2.7.7 $ty' - y = t^2 \cos t, \qquad y(0) = -3$

Solution (i)

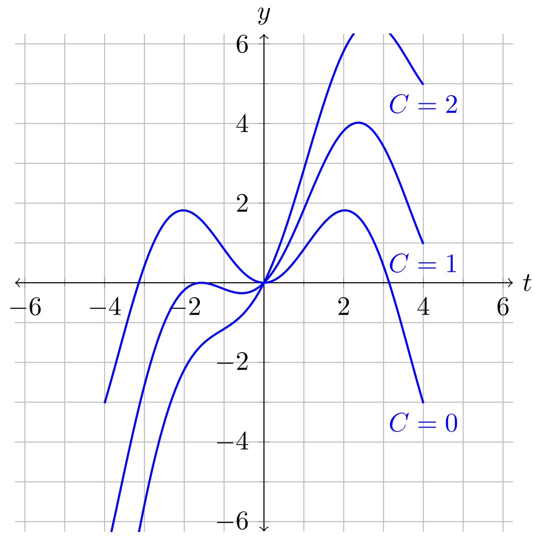
$$y' - \frac{1}{t} y = t \cos t$$

$$\mu(t) = e^{-\int 1/t \, dt} = \frac{1}{t}$$

$$\left(y \cdot \frac{1}{t} \right)' = \cos t$$

$$\frac{y}{t} = \sin t + C$$

$$y = t \sin t + Ct$$



(ii) There is no solution satisfying the given initial condition because every solution will pass through the origin. We can see it by setting $t = 0$ in the differential equation:

$$-y = 0 \Rightarrow y(0) \equiv 0$$

However, this does not contradict the existence theorem because if we solve for y',

$$y' = \frac{t^2 \cos t + y}{t} = f(t, y)$$

we see that f is not continuous at $t = 0$, so the existence theorem does not apply.

2.7.10 Show that $y(t) = 0$ and $y(t) = (1/16)t^4$ are both solutions of the initial value problem $y' = ty^{1/2}$, where $y(0) = 0$. Explain why this fact does not contradict Theorem 7.16.

Solution Let $y_1(t) = 0$ and $y_2(t) = (1/16)t^4$.

	y'	$y(0)$	$y' \stackrel{?}{=} ty^{1/2}$
y_1	0	0	$0 = 0$ ✓
y_2	$\frac{1}{4}t^3$	0	$\frac{1}{4}t^3 = t \cdot \frac{1}{4}t^2 = \frac{1}{4}t^3$ ✓

Both functions satisfy the initial value problem. However, this does not contradict the uniqueness theorem because $\frac{\partial}{\partial y} ty^{1/2} = \frac{t}{2y^{1/2}}$ is not continuous at $(0, 0)$—there exists a singularity there. Therefore, we cannot apply the theorem to this problem.

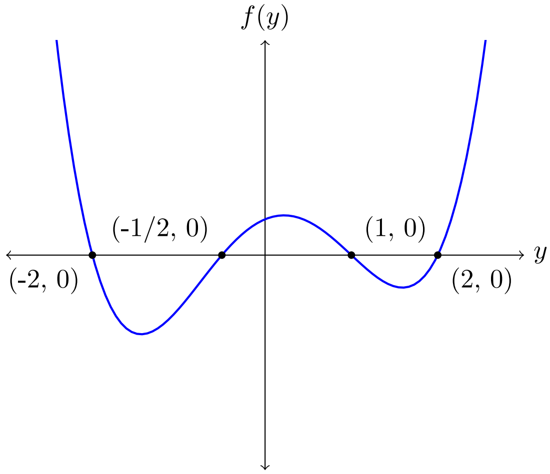
2.7.30 Suppose that x is a solution to the initial value problem

$$x' = \frac{x^3 - x}{1 + t^2 x^2} \quad \text{and} \quad x(0) = 1/2.$$

Show that $0 < x(t) < 1$ for all t for which x is defined.

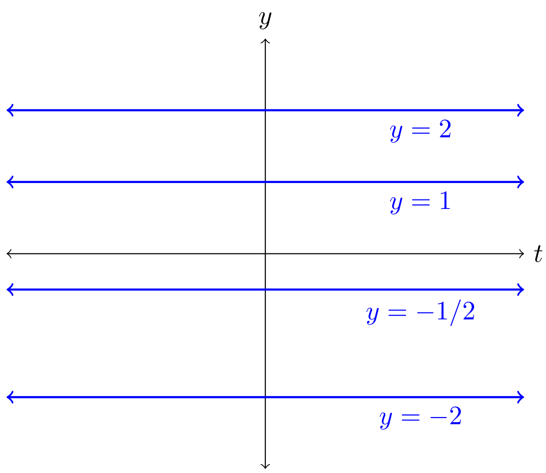
Solution Let $f(t, x) = \frac{x^3 - x}{1 + t^2 x^2}$. f is continuous on $\mathbb{R}^2$ since the numerator is a polynomial and the denominator is never less than 1. Also, $\frac{\partial f}{\partial x}$ is also continuous on $\mathbb{R}^2$ since its numerator will be a polynomial, and its denominator will be $\left(1 + t^2 x^2\right)^2$ which will never be less than 1. Thus, we can apply the uniqueness theorem everywhere. Notice that $x_0(t) = 0$ and $x_1(t) = 1$ are both solutions to the differential equation. Suppose $x(c) = 0$ for some c. $x_0(c) = 0$ also, so by uniqueness, $x(t) = x_0(t)$, which is a contradiction, so such a value of c does not exist. A similar argument holds for $x(t) = 1$. Thus, $x(t)$ is never equal to 0 or 1, and since x is continuous and $x(0) = 1/2$, $0 < x(t) < 1$ for all t such that x is defined.

2.9.10 Indentify the equilibrium points and sketch the equilibrium solutions in the ty-plane. Classify each equilibrium point as either unstable or asymptotically stable.



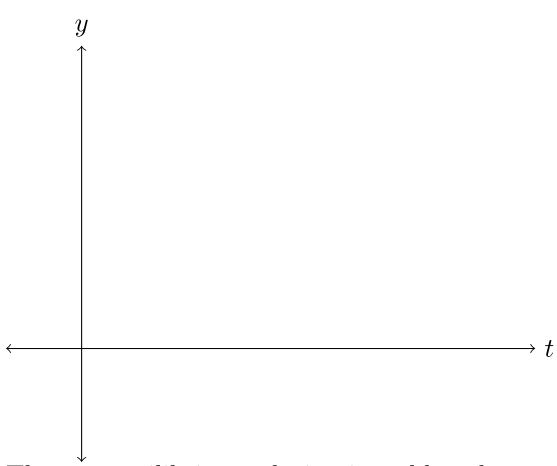
Solution The equilibrium solutions are:

y	$f'(y)$	Classification
$y(t) = -2$	< 0	Stable
$y(t) = -1/2$	> 0	Unstable
$y(t) = 1$	< 0	Stable
$y(t) = 2$	> 0	Unstable



2.9.12 Sketch the remainder of the direction field. Then superimpose the equilibrium solution(s), classifying each as either unstable or asymptotically stable.

Solution



The top equilibrium solution is stable, whereas the bottom one is unstable.

2.9.26 $y' = (3+y)(1-y), \quad y(0) = 2$

Solution (i) Assuming $y \neq -3$ and $y \neq 1$:

$$\frac{\mathrm{d}y}{(3+y)(1-y)} = \mathrm{d}t$$

$$\frac{1}{4}\left(\frac{1}{3+y} + \frac{1}{1-y}\right)\mathrm{d}y = \mathrm{d}t$$

$$\ln|3+y| - \ln|1-y| = 4t + C$$

$$\ln\left|\frac{3+y}{1-y}\right| = 4t + C$$

$$\frac{3+y}{1-y} = Ae^{4t} \text{ (we can ignore absolute values because the result is a solution for any real } A\text{)}$$

$$3 + y = Ae^{4t} - yAe^{4t}$$

$$y(t) = \frac{Ae^{4t} - 3}{1 + Ae^{4t}}$$

$$y(0) = 2 \Rightarrow \frac{A-3}{1+A} = 2 \Rightarrow A = -5$$

$$y(t) = \frac{-5e^{4t} - 3}{1 - 5e^{4t}}$$

(ii)
$$\lim_{t\to\infty} \frac{-5e^{4t} - 3}{1 - 5e^{4t}} = \lim_{t\to\infty} \frac{1 - \frac{3}{-5e^{4t}}}{\frac{1}{-5e^{4t}} + 1} = 1$$

(iii) In the long term, our function will begin to behave like $y(t) = 1$, which is the stable equilibrium solution of the differential equation. Using equilibrium solutions and phase portraits is the easier way to predict the long-term behavior of the solution, since we wouldn't have to solve the equation by hand.