27.22.1 Let R be a domain and a a non-zero non-unit in R. Show that a is irreducible if and only if the principal ideal (a) is maximal in the set $\{(b) \mid b \text{ a non-zero non-unit in } R\}$. In particular, if R is a PID, then every irreducible element in R is a prime element.

Solution "$\implies$"

Let a be irreducible.

Suppose (a) is not maximal in the given set so that there exists $b \neq a$ so that $(b) > (a)$. But this implies that there exists $r \in R$ so that $a = rb$. Since a is irreducible and b is not a unit, we must have that r is a unit. But this means that there exists $s \in R$ such that $sr = 1 \implies sa = b \implies (b) \subseteq (a)$, a contradiction.

"$\impliedby$"

Let (a) be maximal in the given set

Suppose $(a) = xy$, for some non-zero $x, y \in R$, and suppose that x is not a unit. Then $(a) \subseteq (x)$. Since (a) was maximal, $(a) = (x)$, so there exists $r \in R$ such that $x = ra \implies a = (ry)a \implies ry = 1$, so y is a unit.

27.22.2 Produce elements a and b in the domain $R := \{x + 2y\sqrt{-1} \mid x, y \in \mathbb{Z}\}$ having no gcd. Prove your elements do not have a gcd.

Solution We claim that 8 and $4 - 4\sqrt{-1}$ do not have a gcd.

Notice that

$$8 = (2 - 2\sqrt{-1})(2 + 2\sqrt{-1}) = 2 \cdot 4 \quad \text{and} \quad 4 - 4\sqrt{-1} = 2(2 - 2\sqrt{-1}),$$

which mean that 4 and $2 - 2\sqrt{-1}$ divide both numbers.

2 and $1 - 1\sqrt{-1}$ are irreducible:

$$N(2) = N(a + 2b\sqrt{-1})N(c + 2d\sqrt{-1}) \implies 4 = (a^2 + 4b^2)(c^2 + 4d^2).$$

To get a non-trivial factorization, we need $a^2 + 4b^2 = 2$, but this isn't possible. Similarly,

$$N(1 - 1\sqrt{-1}) = N(a + 2b\sqrt{-1})N(c + 2d\sqrt{-1}) \implies 2 = (a^2 + 4b^2)(c^2 + 4d^2).$$

But 2 is prime, so there is no non-trivial factorization. Hence, both of these numbers are irreducible, so 2 and $1 - 1\sqrt{-1}$ are the unique (up to multiplication by a unit) divisors of $4 - 4\sqrt{-1}$.

All the numbers that divide both of them are $1, 2, 4, 1 - 1\sqrt{-1}$, and $2 - 2\sqrt{-1}$. It's clear that our only candidates for the gcd are 4 and $2 - 2\sqrt{-1}$, but

$$4 = 2 \cdot 2 \quad \text{and} \quad 2 - 2\sqrt{-1} = 2 \cdot (1 - \sqrt{-1}).$$

But $1 - \sqrt{-1}$ and 2 are irreducible, which means that these two numbers do not divide each other. Consequence, they do not have a gcd.

27.22.4 Show 1 is a gcd for 2 and t in $\mathbb{Z}[t]$, but there are no polynomials $f, g \in \mathbb{Z}[t]$ satisfying $1 = 2f + tg$.

Solution Let $e \in \mathbb{Z}[t]$ be so that $e \mid 2$ and $e \mid t$, so that there exist $f, g \in \mathbb{Z}[t]$ such that $2 = ef$ and $t = eg$.

If $e \approx 2$, then $t = eg \approx 2g$, but this is impossible since 2 is not a unit in $\mathbb{Z}$.

So, we have $f \approx 2$, which implies that e is a unit, which implies that $e \mid 1$, so 1 is a gcd for 2 and t.

There are no such $f, g \in \mathbb{Z}[t]$ satisfying $1 = 2f + tg$. Otherwise, we require that $g = 0$ so that we have no non-constant terms, which leaves us with $1 = 2f$. But as stated before, 2 is not a unit in $\mathbb{Z}$, so there is no such f that satisfies the equation.

27.22.6 Prove that the three conditions defining a Noetherian ring are indeed equivalent.

Solution (i) $\implies$ (ii)

Let R satisfy the ACC. We wish to show that every ideal is finitely generated.

Let $\mathfrak{A}$ be an ideal in R, and let $a_1 \in \mathfrak{A}$.

If $(a_1) = \mathfrak{A}$, then we're done. If not, pick $a_2 \in \mathfrak{A} \setminus (a_1)$.

If $(a_1, a_2) = \mathfrak{A}$, then we're done. If not, pick $a_3 \in \mathfrak{A} \setminus (a_1, a_2)$.

This process generates a sequence

$$(a_1) \subseteq (a_1, a_2) \subseteq \cdots .$$

So R satisfies the ACC, the process must terminate after finitely many steps, so $\mathfrak{A}$ is finitely generated.

(ii) $\implies$ (iii)

Suppose every ideal is finitely generated. We wish to show that R satisfies the maximal principle.

Let S be a non-empty set of ideals, and let $\mathcal{C}$ be a chain in S. Then $\bigcup \mathcal{C}$ is an upper bound: Indeed, it is an ideal, and by assumption it's finitely generated. Each of its generators must lie in some element in $\mathcal{C}$, and since the chain is totally ordered, there exists an element in $\mathcal{C}$ that contains all of them, so $\bigcup \mathcal{C} \in \mathcal{C} \subseteq S$.

Thus, every chain has an upper bound in S, so by Zorn's lemma, S has a maximal element, so R has the maximal principle.

(iii) $\implies$ (i)

Let R satisfy the maximal principle. We want to show that it satisfies the ACC.

Let $\{\mathfrak{A}_i\}$ be a sequence of ideals in R. Now consider the set $S = \{\mathfrak{A}_i \mid i \geq 1\}$. By assumption, this has a maximal element $\mathfrak{A}_n$. This implies that if $i \geq n$, $\mathfrak{A}_i = \mathfrak{A}_n$, so R satisfies the ACC.

27.22.9 A commutative ring R is called *Artinian* if it satisfies the following condition (called the *descending chain condition* or *DCC*): Any chain of ideals

$$\mathfrak{A}_1 \supseteq \mathfrak{A}_2 \supseteq \cdots \supseteq \mathfrak{A}_n \supseteq \cdots$$

(countable) in R *stabilizes*, i.e., there exists an integer N such that $\mathfrak{A}_{N+i} = \mathfrak{A}_N$ for all $i \geq 0$. Equivalently, there exist no infinite chains

$$\mathfrak{B}_1 > \mathfrak{B}_2 > \cdots > \mathfrak{B}_n > \cdots$$

Show that R is Artinian if and only if it satisfies the *minimal principle* which says that any non-empty collection of ideals in R has a minimal element (under set inclusion).

Solution "$\implies$"

Suppose R is Artinian, and let S be a non-empty collection of ideals in R.

Suppose that S has no minimal element. Then for every $\mathfrak{A}_i \in S$, we can find $\mathfrak{A}_{i+1} < \mathfrak{A}_i$. Since S is non-empty, we can pick any ideal in S to be $\mathfrak{A}_1$. Then we have

$$\mathfrak{A}_1 > \mathfrak{A}_2 > \cdots$$

But this violates the fact that R is Artinian because this sequence must stabilize, a contradiction. Hence, S must contain some minimal element.

"$\impliedby$"

Suppose R satisfies the minimal principle. We would like to show that R is Artinian.

Let $\{\mathfrak{A}_i\}$ be a sequence of ideals with

$$\mathfrak{A}_1 \supseteq \mathfrak{A}_2 \supseteq \cdots$$

Now consider $S = \{\mathfrak{A}_i \mid i \geq 1\}$. This has a minimal element $\mathfrak{A}_n$, by assumption. Hence, if $i \geq n$, $\mathfrak{A}_i = \mathfrak{A}_n$, so the chain stabilizes, i.e., R is Artinian.

27.22.10 If R is a domain, show that it is Artinian if and only if it is a field.

Solution "$\implies$"

Let R be Artinian.

It suffices to show that R is a division ring, i.e., every non-zero element is a unit.

Let $x \in R$. Suppose x is a non-zero non-unit, so that $(x) < R$, and consider the chain

$$(x) \supseteq (x^2) \supseteq \cdots$$

Since R is Artinian, this sequence must stabilize, so there exists $n \geq 1$ such that $x^n = x^{n+1} \implies x^n(x-1) = 0$. Since R is a domain, we have $x^n = 0$ or $x - 1 = 0$. Again, because R is a domain, the first case implies that $x = 0$, which can't happen. But in the other case, $x = 1$, which is impossible since we assumed x to be non-unit. Hence, x must be a unit, so R is a field.

"$\impliedby$"

Let R be a field, and let $\mathfrak{A}_1, \mathfrak{A}_2, \ldots < R$ be a sequence of ideals with

$$\mathfrak{A}_1 \supseteq \mathfrak{A}_2 \supseteq \cdots$$

None of these ideals can be trivial. Otherwise, if $\mathfrak{A}_n = \{0\}$, then $\mathfrak{A}_{n+i} = \{0\}$ for all $i \geq 1$.

Since the sequence does not stabilize, there exists $n \geq 1$ so that $\mathfrak{A}_n > \mathfrak{A}_{n+1}$. In particular, there exists a non-zero element $x \in \mathfrak{A}_n \setminus \mathfrak{A}_{n+1}$. But x is a unit, which implies that $\mathfrak{A}_n = R$, a contradiction. Hence, this sequence must stabilize.

27.22.11 Let R be a Noetherian ring. Show that $\varphi\colon R \to S$ is a ring epimorphism, then S is Noetherian.

Solution Let $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ be an increasing sequence of ideals in S. Since φ is epi, there exist $\mathfrak{B}_1, \mathfrak{B}_2, \ldots$ so that $\varphi(\mathfrak{B}_i) = \mathfrak{A}_i$ for every i.

If $\mathfrak{B}_i$ is an ideal for every i, then $\mathfrak{B}_i$ is an increasing sequence of ideals, which must stabilize since R is Noetherian. This implies that $\varphi(\mathfrak{B}_i) = \mathfrak{A}_i$ must stabilize also, which shows that S is Noetherian. So, it suffices to show that $\mathfrak{B}_i$ is an ideal.

Let $x, y \in \mathfrak{B}_i$. Since $\mathfrak{A}_i$ is an ideal and φ is a morphism, $\varphi(x+y) = \varphi(x) + \varphi(y) \in \mathfrak{A}_i \implies x + y \in \mathfrak{B}_i$.

Now let $r \in R$. Then $\varphi(rx) = r\varphi(x) \in \mathfrak{A}_i \implies rx \in \mathfrak{B}_i$.

Thus, $\mathfrak{B}_i$ is ideal, and we're done.

27.22.13 Let R be a Noetherian domain. Show that any non-trivial ideal of R contains a finite product of non-zero prime ideals, i.e., if $0 < \mathfrak{A} < R$ is an ideal, then there exist non-zero prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ in R such that $\mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_n \subseteq \mathfrak{A}$.

Solution Let $\mathfrak{A}$ be an ideal.

Suppose that the claim is not true, and that there is some ideal which does not contain a product of non-zero prime ideals. Now let S be the set where the claim is false, which is non-empty by assumption.

Any chain in S has an upper bound, i.e., the union of the chain, and this union is clearly contained in S. Otherwise, if $\mathfrak{p}$ is in the union, it must be contained one of the elements of the chain.

By Zorn's lemma, S has a maximal element with respect to $\subseteq$, which we'll call $\mathfrak{A}$.

Now let $xy \in \mathfrak{A}$. Suppose $x, y \notin \mathfrak{A}$. Then $\mathfrak{A} + (x), \mathfrak{A} + (y) > \mathfrak{A}$, which means that $\mathfrak{A} + (x)$ and $\mathfrak{A} + (y)$ each contain some prime ideal $\mathfrak{p}_1, \mathfrak{p}_2$, since $\mathfrak{A}$ was maximal in S. But

$$\mathfrak{p}_1\mathfrak{p}_2 \subseteq (\mathfrak{A} + (x))(\mathfrak{A} + (y)) = \mathfrak{A} + (xy) = \mathfrak{A},$$

a contradiction. Thus, no $\mathfrak{A}$ should have existed, so S must have been empty to begin with.

27.22.15 An ideal $\mathfrak{C}$ in a commutative ring R is called *irreducible* if whenever $\mathfrak{C} = \mathfrak{A} \cap \mathfrak{B}$ for some ideals $\mathfrak{A}$ and $\mathfrak{B}$ in R, then either $\mathfrak{C} = \mathfrak{A}$ or $\mathfrak{C} = \mathfrak{B}$. Show if R is Noetherian, then every ideal $\mathfrak{A} < R$ is a finite intersection of irreducible ideals of R, i.e., $\mathfrak{A} = \mathfrak{C}_1 \cap \cdots \cap \mathfrak{C}_n$, for some irreducible ideals $\mathfrak{C}_i$ in R.

Solution Suppose there exists an ideal which is not a finite intersection of irreducible ideals, and let S be the set of these ideals.

S is non-empty by assumption, and if $\mathcal{C} \subseteq S$ is a chain, then $\bigcup \mathcal{C}$ is an upper bound for $\mathcal{C}$. Indeed, R is Noetherian, so $\mathcal{C}$ must stabilize, which means that $\bigcup \mathcal{C} \in \mathcal{C}$, so every chain has an upper bound in S. Zorn's lemma gives us a maximal element $\mathfrak{A} \in S$ with respect to $\subseteq$.

Then $\mathfrak{A}$ is not irreducible. If it were, then it is a finite intersection with itself, which is a contradiction since $\mathfrak{A} \in S$.

Hence, there exist $\mathfrak{B}, \mathfrak{C} > \mathfrak{A}$ so that $\mathfrak{A} = \mathfrak{B} \cap \mathfrak{C}$. But $\mathfrak{A}$ was maximal in S, which means that $\mathfrak{B}$ and $\mathfrak{C}$ are finite intersections of irreducible ideals. But this implies that $\mathfrak{A}$ is a finite intersection of irreducible ideals, a contradiction. Thus, S must have been empty to begin with.