1 a. Let $k > 1$. Show that $p_n = 2^{-n^k}$ converges to 0, but not quadratically, regardless of k.

b. Given $\alpha > 1$, construct a sequence $\{q_n\}_{n \geq 1}$ converging to 1 of order α.

Solution a. Since $k > 1$, $n^k \xrightarrow{n\to\infty} \infty$, so $p_n \xrightarrow{n\to\infty} 0$, since $2^x \geq x$.

$$\frac{2^{-(n+1)^k}}{\left(2^{-n^k}\right)^2} = 2^{2n^k-(n+1)^k}.$$

Notice that $2n^k - (n+1)^k \xrightarrow{n\to\infty} \infty$, which means that the limit of the above is ∞, so the convergence is not quadratic.

b. Take $q_n = 2^{-\alpha^n}$. Since $\alpha > 0$, this clearly converges to 0, and

$$\frac{2^{-\alpha^{n+1}}}{\left(2^{-\alpha^n}\right)^\alpha} = \frac{2^{-\alpha^{n+1}}}{2^{-\alpha^{n+1}}} = 1 \xrightarrow{n\to\infty} 1,$$

so the convergence is of order α.

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2 Let $f(x) \in C^\infty(\mathbb{R})$. Show that p is a zero of multiplicity 2 of $f(x)$ if and only if $f(p) = f'(p) = 0$ while $f''(p) \neq 0$.

Solution "$\Longrightarrow$"

Let p be a zero of multiplicity 2 of $f(x)$. Then we can write

$$f(x) = (x-p)^2 q(x),$$

for some smooth and bounded q with $q(p) \neq 0$. Then

$$\begin{aligned} f'(x) &= 2(x-p)q(x) + (x-p)^2 q'(x) \implies f'(p) = 0 \\ f''(x) &= 2q(x) + 2(x-p)q'(x) + (x-p)^2 q''(x) \implies f''(p) = 2q(p) \neq 0, \end{aligned}$$

so this direction holds.

"$\Longleftarrow$"

Let $f(p) = f'(p) = 0$, but $f''(p) \neq 0$. Then using the Lagrange remainder,

$$f(x) = f(p) + f'(p)(x-p) + \frac{1}{2}f''(\xi(x))(x-p)^2 = \frac{1}{2}f''(\xi(x))(x-p)^2,$$

where $\xi(x) \in (x, p)$, which is the form we want.

3 a. Let $p_n = e^{-n}/n^2$. Prove that $p_n \to 0$ linearly. Compute $\hat{p}_n$ using Aitken's Δ^2 method and show that $\lim_{n\to\infty} \hat{p}_n/p_n = 0$.

b. We shall apply Steffensen's method to find the fixed point of $g(x) = \cos x$ on $[0,1]$.

(a) Let $p_0 = 1$. Compute $p_1, \ldots, p_4$ using the classic fixed-point iteration. You may use calculators or write some code.

(b) Let $p_0^{(0)} = 1$. Compute $p_1^{(0)}, p_2^{(0)}, p_0^{(1)}, p_1^{(1)}, p_2^{(1)}, p_0^{(2)}$.

Solution a. If we replace n with a continuous variable x, we see that by l'Hôpital's rule,

$$\lim_{n\to\infty} \frac{e^{-n}}{n^2} = \lim_{x\to\infty} \frac{e^{-x}}{x^2} = \lim_{x\to\infty} \frac{-e^{-x}}{2x} = \lim_{x\to\infty} \frac{e^{-x}}{2} = 0.$$

Now we show that it converges linearly:

$$\lim_{n\to\infty} \frac{e^{-(n+1)}}{(n+1)^2} \cdot \frac{n^2}{e^{-n}} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) \frac{1}{e} = \frac{1}{e} < 1.$$

From Aitken's Δ^2 method, we see

$$\begin{aligned}
\hat{p}_n &= p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n} \\
&= \frac{e^{-n}}{n^2} - \frac{\left(\frac{e^{-(n+1)}}{(n+1)^2} - \frac{e^{-n}}{n^2}\right)^2}{\frac{e^{-(n+2)}}{(n+2)^2} - 2\frac{e^{-(n+1)}}{(n+1)^2} + \frac{e^{-n}}{n^2}} \\
&= \frac{e^{-n}}{n^2} - \frac{\frac{e^{-2(n+1)}}{(n+1)^4} - 2\frac{e^{-(n+1)}e^{-n}}{n^2(n+1)^2} + \frac{e^{-2n}}{n^4}}{\frac{e^{-(n+2)}}{(n+2)^2} - 2\frac{e^{-(n+1)}}{(n+1)^2} + \frac{e^{-n}}{n^2}} \\
\frac{\hat{p}_n}{p_n} &= 1 - \frac{n^2}{e^{-n}} \frac{\frac{e^{-2(n+1)}}{(n+1)^4} - 2\frac{e^{-(n+1)}e^{-n}}{n^2(n+1)^2} + \frac{e^{-2n}}{n^4}}{\frac{e^{-(n+2)}}{(n+2)^2} - 2\frac{e^{-(n+1)}}{(n+1)^2} + \frac{e^{-n}}{n^2}} \\
&\sim 1 - \frac{n^2}{e^{-n}} \frac{\frac{e^{-2n}}{n^4}}{\frac{e^{-n}}{n^2}} \\
&= 1 - \frac{n^2}{e^{-n}} \frac{e^{-2n}}{n^4} \frac{n^2}{e^{-n}} \\
&= 1 - 1 \\
&= 0,
\end{aligned}$$

so the limit is proved.

b. The classic fixed-point iteration gives the following output:

n	p_n
0	1
1	0.540 302 305 868 139 8
2	0.857 553 215 846 393 4
3	0.654 289 790 497 779 1
4	0.793 480 358 742 565 6

On the other hand, Steffensen's method gives the following:

n	p_n
$p_0^{(0)}$	1
$p_1^{(0)}$	0.540 302 305 868 139 8
$p_2^{(0)}$	0.857 553 215 846 393 4
$p_0^{(1)}$	0.728 010 361 467 617 1
$p_1^{(1)}$	0.746 499 756 045 220 3
$p_2^{(1)}$	0.734 070 283 736 529 6
$p_0^{(2)}$	0.739 066 966 908 673 8

Steffensen's method seems to converge faster, since it gets close to 0.739085 in 2 steps.

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4 Let $f(x) = \sin \pi x$.

a. Construct its Lagrange interpolating polynomial $P(x)$ using $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$, and $x_3 = 2$.

b. Evaluate $P(3/2)$; it can be viewed as an approximation of $f(3/2)$. Derive a bound for $|f(3/2) - P(3/2)|$.

Solution a. We want $P(x_i) = f(x_i)$ for each i. Notice that

$$f(x_0) = f(x_2) = f(x_3) = 0,$$

so the Lagrange polynomial reduces to

$$\frac{x(x-1)(x-2)}{x_1(x_1-1)(x_1-2)} = \frac{x(x-1)(x-2)}{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2}} = \frac{8}{3}x(x-1)(x-2).$$

b. We get

$$P(3/2) = \frac{8}{3} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot -\frac{1}{2} = -1.$$

By the Lagrange polynomial error bound,

$$|f(3/2) - P(3/2)| = \left| \frac{f^{(4)}(\xi)}{4!} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot -\frac{1}{2} \right| = \frac{3}{8} \left| \frac{f^{(4)}(\xi)}{4!} \right|, \quad \xi \in [0, 2].$$

Notice that

$$|f^{(4)}(\xi)| = \pi^4 |\sin \pi x| \leq \pi^4,$$

so

$$|f(3/2) - P(3/2)| \leq \frac{\pi^4}{64}.$$

5 Consider the Runge function

$$R(x) = \frac{1}{x^2+1}, \quad x \in [-5,5].$$

a. Use the nodes $x_k = -5+k$, $k = 0,1,\ldots,10$, to form an interpolating polynomial $P(x)$ of $R(x)$ on $[-5,5]$. Plot the graphs of $R(x)$ and $P(x)$ on $[-5,5]$.

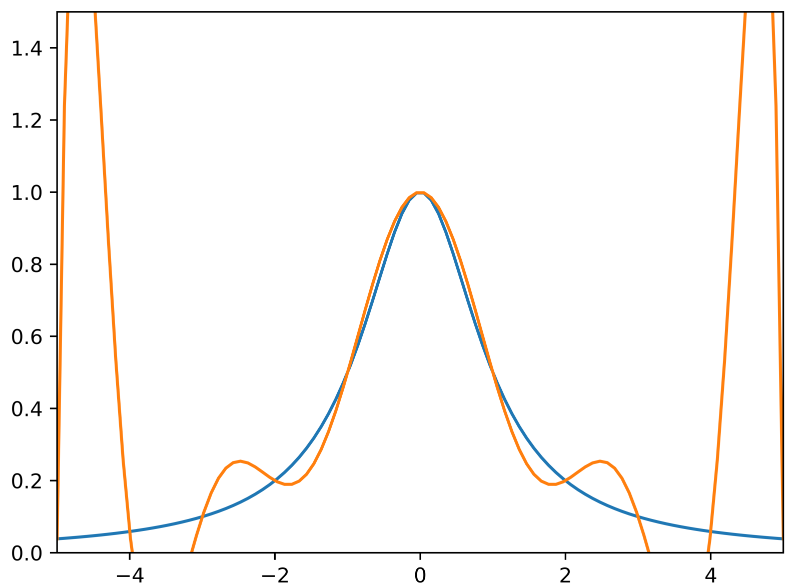
b. What have you observed from the plot? Can you explain why?

c. Now use a new set of nodes

$$\hat{x}_k = 5\cos\left(\frac{k\pi}{10}\right), \quad k = 0,1,\ldots,10$$

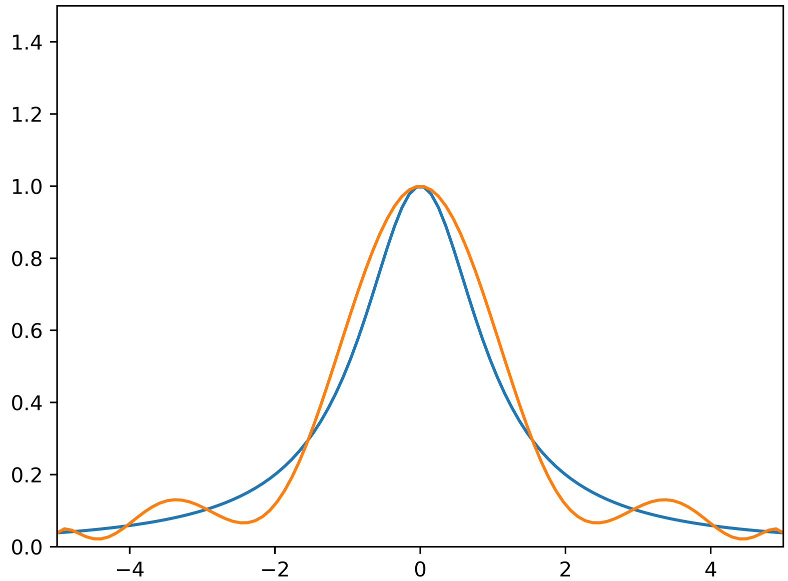
to form an interpolating polynomial $Q(x)$. Plot $R(x)$ and $Q(x)$ on $[-5,5]$. What do you observe this time?

Solution a. The blue is $R(x)$ and the orange is $P(x)$:



b. P matches with R at all the nodes, as expected. P approximates R very well close to 0. On the other hand, close to -5 or 5, P does not approximate R well at all, which is likely due to overfitting; the polynomial focuses too much on making it through these nodes, and essentially ignores what's in between. In addition, the larger the degree of the polynomial, the larger its derivatives are, which means that the polynomial is very unstable. We can see that the oscillation grows worse and worse as $|x|$ grows larger.

c. Like the above, $R(x)$ is blue and $Q(x)$ is orange:



This time, the oscillations are significantly better than P's. While Q's approximation of R close to 0 is not as good as P's, it's safe to say that on average, Q is a better approximation than P on the entire interval $[-5, 5]$.