1 Find the limits and determine the rates of convergence.

a. $\lim_{n\to\infty} \ln(n^2 + (-1)^n n) - 2\ln n$

b. $\lim_{h\to 0} \dfrac{\sin h - he^{-h}}{h}$

Solution a. Using the properties of logarithms, we can rewrite the function as

$$\ln\left(\frac{n^2 + (-1)^n n}{n^2}\right) = \ln\left(1 + \frac{(-1)^n}{n}\right).$$

Since $\ln(\,\cdot\,)$ is continuous, we can pass the limit to the argument of the function to get

$$\lim_{n\to\infty} \ln(n^2 + (-1)^n n) - 2\ln n = \ln\left[\lim_{n\to\infty}\left(1 + \frac{(-1)^n}{n}\right)\right] = \ln 1 = 0.$$

By the mean value theorem,

$$\left|\ln(n^2 + (-1)^n n) - 2\ln n\right| = \left|\ln(n^2 + (-1)^n n) - \ln n^2\right| \le \frac{1}{\xi}\left|(n^2 + (-1)^n n - n^2)\right| = \frac{n}{\xi},$$

for some $\xi \in [n^2 + (-1)^n n, n^2]$ (or the other way around). In either case, have

$$\xi \ge n^2 - n \implies \frac{1}{\xi} \le \frac{1}{n^2 - n} \le \frac{1}{n^2 - n^2/2} = \frac{2}{n^2}.$$

Thus,

$$\left|\ln(n^2 + (-1)^n n) - 2\ln n\right| \le \frac{n}{\xi} \le \frac{2n}{n^2} = 2 \cdot \frac{1}{n},$$

so

$$\left|\ln(n^2 + (-1)^n n) - 2\ln n\right| = \mathcal{O}\left(\frac{1}{n}\right).$$

b. By Taylor's theorem and the fact that $h^3 \le h^2 \implies h^3 = \mathcal{O}(h^2)$ for $|h| \le 1$, we can write

$$\frac{\sin h - he^{-h}}{h} = \frac{(h + \mathcal{O}(h^3)) - (h - \mathcal{O}(h^2))}{h} = \frac{\mathcal{O}(h^2)}{h} = \mathcal{O}(h).$$

This also shows that the limit is 0.

2 Write an algorithm to sum the finite series $\sum_{k=1}^{N} kx_k$ is reverse order.

Solution Here is the algorithm:

Input : $x_1, \dots, x_N$
Output: The sum $\sum kx_k$

Initialize $sum = 0$;
for $i = 0, \dots, N-1$ **do**
  $sum = sum + (N-i)x_{N-i}$
return sum

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3 Let $f(x) = x^2 - 5x + 3$.

a. Find out the exact solution p to $f(x) = 0$ on $[0, 1]$.

b. If we are to use the Bisection method to find the solution of $f(x) = 0$ on $[0, 1]$ accurate to 10^{-4} (i.e., the absolute error of the solution is no greater than 10^{-4}), how many iterations do we need? Provide a reasonable estimate for this.

c. *Manually* implement the Bisection method to solve $f(x) = 0$ on $[0, 1]$. Write your results as fractions. You may stop the iteration when the length of the subinterval is less than 0.1; return the midpoint p^* of the final subinterval as an approximate solution.

d. Calculate the absolute and relative errors of the approximate solution p^* obtained in the previous part.

Solution a. By the quadratic formula,

$$p = \frac{5 - \sqrt{25 - 12}}{2} \approx 0.697224362268.$$

b. The error $|p - p^*|$ after the n-th iteration is bounded by $\dfrac{b-a}{2^n} = \dfrac{1}{2^n}$, so a precision of 10^{-4} occurs at about 14 iterations, since

$$2^{-14} \approx 0.00006103515625 \approx 0.5 \times 10^{-5}.$$

c.

n	a_n	b_n	$f(a_n)$	$f(b_n)$	p_n	$f(p_n)$
0	0	1	3	-1	1/2	1/2
1	1/2	1	3/4	-1	3/4	$-3/16$
2	1/2	3/4	3/4	$-3/16$	5/8	17/64
3	5/8	3/4	17/64	$-3/16$	11/16	9/256
4	11/16	3/4	9/256	$-3/16$	23/32	

Our estimate is $p^* = 23/32 = 0.71875$.

d.

	Error
$\lvert p - p* \rvert$	0.02152563773
$\left\lvert \frac{p - p*}{p} \right\rvert$	0.03087332987

4 We mentioned in class that when implementing the Bisection method on a computer, it is suggested to use

$$p_n = a_n + \frac{b_n - a_n}{2} \quad \text{instead of} \quad p_n = \frac{a_n + b_n}{2}$$

to compute the midpoint of the interval $[a_n, b_n]$, although the former may lead to loss of accuracy when a_n and b_n are very close—due to *subtraction of nearly equal numbers*. The problem of the latter formula in the finite-digit arithmetic, however, is that it may return a number outside the interval $[a_n, b_n]$, which is fatal to the Bisection method. The following is an example.

Let $a = 0.7326$ and $b = 0.7329$.

a. Compute $p = (a + b)/2$ using 4-digit chopping.

b. Compute $p' = (a + b)/2$ using 4-digit rounding.

c. Compute $p'' = a + (b - a)/2$ using 4-digit rounding. (You are also encouraged to try 4-digit chopping.)

Solution a. $p = \dfrac{a+b}{2} = \dfrac{\mathrm{fl}(\mathrm{fl}\, a + \mathrm{fl}\, b)}{2} = \dfrac{1.465}{2} = 0.7325 < a.$

b. $p' = \dfrac{a+b}{2} = \dfrac{\mathrm{fl}(\mathrm{fl}\, a + \mathrm{fl}\, b)}{2} = \dfrac{1.466}{2} = 0.733 > b.$

c. $p'' = a + \dfrac{b-a}{2} = \mathrm{fl}\left[\mathrm{fl}\, a + \dfrac{\mathrm{fl}(\mathrm{fl}\, b - \mathrm{fl}\, a)}{2}\right] = \mathrm{fl}(0.7326 + 0.00015) = 0.7328 \in [a, b].$

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5 Show that $g(x) = \cos x$ has a unique fixed point on $[1/2, 1]$.

Solution We need to show two things: that g has a fixed point, and that the fixed point is unique.

Consider $f(x) = x - \cos x$. Then $f(1/2) < 0$ and $f(1) > 0$, so by the intermediate value theorem, f has a 0 on the interval $[1/2, 1]$, which is equivalent to saying that g has a fixed point on the same interval.

Next, we will show uniqueness. Notice that

$$|g'(x)| = |-\sin x| = |\sin x|.$$

Since $[1/2, 1] \subseteq (0, \pi/2)$. Since $\sin x$ is increasing on $[0, \pi/2]$ and $\sin \pi/2 = 1$, it follows that $|\sin x| < 1$ on $[1/2, 1]$. Moreover, $\sin x$ is continuous and $[1/2, 1]$ is compact, so $\sin x$ obtains a maximum value $0 < k < 1$ on $[1/2, 1]$. Hence, $|g'(x)| \leq k < 1$, so by the uniqueness theorem for the fixed point method, the fixed point of g is unique.