1 Solve the recurrence relation $a_n = \sqrt{\dfrac{a_{n-2}}{a_{n-1}}}$ with initial conditions $a_0 = 8$, $a_1 = \dfrac{1}{2\sqrt{2}}$.

Solution We follow the hint, and let $b_n = \log_2(a_n)$. By taking logarithms on both sides, the relation then becomes

$$b_n = -\frac{1}{2}b_{n-1} + \frac{1}{2}b_{n-2},$$

with initial conditions $b_0 = 3$ and $b_1 = -3/2$.

The characteristic polynomial is $2x^2 + x - 1 = (2x-1)(x+1)$, so we will look for solutions of the form $b_n = a(1/2)^n + b(-1)^n$. By inspection, $a = 1$ and $b = 2$. Thus,

$$a_n = 2^{b_n} = 2^{(1/2)^n + 2(-1)^n}.$$

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2 Find general solutions for the recurrence relations:

a. $a_n = 7a_{n-1} - 10a_{n-2} + 16n$

b. $a_n = 2a_{n-1} + 8a_{n-2} + 81n^2$

c. $2a_n = 7a_{n-1} - 3a_{n-2} + 2^n$

Solution a. We first look for a particular solution of the form $an + b$:

$$an + b = 7a(n-1) + 7b - 10a(n-2) - 10b + 16n \implies an + b = (16 - 3a)n - 3b - 7a + 20a.$$

Equating coefficients, we find that $a = 4$ and $b = 13$.

We now turn our attention to the homogeneous equation $a_n = 7a_{n-1} - 10a_{n-2}$. The characteristic polynomial is $x^2 - 7x + 10 = (x-2)(x-5)$, so the general solution is

$$a_n = a \cdot 2^n + b \cdot 5^n + 4n + 13.$$

b. We look for a solution of the form $an^2 + bn + c$. Expanding and equating coefficients yield

$$a = -9, \quad b = -36, \quad c = -38.$$

For the associated homogeneous equation, the characteristic polynomial is $x^2 - 2x - 8 = (x-4)(x+2)$. So, the general solution is

$$a_n = a \cdot 4^n + b \cdot (-2)^n - 9n^2 - 36n - 38.$$

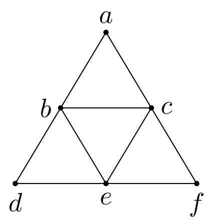
c. We try solutions of the form $a \cdot 2^n$:

$$a \cdot 2^{n+1} = 7a \cdot 2^{n-1} - 3a \cdot 2^{n-2} + 2^n \implies 8a = 14a - 3a + 4 \implies a = -\frac{4}{3}.$$

The characteristic polynomial here is $2x^2 - 7x + 3 = (2x-1)(x-3)$, so the general solution is

$$a_n = a \cdot \left(\frac{1}{2}\right)^n + b \cdot 3^n - \frac{1}{3} \cdot 2^{n+2}.$$

3 Show that the graph has a path from a to a that passes through each edge exactly one time, by finding such a path by inspection:



Solution Here is my path: (a, b), (b, c), (c, e), (e, b), (b, d), (d, e), (e, f), (f, c), (c, a).

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4 a. Find a formula for the number of edges of K_n.

b. Find a formula for the number of edges of $K_{m,n}$.

Solution a. There are $C(n, 2)$ distinct, unordered pairs of points, and we have an edge between each pair of points, so there are

$$C(n, 2) = \frac{n!}{(n-2)! \cdot 2!}$$

edges.

b. For each of the m nodes on our "left" subgraph, we have n edges; one for each of the n nodes on the "right" subgraph. So, by the multiplication principle, there are mn total edges.

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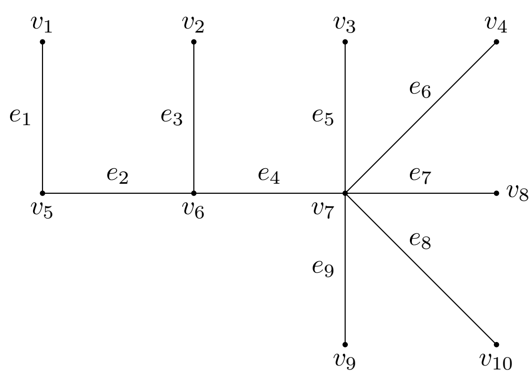
5 Is this graph bipartite:



If so, specify the distinct vertex sets.

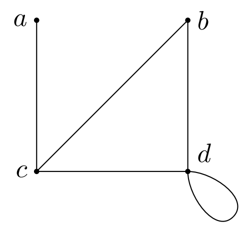
Solution The graph is bipartite with sets $V_1 = \{v_1, v_3, v_4, v_6, v_8, v_9, v_{10}\}$ and $V_2 = \{v_2, v_5, v_7\}$

6 Draw a graph having the given properties, or explain why no such graph exists:

a. Four vertices having degrees $1, 2, 3, 4$.

b. Simple graph; five vertices having degrees $2, 2, 4, 4, 4$.

Solution a. Here is the graph:



$\delta(a) = 1$, $\delta(b) = 2$, $\delta(c) = 3$, and $\delta(d) = 4$.

b. Since the graph is simple, for a vertex to have degree 4, it must be connected to every other vertex in the graph. There are 3 of these vertices, which means that every vertex must have degree at least 3, so this graph is not possible.

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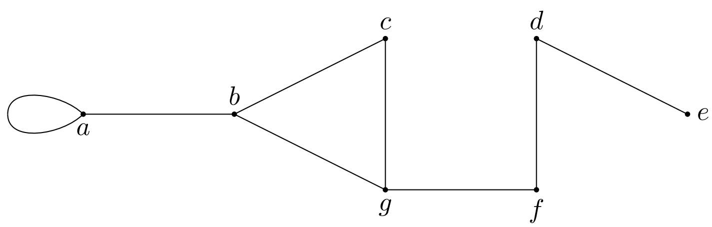
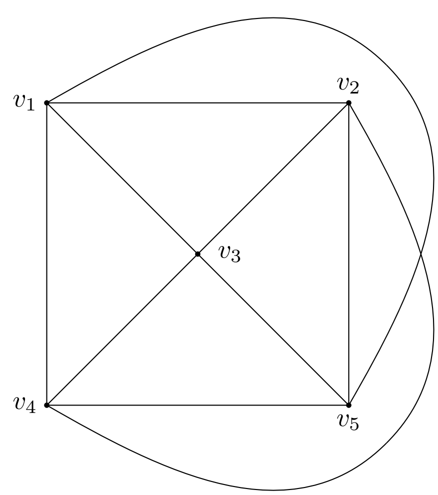
7 Decide whether the graph has an Euler cycle. If the graph has an Euler cycle, exhibit one.

a.



b.



Solution a. This does not have an Euler cycle, since a has degree 3, which is not even.

b. This does have an Euler cycle, and one is (v_1, v_2), (v_2, v_3), (v_3, v_1), (v_1, v_4), (v_4, v_3), (v_3, v_5), (v_5, v_4), (v_4, v_2), (v_2, v_5), (v_5, v_1).

8 a. When does the complete graph K_n contain an Euler cycle?

b. When does the complete bipartite graph $K_{m,n}$ contain an Euler cycle?

Solution a. A graph has an Euler cycle if and only if it is connected and the degree of every vertex is even. Each vertex has $n-1$ edges connected to it, so K_n has an Euler cycle if and only if $n-1$ is even, i.e., if and only if n is odd.

b. For each of the m vertices in one subgraph, there needs to be an even number of edges connected to it, so n needs to be even. Similarly, m must be even also, so $K_{m,n}$ has an Euler cycle if both m and n are even.

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9 Show that the maximum number of edges in a simple, bipartite graph with n vertices is $\lfloor n^2/4 \rfloor$.

Solution A complete bipartite graph is of the form $K_{k,n-k}$, where $1 \le k \le n$. From a previous problem, there are $k(n-k)$ edges, which is a discrete parabola.

Ignoring the fact that k and n are integers for a second, this function is maximized when $k = n/2$, since it's a parabola. So, if n is even, then the maximum number of edges is $(n/2)(n/2) = n^2/4$.

If n is odd, then we can remove one vertex, which means that the maximum for $n-1$ edges is $(n-1)^2/4$. We can add the remaining vertex to "either side" of the bipartite graph and get the same maximum, by symmetry. Doing so adds $(n-1)/2$ more edges, so

$$\frac{(n-1)^2}{4} + \frac{n-1}{2} = \frac{n^2-1}{4} = \frac{n^2}{4} - \frac{1}{4} = \left\lfloor \frac{n^2}{4} \right\rfloor,$$

where the last equality holds because $n^2/4$ and $(n^2-1)/4$ differ by $1/4$, which will disappear when we take the floor.

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10 Let v and w be distinct vertices in K_n. Let p_m denote the number of paths of length m from v to w in K_n.

a. Derive a recurrence relation for p_m.

b. Find an explicit formula for p_m.

Solution a. Suppose we have a path between v and w. If we remove the last edge in our path, then there are two cases:

The previous vertex was v:

If we remove another vertex, then we have a path from v to a vertex with length $m-2$. So, there $n-1$ vertices we could be at, and p_{m-2} paths from v to it. Then there is exactly one path to get to v, and exactly one more path to get to w, so there are $(n-1)p_{m-2}$ paths here.

The previous vertex was not v:

Then there are $n-2$ different vertices we could be at, and for each of these, there are p_{m-1} paths to get from v to one of them, and then exactly one path to get from that vertex to w. So, there are $(n-2)p_{m-1}$ paths in this case.

So, our recurrence relation is $p_m = (n-2)p_{m-1} + (n-1)p_{m-2}$.

For our initial conditions, if $m = 1$, then $p_1 = 1$, since there's only one edge possible. If $m = 2$, then $p_2 = n-2$, since there are $n-2$ vertices which are not v or w, and then exactly one edge from that vertex to w.

b. The characteristic polynomial is $x^2 - (n-2)x - (n-1) = (x-(n-1))(x+1)$, so we need to look for a solution of the form $p_m = a \cdot (n-1)^m + b \cdot (-1)^m$.

By some calculation, we find that $a = 1/n$ and $b = -1/n$, so

$$p_m = \frac{(n-1)^m - (-1)^m}{n}.$$