1 Determine the clamped cubic spline S on $[0,2]$ that interpolates $f(0)=0$, $f(1)=1$, $f(2)=2$, and $f'(0)=f'(2)=2$.

Solution We wish to fit the data with the following function:

$$S(x)=\begin{cases} S_0(x)=a_0+b_0x+c_0x^2+d_0x^3 & \text{if } 0\le x\le 1\\ S_1(x)=a_1+b_1(x-1)+c_1(x-1)^2+d_1(x-1)^3 & \text{if } 1\le x\le 2.\end{cases}$$

We have the following equalities:

$$\begin{aligned} S_0(0)=f(0) &\implies a_0=0\\ S_0'(0)=f'(0) &\implies b_0=2\\ S_1(1)=f(1) &\implies a_1=1\\ S_1'(2)=f'(2) &\implies b_1=2\\ S_0(1)=S_1(1) &\implies 2+c_0+d_0=1\\ S_0'(1)=S_1'(1) &\implies 2+2c_0+3d_0=2\\ S_0''(1)=S_1''(1) &\implies 2c_0+6d_0=2c_1\\ S_1(2)=f(2) &\implies 1+2+c_1+d_1=2 \end{aligned}$$

Put in a matrix, we get the system

$$\begin{pmatrix}1&1&0&0\\2&3&0&0\\2&6&-2&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}c_0\\d_0\\c_1\\d_1\end{pmatrix}=\begin{pmatrix}-1\\0\\0\\-1\end{pmatrix}.$$

This has the unique solution $(c_0,d_0,c_1,d_1)=(-3,2,3,-4)$. This gives the following cubic spline:

$$S(x)=\begin{cases}2x-3x^2+2x^3 & \text{if } 0\le x\le 1\\ 1+2(x-1)+3(x-1)^2-4(x-1)^3 & \text{if } 1\le x\le 2.\end{cases}$$

---

2 a. Use the following data to construct a Hermite interpolating polynomial, and approximate $f(0)$.

x	$f(x)$	$f'(x)$
-0.5	$\frac{29}{16}$	$-\frac{5}{2}$
0.5	$\frac{13}{16}$	$\frac{1}{2}$

b. In fact, the data above is generated using $f(x)=x^4+x^2-x+1$. Derive an error bound for your approximation of $f(0)$.

Solution a. The Hermite polynomials are given by

$$\begin{aligned} H_{n,j}(x)&=(1-2L'_{n,j}(x_j)(x-x_j))L_{n,j}^2(x)\\ \widehat{H}_{n,j}(x)&=(x-x_j)L_{n,j}^2. \end{aligned}$$

The $L_{n,j}$ are given by

$$\begin{aligned} L_{1,0}(x)&=\frac{x-0.5}{-0.5-0.5}=-x+\frac{1}{2}\\ L_{1,1}(x)&=\frac{x+0.5}{0.5+0.5}=x+\frac{1}{2}. \end{aligned}$$

Thus, our Hermite polynomial is given by

$$\begin{aligned}
H_3(x) &= \frac{29}{16}H_{1,0}(x) + \frac{13}{16}H_{1,1}(x) - \frac{5}{2}\widehat{H_{1,0}}(x) + \frac{1}{2}\widehat{H_{1,1}}(x) \\
&= \frac{29}{16}(2x+2)\left(-x+\frac{1}{2}\right)^2 + \frac{13}{16}(2-2x)\left(x+\frac{1}{2}\right)^2 - \frac{5}{2}(x+0.5)\left(-x+\frac{1}{2}\right)^2 + \frac{1}{2}(x-0.5)\left(x+\frac{1}{2}\right)^2
\end{aligned}$$

At $x=0$, we get

$$\frac{29}{16}\cdot 2\cdot\frac{1}{4} + \frac{13}{16}\cdot 2\cdot\frac{1}{4} - \frac{5}{2}\cdot\frac{1}{2}\cdot\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{4} = 0.9375.$$

b. Notice that $|f^{(3)}(x)| = |24x| \le 12$ on $[-0.5, 0.5]$.

Thus, the error bound is given by

$$|f(0) - H_3(0)| = \left|\frac{(0+0.5)^2(0-0.5)^2}{4!}f^{(3)}(\zeta)\right| \le \frac{1}{16\cdot 4!}\cdot 12 = 0.03125.$$

---

3 Given $f \in \mathcal{C}^2([a,b])$ and distinct nodes $a = x_0 < x_1 < \cdots < x_n = b$, define

$$X = \{g \in \mathcal{C}^2([a,b]) \mid g(x_j) = f(x_j),\ j = 0, 1, \ldots, n\}.$$

Let S be the *natural* cubic spline determined by $\{(x_j, f(x_j))\}_{j=0}^n$—obviously, $S \in X$. Prove that for any $h \in X$,

$$\int_a^b |S''(x)|^2 \,\mathrm{d}x \le \int_a^b |h''(x)|^2 \,\mathrm{d}x.$$

Hint: Show that for $h \in X$,

$$\int_a^b S''(x)(h(x) - S(x))'' \,\mathrm{d}x \ge 0.$$

Solution We'll first show the hint.

Notice that $S^{(4)}(x) = 0$ since it's a cubic polynomial. By integration by parts twice and using the fact that $S(a) = h(a)$ and $S(b) = h(b)$,

$$\begin{aligned}
\int_a^b S''(x)(h(x)-S(x))''\,\mathrm{d}x &= S''(x)(h(x)-S(x))'\Big|_a^b - \int_a^b S'''(x)(h(x)-S(x))'\,\mathrm{d}x \\
&= S''(x)(h(x)-S(x))'\Big|_a^b - S'''(x)(h(x)-S(x))\Big|_a^b + \int_a^b S''''(x)(h(x)-S(x))\,\mathrm{d}x \\
&= S''(b)(h'(b)-S'(b)) - S''(a)(h'(a)-S'(a)) \\
&= 0.
\end{aligned}$$

Rearranging, we get

$$\int_a^b S''(x)h''(x)\,\mathrm{d}x = \int_a^b |S''(x)|^2\,\mathrm{d}x.$$

Notice that

$$\begin{aligned}
0 \le \int_a^b (S''(x) - h''(x))^2\,\mathrm{d}x &= \int_a^b |S''(x)|^2 - 2S''(x)h''(x) + |h''(x)|^2\,\mathrm{d}x \\
&= \int_a^b S''(x)(S(x)-h(x))'' - S''(x)h''(x) + |h''(x)|^2\,\mathrm{d}x \\
\implies \int_a^b S''(x)(h(x)-S(x))''\,\mathrm{d}x &+ \int_a^b S''(x)h''(x)\,\mathrm{d}x \le \int_a^b |h''(x)|^2\,\mathrm{d}x \\
\implies \int_a^b |S''(x)|^2\,\mathrm{d}x &\le \int_a^b |h''(x)|^2\,\mathrm{d}x,
\end{aligned}$$

as desired.

4 When performing Hermite interpolation to approximation $f(x)$, we use information of $f(x_j)$ and $f'(x_j)$ at the given nodes x_j, $j = 0, \ldots, n$. Chances are that values of $f'(x)$ are not available at part of the nodes. This motivates us to consider the following problem:

Let $x_j = j$ for $j = 0, 1, 2$. Suppose that we are given the values $f(0)$, $f(1)$, $f(2)$, and $f'(0)$. Find an appropriate polynomial approximation $g(x)$ of f on $[0, 2]$ such that

$$g(j) = f(j), \quad j = 0, 1, 2, \quad \text{and} \quad g'(0) = f'(0).$$

Solution We'll use the Hermite polynomial at 0, and we'll discard the $\widehat{H}$ for the other points.

$$g(x) = f(0)H_{5,0}(x) + f'(0)\widehat{H}_{5,0}(x) + f(1)H_{5,1}(x) + f(2)H_{5,2}(x),$$

where $H_{5,0}$ and $\widehat{H}_{5,0}$ are the Hermite polynomials described in problem 2.

Clearly $f(j) = g(j)$ because all the polynomials are 0 at i except for $H_{5,j}$. We also have $f'(0) = g'(0)$ because $H'_{5,j}$ is 0 at all the i's except for j, and because $\widehat{H}_{5,j}$ is 0 at the i's, unless $i = j$, where it's 1.