0$, as desired.

Proof of Problem 1

Let f be integrable on $[0, 2\pi]$. By Exercise 22,

$$\int_0^{2\pi} f(x)e^{-inx}\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)e^{-inx}\chi_{[0,2\pi]} \xrightarrow{|n|\to\infty} 0,$$

which implies that

$$\int_E f(x)e^{-inx}\,\mathrm{d}x = \int_E f(x)\cos(nx) - if(x)\sin(nx)\,\mathrm{d}x \xrightarrow{n\to\infty} 0,$$

so $\int_E f(x)\cos(nx)\,\mathrm{d}x$ and $\int_E f(x)\sin(nx)\,\mathrm{d}x$ both converge to 0 as $n \to \infty$.

It suffices to show that

$$\int_E \cos^2(nx+u_n) - \frac{1}{2}\,\mathrm{d}x \xrightarrow{n\to\infty} 0,$$

where u_n is any sequence.

$$\begin{aligned}
\left|\int_E \cos^2(nx+u_n) - \frac{1}{2}\,\mathrm{d}x\right| &= \left|\int_E \frac{1}{2}\Big(1+\cos(2nx+u_n)\Big) - \frac{1}{2}\,\mathrm{d}x\right| \\
&= \left|\int_E \cos(2nx+2u_n)\,\mathrm{d}x\right| \\
&= \left|\int_0^{2\pi} \cos(2nx+2u_n)\chi_E\,\mathrm{d}x\right| \\
&= \left|\int_0^{2\pi} \chi_E\cos(2nx)\cos(2u_n) - \chi_E\sin(2nx)\sin(2u_n)\,\mathrm{d}x\right| \\
&\le |\cos(2u_n)|\left|\int_0^{2\pi}\chi_E\cos(2nx)\,\mathrm{d}x\right| + |\sin(2u_n)|\left|\int_0^{2\pi}\chi_E\sin(2nx)\,\mathrm{d}x\right| \xrightarrow{n\to\infty} 0.
\end{aligned}$$

The limit holds because of the first half of the problem.

Thus,

$$\int_E \cos^2(nx+u_n) - \frac{1}{2}\,\mathrm{d}x \xrightarrow{n\to\infty} 0 \implies \int_E \cos^2(nx+u_n)\,\mathrm{d}x \xrightarrow{n\to\infty} \int_E \frac{1}{2}\,\mathrm{d}x \implies \int_E \cos^2(nx+u_n)\,\mathrm{d}x \xrightarrow{n\to\infty} \frac{m(E)}{2},$$

as desired.