1 Consider the sequence defined by $w_n = \dfrac{1}{n} - \dfrac{1}{n+1}$ defined for $n \geq 1$.

a. Find $\displaystyle\sum_{i=1}^{10} w_i$.

b. Find a formula for the sequence defined by $c_n = \displaystyle\sum_{i=1}^{n} w_i$.

c. Is w_n increasing?

d. Is w_n non-increasing?

Solution a. By direct calculation,

$$\begin{aligned}\sum_{i=1}^{10} w_i &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{9} - \frac{1}{10}\right) + \left(\frac{1}{10} - \frac{1}{11}\right) \\ &= \frac{1}{1} + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{9} + \frac{1}{9}\right) + \left(-\frac{1}{10} + \frac{1}{10}\right) - \frac{1}{11} \\ &= 1 - \frac{1}{11} = \frac{10}{11}.\end{aligned}$$

b. We claim that $c_n = 1 - \dfrac{1}{n+1}$, and we will prove this by induction:

Base step: $n = 1$

In this case, $c_n = \dfrac{1}{1} - \dfrac{1}{2} = 1 - \dfrac{1}{1+1}$, so the base case holds.

Inductive step:

Suppose the formula holds for $n = k$. We wish to show that it holds for $n = k+1$. By calculating, we have

$$c_{k+1} = \sum_{i=1}^{k+1} w_i = \sum_{i=1}^{k} w_i + w_{k+1} = c_k + w_{k+1} = 1 - \frac{1}{k+1} + \left(\frac{1}{k+1} - \frac{1}{(k+1)+1}\right) = 1 - \frac{1}{(k+1)+1},$$

so the inductive step holds.

By induction, $c_n = 1 - \dfrac{1}{n+1}$.

c. Consider $f(x) = \dfrac{1}{x} - \dfrac{1}{x+1} = \dfrac{1}{x(x+1)}$ on $[1, \infty)$. We have

$$f'(x) = -\frac{2x+1}{(x(x+1))^2} \leq -\frac{1}{(x(x+1))^2} < 0,$$

so f is strictly decreasing on this interval. Hence, because $f(n) = w_n$ for all $n \geq 1$, it follows that w_n is not increasing.

d. Yes, since w_n is decreasing.

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2 Consider the sequence $\{r_n\}$ defined by $r_n = 3 \cdot 2^n - 4 \cdot 5^n$ for $n \geq 0$. Prove that $\{r_n\}$ satisfies $r_n = 7r_{n-1} - 10r_{n-2}$ for $n \geq 2$.

Solution By calculation,

$$\begin{aligned}7r_{n-1} - 10r_{n-2} &= 7 \cdot 3 \cdot 2^{n-1} - 7 \cdot 4 \cdot 5^{n-1} - 10 \cdot 3 \cdot 2^{n-2} + 10 \cdot 4 \cdot 5^{n-2} \\ &= 7 \cdot 3 \cdot 2^{n-1} - 7 \cdot 4 \cdot 5^{n-1} - 5 \cdot 3 \cdot 2^{n-1} + 2 \cdot 4 \cdot 5^{n-1} \\ &= 2 \cdot 3 \cdot 2^{n-1} - 5 \cdot 4 \cdot 5^{n-1} \\ &= 3 \cdot 2^n - 4 \cdot 5^n \\ &= r_n,\end{aligned}$$

as required.

3 Consider the sequence $\{z_n\}$ defined by $z_n = (2+n)3^n$ for $n \geq 0$. Prove that $\{z_n\}$ satisfies $z_n = 6z_{n-1} - 9z_{n-2}$ for $n \geq 2$.

Solution Substitution yields

$$\begin{aligned} 6z_{n-1} - 9z_{n-2} &= 6(2+n-1)3^{n-1} - 9(2+n-2)3^{n-2} \\ &= (2+2n)3^n - n \cdot 3^n \\ &= (2+n)3^n \\ &= z_n \end{aligned}$$

as desired.

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4 Rewrite the sum $\sum_{i=1}^{n} i^2 r^{n-1}$ replacing the index k by i, where $i = k+1$.

Solution We have

$$\sum_{i=1}^{n} i^2 r^{n-1} = \sum_{k=0}^{n-1} (k+1)^2 r^{n-1}.$$

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5 Let $X = \{a, b\}$ and let X^* be the set of strings over X. For any $\alpha \in X^*$, let $\alpha^R \in X^*$ be the string obtained by reversing α. A *palindrome over* X is a string $\alpha \in X^*$ for which $\alpha = \alpha^R$. Define a function f from X^* to the set of palindromes over X as $f(\alpha) = \alpha\alpha^R$. Is f one-to-one? Is f onto? Prove your answers.

Solution f is one-to-one: Suppose we had $\alpha, \beta \in X^*$ with

$$f(\alpha) = \alpha\alpha^R = \beta\beta^R = f(\beta).$$

Then $\alpha = \beta$: α and β must both be the first half of $\alpha\alpha^R = \beta\beta^R$, i.e., they are both the first half of the same string, so they must be the same string.

f is not onto: *aba* is palindrome, but it is not of the form $\alpha\alpha^R$. Indeed, $\alpha\alpha^R$ must have an even number of characters, but *aba* has an odd number of characters.

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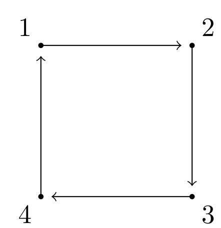
6 Draw the digraph of the relation $R = \{(1,2), (2,3), (3,4), (4,1)\}$ on $\{1,2,3,4\}$.

Solution The digraph is the following:



7 Determine whether each relation defined on $\mathbb{Z}^+$ is reflexive, symmetric, antisymmetric, transitive, and/or a partial order.

a. $(x, y) \in R$ if $x \geq y$.

b. $(x, y) \in R$ if 3 divides $x + 2y$.

c. $(x, y) \in R$ if $|x - y| = 2$.

Solution a. R is reflexive, since $x \geq x$ for all $x \in \mathbb{Z}^+$.
R is not symmetric: $2 \geq 1$ for $1 \ngeq 2$.
R is antisymmetric: if $x \geq y$ and $y \geq x$, then by trichotomy of $\geq$, $x = y$.
R is transitive: $x \geq y$ and $y \geq z$ imply that $x \geq z$.
R is a partial order: R is reflexive, transitive, and antisymmetric.

b. R is reflexive: $x + 2x = 3x$, so $(x, x) \in R$.
R is symmetric:
Notice that if $(x, y) \in R$, then $x + 2y = 4x + 2y - 3x = 2(2x + y) - 3x$ is divisible by 3, i.e., there exists $k \in \mathbb{Z}$ so that

$$2(2x + y) - 3x = 3k \implies 2(2x + y) = 3(x + k) \implies 3 \mid 2(2x + y).$$

Since 2 and 3 are different primes, it follows that $3 \mid 2x + y$, so $(y, x) \in R$.
R is not antisymmetric: for example, $(1, 4), (4, 1) \in R$, but $1 \neq 4$.
R is transitive:
If $(x, y) \in R$ and $(y, x) \in R$, then there exist $a, b \in \mathbb{Z}$ so that $x + 2y = 3a$ and $y + 2z = 3b$. Then

$$(x + 2y) + (y + 2z) = 3a + 3b \implies x + 2z = 3a + 3b - 3y = 3(a + b - y) \implies 3 \mid x + 2z,$$

since $y \in \mathbb{Z}$, so $(x, z) \in R$.
R is not a partial order: R is not antisymmetric.

c. R is not reflexive: $|x - x| = 0 \neq 2$ for all $x \in \mathbb{Z}^+$.
R is symmetric: if $(x, y) \in R$, then $|y - x| = |x - y| = 2$, so $(y, x) \in R$.
R is not antisymmetric: for example, $(1, 3), (3, 1) \in R$, but $1 \neq 3$.
R is not transitive: for example, $(1, 3), (3, 5) \in R$, but $(1, 5) \notin R$.
R is not a partial order: it is not reflexive.

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8 Give examples of relations on $\{1, 2, 3, 4\}$ having the specified properties:

a. Reflexive, not symmetric, and not transitive.

b. Not reflexive, not symmetric, and transitive.

Solution a. $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3)\}$. It's clearly reflexive, it's not symmetric since $(1, 2) \in R$ but $(2, 1) \notin R$, and it's not transitive since $(1, 2), (2, 3) \in R$, but $(1, 3) \notin R$.

b. $R = \{(1, 2), (2, 3), (1, 3)\}$. R is not reflexive since $(1, 1) \notin R$, it's not symmetric since $(1, 2) \in R$, but $(2, 1) \notin R$, but it's transitive: if (x, y) and (y, z) are in R, then necessarily $x = 1$, $y = 2$, and $z = 3$, and $(x, z) = (1, 3) \in R$.

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9 What's wrong with the following argument, which supposedly shows that any relation R on X that is symmetric and transitive is also reflexive?

Let $x \in X$. Using symmetry we have (x, y) and (y, x) both in R. Since $(x, y), (y, x) \in R$, by transitivity we have $(x, x) \in R$. Therefore R is reflexive.

Solution Given $x \in X$, it may not be true that $(x, y) \in R$ for any $y \in X$. For example, take $R = \{(2, 3), (3, 2), (2, 2), (3, 3)\}$ on $\{1, 2, 3\}$: R is symmetric and transitive, but $(1, 1) \notin R$, so R is not reflexive.

10 Determine whether the given relations are equivalence relations on the set of all people:

a. $\{(x, y) \mid x$ and y have, at some time, lived in the same country at the same time$\}$.

b. $\{(x, y) \mid x$ and y have the same color hair$\}$.

Solution a. This is not an equivalence relation. For example, say person a lived in Canada from 2000 to 2005, person b lived there from 2005 to 2010, and person c lived there from 2010 to 2015. Then $(a, b), (b, c) \in R$, but $(a, c) \notin R$. So R is not transitive and hence not an equivalence relation.

b. This is an equivalence relation. Person a obviously has the same color as a, so it is reflexive. If a and b have the same hair color, then b and a have the same hair color. Lastly, if a and b have hair color x, and b and c have hair color y, then because b can only have one hair color, $x = y$, so a and c both have hair color x. Hence, R is transitive, so R is an equivalence relation.