6.4.36 If $f \in$ weak L^p and $\mu(\{x \mid f(x) \neq 0\}) < \infty$, then $f \in L^q$ for all $q < p$. On the other hand, if $f \in$ (weak $L^p) \cap L^\infty$, then $f \in L^q$ for all $q > p$.

Solution Let $f \in$ weak L^p and assume $\mu(\{x \mid f(x) \neq 0\}) < \infty$. Set $E = \{x \mid f(x) \neq 0\}$. We then have

$$\begin{aligned} \|f\|_q^q = q\int_0^\infty \alpha^{q-1}\lambda_f(\alpha)\,\mathrm{d}\alpha &= q\int_0^1 \alpha^{q-1}\lambda_f(\alpha)\,\mathrm{d}\alpha + q\int_1^\infty \alpha^{q-p-1}\alpha^p\lambda_f(\alpha)\,\mathrm{d}\alpha \\ &\le \mu(E)q\int_0^1 \alpha^{q-1}\,\mathrm{d}\alpha + q[f]_p^p\int_1^\infty \alpha^{q-p-1}\,\mathrm{d}\alpha \\ &= \mu(E) + \frac{q[f]_p^p}{q-p} < \infty, \end{aligned}$$

since $q-p-1 < -1$ and $q-1 > -1$. Hence, $f \in L^q$ whenever $q < p$.

Now let $f \in$ (weak $L^p) \cap L^\infty$. Then $\lambda_f(\alpha) = 0$ whenever $\alpha > \|f\|_\infty$, so

$$\begin{aligned} \|f\|_q^q = q\int_0^\infty \alpha^{q-1}\lambda_f(\alpha)\,\mathrm{d}\alpha = q\int_0^{\|f\|_\infty} \alpha^{q-p-1}\alpha^p\lambda_f(\alpha)\,\mathrm{d}\alpha &\le q[f]_p^p\int_0^{\|f\|_\infty} \alpha^{q-p-1}\,\mathrm{d}\alpha \\ &= \frac{q[f]_p^p}{q-p}\|f\|_\infty^{q-p} < \infty, \end{aligned}$$

since $q-p-1 > -1$. Hence $f \in L^q$ whenever $q > p$.

6.4.37 Prove Proposition 6.25:

Proposition. *If f is a measurable function and $A > 0$, let $E(A) = \{x \mid |f(x)| > A\}$, and set*

$$h_A = f\chi_{X\setminus E(A)} + A(\operatorname{sgn} f)\chi_{E(A)}, \quad g_A = f - h_A = (\operatorname{sgn} f)(|f| - A)\chi_{E(A)}.$$

Then

$$\lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) & \text{if } \alpha < A, \\ 0 & \text{if } \alpha \ge A, \end{cases} \quad \lambda_{g_A}(\alpha) = \lambda_f(\alpha + A)$$

Solution Notice that $|h_A(x)| = |f(x)|$ when $x \in X \setminus E(A)$ and $|h_A(x)| = A$ otherwise, so h_A is bounded by A. Hence, if $\alpha \ge A$,

$$\{x \mid |h_A(x)| > \alpha\} \subseteq \{x \mid |h_A(x)| > A\} = \emptyset \implies \lambda_{h_A}(\alpha) = 0.$$

If $\alpha < A$,

$$\begin{aligned} \{x \mid |h_A(x)| > \alpha\} &= \{x \mid \alpha < |h_A(x)| \le A\} \\ &= \{x \mid \alpha < |h_A(x)| < A\} \cup \{x \mid |h_A(x)| = A\} \\ &= \{x \mid \alpha < |f(x)| < A\} \cup \{x \mid |f(x)| \ge A\} \\ &= \{x \mid |f(x)| > \alpha\} \\ \implies \lambda_{h_A}(\alpha) = \lambda_f(\alpha). \end{aligned}$$

As for g_A, notice that if $x \in E(A)$, $||f| - A| = |f| - A$, since $|f| > A$. Hence,

$$\{x \mid |g_A(x)| > \alpha\} = \{x \mid |f(x)| - A > \alpha\} = \{x \mid |f(x)| > \alpha + A\},$$

so $\lambda_{g_A}(\alpha) = \lambda_f(\alpha + A)$, as required.

6.4.38 $f \in L^p$ iff $\sum_{-\infty}^{\infty} 2^{kp}\lambda_f(2^k) < \infty$.

Solution Notice that

$$\|f\|_p^p = p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)\,\mathrm{d}\alpha = p\sum_{k=-\infty}^{\infty}\int_{2^k}^{2^{k+1}} \alpha^{p-1}\lambda_f(\alpha)\,\mathrm{d}\alpha.$$

We then have

$$\|f\|_p^p \le p\sum_{k=-\infty}^{\infty}\lambda_f(2^k)\int_{2^k}^{2^{k+1}} \alpha^{p-1}\,\mathrm{d}\alpha = \sum_{k=-\infty}^{\infty}\lambda_f(2^k)\Big(2^{p(k+1)} - 2^{pk}\Big) = (2^p - 1)\sum_{k=-\infty}^{\infty} 2^{pk}\lambda_f(2^k).$$

Similarly,

$$\begin{aligned}\|f\|_p^p \ge p\sum_{k=-\infty}^{\infty}\lambda_f(2^{k+1})\int_{2^k}^{2^{k+1}} \alpha^{p-1}\,\mathrm{d}\alpha = \sum_{k=-\infty}^{\infty}\lambda_f(2^{k+1})\Big(2^{p(k+1)} - 2^{pk}\Big) &= \frac{2^p-1}{2^p}\sum_{k=-\infty}^{\infty} 2^{p(k+1)}\lambda_f(2^{k+1}) \\ &= \frac{2^p-1}{2^p}\sum_{k=-\infty}^{\infty} 2^{pk}\lambda_f(2^k).\end{aligned}$$

In summary, we have

$$\frac{2^p-1}{2^p}\sum_{k=-\infty}^{\infty} 2^{pk}\lambda_f(2^k) \le \|f\|_p^p \le (2^p-1)\sum_{k=-\infty}^{\infty} 2^{pk}\lambda_f(2^k).$$

Thus, $\|f\|_p < \infty \iff \displaystyle\sum_{-\infty}^{\infty} 2^{kp}\lambda_f(2^k) < \infty$, and the claim follows immediately.