5.2 Prove that homology is an equivalence relation on curves in a region Ω.

Solution Let γ_1, γ_2, and γ_3 be closed curves in Ω, and let $a \notin \bar{\Omega}$.

Reflexivity: $n(\gamma_1 - \gamma_1, a) = n(0, a) = 0$, so $\gamma_1 \sim \gamma_1$.

Symmetry: Let $\gamma_1 \sim \gamma_2$. Then $n(\gamma_1 - \gamma_2, a) = -n(\gamma_2 - \gamma_1, a) = 0$, so $\gamma_2 \sim \gamma_1$.

Transitivity: Let $\gamma_1 \sim \gamma_2$ and $\gamma_2 \sim \gamma_3$. Then

$$n(\gamma_1 - \gamma_2, a) = 0 \quad \text{and} \quad n(\gamma_2 - \gamma_3, a) = 0.$$

Then

$$n(\gamma_1 - \gamma_3, a) = n(\gamma_1 - \gamma_2 + \gamma_2 - \gamma_3, a) = n(\gamma_1 - \gamma_2, a) + n(\gamma_2 - \gamma_3, a) = 0 + 0 = 0$$

so $\gamma_1 \sim \gamma_3$.

Hence, homology is an equivalence relation on curves in Ω.

5.3 The proof of Corollary 2.6(iii) should remind you of the proof that the winding number is an integer. Explain this using the increase in the imaginary part of a continuous determination of $\log(z - a)$ along γ.

Solution Let $\gamma\colon [0, 1] \to \mathbb{C}$ be a closed loop that winds around a counter-clockwise once, and consider the bounded region Ω with $\partial\Omega = \gamma$.

If $a \in \Omega$, then $\frac{1}{z-a}$ is analytic on $\Omega \setminus \{\text{the ray passing through } a \text{ and } \gamma(0)\}$, and its primitive is a branch of $\log(z - a)$. $\log(z - a)$ is analytic (and thus continuous) on the same domain since its derivative exists.

Consider $\log(\gamma(t) - a)$, which is continuous on the same slit domain, since it is the composition of two continuous functions.

The complex part of $\log(\gamma(t) - a)$ gives us the angle that $\gamma(t)$ makes with a, and the real part gives its distance from a. Suppose $\lim_{t\to 0} \log(\gamma(0) - a) = r + i\alpha$. Then as γ is closed, it completes a full loop as $t \to 1$. So,

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z = \frac{1}{2\pi i}\left[\lim_{t\to 1}\log(\gamma(t) - a) - \lim_{t\to 0}\log(\gamma(0) - a)\right] = \frac{1}{2\pi i}\Big[(r + i(\alpha + 2\pi)) - (r + i\alpha)\Big] = 1.$$

We can apply this argument to loops that wind around a multiple times also by splitting into integrals along curves (not necessarily closed) that wind around a once. Summing up these integrals will yield the same result: each loop picks up 1 if it winds counter-clockwise, or it picks up a -1 if it winds clockwise, along with a telescoping sum of distances from a.

5.4 Suppose Ω is a bounded region whose boundary consists of finitely many disjoint piecewise differentiable simple closed curves. Orient $\partial\Omega$ so that the region lies on the left for each boundary component (the inner normal is i times the unit tangent vector). Prove $n(\partial\Omega, a) = 1$ if $a \in \Omega$ and $n(\partial\Omega, a) = 0$ if $a \in \mathbb{C} \setminus \bar{\Omega}$. You may assume that Ω is formed from a simply connected region by removing finitely many pairwise disjoint closures of simply connected subregions.

Solution We can write $\partial\Omega = \sum_{j=1}^{N} \partial\Omega_j$, oriented counter-clockwise, by assumption.

Let $a \in \Omega$.

As $a \in \Omega$, there exists a unique j_0 such that $a \in \Omega_{j_0}$, and if $j \neq j_0$, then $a \notin \Omega_j$. This j_0 is unique as each curve is disjoint. Thus, by Cauchy's integral formula,

$$n(\partial\Omega, a) = \int_{\partial\Omega} \frac{\mathrm{d}\zeta}{\zeta - a} = \sum_{j=1}^{N} \int_{\partial\Omega_j} \frac{\mathrm{d}\zeta}{\zeta - a} = \int_{\partial\Omega_{j_0}} \frac{\mathrm{d}\zeta}{\zeta - a} = 1.$$

Similarly, if $a \notin \bar{\Omega}$, then $a \notin \Omega_j$ for all $1 \leq j \leq N$. Thus,

$$n(\partial\Omega, a) = \int_{\partial\Omega} \frac{\mathrm{d}\zeta}{\zeta - a} = \sum_{j=1}^{N} \int_{\partial\Omega_j} \frac{\mathrm{d}\zeta}{\zeta - a} = 0.$$

5.5 Prove the uniqueness of the Laurent series expansion, that is if

$$\sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n=-\infty}^{\infty} b_n z^n$$

for $r < |z| < R$ then $a_n = b_n$ for all n. Convergence of the series on the region is part of the assumption. Hint: Liouville.

Solution Consider the series

$$0 \equiv f(z) = \sum_{n=-\infty}^{\infty} c_n z^n,$$

where $c_n = a_n - b_n$. Note that f is analytic, so we can write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=-\infty}^{-1} a_n z^n := g(z) + h(z).$$

$f(z) \equiv 0$, so f is entire, which means g and h are entire also.

Note that $\lim_{z\to\infty} f(z) = \lim_{z\to\infty}(g(z) + h(z)) = 0$ and $\lim_{z\to\infty} h(z) = 0$ also by Laurent series. Hence, we must have that $\lim_{z\to\infty} g(z) = 0$.

But this implies that g is bounded. Indeed, for $M > 0$, $|z| \geq M \implies |f(z)| < \epsilon$. Hence, by Liouville's theorem, $f \equiv 0$. By uniqueness of power series, we must have that $a_n = b_n$ for all $n \geq 0$.

Since $f(z) \equiv 0$, $f(1/z) \equiv 0$ also, so $f(1/z)$ is entire. Using the same argument, but for $h(1/z)$ (which can now be written as a power series), we get that $h \equiv 0$ also. By uniqueness of power series, $a_n = b_n$ for all $n \leq -1$.

Thus, $a_n = b_n$ for all n.

5.6 Notice that in the proof of Laurent series expansions we proved that a function f which is analytic on $r < |z| < R$ can be written as $f = f_1 + f_2$ where f_1 is analytic in $|z| < R$ and f_2 is analytic in $|z| > r$ and $f_2(z) \to 0$ as $|z| \to \infty$. Suppose that Ω is a bounded region in $\mathbb{C}$ such that $\partial\Omega$ is a finite union of disjoint (piecewise continuously differentiable) closed curves Γ_j, $j = 1, \ldots, n$. Suppose that f is analytic on $\bar{\Omega}$. Prove that $f = \sum f_j$ where f_j is analytic on the component of $\mathbb{C} \setminus \Gamma_j$ which contains Ω.

Solution Let $D_j \subseteq \mathbb{C} \setminus \Omega$ be such that $\partial D_j = \Gamma_j$, i.e., each D_j is the bounded region in the exterior of Ω whose boundary is Γ_j.

Let Γ_{j_0} be such that Ω is contained within the bounded region whose boundary is Γ_{j_0}. Orient Γ_{j_0} counterclockwise and all the other Γ_j clockwise.

Note that $\sum \Gamma \sim 0$ because of the curves oriented clockwise. Then applying Cauchy's integral formula yields

$$f(z) = \sum_{j=1}^{n} \frac{1}{2\pi i} \oint_{\Gamma_j} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta.$$

Each term in the sum is analytic in z since each Γ_j will never touch $z \in \Omega$, and the product of analytic functions is analytic. Hence, we can write

$$f(z) = \sum_{j=1}^{n} f_j(z).$$

5.7 Prove that if a sequence of analytic polynomials converges uniformly on a region Ω then the sequence converges uniformly on a simply connected region containing Ω. Hint: See Exercise IV.2(b).

Solution Suppose the sequence of polynomials p_n converges uniformly to f on some open set D, with $\Omega \subseteq D$ by assumption. We wish to show that D is simply connected.

Suppose D were not simply connected. Then there exists a closed, bounded component of $\mathbb{C} \setminus D$, which we call K. Notice that K is compact.

Consider a point z_0 on ∂K, which is a point on $\bar{D}$. We define $f(z_0)$ to be the limit of $p_n(z_0)$ as n approaches infinity.

Suppose $f(z_0) = \infty$ for all $z_0 \in \partial K$. Then the set $\{z \in \mathbb{C} \mid f(z) = \infty\}$ admits an accumulation point, which means that $f(z) \equiv \infty$, so p_n converges uniformly everywhere in $\mathbb{C}^*$, which is compact and simply connected.

Then if $f(z_0) < \infty$, then p_n converges uniformly on $D \cup \{z_0\}$. But $z_0 \in K$ also, and since D and K are both connected, $D \cup K$ must be connected also. This is a contradiction, since K is the maximal closed component of $\mathbb{C} \setminus D$. Hence, no such K exists, so D must be simply connected.

5.8 We define the singularity at ∞ of $f(z)$ to be the singularity at 0 of $g(z) = f(1/z)$. Find the singularity at ∞ of the following functions. If the singularity is removable, give the value. If the singularity is a zero or pole, give the order.

a. $\dfrac{z^2+1}{e^z}$

b. $\dfrac{1}{e^{1/z}-1} - z$

c. $e^{z/(1-z)}$

d. $ze^{1/z}$

e. $z^2 - z$

f. $\dfrac{1}{z^3}e^{1/z}$

Solution a. $g(z) = \dfrac{\frac{1}{z^2}+1}{e^{1/z}} = \dfrac{z^2+1}{z^2 e^{1/z}} = \left(1 + \dfrac{1}{z^2}\right)\left(-\displaystyle\sum_{n=-\infty}^{-1} \dfrac{1}{n!} \cdot z^n\right)$. The singularity is essential since $e^{1/z}$ has an essential singularity there will still be infinitely many negative coefficients after multiplication.

b. $g(z) = \dfrac{1}{e^z - 1} - \dfrac{1}{z} = -\dfrac{1}{2} + \dfrac{z}{12} + \cdot$. This has a removable singularity, and $g(z) = -1/2$.

c. $g(z) = \exp \dfrac{1}{z(1-\frac{1}{z})} = \exp \dfrac{1}{z-1}$. $g(0) = 1/e$, so the singularity is removable.

d. $g(z) = \dfrac{e^z}{z} = \dfrac{1}{z} + 1 + \dfrac{z}{2} + \cdots$. This has a pole of degree 1 since $a_n = 0$ for all $n < -1$.

e. $g(z) = \dfrac{1}{z^2} - \dfrac{1}{z}$. This has a pole of degree 2.

f. $g(z) = z^3 e^z = \displaystyle\sum_{n=0}^{\infty} a_n z^{n+3} = \sum_{n=3}^{\infty} a_{n-3} z^n$. This has a zero of degree 3.

5.9 Find the expansion in powers of z for

$$\frac{z}{(z^2+4)(z-3)^2(z-4)}$$

which converges in $3 < |z| < 4$.

Solution By partial fractions, the function is equal to

$$\left(\frac{-\frac{34}{169}}{z-3} + \frac{\frac{1}{1690} - i\frac{29}{3380}}{z-2i} + \frac{\frac{1}{1690} + i\frac{29}{3380}}{z+2i} + \frac{-\frac{3}{13}}{(z-3)^2}\right) + \frac{\frac{1}{5}}{z-4}$$

The term in the parentheses is analytic on $|z| > 3$, and its limit as z tends to infinity is 0. So, we can find a power series expansion for that region. Additionally, the last term is analytic on $|z| < 4$, so we can find a power series expansion for that also.

By Laurent series, we can find an expansion in powers of z which converges in $3 < |z| < 4$. The calculation is messy, but trivial since we just need to take contours around each pole, so it will not be typed out.