1 Let (X, d) be a metric space. Prove that if a sequence $\{x_n\}_{n\geq 1} \subseteq X$ converges in X, then its limit is unique.

Solution Suppose $\{x_n\}_{n\geq 1}$ converges to both a and b in X. Then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d(a, b) \leq d(x_n, a) + d(x_n, b) < \epsilon.$$

As this holds for all ϵ, it follows that $a = b$. If this were not true, then taking $\epsilon = d(a, b) > 0$ gives a contradiction.

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2 Let (X, d) be a metric space. Prove that a sequence $\{x_n\}_{n\geq 1} \subseteq X$ converges to some $x \in X$ if and only if every subsequence of $\{x_n\}_{n\geq 1}$ converges to x.

Solution "$\Longrightarrow$"

Let $\{x_{k_n}\}_{n\geq 1}$ be a subsequence of $\{x_n\}_{n\geq 1}$. Then as $\{x_n\}_{n\geq 1}$ converges to x, then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d(x_n, x) < \epsilon$$

By definition, $k_n \geq n \geq N$, so

$$d(x_{k_n}, x) < \epsilon$$

Thus, by definition, any subsequence of $\{x_n\}_{n\geq 1}$ will converge to x.

"$\Longleftarrow$"

$\{x_n\}_{n\geq 1}$ is a subsequence of itself with $k_n = n$ for all $n \geq 1$. Thus, if all subsequences converge, $\{x_n\}_{n\geq 1}$ converges.

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3 Let (X, d) be a metric space and let $\{x_n\}_{n\geq 1} \subseteq X$ be a convergent sequence. Prove that $\{x_n\}_{n\geq 1}$ is bounded, that is, there exist $a \in X$ and $r > 0$ such that $\{x_n\}_{n\geq 1} \subseteq B_r(a)$.

Solution Let $\lim_{n\to\infty} x_n = x$. Then by definition, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$d(x_n, x) < \epsilon$$

Take $\epsilon = 1$. Then $d(x_n, x) < 1$ for all $n \geq N$. Let $a = x$ and $r > \max\{d(x_1, x), d(x_2, x), \ldots, d(x_N, x), 1\}$. Then $\{x_n\}_{n\geq 1} \subseteq B_r(a)$ because $d(x_n, x) < r$ for all n by construction.

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4 Let (X, d) be a metric space and let $A \subseteq X$ be complete. Show that A is closed.

Solution Let $a \in \bar{A}$. By definition, for all $r > 0$, $B_r(a) \cap A \neq \emptyset$. Take $r_1 = 1$. Then let $x_1 \in B_{r_1}(a) \cap A$.

Let $r_2 < \frac{1}{2}$. Then as $a \in \bar{A}$, $B_{r_2}(a) \cap A \neq \emptyset$. Let $x_2 \in B_{r_2}(a) \cap A$.

We proceed inductively.

Suppose we have $x_1, \ldots, x_n$ as defined above. Then let $r_{n+1} < \frac{1}{n+1}$. Since $a \in \bar{A}$, then $B_{r_{n+1}}(a) \cap A \neq \emptyset$. Then define $x_{n+1} \in B_{r_{n+1}}(a) \cap A$.

Thus, we have a sequence $\{x_n\}_{n\geq 1}$ such that $d(x_n, a) < \frac{1}{n}$ and $x_n \in A$ for all $n \geq 1$.

$\{x_n\}_{n\geq 1}$ clearly converges to a, since by the Archimedean principle, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Thus, for all $n \geq N$, $d(x_n, a) < \frac{1}{n} \leq \frac{1}{N} < \epsilon$. Since A is complete, $\{x_n\}_{n\geq 1}$ converges in A, so $a \in A$. Thus, $A = \bar{A}$.

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5 Let (X, d) be a complete metric space and let $F \subseteq X$ be a closed set. Show that F is complete.

Solution Let $\{x_n\}_{n\geq 1} \subseteq F$. As X is complete, $\{x_n\}_{n\geq 1}$ converges to some $x \in X$. We will show that $x \in F$. Fix $\epsilon > 0$. Then by definition, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$d(x_n, x) < \epsilon$$

In particular, $B_\epsilon(x) \cap F \neq \emptyset$ for all $\epsilon > 0$. Then by definition, $x \in \bar{F} = F$. Hence, F is complete.

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6 Let

$$l^\infty = \{\{x_n\}_{n\geq 1} \subseteq \mathbb{R} \mid \sup_{n\geq 1} |x_n| < \infty\}.$$

Define $d_\infty \colon l^\infty \times l^\infty \to \mathbb{R}$ as follows: for any $x = \{x_n\}_{n\geq 1} \in l^\infty, y = \{y_n\}_{n\geq 1} \in l^\infty$,

$$d_\infty(x, y) = \sup_{n\geq 1} |x_n - y_n|.$$

Show that (l^∞, d_∞) is a complete metric space.

Solution We first show that (l^∞, d_∞) is a metric space. The first three axioms trivially hold. So, we only need to show that the triangle inequality holds. By Minkowski,

$$d_\infty(x, y) = \sup_{n\geq 1} |x_n - y_n - z_n + z_n| \leq \sup_{n\geq 1} |x_n - z_n| + \sup_{n\geq 1} |y_n - z_n| = d_\infty(x_n, z_n) + d_\infty(z_n, y_n)$$

Thus, by definition, (l^∞, d_∞) is a metric space.

Let $x^{(k)} = \{x_n^{(k)}\}_{k\geq 1} \subseteq l^\infty$ be Cauchy. We wish to show that it converges, and that its limit, denoted by $x^{(k)}$, exists in l^∞. By definition, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $k, l \geq N$, we have

$$d_\infty(x^{(k)}, x^{(l)}) = \sup_{n\geq 1} |x_n^{(k)} - x_n^{(l)}| < \epsilon$$

$$\left| \sup_{n\geq 1} |x_n^{(k)}| - \sup_{n\geq 1} |x_n^{(l)}| \right| < \epsilon$$

Thus, $\{\sup_{n\geq 1} |x_n^{(k)}|\}_{k\geq 1}$ is Cauchy, and it converges in $(\mathbb{R}, |\cdot|)$, which is complete. Then $\lim_{k\to\infty} \sup_{n\geq 1} |x_n^{(k)}| < \infty$. Since the limit of the supremum is finite, then all the terms of $\lim_{k\to\infty} |x_n^{(k)}|$ are finite, so the sequence $x = \{x_n\}_{n\geq 1} = \lim_{k\to\infty} \{x_n^{(k)}\}_{n\geq 1}$ exists in l^∞.

Hence, l^∞ is a complete metric space.

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7 Let $\mathbb{R}^n$ be endowed with the Euclidean metric d_2. Let S be a non-empty subset of $\mathbb{R}^n$; in particular, $(S, d_2 |_{S\times S})$ is a metric space.

a. Given $x \in S$, is the set $\{y \in S \mid d_2(x, y) \geq r\}$ closed in S?

b. Given $x \in S$, is the set $\{y \in S \mid d_2(x, y) \geq r\}$ contained in the closure of $\{y \in S \mid d_2(x, y) > r\}$ in S?

Solution

a. Yes. Consider the set $\{y \in X \mid d_2(x, y) \geq r\}$. Its complement is the ball $B_r(x)$, which is open. So, by definition, $\{y \in X \mid d_2(x, y) \geq r\}$ is closed. Then $\{y \in X \mid d_2(x, y) \geq r\} \cap S = \{y \in S \mid d_2(x, y) \geq r\}$, so by definition, $\{y \in X \mid d_2(x, y) \geq r\}$ is closed in S.

b. No, not in general. Consider $\mathbb{R}$, $r = 1$, $x = 0$, and $S = (-2, 0] \cup \{1\}$. Then

$$\{y \in S \mid d_2(x, y) \geq r\} = (-2, -1] \cup \{1\}$$
$$\overline{\{y \in S \mid d_2(x, y) > r\}} = \overline{(-2, -1)} = [-2, -1]$$

1 is clearly not in the second set, so $\{y \in S \mid d_2(x, y) \geq r\}$ is not contained within the closure of $\{y \in S \mid d_2(x, y) > r\}$.

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8 Let (X, d) be a complete metric space and let $\{F_n\}_{n\geq 1}$ be a sequence of non-empty closed subsets of X such that $F_{n+1} \subseteq F_n$ for all $n \geq 1$ and $\delta(F_n) \to 0$. Show that there exists $x \in X$ such that $\bigcap_{n\geq 1} F_n = \{x\}$.

Solution As $\delta(F_n) \to 0$, then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\delta(F_n) < \epsilon.$$

Let $r_1 \in \mathbb{N}$ be such that $\delta(F_{r_1}) < 1$. Then define $x_1 \in F_{r_1}$.

Let $r_1 < r_2 \in \mathbb{N}$ be such that $\delta(F_{r_2}) < \min\{1/2, \delta(F_{r_1})\}$. Let $x_2 \in F_{r_2}$. Note that $F_{r_2} \subseteq F_{r_1}$, so $d(x_1, x_2) < 1$.

We proceed inductively.

Suppose we have $x_1, \ldots, x_n$ and $r_1 < r_2 < \ldots < r_n$ such that $\delta(F_{r_n}) < \min\{\frac{1}{n}, \delta(F_{r_{n-1}})\}$. Let $r_n < r_{n+1} \in \mathbb{N}$ be such that $\delta(F_{r_{n+1}}) < \min\{\frac{1}{n+1}, \delta(F_{r_n})\}$. Define $x_{n+1} \in F_{r_{n+1}}$.

Thus, we have defined $\{x_n\}_{n\geq 1}$. By construction, if $x_m, x_n \in F_{r_i}$, where $i = \min\{n, m\}$, we have that $d(x_n, x_m) < \max\{\frac{1}{n}, \frac{1}{m}\}$. By the Archimedean principle, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Thus, for all $n, m \geq N$ we have

$$d(x_n, x_m) < \max\left\{\frac{1}{n}, \frac{1}{m}\right\} \leq \frac{1}{N} < \epsilon$$

so by definition, $\{x_n\}_{n\geq 1}$ is Cauchy, and since (X, d) is complete, the sequence converges to some $x \in X$.

Since X is complete and F_n is closed for all $n \geq 1$, then by exercise 5, F_n is complete. Thus, for all $m \geq 1$, there exists $i \in \mathbb{N}$ such that $r_i \geq m$, so we have that $\{x_n\}_{n\geq i} \subseteq F_{r_i} \implies x \in F_{r_i} \subseteq F_m$. It follows that $x \in \bigcap_{n\geq 1} F_n$.