1 Let $q \geq 2$ be a prime number. Recall the equivalence relation on $\mathbb{Z}$ defined as follows: for $m, n \in \mathbb{Z}$, we write $m \sim n$ if $q \mid (m - n)$. For $n \in \mathbb{Z}$, denote by $C(n)$ the equivalence class of n. Let $\mathbb{Z}/q\mathbb{Z}$ denote the set of equivalence classes. Define addition and multiplication on $\mathbb{Z}/q\mathbb{Z}$ as follows:

$$C(n) + C(m) = C(n+m) \qquad \text{and} \qquad C(n) \cdot C(m) = C(nm).$$

a. Prove that addition and multiplication are well defined, that is, the result is independent of the representatives chosen from the equivalence classes.

b. Verify that with these operations $\mathbb{Z}/q\mathbb{Z}$ is a field.

c. Show that there is no order relation on $\mathbb{Z}/q\mathbb{Z}$ that makes it an ordered field.

Solution a. Let $a, a' \in C(n)$ and $b, b' \in C(m)$.

We wish to show that $a + b \sim a' + b'$. By definition, we have $a - a' = qc$ for $c \in \mathbb{Z}$ and $b - b' = qd$ for $d \in \mathbb{Z}$. Adding the equalities yields $a + b - (a' + b') = q(c + d)$. Since the integers are closed under addition, $c + d \in \mathbb{Z}$, so by definition, $a + b \sim (a' + b')$.

We now wish to show that $a \cdot b \sim a' \cdot b'$. Once again, by definition, $a - a' = qc \implies a = a' + qc$ and $b - b' = qd \implies b = b' + qd$. Then

$$ab - a'b' = (a' + qc)(b' + qd) - a'b' = a'b' + a'qd + b'qc + q^2cd - a'b' = q(a'd + b'c + qcd).$$

$a'd + b'c + qcd \in \mathbb{Z}$, so by definition, $ab \sim a'b'$ as desired.

b. Let $l, m, n \in \mathbb{Z}$.

(A1) $C(n) + C(m) = C(n + m)$. $n + m \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under addition. So, $\mathbb{Z}/q\mathbb{Z}$ is also closed under addition.

(A2) $C(n) + C(m) = C(n + m) = C(m + n) = C(m) + C(n)$

(A3) $C(n) + [C(m) + C(l)] = C(n) + C(m + l) = C(n + (m + l)) = C((n + m) + l) = C(n + m) + C(l) = [C(n) + C(m)] + C(l)$

(A4) $0 = C(0)$ is the identity: $C(n) + 0 = C(n) + C(0) = C(n + 0) = C(n)$.

(A5) Given $C(n)$, its inverse is $-C(n) = C(-n)$. $C(n) - C(n) = C(n) + C(-n) = C(n - n) = C(0) = 0$.

(M1) $C(n) \cdot C(m) = C(nm)$. $nm \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under multiplication. So, $\mathbb{Z}/q\mathbb{Z}$ is also closed under multiplication.

(M2) $C(n) \cdot C(m) = C(nm) = C(mn) = C(m) \cdot C(n)$.

(M3) $C(n) \cdot [C(m) \cdot C(l)] = C(n) \cdot C(ml) = C(n(ml)) = C((nm)l) = C(nm) \cdot C(l) = [C(n) \cdot C(m)] \cdot C(l)$.

(M4) $1 = C(1)$ is an identity. $C(n) \cdot 1 = C(n1) = C(n)$.

(M5) An inverse for any element $C(n)$ of $\mathbb{Z}/q\mathbb{Z}$ can be found using the Euclidean algorithm since q and n must be coprime.

(D) $C(n) \cdot [C(m) + C(l)] = C(n) \cdot C(m + l) = C(n(m + l)) = C(nm + nl) = C(nm) + C(nl) = C(n) \cdot C(m) + C(n) \cdot C(l)$

Thus all the axioms hold, so $\mathbb{Z}/q\mathbb{Z}$ is a field.

c. Suppose we have an order relation $<$ on $\mathbb{Z}/q\mathbb{Z}$ that makes it an ordered field. Note that $q \sim 0$, so $C(q) = C(0)$.

From a proposition proved in class, we have $1 > 0 \implies C(1) > C(0)$. Then by adding the inequality to itself q times, we have $C(1) + C(1) + \cdots + C(1) = C(q) > C(0)$. But $C(q) = C(0)$, which is a contradiction. Hence, there is no order relation such that $\mathbb{Z}/q\mathbb{Z}$ is an ordered field.

2 Define two internal laws of composition on $R = \mathbb{R} \times \mathbb{R}$ as follows:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$
$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1).$$

a. Show that with these operations $\mathbb{R}$ is a field.

b. Show that there is no order relation on R that makes R an ordered field.

Solution a. Let $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in R$. This proof will make heavy use of the fact that $\mathbb{R}$ is a field.

(A1) $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$. $a_1, b_1, a_2, b_2 \in \mathbb{R}$, so the sum of the tuples is in R.

(A2) $(a_1, a_2) = (b_1, b_2) = (a_1 + b_1, a_2 + b_2) = (b_1 + a_1, b_2, a_2) = (b_1, b_2) + (a_1 + a_2)$

(A3) $(a_1, a_2) + [(b_1, b_2), (c_1, c_2)] = (a_1, a_2) + (b_1 + c_1, b_2 + c_2) = (a_1 + (b_1 + c_1), a_2 + (b_2 + c_2)) = ((a_1 + b_1) + c_1, (a_2 + b_2) + c_2) = (a_1 + b_1, a_2 + b_2) + (c_1, c_2) = [(a_1, a_2) + (b_1, b_2)] + (c_1, c_2)$

(A4) $0 = (0, 0)$ is the identity of the field.

(A5) $-(a_1, a_2) = (-a_1, -a_2)$ is the inverse of an element in the field.

(M1) $(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1)$. $a_1, b_1, a_2, b_2 \in \mathbb{R}$, so the field is closed under the defined multiplication.

(M2) $(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1) = (b_1 a_1 - b_2 a_2, b_2 a_1 + b_1 a_2) = (b_1, b_2) \cdot (a_1, a_2)$.

(M3)
$$\begin{aligned} (a_1, a_2) \cdot [(b_1, b_2) \cdot (c_1, c_2)] &= (a_1, a_2) \cdot (b_1 c_1 - b_2 c_2, b_1 c_2 + b_2 c_1) \\ &= (a_1(b_1 c_1 - b_2 c_2) - a_2(b_1 c_2 + b_2 c_1), a_1(b_1 c_2 + b_2 c_1) + a_2(b_1 c_1 - b_2 c_2)) \\ &= ((a_1 b_1 - a_2 b_2) c_1 - (a_1 b_2 + a_2 b_1) c_2, (a_1 b_1 - a_2 b_2) c_2 + (a_1 b_2 - a_2 b_2) c_2) \\ &= (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1) \cdot (c_1, c_2) \\ &= [(a_1, a_2) \cdot (b_1, b_2)] \cdot (c_1, c_2) \end{aligned}$$

(M4) $(1, 0)$ is the multiplicative identity of the field.

(M5) $\left(\frac{a}{\sqrt{a^2+b^2}}, -\frac{b}{\sqrt{a^2+b^2}} \right)$ is the multiplicative inverse of an element of the field.

(D)
$$\begin{aligned} (a_1, a_2) \cdot [(b_1, b_2) + (c_1, c_2)] &= (a_1, a_2) \cdot (b_1 + c_1, b_2 + c_2) \\ &= (a_1(b_1 + c_1) - a_2(b_2 + c_2), a_1(b_2 + c_2) + a_2(b_1 + c_1)) \\ &= (a_1 b_1 - a_2 b_2 + a_1 c_1 - a_2 c_2, a_1 b_2 + a_2 b_1 + a_1 c_2 + a_2 c_1) \\ &= (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1) + (a_1 c_1 - a_2 c_2, a_1 c_2 + a_2 c_1) \\ &= (a_1, a_2) \cdot (b_1, b_2) + (a_1, a_2) \cdot (c_1, c_2) \end{aligned}$$

Thus, R is a field with these operations.

b. Suppose there is an order relation $<$ on R such that it becomes an ordered field. Then we must have $1 > 0 \implies (1, 0) > (0, 0) \implies (-1, 0) < (0, 0)$. The multiplicative inverse of $(0, 1)$ is $(0, -1)$, which is also its additive inverse. Then we have either $(0, 1) < 0 < (0, -1)$ or $(0, -1) < 0 < (0, 1)$. In both cases, we have a contradiction, because according to a proposition we proved, $0 < (0, 1) \implies 0 < (0, 1)^{-1}$. But $(0, 1)$ and its additive inverse have different signs, so no matter what order relation we impose on R, it will never be an ordered field.

3 Show that if a sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers converges to a, then the sequence $\{|a_n|\}_{n\in\mathbb{N}}$ converges to $|a|$. Show (via an example) that the converse is not true.

Solution We first prove a lemma: $\Big||x|-|y|\Big| \le |x-y|$. From the triangle inequality, we have

$$\begin{aligned}|x-y+y| &\le |x-y|+|y|\\ |x| &\le |x-y|+|y|\\ |x-y| &\ge |x|-|y|\end{aligned}$$

Then consider $\Big||x|-|y|\Big|$. If $|x|-|y|\ge 0$, then $\Big||x|-|y|\Big| = |x|-|y| \le |x-y|$. Otherwise, $\Big||x|-|y|\Big| = |y|-|x| \le |y-x| = |x-y|$. Thus, in either case, $\Big||x|-|y|\Big| \le |x-y|$.

We now apply the lemma to the problem.

Let $\epsilon > 0$. We wish to find $N\in\mathbb{N}$ such that $\Big||a|-|a_n|\Big| < \epsilon$ for all $n > N$.

Since $\{a_n\}$ converges to a, we can find $N_0\in\mathbb{N}$ such that for all $n \ge N_0$, we have $|a-a_n| < \epsilon$. By our lemma, we have

$$\Big||a|-|a_n|\Big| \le |a-a_n| < \epsilon.$$

Thus, if we take $N = N_0$, then for all $n > N$, we have $\Big||a|-|a_n|\Big| < \epsilon$ for all $\epsilon > 0$. Hence, $\{|a_n|\}_{n\in\mathbb{N}}$ converges to $|a|$.

The converse to the statement is that if the sequence $\{|a_n|\}_{n\in\mathbb{N}}$ converges to $|a|$, then $\{a_n\}_{n\in\mathbb{N}}$ converges to a. A simple counterexample is the sequence $\{a_n\}_{n\in\mathbb{N}}$ where $a_n = (-1)^n$. $\{|a_n|\}_{n\in\mathbb{N}}$ converges to 1, but $\{a_n\}_{n\in\mathbb{N}}$ does not converge.

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4 Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of rational numbers defined as follows:

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = a_n + \frac{1}{3^n} \quad \text{for all} \quad n \ge 1.$$

Show that the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

Solution We will prove by induction that $a_n = \frac{1}{3^0}+\frac{1}{3^1}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{n-1}}$ for $n \ge 1$.

Base step:

$a_1 = 1 = \frac{1}{3^0}$, so the base step holds.

Inductive step:

Suppose $a_n = \frac{1}{3^0}+\frac{1}{3^1}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{n-1}}$. Then we wish to show that $a_{n+1} = \frac{1}{3^0}+\frac{1}{3^1}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{n-1}}+\frac{1}{3^n}$. By the definition of the given sequence,

$$\begin{aligned}a_{n+1} = a_n + \frac{1}{3^n} &= \left(\frac{1}{3^0}+\frac{1}{3^1}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{n-1}}\right)+\frac{1}{3^n}\\ &= \frac{1}{3^0}+\frac{1}{3^1}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{n-1}}+\frac{1}{3^n}\end{aligned}$$

Thus, the inductive step holds.

Taking both steps and invoking the principle of mathematical induction, we can conclude that $\forall n \ge 1$, $a_n = \frac{1}{3^0}+\frac{1}{3^1}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{n-1}}$.

Notice that

$$\begin{aligned}\frac{1}{3}a_n = \frac{1}{3^1}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{n-1}}+\frac{1}{3^n} &= a_n - \frac{1}{3^0}+\frac{1}{3^n}\\ \frac{2}{3}a_n &= 1-\frac{1}{3^n}\\ a_n = \frac{1-\frac{1}{3^n}}{\frac{2}{3}} &= \frac{3-\frac{1}{3^{n-1}}}{2}\end{aligned}$$

We now wish to show that $\{a_n\}_{n\in\mathbb{N}}$ converges to $\frac{3}{2}$. If the limit exists, then for any $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $n \geq N$, $\left|\frac{3-\frac{1}{3^{n-1}}}{2} - \frac{3}{2}\right| < \epsilon$. Let $N = \max\{1 - \log_3 \epsilon, 1\}$.

If $\epsilon < 1$, then for all $n \geq N = 1 - \log_3 \epsilon \implies \frac{1}{3^{n-1}} \leq \epsilon$, we have

$$\left|\frac{3 - \frac{1}{3^{n-1}}}{2} - \frac{3}{2}\right| = \left|\frac{\frac{1}{3^{n-1}}}{2}\right| = \left|\frac{1}{2(3^{n-1})}\right| < \left|\frac{1}{3^{n-1}}\right| \leq \epsilon$$

Thus, by definition, $\{a_n\}_{n\in\mathbb{N}}$ converges to $\frac{3}{2}$.

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5 Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of real numbers such that $\{a_n\}_{n\geq 1}$ is bounded and $\{b_n\}_{n\geq 1}$ converges to 0. Show that the sequence $\{a_n b_n\}_{n\geq 1}$ converges to 0.

Solution Since $\{a_n\}_{n\geq 1}$ is bounded, then there exist real numbers a and A such that $a \leq a_n \leq A \leq |A|$ for all $n \geq 1$. If we multiply the inequality by -1, we get $-A \leq -a_n \leq -a \leq |a|$. If $|A| \geq |a|$ then $a_n \leq |A|$ and $-a_n \leq |A| \implies |a_n| \leq |A|$. Otherwise, we can apply the same argument but with $|A|$ and $|a|$ switched, and end up with $|a_n| \leq |a|$. Thus, $a_n \leq \max\{|A|, |a|\}$. Let $M = \max\{|A|, |a|\}$. Since both $|A|$ and $|a|$ are non-negative, $M = |M|$. Then we have two cases:

$M = 0$:

We have $|a_n| \leq 0 \implies a_n = 0$ for all $n \geq 1$. Thus, $a_n b_n = 0$ for all $n \geq 1$. Then the sequence $\{a_n b_n\}_{n\geq 1}$ obviously converges to 0.

$M \neq 0$:

Let $\epsilon > 0$. We wish to find that $N \in \mathbb{N}$ such that $|a_n b_n - 0| = |a_n b_n| < \epsilon$ for all $n \geq N$. Since $\{b_n\}_{n\geq 1}$ converges to 0, we can find $N_b \in \mathbb{N}$ such that if $n \geq N_b$, then $|b_n - 0| = |b_n| < \frac{\epsilon}{M}$. Thus,

$$|a_n b_n| = |a_n||b_n| \leq |M||b_n| < M\frac{\epsilon}{M} = \epsilon.$$

If we let $N = N_b$, then we have shown that for an arbitrary $\epsilon > 0$, we have $|a_n b_n - 0| < \epsilon$ for all $n \geq N$. Thus, $\{a_n b_n\}_{n\geq 1}$ converges to 0.

In all cases, $\{a_n b_n\}_{n\geq 1}$ converges to 0 as desired.

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6 Let $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$, and $\{c_n\}_{n\geq 1}$ be three convergent sequences of real numbers such that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n \quad \text{and} \quad a_n \leq b_n \leq c_n \text{ for all } n \geq 1.$$

Show that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$.

Solution We will first prove a lemma: $|x| < y \iff -y < x < y$ for some $y > 0$. We will prove the $\implies$ direction first.

Let $|x| < y$. Then there are two cases:

$x \geq 0$:

If $x \geq 0$, then $x = |x|$. Thus, $|x| = x < y$.

$x < 0$:

If $x < 0$, then $|x| = -x$. So, $|x| = -x < y \implies -y < x$.

Combining the inequalities, we have $|x| < y \implies -y < x < y$. We now prove the $\impliedby$ direction.

Let $-y < x < y$. Once again, we break this into two cases.

$x \geq 0$:

Then $|x| = x$. So, $-y < |x| < y$.

$x < 0$:

Then $x < 0 < -x = |x|$. Thus, $-y < x < |x| < y$.

In both cases, $|x| < y$, as desired.

Let $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$. Then for some $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - L| < \epsilon \implies -\epsilon < a_n - L < \epsilon \implies L - \epsilon < a_n < L + \epsilon$$
$$|c_n - L| < \epsilon \implies -\epsilon < c_n - L < \epsilon \implies L - \epsilon < c_n < L + \epsilon$$

Then,

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon$$
$$L - \epsilon < b_n < L + \epsilon$$
$$-\epsilon < b_n - L < \epsilon$$
$$|b_n - L| < \epsilon$$

Thus, given an arbitrary $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $n \geq N$, $|b_n - L| < \epsilon$. Therefore, $\{b_n\}_{n\geq 1}$ converges to L, so by definition, $\lim_{n\to\infty} b_n = L = \lim_{n\to\infty} a_n$.

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7 Prove that

$$\lim_{n\to\infty} \sqrt{4n^2 + n} - 2n = \frac{1}{4}.$$

Solution Let $\epsilon > 0$. We wish to find $N \in \mathbb{N}$ such that whenever $n \geq N$, we have $\left|\sqrt{4n^2 + n} - 2n - \frac{1}{4}\right| < \epsilon$. Let $N = \frac{1}{\epsilon}$. Then if $n \geq N = \frac{1}{\epsilon} \implies \epsilon \geq \frac{1}{n}$, we have

$$\left|\sqrt{4n^2 + n} - 2n - \frac{1}{4}\right| = \left|\frac{4n^2 + n - 4n^2 - n - \frac{1}{16}}{\sqrt{4n^2 + n} + 2n + \frac{1}{4}}\right| = \left|\frac{1}{16\sqrt{4n^2 + n} + 32n + 4}\right|$$
$$\leq \left|\frac{1}{16\sqrt{4n^2}}\right|$$
$$< \left|\frac{1}{n}\right| \leq \epsilon$$

Thus, by definition, $\lim_{n\to\infty} \sqrt{4n^2 + n} - 2n = \frac{1}{4}$.

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8 Let $\{a_n\}_{n\geq 1}$ be a convergent sequence of real numbers.

a. Show that if for all but finitely many a_n we have $a_n \geq a$, then $\lim_{n\to\infty} a_n \geq a$.

b. Show that if for all but finitely many a_n we have $a_n \leq b$, then $\lim_{n\to\infty} a_n \leq b$.

c. Conclude that if all but finitely many a_n belong to the interval $[a, b]$, then $\lim_{n\to\infty} a_n \in [a, b]$.

Solution a. Let A be the set of the values of i such that $a_i < a$. Since A has finitely many elements, we can find its largest value. Let $N_1 = \max A$. Then if $n \geq N_1$, we have $a_n \geq a$. Let $\lim_{n\to\infty} a_n = L$. Since the limit exists, then for any $\epsilon > 0$, we can find $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, we have $|a_n - L| < \epsilon$. Let $N = \max\{N_1, N_2\}$. Suppose $L < a \implies a - L > 0$. Let $\epsilon = a - L$. If $n \geq N$, then by the lemma proved in problem (6), we have

$$|a_n - L| < \epsilon \implies a \leq a_n < L + \epsilon = L + a - L = a$$

We have a contradiction, since a cannot be less than itself. Thus, we must have $L \geq a$.

b. The argument is similar to the above, but with a and b switched, as well as some inequality signs. The difference is the assumption: Suppose $L > b \implies L - b > 0$. Then let $\epsilon = L - b$. If $n \geq N = \max\{N_1, N_2\}$, then by the lemma in problem (6), we have

$$|a_n - L| < \epsilon \implies -\epsilon + L = b - L + L = b < a_n < b$$

Once again, we have a contradiction as b cannot be less than itself. Thus, we must have $L \leq b$.

c. Taking parts (a) and (b) together, we have that if all but finitely many a_n belong to the interval $[a, b] \iff a \leq a_n \leq b$, then $a \leq \lim_{n\to\infty} a_n \leq b \iff \lim_{n\to\infty} a_n \in [a, b]$.

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9 Let $\{a_n\}_{n\geq 1}$ be a convergent sequence of real numbers and let $a \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n > a$. Show that there exists $n_0 \in \mathbb{N}$ such that $a_n > a$ for all $n \geq n_0$.

Solution Let $\lim_{n\to\infty} a_n = L > a$. As the limit exists, then for any $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ whenever $n \geq n_0$. Let $\epsilon = |L - a| = L - a$. Then for all $n \geq n_0$, we have by the lemma proved in problem (6) that

$$|a_n - L| < \epsilon \implies -\epsilon < a_n - L \implies L - \epsilon = L - L + a = a < a_n$$

as desired.