53.5 When the polynomial $z(p)$ has distinct real zeros a and b, so that

$$\frac{1}{z(p)} = \frac{1}{(p-a)(p-b)} = \frac{A}{p-a} + \frac{B}{p-b}$$

for suitable constants A and B, then

$$h(t) = Ae^{at} + Be^{bt}$$

and (15) takes the form

$$y(t) = \int_0^t f(\tau)\left[Ae^{a(t-\tau)} + Be^{b(t-\tau)}\right] \mathrm{d}\tau.$$

This is sometimes called the *Heaviside expansion theorem.*

a. Use this theorem to write the solution of $y'' + 3y' + 2y = f(t)$, $y(0) = y'(0) = 0$.

b. Give an explicit evaluation of the solution in (a) for the cases $f(t) = e^{3t}$ and $f(t) = t$.

c. Find the solutions in (b) by using the superposition principle (13).

Solution a. In this case, we have

$$\frac{1}{z(p)} = \frac{1}{(p+1)(p+2)} = \frac{1}{p+1} - \frac{1}{p+2},$$

so $h(t) = e^{-t} - e^{-2t}$. Then

$$y(t) = \int_0^t f(\tau)\left[e^{-(t-\tau)} - e^{-2(t-\tau)}\right] \mathrm{d}\tau.$$

b. If $f(t) = e^{3t}$, then

$$\begin{aligned} y(t) &= \int_0^t e^{3\tau}\left[e^{-(t-\tau)} - e^{-2(t-\tau)}\right] \mathrm{d}\tau \\ &= \int_0^t e^{-t}e^{4\tau} - e^{-2t}e^{5\tau} \, \mathrm{d}\tau \\ &= \frac{e^{-t}}{4}\left(e^{4t} - 1\right) - \frac{e^{-2t}}{5}\left(e^{5t} - 1\right) \\ &= \frac{e^{3t}}{20} - \frac{e^{-t}}{4} + \frac{e^{-2t}}{5}. \end{aligned}$$

If $f(t) = t$, then

$$\begin{aligned} y(t) &= \int_0^t \tau e^{-t}e^{\tau} - \tau e^{-2t}e^{2\tau} \, \mathrm{d}\tau \\ &= \frac{t}{2} - \frac{e^{-2t}}{4} + e^{-t} - \frac{3}{4}. \end{aligned}$$

c. With the superposition principle, we have

$$\mathcal{L}[A] = \frac{1}{pz(p)} = \frac{1}{p(p+1)(p+2)} = \frac{1}{2p} - \frac{1}{p+1} + \frac{1}{2(p+2)},$$

so

$$A(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t},$$

which gives us

$$y(t) = \int_0^t f(\tau)A'(t-\tau) \, \mathrm{d}\tau.$$

It is known that $h(t) = A'(t)$, so the integrals will evaluate to the same functions.

69.2 Find the exact solution of the initial value problem

$$y' = 2x(1+y), \qquad y(0) = 0.$$

Starting with $y_0(x) = 0$, calculate $y_1(x), y_2(x), y_3(x), y_4(x)$ and compare these results with the exact solution.

Solution By separating variables,

$$\frac{\mathrm{d}y}{1+y} = 2x\,\mathrm{d}x \implies \log(1+y) = x^2 + C \implies y = Ce^{x^2} - 1.$$

Since $y(0) = 0$, we see that $C = 1$, so

$$y(x) = e^{x^2} - 1.$$

We iterate via

$$y_n(x) = \int_0^x 2t(1 + y_{n-1}(t))\,\mathrm{d}t.$$

With $y_0(x) = 0$, we get

$$\begin{aligned}
y_0(x) &= 0\\
y_1(x) &= \int_0^x 2t\,\mathrm{d}t = x^2\\
y_2(x) &= \int_0^x 2t(1+t^2)\,\mathrm{d}t = x^2 + \frac{1}{2}x^4\\
y_3(x) &= \int_0^x 2t\left(1 + t^2 + \frac{1}{2}t^4\right)\mathrm{d}t = x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6\\
y_4(x) &= \int_0^x 2t\left(1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6\right)\mathrm{d}t = x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8.
\end{aligned}$$

Notice that in general,

$$y_n(x) = \sum_{k=1}^{n} \frac{(x^2)^n}{n!} \xrightarrow{n\to\infty} \left(\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}\right) - 1 = e^{x^2} - 1,$$

which is the same as the solution to the initial value problem.

70.1 Let (x_0, y_0) be an arbitrary point in the plane and consider the initial value theorem

$$y' = y^2, \qquad y(x_0) = y_0.$$

Explain why Theorem A guarantees that this problem has a unique solution on some interval $|x - x_0| \le h$. Since $f(x,y) = y^2$ and $\partial f/\partial y = 2y$ are continuous on the entire plane, it is tempting to conclude that this solution is valid for all x. By considering the solutions through the points $(0,0)$ and $(0,1)$, show that this conclusion is sometimes true and sometimes false, and that therefore the inference is not legitimate.

Solution In this problem, $f(x,y) = y^2$, which is continuous and has a continuous y-derivative on $\mathbb{R}^2$. So, there exists an interval $|x - x_0| \le h$ such that the equation has a unique solution that stays in $\mathbb{R}^2$.

However, the theorem does not guarantee that the solution around (x_0, y_0) is the same for all values of x.

Consider $(x_0, y_0) = (0,0)$. Then the solution on an interval around the origin is given by $y(x) \equiv 0$.

On the other hand, if $(x_0, y_0) = (0,1)$, then the solution on an interval about this point is given by $y(x) = 1/(1-x)$, by inspection.

In the first case, the solution is valid on all of $\mathbb{R}$, whereas in the second case, the solution is only valid on $\mathbb{R} - \{1\}$.

70.2 Show that $f(x,y) = y^{1/2}$

a. does not satisfy a Lipschitz condition on the rectangle $|x| \le 1$ and $0 \le y \le 1$

b. does satisfy a Lipschitz condition on the rectangle $|x| \le 1$ and $c \le y \le d$, where $0 < c < d$.

Solution a. Notice that

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}} \xrightarrow{y \to 0} \infty.$$

In other words, the y-derivative is unbounded. If we fix $x \in [-1,1]$, then we can consider f as a differentiable function of y. Then applying the mean value theorem on $(0,y)$, we see that

$$\left|\frac{f(x,y) - f(x,0)}{y}\right| = \left|\frac{1}{2\sqrt{\xi_y}}\right| \qquad \xi_y \in (0,y).$$

But if we take y sufficiently close to 0, we see that the difference quotient becomes unbounded, since we can make ξ_y arbitrarily close to 0, so the function does not satisfy a Lipschitz condition.

b. Notice that since $c > 0$, f_y is continuous on the rectangle $[-1,1] \times [c,d]$, so it is bounded by some $M > 0$. Applying the mean value theorem on the interval (y_1, y_2), we see

$$\left|\frac{f(x,y_2) - f(x,y_1)}{y_2 - y_1}\right| = \left|\frac{1}{2\sqrt{\xi_y}}\right| \le M,$$

so f satisfy a Lipschitz condition in the variable y.

70.3 Show that $f(x,y) = x^2|y|$ satisfies a Lipschitz condition on the rectangle $|x| \le 1$ and $|y| \le 1$, but that $\partial f/\partial y$ fails to exist at many points of this rectangle.

Solution $\partial f/\partial y$ is discontinuous on the line $y = 0$, since the function $|\cdot|$ is not differentiable at 0. However, if we fix y, then applying the mean value theorem on x^2 on (x_1, x_2),

$$\frac{x_2^2|y| - x_1^2|y|}{x_2 - x_1} = |y|\left(\frac{x_2^2 - x_1^2}{x_2 - x_1}\right) = |y||2x| \le 1 \cdot 2 \cdot 1 = 2,$$

so f satisfies a Lipschitz condition in the x variable.

70.5 Show that $f(x,y) = xy$

a. satisfies a Lipschitz condition on any rectangle $a \le x \le b$ and $c \le y \le d$

b. satisfies a Lipschitz condition on any strip $a \le x \le b$ and $-\infty < y < \infty$

c. does not satisfy a Lipschitz condition on the entire plane.

Solution First notice that if we fix $x \in [a,b]$ and let $y_1, y_2 \in [c,d]$,

$$\left|\frac{f(x,y_2) - f(x,y_1)}{y_2 - y_1}\right| = |x|\left|\frac{y_2 - y_1}{y_2 - y_1}\right| = |x| < b.$$

This is independent of y, so we have shown (a) and (b), since f satisfies a Lipschitz condition in the x variable.

If x takes on values from $(-\infty, \infty)$, then we see from the above that f does not satisfy a Lipschitz condition in x.

Similarly, if we fix y and let $x_1, x_2 \in (-\infty, \infty)$, we see that the absolute value of the difference quotient is given by $|y|$, which is unbounded, so f does not satisfy a Lipschitz condition in y either.

So, f does not satisfy a Lipschitz condition, and this shows (c).

70.6 Consider the initial value problem

$$y' = y|y|, \quad y(x_0) = y_0.$$

a. For what points (x_0, y_0) does Theorem A imply that this problem has a unique solution on some interval $|x - x_0| \leq h$?

b. For what points (x_0, y_0) does this problem actually have a unique solution on some interval $|x - x_0| \leq h$?

Solution a. In this problem, $f(x, y) = y|y|$, and

$$\frac{\partial f}{\partial y} = |y| + \frac{y^2}{|y|},$$

which is discontinuous whenever $y = 0$. So the theorem implies that this problem has a unique solution around the points in $\mathbb{R}^2 - \{(x, 0) \mid x \in \mathbb{R}\}$, i.e., the plane cut in half at the x-axis.

b. For any point $(x_0, 0)$ on the x-axis, we have a unique solution: $y(x) \equiv 0$. Indeed, $0 \equiv y' \equiv 0 \cdot |0| \equiv 0$, and $y(x_0) = 0$. So, we actually have a unique solution on an interval around every point in $\mathbb{R}^2$.

70.7 For what points (x_0, y_0) does Theorem A imply that the initial value problem

$$y' = y|y|, \quad y(x_0) = y_0$$

has a unique solution on some interval $|x - x_0| \leq h$?

Solution As we saw from the previous problem, $\mathbb{R}^2 - \{(x, 0) \mid x \in \mathbb{R}\}$.