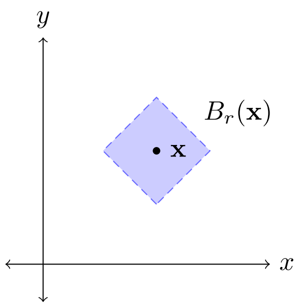
20.1 a. In $\mathbb{R}^n$, define
$$d'(\mathbf{x},\mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$
Show that d' is a metric that induces the usual topology of $\mathbb{R}^n$. Sketch the basis elements under d' when $n = 2$.

b. More generally, given $p \geq 1$, define
$$d'(\mathbf{x},\mathbf{y}) = \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p}$$
for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Assume that d' is a metric. Show that it induces the usual topology on $\mathbb{R}^n$.

Solution a. (i) Note that $\sum_{i=1}^{n} |x_i - y_i| = 0 \iff |x_i - y_i| = 0$ for all $1 \leq i \leq n$ because each term in the sum is nonnegative. If one of the terms were positive, then the entire sum would be positive. Hence,
$$d'(\mathbf{x},\mathbf{y}) = 0 \iff \sum_{i=1}^{n} |x_i - y_i| = 0 \iff |x_i - y_i| = 0 \ \forall i \iff x_i = y_i \ \forall i \iff \mathbf{x} = \mathbf{y}.$$

(ii) $d'(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |y_i - x_i| = d'(\mathbf{y},\mathbf{x})$.

(iii) $d'(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i| \leq \sum_{i=1}^{n} (|x_i - z_i| + |z_i - y_i|) = \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |z_i - y_i| = d'(\mathbf{x},\mathbf{z}) + d'(\mathbf{z},\mathbf{y})$.



b. It suffices to show that a ball in the metric d', $B'_r(\mathbf{x})$ contains a regular ball (i.e., a ball given by the Euclidean distance) and is contained in a regular ball in $\mathbb{R}^n$.

Fix $\mathbf{x} \in \mathbb{R}^n$ and pick a point $\mathbf{y} \in B'_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n \mid d'(\mathbf{x},\mathbf{y}) < r\}$. Notice that
$$\begin{aligned} d'(\mathbf{x},\mathbf{y}) &= \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p} \\ &\leq \left[\sum_{i=1}^{n} \max_{1 \leq i \leq n} |x_i - y_i|^p\right]^{1/p} \\ &= n^{1/p} \max_{1 \leq i \leq n} |x_i - y_i| \coloneqq M_{\mathbf{x}}(\mathbf{y}). \end{aligned}$$

Also note that $d'(\mathbf{x},\mathbf{y})$ is continuous in $\mathbf{y}$ in $(\mathbb{R}^n, d)$ (for some $\varepsilon > 0$, take $\delta < \varepsilon n / n^{1/p}$), so $B'_r(\mathbf{x}) = (d'(\mathbf{x},\mathbf{y}))^{-1}((-\infty, r))$ is open in $(\mathbb{R}^n, d)$. Hence, there exists m such that $B_m(\mathbf{x}) \subseteq B'_r(\mathbf{x})$.

Define $M = \sup\{M_{\mathbf{x}}(\mathbf{y}) \mid \mathbf{y} \in B'_r(\mathbf{x})\}$, which is finite since $B'_r(\mathbf{x})$ is bounded.

Hence, $B_m(\mathbf{x}) \subseteq B'_r(\mathbf{x}) \subseteq B_M(\mathbf{x})$, so the two metrics give the same topology on $\mathbb{R}^n$.

20.10 Let X denote the subset of $\mathbb{R}^\omega$ consisting of all sequences $(x_1, x_2, \ldots)$ such that $\sum x_i^2$ converges.

a. Show that if $\mathbf{x}, \mathbf{y} \in X$, then $\sum |x_i y_i|$ converges.

b. Let $c \in \mathbb{R}$. Show that if $\mathbf{x}, \mathbf{y} \in X$, then so are $\mathbf{x} + \mathbf{y}$ and $c\mathbf{x}$.

c. Show that

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{1/2}$$

is a well-defined metric on X.

Solution a. Note that for any real numbers $a, b \in \mathbb{R}$, $(a-b)^2 \geq 0 \implies a^2 + b^2 \geq 2|ab| \geq |ab|$.
Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \sum_{i=1}^{\infty} (x_i^2 + y_i^2) = \sum_{i=1}^{\infty} x_i^2 + \sum_{i=1}^{\infty} y_i^2 < \infty,$$

by hypothesis. Hence, $|x_i y_i|$ converges by the monotone convergence theorem.

b. Since $\sum x_i^2$ and $\sum y_i^2$ are convergent,

$$\sum_{i=1}^{\infty} (x_i^2 + y_i^2) = \sum_{i=1}^{\infty} x_i^2 + \sum_{i=1}^{\infty} y_i^2 < \infty \iff \mathbf{x} + \mathbf{y} \in X.$$

Also,

$$\sum_{i=1}^{\infty} c x_i^2 = c \sum_{i=1}^{\infty} x_i^2 < \infty \iff c\mathbf{x} \in X.$$

c. (i) Note that $\sum x_i^2 = 0 \iff \mathbf{x} = \mathbf{0} = (0, 0, \ldots)$ by a similar argument as in 20.1(a). Hence,

$$d(\mathbf{x}, \mathbf{y}) = 0 \iff \left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{1/2} = 0 \iff x_i = y_i \ \forall i \iff \mathbf{x} = \mathbf{y}.$$

(ii) $d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{1/2} = \left[\sum_{i=1}^{\infty} (y_i - x_i)^2\right]^{1/2} = d(\mathbf{x}, \mathbf{y})$.

(iii) The triangle inequality holds since for $0 < n < \infty$,

$$\left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2} = \left[\sum_{i=1}^{n} (x_i - z_i)^2 + (z_i - y_i)^2\right]^{1/2}$$

since the Euclidean distance is a metric on $\mathbb{R}^n$. Taking $n \to \infty$, we get $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$.
The metric is also well-defined since $(x_i - y_i)^2 = x_i^2 + y_i^2 - 2x_i y_i$.
Since $\mathbf{x}, \mathbf{y} \in X$, $\sum x_i^2$ and $\sum y_i^2$ both converge. By absolute convergence of $\sum x_i y_i$, it converges.
Combining the above with part (b), we get that

$$\sum_{i=1}^{\infty} (x_i - y_i)^2$$

converges, so the metric is well-defined on X.

21.3 Let X_n be a metric space with the metric d_n for $n \in \mathbb{Z}_+$.

a. Show that
$$\rho(x,y) = \max\{d_1(x_1,y_1),\ldots,d_n(x_n,y_n)\}$$
is a metric for the product space $X_1 \times \cdots \times X_n$.

b. Let $\overline{d}_i = \min\{d_i, 1\}$. Show that
$$D(x,y) = \sup\{\overline{d}_i(x_i,y_i)/i\}$$
is a metric for the product space $\prod X_i$.

Solution Let $X = \prod X_i$.

a. We'll first show that ρ is a metric.

(i) $d(x,y) = 0 \iff \max\{d_1(x_1,y_1),\ldots,d_n(x_n,y_n)\} = 0 \iff d_i(x_i,y_i) = 0\ \forall i \iff x_i = y_i\ \forall i \iff x = y$.

(ii) ρ is clearly symmetric since each d_i is symmetric.

(iii) ρ also satisfies the triangle inequality since each d_i satisfies the triangle inequality.

To show that this metric gives the product topology, it suffices to show that every ball under this metric is part of the product topology, and that given an open set in the product topology, we can write it as the union of balls.

Let $B_r(x)$ be an open ball in X. Then we can write
$$B_r(x) = \{y \in X \mid d_i(x_i,y_i) < r\ \forall i\} = \bigcap_{i=1}^{n} \pi_i^{-1}(\{y_i \in X_i \mid d_i(x_i,y_i) < r\})$$
which is an open set in the product topology since topologies are closed under finite unions.

Consider a basis element of the topology of a product space, which is of the form $\pi_i^{-1}(U_i)$, where $U_i \subseteq X_i$ is an open set.

Let $x \in \pi_i^{-1}(U_i)$.

Since U_i is open in a metric space, there exists a ball centered at x_i of radius r_x that is contained within U_i. We claim that a ball of the same radius centered at x under the metric ρ is contained within $\pi_i^{-1}(U_i)$. Let $y \in B_{r_x}(x)$. Then $d(x_i,y_i) < r_x \implies y_i \in U_i \implies y \in \pi_i^{-1}(U_i)$. Hence, $B_{r_x}(x) \subseteq \pi_i^{-1}(U_i)$, so we can write
$$\pi_i^{-1}(U_i) = \bigcup_{x \in \pi_i^{-1}(U_i)} B_{r_x}(x).$$

Thus, the metric induces the product topology on a product space because every ball is in the product topology and any open set in the product topology is a union of these open balls.

b. $\overline{d}_i$ is a metric, as discussed in class. Moreover, it gives rise to the same topology as d_i.

We'll first show that D is a metric.

(i) $D(x,y) = 0 \iff \sup\{\overline{d}_i(x_i,y_i)/i\} = 0 \iff \overline{d}_i(x_i,y_i)/i = 0\ \forall i \iff x_i = y_i\ \forall i \iff x = y$.

(ii) D is symmetric since $\overline{d}_i$ is symmetric for each i.

(iii) $D(x,y) = \sup\{\overline{d}_i(x_i,y_i)/i\} \le \sup\{(\overline{d}_i(x_i,z_i) + \overline{d}_i(z_i,y_i))/i\} \le \sup\{(\overline{d}_i(x_i,z_i))\} + \sup\{(\overline{d}_i(z_i,y_i))\} = d(x,z) + d(z,y)$.

Let $\overline{\rho} = \max \overline{d}_1(x_1,y_1),\ldots,\overline{d}_n(x_n,y_n)$. We can use the same argument from part (a) to show that this is a metric. Moreover, since each $\overline{d}_i$ gives the same topology as d_i, $\overline{\rho}$ gives the same topology as ρ.

Notice that $D(x,y) \le \overline{\rho}(x,y)$, where ρ is the metric from part (a), since $i \ge 1$. Thus, D gives the same topology as $\overline{\rho}$, which induces the same topology as ρ, which is the product topology.

Indeed, we can fit a ball B_D under the metric D inside a ball $B_{\overline{\rho}}$ under the metric $\overline{\rho}$. So, we can write any open set as a union of open balls B_D, which means that the topologies are the same.

18.2 Suppose that $f\colon X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

Solution No. Consider $X = [0,1] \cup [2,3]$ with the Euclidean metric, Y with the Euclidean metric, and

$$f(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 2 & \text{if } x \in [2,3]. \end{cases}$$

2 is a limit point of X, since it is the limit of the sequence $(2 + \frac{1}{n})_{n \geq 1}$, but $f(2) = 2$ is not a limit point of $f(A) = [0,1] \cup \{2\}$, since no open neighborhood of 2 of radius smaller than 1 in Y intersects $f(A) \setminus \{2\}$.

18.13 Let $A \subseteq X$; let $f\colon A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g\colon \overline{A} \to Y$, then g is uniquely determined by f.

Solution For $x \in A$, $f(x) = g(x)$.

If $\overline{A} \setminus A = \emptyset$, then $A = \overline{A}$, so g is uniquely determined by f since $f(x) = g(x)$ for all $x \in A = \overline{A}$.

Assume from now on that $\overline{A} \setminus A \neq \emptyset$. Let $y \in \overline{A} \setminus A$. For g to be continuous, we need $g(\overline{A}) \subseteq \overline{g(A)}$, so we need $g(y) \in \overline{g(A)}$. Moreover, since $y \notin A$, $g(y) \notin g(A) \implies g(y) \in \overline{g(A)} \setminus g(A)$. Hence, there exists a sequence $(y_n)_{n \geq 1}$ in $g(A)$ such that $y_n \xrightarrow{n \to \infty} g(y)$. Since each $y_n \in g(A) = f(A)$, this means there exists a sequence $(x_n)_{n \geq 1}$ in A such that $f(x_n) \xrightarrow{n \to \infty} g(y)$.

This limit is unique since Y is Hausdorff. Indeed, suppose there existed some $y' \in \overline{g(A)}$ such that $f(x_n) \xrightarrow{n \to \infty} y' \neq g(y)$. Y Hausdorff $\implies$ there exists $V \in Y$ open such that $g(y) \in V$ but $y' \notin V$. But this means that there is no $N \in \mathbb{N}$ such that $f(x_n)$ is in both a neighborhood of y' and $g(y)$ for $n \geq N$. Hence, $y' = g(y)$.

This means that g is uniquely determined by f since on A, $f(x) = g(x)$, and on $\overline{A} \setminus A$, $g(y)$ is the limit of some sequence $(f(x_n))_{n \geq 1}$.

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1 Let X be the set of nonnegative integers, and consider the topology $\mathcal{T}$ consisting of the empty set and all complements in X of finite subsets of the positive integers. Find all limits of the convergent sequences $(n)_{n \geq 1}$, $(0)_{n \geq 1}$, and $(1)_{n \geq 1}$.

Solution All open sets of X are of the form $X \setminus \{n_1, n_2, \ldots, n_N\}$, where $n_1, n_2, \ldots, n_N > 0$ and N is finite.

$(n)_{n \geq 1}$

This converges to all numbers in X.

Let $m \in X$.

If $m = 0$, then this is clearly true, since 0 belongs to every open set in X.

If $m > 0$, then let U_m be an open neighborhood of m. Then we can write $U_m = X \setminus A$ for some finite $A \subseteq \mathbb{Z}_{>0}$.

Let $N = \max A + 1$. Then for all $n \geq N$, $n \in U_m$, so $(n)_{n \geq 1}$ converges to m.

$(0)_{n \geq 1}$

This converges to all numbers in X, since 0 is in every open set of X. Hence, for any $m \in X$, 0 is in every open neighborhood of m for all $n \geq 1$.

$(1)_{n \geq 1}$

This converges to 1.

Every open neighborhood of 1 contains 1, so the sequence clearly converges to 1.

For any $m \in X \setminus \{1\}$, $X \setminus \{1\}$ itself is an open neighborhood of m. But $1 \notin X \setminus \{1\}$, so it cannot converge to any number other than 1.

2 Let X be the topological space of problem 1. Precisely under what conditions on $f^{-1}(\{n\})$ for $n \geq 0$ must one put on $f\colon X \to X$ for f to be nonconstant and continuous?

Solution For $n \geq 1$, we need $f^{-1}(\{n\})$ to be closed for f to be continuous. This is because the closed sets in X are complements of open sets, which are complements of finite subsets of $X \setminus \{0\}$. Hence, we need $f^{-1}(\{n\})$ to be finite and positive. This also ensures that f is not constant since there are infinitely many values of $m \in X$ such that $f(m) \neq n$.

For $n = 0$, we must have $0 \in f^{-1}(\{0\}) \neq X$ to ensure the above. Otherwise, $0 \in f^{-1}(\{n\})$ for some $n \geq 1$ or if $f^{-1}(\{0\}) = X$, then f is constant.

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3 Suppose that $X \neq \emptyset$ is a topological space that is a disjoint union of nonempty open sets that form a basis for the topology on X. Under exactly what condition on these open sets is X Hausdorff?

Solution For any $x \neq y$, there exist basic open neighborhoods $U \ni x$ and $V \ni y$ such $U \cap V = \emptyset$.