2.1.8

(a) Use implicit differentiation to show that $t^2 + y^2 = C^2$ implicitly defines solutions of the differential equation $t + yy' = 0$.

(b) Solve $t^2 + y^2 = C^2$ for y in terms of t to provide explicit solutions. Show that these functions are also solutions of $t + yy' = 0$.

(c) Discuss the interval of existent for each of the solutions in part (b).

(d) Sketch the solutions in part (b) for $C = 1, 2, 3, 4$.

Solution

(a)
$$\begin{aligned} t^2 + y^2 &= C^2 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(t^2 + y^2\right) &= \frac{\mathrm{d}}{\mathrm{d}t} C^2 \\ 2t + 2y\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ 2t + 2yy' &= 0 \\ t + yy' &= 0 \end{aligned}$$

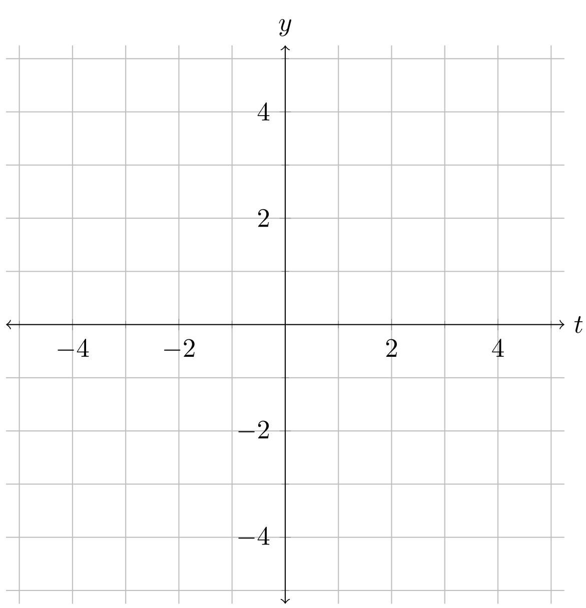
The differential equation is satisfied.

(b) $t^2 + y^2 = C^2$
$$\begin{aligned} y^2 &= C^2 - t^2 \\ y &= \pm\sqrt{C^2 - t^2} \\ y' &= \pm\frac{1}{2\sqrt{C^2 - t^2}}(-2t) = \frac{-2t}{2 \cdot \pm\sqrt{C^2 - t^2}} = -\frac{t}{y} \end{aligned}$$
$$t + yy' = t + y\left(-\frac{t}{y}\right) = t - t = 0 \Rightarrow t + yy' = 0$$

The functions indeed satisfy the differential equation.

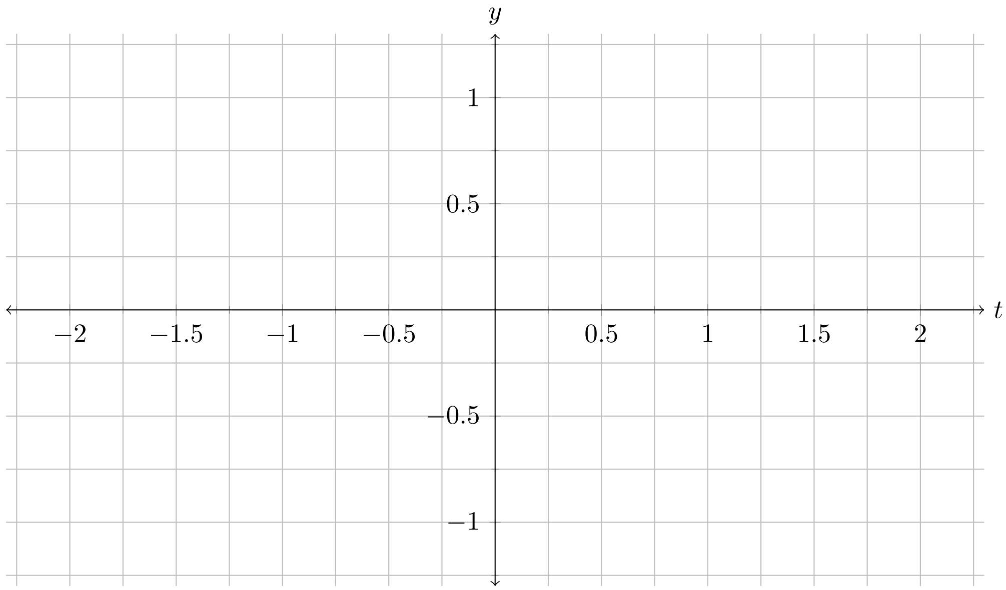
(c) For both functions, y' does not exist when $t = C$ or $t = -C$, and y does not exist for $|t| > C$. So, the interval of convergence for both functions is $(-C, C)$.

(d)



2.1.18 Plot the direction field for $y' = y^2 - t$ where $-2 \leq t \leq 2$ and $-1 \leq y \leq 1$.

Solution



2.2.2 $xy' = 2y$

Solution

$$
\begin{aligned}
xy' &= 2y \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2y}{x} \\
\frac{1}{y}\,\mathrm{d}y &= \frac{2}{x}\,\mathrm{d}x \quad (x, y \neq 0) \\
\int \frac{1}{y}\,\mathrm{d}y &= \int \frac{2}{x}\,\mathrm{d}x \\
\ln|y| &= 2\ln|x| + C_0 \\
|y| &= Cx^2, \text{ where } C = e^{C_0} \text{ and } x \neq 0
\end{aligned}
$$

However, when $x = 0, y = 0$ from the differential equation. So, the solution will pass through the origin. Thus, for the case where $y(x) \neq 0$, we have

$$
y(x) = \begin{cases} Cx^2 & \text{for } y > 0 \\ -Cx^2 & \text{for } y < 0 \end{cases}
$$

Consider the case where $y(x) = 0$. Then $y' = 0$, and the differential equation is obviously satisfied. So, the general solution for the differential equation is

$$
y(x) = Cx^2, C \in \mathbb{R}.
$$

2.2.18 $y' = \dfrac{x}{1+2y}$, $y(-1) = 0$

Solution

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x}{1+2y} \\ (1+2y)\,\mathrm{d}y &= x\,\mathrm{d}x \\ \int 1+2y\,\mathrm{d}y &= \int x\,\mathrm{d}x \\ y + y^2 &= \frac{1}{2}x^2 + C \end{aligned}$$

Solving for C,

$$\begin{aligned} y(-1) = 0 \Rightarrow 0 &= \frac{1}{2} + C \\ C &= -\frac{1}{2} \\ y + y^2 &= \frac{1}{2}x^2 - \frac{1}{2} \end{aligned}$$

Solving for $y(x)$ explicitly,

$$\begin{aligned} y^2 + y - \left(\frac{1}{2}x^2 - \frac{1}{2}\right) &= 0 \\ y_{1/2}(x) &= -\frac{1}{2} \pm \frac{1}{2}\sqrt{2x^2 - 1} \end{aligned}$$

$y = -\frac{1}{2} - \frac{1}{2}\sqrt{2x^2-1}$ does not satisfy the initial condition, so our explicit solution is

$$\boxed{y = -\frac{1}{2} + \frac{1}{2}\sqrt{2x^2-1}}.$$

y' does not exist when $y = -\frac{1}{2}$, which occurs when $x = \pm\frac{\sqrt{2}}{2}$. Therefore, the domain of definition is $\mathbb{R} - \left\{-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\}$. $-1 < -\frac{\sqrt{2}}{2}$, so the interval of existence is $\boxed{\left(-\infty, -\frac{\sqrt{2}}{2}\right)}$.

2.2.24 The half-life of ^{238}U is 4.47×10^7 yr.

(a) Use equation (2.38) to compute the ***decay constant*** λ for ^{238}U.

(b) Suppose that 1000 mg of ^{238}U are present initially. Use the equation $N = N_0 e^{-\lambda t}$ and the decay constant determined in part (a) to determine the time for this sample to decay to 100 mg.

Solution

(a)
$$\begin{aligned} T_{1/2} &= \frac{\ln 2}{\lambda} \\ \lambda &= \frac{\ln 2}{4.47 \times 10^7 \text{ yr}} \approx \boxed{1.55 \times 10^{-8}\,\frac{1}{\text{yr}}} \end{aligned}$$

(b)
$$\begin{aligned} N &= N_0 e^{-\lambda t} \\ 100 \text{ mg} &= (1000 \text{ mg})\, e^{\left(-1.55\times10^{-8}\,1/\text{yr}\right)t} \\ t &= -\frac{1}{1.55 \times 10^{-8}\,\frac{1}{\text{yr}}} \ln\left(\frac{1}{10}\right) \approx \boxed{1.49 \times 10^8 \text{ yr}} \end{aligned}$$

2.2.33 A murder victim is discovered at midnight and the temperature of the body is recorded at 31°C. One hour later, the temperature of the body is 29°C. Assume the surrounding air temperature remains constant at 21°C. Use Newton's law of cooling to calculate the victim's time of death. *Note:* The "normal" temperature of a living human being is approximately 37°C.

Solution Newton's law of cooling tells us that

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k\left(T - A\right),$$

where T is the temperature of the object, t is time, k is a positive constant, and A is the ambient temperature. The equation is separable, and the solution to the differential equation is

$$T\left(t\right) = A + \left(T_0 - A\right)e^{-kt}$$

where T_0 is the initial temperature of our object. For this problem, we have

$$\begin{cases} T\left(t_0\right) = 31°\mathrm{C} \\ T\left(t_0 + 1\right) = 29°\mathrm{C} \\ A = 21°\mathrm{C} \\ T_0 = 37°\mathrm{C} \end{cases}$$

where t_0 is the time between death and midnight. Substituting,

$$\begin{aligned} T\left(t_0\right) &= 31°\mathrm{C} = 21°\mathrm{C} + \left(16°\mathrm{C}\right)e^{-kt_0} \\ T\left(t_0 + 1\right) &= 29°\mathrm{C} = 21°\mathrm{C} + \left(16°\mathrm{C}\right)e^{-k(t_0+1)} \\ \Rightarrow 10°\mathrm{C} &= \left(16°\mathrm{C}\right)e^{-kt_0} \\ 8°\mathrm{C} &= \left(16°\mathrm{C}\right)e^{-kt_0-k} \end{aligned}$$

Dividing the first equation by the second,

$$\frac{10}{8} = e^k \Rightarrow k = \ln\frac{5}{4}$$

Substituting back into the first equation,

$$10°\mathrm{C} = \left(16°\mathrm{C}\right)\left(\frac{4}{5}\right)^{t_0} \Rightarrow t_0 = \frac{\ln\frac{5}{8}}{\ln\frac{4}{5}} \approx 2.11 \text{ hours}$$

The murder happened approximately $\boxed{2.11 \text{ hours}}$ before midnight.

2.2.36 Consider the equation

$$y' = f(at + by + c),$$

where a, b, and c are constants. Show that the substitution $x = at + by + c$ changes the equation to the separable equation $x' = a + bf(x)$. Use this method to find the general solution of the equation $y' = (y + t)^2$.

Solution With $x = at + by + c$, we have

$$\begin{aligned} x' &= a + by' \\ &= a + bf(at + by + c) \\ &= a + bf(x) \end{aligned}$$

which is indeed separable. For $y' = (y + t)^2$, we can use the substitution $x = y + t$, which is the case where $a = 1$ and $b = 1$. We also have $f(u) = u^2$. Then

$$\begin{aligned} x' &= a + bf(x) \\ &= 1 + x^2 \\ \arctan x &= t + C \\ x &= \tan(t + C) \\ t + y &= \tan(t + C) \\ &\boxed{y = \tan(t + C) - t} \end{aligned}$$