28 Let E be a subset of $\mathbb{R}$ with $m_*(E) > 0$. Prove that for each $0 < \alpha < 1$, there exists an open interval I so that

$$m_*(E \cap I) \geq \alpha m_*(I).$$

[Hint: Choose an open set $\mathcal{O}$ that contains E and such that $m_*(E) \geq \alpha m_*(\mathcal{O})$. Write $\mathcal{O}$ as the countable union of disjoint open intervals, and show that one these intervals must satisfy the desired property.]

Solution Fix $0 < \alpha < 1$.

For any $\varepsilon > 0$, there exists an open subset of $\mathbb{R}$ containing E such that

$$m_*(E) + \varepsilon \geq m_*(U).$$

Take $\varepsilon = \left(\frac{1-\alpha}{\alpha}\right)m_*(E)$, which is positive since $0 < \alpha < 1$ and $m_*(E) > 0$. This gives us an open set $\mathcal{O}$ such that

$$m_*(\mathcal{O}) \leq m_*(E) + \left(\frac{1-\alpha}{\alpha}\right)m_*(E) = \frac{1}{\alpha}m_*(E) \implies \alpha m_*(O) \leq m_*(E).$$

Since $\mathcal{O}$ is open, we can write it as a disjoint union of open intervals $U_i = (a_i, b_i)$, with $a_i < b_i$ for all $1 \leq i \leq n$.

Then at least one of these these U_i must satisfy the desired property.

Suppose otherwise, and that for each i, we have $m_*(E \cap U_i) < \alpha m_*(U_i)$.

Note that $\bigcup_{i=1}^{n}(E \cap U_i) = E \cap \mathcal{O} = \mathcal{O}$. So, by subadditivity of the outer measure,

$$m(\mathcal{O}) = m_*\left(\bigcup_{i=1}^{n}(E \cap U_i)\right) \leq \sum_{i=1}^{n} m_*(E \cap U_i).$$

Since each U_i is open and disjoint, they are measurable, so by the additivity of Lebesgue measure,

$$\alpha m(\mathcal{O}) = \sum_{i=1}^{n} \alpha m(U_i) > \sum_{i=1}^{n} m_*(E \cap U_i) \geq m(\mathcal{O}),$$

which is a contradiction since $0 < \alpha < 1$. Hence, at least one U_i must satisfy the given property.

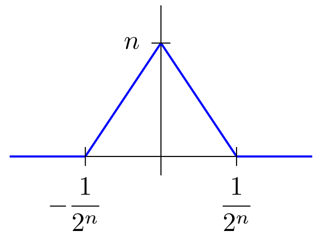
6 Integrability of f on $\mathbb{R}$ does not necessarily imply the converge of $f(x)$ to 0 as $x \to \infty$.

a. There exists a positive continuous function f on $\mathbb{R}$ so that f is integrable on $\mathbb{R}$, but yet $\limsup_{x\to\infty} f(x) = \infty$.

b. However, if we assume that f is uniformly continuous on $\mathbb{R}$ and integrable, then $\lim_{|x|\to\infty} f(x) = 0$.

Solution a. Consider the "spike" function, which we define as

$$s_n(x) = \begin{cases} 2^n n\left(x + \dfrac{1}{2^n}\right) & \text{if } x \in \left[-\dfrac{1}{2^n}, 0\right] \\ -2^n n\left(x - \dfrac{1}{2^n}\right) & \text{if } x \in \left[0, \dfrac{1}{2^n}\right] \\ 0 & \text{otherwise.} \end{cases}$$



Then

$$\int s_n = \int s_n \chi_{[-\frac{1}{2^n}, \frac{1}{2^n}]} = \frac{1}{2} \cdot \frac{2}{2^n} \cdot n = \frac{n}{2^n}.$$

Thus, we define $f(x)$ via

$$f(x) = \sum_{k=1}^{\infty} s_k(x-k),$$

which is nonzero since each s is nonzero. It is also clearly continuous, so it's Riemann integrable and thus Lebesgue integrable.

For each $n \in \mathbb{N}$, define

$$f_n(x) = \sum_{k=1}^{n} s_k(x-k)$$

This is an increasing sequence of functions since each s_k is non-negative. Moreover, $f_n \xrightarrow{n\to\infty} f$ everywhere.

Note that none of the spikes overlap since the centers of the spikes are at least 1 apart, and the largest spike has radius 1, so by translation invariance of area,

$$\int f_n = \sum_{k=1}^{n} \int s_k(x-k) = \sum_{k=1}^{n} \frac{k}{2^k}.$$

By the root test, the series $\sum k/2^k$ converges:

$$\left(\frac{k}{2^k}\right)^{1/k} = \frac{k^{1/k}}{2} \xrightarrow{k\to\infty} \frac{1}{2} < 1.$$

Thus,

$$\lim_{n\to\infty} \int f_n = \sum_{k=1}^{\infty} \frac{k}{2^k} < \infty$$

Hence, by the Monotone Convergence Theorem,

$$\int f = \int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n < \infty.$$

To find a positive function for the problem, we can simply add the Gaussian e^{-x^2} to f. Then

$$\int (f + e^{-x^2}) = \sum_{k=1}^{\infty} \frac{k}{2^k} + \sqrt{\pi} < \infty.$$

e^{-x^2} is positive and continuous, and f is non-negative and continuous, so the sum, which we will name g, is positive and continuous.

Moreover, $g(n) \geq f(n) = n$ for all $n \geq 1$, so $\limsup_{x\to\infty} g(x) = \infty$.

b. Let f be uniformly continuous and integrable on $\mathbb{R}$.
Since f is integrable, then by definition, $|f|$ is integrable, i.e.,

$$\int |f| = \sup_g \int g < \infty,$$

where $0 \leq g \leq f$, g is measurable, and supported on a set of finite measure.
Moreover, $|f|$ is also uniformly continuous, since by the triangle inequality,

$$\big||f(x)| - |f(y)|\big| < |f(x) - f(y)|.$$

So given $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \implies \big||f(x)| - |f(y)|\big| < \varepsilon$.
It suffices to show that $\lim_{|x|\to\infty}|f(x)| = 0$.
Suppose, without loss of generality, that $\lim_{x\to\infty}|f(x)| \neq 0$. Then there exists $\varepsilon_0 > 0$ and a strictly increasing sequence $\{x_n\}_{n\geq 1}$ satisfying $|f(x_n)| \geq \varepsilon_0$ and $|x_n - x_m| > \delta$. Indeed, after finding x_1, we can find $x_2 > x_1 + \delta$ such that $|f(x_2)| \geq \varepsilon_0$, and so on.
Since $|f|$ is uniformly continuous, there exists $\delta > 0$ such that $\big||f(x)| - |f(y)|\big| < \varepsilon_0/2$ whenever $|x - y| < \delta$.
Define $E_n = [x_n - \delta, x_n + \delta]$. Then for $x \in E_n$, we have

$$|x - x_n| < \delta \implies \big||f(x)| - |f(x_n)|\big| \leq \frac{\varepsilon_0}{2} \implies |f(x)| \geq |f(x_n)| - \frac{\varepsilon_2}{2} \geq \varepsilon_0 - \frac{\varepsilon_2}{2} = \frac{\varepsilon_0}{2}$$

$$\int |f| \geq \sum_{n=1}^{\infty} \int_{E_n} |f(x)| \geq \sum_{n=1}^{\infty} \frac{\varepsilon_0}{2} \cdot 2\delta = \sum_{n=1}^{\infty} \varepsilon_0 \delta = \infty,$$

but this is a contradiction as we assumed $|f|$ to be integrable. Hence, we must have that

$$\lim_{x\to\infty} |f(x)| = 0 \implies \lim_{x\to\infty} f(x) = 0.$$

For $x \to -\infty$, we can perform the same argument, but by finding a strictly decreasing sequence $\{x_n\}_{n\geq 1}$ with the same property. Then the rest of the argument holds. Thus, we have $\lim_{|x|\to\infty} f(x) = 0$.

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8 If f is integrable on $\mathbb{R}$, show that $F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$ is uniformly continuous.

Solution Let f be integrable.
Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that whenever $m(E) < \delta$, we have $\int_E |f| < \varepsilon$.
Then if $|x - y| < \delta$, the open interval (x, y) (or the other way around) has measure $|x - y| < \delta$, so we get

$$|F(x) - F(y)| = \left|\int_y^x f(t)\,\mathrm{d}t\right| \leq \int_y^x |f(t)|\,\mathrm{d}t < \varepsilon,$$

so F is uniformly continuous.

10 Suppose $f \geq 0$, and let $E_{2^k} = \{x \mid f(x) > 2^k\}$ and $F_k = \{x \mid 2^k < f(x) \leq 2^{k+1}\}$. If f is finite almost everywhere, then

$$\bigcup_{k=-\infty}^{\infty} F_k = \{f(x) > 0\},$$

and the sets F_k are disjoint.

Prove that f is integrable if and only if

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty, \quad \text{if and only if} \quad \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty.$$

Use this result to verify the following assertions. Let

$$f(x) = \begin{cases} |x|^{-a} & \text{if } |x| \leq 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} |x|^{-b} & \text{if } |x| > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is integrable on $\mathbb{R}^d$ if and only if $a < d$; also g is integrable on $\mathbb{R}^d$ if and only if $b > d$.

Solution Define $\psi(x) \coloneqq \sum_{k=-\infty}^{\infty} 2^k \chi_{E_{2^k}}(x)$. Then

$$f \leq \psi \leq 2f.$$

Indeed, whenever $2^k < f(x) \leq 2^{k+1}$, we have

$$f(x) \leq \psi(x) = \sum_{n=-\infty}^{k} 2^n = \frac{2^k}{1 - \frac{1}{2}} = 2^{k+1} < 2f(x).$$

Hence, by monotonicity,

$$\int f \leq \int \psi = \sum_{k=-\infty}^{\infty} 2^k m(E_k) \leq 2 \int f,$$

which shows that f is integrable if and only if $\sum_{k=-\infty}^{\infty} 2^k m(E_k) < \infty$.

On each F_k, we have $2^k < f(x) \leq 2^{k+1}$, so

$$2^k m(F_k) = \int_{F_k} 2^k \leq \int_{F_k} f \leq \int_{F_k} 2^{k+1} = 2^{k+1} m(F_k).$$

Taking sums yields

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) \leq \int f \leq \sum_{k=-\infty}^{\infty} 2^{k+1} m(F_k),$$

which shows that f is integrable if and only if $\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$, so we're done.

For $a < 0$, $|x|^{-a}$ is a strictly increasing function in the norm, $|x|$, so $|x|^a$ is a strictly decreasing function.

For $k \geq 0$, E_{2^k} is empty and for $k \leq 1$, it has maximum measure proportional to 1^d, since $|x| \leq 1$ in the definition of f. Then

$$\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) \leq \sum_{k=-\infty}^{1} 2^k c = \frac{2c}{1 - \frac{1}{2}} = 4c < \infty$$

for some $c > 0$. For $a = 0$, $f(x) = 1$ when $|x| \leq 1$, so $E_{2^k} = \emptyset$ when $k \geq 0$, and has measure proportional to

1^d otherwise. So, we can use the same sum as above to see that the series converges.

For $a > 0$, $|x|^{-a}$ is a strictly decreasing function, so for some k,

$$|x|^{-a} > 2^k \iff |x| > 2^{-k/a}.$$

For $k \leq 0$, $2^{-k/a} > 1$, so E_{2^k} is empty for those k. So E_{2^k} is non-empty for $k > 0$, and has measure proportional to $1^d - 2^{-dk/a}$. So,

$$\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) = c \sum_{k=1}^{\infty} 2^k (1 - 2^{-dk/a})$$

Using the ratio test,

$$\frac{2^{k+1}(1 - 2^{-dk/a})}{2^k(1 - 2^{-d(k+1)/a})} \overset{H}{=} 2 \frac{2^{-dk/a}}{2^{-d(k+1)/a}} = 2 \cdot 2^{-d/a} \xrightarrow{k \to \infty} 2^{1-\frac{d}{a}} < 1 \iff \frac{d}{a} > 1 \iff a < d.$$

Thus, f is integrable if and only if $a < d$.

A similar argument can be used to show the similar result for g.

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11 Prove that if f is integrable on $\mathbb{R}^d$, real-valued, and $\int_E f(x)\,\mathrm{d}x \geq 0$ for every measurable E, then $f(x) \geq 0$ a.e. x. As a result, if $\int_E f(x)\,\mathrm{d}x = 0$ for every measurable E, then $f(x) = 0$ a.e.

Solution Since f is integrable, it is measurable. In particular, the set $E_a \coloneqq \{f \leq a\}$ is measurable, where $a < 0$. Then by assumption,

$$0 \leq \int_{E_a} f \leq \int_{E_a} a = am(E_a),$$

so $m(E_a) = 0$ for every negative a. Thus, $m(\{f < 0\}) = 0$, which proves the claim.

We can use the same argument in the case that $\int_E f = 0$ for every measurable E on the sets $\{f \leq 0\}$ and $\{f \geq 0\}$ to get that $0 \leq f(x) \leq 0$ almost everywhere, so $f(x) = 0$ a.e.

15 Consider the function defined over $\mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

For a fixed enumerate $\{r_n\}_{n \geq 1}$ of the rationals $\mathbb{Q}$, let

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).$$

Prove that F is integrable, hence the series defining F converges for almost every $x \in \mathbb{R}$. However, observe that this series is unbounded on every interval, and in fact, any function $\tilde{F}$ that agrees with F a.e. is unbounded in any interval.

Solution First note that

$$\int_0^1 x^{-1/2}\,\mathrm{d}x = 2x^{1/2}\Big|_0^1 = 2,$$

since Lebesgue integrals coincide with Riemann integrals, for Riemann integrable functions. Horizontal shifts also preserve the integral. In particular, shifting by a rational number will still preserve the integral.

Notice that $2^{-n} f(x - r_n) \geq 0$ for all x. Thus,

$$F_k(x) = \sum_{n=1}^{k} 2^{-n} f(x - r_n)$$

is an increasing sequence of non-negative functions which converge to F everywhere. Then

$$\int F_k = \sum_{n=1}^{k} 2^{-n} \int f(x - r_n) = \sum_{n=1}^{k} 2^{-n} \cdot 2 = \sum_{n=1}^{k} 2^{1-n} < \infty.$$

Then by the monotone convergence theorem,

$$\int F = \lim_{k \to \infty} \int F_k = \sum_{n=1}^{\infty} 2^{1-n} = \frac{1}{1 - \frac{1}{2}} = 2 < \infty,$$

so F is integrable.

Observe that F is unbounded in a neighborhood of a rational number. Indeed, given r_n and $\delta > 0$, then the n-th term in the series for $F(r_n + \delta)$ has the form $2^{-n} f(\delta) = 2^{-n}\delta^{-1/2}$. Then as $\delta \searrow 0$, $2^{-n}\delta^{-1/2} \to \infty$.

Hence, since $\mathbb{Q}$ is dense in $\mathbb{R}$, any open neighborhood contains a rational number, so F is unbounded in that neighborhood.

Suppose $\tilde{F}$ agrees with F on the set E. Then given an open interval around a rational number r, E must contain points arbitrarily close to r. If not, then E is missing an entire open neighborhood around r, which has positive Lebesgue measure since it must contain at least one non-empty open interval.