3.4.2 If $f(z)$ is analytic and $\operatorname{Im} f(z) \geq 0$ for $\operatorname{Im} z > 0$, show that

$$\frac{|f(z) - f(z_0)|}{|f(z) - \overline{f(z_0)}|} \leq \frac{|z - z_0|}{|z - \overline{z_0}|}$$

and

$$\frac{|f'(z)|}{\operatorname{Im} f(z)} \leq \frac{1}{y} \quad (z = x + iy).$$

Solution Notice that if $\operatorname{Im} f(z) = 0$ for any value of z, then $f(z)$ is constant by the maximum principle, and there is nothing to prove. So, assume from now on that $\operatorname{Im} f(z) > 0$.

Our first claim is that the Möbius transformation

$$T(z) = \frac{z - z_0}{z - \overline{z_0}}$$

maps the upper-half plane to the unit disk. Indeed,

$$|T(z)|^2 = \frac{|z|^2 + |z_0|^2 - z\overline{z_0} - \overline{z}z_0}{|z|^2 + |z_0|^2 - \overline{z}\overline{z_0} - zz_0} = \frac{|z|^2 + |z_0|^2 - 2\operatorname{Re}(z\overline{z_0})}{|z|^2 + |z_0|^2 - 2\operatorname{Re}(zz_0)} < 1;$$

the numerator is larger because if $z = a + bi$, $z_0 = \alpha + \beta i$, where $b, \beta > 0$, we have

$$\operatorname{Re}(z\overline{z_0}) = a\alpha + b\beta \quad \text{and} \quad \operatorname{Re}(zz_0) = a\alpha - b\beta \implies \operatorname{Re}(z\overline{z_0}) > \operatorname{Re}(zz_0).$$

Then if $f(z_0) = w_0$, consider

$$S(w) = \frac{w - w_0}{w - \overline{w_0}}.$$

which maps the upper-half plane to the unit disk also. By assumption, $\operatorname{Im} f(z) \geq 0$, so the image of f gets mapped back to $\mathbb{D}$. Moreover,

$$SfT^{-1}(0) = Sf(z_0) = 0,$$

so the function satisfies the hypotheses of Schwarz's lemma. Thus, for all z in the unit disk,

$$|Sf| \leq |Tz| \implies \frac{|f(z) - f(z_0)|}{|f(z) - \overline{f(z_0)}|} \leq \frac{|z - z_0|}{|z - \overline{z_0}|},$$

as desired.

If we rearrange the inequality,

$$\frac{|f(z) - f(z_0)|}{|z - z_0||f(z) - \overline{f(z_0)}|} \leq \frac{1}{|z - \overline{z_0}|}.$$

Taking the limit as $z_0 \to z$ and using the fact that $\operatorname{Im} f(z) \geq 0$ and $\operatorname{Im} z > 0$,

$$\frac{|f'(z)|}{2\operatorname{Im} f(z)} \leq \frac{1}{2\operatorname{Im} z} \implies \frac{|f'(z)|}{\operatorname{Im} f(z)} \leq \frac{1}{y},$$

as desired.

3.4.4 Derive corresponding inequalities if $f(z)$ maps $|z| < 1$ into the upper half plane.

Solution Let $w_0 = f(z_0)$. It has a positive imaginary part by the assumptions in the first problem.

Let T map z_0 to 0 and send the unit disk to itself:

$$T(z) = \frac{z - z_0}{1 - \overline{z_0}z}.$$

By the same argument as in the first problem, the map

$$S(w) = \frac{w - w_0}{w - \overline{w_0}}$$

maps the upper-half plane to the unit disk.

Then SfT is a map from the unit disk to the unit disk with

$$SfT^{-1}(0) = Sf(z_0) = 0,$$

so it satisfies the hypotheses of the Schwarz lemma. Thus,

$$\begin{aligned} |Sf(z)| \leq |Tz| &\implies \left|\frac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}}\right| \leq \left|\frac{z - z_0}{1 - \overline{z_0}z}\right| \\ &\implies \left|\frac{f(z) - f(z_0)}{z - z_0}\right| \leq \left|\frac{f(z) - \overline{f(z_0)}}{1 - \overline{z_0}z}\right|. \end{aligned}$$

Let $z_0 \to z$ and we get

$$|f'(z)| \leq \frac{2 \operatorname{Im} f(z)}{|1 - |z|^2|}.$$

3.4.5 Prove by use of Schwawrz's lemma that every one-to-one conformal mapping of a disk onto another (or a half plane) is given by a linear transformation.

Solution As in the book, let T be a linear transformation that maps the disk $|z| < R$ to the unit disk and sends z_0 to the origin, S be a linear transformation that maps the disk $|w| < M$ to the unit disk and sends $f(z_0)$ to the origin.

Then $F := SfT^{-1}$ satisfies the conditions of Schwarz's lemma. Similarly, so does F^{-1}.

Pick $z \in F(\mathbb{D})$. f is one-to-one, so F is a bijection on its image. Thus, since $|F(w)| < 1$ for all $w \in \mathbb{D}$,

$$|z| = \left|F^{-1}(F(z))\right| \leq |F(z)| \leq |z|,$$

so $|F(z)| = |z|$. By Schwarz's lemma, $F(z) = cz$ for some c with $|c| = 1$, which applies for all z. But this is a Möbius transformation, so

$$f(z) = S^{-1} \circ (cz) \circ T$$

is a Möbius transformation, as desired.

We can perform the same argument with half planes, since transformations from disks to half planes and vice versa are given by Möbius transformations also.

3.4.8 How should noneuclidean length in the upper half plane be defined?

Solution The hyperbolic length of γ in the unit disk is given by

$$\int_\gamma \frac{|\mathrm{d}z|}{1-|z|^2}.$$

Noneuclidean length on the upper half plane should be defined by mapping γ from the half plane to the unit disk with a linear transformation like

$$T(z) = \frac{z-i}{z+i},$$

and then applying the definition like normal. With this map, we get

$$z \mapsto T(w) = \frac{w-i}{w+i},$$

which gives us

$$\mathrm{d}z = \frac{2wi}{(w+i)^2}\,\mathrm{d}w \implies |\mathrm{d}z| = \frac{2|w|}{|w+i|^2}\,|\mathrm{d}w|$$

and substituting gives us the formula

$$\int_{T(\gamma)} \frac{2|w|}{|w+i|^2} \frac{|\mathrm{d}w|}{1-\left|\frac{w-i}{w+i}\right|} = \int_{T(\gamma)} \frac{2|w|}{|w+i|^2 - |w^2+1|}\,|\mathrm{d}w|.$$

4.7.2 Prove that the region obtained from a simply connected region by removing m points has the connectivity $m+1$, and find a homology basis.

Solution Let Ω be simply connected.

Let the removed points be $z_1, \dots, z_m$. Since regions are open and each $z_i \in \Omega$, they are contained in an open ball B_i in Ω. In other words, they are isolated points in Ω^c, so they are their own connected components. Thus, after removing the z_i, Ω has $m+1$ connected components in its complement: $\{z_1\}, \dots, \{z_m\}$, and the unbounded component of Ω, which is connected since Ω is simply connected.

For each open B_i, we can shrink it so that $\partial B_i \subseteq \Omega$ and contains no other z_j, since they're isolated. Orient them counter-clockwise, and this gives us our homology basis.

4.7.3 Show that the bounded regions determined by a closed curve are simply connected, while the unbounded region is doubly connected.

Solution Let $\{U_i\}_{i\in I}$ be the regions bounded by a closed curve γ, with I possibly being infinite.

Notice that $\gamma \cap \overline{U_i} \neq \emptyset$. In particular, $\partial U_i \cap \gamma \neq \emptyset$ for all i. Thus, by analysis, for any subset $J \subseteq I$,

$$\gamma \cup \bigcup_{i\in J} U_i \tag{$*$}$$

is connected. Moreover, they're all disjoint since each U_i is a connected component. If any of them intersect, then they are the same set, by connectedness. So, the sets partition $\mathbb{C}^*$.

If U_{i_0} is bounded, then its complement in $\mathbb{C}^*$ is exactly

$$\gamma \cup \bigcup_{i\in I\setminus\{i_0\}} U_i,$$

which is connected by $(*)$, so U_{i_0} is simply connected.

Consider U_{i_∞}, which is unbounded. Then its complement is

$$\gamma \cup \bigcup_{i\in I\setminus\{i_\infty\}} U_i \cup \{\infty\}.$$

Since each U_i is a component in $\mathbb{C}$, none of them contain ∞. Moreover, γ along with the other U_i are connected because of $(*)$. Finally, $\{\infty\}$ is its own component, so U_{i_∞} is doubly connected.

4.7.4 Show that single-valued analytic branches of $\log z$, z^α, and z^z can be defined in any simply connected region which does not contain the origin.

Solution Let Ω be a simply connected region not containing the origin.

By a corollary in the book, since $z \neq 0$ in Ω, we can define a single-valued analytic branch of $\log z$.

Then the following identities are valid:

$$z^\alpha = e^{\alpha \log z} \quad \text{and} \quad z^z = e^{z \log z}.$$

Since e^z is analytic, $\log z$ is analytic, products and compositions of analytic functions are analytic, it follows that the branches of z^α and z^z, which are determined by the branch of $\log z$ we chose, are analytic.

4.7.5 Show that a single-valued analytic branch of $\sqrt{1-z^2}$ can be defined in any region such that the points ± 1 are in the same component of the complement. What are the possible values of

$$\int \frac{\mathrm{d}z}{\sqrt{1-z^2}}$$

over a closed curve in the region?

Solution Let Ω be a region whose complement contains a component U which houses -1 and 1 in it.

Then for any closed curve $\gamma \subseteq \Omega$,

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-1} - \frac{1}{z+1}\,\mathrm{d}z = 0,$$

since if γ looped around U, we would get $n(\gamma,-1) - n(\gamma,1) = 0$. Otherwise, the integrand is holomorphic in the region bounded by γ, so the integral is 0.

Thus, the integrand has a primitive, which we'll call F. This determines our branch for the function.

Consider $f(z) = (z+1)e^{F(z)/2}$. Next, consider

$$g(z) = \frac{f(z)^2}{z^2-1} = \frac{z+1}{z-1}e^{F(z)}.$$

Then if we use "logarithmic differentiation," we see

$$\frac{g'(z)}{g(z)} = \frac{1}{z+1} - \frac{1}{z-1} + F'(z) = 0 \implies g'(z) \equiv 0.$$

Indeed, $-1 \notin \Omega$, and this implies that $g(z)$ is constant, which gives a constant $-c$ so that

$$f(z)^2 = c(1-z^2).$$

Thus, since f is holomorphic ($F(z)$ is holomorphic), $f(z) = \sqrt{c(1-z^2)}$ is holomorphic. Scaling does affect differentiability, so $f(z) = \sqrt{1-z^2}$ is holomorphic.

We can extend f analytically so that f is analytic on $\mathbb{C} - [-1,1]$. Consider a rectangle that contains $[-1,1]$. Since f is analytic, integrals on the boundaries of rectangles whose closures don't touch $[-1,1]$ are 0, so

$$\int_{\partial R} \frac{\mathrm{d}z}{\sqrt{1-z^2}} = \int_{\partial S} \frac{\mathrm{d}z}{\sqrt{1-z^2}},$$

for a smaller rectangle such that $[-1,1] \subseteq S \subseteq R$. Shrink S so that ∂S is two segments on $[-1,1] \subseteq \mathbb{R}$ going from -1 to 1 and back. Then this gives us

$$\begin{aligned}\int_{\partial R} \frac{\mathrm{d}z}{\sqrt{1-z^2}} &= \int_{-1}^{1} \frac{\mathrm{d}z}{\sqrt{1-z^2}} + \int_{1}^{-1} \frac{\mathrm{d}z}{\sqrt{1-z^2}} \\ &= \arcsin(1) - \arcsin(-1) + \arcsin(1) - \arcsin(-1) + 2\pi k \\ &= 2\pi k.\end{aligned}$$

However, k depends on the branch we pick, so we see that the integral can take on values in $2\pi\mathbb{Z}$.

5.2.1 How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $|z| < 1$? *Hint*: Look for the biggest term when $|z| = 1$ and apply Rouché's theorem.

Solution When $|z| = 1$,

$$|(z^7 - 2z^5 + 6z^3 - z + 1) - 6z^3| \leq \left|z^7 - 2z^5 - z + 1\right| \leq 1 + 2 + 1 + 1 = 5 < |6z^3|.$$

Thus, by Rouché's theorem, the polynomial has as many roots as $6z^3$, counting multiplicity, so it has 3 roots.

5.2.2 How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2?

Solution We first look at $|z| = 2$. We'll compare the polynomial to z^4:

$$|(z^4 - 6z + 3) - z^4| = |-6z + 3| \leq 12 + 3 = 15 < 16 = |z^4|.$$

So, by Rouché's, $z^4 - 6z + 3$ has 4 roots with modulus less than 2.

Next, on $|z| = 1$, we'll compare it to $-6z$:

$$|(z^4 - 6z + 3) + 6z| = |z^4 + 3| \leq 1 + 3 = 4 < 6 = |-6z|,$$

so by Rouché's theorem again, $z^4 - 6z + 3$ has 1 root with modulus less than 1.

Thus, $z^4 - 6z + 3$ must have $4 - 1 = 3$ roots with modulus between 1 and 2.

5.2.3 How many roots of the equation $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half plane? *Hint*: Sketch the image of the imaginary axis and apply the argument principle to a large half disk.

Solution Polynomials have no poles, so the argument principle will yield the number of zeroes of $f(z) = z^4 + 8z^3 + 3z^2 + 8z + 3$.

Consider a disk of radius R centered at the origin. We will integrate along the right side of the circle, and then connect the top and bottom by a line across the imaginary axis, and we'll call this curve γ.

We'll be taking the limit as $R \to \infty$, and the term along the imaginary axis will drop off since it's the integral of something $o(1/R)$. So, we'll only look at what happens on the semi-circle.

By taking the substitution $z \mapsto Re^{i\theta}$ and using the bounded convergence theorem,

$$\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{4R^4e^{4i\theta} + 24R^3e^{3i\theta} + 6R^2e^{2i\theta} + 8Re^{i\theta}}{R^4e^{4i\theta} + 8R^3e^{3i\theta} + 3R^2e^{2i\theta} + 8Re^{i\theta} + 3}\, \mathrm{d}\theta \xrightarrow{R\to\infty} \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 4\, \mathrm{d}\theta = 2,$$

which means that

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, \mathrm{d}z = 2,$$

so the polynomial has 2 zeroes.