32.2 Show that if $\prod X_\alpha$ is Hausdorff, or regular, or normal, then so is X_α. (Assume that each X_α is non-empty).

Solution Define $X \coloneqq \prod X_\alpha$.

Let X be Hausdorff.

Let $x \neq y \in X_\alpha$. Then consider the two points x' and y' which are equal everywhere except the α-th coordinate, where $x'_\alpha = x$ and $y'_\alpha = y$.

Since X is Hausdorff, we can separate x' and y' by U and V open and disjoint. $U_\alpha \cap V_\alpha = \emptyset$ necessarily, since that is the only coordinate where x' and y' differ. Since $x \in U_\alpha$ and $y \in V_\alpha$, this shows that X_α is also Hausdorff.

Let X be regular.

Let $x \in X_\alpha$ and $A \subseteq X_\alpha$ be closed with $x \notin A$. Consider $x' \in X$ with $x'_\alpha = x$, and $A' = A \times \prod_{\beta \neq \alpha} X_\beta$, which is closed by the same argument as below.

Since X is regular, we can separate x' and A' with open disjoint sets U and V. $U_\alpha \cap V_\alpha = \emptyset$ necessarily, since $V_\beta = X_\beta$ for all $\beta \neq \alpha$, which means that $U_\beta \cap V_\beta \neq \emptyset$. Hence, for $U \cap V = \emptyset$, we need $U_\alpha \cap V_\alpha = \emptyset$.

Then $x \in U_\alpha$ and $A \subseteq V_\alpha$, so X is regular.

Let X be normal.

Let A and B be disjoint closed sets in X_α. Then notice that the sets

$$A' = A \times \prod_{\beta \neq \alpha} X_\beta \quad \text{and} \quad B' = B \times \prod_{\beta \neq \alpha} X_\alpha$$

are disjoint, since A and B are disjoint. Moreover, A' and B' are both closed, since

$${}^{\mathrm{c}}A' = {}^{\mathrm{c}}A \times \prod_{\beta \neq \alpha} X_\beta \quad \text{and} \quad {}^{\mathrm{c}}B' = {}^{\mathrm{c}}B \times \prod_{\beta \neq \alpha} X_\beta$$

are basic open sets in the product topology.

Since X is regular, there exist open sets U and V open and disjoint which separate A' and B'. We necessarily have that $U_\alpha \cap V_\alpha = \emptyset$ since every other component U_β and V_β are all of X_β. Then $A \subseteq U_\alpha$ and $B \subseteq V_\alpha$, so X_α is regular.

32.3 Show that every locally compact Hausdorff space is regular.

Solution Let X be a locally compact Hausdorff space, and let $x \in X$ and $A \subseteq X$ be a closed set which doesn't contain x.

Since X is locally compact, there exists $C \subseteq X$ compact and $U_1 \subseteq X$ open such that $x \in U_1 \subseteq C$.

Because X is Hausdorff, C is closed, so ${}^{\mathrm{c}}C$ is open and contains "most" if not all of A. Also note that $U_1 \subseteq C \implies U_1 \cap {}^{\mathrm{c}}C = \emptyset$.

If $A \cap C$ is non-empty, then it is a closed subset of C compact, so it is compact also and doesn't contain x. Since X is Hausdorff, we can separate x and $A \cap C$ by disjoint open sets $U_2 \ni x$ and $W \supseteq A \cap C$.

Thus, if we take $U = U_1 \cap U_2$ and $V = {}^{\mathrm{c}}C \cup W$, then these sets separate x and A.

They are both sets since open sets are closed under finite unions and intersections. Moreover,

$$U \cap V = (U_1 \cap U_2) \cap ({}^{\mathrm{c}}C \cup W) \subseteq (U_1 \cap {}^{\mathrm{c}}C) \cup (U_2 \cap W) = \emptyset,$$

so X is regular.

32.6 A space X is said to be completely normal if every subspace of X is normal. Show that X is completely normal if and only if for every pair A, B of separated sets in X (that is, sets such that $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$), there exist disjoint open sets containing them.

[*Hint:* If X is completely normal, consider $X - (\overline{A} \cap \overline{B})$.]

Solution "$\Longrightarrow$"

Let X be completely normal.

Let A, B be separated sets in X, and consider the subspace $X - (\overline{A} \cap \overline{B}) = X \cap \big({}^{\mathrm{c}}(\overline{A}) \cup {}^{\mathrm{c}}(\overline{B})\big)$. By definition, this subspace is normal. Notice that

$$\overline{A} \cap X \cap \big({}^{\mathrm{c}}(\overline{A}) \cup {}^{\mathrm{c}}(\overline{B})\big) = \overline{A} \cap {}^{\mathrm{c}}(\overline{B}) = \overline{A} - \overline{B}.$$

Similarly, the intersection of $\overline{B}$ with this subspace is $\overline{B} - \overline{A}$. By definition, these two sets are closed in the subspace. Moreover,

$$(\overline{A} - \overline{B}) \cap (\overline{B} - \overline{A}) = \emptyset,$$

so since the subspace $X - (\overline{A} \cap \overline{B})$ is normal, there exist open sets U' and V' in X such that

$$U' \cap V' \cap (X - (\overline{A} \cap \overline{B})) = (U' \cap V') - (\overline{A} \cap \overline{B}) = \emptyset,$$

$\overline{A} - \overline{B} \subseteq U'$, and $\overline{B} - \overline{A} \subseteq V'$.

Thus, we can take $U = U' - \overline{B}$ and $V = V' - \overline{A}$, which are open sets in X, since U' and V' are open in X, and because $\overline{A}$ and $\overline{B}$ are closed in X.

Since A and B are separated, $A \subseteq {}^{\mathrm{c}}(\overline{B})$, so

$$A \cap U = A \cap U' \cap {}^{\mathrm{c}}(\overline{B}) = A \cap {}^{\mathrm{c}}(\overline{B}) = A \implies A \subseteq U.$$

Similarly, we have that $B \subseteq V$. Lastly,

$$U \cap V = U' \cap V' \cap {}^{\mathrm{c}}(\overline{A}) \cap {}^{\mathrm{c}}(\overline{B}) = (U' \cap V') - (\overline{A} \cup \overline{B}) \subseteq (U' \cap V') - (\overline{A} \cap \overline{B}) = \emptyset,$$

so U and V satisfy the conditions of the problem.

"$\Longleftarrow$"

Let $S \subseteq X$ be a subspace of X, and let A, B be closed, disjoint sets in S. By definition, there exist closed sets $C, D \subseteq X$ such that $A = C \cap S$ and $B = D \cap S$.

We claim that A and B are separated.

Suppose they were not. Assume, without loss of generality, that $\overline{A} \cap B \neq \emptyset$, where $\overline{A}$ is the closure of A is X. We can switch A and B and form the same following argument for $A \cap \overline{B} \neq \emptyset$.

Notice that $\overline{A} \subseteq C$, since C is a closed set containing A. But this implies that $\overline{A} \cap B \subseteq C$, which means

$$A \cap B = S \cap C \cap B \supseteq S \cap \overline{A} \cap B = \overline{A} \cap B \neq \emptyset.$$

This is a contradiction, since we assumed A and B to be disjoint. Hence, $\overline{A} \cap B = \emptyset$.

By the same argument, we find that $A \cap \overline{B} = \emptyset$ also, so A and B are separated in X. Hence, by assumption, there exist disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$.

The corresponding disjoint open sets in S are $S \cap U$ and $S \cap V$, which also satisfy $A \subseteq S \cap U$, $B \subseteq S \cap V$, and $S \cap U \cap V = \emptyset$, so S is normal.

Thus, X is completely normal, since S was arbitrary.

33.4 Recall that A is a "G_δ set" in X if A is the intersection of a countable collection of open sets of X.

Theorem. Let X be normal. There exists a continuous function $F\colon X \to [0,1]$ such that $f(x) = 0$ for $x \in A$, and $f(x) > 0$ for $x \notin A$, if and only if A is a closed G_δ set in X.

Solution "$\implies$"

Let $F\colon X \to [0,1]$ be a continuous function such that $f(x) = 0 \iff x \in A$.

Notice that for each $n \in \mathbb{N}$, the set $[0, 1/n)$ is open in $[0,1]$. Hence, since F is continuous, $F^{-1}([0,1/n))$ is open. Then

$$A = F^{-1}(\{0\}) = F^{-1}\left(\bigcap_{n=1}^{\infty}\left[0, \frac{1}{n}\right)\right) = \bigcap_{n=1}^{\infty} F^{-1}\left(\left[0, \frac{1}{n}\right)\right),$$

so A is G_δ.

Moreover $\{0\}$ is closed in $[0,1]$, so since F is continuous, we have $F^{-1}(\{0\}) = A$ is closed also.

"$\impliedby$"

Let A be a closed G_δ set in X. By definition, we can write $A = \bigcap_{n=1}^{\infty} U_n'$, for some open sets $U_n' \subseteq X$.

We can define U_n as follows:

Let $U_1 = U_1'$. By normality of X, for each $n \geq 1$, we can find U_n so that

$$A \subseteq U_n = \bigcap_{i=1}^{n} U_i \subseteq \overline{U_n} \subseteq \bigcap_{i=1}^{n} U_i'.$$

In other words, at each step, we can find $U_n \subseteq U_{n-1}$ such that U_n and its closure lies between A and $\bigcap_{i=1}^{n} U_i'$.

Notice that with this construction, we have

$$A \subseteq \bigcap_{n=1}^{\infty} U_n \subseteq \bigcap_{n=1}^{\infty} U_n' = A,$$

so we have a new representation of A as a G_δ set.

Take $V_{1/n} = U_n$ for all $n \geq 1$. As in the proof of Urysohn's lemma, for each interval $[1/(n+1), 1/n]$, we can find V_q open such that $\overline{V_q} \subseteq V_r$ whenever $1/(n+1) \leq q < r \leq 1/n$. Thus, we have a collection $(V_q)_{q\in\mathbb{Q}}$, which satisfies $A \subseteq V_q$ $\forall q$ and $A \subseteq \overline{V_q} \subseteq V_r$ whenever $q < r$.

Define $F\colon X \to [0,1]$ via $F(x) = \inf\{q \in \mathbb{Q} \cap (0,1] \mid x \in U_q\}$, which is continuous, using the same argument as in Urysohn's lemma. We claim that F satisfies the statement theorem.

Let $x \in A$. Then $x \in V_q$ for every q. In particular, $x \in V_{1/n}$ for every n, so $0 \leq F(x) \leq 1/n$ for all n, which implies that $F(x) = 0$.

If $x \notin A$, then there exists $n \geq 1$ such that $x \notin \overline{V_{1/n}}$, so $F(x) \geq 1/n > 0$.

Thus, F satisfies the conditions of the theorem, as desired.

33.8 Let X be completely regular; let A and B be disjoint closed subsets of X. Show that if A is compact, there is a continuous function $f\colon X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Solution By definition, singleton sets are closed, and for each $x \in X$ and $A \subseteq X$ closed not containing x, there exists a continuous function $f\colon X \to [0,1]$ such that $f(x) = 1$ and $f(A) = \{0\}$.

Since X is completely regular, for each $a \in A$, there exists a continuous function $f_a\colon X \to [0,1]$ such that $f_a(a) = 0$ and $f(B) = \{1\}$.

Define $U_a = f_a^{-1}([0, 1/2))$, which are open sets that cover A, since $f_a(a) = 0$ for all $a \in A$. Since A is compact, there exist $a_1, \ldots, a_n$ such that $U_{a_1}, \ldots, U_{a_n}$ cover A.

Consider $g(x) = \min_{1 \leq i \leq n} f_{a_i}(x)$. By definition of U_a, we have that $0 \leq g(x) < 1/2$ for all $a \in A$. Moreover, g is continuous, since the minimum of a finite number of continuous functions is continuous. Indeed, if for some continuous functions f, g, if we have $f(x) < g(x) \iff f(x) - g(x) < 0$, then since a difference of continuous functions is continuous, $f < g$ on an open neighborhood around x. We can apply this inductively to $f_{a_1}, \ldots, f_{a_n}$ to see that g is continuous also.

Finally, take $f(x) = 2\max\{g(x), 1/2\} - 1$. Notice that on B, g is identically 1, so B, $f(x) = 2 - 1 = 1$. Moreover, on A, $0 \leq g(x) < 1/2$, so $f(x) = 1 - 1 = 0$. As minimums over finitely many continuous functions are continuous, f is continuous, as desired.

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1 Consider the set $H = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$, endowed with the unique topology such that the subspace topology on $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ has the subspace topology on $\mathbb{R}^2$ and for which any $(x, 0)$ has a basis of open neighborhoods of the form $\{(x, 0)\} \cup B((x, \frac{r}{2}), \frac{r}{2})$, where $B((x, y), r)$ is the usual open ball of radius r about (x, y) in $\mathbb{R}^2$. Prove or disprove each of the following:

a. H is regular,

b. H is separable: i.e., it has a countable dense subset,

c. H is Lindelöf: i.e., every open cover of H has a countable subcover,

d. H is normal,

e. H is first-countable,

f. H is locally compact.

Solution a. $\mathring{H}$ is regular since $\mathbb{R}^2$ is regular, so we only need to prove this for when our point or our closed set belongs to $\mathbb{R} \times \{0\}$.

$(a, b) \in \mathring{H}$ and $A \subseteq \mathbb{R} \times \{0\}$:

Take $B((a, b), r)$ such that the ball lies completely in $\mathring{H}$. Then for each $(x, 0) \in A$, we can find a basic open neighborhood whose ball is tangent to our original ball by shrinking the radius. Taking $U = B((a, b), r)$ and V to be the union of those balls, these give us our separation.

$(a, b) \in \mathbb{R} \times \{0\}$ and $A \subseteq \mathring{H}$:

In $\mathbb{R}^2$, we can separate (a, b) and A with open sets U and V. Projecting these sets onto H, we can then find a basic open neighborhood W of (a, b) contained in U by making the radius of the ball sufficiently small. Then $W \cap V \subseteq V \cap W = \emptyset$, so they give us our separation.

$(a, b) \in R \times \{0\}$ and $A \subseteq \mathbb{R} \times \{0\}$:

We can use the same argument as the first case.

Hence, H is regular.

b. Consider $\mathbb{Q}^2 \cap [0, \infty)$, which is countable and dense in H, since $\mathbb{Q}$ is countable and dense in $\mathbb{R}$.

c. Consider the open cover $\{\mathring{H}\} \cup \bigcup_{x \in \mathbb{R}} \{\{(x, 0)\}\} \cup \{B((x, 1), 1)\}$. This has no countable subcover because in order to get the line $\mathbb{R} \times \{0\}$ in any union, we must union over all the real numbers (which are uncountable), since the only open covers of $(x, 0)$ contain no other points on the set $\mathbb{R} \times \{0\}$.

d. H is not normal.

e. Let $(x, y) \in \mathring{H}$. Then we can take the balls $B((x, y), 1/n)$ for all $n \geq 1$ to get a countable neighborhood basis of x, since open balls form a basis in $\mathbb{R}^2$.

Let $(x, 0) \in H$. We can take the sets $\{(x, 0)\} \cup B((x, 1/n), 1/n)$ for all $n \geq 1$. This is a countable neighborhood basis since if $r_1 < r_2$, then $B((x, r_1), r_1) \subseteq B((x, r_2), r_2)$, by the triangle inequality.

In particular, given any open neighborhood U of $(x, 0)$, there exists $r > 0$ such that $\{(x, 0)\} \cup B((x, r), r) \subseteq U$, since these sets form a basis of open neighborhoods for $(x, 0)$. By the Archimedean principle, there exists $n \geq 1$ such that $1/n < r$, so which means $B((x, 1/n), 1/n) \subseteq U$, so $\{(x, 0)\} \cup B((x, 1/n), 1/n)$ is a countable neighborhood basis.

Hence, H is first-countable.

f. Consider the point $(0, 0)$. Suppose H is locally compact, which means that there exists a compact neighborhood C of $(0, 0)$. Then there exists some $r > 0$ such that $\{(0, 0)\} \cup B((0, r), r) \subseteq C$.

Notice that

$$\partial B((0, r/2), r/2) = \overline{B((0, r/2), r/2)} - B((0, r/2), r/2) \subseteq B((0, r), r),$$

so $\partial B((0, r/2), r/2)$ is a closed subset of C compact, which means that $\partial B((0, r/2), r/2)$ is compact also.

Note that the boundary of a ball with a point removed is not compact. Indeed, it is homeomorphic to $\mathbb{R}$ via stereographic projection, which is not compact.

Then consider $\{(0, 0)\} \cup B((0, r/4), r/4)$, which only covers the point $(0, 0)$ on the boundary. The rest of the circle is not compact, so we can find an open cover which does not admit a finite subcover, which shows that $\partial B((0, r/2), r/2)$ is not compact. This is a contradiction, so H is not locally compact.