1 Suppose $\varphi\colon G \to G_1$ is a homomorphism of abelian groups. If φ is onto, show that if G is Noetherian, then G_1 is Noetherian.

Solution Let G be Noetherian.

Let $\{H_n\}_{n\geq 1}$ be an increasing sequence of subgroups of G_1. Then $\{\varphi^{-1}(H_n)\}_{n\geq 1}$ is an increasing sequence of subgroups of G. Since G is Noetherian, there exists $N \geq 1$ such that $n \geq N \implies \varphi^{-1}(H_n) = \varphi^{-1}(H_N)$.

Since φ is onto, $\varphi(\varphi^{-1}(H)) = H$, so we have that $H_n = H_N$ for $n \geq N$, so G_1 is Noetherian.

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2 Suppose $\varphi\colon G \to G_1$ is a homomorphism of abelian groups. Show that if G is Noetherian, then $\ker\varphi$ is Noetherian.

Solution Let G be Noetherian.

Let $\{H_n\}_{n\geq 1}$ be an increasing sequence of subgroups of $\ker\varphi$. Then it is an increasing sequence of subgroups of G, which is Noetherian. It follows that there is an $N \geq 1$ such that $n \geq N \implies H_n = H_N$, so $\ker\varphi$ is Noetherian.

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3 Prove that $\mathbb{Z}$ is Noetherian.

Solution Let $\{H_n\}_{n\geq 1}$ be an increasing sequence of subgroups of $\mathbb{Z}$, which are of the form $n\mathbb{Z}$, where $n \in \mathbb{Z}^+$. So, we can write $H_n = k_n\mathbb{Z}$, where $k_n \in \mathbb{Z}_{\geq 0}$, for all $n \geq 1$.

If $H_n = \{0\}$ for all $n \geq 1$, then we're done. Assume from now on that there is n so that $k_n > 0$.

Notice that if $H_n \subseteq H_{n+1}$, then $k_{n+1} \leq k_n$; otherwise, $k_{n+1}\mathbb{Z}$ would not contain $k_n \in k_n\mathbb{Z}$.

Thus, consider the set $S = \{k_n \in \mathbb{Z}^+ \mid n \geq 1\}$. $S \neq \emptyset$ since subgroups cannot be empty, so by well-ordering, S has a minimal element k_{n_0}. If $n \geq n_0$, then $H_n = H_{n_0}$. Indeed, if $H_{n_0} \subsetneq H_n$ for some $n > n_0$, then this implies that $k_n < k_{n_0}$, which cannot happen.

Hence, $\mathbb{Z}$ is Noetherian.

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4 Prove that if an abelian group G is Noetherian, then G is finitely generated.

Solution Let G be a Noetherian abelian group.

Suppose G is not finitely generated, i.e., given $\{g_1, \ldots, g_n\}$ for any $n \geq 1$, there exists $g_{n+1} \in G$ so that g_{n+1} is not a linear combination of the g_i.

Thus, consider the sequence $\{H_n\}_{n\geq 1}$, given by $H_n = \langle g_1, \ldots, g_n\rangle$. This sequence never becomes constant, since $H_n \subsetneq H_{n+1}$ for every $n \geq 1$, which contradicts the fact that G is Noetherian. Hence, G must be finitely generated.

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5 Prove that if an abelian group G is finitely generated, then there is a positive integer r and an onto homomorphism $\varphi\colon \mathbb{Z}^r \to G$.

Solution Let G be a finitely generated abelian group. Then there exist $g_1, \ldots, g_r \in G$ so that $\langle g_1, \ldots, g_r\rangle = G$. Consider the map $(k_1, \ldots, k_r) \overset{\varphi}{\longmapsto} k_1g_1 + \cdots + k_rg_r$. This is clearly onto, since G is finitely generated.

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6 Suppose $\varphi\colon G \to G_1$ is a homomorphism of abelian groups, G_2 and G_3 are subgroups of G, and that $G_2 \subseteq G_3$. Let H be the kernel of φ. Suppose that $\varphi(G_2) = \varphi(G_3)$ and that $H \cap G_2 = H \cap G_3$. Show $G_2 = G_3$.

Solution Let $g_3 \in G_3$. Since $\varphi(G_2) = \varphi(G_3)$, there exists $g_2 \in G_2$ such that $\varphi(g_2) = \varphi(g_3) \implies g_2^{-1}g_3 \in \ker\varphi = H$. Since $g_2 \in G_3$, we have

$$g_2^{-1}g_3 \in H \cap G_3 = H \cap G_2 \implies g_2^{-1}g_3 \in G_2 \implies g_3 \in G_2,$$

so $G_2 = G_3$, as desired.

7 Prove that $\mathbb{Z} \oplus \mathbb{Z}$ is Noetherian.

Solution See problem 9.

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8 Suppose $\varphi\colon G \to G_1$ is an onto homomorphism, where G is abelian. Suppose G_1 and $\ker\varphi$ are Noetherian. Show G is Noetherian.

Solution Let $\{H_n\}_{n\geq 1}$ be an increasing sequence of subgroups of G.

Consider $\{H_n \cap \ker\varphi\}_{n\geq 1}$, which is an increasing sequence of subgroups of $\ker\varphi$. Since $\ker\varphi$ is Noetherian, there exists n_0 so that $H_n \cap \ker\varphi = H_{n_0} \cap \ker\varphi$ for all $n \geq n_0$. Let K be G after we remove elements of $\ker\varphi$ that are not in $H_{n_0} \cap \ker\varphi$.

Consider $\varphi\big|_K$, which is φ restricted to K. This is a homomorphism, since φ is a homomorphism. Since G_1 is Noetherian, there exists $n_1 \geq n_0$ so that $\varphi\big|_K(H_n)$ is constant when $n \geq n_1$.

We'll show that $\left(\varphi\big|_K\right)^{-1}(\varphi\big|_K(H_n)) = H_n$ for $n \geq n_1$.

It's clear that $H_n \subseteq \left(\varphi\big|_K\right)^{-1}(\varphi\big|_K(H_n))$ and that $\varphi\big|_K(H_n) = \varphi\big|_K\Big(\left(\varphi\big|_K\right)^{-1}(\varphi\big|_K(H_n))\Big)$. Moreover, we have

$$\ker\varphi\big|_K \cap H_n = \ker\varphi \cap H_{n_0} \cap H_n = \ker\varphi \cap H_{n_0} = \ker\varphi\big|_K,$$

since $\ker\varphi \cap H_{n_0}$ is constant when $n \geq n_1$. Moreover,

$$\ker\varphi\big|_K \cap \left(\varphi\big|_K\right)^{-1}\left(\varphi\big|_K(H_n)\right) = \ker\varphi\big|_K \cap \left(\varphi\big|_K\right)^{-1}(\{e\}) = \ker\varphi\big|_K.$$

Thus, by problem 6, $\left(\varphi\big|_K\right)^{-1}(\varphi\big|_K(H_n)) = H_n$ for $n \geq n_1$, so

$$H_n = \left(\varphi\big|_K\right)^{-1}(\varphi\big|_K(H_n)) = \left(\varphi\big|_K\right)^{-1}(\varphi\big|_K(H_{n_1})).$$

Thus, G is Noetherian.

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9 Show $\mathbb{Z}^r$ is Noetherian.

Solution Let $\{H_n\}_{n\geq 1}$ be an increasing sequence of subgroups of $\mathbb{Z}^r$.

Notice that if we project the ℓ-th coordinate, we get an increasing sequence of subgroups $\{H_n^{(\ell)}\}_{n\geq 1}$ of $\mathbb{Z}$, since each coordinate must satisfy the subgroup axioms. Since $\mathbb{Z}$ is Noetherian, there exists n_ℓ so that $n \geq n_\ell \implies H_n^{(\ell)} = H_{n_\ell}^{(\ell)}$.

Take $n_0 = \max\{n_\ell \mid 1 \leq \ell \leq r\}$. Then if $n \geq n_0$, we have $n \geq n_\ell$ for each ℓ, so $H_n^{(\ell)} = H_{n_\ell}^{(\ell)}$. Thus, for $n \geq n_0$, each component becomes constant, so $\mathbb{Z}^r$ is Noetherian.

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10 Show that if an abelian group G is finitely generated, then G is Noetherian.

Solution Let G be finitely generated, so that $\langle g_1, \ldots, g_r\rangle = G$.

Let $\{H_n\}_{n\geq 1}$ be an increasing sequence of subgroups of G. Each H_n is generated by a subset of $\{g_1, \ldots, g_r\}$. Thus, eventually, the sequence given by the number of generators in H_n must become constant. Indeed, it is an increasing sequence bounded above by r, so there exists $n_0 \geq 1$ so that if $n \geq n_0$, we get

$$H_n = \langle g_{\sigma(1)}, g_{\sigma(2)}, \ldots, g_{\sigma(m)}\rangle,$$

for some permutation σ, so G is finitely generated.

11 Show that if an abelian group G is finitely generated, then we can find integers s, r and a homomorphism $\Phi\colon \mathbb{Z}^s \to \mathbb{Z}^r$ so that G is isomorphic to

$$\mathbb{Z}^r/\Phi(\mathbb{Z}^s).$$

Solution Let G be a finitely generated abelian group, so that $G = \langle g_1, \ldots, g_r \rangle$, and consider the surjective homomorphism $\varphi\colon \mathbb{Z}^r \to G$ given by $\varphi(n_1, \ldots, n_r) = n_1 g_1 + \cdots + n_r g_r$.

Notice that $\ker\varphi$ is a linear subspace of $\mathbb{Z}^s$ over $\mathbb{R}$, since φ is linear, and that $\ker\varphi$ has dimension $s \leq r$ for some s. If we write an r-tuple as $\mathbf{n}$, we can find s linearly independent r-tuples $\mathbf{n_1}, \ldots, \mathbf{n_s}$. Our homomorphism can then be

$$\Phi(m_1, \ldots, m_s) = m_1\mathbf{n_1} + \cdots + m_s\mathbf{n_s}.$$

Since our group is abelian, $\varphi(\Phi(m_1, \ldots, m_s)) = m_1\varphi(\mathbf{n_1}) + \cdots + m_s\varphi(\mathbf{n_s}) = e$, so Φ maps to $\ker\varphi$, and the map is onto since the dimension of $\ker\varphi$ and $\mathbb{Z}^s$ is s. Thus, by the first isomorphism theorem,

$$G \simeq \mathbb{Z}^r/\ker\varphi = \mathbb{Z}^r/\Phi(\mathbb{Z}^s).$$

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12 Suppose A is an $r \times s$ integer-valued matrix. Show that there is a homomorphism

$$L_A\colon \mathbb{Z}^s \to \mathbb{Z}^r$$

defined by the usual matrix multiplication. Show that any homomorphism $\Phi\colon \mathbb{Z}^s \to \mathbb{Z}^r$ is of the form L_A for some $r \times s$ integer-valued matrix A.

Solution Φ is linear because it is a homomorphism and because $\mathbb{Z}^s$ is abelian, i.e., $\Phi(an_1 + bn_1', \ldots, an_s + bn_s') = a\Phi(n_1, \ldots, n_s) + b\Phi(n_1', \ldots, n_s")$. By linear algebra, any linear transformation can be associated with a matrix, so any homomorphism $\Phi\colon \mathbb{Z}^s \to \mathbb{Z}^r$ can be represented with an $r \times s$ matrix with integer entries. Indeed, we can find the matrix by finding the image of the canonical basis vectors under Φ.

Consider the matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rs}, \end{pmatrix}$$

where $A_{ij} \in \mathbb{Z}$. It is well-known that the transformation given by $L_A(\mathbf{n}) = A\mathbf{n}$ is linear, i.e.,

$$L_A(\mathbf{n} + \mathbf{m}) = A(\mathbf{n} + \mathbf{m}) = A\mathbf{n} + A\mathbf{m} = L_A(\mathbf{n}) + L_A(\mathbf{m}).$$

It's also clear from the definitions of matrix multiplication and the integers that this maps to $\mathbb{Z}^r$, so L_A is a homomorphism from $\mathbb{Z}^s$ to $\mathbb{Z}^r$.

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13 If B is an integer-valued $r \times r$ matrix with $\det B = \pm 1$, show $L_B\colon \mathbb{Z}^r \to \mathbb{Z}^r$ is an isomorphism.

Solution A result of linear algebra is Cramer's rule, which tells us that if B is invertible, then $B(n_1, \ldots, n_r) = (m_1, \ldots, m_r)$ has a solution which is given by

$$n_i = \frac{1}{\det B} \det \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1(i-1)} & m_1 & A_{1(i+1)} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2(i-1)} & m_2 & A_{1(i+1)} & \cdots & A_{2s} \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{r(i-1)} & m_r & A_{1(i+1)} & \cdots & A_{rs}, \end{pmatrix}.$$

The determinant is a sum of products of the entries of A, which are all integers, so the determinant will also be an integer, since $\mathbb{Z}$ is a ring. $1/\det B$ will multiply the result by 1 or -1, so the result is still integer. Moreover, the relation given above defines an inverse for L_B, so it is an isomorphism. In particular, it's a homomorphism, so it can also be represented by an integer-valued matrix.

14 Suppose A is an $r \times s$ integer-valued matrix. If B is an $r \times r$ matrix with $\det B = \pm 1$, show that the cokernel of L_{BA} is isomorphic to the cokernel of L_A.

Solution We need to show that $\mathbb{Z}^r / L_{BA}(\mathbb{Z}^r) \simeq \mathbb{Z}^r / L_A(\mathbb{Z}^r)$.

Consider the map $\mathbf{n} + L_{BA}(\mathbb{Z}^r) \stackrel{\varphi}{\longmapsto} L_B^{-1}(\mathbf{n}) + L_A(\mathbb{Z}^r)$.

It's a homomorphism, since L_B^{-1} is another matrix and is thus linear:

$$\varphi(\mathbf{n} + \mathbf{m} + L_{BA}(\mathbb{Z}^r)) = L_B^{-1}(\mathbf{n}) + L_A(\mathbb{Z}^r) + L_B^{-1}(\mathbf{m}) + L_A(\mathbb{Z}^r) = \varphi(\mathbf{n} + L_A(\mathbb{Z}^r)) + \varphi(\mathbf{m} + L_A(\mathbb{Z}^r)).$$

It is also injective. Notice that $L_B \circ L_A = L_{BA}$. Then by linearity,

$$\begin{aligned} &\varphi(\mathbf{n} + L_A(\mathbb{Z}^r)) = \varphi(\mathbf{m} + L_A(\mathbb{Z}^r)) \\ \implies &L_B^{-1}(\mathbf{n}) + L_A(\mathbb{Z}^r) = L_B^{-1}(\mathbf{m}) + L_A(\mathbb{Z}^r) \\ \implies &L_B\left[L_B^{-1}(\mathbf{n}) + L_A(\mathbb{Z}^r)\right] = L_B\left[L_B^{-1}(\mathbf{m}) + L_A(\mathbb{Z}^r)\right] \\ \implies &\mathbf{n} + L_{BA}(\mathbb{Z}^r) = \mathbf{m} + L_{BA}(\mathbb{Z}^r), \end{aligned}$$

so φ is injective.

φ is also surjective because L_B^{-1} is bijective, i.e., given $\mathbf{m} + L_A(\mathbb{Z}^r)$, there exists $\mathbf{n}$ so that $\mathbf{n} = L_B^{-1}(\mathbf{m})$.

Thus, φ is an isomorphism, so $\mathbb{Z}^r / L_{BA}(\mathbb{Z}^r) \simeq \mathbb{Z}^r / L_A(\mathbb{Z}^r)$.

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15 Suppose A is an $r \times s$ integer-valued matrix. If C is an integer-valued $s \times s$ matrix with $\det C = \pm 1$, show that the cokernel of L_{AC} equals to the cokernel of L_A.

Solution Notice that $L_{AC}(\mathbb{Z}^s) = L_A(\mathbb{Z}^s)$. Indeed, L_C is an automorphism on $\mathbb{Z}^s$ by problem 13, so $L_{AC}(\mathbb{Z}^s) = L_A(L_C(\mathbb{Z}^s)) = L_A(\mathbb{Z}^s)$. Thus, $\mathbb{Z}^r / L_{AC}(\mathbb{Z}^s) \simeq \mathbb{Z}^r / L_A(\mathbb{Z}^s)$.

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16 Show that if A' is the result of a row operation on A, then there is an $r \times r$ matrix B with $\det B = \pm 1$ so that $A' = BA$. Also show that if A_1 is the result of a column operation on A, then there is an $s \times s$ matrix C with $\det C = \pm 1$ so that $A_1 = AC$. B and C are usually called elementary matrices.

Solution If we perform the matrix multiplication EA, it is easy to see that the rows of E puts the rows of A in linear combinations. E.g., multiplying by $\begin{pmatrix} 1 & 2 & 0 \end{pmatrix}$

$$(1 \times \text{Row 1}) + (2 \times \text{Row 2}) + (0 \times \text{Row 3}).$$

The elementary matrix that switches the i-th and j-th row on A is given by the identity matrix with the i-th and j-th row switched, and the matrix is applied to the left of A. When calculating the determinant, each cofactor expansion on a row only has 1 as a non-zero term, so the determinant is ± 1.

Multiplying the i-th row by -1 is given by the identity matrix, but with -1 as the i-th entry on the diagonal instead of 1. This gives a diagonal matrix, so its determinant is given by the product of the diagonals, which gives -1.

Adding λ times the j-th row to the i-th row is given by the identity matrix, but with $I_{ij} = \lambda$ instead of 0. This gives an upper or lower-triangular matrix, and its determinant is also given by the product of the diagonal entries, which gives 1.

Compositions of these elementary matrices give us all possible row operations. Moreover, the determinant of a product of matrices is the product of the determinants, so a general row operation B has $\det B = \pm 1$.

For column operations, we can just perform $(EA^\top)^\top = AE^\top$. Transposing A reduces the problem to performing row operations on the transposed columns, e.g., swapping the i-th and j-th row is equivalent to swapping the i-th and j-th column after transposition. Transposition doesn't affect the determinant so such a matrix also has determinant ± 1.

17 Suppose A is an $r \times s$ integer-valued matrix. Show that you can perform a sequence of row and column operations on A to reduce A to a special matrix.

Solution We wish to show that we can reduce A to a matrix with non-zero entries only on the first diagonal entries, and zero everywhere else.

We first swap rows so that the first rows are non-zero, and the remaining rows are zero. The algorithm is given as follows:

Consider the first column. The first row has entry n and the second row has entry m. There exist q, r with $0 \leq r < m$ so that $n = qm + r$. So, we can subtract q times the second row to end up with r in the first row, which is strictly less than m. We perform the division algorithm again to get $m = q_1 r + r_1$, and subtract q_1 times the first row from the second row to get $r_1 < r$. This causes the entries to strictly decrease, so we can repeat the process until we get a zero entry in one of the rows. If it's the first row, then we move on. Otherwise, we swap the first two rows so that the first row of the first column has a non-zero entry.

$$\begin{pmatrix} n & m & \cdots \end{pmatrix} = \begin{pmatrix} qm + r & m & \cdots \end{pmatrix} \longrightarrow \begin{pmatrix} r & m & \cdots \end{pmatrix} = \begin{pmatrix} r & q_1 r + r_1 & \cdots \end{pmatrix} \longrightarrow \begin{pmatrix} r & r_1 & \cdots \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & r_k & \cdots \end{pmatrix}$$

We repeat this process with the first row and the i-th row until the first column has a non-zero first row, and zeroes everywhere else.

We can repeat this process with the transpose of A, and after transposing again, A has a non-zero entry in the first diagonal, and zeroes everywhere else in the first row and column.

Now rearrange the rows and columns again, keeping the first row and column fixed, so that the second column has a non-zero entry in the second row. If we consider the submatrix we get by ignoring the first row and column, we can perform the same algorithm, since the algorithm does not depend on the size of the matrix A.

This algorithm gives us a special matrix, since we eliminate every element except on the diagonal. Additionally, each elementary matrix we used has determinant ± 1.

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18 Show that if A is special, then the cokernel of A is isomorphic to a direct sum of cyclic groups.

Solution Let A be an $r \times s$ special matrix, so that $A_{ii} := a_i > 0$ for $i \leq k$ for some k, and $A_{ij} = 0$ otherwise. We can do this by multiplying the rows of A by a unit.

We wish to show that $\mathbb{Z}^r / A(\mathbb{Z}^s) \simeq \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_r\mathbb{Z}$.

Notice that $A(\mathbb{Z}^s) = a_1\mathbb{Z} \oplus \cdots \oplus a_r\mathbb{Z} \oplus \{0\} \oplus \cdots \oplus \{0\}$, since A is a diagonal matrix. It suffices to show that each component is cyclic.

Consider the unit vector $\mathbf{e_k}$, where the k-th entry is 1 and has zeroes everywhere else. Then $a_k(\mathbf{e_k} + A(\mathbb{Z}^s)) = a_k\mathbf{e_k} + A(\mathbb{Z}^s)$, since $A(\mathbb{Z}^s) + \cdots + A(\mathbb{Z}^s) = A(\mathbb{Z}^s)$. Then

$$a_k\mathbf{e_k} \in \prod_{i=1}^{k-1} \{0\} \times a_k\mathbb{Z} \times \prod_{i=k+1}^{r} \{0\} \subseteq A(\mathbb{Z}^s) \implies a_k\mathbf{e_k} + A(\mathbb{Z}^s) = A(\mathbb{Z}^s),$$

so $\mathbf{e_k} + A(\mathbb{Z}^s)$ has order a_k; anything smaller will not be a multiple of a_k. Thus, each component is cyclic, so we have

$$\mathbb{Z}^r / A(\mathbb{Z}^s) \simeq \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_r\mathbb{Z}.$$

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19 Any finitely generated abelian group is isomorphic to a direct sum of cyclic groups.

Solution By problem 11, we know that there exist r, s integers and a homomorphism $\Phi \colon \mathbb{Z}^s \to \mathbb{Z}^r$ so that $G \simeq \mathbb{Z}^r / \Phi(\mathbb{Z}^s)$. By problem 12, we know that Φ can be represented by a matrix A'. By 15 and 17, we know that the cokernel of Φ is isomorphic to the cokernel as a special matrix A, by perform row operations, which can be represented by an elementary matrix with determinant ± 1. Lastly, by 18, the cokernel of a special matrix is isomorphic to the direct sum of cyclic groups. Symbolically,

$$G \simeq \mathbb{Z}^r / \Phi(\mathbb{Z}^s) = \mathbb{Z}^r / A'(\mathbb{Z}^s) \simeq \mathbb{Z}^r / A(\mathbb{Z}^s) \simeq \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_r\mathbb{Z}.$$