4.1.10 In an experiment, a 5-kg mass is suspended from a spring. The displacement of the spring-mass equilibrium from the spring equilibrium is measured to be 75 cm. If the mass is then displaced 36 cm downward from its spring-mass equilibrium and then given a sharp downward tap, imparting an instantaneous downward velocity of 0.45 m/s. Set up (but do not solve) the initial value problem that models this experiment. Assume no damping is present.

Solution The spring constant k of the spring is $k = \frac{mg}{x} = \frac{(5\text{ kg})\left(9.8\text{ m/s}^2\right)}{0.75\text{ m}} = 65.\bar{3}$ N/m. By Newton's Second Law, we have the differential equation

$$mx'' = -kx$$
$$x'' + \frac{k}{m}x = 0$$
$$x'' + \frac{196}{15}x = 0$$

where x is the displacement from the spring-mass equilibrium with the upwards direction positive. We also have the initial conditions $x(0) = 36\text{ cm}, x'(0) = -0.45\text{ m/s}$. Thus, our initial value problem is

$$\begin{cases} x'' + \frac{196}{15}x = 0 \\ x(0) = 36\text{ cm} \\ x'(0) = -0.45\text{ m/s} \end{cases}$$

4.1.24 Show that $y_1(t) = e^{-t}\cos 2t$ and $y_2(t) = e^{-t}\sin 2t$ form a fundamental set of solutions for $y'' + 2y' + 5y = 0$, then find a solution satisfying $y(0) = -1$ and $y'(0) = 0$.

Solution

$$\begin{aligned}
&y_1'' + 2y_1' + 5y_1 \\
&= 4e^{-t}\sin 2t - 3e^{-t}\cos 2t - 4e^{-t}\sin 2t - 2e^{-t}\cos 2t + 5e^{-t}\cos 2t \\
&= \left(4e^{-t}\sin 2t - 4e^{-t}\sin 2t\right) + \left(-5e^{-t}\cos 2t + 5e^{-t}\cos 2t\right) \\
&= 0 \Rightarrow y_1 \text{ is a solution to the differential equation.} \\
&y_2'' + 2y_2' + 5y_2 \\
&= -3e^{-t}\sin 2t - 4e^{-t}\cos 2t + 4e^{-t}\cos 2t - 2e^{-t}\sin 2t + 5e^{-t}\sin 2t \\
&= \left(-5e^{-t}\sin 2t + 5e^{-t}\sin 2t\right) + \left(-4e^{-t}\cos 2t + 4e^{-t}\cos 2t\right) \\
&= 0 \Rightarrow y_2 \text{ is a solution to the differential equation.}
\end{aligned}$$

Notice that $y_2 = y_1 \tan 2t$, meaning the two functions are not scalar multiples of each other because $\tan 2t$ is not constant. The two functions satisfy the ODE and are linearly independent, so they form a fundamental set of solutions for the ODE. Thus, the general solution is $y = c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t$ where $c_1, c_2 \in \mathbb{R}$.

$$y(0) = -1 \Rightarrow c_1 = -1$$
$$y'(0) = 0 \Rightarrow 2c_2 - c_1 = 0 \Rightarrow c_2 = -\frac{1}{2}$$

Thus, the particular solution with $y(0) = -1$ and $y'(0) = 0$ is

$$-e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t.$$

4.1.26 Unfortunately Theorem 1.23 does not show us how to find two independent solutions. However, there is a technique that can be used to find a second solution when one solution is known.

a. Show that $y_1(t) = t^2$ is a solution of

$$t^2y'' + ty' - 4y = 0. \tag{1.32}$$

b. Let $y_2(t) = vy_1 t = vt^2$, where v is a yet to be determined function of t. Note that if $y_2/y_1 = v$ and v is nonconstant, then y_1 and y_2 are independent. Show that the substitution $y_2 = vt^2$ reduces equation (1.32) to the sparable equation

$$5v' + tv'' = 0. \tag{1.33}$$

Solve equation (1.33) for v, form the solution $y_2 = vt^2$, and then state the general solution of equation (1.32).

Solution a. $t^2 y_1'' + ty_1' - 4y_1$
$= 2t^2 + 2t^2 - 4t^2$
$= 0 \Rightarrow y_1$ satisfies equation (1.32).

b.

$$\begin{aligned} t^2 y_2'' + ty_2' - 4y_2 &= 0 \\ t^2\big(t^2 v'' + 4tv' + 2v\big) + t\big(t^2 v' + 2tv\big) - 4vt^2 &= 0 \\ t^4 v'' + 4t^3 v' + 2t^2 v + t^3 v' + 2t^2 v - 4t^2 v &= 0 \\ t^4 v'' + 5t^3 v' &= 0 \\ tv'' + 5v' &= 0 \end{aligned}$$

as desired. Solving the ODE,

$$\begin{aligned} tv'' + 5v' &= 0 \\ \frac{\mathrm{d}(v')}{v'} &= -\frac{5}{t} \\ \ln v' &= -5\ln t + A_0 \\ v' &= A_1 t^{-5} \\ v &= At^{-4} + B \end{aligned}$$

$vt^2 = At^{-2} + Bt^2$. Note that the second term is a scalar multiple of y_1. Then our general solution for (1.32) is $y = c_1 t^2 + c_2 t^{-2}$, where $c_1, c_2 \in \mathbb{R}$.

4.1.30 $t^2 y'' + 4ty' + 2y = 0, \quad y_1(t) = 1/t$

Solution $t^2 y_1'' + 4ty_1' + 2t_1$
$= \dfrac{2}{t} - \dfrac{4}{t} + \dfrac{2}{t}$
$= 0 \Rightarrow y_1$ is a solution to the ODE.

Let $y_2 = \frac{v}{t}$. Substituting into the ODE,

$$\begin{aligned} t^2 y_2'' + 4ty_2' + 2y_2 &= 0 \\ t^2\left(2\frac{v}{t^3} - 2\frac{v'}{t^2} + \frac{v''}{t}\right) + 4t\left(\frac{v'}{t} - \frac{v}{t^2}\right) + 2\frac{v}{t} &= 0 \\ \frac{2v}{t} - 2v' + tv'' + 4v' - \frac{4v}{t} + \frac{2v}{t} &= 0 \\ tv'' + 2v' &= 0 \\ \frac{\mathrm{d}(v')}{v'} &= -\frac{2}{t} \\ \ln v' &= -2\ln t + A_0 \\ v' &= A_1 t^{-2} \\ v &= At^{-2} + B \end{aligned}$$

Our general solution is $y(t) = c_1 \dfrac{1}{t} + c_2 \dfrac{1}{t^2}$, where $c_1, c_2 \in \mathbb{R}$.

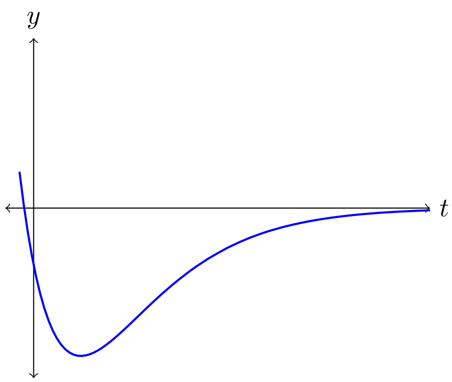
4.2.3 $y'' + 3y' + 4y = 2\cos 2t$

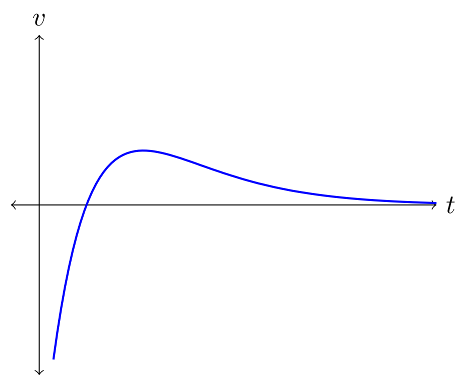
Solution $v = y' \Rightarrow y'' = v'$

$$\begin{cases} y' = v \\ v' = 2\cos 2t - 3v - 4y \end{cases}$$

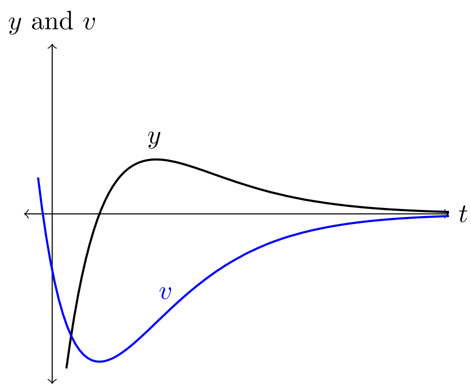
4.2.14 $m = 1$ kg, $\mu = 2$ kg/s, $k = 1$ kg/s^2, $y(0) = -1$ m, $y'(0) = -5$ m/s

Solution (i)

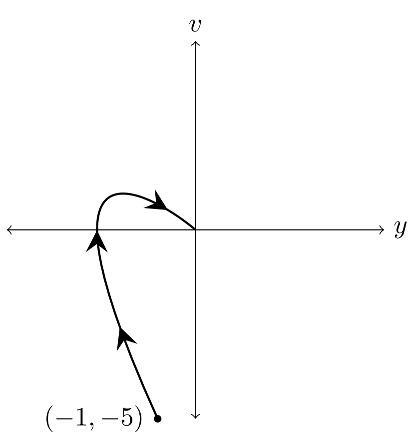




(ii)



(ii)



The best viewing window is the combined plots of y and v vs. t. (ii)