17.1 Find the general solution of each of the following equations:

(a) $y'' + y' - 6y = 0$

(d) $2y'' - 4y' + 8y = 0$

(g) $2y'' + 2y' + 3y = 0$

(r) $y'' + 4y' - 5y = 0$

Solution (a) $\lambda^2 + \lambda - 6 = 0 \implies \lambda_1 = -3,\ \lambda_2 = 2.$

$\boxed{y(t) = c_1 e^{-3t} + c_2 e^{2t}.}$

(d) $2\lambda^2 - 4\lambda + 8 = 0 \implies \lambda^2 - 2\lambda + 4 = 0 \implies \lambda_{1/2} = \dfrac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\sqrt{3}.$

The solution over $\mathbb{C}$ is then given by

$$\begin{aligned} y(t) &= c_1 e^{(1+i\sqrt{3})t} + c_2 e^{(1-i\sqrt{3})t} \\ &= c_1 e^t(\cos\sqrt{3}t + i\sin\sqrt{3}t) + c_2 e^t(\cos\sqrt{3}t - i\sin\sqrt{3}t). \end{aligned}$$

Since the real and imaginary parts are linear combinations of y, the general solution over $\mathbb{R}$ is

$$\boxed{y(t) = c_1 e^t \cos\sqrt{3}t + c_2 e^t \sin\sqrt{3}t.}$$

(g) $2\lambda^2 + 2\lambda + 3 = 0 \implies \lambda_{1/2} = \dfrac{-2 \pm \sqrt{4-24}}{4} = -\dfrac{1}{2} \pm i\dfrac{\sqrt{5}}{2}.$

The solution over $\mathbb{C}$ is then

$$\begin{aligned} y(t) &= c_1 e^{(-1+\sqrt{5})t/2} + c_2 e^{(-1-\sqrt{5})t/2} \\ &= c_1 e^{-t/2}\left(\cos\frac{\sqrt{5}}{2}t + i\sin\frac{\sqrt{5}}{2}t\right) + c_2 e^{-t/2}\left(\cos\frac{\sqrt{5}}{2}t - i\sin\frac{\sqrt{5}}{2}t\right). \end{aligned}$$

So the general solution over $\mathbb{R}$ is

$$\boxed{y(t) = c_1 e^{-t/2}\cos\frac{\sqrt{5}}{2}t + c_2 e^{-t/2}\sin\frac{\sqrt{5}}{2}t.}$$

(r) $\lambda^2 + 4\lambda - 5 = 0 \implies \lambda_1 = -5,\ \lambda_2 = 1.$

$\boxed{y(t) = c_1 e^{-5t} + c_2 e^{t}.}$

17.2 Find the solutions of the following initial value problems:

(a) $y'' - 5y' + 6y = 0,\ y(1) = e^2$ and $y'(1) = 3e^2$

(c) $y'' - 6y + 9y = 0,\ y(0) = 0$ and $y'(0) = 5$

(e) $y'' + 4y' + 2y = 0,\ y(0) = -1$ and $y'(0) = 2 + 3\sqrt{2}$

Solution (a) $\lambda^2 - 5\lambda + 6 = 0 \implies \lambda_1 = 3,\ \lambda_2 = 2.$

$y(t) = c_1 e^{3t} + c_2 e^{2t}.$

$$\begin{aligned} y(1) = e^2 &\implies \quad c_1 e^3 + \ \ c_2 e^2 = \ \ e^2 \\ y'(1) = 3e^2 &\implies 3c_1 e^3 + 2c_2 e^2 = 3e^2 \end{aligned}$$

By inspection, $c_1 = e^{-1}$ and $c_2 = 0$, so the solution is $\boxed{y(t) = e^{3t-1}.}$

(c) $\lambda^2 - 6\lambda + 9 = 0 \implies \lambda_1 = \lambda_2 = 3$. We check if the following is a solution: $y(t) = te^{3t}$:

$$\begin{aligned} y'(t) &= e^{3t} + 3te^{3t} \\ y''(t) &= 3e^{3t} + 3e^{3t} + 9te^{3t} = 6e^{3t} + 9te^{3t} \\ y'' - 6y' + 9y &= 6e^{3t} + 9te^{3t} - 6e^{3t} - 18te^{3t} + 9te^{3t} \\ &= 0. \end{aligned}$$

It is, so the general solution is given by $y(t) = c_1 e^{3t} + c_2 te^{3t}$.

$$\begin{aligned} y(0) = 0 &\implies c_1 = 0 \\ y'(0) = 5 &\implies 3c_1 + c_2 = c_2 = 5. \end{aligned}$$

Hence, the solution is $\boxed{y(t) = 5te^{3t}.}$

(e) $\lambda^2 + 4\lambda + 2 = 0 \implies \lambda_{1/2} = \dfrac{-4 \pm \sqrt{16-8}}{2} = -2 \pm \sqrt{2}.$

$y(t) = c_1 e^{(-2+\sqrt{2})t} + c_2 e^{(-2-\sqrt{2})t}.$

$$\begin{aligned} y(0) = -1 &\implies c_1 + c_2 = -1 \\ y'(0) = 2 + 3\sqrt{2} &\implies (-2+\sqrt{2})c_1 + (-2-\sqrt{2})c_2 = 2 + 3\sqrt{2}. \end{aligned}$$

By inspection, $c_1 = 1$ and $c_2 = -2$, so the solution is $\boxed{y(t) = e^{(-2+\sqrt{2})t} - 2e^{(-2-\sqrt{2})t}.}$

17.3 Show that the general solution of equation (1) approaches 0 as $x \to \infty$ if and only if p and q are both positive.

Solution "$\implies$"

Let $\lim_{x\to\infty} y(x) = 0$.

$\lambda^2 + p\lambda + q$ has real roots:

Then the general solution is $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

We must have $m_1, m_2 \le 0$, since $e^{at} \xrightarrow{t\to\infty} \infty$ if $a > 0$.

Assume without loss of generality that $m_1 < 0$ and $m_2 = 0$. We can switch m_1 and m_2 and repeat the following argument.

Then $y(x) \xrightarrow{x\to\infty} c_2$, which is not necessarily 0 in general.

If $m_1 = m_2 = 0$, then $y(x) \xrightarrow{x\to\infty} c_1 + c_2$, which is also not necessarily 0 in general.

Hence, we must have $m_1, m_2 < 0$. Since m_1 and m_2 are roots of $\lambda^2 + p\lambda + q$, we know that $p = -(m_1 + m_2)$ and $q = m_1 m_2$. Thus, p and q must both be positive, since m_1 and m_2 are negative.

$\lambda^2 + p\lambda + q$ has complex roots:

Then the general solution is $y(x) = e^{ax}(c_1 \cos bx + ic_2 \sin bx)$.

Since $\cos bx$ and $\sin bx$ are bounded functions, the convergence of y does not depend on b, so we only need to look at a.

We must have $a < 0$ since $e^{at} \xrightarrow{t \to \infty} 0$ if and only if $a < 0$. If $a = 0$, then our solution oscillates in general. If $a > 0$, then $y(x) \xrightarrow{x \to \infty} \infty$. Hence, $a < 0$.

Since $a + bi$ and $a - bi$ are roots of $\lambda^2 + p\lambda + q$, we have $p = -2a > 0$ and $q = a^2 + b^2 > 0$.

$\lambda^2 + p\lambda + q$ has a repeated root:

In this case, the general solution is $y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$.

Similar to the case with two distinct real roots, we must have $\lambda < 0$. Then since λ is a root of the characteristic polynomial, $p = -\lambda/2 > 0$ and $q = \lambda^2/4 > 0$.

"$\Longleftarrow$"

Let p and q be positive. By the quadratic formula, the roots of the characteristic polynomial are given by

$$\lambda_{1/2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

If the roots are real and distinct, then we have $\sqrt{p^2 - 4q} < \sqrt{p^2} = p$, so

$$\begin{aligned} -p + \sqrt{p^2 - 4q} &< -p + p = 0 \\ -p - \sqrt{p^2 - 4q} &< 0, \end{aligned}$$

so both roots are negative. Hence, by the argument in the first part, $y(x) \xrightarrow{x \to \infty} 0$.

If the roots are complex, the real part is $-p/2 < 0$, so by the argument above, $y(x)$ converges to 0.

If the roots are repeated, then $p^2 - 4q = 0 \implies \lambda = -p/2 < 0$, so the solution also converges to 0.

17.5 The equation

$$x^2y'' + pxy' + qy = 0,$$

where p and q are constants, is called *Euler's equidimensional equation*. Show that the change of independent variable given by $x = e^z$ transforms it into an equation with constant coefficients, and apply this technique to find the general solution of each of the following equations:

(a) $x^2y'' + 3xy' + 10y = 0$

(c) $x^2y'' + 2xy' - 12y = 0$

(e) $x^2y'' - 3xy' + 4y = 0$

Solution Substituting e^z for x, we get

$$\begin{aligned} u(z) &:= y(e^z) \\ u' &= (y(e^z))' = e^z y'(e^z) \\ u'' &= (y(e^z))'' = e^{2z}y''(e^z) + e^z y'(e^z) = e^{2z}y''(e^z) + u'. \end{aligned}$$

Then we get

$$\begin{aligned} e^{2z}y''(e^z) + pe^z y'(e^z) + qy(e^z) &= 0 \\ (u'' - u') + pu' + qu &= 0 \\ u'' + (p-1)u' + qu &= 0. \end{aligned}$$

(a) After substitution, the problem becomes $u'' + 2u + 10u = 0$, whose characteristic roots are given by

$$\lambda_{1/2} = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3i,$$

so the general solution in u is given by $u(z) = e^{-z}(c_1 \cos 3z + c_2 \sin 3z)$. Since $u = y(e^z)$, we can undo the transformation with $z = \log x$, which gives

$$\boxed{y(x) = \frac{c_1 \cos(3\log x) + c_2 \sin(3\log x)3}{x}.}$$

(c) After transformation, the differential equation becomes $u'' + u' - 12u = 0$, which has characteristic roots $\lambda_1 = -4$ and $\lambda_2 = 3$. Hence, the general solution in u is $u(z) = c_1e^{-4z} + c_2e^{3z}$. Undoing the transformation, we get

$$\boxed{y(x) = c_1x^{-4} + c_2x^3.}$$

(e) Applying the substitution yields $u'' - 4u' + 4u = 0$, which has the repeated root 2.

Thus, the general solution in u is $u(z) = c_1e^{2z} + c_2ze^{2z}$. Undoing the transformation finally gives

$$\boxed{y(x) = c_1x^2 + c_2x^2 \log x.}$$

17.8 In this problem we present another way of discovering the second linearly independent solution of (1) when the roots of the auxiliary equation are real and equal.

a. If $m_1 \neq m_2$, verify that the differential equation

$$y'' - (m_1 + m_2)y' + m_1 m_2 y = 0$$

has

$$y = \frac{e^{m_1 x} - e^{m_2 x}}{m_1 - m_2}$$

as a solution.

b. Think of m_2 as fixed and use l'Hôpital's rule to find the limit of the solution in part (a) as $m_1 \to m_2$.

c. Verify that the limit in part (b) satisfies the differential equation obtained from the equation in part (a) by replacing m_1 by m_2.

Solution a. The general solution to the differential equation is given by $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$, since m_1 and m_2 are roots to the characteristic polynomial. Taking $c_1 = 1/(m_1 - m_2)$ and $c_2 = -1/(m_1 - m_2)$ gives us the desired solution.

b. The numerator and denominator both approach 0 as $m_1 \to m_2$, so we can apply l'Hôpital's to get

$$\lim_{m_1 \to m_2} \frac{e^{m_1 x} - e^{m_2 x}}{m_1 - m_2} = \lim_{m_1 \to m_2} \frac{x e^{m_1 x}}{1} = x e^{m_2 x}.$$

c. Letting $m_1 = m_2 = \lambda$, the differential equation becomes $y'' - 2\lambda y' + \lambda^2 y = 0$. We wish to verify that $y(x) = x e^{\lambda x}$ is a solution:

$$\begin{aligned} y' &= e^{\lambda x} + \lambda x e^{\lambda x} \\ y'' &= \lambda e^{\lambda x} + \lambda e^{\lambda x} + \lambda^2 x e^{\lambda x} = \lambda e^{\lambda x} + \lambda y'. \end{aligned}$$

Substituting gives

$$\begin{aligned} y'' - 2\lambda y' + \lambda^2 y &= \lambda e^{\lambda x} + \lambda y' - 2\lambda y' + \lambda^2 x e^{\lambda x} \\ &= \lambda e^{\lambda x} - \lambda e^{\lambda x} - \lambda^2 x e^{\lambda x} + \lambda^2 x e^{\lambda x} \\ &= 0, \end{aligned}$$

so $y(x) = x e^{\lambda x}$ indeed satisfies the differential equation.