1 a. Let u be harmonic in the disc $|x| < R$ in $\mathbb{R}^2$ and assume that $u \geq 0$. Show that

$$u(0)\frac{R-|x|}{R+|x|} \leq u(x) \leq u(0)\frac{R+|x|}{R-|x|}, \quad |x| < R.$$

b. Let $\Omega \subseteq \mathbb{R}^2$ be open connected and let $K \subseteq \Omega$ be compact. Show that there exists a constant $C = C_{\Omega,K} > 0$ such that

$$u(x_1) \leq Cu(x_2)$$

for all $x_1, x_2 \in K$ and every positive harmonic function u on Ω.

c. Let $\Omega \subseteq \mathbb{R}^2$ be open connected and let u_n be harmonic in Ω with $u_1 \leq u_2 \leq u_3 \leq \cdots$. Show that unless $u_n \to +\infty$ locally uniformly in Ω, we have: u_n converges locally uniformly in Ω to a harmonic function.

Solution a. Using the Poisson representation formula, if $0 < r < R$, we have for $|x| < r$ that

$$u(x) = \frac{1}{2\pi r}\int_{|y|=r} \frac{r^2-|x|^2}{|y-x|^2} u(y)\,\mathrm{d}s(y).$$

Notice that by optimizing the distance from y to x, we have

$$\frac{r-|x|}{r+|x|} = \frac{r^2-|x|^2}{(r+|x|)^2} \leq \frac{r^2-|x|^2}{|y-x|^2} \leq \frac{r^2-|x|^2}{(r-|x|)^2} = \frac{r+|x|}{r-|x|}.$$

Since u is non-negative and harmonic, we have

$$\frac{r-|x|}{r+|x|}\frac{1}{2\pi r}\int_{|y|=r} u(y)\,\mathrm{d}s(y) \leq u(x) \leq \frac{r+|x|}{r-|x|}\frac{1}{2\pi r}\int_{|y|=r} u(y)\,\mathrm{d}s(y)$$
$$u(0)\frac{r-|x|}{r+|x|} \leq u(x) \leq u(0)\frac{r+|x|}{r-|x|}.$$

Taking $r \nearrow R$, we get the desired inequality.

b. Let $K \subseteq \Omega$ be compact. Without loss of generality, let K be connected. Otherwise because Ω is (path) connected, we may connect the components of K together via a path.

For every $z \in K$, there exists $R_z > 0$ so that $\overline{B(z, 2R_z)} \subseteq \Omega$. By compactness, there are $z_1, \ldots, z_n \in K$ so that the $B(z_i, R_{z_i})$ cover K.

By (a) and translation, given any $1 \leq i \leq n$ and $x \in B(z_i, R_{z_i}) \subseteq \overline{B(z, 2R_z)}$, we have the inequality

$$\frac{1}{3}u(z_i) = u(z_i)\frac{2R_{z_i}-R_{z_i}}{2R_{z_i}+R_{z_i}} \leq u(x) \leq u(z_i)\frac{2R_{z_i}+R_{z_i}}{2R_{z_i}-R_{z_i}} = 3u(z_i).$$

Now let $x_1, x_2 \in K$. Since K is connected, there is a path connecting the two points. By construction, there exist $z_{i_1}, \ldots, z_{i_m}$ with $m \leq n$ so that $\bigcup_{j=1}^m B(z_{i_j}, R_{z_{i_j}})$ is a connected set containing x_1 and x_2 so that $x_1 \in B(z_{i_1}, R_{z_{i_1}})$, $x_2 \in B(z_{i_m}, R_{z_{i_m}})$, and $B(z_{i_j}, R_{z_{i_j}}) \cap B(z_{i_{j+1}}, R_{z_{i_{j+1}}})$ for $1 \leq j \leq m-1$. In other words, we picked the z_{i_j} in a nice order.

Thus, if $y_j \in B(z_{i_j}, R_{z_{i_j}}) \cap B(z_{i_{j+1}}, R_{z_{i_{j+1}}})$, we have

$$\frac{1}{3}u(z_{i_j}) \leq u(y_j) \leq 3u(z_{i_{j+1}}) \implies u(z_{i_j}) \leq 3^2 u(z_{i_{j+1}}).$$

Hence,

$$u(x_1) \leq 3u(z_{i_1}) \leq 3 \cdot 3^2 u(z_{i_2}) \leq \cdots \leq 3 \cdot 3^{2m-1} u(z_{i_m}) \leq 3^{2m} u(x_2).$$

In the worst case scenario, we need to use all n balls, so we get, for any $x_1, x_2 \in K$, that

$$u(x_1) \leq 3^{2n} u(x_2),$$

which completes this part.

c. Suppose there exists $a \in \Omega$ so that $u_n(a) \xrightarrow{n\to\infty} \infty$. Let $\overline{B(a,R)} \subseteq \Omega$ for some $R > 0$. Then by part (a), for $|x-a| < R$,

$$u_n(a)\frac{R-|x-a|}{R+|x-a|} \leq u_n(x),$$

so u_n converges converges uniformly to ∞ on this ball. If we let E be the set where $u_n(z) \xrightarrow{n\to\infty} \infty$ on a compact neighborhood of Ω, then it's clear that E is open by the above argument. Moreover, it is closed in Ω: Let $\{z_n\}$ be a sequence in E which converges to z_0 in Ω. If $u(z_0) \neq \infty$, then by the same inequality, we get

$$u(z) \leq u(z_0)\frac{R+|z-z_0|}{R-|z-z_0|} < \infty,$$

for some $R > 0$. But because $z_n \to z_0$, the inequality applies to z_n for large z_n, which is a contradiction since $u(z_n) = \infty$. Thus, $u(z_0) = \infty$, and by the same argument as the above, $z_0 \in E$. Thus, E is open and closed, and non-empty since $a \in E$, so by connectedness, $E = \Omega$. This shows that u_n converges locally uniformly to ∞ in Ω if and only if there exists $a \in \Omega$ where $u(z_n) \xrightarrow{n\to\infty} \infty$.

Now suppose that u_n does not converge to ∞ at any point in Ω, and let $K \subseteq \Omega$ be compact. u_n converges pointwise to some function u, since it is a monotone sequence of functions.

By assumption, there exists $a \in K$ so that $u_n(a) \xrightarrow{n\to\infty} u(a)$, i.e., $\{u_n(a)\}$ is a Cauchy sequence. Moreover, $u_m - u_n \geq 0$, since the sequence is monotone, so we may apply Harnack's inequality. Thus, there exists $C > 0$ so that for any $x \in K$, we have

$$|u_m(x) - u_n(x)| = u_m(x) - u_n(x) \leq C(u_m(a) - u_n(a)) = C|u_m(a) - u_n(a)| \xrightarrow{n\to\infty} 0,$$

so $\{u_n\}$ is uniformly Cauchy. Hence u is a uniform limit of continuous functions, so u is continuous also.

Lastly, by applying the monotone convergence theorem to the mean value of u_n, we see that u also has the mean value property, so u is harmonic.

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2 Let $\Omega \subseteq \mathbb{C}$ be open and let $h\colon \Omega \to \mathbb{R}$ be a harmonic function not vanishing identically. Show that the set $h^{-1}(0) \subseteq \Omega$ is of Lebesgue measure zero.

Solution By working locally, we may assume that Ω is simply connected (e.g., an open ball) so that h is the real part of some holomorphic function u with $u = h + ig$ for some harmonic function g. Notice that $u' = h_x - ih_y$, so $u' = 0 \iff \partial_z h = 0$.

Let $E := \{x \in \Omega \mid \nabla h(x) = 0 \text{ and } \partial_z h = 0\}$. E is a subset of the set of the zeroes of u, which must be discrete since h is non-constant. Hence, F must also be a discrete set, hence a Lebesgue null set.

Now consider the set $F := \{x \in \Omega \mid \nabla h(x) \neq 0 \text{ and } \partial_z h = 0\}$. By the implicit function theorem, this set is locally a graph, which also has Lebesgue measure zero, by Fubini's theorem.

Thus, $E \cup F = \{x \in \Omega \mid \partial_z h = 0\}$ has Lebesgue measure zero. Thus, if $h^{-1}(0)$ had positive measure, then u must vanish, which implies that $E \cup F$ has positive measure also, a contradiction. Thus, $h^{-1}(0)$ has Lebesgue measure zero.

3 Let $u(z)$ be subharmonic in $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$, upper semicontinuous in the closure, and assume that

$$u(z) \le C + a|z|, \quad \operatorname{Re} z \ge 0.$$

Assume furthermore that $u(iy) \le C - b|y|$, for all $y \in \mathbb{R}$, where $b > 0$. Show that $u \equiv -\infty$.

Solution We follow the hint, and will show that $u(z) \le C + ax - b|y|$ for $\operatorname{Re} z \ge 0$.

Set $v(z) \coloneqq u(z) - ax + b|y|$, which is subharmonic on $\operatorname{Re} z > 0$ and upper semicontinuous in the closure.

Notice for $0 < \arg z < \pi/2$ and $-\pi/2 < \arg z < 0$, $|z|^k$ is a PL function if $0 < k < 2$. Thus, if $\operatorname{Re} z > 0$ and $\operatorname{Im} z > 0$ (or $\operatorname{Im} z < 0$), v is dominated by $|z|$ for large $|z|$, so we can apply Phragmén-Lindelöf to v on these regions.

When $\operatorname{Re} z = 0$, we write $z = iy$ and get

$$v(iy) = u(iy) + b|y| \le C.$$

When $\operatorname{Im} z = 0$, we write $z = x$ and get

$$v(x) = u(x) - ax \le C.$$

Thus, $v \le C$ on the boundary of the quarter-plane and v is dominated by a PL function on it, so by Phragmén-Lindelöf, $v \le C$ on the quarter-plane. We can apply the same argument to the other quarter-plane, which proves the hint.

Now consider $u(z) + cx$ for $c > 0$. This satisfies the same conditions as the problem, so $u(z) + cx \le C$ for all z in the right-half plane. In particular, the inequality is satisfied in the limit as $c \to \infty$ for any z. The only way this can happen is when $u \equiv -\infty$, or else $u(z) + cx$ becomes positively unbounded. Hence, $u \equiv -\infty$, as required.

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4 Let $\{z_n\}_{n\ge1}$ be a countable dense subset of the closed unit disc $\overline{D} = \{z \mid |z| \le 1\}$ and let us set

$$u(z) = \sum_{n=1}^{\infty} 2^{-n} \log|z - z_n|.$$

Show that u is subharmonic in $\mathbb{C}$. Show also that $u = -\infty$ on an uncountable dense subset of $\overline{D}$ and that u is discontinuous almost everywhere on $\overline{D}$.

Solution Consider the measure $\mu \colon \mathcal{P}(\mathbb{C}) \to \mathbb{R}_{\ge 0}$ on $\mathbb{C}$ given by

$$\mu(\{z\}) = \begin{cases} 2^{-n} & \text{if } z = z_n, \text{ for some } n \ge 1, \\ 0 & \text{otherwise} \end{cases}$$

and extending it to any subset of $\mathbb{C}$ in the obvious way.

It is easy to see that this is indeed a measure. Thus, we can represent u via

$$u(z) = \int_{\mathbb{C}} \log|z - w| \, \mathrm{d}\mu(w).$$

By Fatou's lemma,

$$\limsup_{z\to y} u(z) = \limsup_{z\to y} \int_{\mathbb{C}} \log|z - w| \, \mathrm{d}\mu(w) \le \int_{\mathbb{C}} \limsup_{z\to y} \log|z - w| \, \mathrm{d}\mu(w) = u(y),$$

so u is upper semicontinuous.

By Fubini's theorem and the fact that the logarithm is subharmonic, we have

$$\begin{aligned}
\frac{1}{2\pi R}\int_{|y|=R} u(z+y)\,\mathrm{d}s(y) &= \frac{1}{2\pi R}\int_{|y|=R}\left(\int_{\mathbb{C}} \log|z+y-w|\,\mathrm{d}w\right)\mathrm{d}s(y) \\
&= \int_{\mathbb{C}}\left(\frac{1}{2\pi R}\int_{|y|=R} \log|z+y-w|\,\mathrm{d}s(y)\right)\mathrm{d}w \\
&\geq \int_{\mathbb{C}} \log|z-w|\,\mathrm{d}w \\
&= u(z),
\end{aligned}$$

so u is subharmonic.

Certainly $u(z) = -\infty$ on a countable dense subset. We just need to show that this is uncountable.

Set $E = \left\{z \in \overline{D} \mid u(z) = -\infty\right\}$. Suppose E were countable. Then on $\overline{D} \setminus E$, u is pointwise bounded, so we can write

$$\overline{D} \setminus E = \bigcup_{n\geq 1} \left\{z \in \overline{D} \mid u(z) \geq -n\right\} := \bigcup_{n \geq 1} E_n.$$

Each E_n is closed, since u is upper semicontinuous. Because $\overline{D}$ is uncountable, there exists $n \geq 1$ so that E_n has non-empty interior, by Baire category. But $\{z_n\}$ is dense in $\overline{D}$, so there exists $z_k \in E_n$, which is a contradiction, since $u(z_k) = -\infty$. Thus, E must have been uncountable to begin with, as needed.

Points of continuity of u must take on the value $-\infty$, since the z_n are dense in $\overline{D}$. By a theorem in class, if $u \not\equiv -\infty$, then the set $\{u(z) = -\infty\}$ has Lebesgue measure zero, so the points of continuity of u must also be a Lebesgue null set.

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5 Let $\Omega \subseteq \mathbb{C}$ be open and let $u \in C(\Omega)$. Show the following version of Morera's theorem: assume that

$$\int_{\partial D} u(z)\,\mathrm{d}z = 0,$$

for all discs D with $\overline{D} \subseteq \Omega$. Show that $u \in \mathrm{Hol}(\Omega)$.

Solution We first consider the case where u is C^1. Let $z_0 \in \Omega$ and $R > 0$ so that $\overline{B(z_0, R)} \subseteq \Omega$. Then, writing $u = f + ih$, we have

$$\begin{aligned}
0 &= \int_{\partial B(z_0,R)} u(z)\,\mathrm{d}z \\
&= \int_{\partial B(z_0,R)} (f+ih)\,\mathrm{d}(x+iy) \\
&= \left(\int_{\partial B(z_0,R)} f\,\mathrm{d}x - h\,\mathrm{d}y\right) + i\left(\int_{\partial B(z_0,R)} f\,\mathrm{d}y + h\,\mathrm{d}x\right).
\end{aligned}$$

Thus, the two integrals must vanish. By Green's theorem and dividing by πR^2, we get

$$\begin{aligned}
0 &= \frac{1}{\pi R^2}\iint_{B(z_0,R)} \left(\frac{\partial f}{\partial y} + \frac{\partial h}{\partial x}\right)\mathrm{d}x\,\mathrm{d}y \\
0 &= \frac{1}{\pi R^2}\iint_{B(z_0,R)} \left(\frac{\partial f}{\partial x} - \frac{\partial h}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Taking $R \to 0$, continuity tells us that

$$\frac{\partial f}{\partial y} = -\frac{\partial h}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial x} = \frac{\partial h}{\partial y} \implies \frac{\partial u}{\partial x} = i\frac{\partial u}{\partial y},$$

so u satisfies the Cauchy-Riemann equations. Since u is C^1, it follows that u is holomorphic.

Now we will prove it for $u \in C(\Omega)$ which may not be differentiable.

Let φ be a C^1 with compact support with total mass 1. For example,

$$\varphi(z) = \begin{cases} C(1-|z|^2)^2 & \text{if } |z| \leq 1 \\ 0 & \text{if } |z| \geq 1 \end{cases}$$

for an appropriate value of B. Set

$$\varphi_n(z) = nf(nx)$$

so that φ_n is C^1, has total mass 1, and has support on $|z| \leq 1/n$.

Now consider the convolution

$$u_n(z) \coloneqq \iint_{\mathbb{C}} u(w)\varphi_n(z-w)\,\mathrm{d}w = \iint_{\mathbb{C}} u(z-w)\varphi_n(z)\,\mathrm{d}w.$$

Notice that φ_n' is continuous and has compact support, so it is $L^1(\mathbb{C})$. Thus, we may pass the derivative through the integral sign without issue to see that u_n is C^1 also.

Now let z_0 and R be as before. Since the integrand is L^1, we may apply Fubini's theorem along with a change of variables to get

$$\begin{aligned}
\int_{\partial B(z_0,R)} u_n(z)\,\mathrm{d}z &= \int_{\partial B(z_0,R)} \left(\iint_{\mathbb{C}} u(z-w)\varphi_n(z)\,\mathrm{d}w\right)\mathrm{d}z \\
&= \iint_{\mathbb{C}} \left(\int_{\partial B(z_0,R)} u(z-w)\,\mathrm{d}z\right)\varphi_n(w)\,\mathrm{d}w \\
&= \iint_{\mathbb{C}} \left(\int_{\partial B(z_0-w,R)} u(z)\,\mathrm{d}z\right)\varphi_n(w)\,\mathrm{d}w \quad (z \mapsto z-w) \\
&= \iint_{\mathbb{C}} 0 \cdot \varphi_n(w)\,\mathrm{d}w \\
&= 0.
\end{aligned}$$

Thus, by the first part of the problem, each u_n is C^1, so each u_n is holomorphic.

We will now show that u_n converges to u locally uniformly. Then u is (locally) a uniform limit of holomorphic functions, which shows that it is holomorphic, as required.

Let $K \subseteq \Omega$ be compact. Since u is continuous on K, compactness tells us that u is uniformly continuous on K. Thus, for any $\varepsilon > 0$, there exists $\delta > 0$ so that $|u(w) - u(z)| < \varepsilon$ whenever $|z - w| < \delta$. Moreover, for n large, $\varphi_n(z-w)$ has support contained in $B(z,\delta)$. Then for n large,

$$\begin{aligned}
|u_n(z) - u(z)| &\leq \iint_{\mathbb{C}} |u(w) - u(z)|\varphi_n(z-w)\,\mathrm{d}w \\
&= \iint_{B(z,\delta)} |u(w) - u(z)|\varphi_n(z-w)\,\mathrm{d}w \\
&< \iint_{B(z,\delta)} \varepsilon\varphi_n(z-w)\,\mathrm{d}w \\
&= \varepsilon,
\end{aligned}$$

which shows uniform convergence on K, so $u_n \to u$ locally uniformly. Thus, u is holomorphic in Ω.

6 Let $f \in C_0^1(\mathbb{C})$. Show that the equation

$$\frac{\partial u}{\partial \overline{z}}(z) = f(z)$$

has a compactly supported solution if and only if

$$\int_{\mathbb{C}} z^n f(z)\, L(\mathrm{d}z) = 0, \quad n = 0, 1, 2, \ldots.$$

Solution "$\Longrightarrow$"

Let the given equation have a compact supported solution u.

Since u has compact support, there exists $R > 0$ so that the support of u is contained within $B(0, R)$. Then

$$\int_{\mathbb{C}} z^n f(z)\, L(\mathrm{d}z) = \int_{\mathbb{C}} z^n \frac{\partial u}{\partial \overline{z}}(z)\, L(\mathrm{d}z) = \int_{B(0,R)} z^n \frac{\partial u}{\partial \overline{z}}(z)\, L(\mathrm{d}z).$$

By Stokes' theorem,

$$\int_{B(0,R)} z^n \frac{\partial u}{\partial \overline{z}}(z)\, L(\mathrm{d}z) = \int_{\partial B(0,R)} z^n \frac{\partial u}{\partial \overline{z}}(z)\, \mathrm{d}z = 0,$$

since u vanishes outside $B(0, R)$.

"$\Longleftarrow$"

Recall that

$$u(z) = -\frac{1}{\pi} \int_{\mathbb{C}} f(z) \frac{L(\mathrm{d}\zeta)}{\zeta - z}$$

is a solution to the differential equation.

Expanding out in a power series, we have

$$u(z) = \frac{1}{\pi} \int_{\mathbb{C}} \sum_{n=0}^{\infty} \left(\frac{\zeta}{z}\right)^n \frac{f(\zeta)}{z}\, L(\mathrm{d}\zeta).$$

Since f has compact support, there exists $R > 0$ so that if $|\zeta| > R$, then $f(\zeta) = 0$, so we may write

$$u(z) = \frac{1}{\pi} \int_{B(0,R)} \sum_{n=0}^{\infty} \left(\frac{\zeta}{z}\right)^n \frac{f(\zeta)}{z}\, L(\mathrm{d}\zeta).$$

Thus, the series converges uniformly for $|z| \geq r > R$, so we may interchange the integral and the sum for these values of z:

$$u(z) = \frac{1}{\pi z^{n+1}} \sum_{n=0}^{\infty} \int_{B(0,R)} \zeta^n f(\zeta)\, L(\mathrm{d}\zeta) = 0,$$

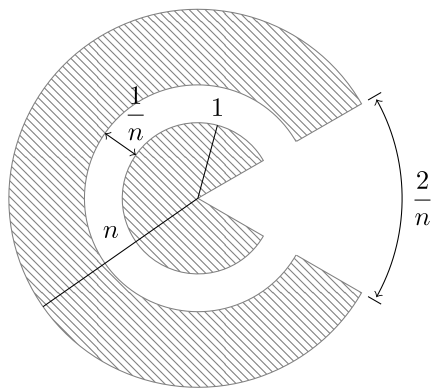
so u has compact support in the set $B(0, R)$.

7 Show that there exists a sequence of polynomials p_n such that $p_n(z) \to 1$ for all $|z| \leq 1$ and $p_n(z) \to 0$ for all $|z| > 1$.

Solution Consider the following compact set:

$$K_n := \left(\{|z| \leq 1\} \cup \left\{ 1 + \frac{1}{n} \leq |z| \leq n \right\} \right) \cap \left\{ |\arg z| \geq \frac{1}{n} \right\},$$

where the argument is measured from the positive real axis.



Set f_n to be 1 on $\overline{D}$ and 1 on $\{1 + 1/n \leq |z| \leq n\}$. Then f_n is analytic and $\mathbb{C} \setminus K_n$ is connected, by the picture. Thus, by Runge's theorem, there exists a polynomial p_n so that $\sup_{K_n} |f_n - p_n| < 2^{-n}$.

Notice that $f_n \xrightarrow{n \to \infty} \chi_{\overline{D}}$, where $\chi_{\overline{D}}$ is the indicator function on the unit disk, by construction. Thus,

$$|\chi_{\overline{D}}(x) - p_n(x)| \leq |\chi_{\overline{D}}(x) - f_n(x)| + |f_n(x) - p_n(x)| \xrightarrow{n \to \infty} 0$$

pointwise, as desired.

8 For a compact set $K \subseteq \mathbb{C}$, define the polynomially convex hull

$$\hat{K} = \left\{ z \in \mathbb{C} \mid |p(z)| \leq \sup_K |p| \text{ for all polynomials } p \right\}.$$

Note that $\hat{K} \supseteq K$. A compact set K is said to be *polynomially convex* if $\hat{K} = K$. Show that K is polynomially convex if and only if $\mathbb{C} \setminus K$ is connected.

Solution "$\Longrightarrow$"

Let $K = \hat{K}$. Since K is compact, $\mathbb{C} \setminus K$ has precisely one unbounded component.

Now suppose that $\mathbb{C} \setminus K$ is not connected so that $\mathbb{C} \setminus K$ has a bounded component, which we'll call C.

Let $z \in C$. Now let p be a polynomial, which is continuous on $\overline{C}$. Moreover, C is bounded and open by assumption, so by the maximum principle, p attains its maximum on ∂C, so for any $z \in C$,

$$|p(z)| \leq \sup_{\partial C} |p| \leq \sup_K |p|,$$

since $\partial C \subseteq K$. Thus, $z \in \hat{K} \implies C \subseteq K$, but this contradicts the fact that $C \subseteq \mathbb{C} \setminus K$. Thus, $\mathbb{C} \setminus K$ cannot have a bounded component, so it only has one component, i.e., $\mathbb{C} \setminus K$ is connected.

"$\Longleftarrow$"

Suppose $\mathbb{C} \setminus K$ is connected, and suppose there exists $z_0 \in \hat{K} \setminus K$. Since $\mathbb{C}$ is regular, we can separate $\{z_0\}$ and K by disjoint open sets U, V, where $K \subseteq U$ and $z_0 \in V$. Then consider

$$f(z) = \begin{cases} 0 & \text{if } z \in U \\ 1 & \text{if } z \in V. \end{cases}$$

By Runge's theorem, there exists a polynomial p with $\sup_{U \cup V} |f - p| < 1/2$. Then for $z \in K$,

$$|p(z)| < \frac{1}{2} \implies \sup_K |p| \leq \frac{1}{2},$$

but this means for $z = z_0$, we get

$$|p(z_0) - 1| < \frac{1}{2} \implies |p(z_0)| > \frac{1}{2} \geq \sup_K |p| \implies z_0 \notin \hat{K},$$

a contradiction. Thus, no such z_0 exists, so $K = \hat{K}$ as required.

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9 Let u be a subharmonic in all of $\mathbb{R}^2$ such that

$$u(x) \leq o(\log |x|), \quad |x| \to \infty.$$

Show that u is a constant.

Solution For $r > 0$, set $M(r) := \sup_{|x|=R} |u|$. This is a convex function in the logarithm, by Hadamard's three circle theorem, i.e., if $0 < r_1 < r_2 < r$, we have

$$\log\left(\frac{r}{r_1}\right) M(r_2) \leq \log\left(\frac{r}{r_2}\right) M(r_1) + \log\left(\frac{r_2}{r_1}\right) M(r)$$
$$\implies M(r_2) \leq \frac{\log(r/r_2)}{\log(r/r_1)} M(r_1) + \frac{\log(r_2/r_1)}{\log(r/r_1)} M(r)$$

On the other hand, our assumption tells us that for $|x|$ large,

$$M(|x|) \leq o(\log |x|).$$

Thus, taking $r \to \infty$ in our inequality, we get $M(r_2) \leq M(r_1)$. Since $M(r)$ is an increasing function, it follows that $M(r_1) = M(r_2)$, so by the maximum principle, u is constant on $r_1 < |x| < r_2$. Taking r_1 to 0 and r_2 to ∞ shows that u is constant, as desired.

10 Let A, B be positive definite $n \times n$ real symmetric matrices such that with the Euclidean norm on $\mathbb{R}^n$, we have

$$\|BA^{-1}x\| \leq \|x\|, \quad x \in \mathbb{R}^n.$$

Show that for $0 < \theta < 1$, we have

$$\|B^\theta A^{-\theta}x\| \leq \|x\|, \quad x \in \mathbb{R}^n.$$

Solution Consider $z \mapsto \langle B^z A^{-z}x, y\rangle$, for $x, y \in \mathbb{R}^n$ and for $0 < \operatorname{Re} z < 1$. This is a holomorphic function, since $0 < \operatorname{Re} z < 1$ is a simply connected region excluding 0, so λ^z is a holomorphic function for any $\lambda \in \mathbb{R}$. Indeed, the components of the diagonal matrices similar to B^z and A^{-z} are simply the eigenvectors (which are real because A and B are self-adjoint) raised to the power z, so any linear combination of them is also holomorphic.

Note that if $\operatorname{Re} z = 0$, we may write $z = bi$. Then if $\lambda \in \mathbb{R}$, $|\lambda^{bi}| = 1$. It follows that A^{-z} and B^z have operator norm 1 on this line, so by Cauchy-Schwarz,

$$\left|\langle B^{-z}A^z x, y\rangle\right| \leq \|B^z\|\|A^{-z}\|\|x\|\|y\| \leq \|x\|\|y\|$$

For $\operatorname{Re} z = 1$, write $z = 1 + bi$ so that

$$\|B^{1+bi}A^{-1-bi}\| \leq \|B^{bi}\|\|BA^{-1}\|\|A^{-bi}\| \leq 1.$$

The outer matrices have operator norm 1 because their power has real part 0, and the inner matrix has operator norm 1 by assumption. Thus,

$$\left|\langle B^{1+bi}A^{-1-bi}x, y\rangle\right| \leq \|B^{1+bi}A^{-1-bi}x\|\|y\| \leq \|x\|\|y\|$$

also.

Notice $|\langle B^{-z}A^z x, y\rangle|$ is bounded because each λ^z is: $|\lambda^z| = \lambda^{\operatorname{Re} z} \leq \lambda$, so we can simply take the maximal eigenvalue as the upper bound. Positive constant functions are a PL function, so it follows that

$$\left|\langle B^z A^{-z}x, y\rangle\right| \leq \|x\|\|y\|$$

whenever $0 < \operatorname{Re} z < 1$. In particular, it holds for when $z = \theta \in (0, 1)$. Thus, if we take $y = B^\theta A^{-\theta}x / \|B^\theta A^{-\theta}x\|$, we have

$$\|B^\theta A^{-\theta}x\| = \left|\left\langle B^\theta A^{-\theta}x, \frac{B^\theta A^{-\theta}x}{\|B^\theta A^{-\theta}x\|}\right\rangle\right| \leq \|x\|,$$

as needed.