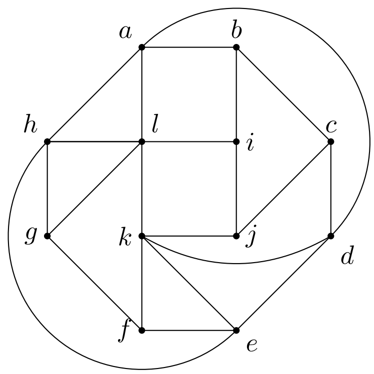
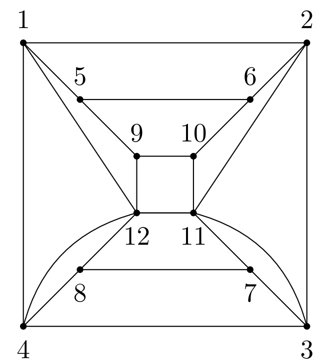
1 Determine whether the graphs G_1 and G_2 are isomorphic. Prove your answer.

a.

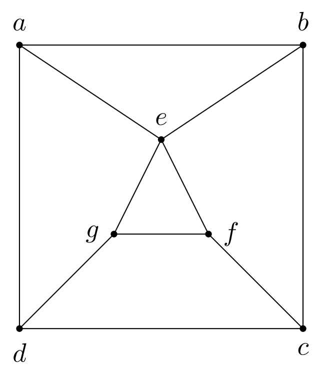


G_1

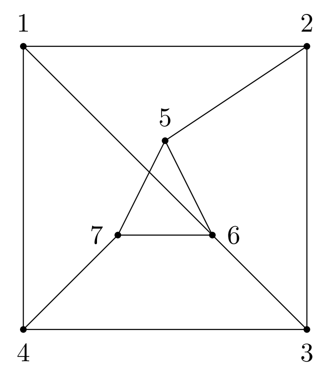


G_2

b.



G_1



G_2

Solution a. G_1 has four 3-cycles: (a, h, l, a), (g, h, l, a), (k, f, e, k), and (k, d, e, k). On the other hand, G_2 has six 3-cycles; $(1, 9, 12, 1)$, $(2, 10, 11, 2)$, $(3, 7, 11, 3)$, $(4, 8, 12, 4)$, $(1, 4, 12, 1)$, and $(2, 3, 11, 2)$. Thus, the two graphs cannot be isomorphic.

b. These graphs are not isomorphic. Notice that with the 4 vertices adjacent to e, we can split them into adjacent pairs, i.e., a and b are adjacent; g and f are adjacent.

e and 6 are the only elements of degree 4 in their respective graphs, so any isomorphism needs to send $e \mapsto 6$. We see that 5 and 7 are adjacent, but 1 and 3 are not. Thus, G_1 cannot be isomorphic to G_2.

2 Show that the given property is an invariant:

a. Has n simple cycles of length k.

b. Has an edge (v, w), where $\delta(v) = i$ and $\delta(w) = j$.

Solution a. Let $(v_0, e_1, v_1, \ldots, e_k, v_0)$ be a simple cycle of length k, and let $f\colon V \to V'$, $g\colon E \to E'$ be an isomorphism between the graphs G to G'. We claim that $(f(v_0), g(e_1), f(e_1), \ldots, g(e_k), f(v_0))$ is a simple cycle of length k.

Since f and g form an isomorphism, this is a cycle of length k: $g(e_i)$ is incident on $f(v_{i-1})$ and $f(v_i)$ for all $1 \leq i \leq k$, by definition of an isomorphism. Moreover, no edges are repeated, since $\{e_i\}_i$ is pairwise disjoint, which means that $\{g(e_i)\}$ is pairwise disjoint, because g is a bijection. Thus, the cycle is simple.

If G has n distinct simple cycles, then G' must have n distinct simple cycles as well. Indeed, two different simple cycles must differ at some vertex or edge, which means that the image of the cycles differ at some vertex or edge also, because f and g are injective.

Thus, the property is invariant.

b. Let f, g be an isomorphism from G to G' as before. Let v, w be so that $\delta(v) = i$, $\delta(w) = j$, and so there is an edge (v, w). By definition of an isomorphism, G_2 must have an edge $(f(v), f(w))$. Moreover, $\delta(f(v)) = i$ and $\delta(f(w)) = j$, since isomorphisms preserve the degree of a vertex.

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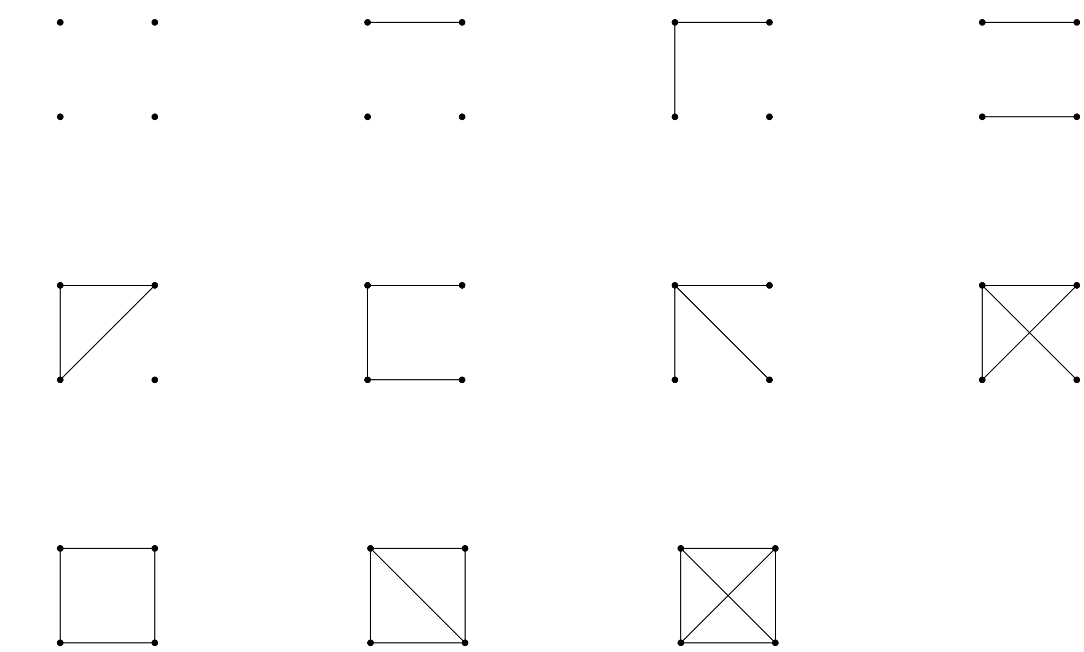
3 In each case below, draw all nonisomorphic graphs having the listed properties.

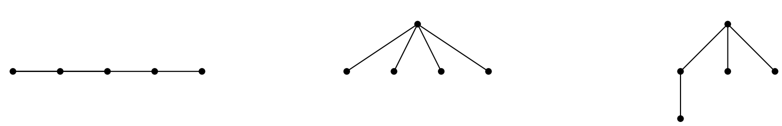
a. Simple graphs having four vertices.

b. Cycle-free connected graphs (i.e., trees) having five vertices.

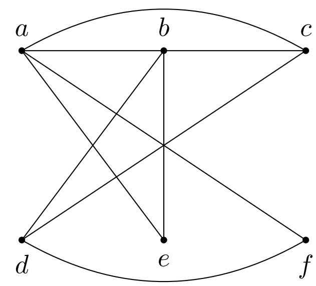
Solution a.



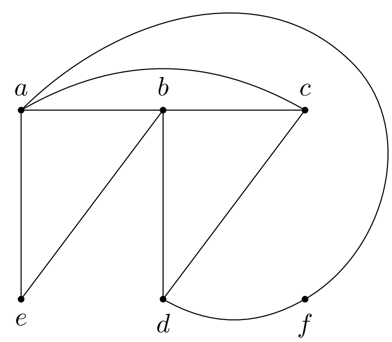
b.



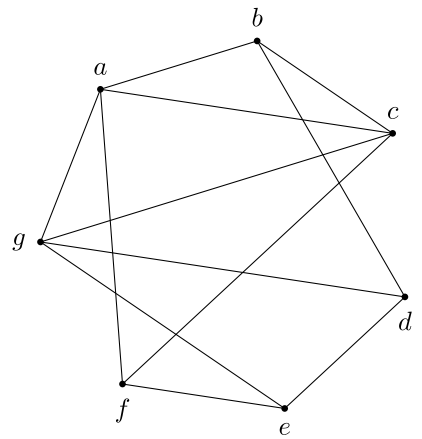
4 Show that the graph is planar by redrawing it so that no edges cross:



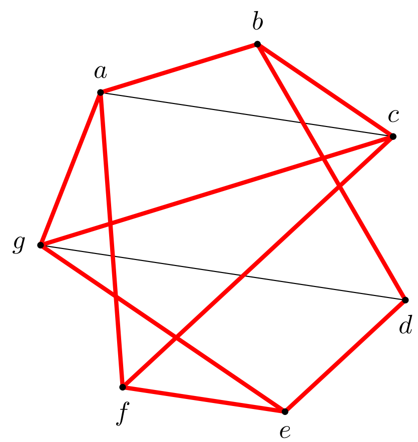
Solution

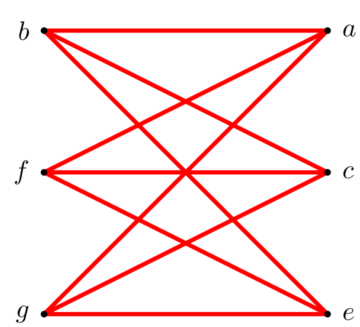


5 Show that the graph that is not planar by finding a subgraph homeomorphic to either K_5 or $K_{3,3}$:



Solution The graph has a subgraph homeomorphic to $K_{3,3}$. If we delete the edges (a, c) and (g, d), and remove the vertex d,





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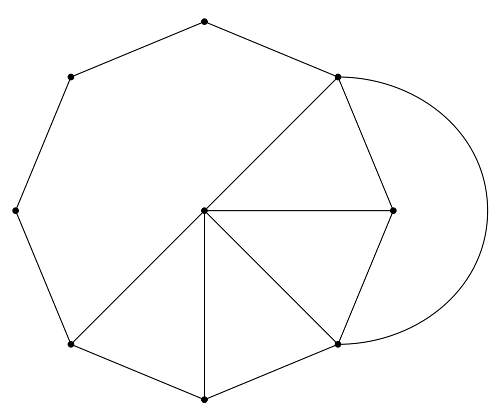
6 A connected, planar graph has nine vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4, and 5. How many edges are there? How many faces are there?

Solution The graph looks as follows:



So, it has 14 edges and 7 faces.

7 a. Show that in any simple connected planar graph, $e \leq 3v - 6$.

b. Give an example of a simple connected, nonplanar graph for which $e \leq 3v - 6$.

c. Use part (a) to show that K_5 is not planar.

Solution a. Any face must have at least 3 edges and vertices bordering it, since the graph is simple. Since an edge can enclose at most 2 faces, we get $2e \geq 3f$, so by Euler's formula,

$$2e \geq 3(2 + e - v) = 6 + 3e - 3v \implies e \leq 3v - 6.$$

b. $K_{3,3}$ is one. It has 6 vertices and 9 edges, so

$$9 \leq 12 = 3(6) - 6.$$

c. K_5 has 5 vertices and $C(5, 2) = 45$ edges. If K_5 is planar, then

$$45 \leq 3(5) - 6 = 9,$$

which is impossible.

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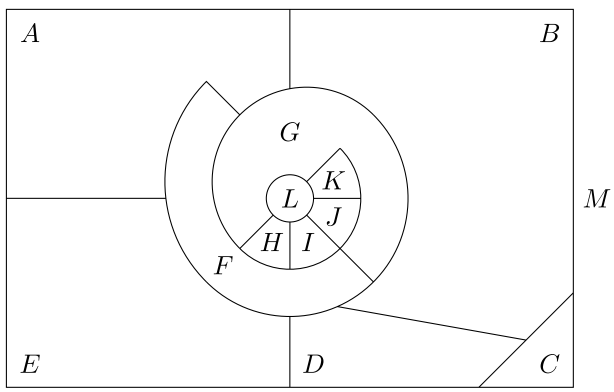
8 Consider the following planar map:

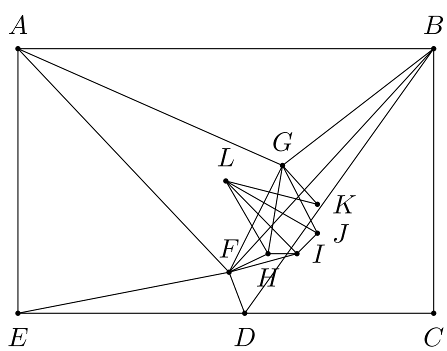


a. Find the dual of this map.

b. Show that any coloring of this map, excluding the bounded region, requires at least four colors.

c. Color the map, excluding the bounded region, using four colors.

Solution a.



b. Because L, H, and I are adjacent to each other, they must all have different colors, so color L with color 1, H with 2, and I with 3. Now suppose G is colored with color 3. J cannot be color 1 or color 3, so color it color 2. Then K cannot be 1, 2, or 3, so it must be color 4. Thus, a coloring of this map requires at least four colors.

c. Use the following coloring:

$$\begin{array}{cccc} A \mapsto 1 & B \mapsto 2 & C \mapsto 1 & D \mapsto 3 \\ E \mapsto 2 & F \mapsto 4 & G \mapsto 3 & H \mapsto 2 \\ I \mapsto 3 & J \mapsto 2 & K \mapsto 1 & L \mapsto 4. \end{array}$$

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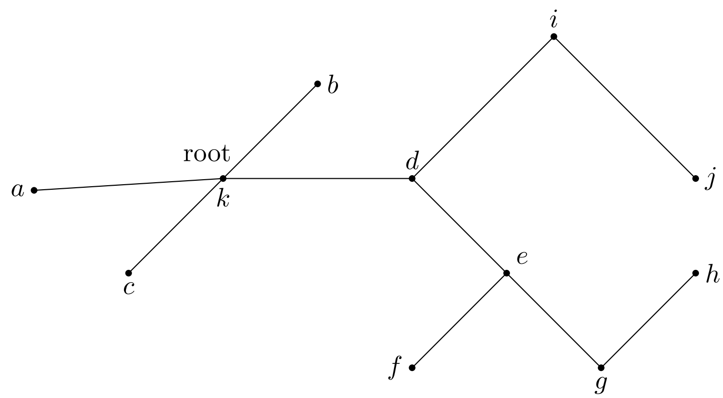
9 Consider the following tree:



a. Find the level of each vertex of the tree.

b. Find the height of the tree.

Solution a.

a	b	c	d	e	f	g	h	i	j
1	1	1	1	2	3	3	4	2	3

b. The height is 4.