48.3 Find $\mathcal{L}\left[\sin^2 ax\right]$ and $\mathcal{L}\left[\cos^2 ax\right]$ without integrating. How are these two transforms related to one another?

Solution We can employ the following identity:

$$\sin^2 ax = \frac{1 - \cos 2ax}{2}.$$

Then the problem becomes finding

$$\mathcal{L}\left[\frac{1 - \cos 2ax}{2}\right] = \mathcal{L}\left[\frac{1}{2}\right] - \frac{1}{2}\,\mathcal{L}[\cos 2ax] = \frac{1}{2p} - \frac{p}{2(p^2 + 4a^2)}.$$

We can also employ the Pythagorean theorem to see that

$$\mathcal{L}\left[\cos^2 ax\right] = \mathcal{L}\left[1 - \sin^2 ax\right] = \mathcal{L}[1] - \mathcal{L}\left[\sin^2 ax\right] = \frac{1}{p} - \left[\frac{1}{2p} - \frac{p}{2(p^2 + 4a^2)}\right] = \frac{1}{2p} + \frac{p}{2(p^2 + 4a^2)}.$$

48.4 Use the formulas given in the text to find the transform of each of the following functions:

(a) 10

(c) $2e^{3x} - \sin 5x$

(e) $x^6 \sin^2 3x + x^6 \cos^2 3x$

Solution (a) $\mathcal{L}[10] = 10\,\mathcal{L}[1] = \dfrac{10}{p}$.

(c) $\mathcal{L}\left[2e^{3x} - \sin 5x\right] = 2\,\mathcal{L}\left[e^{3x}\right] - \mathcal{L}[\sin 5x] = \dfrac{2}{p-3} - \dfrac{5}{p^2 + 25}$.

(e) $\mathcal{L}\left[x^6 \sin^2 3x + x^6 \cos^2 3x\right] = \mathcal{L}\left[x^6\right] = \dfrac{6!}{p^7}$.

48.5 Find a function $f(x)$ whose transform is

(a) $\dfrac{30}{p^4}$

(c) $\dfrac{4}{p^3} + \dfrac{6}{p^2 + 4}$

(e) $\dfrac{1}{p^4 + p^2}$

Solution (a) $\dfrac{30}{p^4} = 5 \cdot \dfrac{3!}{p^4} = 5\,\mathcal{L}\left[x^3\right] \implies f(x) = 5x^3$.

(b) $\dfrac{4}{p^3} + \dfrac{6}{p^2 + 4} = 2 \cdot \dfrac{2!}{p^3} + 3 \cdot \dfrac{2}{p^2 + 2^2} = 2\,\mathcal{L}\left[x^2\right] + 3\,\mathcal{L}[\sin 2x] \implies f(x) = 2x^2 + 3\sin 2x$.

(c) $\dfrac{1}{p^4 + p^2} = \dfrac{1}{p^2} - \dfrac{1}{p^2 + 1} = \mathcal{L}[x] - \mathcal{L}[\sin x] \implies f(x) = x - \sin x$.

49.1 If f denotes the integral in (4), then (s being a dummy variable) we can write

$$I^2 = \left(\int_0^\infty e^{-x^2}\,\mathrm{d}x\right)\left(\int_0^\infty e^{-y^2}\,\mathrm{d}y\right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y.$$

Evaluate this double integral by changing to polar coordinates, and thereby show that $I = \sqrt{\pi}/2$.

Solution The Jacobian determinant of the transformation $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$ is given by

$$\det\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix} = r,$$

and the quarter-plane in Cartesian coordinates $[0,\infty]\times[0,\infty]$ maps to the polar square $[0,\infty]\times\left[0,\frac{\pi}{2}\right]$, so by the change of variables formula,

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\pi/2}\int_0^\infty re^{-r^2}\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi/2}\frac{1}{2}\,\mathrm{d}\theta = \frac{\pi}{4}.$$

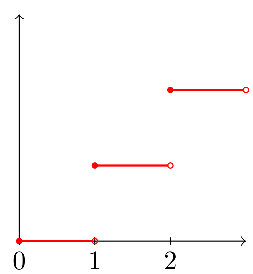
Hence, $I = \sqrt{\pi}/2$.

49.2 In each of the following cases, graph the function and find its Laplace transform:

(b) $f(x) = [x]$ where $[x]$ denotes the greatest integer $\le x$

(d) $f(x) = \begin{cases}\sin x & \text{if } 0 \le x \le \pi \\ 0 & \text{if } x > \pi.\end{cases}$

Solution (b)



Note that we can write

$$[x] = \sum_{n=0}^{\infty} n\chi_{[n,n+1)}(x),$$

where

$$\chi_E(x) = \begin{cases}0 & \text{if } x \in E \\ 1 & \text{if } x \notin E.\end{cases}$$

By linearity of the Laplace transformation and the fact that points have measure 0, we can express $\mathcal{L}[f]$ as the following:

$$\begin{aligned}\mathcal{L}[f](s) &= \sum_{n=0}^{\infty} \mathcal{L}\big[n\chi_{[n,n+1)}\big](s)\\ &= \sum_{n=0}^{\infty} \int_0^{\infty} e^{-sx} n\chi_{[n,n+1)}(x)\,\mathrm{d}x\\ &= \sum_{n=0}^{\infty} \int_n^{n+1} e^{-sx} n\,\mathrm{d}x\\ &= \sum_{n=0}^{\infty} -\frac{n}{s}\Big(e^{-s(n+1)} - e^{-sn}\Big)\\ &= -\frac{1}{s}\sum_{n=1}^{\infty} n\Big(e^{-s(n+1)} - e^{-sn}\Big)\end{aligned}$$

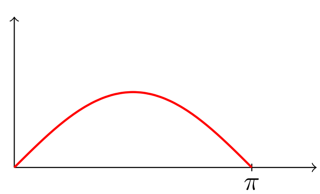
Note that the sum telescopes so that the partial sum is equal to

$$\begin{aligned}\sigma_n(s) &= e^{-2s} - e^{-s} + 2e^{-3s} - 2e^{-2s} + 3e^{-4s} - 3e^{-3s} + \cdots + (n-1)e^{-ns} - (n-1)e^{-(n-1)s} + ne^{-(n+1)s} - ne^{-ns}\\ &= -e^{-s} + (e^{-2s} - 2e^{-2s}) + (2e^{-3s} - 3e^{-3s}) + \cdots + ((n-1)e^{-ns} - ne^{-ns}) + ne^{-(n+1)s}\\ &= -e^{-s} - e^{-2s} - e^{-3s} - \cdots - e^{-ns} + ne^{-(n+1)s}\\ &= -e^{-s}\frac{1-e^{-ns}}{1-e^{-s}} + ne^{-(n+1)s}.\end{aligned}$$

Thus, taking the limit as $n \to \infty$, we get

$$\mathcal{L}[f](s) = -\frac{1}{s}\lim_{n\to\infty}\sigma_n(s) = \frac{1}{s(1-e^{-s})}e^{-s} = \frac{1}{s(e^s - 1)}.$$

(d)



By definition,

$$\mathcal{L}[f](s) = \int_0^{\pi} e^{-sx}\sin x\,\mathrm{d}x.$$

By integration by parts twice, we eventually get that

$$\mathcal{L}[f](s) = \int_0^{\pi} e^{-sx}\sin x\,\mathrm{d}x = \frac{e^{-\pi s} + 1}{s^2 + 1}.$$

49.4 Show explicitly that $\mathcal{L}\big[x^{-1}\big]$ does not exist.

Solution By definition,

$$\mathcal{L}\big[x^{-1}\big](s) = \int_0^\infty e^{-st}\frac{1}{t}\,\mathrm{d}t,$$

if the integral converges. We'll show that it does not converge. Note that

$$\int_0^\infty e^{-st}\frac{1}{t}\,\mathrm{d}t = \int_0^1 e^{-st}\frac{1}{t}\,\mathrm{d}t + \int_1^\infty e^{-st}\frac{1}{t}\,\mathrm{d}t.$$

On $[0,1]$, $e^{-st} \geq e^{-s}$ for all s. Then

$$\int_0^1 \frac{e^{-st}}{t}\,\mathrm{d}t \geq e^{-s}\int_0^1 \frac{1}{t}\,\mathrm{d}t = \infty,$$

so the integral must diverge. Hence, $\mathcal{L}\big[x^{-1}\big]$ does not exist.

49.5 Let ε be a positive number and consider the function $f_\varepsilon(x)$ defined by

$$f_\varepsilon(x) = \begin{cases} 1/\varepsilon & \text{if } 0 \leq x \leq \varepsilon \\ 0 & \text{if } x > \varepsilon. \end{cases}$$

It is clear that for every $\varepsilon > 0$ we have $\int_0^\infty f_\varepsilon(x)\,\mathrm{d}x = 1$. Show that

$$\mathcal{L}[f_\varepsilon(x)] = \frac{1 - e^{-p\varepsilon}}{p\varepsilon}$$

and

$$\lim_{\varepsilon\to 0}\mathcal{L}[f_\varepsilon(x)] = 1.$$

Strictly speaking, $\lim_{\varepsilon\to 0} f_\varepsilon(x)$ does not exist as a function, so $\mathcal{L}\big[\lim_{\varepsilon\to 0 f_\varepsilon(x)}\big]$ is not defined; but if we throw caution to the winds, then

$$\delta(x) = \lim_{\varepsilon\to 0} f_\varepsilon(x)$$

is seen to be some kind of quasi-function that is infinite at $x = 0$ and zero for $x > 0$ and has the properties

$$\int_0^\infty \delta(x)\,\mathrm{d}x = 1 \quad \text{and} \quad \mathcal{L}[\delta(x)] = 1.$$

This quasi-function $\delta(x)$ is called the *Direc delta function* or *unit impulse function.*

Solution By definition,

$$\mathcal{L}[f_\varepsilon(x)](s) = \int_0^\infty e^{-sx} f_\varepsilon(x)\,\mathrm{d}x = \int_0^\varepsilon \frac{e^{-sx}}{\varepsilon}\,\mathrm{d}x = -\frac{1}{s\varepsilon}\big(e^{-s\varepsilon} - 1\big) = \frac{1 - e^{-s\varepsilon}}{s\varepsilon}.$$

Then by the definition of the derivative,

$$\lim_{\varepsilon\to 0}\mathcal{L}[f_\varepsilon(x)] = \lim_{\varepsilon\to 0} -\frac{e^{-s\varepsilon} - 1}{s\varepsilon} = \frac{\mathrm{d}}{\mathrm{d}t} -e^{-t}\Big|_{t=0} = 1.$$

50.1 Find the Laplace transforms of

(a) $x^5 e^{-2x}$

(c) $e^{3x} \cos 2x$

Solution (a) $\mathcal{L}\left[x^5 e^{-2x}\right](s) = \mathcal{L}\left[x^5\right](s+2) = \dfrac{5!}{(s+2)^6}$.

(c) $\mathcal{L}\left[e^{3x} \cos 2x\right](s) = \mathcal{L}[\cos 2x](s-3) = \dfrac{s-3}{(s-3)^2+4}$.

50.2 Find the inverse Laplace transforms of

(a) $\dfrac{6}{(p+2)^2+9}$

(c) $\dfrac{p+3}{p^2+2p+5}$

Solution (a) $\dfrac{6}{(p+2)^2+9} = 2 \cdot \dfrac{3}{(p+2)^2+3^2} = 2\,\mathcal{L}\left[e^{-2x} \sin 3x\right](s)$, so its inverse Laplace transform is $2e^{-2x} \sin 3x$.

(b) $\dfrac{p+3}{p^2+2p+5} = \dfrac{p+1}{(p+1)^2+2^2} + \dfrac{2}{(p+1)^2+2^2} = \mathcal{L}[e^{-x} \cos 2x](s) + \mathcal{L}[e^{-x} \sin 2x](s)$, so the inverse Laplace transform is $e^{-x}(\cos 2x + \sin 2x)$.