2.4.8 For what values of z is

$$\sum_{n=0}^{\infty}\left(\frac{z}{1+z}\right)^n$$

convergent?

Solution This is a geometric series, so it converges if and only if

$$\left|\frac{z}{1+z}\right| < 1 \implies |z|^2 < 1 + z + \overline{z} + |z|^2 \implies \operatorname{Re} z = \frac{z+\overline{z}}{2} > -\frac{1}{2}.$$

2.4.9 Same question for

$$\sum_{n=0}^{\infty}\frac{z^n}{1+z^{2n}}.$$

Solution Notice that if $|z| = 1$, then the series does not converge because the terms oscillate and do not converge to 0.

If $z = 0$, then the terms all vanish, so also get convergence. Assume from now on that $0 < |z| \neq 1$.

By partial fractions, we see

$$\frac{z^n}{1+z^{2n}} = \frac{1}{z^n+i} + \frac{1}{z^n-i}.$$

Notice that if we treat n as a continuous variable, using the fact that $\exp(\cdot)$ is continuous, and applying L'Hôpital's, we get

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{|z^n+a|^{1/n}} &= \exp\left(\lim_{n\to\infty} -\frac{\log|z^n+a|}{n}\right) \\
&= \exp\left(\lim_{n\to\infty} -\frac{nz^{n-1}}{z^n+a}\right) \\
&= \exp\left(\lim_{n\to\infty} -\frac{z^{n-1}+n(n-1)z^{n-2}}{nz^{n-1}}\right) \\
&= \exp\left(\lim_{n\to\infty} -\left(\frac{1}{n}+\frac{n-1}{z}\right)\right) \\
&= \exp(-\infty) \\
&= 0 < 1.
\end{aligned}$$

Thus, by the ratio test, both

$$\sum_{n=0}^{\infty}\frac{1}{z^n+i} \quad \text{and} \quad \sum_{n=0}^{\infty}\frac{1}{z^n-i}$$

converge, so our original series converges if $|z| \neq 1$.

3.2.2 The hyperbolic cosine and sine are defined by $\cosh z = (e^z + e^{-z})/2$, $\sinh z = (e^z - e^{-z})/2$. Express them through $\cos iz, \sin iz$. Derive the addition formulas, and formulas for $\cosh 2z, \sinh 2z$.

Solution Notice that

$$\sin iz = \sum_{n=0}^{\infty} \frac{(-1)^n (iz)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n i z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} i \frac{z^{2n+1}}{(2n+1)!}$$

and

$$\cos iz = \sum_{n=0}^{\infty} \frac{(-1)^n (iz)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.$$

Thus, we get

$$\sin iz + i \cos iz = i \sum_{n=0}^{\infty} \frac{z^n}{n!} = ie^z \implies e^z = \cos iz - i \sin iz.$$

Then

$$\begin{aligned} \cosh z &= \frac{e^z + e^{-z}}{2} = \frac{1}{2}(\cos iz - i \sin iz + \cos iz + i \sin iz) = \quad \cos iz \\ \sinh z &= \frac{e^z - e^{-z}}{2} = \frac{1}{2}(\cos iz - i \sin iz - \cos iz - i \sin iz) = -i \sin iz. \end{aligned}$$

The addition formulas are thus

$$\begin{aligned} \cosh(z+w) &= \quad \cos(iz+iw) = \cos(iz)\cos(iw) - \sin(iz)\sin(iw) &&= \cosh(z)\cosh(w) + \sinh(z)\sinh(w) \\ \sinh(z+w) &= \quad -i\sin(iz+iw) = -i[\sin(iz)\cos(iw) + \sin(iw)\cos(iz)] &&= \sinh(z)\cosh(w) + \cosh(z)\sinh(z). \end{aligned}$$

Finally, the double angle identities are

$$\begin{aligned} \cosh(2z) &= \cosh^2(z) + \sinh^2(z) \\ \sinh(2z) &= 2\sinh(z)\cosh(z). \end{aligned}$$

3.2.4 Show that

$$|\cos z|^2 = \sinh^2 y + \cos^2 x = \cosh^2 y - \sin^2 x = \frac{1}{2}(\cosh 2y + \cos 2x)$$

and

$$|\sin z|^2 = \sinh^2 y + \sin^2 x = \cosh^2 y - \cos^2 x = \frac{1}{2}(\cosh 2y - \cos 2x).$$

Solution First notice the following properties:

$$\begin{gathered} \cosh^2 z - \sinh^2 z = \cos^2 iz + \sin^2 iz = 1 \\ \cosh 2z = \cosh^2 z + \sinh^2 z = 2\cosh^2 z - 1 = 2\sinh^2 z + 1 \\ \cos(x+iy) = \cos(z)\cos(iy) - \sin(x)\sin(iy) = \cos(z)\cosh(y) + i\sin(x)\sinh(y) \\ \sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y) \end{gathered}$$

Thus,

$$\begin{aligned}
|\cos z|^2 &= \cos^2(x)\cosh^2(y) + \sin^2(x)\sinh^2(y) \\
&= \cos^2(x)\cosh^2(y) + \left[1 - \cos^2(x)\right]\sinh^2(y) \\
&= \sinh^2(y) + \cos^2(x)\left[\cosh^2(y) - \sinh^2(y)\right] \\
&\boxed{= \sinh^2(y) + \cos^2(x)} \\
&= \left[\cosh^2(y) - 1\right] + \left[1 - \sin^2(x)\right] \\
&\boxed{= \cosh^2(y) - \sin^2(x)} \\
&= \left[\frac{\cosh(2y)}{2} + \frac{1}{2}\right] - \left[\frac{1}{2} - \frac{\cos(2x)}{2}\right] \\
&\boxed{= \frac{1}{2}(\cosh(2y) + \cos(2x)).}
\end{aligned}$$

Finally,

$$\begin{aligned}
|\sin z|^2 &= \sin^2(x)\cosh^2(y) + \cos^2(x)\sinh^2(y) \\
&= \sin^2(x)\cosh^2(y) + \left[1 - \sin^2(x)\right]\sinh^2(y) \\
&= \sinh^2(y) + \sin^2(x)\left[\cosh^2(y) - \sinh^2(y)\right] \\
&\boxed{= \sinh^2(y) + \sin^2(x)} \\
&= \left[\cosh^2(y) - 1\right] + \left[1 - \cos^2(x)\right] \\
&\boxed{= \cosh^2(y) - \cos^2(x)} \\
&= \left[\frac{\cosh(2y)}{2} + \frac{1}{2}\right] - \left[\frac{1}{2} + \frac{\cos(2x)}{2}\right] \\
&\boxed{= \frac{1}{2}(\cosh(2y) - \cos(2x)).}
\end{aligned}$$

3.1.2 If

$$T_1 z = \frac{z+2}{z+3}, \qquad T_2 z = \frac{z}{z+1},$$

find T_1T_2z, T_2T_1z, and $T_1^{-1}T_2z$.

Solution We will use the fact that $\mathrm{PGL}(2,\mathbb{C}) \simeq \mathcal{M}$, where $\mathcal{M}$ is the group of Möbious transformations with the operation "$\circ$", i.e., function composition. In particular, the map

$$\mathrm{PGL}(2,\mathbb{C}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{\varphi}{\longmapsto} \frac{az+b}{cz+d} \in \mathcal{M}$$

is an isomorphism.

$$\begin{aligned}
T_1T_2z &= \varphi\left[\varphi^{-1}(T_1)\varphi^{-1}(T_2)\right]z = \varphi\left[\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right]z = \varphi\left[\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}\right]z = \frac{3z+2}{4z+3} \\
T_2T_1z &= \varphi\left[\varphi^{-1}(T_2)\varphi^{-1}(T_1)\right]z = \varphi\left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\right]z = \varphi\left[\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}\right]z = \frac{z+2}{2z+5} \\
T_1^{-1}T_2z &= \varphi\left[\varphi^{-1}(T_1^{-1})\varphi^{-1}(T_2)\right]z = \varphi\left[\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right]z = \varphi\left[\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\right]z = z - 2.
\end{aligned}$$

3.1.3 Prove that the most general transformation which leaves the origin fixed and preserves all distances is either a rotation or a rotation followed by reflexion in the real axis.

Solution Let $f\colon \mathbb{C} \to \mathbb{C}$ be a transformation that describes the above.

Notice that $\|x\| = \|x - 0\| = \|f(x) - f(0)\| = \|f(x)\|$, for any $x \in \mathbb{R}^2$.

We will first show that f preserves angles. To do so, we will treat f as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. By the law of cosines, we know that the angle between $x, y \in \mathbb{R}^2$ is given by

$$\cos\theta = \frac{\|x\|^2 + \|y\|^2 - \|x-y\|^2}{2\|x\|\|y\|}.$$

To prove that f is angle-preserving, we need to show that the angle between $f(x)$ and $f(y)$ is also θ. Let α be the angle between $f(x)$ and $f(y)$. Then by the law of cosines and the fact that f is an isometry,

$$\cos\alpha = \frac{\|f(x)\|^2 + \|f(y)\|^2 - \|f(x) - f(y)\|^2}{2\|f(x)\|\|f(y)\|} = \frac{\|x\|^2 + \|y\|^2 - \|x-y\|^2}{2\|x\|\|y\|} = \cos\theta.$$

Since α and θ are taken to be in the interval $[0, \pi]$, we have $\alpha = \theta$, so f preserves angles.

We will now treat f as a complex function again.

Notice that $|f(1)| = 1 \implies f(1) = e^{i\theta}$, for some θ. Since f is preserves angle but not necessarily orientation, we have two cases:

$$\arg\left(\frac{f(z)}{f(w)}\right) = \arg\left(\frac{z}{w}\right) \quad \text{or} \quad \arg\left(\frac{f(z)}{f(w)}\right) = \arg\left(\frac{w}{z}\right).$$

Case 1:

Take $w = 1$ and we have

$$\frac{f(z)}{f(1)} = \frac{z}{1} \implies f(z) = e^{i\theta}z,$$

since f preserves distances.

Thus, in this case, f is a rotation.

Case 2:

Again, we take $y = 1$ to see

$$\frac{f(z)}{f(1)} = \frac{c}{z},$$

for some $c \in \mathbb{R}$. Since f preserves distances,

$$|z| = |f(z)| = \frac{|c|}{|z|} \implies c = |z|^2 \implies c = e^{i\alpha}z\overline{z},$$

for some $\alpha \in \mathbb{R}$.

We can ignore the case where $c < 0$ by adding π to the argument. Thus,

$$f(z) = f(1)\frac{c}{z} = e^{i(\theta+\alpha)}\overline{z},$$

which is a reflection across the real axis followed by a rotation, as desired.

3.1.4 Show that any linear transformation which transforms the real axis into itself can be written with real coefficients.

Solution Since Möbius tranformations are bijective (their inverse exists), if we show that such a transformation can be written with real coefficients, we will have shown that it maps $\mathbb{R}$ to $\mathbb{R}$. Indeed, its inverse will also have real coefficients, so its image will be a real number.

Every fractional linear transformation can be written as compositions of matrices of the form

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

i.e., translations, rotations and scalings, and inversions. Thus, it suffices to check compositions of these functions.

Inversions already have real entries, so it suffices to check compositions of translations and rotation-scalings, so there are two to look at.

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k & \alpha \\ 0 & 1 \end{pmatrix},$$

which corresponds to the Möbius transformation

$$kz + \alpha.$$

If $z = x \in \mathbb{R}$, then we check the imaginary part of its image:

$$\operatorname{Im}(kx + \alpha) = \frac{kx + \alpha - \overline{k}x - \overline{\alpha}}{2} = \frac{x(k - \overline{k}) + (\alpha - \overline{\alpha})}{2} = \frac{x \operatorname{Im} k + \operatorname{Im} \alpha}{2}.$$

If k or α have imaginary parts, then the image of x is not real. Thus, in this case, k and α must be real.

Next, we look at

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k & k\alpha \\ 0 & 1 \end{pmatrix},$$

and this corresponds to the transformation

$$kz + k\alpha.$$

If $z = x \in \mathbb{R}$, then we again look at the imaginary part:

$$\operatorname{Im}(kx + k\alpha) = \frac{kx + k\alpha - \overline{k}x - \overline{k\alpha}}{2} = \frac{x(k - \overline{k}) + (k\alpha - \overline{k\alpha})}{2} = \frac{x \operatorname{Im} k + \operatorname{Im} k\alpha}{2}.$$

Once again, the image of x is not always real unless k and $k\alpha$ have zero imaginary part. Thus, we must have that k and $k\alpha$ are real.

Since all Möbius transformations are a composition of these matrices and the fact that $\mathbb{R}$ is closed under addition and multiplication (i.e., matrix multiplication with real entries yields matrices with real entries), the Möbius transformations described in the problem can be written with real coefficients.

3.2.2 Express the cross ratios corresponding to the 24 permutations of four points in term of $\lambda = (z_1, z_2, z_3, z_4)$.

Solution For simplicity, we will only consider the case where none of the points are infinity. In those cases, the calculations are the same, but simpler.

Let T be the transformation which sends (z_2, z_3, z_4) to $(1, 0, \infty)$.

Recall that the cross ratio is given by

$$Tz_1 = \frac{\left(\dfrac{z_1 - z_3}{z_1 - z_4}\right)}{\left(\dfrac{z_2 - z_3}{z_2 - z_4}\right)} = \frac{|z_1 - z_3|}{|z_1 - z_4|} \cdot \frac{|z_2 - z_4|}{|z_2 - z_3|}.$$

Notice that it's easy to see that the following permutations do not change the cross ratio:

switching nothing		
switching 1 and 2	and	switching 3 and 4
switching 1 and 3	and	switching 2 and 4
switching 1 and 4	and	switching 2 and 3

We next consider the Möbius transformations that do the following:

$$\begin{pmatrix}1\\0\\\infty\end{pmatrix} \mapsto \begin{pmatrix}1\\0\\\infty\end{pmatrix}, \; S_{234} = z \qquad \begin{pmatrix}1\\\infty\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\0\\\infty\end{pmatrix}, \; S_{243} = \frac{1}{z}$$

$$\begin{pmatrix}0\\1\\\infty\end{pmatrix} \mapsto \begin{pmatrix}1\\0\\\infty\end{pmatrix}, \; S_{324} = 1 - z \qquad \begin{pmatrix}0\\\infty\\1\end{pmatrix} \mapsto \begin{pmatrix}1\\0\\\infty\end{pmatrix}, \; S_{342} = \frac{1}{1-z}$$

$$\begin{pmatrix}\infty\\1\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\0\\\infty\end{pmatrix}, \; S_{423} = \frac{z-1}{z} \qquad \begin{pmatrix}\infty\\0\\1\end{pmatrix} \mapsto \begin{pmatrix}1\\0\\\infty\end{pmatrix}, \; S_{432} = \frac{z}{z-1}.$$

The subscript indicates a permutation of the images of z_2, z_3, and z_4 under T and does the following:

$$\begin{pmatrix}z_i\\z_j\\z_k\end{pmatrix} \xmapsto{T} \begin{pmatrix}Tz_i\\Tz_2\\Tz_3\end{pmatrix} \xmapsto{S_{ijk}} \begin{pmatrix}1\\0\\\infty\end{pmatrix} \implies S_{ijk}T\begin{pmatrix}z_i\\z_j\\z_k\end{pmatrix} = \begin{pmatrix}1\\0\\\infty\end{pmatrix}.$$

Indeed, by construction, S_{ijk} sends Tz_i to $Tz_2 = 1$, etc., so by the definition of the cross ratio,

$$(z_1, z_i, z_j, z_k) = S_{ijk}Tz_1 = S_{ijk}\lambda.$$

These allow us to calculate many of the permutations quickly:

$$\begin{aligned}
(z_1, z_2, z_3, z_4) &= (z_2, z_1, z_4, z_3) = (z_3, z_4, z_1, z_2) = (z_4, z_3, z_2, z_1) = S_{234}\lambda = \lambda\\
(z_1, z_2, z_4, z_3) &= (z_2, z_1, z_3, z_4) = (z_4, z_3, z_1, z_2) = (z_3, z_4, z_2, z_1) = S_{243}\lambda = \frac{1}{\lambda}\\
(z_1, z_3, z_2, z_4) &= (z_3, z_1, z_4, z_2) = (z_2, z_4, z_1, z_3) = (z_4, z_2, z_3, z_1) = S_{324}\lambda = 1 - \lambda\\
(z_1, z_3, z_4, z_2) &= (z_3, z_1, z_2, z_4) = (z_4, z_2, z_1, z_3) = (z_2, z_4, z_3, z_1) = S_{342}\lambda = \frac{1}{1-\lambda}\\
(z_1, z_4, z_2, z_3) &= (z_4, z_1, z_3, z_2) = (z_2, z_3, z_1, z_4) = (z_3, z_2, z_4, z_1) = S_{423}\lambda = \frac{\lambda - 1}{\lambda}\\
(z_1, z_4, z_3, z_2) &= (z_4, z_1, z_2, z_3) = (z_3, z_2, z_4, z_1) = (z_2, z_3, z_4, z_1) = S_{432}\lambda = \frac{\lambda}{\lambda - 1}
\end{aligned}$$

The first column corresponds to S_{ijk}, and the other columns are permutations of (z_1, z_i, z_j, z_k) which do nothing to the cross ratio.

3.2.3 If the consecutive vertices z_1, z_2, z_3, z_4 of a quadrilateral lie on a circle, prove that

$$|z_1 - z_3| \cdot |z_2 - z_4| = |z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4|$$

and interpret the result geometrically.

Solution Recall that

$$(z_1, z_2, z_3, z_4) = \frac{|z_1 - z_3|}{|z_1 - z_4|} \cdot \frac{|z_2 - z_4|}{|z_2 - z_3|}.$$

First notice that

$$\begin{aligned}
\frac{|z_1 - z_2|}{|z_1 - z_4|} &= \frac{|z_1 - z_2|}{|z_1 - z_4|} \cdot \frac{|z_3 - z_4|}{|z_3 - z_4|} \\
&= \frac{|z_1 - z_2|}{|z_1 - z_4|} \cdot \frac{|z_3 - z_4|}{|z_3 - z_2|} \cdot \frac{|z_3 - z_2|}{|z_3 - z_4|} \\
&= \frac{|z_3 - z_2|}{|z_3 - z_4|}(z_1, z_3, z_2, z_4)
\end{aligned}$$

By the previous problem, we get that this is equal to

$$\frac{|z_3 - z_2|}{|z_3 - z_4|}(1 - \lambda).$$

where $\lambda = (z_1, z_2, z_3, z_4)$.

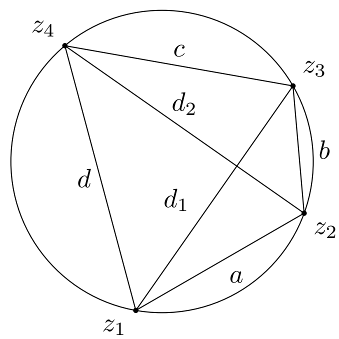
Then, multiplying both sides by $|z_1 - z_4| \cdot |z_3 - z_4|$, we get

$$\begin{aligned}
|z_1 - z_2| \cdot |z_3 - z_4| &= |z_3 - z_2| \cdot |z_1 - z_4| \cdot (1 - \lambda) \\
|z_1 - z_3| \cdot |z_2 - z_4| + |z_1 - z_2| \cdot |z_3 - z_4| &= |z_3 - z_2| \cdot |z_1 - z_4|.
\end{aligned}$$

Our argument did not care about the order of z_1, z_2, z_3, z_4, so we will switch z_1 and z_2 to get

$$|z_1 - z_3| \cdot |z_2 - z_4| = |z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4|,$$

as desired.



Geometrically, the equation tells us that the product of the length of the diagonals is equal to the sum of the products of the lengths of opposite sides of the quadrilateral.

In other words, if the sides of the quadrilateral are a, b, c, d and the diagonals are d_1, d_2, then the equation says

$$d_1 d_2 = ac + bd.$$

3.2.4 Show that any four distinct points can be carried by a linear transformation to positions $1, -1, k, -k$, where the value of k depends on the points. How many solutions are there, and how are they related?

Solution Let z_1, z_2, z_3, z_4 be four distinct points.

Let S and T be linear transformations that do the following:

$$\begin{pmatrix} z_2 \\ z_3 \\ z_4 \end{pmatrix} \xmapsto{S} \begin{pmatrix} 1 \\ 0 \\ \infty \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ k \\ -k \end{pmatrix} \xmapsto{T} \begin{pmatrix} 1 \\ 0 \\ \infty \end{pmatrix}.$$

Then we get

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \xmapsto{S} \begin{pmatrix} Sz_1 \\ 1 \\ 0 \\ \infty \end{pmatrix} \xmapsto{T^{-1}} \begin{pmatrix} T^{-1}Sz_1 \\ -1 \\ k \\ -k \end{pmatrix}.$$

So $T^{-1}S$ almost sends the points to $1, -1, k, -k$. Moreover, since Möbius transformations taking points to $1, 0, \infty$ are unique, this is the only possible linear transformation that does we what we need.

If we have $T^{-1}Sz_1 = 1 \implies Sz_1 = T1$, then we're done. In other words, we need to see under what conditions is the cross ratio conserved.

Notice that

$$T1 = (1, -1, k, -k) = \frac{1-k}{1+k} \cdot \frac{-1+k}{-1-k} = \left(\frac{1-k}{1+k}\right)^2$$

and that

$$Sz_1 = (z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3} := \lambda.$$

A linear transformation that carries z_1, z_2, z_3, z_4 to $1, -1, k, -k$ must preserve the cross ratio. This allows us to solve for the two possible values of k:

$$\left(\frac{1-k}{1+k}\right)^2 = \lambda \implies k = \frac{1 \mp \sqrt{\lambda}}{1 \pm \sqrt{\lambda}}$$

$$k = \frac{1 - \sqrt{\dfrac{z_1 - z_3}{z_1 - z_4} \cdot \dfrac{z_2 - z_4}{z_2 - z_3}}}{1 + \sqrt{\dfrac{z_1 - z_3}{z_1 - z_4} \cdot \dfrac{z_2 - z_4}{z_2 - z_3}}} \quad \text{or} \quad k = \frac{1 + \sqrt{\dfrac{z_1 - z_3}{z_1 - z_4} \cdot \dfrac{z_2 - z_4}{z_2 - z_3}}}{1 - \sqrt{\dfrac{z_1 - z_3}{z_1 - z_4} \cdot \dfrac{z_2 - z_4}{z_2 - z_3}}}.$$

These values of k are reciprocals of each other, and when k is either of these values, $T^{-1}Sz_1 = 1$, which means that

$$T^{-1}S \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ k \\ -k \end{pmatrix},$$

as desired.