1 Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{3, 4\}$. Let $\mathcal{P}(X)$ be the power set of X. Define a relation R on $\mathcal{P}(X)$ by ARB if $A \cup Y = B \cup Y$.

a. Show that R is an equivalence relation.

b. How many distinct equivalence classes are there?

Solution a. Reflexivity:

Clearly $A \cup Y = A \cup Y$, so ARA for any $A \in \mathcal{P}(X)$.

Symmetry:

If ARB, then
$$A \cup Y = B \cup Y = A \cup Y \implies BRA.$$

Transitivity:

If ARB and BRC, then by transitivity of "=",
$$A \cup Y = B \cup Y = C \cup Y \implies ARC.$$

Thus, R is an equivalence relation.

b. We claim that $ARB \iff A \setminus Y = B \setminus Y$.

"$\implies$"

If ARB, then
$$A \cup Y = B \cup Y \implies A \setminus Y = (A \cup Y) \setminus Y = (B \cup Y) \setminus Y = B \setminus Y.$$

"$\impliedby$"

Suppose $A \setminus Y = B \setminus Y$. Then
$$A \cup Y = (A \setminus Y) \cup Y = (B \setminus Y) \cup Y = B \cup Y \implies ARB.$$

Thus, given $A \in \mathcal{P}(X)$, ARB if A and B agree outside of Y. We can identify the equivalence classes with $\mathcal{P}(X \setminus Y) = \mathcal{P}(\{1, 2, 5\})$, since they are
$$[\emptyset], [\{1\}], [\{2\}], [\{5\}], [\{1, 2\}], [\{2, 5\}], [\{1, 5\}], [\{1, 2, 5\}].$$

So there are 8 equivalence classes.

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2 Let $\mathbb{R}^{\mathbb{R}}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Define a relation R on $\mathbb{R}^{\mathbb{R}}$ by fRg if $f(0) = g(0)$. Prove that R is an equivalence relation. Let $f \colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x$ for all $x \in \mathbb{R}$. Describe the set $[f]$.

Solution Reflexivity:

$f(0) = f(0)$, so fRf.

Symmetry:

If fRg, then $f(0) = g(0) = f(0)$, by symmetry of "=", so gRf.

Transitivity:

If fRg and gRh, then by transitivity of "=", $f(0) = g(0) = h(0)$, so fRh.

Since $f(0) = 0$, $[f] = \{g \in \mathbb{R}^{\mathbb{R}} \mid g(0) = 0\}$, i.e., all functions whose graph contains the origin.

3 Let A and X be sets. For any equivalence relation R, let $A/R = \{[a] \mid a \in A\}$ be the set of equivalence classes of A.

a. Let $f\colon A \to X$ be a function. Define a relation R on A by xRy if $f(x) = f(y)$. Show that R is an equivalence relation on A.

b. Let R be an equivalence relation on A. Define a function $p\colon A \to A/R$ by $p(x) = [x]$. When do we have $p(x) = p(y)$?

c. Let R be an equivalence relation on A. Suppose that f is a function $f\colon A \to X$ having the property that if xRy then $f(x) = f(y)$. Show that $h([x]) = f(x)$ defines a function $h\colon A/R \to X$.

d. Let $f\colon A \to X$ be any function, and let R be the equivalence relation from part (a). Let $p\colon A \to A/R$ and $h\colon A/R \to X$ be the functions from parts (b) and (c). Prove that $f = h \circ p$. Also prove that p is onto and h is one-to-one.

Solution a. Reflexivity:

If $x \in A$, then $f(x) = f(x)$, so xRx.

Symmetry:

If xRy, then $f(x) = f(y) = f(x)$, so yRx.

Transitivity:

If xRy and yRz, then $f(x) = f(y) = f(z)$, so xRz.

Hence, R is an equivalence relation on A.

b. We have $p(x) = p(y)$ if and only if $[x] = [y]$ if and only if xRy.

c. We need to show that if $[x] = [y]$, then $h([x]) = h([y])$. Since $[x] = [y] \iff xRy \implies f(x) = f(y)$, we have

$$h([x]) = f(x) = f(y) = h([y]).$$

Also, every element of A/R has an image under h, so h is a function.

d. Let $x \in A$. Then $h(p(x)) = h([x]) = f(x)$. Since x was arbitrary, it follows that $f = h \circ p$.

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4 How many different car license plates can be constructed if the license contains three letters followed by two digits if repetitions are allowed? If repetitions are not allowed?

Solution We have 26^3 ways to choose the three letters and 10^2 ways to choose the digits, so there are $26^3 \cdot 10^2$ licenses, if repetitions are allowed.

If repetitions are not allowed, then we have $26 \cdot 25 \cdot 24$ ways to choose the three letters and $10 \cdot 9$ ways to choose the numbers. Hence, there are $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9$ licenses.

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5 Strings of length 5 are formed using the letters $ABCDEFG$ without repetition. How many strings do not begin with either AB (in that order) or D?

Solution We break this up into three cases:

Strings starting with A:

The second letter cannot be B, so we have 5 choices for the second letter. For the last three letters, we don't have any restrictions other than no repetitions, so we have $5 \cdot 4 \cdot 3$ ways to choose the remaining letters. In this case, there are $5 \cdot 5 \cdot 4 \cdot 3 = 300$ strings.

Strings starting with D:

In this case, we have no restrictions on the rest of the letters, so there are $6 \cdot 5 \cdot 4 \cdot 3 = 360$ strings.

Strings not starting with A or D:

We have 5 choices for our first letter, and no restrictions on the rest, so we have $5 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 1800$ strings.

In total, we have $300 + 360 + 1800 = 2460$ strings.

6 Let X be an n-element set. How many antisymmetric relations are there on X?

Solution Let R be an antisymmetric relation.

If $x \neq y$, then one of following three must hold:

i. $(x, y) \in R$ but $(y, x) \notin R$

ii. $(x, y) \notin R$ but $(y, x) \in R$

iii. $(x, y) \notin R$ and $(y, x) \notin R$

So for each unordered pair x, y with $x \neq y$, we have three choices to make.

If $x = y$, then either $(x, x) \in R$ or $(x, x) \notin R$, so we have two choices to make.

There are $C(n, 2)$ pairs $(x, y) \in X \times X$ with $x \neq y$, and there are n pairs $(x, x) \in X \times X$, so we have

$$3^{C(n,2)} \cdot 2^n$$

antisymmetric relations on X.

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7 Recall the inclusion-exclusion principle for two finite sets: $|X \cup Y| = |X| + |Y| - |X \cap Y|$.

a. Prove the inclusion-exclusion principle for three finite sets:

$$|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| + |X \cap Y \cap Z|$$

b. Among a group of 165 students:

i. 79 are taking calculus

ii. 83 are taking psychology

iii. 63 are taking computer science

iv. 33 are taking both calculus and computer science

v. 20 are taking both calculus and psychology

vi. 24 are taking both psychology and computers science

vii. 8 are taking all three of calculus, psychology, and computer science.

Use the inclusion-exclusion principle for three sets to find the number of students taking none of the three subjects.

Solution a. Applying the inclusion-exclusion principle on $X \cup (Y \cup Z)$, we have

$$\begin{aligned} |X \cup (Y \cup Z)| &= |X| + |Y \cup Z| - |X \cap (Y \cup Z)| \\ &= |X| + |Y| + |Z| - |Y \cap Z| - |(X \cap Y) \cup (X \cap Z)| \\ &= |X| + |Y| + |Z| - |Y \cap Z| - (|X \cap Y| + |X \cap Z| - |X \cap Y \cap Z|) \\ &= |X| + |Y| + |Z| - |X \cap Y| - |Y \cap Z| - |X \cap Z| + |X \cap Y \cap Z|, \end{aligned}$$

as required.

b. Let X be the students taking calculus, Y be the students taking psychology, and Z be the students taking computer science. The total number of students taking a class is

$$\begin{aligned} |X \cup Y \cup Z| &= |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| + |X \cap Y \cap Z| \\ &= 79 + 83 + 63 - 20 - 33 - 24 + 8 = 156. \end{aligned}$$

Thus, the number of students taking none of these is $165 - 156 = 9$.

8 Determine how many strings (of length 5) can be formed by ordering the letters $ABCDE$ subject to the conditions given:

a. Contains the letters ACE together in any order.

b. A appears before C and C appears before E.

Solution a. If we treat ACE as a single character, we have

$$\begin{aligned}(\#\text{ with } ACE \text{ in any order}) &= (\#\text{ with } ACE \text{ in that order}) \cdot (\#\text{ ways to permute } ACE) \\ &= 3! \cdot 3! = 36.\end{aligned}$$

b. We can count this by first placing down ACE in that order, and figuring out where to place the rest of the letters. There are 4 places to place the B, and then there are 5 places to place the D, which gives 20 strings total.

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9 In how many ways can we select a committee of four from a group of 12 persons?

Solution There are

$$C(12,4) = \frac{12!}{4! \cdot 8!} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} = 11 \cdot 5 \cdot 9 = 495$$

ways.

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10 Find the number of (unordered) five-card poker hands that can be selected from an ordinary 52-card deck, having the properties indicated:

a. Containing four-of-a-kind, that is, four cards of the same denomination.

b. Containing all spades.

c. Containing a five-card straight, i.e., five consecutive cards (assume that ace is the lowest denomination).

Solution a. We first pick a denomination from the 13. We only have 1 way to get all four cards, and for the remaining card, we have 48 choices, which gives $13 \cdot 1 \cdot 48 = 624$.

b. There are 13 spades total, so there are $C(13,5) = 1287$ hands.

c. We can only form straights with lower denominations ace, 2, ..., 9. Starting at 10, there aren't enough denominations to form a straight.

For each of these denominations, there are 4 choices for each of the 5 cards; 1 for each suit. Thus, there are

$$9 \cdot 4^5 = 9216$$

straights (including straight flushes).

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11 Show that the number of n-bit strings of 0's and 1's having exactly k 0's with no two 0's consecutive is $C(n-k+1,k)$.

Solution We start by placing the k 0's down, and then placing a 1 between each pair of 0's. This fixes $k+(k-1) = 2k-1$ of our bits. For the remaining $n-2k+1$ 1's, we can place them at the end or in between any of the zeros, so the rest of the problem is equivalent to finding the number of ways to separate $n-2k+1$ 1's into $k+1$ groups, which is

$$\binom{(n-2k+1)+(k+1)-1}{(k+1)-1} = \binom{n-k+1}{k},$$

as required.