1 Let $C^1([0,1])$ be the vector space of continuous functions $f\colon [0,1] \to \mathbb{R}$ such that (i) f is continuously differentiable on $(0,1)$ and (ii) f' tends to well-defined finite limits at 0 and 1. Let ℓ be a linear functional on $C^1([0,1])$ for which there exists $C > 0$ such that

$$|\ell(f)| \leq C(\|f\|_u + \|f'\|_u) \quad \forall f \in C^1([0,1]),$$

where, as usual, $\|\cdot\|$ denotes the uniform norm.

Prove that there exist a finite signed Borel measure μ on $[0,1]$ and a real constant a such that

$$\ell(f) = \int f' \,\mathrm{d}\mu + af(0) \quad \forall f \in C^1([0,1]).$$

Solution By assumption, ℓ is bounded, so we may regard it as a bounded linear functional on $C([0,1])$. Thus, by Riesz representation, there exists a finite signed Borel measure ν so that

$$\ell(f) = \int f \,\mathrm{d}\nu,$$

for any $f \in C^1([0,1])$.

By the fundamental theorem of calculus, we have

$$f(x) = \int_0^x f'(t)\,\mathrm{d}t + f(0).$$

Integrating with respect to ν on both sides, we get

$$\ell(f) = \int_{[0,1]} \int_0^x f'(t)\,\mathrm{d}t\,\mathrm{d}\nu(x) + \nu([0,1])f(0).$$

Since ν is a finite measure, $a \coloneqq \nu([0,1])$ is a real number. Moreover,

$$g \mapsto \int_{[0,1]} \int_0^x g(t)\,\mathrm{d}t\,\mathrm{d}\nu(x)$$

is a bounded linear functional, since g is continuous (hence bounded), and because ν and the Lebesgue measure are finite on the interval $[0,1]$. Hence again by Riesz representation, there exists a finite signed Borel measure μ so that the functional can be written as

$$g \mapsto \int_{[0,1]} g\,\mathrm{d}\mu(x).$$

Thus,

$$\ell(f) = \int_{[0,1]} f'\,\mathrm{d}\mu(x) + af(0),$$

as required.

2 Let $\varphi\colon X \to Y$ be a continuous surjection from one compact metric space to another.

a. Prove that the composition map

$$\Phi\colon f \mapsto f \circ \varphi$$

is a linear and isometric operator from $C(Y, \mathbb{R})$ to a subspace of $C(X, \mathbb{R})$.

b. Let $\nu \in P(Y)$. Prove that there exists $\mu \in P(X)$ such that

$$\int f \,\mathrm{d}\nu = \int f \circ \varphi \,\mathrm{d}\mu \quad \forall f \in C(Y, \mathbb{R}).$$

Solution a. Let $f, g \in C(Y, \mathbb{R})$ and $\lambda \in \mathbb{R}$. Then

$$\Phi(f + \lambda g) = (f + \lambda g) \circ \varphi = f \circ \varphi + \lambda(g \circ \varphi) = \Phi(f) + \lambda\Phi(g),$$

so Φ is linear. This also shows that its image is a subspace of $C(X, \mathbb{R})$.
We now need to show that it's an isometry between itself and its image.
It's clear that

$$\|f \circ \varphi\|_u \leq \|f\|_u.$$

On the other hand, because X is compact, f attains its sup at $y \in Y$. Since φ is surjective, there exists $x \in X$ so that $\varphi(x) = y$, so that

$$|(f \circ \varphi)(x)| = |f(y)|,$$

so Φ is an isometry.

b. Since Φ is an isometry, it is an isomorphism between $C(Y, \mathbb{R})$ and its image. If we write $\ell_\nu(f) = \int f \,\mathrm{d}\nu$, then $\ell_\nu \circ \Phi^{-1}$ is a positive linear functional on a subspace of $C(X, \mathbb{R})$, since ν is positive. Hence, by Riesz-Markov, there exists a finite positive Borel measure μ so that $\int \cdot \,\mathrm{d}\mu = \ell_\nu \circ \Phi^{-1}$. Applying this to $g \in C(X, \mathbb{R})$, we have

$$\int g \,\mathrm{d}\mu = \int \Phi^{-1}(g) \,\mathrm{d}\nu.$$

If we set $\Phi^{-1}(g) = f$, we get

$$\int \Phi(f) \,\mathrm{d}\mu = \int f \circ \varphi \,\mathrm{d}\mu = \int f \,\mathrm{d}\nu.$$

Lastly, we need to show that $\mu \in P(X)$. In the above equation, set $f = \chi_Y$. Then

$$1 = \int \,\mathrm{d}\nu = \int \,\mathrm{d}\mu,$$

so μ is a probability measure.

3 Let X be a compact metric space. Prove that a sequence $\{f_n\}$ in $C(X, \mathbb{R})$ converges weakly to $f \in C(X, \mathbb{R})$ if and only if

$$\sup_n \|f_n\|_u < \infty \quad \text{and} \quad f_n \to f \text{ pointwise.}$$

Solution "$\implies$"

Let $\{f_n\}$ converge to f weakly.

By the Riesz representation theorem, X being a compact metric space implies that $C(X, \mathbb{R})^* = M(X, \mathbb{R})$, so weak convergence implies that for any $\mu \in M(X, \mathbb{R})$, we have

$$\int f_n \, \mathrm{d}\mu \xrightarrow{n \to \infty} \int f \, \mathrm{d}\mu.$$

It is easy to see pointwise convergence by using the Dirac delta measures δ_x.

Consider the isometric embedding of f_n into the double dual, $\hat{f}_n$. Then we have for every finite signed measure μ that $\{\hat{f}_n(\mu)\}$ is bounded, since

$$\hat{f}_n(\mu) = \int f_n \, \mathrm{d}\mu \xrightarrow{n \to \infty} \int f \, \mathrm{d}\mu < \infty.$$

Moreover, $C(X, \mathbb{R})$ and its double dual are both Banach spaces, so by the uniform boundedness principle and the fact that the embedding is an isometry,

$$\sup_n \|f_n\| = \sup_n \|\hat{f}_n\| < \infty,$$

as required.

"$\impliedby$"

We wish to show that $\{f_n\}$ converges to f weakly. By Riesz representation, we can just show that $\int f_n \, \mathrm{d}\mu \to \int f \, \mathrm{d}\mu$ for all measures μ.

Fix a finite measure μ. Let $M = \sup_n \|f_n\|_u$, which is finite. Because μ is finite, $M\chi_X$ is in $L^1(\mu)$. Moreover, $|f_n| \leq M\chi_X$ for every $x \in X$, and $f_n \to f$ pointwise, so by dominated convergence,

$$\int f_n \, \mathrm{d}\mu \xrightarrow{n \to \infty} \int f \, \mathrm{d}\mu.$$

Since μ was arbitrary, it follows that $f_n \to f$ weakly.

4 Let X be a compact metric space, and let $\mu \in P(X)$ have the property that $\mu(U) > 0$ for every nonempty open set (e.g., Lebesgue measure on $[0,1]$).

Prove that for every $x \in X$, there is a sequence $\{f_n\}$ in $L^1(\mu)$ such that the measures

$$\mathrm{d}\mu_n \coloneqq f_n \,\mathrm{d}\mu$$

are all members of $P(X)$ and converge weakly* to δ_x.

Solution Consider the functions

$$f_n = \frac{1}{\mu(B(x,1/n))}\chi_{\mu(B(x,1/n))}.$$

These functions are well-defined since nonempty open sets have positive measure.

For $E \in \mathcal{M}$, we can write

$$\mu_n(E) = \frac{1}{\mu(B(x,1/n))}\int_{E\cap B(x,1/n)} \mathrm{d}\mu = \frac{\mu(E\cap B(x,1/n))}{\mu(B(x,1/n))}.$$

It follows that E is a probability measure, since μ is positive.

We wish to show that for every $f \in C_0(X)$, we have

$$\int f\,\mathrm{d}\mu_n \xrightarrow{n\to\infty} \int f\,\mathrm{d}\delta_x = f(x).$$

This is a result of the Riesz representation theorem, as all bounded linear functionals arise from integration from a measure.

Calculation gives us

$$\int f\,\mathrm{d}\mu_n = \int f f_n\,\mathrm{d}\mu = \frac{1}{\mu(B(x,1/n))}\int_{B(x,1/n)} f\,\mathrm{d}\mu.$$

Since f is continuous, the Lebesgue differentiation theorem holds, so the limit is $f(x)$, as required.

Folland Exercises

6.9 Suppose $1 \le p < \infty$. If $\|f_n - f\|_p \to 0$, then $f_n \to f$ in measure, and hence some subsequence converges to f a.e. On the other hand, if $f_n \to f$ in measure and $|f_n| \le g \in L^p$ for all n, then $\|f_n - f\|_p \to 0$.

Solution Let $\varepsilon > 0$, and consider the set $E_n \coloneqq \{x \mid |f_n(x) - f(x)| \ge \varepsilon\}$. Then

$$\varepsilon^p\mu(E_n) \le \int_{E_n}|f_n - f|^p \le \int |f_n - f|^p \implies \mu(E_n) \le \frac{1}{\varepsilon^p}\int|f_n - f|^p \xrightarrow{n\to\infty} 0.$$

Since ε was arbitrary, it follows that $f_n \to f$ in measure. By a previous theorem, this implies that some subsequence converges to f a.e.

Now suppose $f_n \to f$ in measure and $|f_n| \le g \in L^p$ for each $n \ge 1$. By a previous theorem, there exists $\{f_{n_k}\}$ which converges pointwise to f a.e., i.e., $|f_{n_k}(x) - f(x)| \xrightarrow{k\to\infty} 0$ for a.e. x. Moreover, $|f_{n_k} - f| \le 2g \in L^p$, so by dominated convergence,

$$\|f_{n_k} - f\|_p^p = \int |f_{n_k} - f|^p \xrightarrow{k\to\infty} 0.$$

Now suppose that $\|f_n - f\|_p \not\to 0$. Then there is $\varepsilon > 0$ and a subsequence $\{f_{n_\ell}\}$ so that $\|f_{n_\ell} - f\|_p \ge \varepsilon$ for all $\ell \ge 1$. But because $f_n \to f$ in measure, so does f_{n_ℓ}.

By the same argument as above, there is a further subsequence $\{f_{n_{\ell_m}}\}$ of $\{f_{n_\ell}\}$ so that $\|f_{n_{\ell_m}} - f\|_p \to 0$. But this is impossible, as $\|f_{n_\ell} - f\|_p \ge \varepsilon > 0$. Hence, no such subsequence existed in the first place, so $\|f_n - f\|_p \to 0$, as required.

6.13 $L^p(\mathbb{R}^n, m)$ is separable for $1 \leq p < \infty$. However, $L^\infty(\mathbb{R}^n, m)$ is not separable.

Solution Recall that integrable simple functions are dense in L^p, for $1 \leq p < \infty$, so it suffices to find a countable family of functions which can approximate them in L^p.

We may replace the coefficients of simple functions with rational numbers and preserve density. So, we just need to be able to approximate characteristic functions by a countable collection of characteristic functions.

By regularity of the Lebesgue measure, a measurable set E differs from a G_δ set by a set of measure zero, so we can approximate χ_E by a decreasing sequence of functions χ_{U_n}, where $U_1 \supseteq U_2 \supseteq \cdots$ are open sets. Hence, it suffices to be able to approximate χ_U, where U is open, by a countable family of characteristic functions. But this is easy because $\mathbb{R}^n$ is second countable, so we can write open sets as a countable union (from below) of open balls of rational radii and rational centers. Thus, if $\mathcal{B}$ is a countable base for $\mathbb{R}^n$, we have that

$$\left\{ \sum_{i=1}^{N} q_i \chi_{B_i} \mid N \in \mathbb{N},\ q_1, \ldots, q_N \in \mathbb{Q},\ B_1, \ldots, B_N \in \mathcal{B} \right\}$$

is dense and countable in $L^p(\mathbb{R}^n, m)$, so it is separable.

As for $L^\infty(\mathbb{R}^n, m)$, associate for each $r \in (0, \infty)$ the function

$$f_r \coloneqq \chi_{[0,r]^n}.$$

Then if $r < s$, $|f_s - f_r|$ is the characteristic function on a square with a strictly smaller square removed, which has L^∞ norm 1. If $\mathcal{F}$ is the collection of all of these functions, then $\mathcal{F}$ is uncountable, and the L^∞ norm between any two different ones is 1.

Now suppose that $L^\infty(\mathbb{R}^n, m)$ is separable, and let C be a countable dense subset. Then for each $f \in \mathcal{F}$, there exists $\varphi \in C$ so that $\|f - \varphi\|_\infty < 1/2$. Then if $g \in \mathcal{F}$ is different from f,

$$\|f - g\|_\infty \leq \|f - \varphi\|_\infty + \|g - \varphi\|_\infty \implies \|g - \varphi\|_\infty \geq \|f - g\|_\infty - \|f - \varphi\|_\infty > \frac{1}{2}.$$

Thus, such a φ can only be close to exactly one f. But this is impossible, as there are uncountably many f, whereas there are only countably many φ. Hence, no such C exists, so $L^\infty(\mathbb{R}^n, m)$ is not separable.