1 Calculate

$$\int_0^1 \frac{1}{1+x}\,\mathrm{d}x,$$

and *numerically* apply the composite trapezoidal rule to compute it with evenly spaced nodes $0 = x_0 < x_1 < \cdots < x_n = 1$, where $n = 10, 20, 40, 80$. Compute the absolute errors.

Solution The exact solution is given by

$$\log(1+x)\Big|_0^1 = \log 2 \approx 0.6931471806.$$

Then the composite trapezoidal rule yields

n	Estimate	Absolute error	Absolute error $\times\, n^2$
10	0.6937714031754278	0.0006242226154825614	0.06242226154825614
20	0.6933033817926941	0.00015620123274884268	0.06248049309953707
40	0.6931862400091408	$3.9059449195466556 \times 10^{-5}$	0.06249511871274649
80	0.6931569459942255	$9.76543428021781 \times 10^{-6}$	0.06249877939339399

The absolute error $\times$ n^2 is roughly the same for each value of n. This is because the absolute error is $O(h^2) = O(1/n^2)$, so the error term is unaffected by n.

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2 Again, the exact solution is $\log 2 \approx 0.6931471806$. The composite Simpson's rule yields the following:

n	Estimate	Absolute error	Absolute error $\times\, n^4$
10	0.6931473746651161	$1.9410517082540935 \times 10^{-7}$	0.0019410517082540935
20	0.6931471927479559	$1.2188010600766574 \times 10^{-8}$	0.001950081696122652
40	0.693147181322587	$7.626417275474751 \times 10^{-10}$	0.0019523628225215361
80	0.6931471806076244	$4.767908290403966 \times 10^{-11}$	0.0019529352357494645

Like the composite trapezoidal rule, the absolute error $\times$ n^4 stays roughly the same, since the error term is $O(h^4) = O(1/n^4)$.

Solution

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4 In class, we discussed the 2-point Gaussian quadrature formula on $[-1, 1]$, given by

$$\int_{-1}^1 f(x)\,\mathrm{d}x \approx f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).$$

Determine the 2-point Gaussian quadrature on $[a, b]$.

Solution Consider the change of variables

$$u = \frac{2(x-a)}{b-a} - 1 \implies x = \frac{b-a}{2}(u+1) + a,\ \mathrm{d}u = \frac{2}{b-a}\,\mathrm{d}x.$$

This gives us the integral

$$\frac{b-a}{2}\int_{-1}^1 f\left(\frac{b-a}{2}(u+1)+a\right)\mathrm{d}u \approx \frac{b-a}{2} f\left[\frac{b-a}{2}\left(-\frac{\sqrt{3}}{3}+1\right)+a\right] + \frac{b-a}{2} f\left[\frac{b-a}{2}\left(\frac{\sqrt{3}}{3}+1\right)+a\right].$$

This gives us

$$c_1 = c_2 = \frac{b-a}{2}$$
$$x_1 = \frac{b+a}{2} - \frac{\sqrt{3}}{6}(b-a)$$
$$x_2 = \frac{b+a}{2} + \frac{\sqrt{3}}{6}(b-a).$$

5 You are given the following quadrature formula on $[0, 2]$ with undetermined nodes and coefficients

$$\int_0^2 f(x)\,\mathrm{d}x \approx Af(0) + \frac{4}{3}f(x_1) + Bf(2).$$

Determine $A, B \in \mathbb{R}$ and $x_2 \in [0, 2]$ such that this quadrature formula achieves the greatest degree of accuracy. What is the degree of accuracy of the quadrature formula you eventually obtain?

Solution We expect the degree of accuracy to be 2. This gives us

$$\int_0^2 \mathrm{d}x = 2 = A + \frac{4}{3} + B$$
$$\int_0^2 x\,\mathrm{d}x = 2 = \frac{4}{3}x_1 + 2B$$
$$\int_0^2 x^2\,\mathrm{d}x = \frac{8}{3} = \frac{4}{3}x_1^2 + 4B.$$

Performing (equation 3) $-$ (2 $\times$ equation 2) gives

$$-\frac{4}{3} = \frac{4}{3}x_1^2 - \frac{8}{3}x_1 \implies x_1^2 - 2x_1 + 1 \implies x_1 = 1.$$

Substituting into equation 2 gives us

$$B = \frac{1}{3}.$$

Substituting into equation 1 gives

$$A = \frac{1}{3},$$

so our quadrature is

$$\int_0^2 f(x)\,\mathrm{d}x \approx \frac{1}{3}f(0) + \frac{4}{3}f(1) + \frac{1}{3}f(2).$$

The degree of the quadrature we obtain is actually 3:

$$\int_0^2 x^3\,\mathrm{d}x = 4 = \frac{4}{3} + \frac{1}{3} \cdot 8.$$

By linearity of the integral and the quadrature, we know that this works for any third degree polynomial.