6.2.1 If u is harmonic and bounded in $0 < |z| < \rho$, show that the origin is a removable singularity in the sense that u becomes harmonic in $|z| < \rho$ when $u(0)$ is properly defined.

Solution Define $u(0)$ by

$$u(0) = \frac{1}{2\pi r}\int_{|z|=r} u(z)\,\mathrm{d}z = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,\mathrm{d}\theta,$$

for $0 < r < \rho$. By theorem 20, we know that $u(0) = \alpha \log r + \beta$, where

$$\alpha = \int_{|z|=r} r\frac{\partial u}{\partial r}\,\mathrm{d}\theta,$$

which doesn't depend on r.

Since $u(z)$ is bounded and $\log r$ is unbounded in a neighborhood of 0, it follows that $\alpha = 0$, which means that $u(0)$ well-defined.

Next note that since we're on a circle,

$$\alpha = 0 \implies \int_{|z|=\rho/2} {}^*\mathrm{d}u = \int_{|z|=\rho/2} \frac{\partial u}{\partial n}\,|\mathrm{d}z| = \int_{|z|=\rho/2} \frac{\partial u}{\partial r}\,|\mathrm{d}z| = \int_0^{2\pi} \frac{\rho}{2}\frac{\partial u}{\partial r}\,\mathrm{d}\theta = 0.$$

Thus, since $\{z \mid |z| = \rho/2\}$ is a homology basis for the unit disk, it follows that for any $\gamma \subseteq \mathbb{D}$,

$$\int_\gamma {}^*\mathrm{d}u = 0,$$

so u has a conjugate function v, i.e., there exists f analytic on the punctured disk so that $\operatorname{Re} f = u$. By a previous problem, since $\operatorname{Re} f$ is bounded, f has a removable singularity at 0, so f extends to be analytic on the whole disk, which shows that u extends to be harmonic at 0.

6.2.2 Suppose that $f(z)$ is analytic in the annulus $r_1 < |z| < r_2$ and continuous on the closed annulus. If $M(r)$ denotes the maximum of $f(z)$ for $|z| = r$, show that

$$M(r) \le M(r_1)^\alpha M(r_2)^{1-\alpha}$$

where $\alpha = \log(r_2/r) : \log(r_2/r_1)$. Discuss cases of equality.

Solution Since $r_1 < r < r_2$, it follows that $\alpha > 0$, and α is maximal when $r = r_1$, which gives $\alpha = 1$, so $0 < \alpha < 1$.

Consider $h(z) = z^\lambda f(z)$, which is analytic. The maximum modulus principle tells us that h achieves its maximum on its boundary, so

$$|h(z)| \le \max\{|h(r_1)|, |h(r_2)|\}.$$

We will find λ so that the maximum values are the same.

$$|r_1|^\lambda M(r_1) = |r_2|^\lambda M(r_2) \implies \lambda = -\frac{\log(M(r_2)/M(r_1))}{\log(r_2/r_1)}.$$

Thus, using the fact that $\log x$ is strictly increasing, for z such that $|f(z)| = |M(r)|$ (which exists by continuity and compactness),

$$\begin{aligned}
\log|z^\lambda f(z)| = \lambda\log r + \log M(r) &\le \log|r_2^\lambda f(r_2)| \\
&\le \lambda\log r_2 + \log M(r_2) \\
\implies \log M(r) &\le \lambda\log\frac{r_2}{r} + \log M(r_2) \\
&= -\frac{\log(M(r_2)/M(r_1))}{\log(r_2/r_1)}\log\frac{r_2}{r} + \log M(r_2) \\
&= \frac{\log r_2/r}{\log r_2/r_1}\log M(r_1) + \left(1 - \frac{\log r_2/r}{\log r_2/r_1}\right)\log M(r_2) \\
&= \alpha\log M(r_1) + (1-\alpha)\log M(r_2) \\
&= \log M(r_1)^\alpha M(r_1)^{1-\alpha},
\end{aligned}$$

which is equivalent to what we wanted to show, since the logarithm is strictly increasing.

6.4.1 Assume that $U(\xi)$ is piecewise continuous and bounded for all real ξ. Show that

$$P_U(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} U(\xi)\,\mathrm{d}\xi$$

represents a harmonic function in the upper half plane with boundary values $U(\xi)$ at points of continuity.

Solution First notice that if $z = x + iy$,

$$-\operatorname{Im} \frac{1}{z-\xi} = \frac{1}{2i}\left(\frac{1}{\overline{z}-\xi} - \frac{1}{z-\xi}\right) = \frac{1}{2i}\frac{z-\overline{z}}{|z-\xi|^2} = \frac{y}{|z-\xi|^2},$$

so the integral becomes

$$P_U(z) = \operatorname{Im}\left[\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{U(\xi)}{\xi - z}\,\mathrm{d}\xi\right],$$

so it suffices to show that the bracketed term is analytic.

Let z_0 be in the upper half-plane. Let $\delta < \frac{1}{2}\operatorname{dist}(z_0, \mathbb{R}) = \frac{1}{2}\operatorname{Im}(z_0)$. Then for $z \in B(z_0, \delta)$ and $\xi \in \mathbb{R}$, we have $|z - z_0| \leq |z - \xi|$, so the following geometric sum is uniformly convergent on circles of radius $0 < r < \delta$ and center z_0, which means we can interchange the sum and integral:

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty} \frac{U(\xi)}{(\xi - z_0)^{n+1}}(z-z_0)^n\,\mathrm{d}\xi = \sum_{n=0}^{\infty}\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{U(\xi)}{(\xi-z_0)^{n+1}}\,\mathrm{d}\xi\right)(z-z_0)^n$$

If $U(\xi)$ decays sufficiently quickly, then the integral converges for all $n \geq 0$. Thus, the bracketed part is analytic, so $P_U(z)$ is the imaginary part of an analytic function, i.e., it is harmonic.

6.4.4 If C_1 and C_2 are complementary arcs on the unit circle, set $U = 1$ on C_1, $U = 0$ on C_2. Find $P_U(z)$ explicitly and show that $2\pi P_U(z)$ equals the length of the arc, opposite to C_1, cut off by the straight lines through z and the end points of C_1.

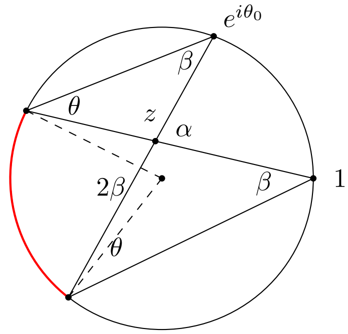
Solution We can rotate so that C_1 starts at $\theta = 0$ and ends at $\theta = \theta_0$, i.e., one of its endpoints is 1. Then if we apply the substitution $u \mapsto e^{i\theta} - z$, we get

$$\begin{aligned}
P_U(z) &= \operatorname{Re}\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} U(\theta)\,\mathrm{d}\theta\right] \\
&= \operatorname{Re}\left[\frac{1}{2\pi}\int_0^{\theta_0} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,\mathrm{d}\theta\right] \\
&= \operatorname{Re}\left[\frac{1}{2\pi i}\int_{1-z}^{e^{i\theta_0}-z} \frac{2}{u} - \frac{1}{u+z}\,\mathrm{d}u\right] \\
&= \operatorname{Re}\left[\frac{1}{2\pi i}(2\log u - \log(u+z))\Big|_{1-z}^{e^{i\theta_0}-z}\right] \\
&= \operatorname{Re}\left[\frac{1}{2\pi i}\left(2\log(e^{i\theta_0}-z) - \log(e^{i\theta_0}) - 2\log(1-z) - \log 1\right)\right] \\
&= \operatorname{Re}\left[\frac{1}{2\pi i}\left(2\log\frac{e^{i\theta_0}-z}{1-z} - i\theta_0\right)\right].
\end{aligned}$$

So, we get

$$2\pi P_U(z) = 2\arg\frac{e^{i\theta_0}-z}{1-z} - \theta_0.$$

Write $\alpha = \arg \dfrac{e^{i\theta_0} - z}{1 - z}$ and consider the following figure:



The angles are calculated from geometry, and the arc-length removed is drawn in red, which subtends an angle of 2β, so the removed arc-length is 2β.

We know that $2\theta = \theta_0$, from the construction, and that $\beta + \theta = \alpha$. Thus,

$$2\theta + 2\beta = 2\alpha \implies 2\beta = 2\alpha - \theta_0,$$

which is the result we got.

1.1.4 As a generalization of Theorem 2, prove that if the $f_n(z)$ have at most m zeroes in Ω, then $f(z)$ is either identically zero or has at most m zeros.

Solution Let $z_0 \in \Omega$ and let $\delta > 0$ so that $B(z_0, \delta) \subseteq \Omega$ and $f(z) \neq 0$ on the boundary. Then by the argument principle,

$$\frac{1}{2\pi i} \int_{|z-z_0|=\delta} \frac{f_n'(z)}{f_n(z)} \, \mathrm{d}z \leq m.$$

On the circle $C = \{z \mid |z - z_0| = \delta\}$, f_n is continuous, so $|f_n|$ is bounded and non-zero on C, so $1/f_n \xrightarrow{n\to\infty} 1/f$ uniformly on C. Moreover, by Cauchy's integral formula, and uniform convergence of f_n, it follows that f_n' converges uniformly to f'. Thus,

$$\frac{1}{2\pi i} \int_{|z-z_0|=\delta} \frac{f_n'(z)}{f_n(z)} \, \mathrm{d}z \xrightarrow{n\to\infty} \frac{1}{2\pi i} \int_{|z-z_0|=\delta} \frac{f'(z)}{f(z)} \, \mathrm{d}z \implies \frac{1}{2\pi i} \int_{|z-z_0|=\delta} \frac{f'(z)}{f(z)} \, \mathrm{d}z \leq m.$$

Thus, f has at most m zeroes, by the argument principle.

1.1.5 Prove that

$$\sum_{n=1}^{\infty} \frac{nz^n}{1 - z^n} = \sum_{n=1}^{\infty} \frac{z^n}{(1 - z^n)^2}$$

for $|z| < 1$.

Solution Notice that we can rewrite the left-hand side as

$$\sum_{n=1}^{\infty} nz^n \sum_{k=0}^{\infty} z^{nk} = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} nz^{nk} z^n = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} n\left(z^{k+1}\right)^n$$

since the sum converges absolutely on $|z| < 1$. Indeed, by the ratio test,

$$\lim_{n\to\infty} \left| \frac{z^{n+1}}{1 - z^{n+1}} \cdot \frac{1 - z^n}{z^n} \right| = \lim_{n\to\infty} |z| \left| \frac{1 - z^n}{1 - z^{n+1}} \right| = |z| < 1.$$

Moreover, the geometric series converges uniformly on any closed disk with radius $r < 1$, so we can differentiate term by term.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \implies \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} \implies \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n.$$

Substituing z^{k+1} for x gives

$$\sum_{n=1}^{\infty} n\left(z^{k+1}\right)^n = \frac{z^{k+1}}{(1-z^{k+1})^2}.$$

Thus, reindexing gives

$$\sum_{k=0}^{\infty}\sum_{n=1}^{\infty} n\left(z^{k+1}\right)^n = \sum_{k=0}^{\infty} \frac{z^{k+1}}{(1-z^{k+1})^2} = \sum_{n=1}^{\infty} \frac{z^n}{(1-z^n)^2},$$

as desired.

5.4.1 Prove that in any region Ω, the family of analytic functions with positive real part is normal. Under what added condition is it locally bounded?

Solution Let $\mathfrak{F} = \{f : \Omega \to C \mid \operatorname{Re} f > 0\}$.

Consider the Möbius transformation

$$Tz = \frac{z-1}{z+1}.$$

This maps the right-half plane to the unit disk.

Let $f \in \mathfrak{F}$. Then

$$T \circ f = \frac{f(z)-1}{f(z)+1} \in \mathbb{D},$$

since f is a map from Ω to the right-half plane. In particular, $|Tf|$ is bounded by 1, and this bound works for any $f \in \mathfrak{F}$ and for any subset of Ω. Hence, $T(\mathfrak{F}) \coloneqq \{T \circ f \mid f \in \mathfrak{F}\}$ is a normal family.

Let $\{f_n\}_{n\geq 1} \subseteq \mathfrak{F}$, and let $K \subseteq \Omega$ be compact.

Then $\{T \circ f_n\}_{n\geq 1}$ admits a subsequence $\{T \circ f_{n_k}\}_{k\geq 1}$ that converges uniformly to some $f : \Omega \to \mathbb{D}$ on K. $S \coloneqq T^{-1}$ is also a Möbius transformation and is analytic, so its derivative S' is also analytic. In particular, it's continuous on K compact, so its magnitude attains a maximum M_K. Thus,

$$|STf_{n_k}(z) - Sf(z)| \leq M_K|Tf_{n_k}(z) - f(z)| \implies |f_{n_k}(z) - Sf(z)| \leq M_K|Tf_{n_k}(z) - f(z)|.$$

By taking the supremum on both sides, it follows that f_{n_k} converges uniformly to f also, so $\mathfrak{F}$ is normal.

5.4.2 Show that the functions z^n, n a nonnegative integer, form a normal family in $|z| < 1$, also in $|z| > 1$, but not in any region that contains a point on the unit circle.

Solution On $|z| < 1$, $|z|^n \leq 1$ for all $n \geq 1$, so the family is uniformly bounded on any compact set of the unit disk. Hence, it is normal.

For $|z| > 1$, notice that $1/z^n$ is analytic for all $n \geq 1$, since $z \neq 0$. Moreover, $1/|z|^n \leq 1$, so the family is uniformly bounded also on any compact subset outside the closed unit disk. Thus, the family is normal, since uniform convergence of reciprocals of bounded functions implies uniform convergence of the functions.

Consider a region Ω which contains a point ζ on the unit circle. Since Ω is open, there exists $\delta > 0$ so that $B(\zeta, 2\delta) \subseteq \Omega$. Notice that $B(\zeta, \delta)$ intersects $\{z \mid |z| < 1\}$ and $\{z \mid |z| > 1\}$.

Take $K = \overline{B(\zeta, \delta)}$ to be our compact subset of Ω. Suppose our family is normal, so that we get a subsequence $\{z^{n_k}\}_{k\geq 1}$ which converges uniformly to some f. Then for $|z| > 1$ in K, $z^{n_k} \xrightarrow{k\to\infty} \infty$; for $|z| < 1$ in K, we get $z^{n_k} \xrightarrow{k\to\infty} 0$; and for $|z| = 1$ in K, $z^{n_k} \xrightarrow{k\to\infty} \xi$ with $|\xi| = 1$. Such a function f is not continuous, which is a contradiction since each z^n is continuous. Thus, the family of functions is not normal.

5.4.4 If that family $\mathfrak{F}$ of analytic (or meromorphic) functions is not normal in Ω, show that there exists a point z_0 such that $\mathfrak{F}$ is not normal in any neighborhood of z_0.

Solution Suppose that for all $z \in \Omega$, there exist open neighborhoods U_z, V_z so that $\overline{U_z} \subseteq V_z$, where $\mathfrak{F}$ is normal in V_z and $\overline{U_z}$ is compact.

Let $\{f_n\}_{n\geq 1} \subseteq \mathfrak{F}$, and let K be a compact subset of Ω. By compactness, there exist $z_1, \dots, z_n$ so that

$$K \subseteq \bigcup_{k=1}^{n} U_{z_k}.$$

Since $\mathfrak{F}$ is normal in V_{z_1}, we can extract a subsequence $\{f_{n_k^{(1)}}\}_{k\geq 1}$ which is uniformly convergent on $\overline{U_{z_1}} \cap K$. Similarly, Ssince $\mathfrak{F}$ is normal in V_{z_2}, we can extract a subsequence of that subsequence $\{f_{n_k^{(2)}}\}_{k\geq 1}$ which is uniformly convergent on $\overline{U_{z_2}} \cap K$. We can repeat this process finitely many times for all the z_k, so that we get a subsequence $\{f_{n_k}\}_{k\geq 1}$ which is uniformly convergent on

$$\bigcup_{k=1}^{n} \overline{U_{z_k}} \cap K \supseteq K,$$

but this means that $\{f_{n_k}\}_{k\geq 1}$ converges uniformly on K, which contradicts the non-normality of $\mathfrak{F}$. Hence, there exists $z_0 \in \Omega$ so that $\mathfrak{F}$ is not normal in any neighborhood of z_0.

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10 Use Runge's theorem to prove there exists an analytic function $f(z)$ on the unit disk $\mathbb{D} = \{|z| < 1\}$ such that for all $\zeta \in \partial\mathbb{D}$,

$$\limsup_{\mathbb{D}\ni z\to\zeta} |f(z)| = \infty$$

but

$$\liminf_{\mathbb{D}\ni z\to\zeta} |f(z)| = 0.$$

Solution Let $r_n < s_n < t_n < 1$, $r_{n+1} = t_n \xrightarrow{n\to\infty} 1$,

$$K_n = \{z \mid |z| \leq r_n\} \cup \{z \mid |z| = s_n,\ |\arg z| \geq 1/n\} \cup \{z \mid |z| = t_n,\ |\arg z| \leq \pi - 1/n\}.$$

Let f_1 be analytic on a neighborhood of K_1 and let P_1 be a polynomial with $\sup_{K_1} |f_1 - P_1| < 1$.

By induction, let f_n be analytic on a neighborhood of K_n such that $f_n = P_{n-1}$ on $\{z \mid |z| \leq r_n\}$. For even n, let $f_n = n$ on $K_n \cap \{z \mid |z| = s_n\}$, $f_n = 0$ on $K_n \cap \{z \mid |z| = t_n\}$. For odd n, switch s_n with t_n. Then let P_n be a polynomial with $\sup_{K_n} |f_n - P_n| < 2^{-n}$.

The polynomials we used for approximation exist because of Runge's theorem.

Take $f = \lim_{n\to\infty} P_n$. f is analytic since the P_n are.

Let $\zeta = e^{i\theta} \neq 1$, with $\theta \in [-\pi, pi]$, on $\partial\mathbb{D}$. There exists $N \in \mathbb{N}$ so that $n \geq N \implies |\theta| \geq 1/n$. Consider $\{s_n e^{i\theta}\}_{n\geq 1}$. By definition, we have that $P_{2n}(s_n e^{i\theta}) = 2n \xrightarrow{n\to\infty} \infty$.

Since the P_n converge uniformly to f, we can find a sequence $\{z_n\}_{n\geq 1}$ so that $|z_n - s_n e^{i\theta}| < 2^{-n}$ with $|f(z_n) - P_{2n}(s_n e^{i\theta})| < 2^{-n}$, which means that $f(z_n) \xrightarrow{n\to\infty} \infty$ along with $z_n \xrightarrow{n\to\infty} \zeta$ also. Indeed, we just need to be able to bound

$$|f(z_n) - P_{2n}(s_n e^{i\theta})| \leq |f(z_n) - f(s_n e^{i\theta})| + |f(s_n e^{i\theta}) - P_{2n}(s_n e^{i\theta})|,$$

which we can easily do by using continuity of f and uniform convergence.

Similarly, $P_{2n+1}(s_n e^{i\theta}) = 0 \xrightarrow{n\to\infty} 0$, so we can find another sequence so that $f(z_n) \xrightarrow{n\to\infty} 0$, which proves the claim for $\zeta \neq 1$.

For $\zeta = 1$, we can perform the same argument, but using t_n as our sequence instead. Then $P_{2n}(t_n) \xrightarrow{n\to\infty} 0$ and $P_{2n+1}(t_n) \xrightarrow{n\to\infty} \infty$, and we can find sequences as before.