3.1 a. Show geometrically why the maximum principle holds using a "walking the dog" argument. Make it rigorous by imitating the last half of the proof of the fundamental theorem of algebra.

b. Use the maximum principle to prove the fundamental theorem of algebra by applying to $1/p$.

Solution a. Let f be analytic on a domain Ω and suppose there exists $z_0 \in \Omega$ such that $|f(z_0)| = \sup_{z\in\Omega}|f(z)|$. Then as f is analytic at z_0, we can write $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ on $|z-z_0| < R$ for some $R > 0$.

Suppose f is not identically $f(z_0)$. Let $0 < r < R$ such that $f(z) \neq f(z_0)$ when $0 < |z-z_0| \leq r$. We can do this because if there were no such r, then $f(z) - f(z_0)$ admits a non-isolated zero, which implies that $f(z) \equiv f(z_0)$.

As f is not constant, not all a_n are 0. Let a_N be the first non-zero coefficient of f. Then

$$f(z) = f(z_0) + a_N(z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots$$

Define $F(z) \coloneqq f(z_0) + a_N(z-z_0)^N$ and $R(z) \coloneqq \sum_{n=1}^{\infty} a_{N+i}(z-z_0)^{N+i}$. Note that R is also analytic since it is a sub-sum of f. Then by walking the dog, we have that there exists z^* such that

$$|F(z^*)| \geq |f(z_0)| + |a_N|r^N$$
$$|R(z)| \leq \sum_{i=1}^{\infty}|a_{N+i}|r^{N+i} < \infty$$

But then

$$\begin{aligned}
|f(z^*)| &= |F(z^*) + R(z^*)| \\
&\geq \big||F(z^*)| - |R(z^*)|\big| \\
&= |f(z_0)| + |a_N|r^N - \sum_{i=1}^{\infty}|a_{N+i}|r^{N+i} \\
&= |f(z_0)| + |a_N|r^N\left(1 - \sum_{i=1}^{\infty}\frac{|a_{N+i}|}{|a_N|}r^i\right)
\end{aligned}$$

We can make $r > 0$ sufficiently small so that $1 - \sum_{i=1}^{\infty}\frac{|a_{N+i}|}{|a_N|}r^i > M$ for some $M > 0$. But then we get the contradiction

$$|f(z^*)| \geq |f(z_0)| + M|a_N|r^N > |f(z_0)| = \sup_{z\in\Omega}|f(z)|.$$

Hence, the maximum principle holds.

b. Let $p(z) = a_0 + \cdots + a_n z^n$, where a_0 and a_n are both non-zero. Suppose $|p(z)| > 0$ for all $z \in \mathbb{C}$. Then $\frac{1}{p}$ is analytic on all of $\mathbb{C}$.

As $\mathbb{R}$ has the greatest lower bound property, $\inf_{z\in\mathbb{C}}|f(z)| \in \mathbb{R}$. Let m be this lower bound. Then

$$\frac{1}{|p(z)|} \leq \frac{1}{m}$$

But by Liouville's theorem, $\frac{1}{p(z)}$ is constant, meaning $a_n = 0$, which is a contradiction. Hence, $p(z)$ admits a zero in $\mathbb{C}$.

3.3 Suppose f is analytic in a connected open set U. If $|f(z)|$ is constant on U, prove that f is constant on U. Likewise, prove that f is constant if $\operatorname{Re} f$ is constant.

Solution As $|f(z)|$ constant, then for all $z_0 \in U$, we have that $|f(z_0)| = \sup_{z\in U}|f(z)|$. Hence, by the maximum principle, f is constant on U.

Let $M = \operatorname{Re} f$ and $z_0 \in U$. Since f is analytic on U, we can write $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ for $|z-z_0| < R$ for some $R > 0$.

Consider $g(z) \coloneqq f(z) - i \operatorname{Im} a_0$ and notice that

$$|g(z)| = \sqrt{(\operatorname{Re} f(z))^2 + (\operatorname{Im} f(z) - \operatorname{Im} a_0)^2} \geq \operatorname{Re} f(z)$$

If $\operatorname{Re} a_0 \neq 0$, $\frac{1}{g}$ is analytic on U. Then since $g(z_0) = \operatorname{Re} a_0$ and

$$\frac{1}{|g(z)|} \leq \frac{1}{\operatorname{Re} f(z)} = \frac{1}{\operatorname{Re} a_0},$$

then by the maximum principle, $\frac{1}{g(z)} \equiv \frac{1}{\operatorname{Re} a_0} \implies f(z) \equiv \operatorname{Re} a_0 + i \operatorname{Im} a_0 = a_0$.

If $\operatorname{Re} a_0 = 0$, then consider $g(z) \coloneqq f(z) - i \operatorname{Im} a_0 + 1$. We can apply the same argument above, but with $g(z_0) = 1$ instead to get that $g(z) \equiv 1 \implies f(z) \equiv i \operatorname{Im} a_0 = a_0$.

3.4 Suppose f and g are analytic on $\mathbb{C}$ and $|f(z)| \leq |g(z)|$ for all z. Prove there exists a constant c so that $f(z) = cg(z)$ for all z.

Solution Suppose $g(z) \equiv 0$. Then $|f(z)| \leq |g(z)| = 0 \implies f(z) \equiv 0$. Then in this case, $c = 0$.

Now assume that $g(z) \not\equiv 0$. Note that whenever $g(z) = 0$, we have that $f(z) = 0$ also. Consider

$$h(z) \coloneqq \begin{cases} \dfrac{f(z)}{g(z)}, & g(z) \neq 0 \\ 0, & g(z) = 0. \end{cases}$$

h is analytic on $\mathbb{C}$. Moreover, as $|f(z)| \leq |g(z)|$ for all z, $h(z) \leq 1$ in $\mathbb{C}$. Hence, by Liouville's theorem, $h(z) \equiv c$ for some $c \in \mathbb{C}$. Thus, $f(z) = cg(z)$ for all $z \in \mathbb{C}$.

3.5 Prove that if f is non-constant and analytic on all of $\mathbb{C}$ then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Solution Suppose $f(\mathbb{C})$ were not dense in $\mathbb{C}$. Then there exists $w \in \mathbb{C}$ such that there exists $r > 0$ so that $B_r(w) \cap f(\mathbb{C}) = \emptyset$. In other words, $f(z) - w \geq r$ for all $z \in \mathbb{C}$. Thus, $\frac{1}{f(z)-w}$ is analytic on $\mathbb{C}$. Moreover,

$$\left| \frac{1}{f(z) - w} \right| \leq \frac{1}{r}$$

on $\mathbb{C}$. Then by Liouville's theorem $\frac{1}{f(z)-w}$ is constant, which implies that f is constant. This is a contradiction, so no such w exists. Hence, $f(\mathbb{C})$ is dense in $\mathbb{C}$.

3.6 Let f be analytic in $\mathbb{D}$ and suppose $|f(z)| < 1$ on $\mathbb{D}$. Let $a = f(0)$. Show that f does not vanish in $\{z \mid |z| < |a|\}$.

Solution Consider $g(z) \coloneqq f(z) - a$. Then $f(0) = 0$ and f is analytic on $\mathbb{D}$. Thus, by the Schwarz lemma, $|f(z) - a| \leq |z|$.

Suppose there exists z_0 such that $f(z_0) = 0$ and $|z_0| \leq |a|$. Then

$$|f(z_0) - a| = |a| \leq |z_0| < |a|$$

which is a contradiction. Hence, f does not vanish on $\{z \mid |z| < |a|\}$.

3.7 Prove that if f is a one-to-one (two-to-two!) analytic map of an open set Ω onto $f(\Omega)$ and if $z_n \in \Omega \to \partial\Omega$, then $f(z_n) \to \partial f(\Omega)$, in the sense that $f(z_n)$ eventually lies outside each compact subset of $f(\Omega)$. A function with this property is called **proper**.

Solution Let $\{z_n\}_{n\geq 1} \subseteq \Omega$ be such that $z_n \neq z_m$ for all $n \neq m \geq 1$ and $z_n \xrightarrow{n\to\infty} \partial\Omega$. Since f is one-to-one, $f(n) \neq f(m)$ for all $n \neq m \geq 1$.

Suppose there exists $K \subseteq f(\Omega)$ compact such that infinitely many $f(z_n)$ lie in. Then $f(z_n)$ converges in K to a point $w \in K$. Since $w \in f(\Omega)$, there exists a unique $z_0 \in \Omega$ such that $f(z_0) = w$.

This implies that $z_n \xrightarrow{n\to\infty} z_0$, since z_0 is unique. This is a contradiction because we assumed that z_n converges outside each compact subset of Ω. Hence, f is proper.

3.8 a. Prove that φ is a one-to-one analytic map of $\mathbb{D}$ onto $\mathbb{D}$ if and only if

$$\varphi(z) = c\left(\frac{z-a}{1-\bar{a}z}\right),$$

for some constants c and a, with $|c| = 1$, and $|a| < 1$. What is the inverse map?

b. Let f be analytic in $\mathbb{D}$ and satisfy $|f(z)| \to 1$ as $|z| \to 1$. Prove f is rational.

Solution a. "$\Longleftarrow$"

Let $\varphi(z) = c\left(\frac{z-a}{1-\bar{a}z}\right)$. We first show that it is one-to-one.

Let $z, w \in \mathbb{D}$ such that $f(z) = f(w)$. Then

$$\begin{aligned}
c\left(\frac{z-a}{1-\bar{a}z}\right) &= c\left(\frac{w-a}{1-\bar{a}w}\right) \\
z - \bar{a}wz - a + |a|^2 w &= w - \bar{a}wz - a + |a|^2 z \\
z - |a|^2 z &= w - |a|^2 w \\
z &= w
\end{aligned}$$

The last step holds since $|a| < 1 \implies 1 - |a|^2 > 0$.

φ is one-to-one because $z - a$ is analytic, and $1 - \bar{a}z$ is analytic $\implies \frac{1}{1-\bar{a}z}$ is analytic. This is because the only zero of $1 - \bar{a}z$ occurs when $z = \frac{1}{\bar{a}}$, which lies outside of the unit disk.

Lastly, we need to show that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Let $|z| = 1$. Then

$$\begin{aligned}
|\varphi(z)| &= \left|\frac{z-a}{1-\bar{a}z}\right| \cdot |\bar{z}| \\
&= \left|\frac{|z|^2 - a\bar{z}}{1-\bar{a}z}\right| \\
&= \left|\frac{1 - a\bar{z}}{1-\bar{a}z}\right| = 1
\end{aligned}$$

It follows that if $|z| < 1$, then $|\varphi(z)| < 1$ also. Thus, $\varphi\colon \mathbb{D} \to \mathbb{D}$.

"$\implies$"

Let φ be a one-to-one analytic map of $\mathbb{D}$ onto $\mathbb{D}$. Assume $\varphi(0) = 0$ so that $|\varphi(z)| \leq |z|$. Then as φ is one-to-one, $F(z) := \frac{z}{\varphi(z)}$ is also analytic on $\mathbb{D}$. Note that $|F(z)| \geq 1$.

By exercise 7, as $|z| \to 1$, $|F(z)| \to 1$ also. Then by the maximum principle, $|F(z)| \leq 1$ on $\mathbb{D}$. Thus, $|F(z)| = 1$, so $\varphi(z) = cz$ for some $|c| = 1$.

Pick a point $a \in \mathbb{D}$ and transform coordinates using T_a so that we get

$$\varphi(z) = cT_a = c\left(\frac{z-a}{1-\bar{a}z}\right)$$

as desired.

b. Suppose $|z| \to 1 \implies |f(z)| \to 1$. If f achieves its maximum in $\mathbb{D}$, then f is constant, and is obviously a rational function.

Otherwise, f attains its maximum on $\partial\mathbb{D}$ by the maximum principle. But as $|z| \to 1$, $|f(z)| \to 1$, so 1 is the maximum value of $|f|$.

By corollary 3.4, if $f(z_j) = 0$ for all j, we can write

$$f(z) = \prod_{j=1}^{n} \left(\frac{z - z_j}{1 - \bar{z}_j z} \right) g(z)$$

Notice that as $|z| \to 1$, $|g(z)| \to 1$ also. Thus, by the maximum principle, its maximum it 1, so $\frac{1}{g}$ is analytic. Moreover, as $|z| \to 1$, $\frac{1}{|g(z)|} \to 1$, so applying the maximum principle again, $\frac{1}{|g(z)|}$ is bounded above by 1. Thus, $|g(z)| = \lambda$, where $|\lambda = 1|$, and so

$$f(z) = \prod_{j=1}^{n} \left(\frac{z - z_j}{1 - \bar{z}_j z} \right) \lambda.$$

Hence, f is rational.