1 Let $\{\mathbf{x}\}_{k=1}^{\infty} \subseteq \mathbb{R}^n$ be a convergent sequence, prove that there exists $B > 0$ such that $\|\mathbf{x}^k\| \leq B$ for all k.

Solution Since $\{\mathbf{x}\}_{k=1}^{\infty}$ is convergent, its limit $\mathbf{x} \in \mathbb{R}^n$ exists. Then by definition of convergence, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $\|\mathbf{x}^k - \mathbf{x}\| < 1$. Then by the triangle inequality,

$$\|\mathbf{x}^k\| - \|\mathbf{x}\| \leq \|\mathbf{x}^k - \mathbf{x}\| < 1 \implies \|\mathbf{x}^k\| \leq 1 + \|\mathbf{x}\|$$

for all $k \geq N$. Choose $B = \max\left\{\|\mathbf{x}\|^1, \|\mathbf{x}\|^2, \ldots, \|\mathbf{x}\|^{N-1}, 1 + \|\mathbf{x}\|\right\}$. Then $\|\mathbf{x}^k\| \leq B$ for all k.

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2 Let $\mathbf{x}, \mathbf{x} \in \mathbb{R}^n$, consider the functions $f(\mathbf{x}) = \mathbf{a}^\top \mathbf{x}$ and $g(\mathbf{x}) = (\mathbf{a}^\top \mathbf{x})^2$.

a. Find $\nabla f(\mathbf{x})$ and the Hessian $H_f(\mathbf{x})$.

b. Show that $g(\mathbf{x})$ is a quadratic form.

c. Find $\nabla g(\mathbf{x})$ and $H_g(\mathbf{x})$ using part (b).

Solution a. Note that $f(\mathbf{x}) = \sum_{n=1}^{n} a_n x_n$. $\frac{\partial f}{\partial x_i} = a_i$ and $\frac{\partial^2 f}{\partial x_i \partial x_2} = 0$. Then $\nabla f(\mathbf{x}) = \mathbf{a}$ and $H_f(\mathbf{x}) = \mathbf{0}$.

b. Note that since $\mathbf{a}^\top \mathbf{x} \in \mathbb{R}$, $\mathbf{a}^\top \mathbf{x} = (\mathbf{a}^\top \mathbf{x})^\top = \mathbf{x}^\top \mathbf{a}$. Thus, $g(\mathbf{x}) = (\mathbf{a}^\top \mathbf{x})^2 = \mathbf{x}^\top (\mathbf{a}\mathbf{a}^\top)\mathbf{x}$. $\mathbf{a}\mathbf{a}^\top$ is an $n \times n$ matrix, so $g(\mathbf{x})$ is a quadratic form.

c. Note that $\mathbf{a}\mathbf{a}^\top$ is symmetric since $(\mathbf{a}\mathbf{a}^\top)^\top = \mathbf{a}\mathbf{a}^\top$. Thus, $\nabla g(\mathbf{x}) = 2\mathbf{a}\mathbf{a}^\top \mathbf{x}$ and $H_g(\mathbf{x}) = 2\mathbf{a}\mathbf{a}^\top$.

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3 Let $f(\mathbf{x}) = \mathbf{x}^\top Q \mathbf{x}$ be a quadratic form, where $Q \in \mathbb{R}^{n \times n}$ is **NOT** symmetric. Show that the Hessian matrix is $H_f(\mathbf{x}) = Q + Q^T$. You may use the result for symmetric matrices without proof.

Solution First notice that since $f(\mathbf{x}) \in \mathbb{R}$, $f(\mathbf{x}) = (f(\mathbf{x}))^\top = \mathbf{x}^\top Q^\top \mathbf{x}$.

Consider the quadratic form $2f(\mathbf{x}) = \mathbf{x}^\top Q \mathbf{x} + \mathbf{x}^\top Q^\top \mathbf{x} = \mathbf{x}^\top (Q + Q^\top)\mathbf{x}$. Note that $Q + Q^\top$ is symmetric, so

$$H_{2f}(\mathbf{x}) = 2H_f(\mathbf{x}) = 2(Q + Q^\top) \implies H_f(\mathbf{x}) = Q + Q^\top.$$

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4 Consider the function $f \colon \mathbb{R}^2 \to \mathbb{R}$,

$$f(\mathbf{x}) = \mathbf{x}^\top \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 \\ 8 \end{pmatrix}^\top \mathbf{x} + 10$$

and the point $\mathbf{x}^* = (1, 1)^\top$.

a. Find $\nabla f(\mathbf{x}^*)$ and $H_f(\mathbf{x}^*)$.

b. Find the **unit-normed** vector (direction) $\mathbf{d}$ that minimizes the directional derivative $\frac{\partial f}{\partial \mathbf{d}}(\mathbf{x}^*)$.

c. Find a point that satisfies the FONC for f. Does this point satisfy the SONC?

Solution a.

$$\nabla f(\mathbf{x}) = 2\begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 4 \\ 8 \end{pmatrix} \implies \nabla f(\mathbf{x}^*) = \begin{pmatrix} 12 \\ 24 \end{pmatrix}$$

$$H_f(\mathbf{x}) = 2\begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 6 & 10 \end{pmatrix} \implies H_f(\mathbf{x}^*) = \begin{pmatrix} 2 & 6 \\ 6 & 10 \end{pmatrix}$$

b. $\frac{\partial f}{\partial \mathbf{d}}(\mathbf{x}^*) = \nabla f(\mathbf{x}^*) \cdot \mathbf{d} = \left\|\begin{pmatrix} 12 \\ 24 \end{pmatrix}\right\| \|\mathbf{d}\| \cos\theta$ is minimized when $\theta = \pi \implies \mathbf{d} = -\frac{1}{12\sqrt{5}}\begin{pmatrix} 12 \\ 24 \end{pmatrix}$.

c. Our feasible set is the interior of $\mathbb{R}^2$ (i.e., the whole space), so we want $\nabla f(\mathbf{x}^\dagger) = 0$. This occurs when $\mathbf{x}^\dagger = -(\frac{1}{2}, \frac{1}{2})^\top$:

$$\nabla f(\mathbf{x}^\dagger) = 2\begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} 4 \\ 8 \end{pmatrix} = -\begin{pmatrix} 4 \\ 8 \end{pmatrix} + \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

However, it does not satisfy the SONC since not all the eigenvalues of $H_f(\mathbf{x}^\dagger)$ are positive; in particular, one of its eigenvalues is $\lambda = -2(\sqrt{13} - 3) < 0$.

5 Consider the problem

$$\min x_1^2 + x_2^2 - 2x_2 + 2 \quad \text{subject to} \quad x_1, x_2 \geq 0.$$

a. Is the FONC satisfied at $\mathbf{x}^* = (0,0)^\top$?

b. Is the FONC satisfied at $\mathbf{x}^* = (0,1)^\top$? Is the SONC satisfied at $\mathbf{x}^*$?

Solution a. Let $f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_2 + 2$. Then $\nabla f(\mathbf{x}) = (2x_1, 2x_2 - 2)^\top \implies \nabla f(\mathbf{x}^*) = (0,-2)^\top$. Then if $\mathbf{d} = (\cos\theta, \sin\theta)^\top$ is a feasible direction, $0 \leq \theta \leq \frac{\pi}{2}$. Hence,

$$\frac{\partial f}{\partial \mathbf{d}} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = -2\sin\theta \leq 0.$$

In particular, the directional derivative is negative if $0 < \theta \leq \frac{\pi}{2}$. Hence, $\mathbf{x}^*$ does not satisfy the FONC.

b. $\nabla f(\mathbf{x}^*) = (0,0)^\top$, so it satisfies the FONC. Note that $H_f(\mathbf{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \implies H_f(\mathbf{x}^*) = 2I_2 \succ 0$ since both of its eigenvalues are $2 > 0$. Hence, it satisfies the SONC also.

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6 Suppose $f \colon \mathbb{R}^n \to \mathbb{R}$ is differentiable. Let $\mathbf{d} \in \mathbb{R}^n$ with $\|d\| = 1$. Show that

$$\langle \nabla f(\mathbf{x}), \mathbf{d} \rangle \geq -\|\nabla f(\mathbf{x})\|.$$

For what $\mathbf{d}$ does the equality hold?

Solution Using the standard inner product, we have that

$$\langle \nabla f(\mathbf{x}), \mathbf{d} \rangle = \nabla f(\mathbf{x}) \cdot \mathbf{d} = \|\nabla f(\mathbf{x})\| \|\mathbf{d}\| \cos\theta = \|\nabla f(\mathbf{x})\| \cos\theta,$$

where θ is the angle between $\nabla f(\mathbf{x})$ and $\mathbf{d}$. The minimum value is attained when $\cos\theta = -1 \implies \theta = \pi$, which gives us

$$\mathbf{d} = -\frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|} \implies \langle \nabla f(\mathbf{x}), \mathbf{d} \rangle = -\|\nabla f(\mathbf{x})\|.$$

Since that is the minimum value, we have the desired inequality

$$\langle \nabla f(\mathbf{x}), \mathbf{d} \rangle \geq -\|\nabla f(\mathbf{x})\|.$$