2.6.10 $(2x + y)\,\mathrm{d}x + (x - 6y)\,\mathrm{d}y = 0$

Solution We check if the partial derivatives are equal:

$$\frac{\partial(2x+y)}{\partial y} = 1, \qquad \frac{\partial(x-6y)}{\partial x} = 1$$

The partial derivatives are equal, so the equation is exact. Solving,

$$\begin{aligned} F(x,y) &= \int 2x + y\,\mathrm{d}x = x^2 + xy + \phi(y) \\ \frac{\partial F}{\partial y} &= x + \phi'(y) = x - 6y \\ &\Rightarrow \phi(y) = -3y^2 \end{aligned}$$

Thus, the solutions to the differential equation is implicitly given by

$$\boxed{F(x,y) = x^2 + xy - 3y^2 = C}$$

2.6.26 Suppose that $y\,\mathrm{d}x + \left(x^2 y - x\right)\mathrm{d}y = 0$ has an integrating factor that is a function of x alone (i.e., $\mu = \mu(x)$). Find the integrating factor and use it to solve the differential equation.

Solution If μ is an integrating factor of the differential equation, then $\mu y\,\mathrm{d}x + \mu\left(x^2y - x\right)\mathrm{d}y = 0$ is exact. That means

$$\begin{aligned} \frac{\partial(\mu y)}{\partial y} &= \frac{\partial\left(\mu\left(x^2y - x\right)\right)}{\partial x} \\ \mu &= \frac{\mathrm{d}\mu}{\mathrm{d}x}\left(x^2y - x\right) + \mu(2xy - 1) \\ \frac{\mathrm{d}\mu}{\mathrm{d}x} &= \mu\frac{(2-2xy)}{x(xy-1)} \\ \frac{\mathrm{d}\mu}{\mathrm{d}x} &= -\mu\frac{2}{x} \\ \mu &= e^{-\int 2/x\,\mathrm{d}x} = \frac{1}{x^2} \end{aligned}$$

Substituting,

$$\begin{aligned} \frac{y}{x^2}\,\mathrm{d}x + \frac{1}{x^2}\left(x^2y - x\right)\mathrm{d}y &= 0 \\ \frac{y}{x^2}\,\mathrm{d}x + \left(y - \frac{1}{x}\right)\mathrm{d}y &= 0 \\ F(x,y) &= \int y - \frac{1}{x}\,\mathrm{d}y \\ &= \frac{1}{2}y^2 - \frac{y}{x} + \phi(x) \\ \frac{\partial F}{\partial x} = \frac{y}{x^2} + \phi'(x) &= \frac{y}{x^2} \\ \Rightarrow \phi(x) &= 0 \end{aligned}$$

Thus, the solution to the differential equation is implicitly given by

$$\boxed{F(x,y) = \frac{y^2}{2} - \frac{y}{x} = C}$$

2.6.30 Consider the differential equation $2y\,\mathrm{d}x + 3x\,\mathrm{d}y = 0$. Determine conditions on a and b so that $\mu(x,y) = x^a y^b$ is an integrating factor. Find a particular integrating factor and use it to solve the differential equation.

Solution If μ is an integrating factor, then the differential equation $2y\mu\,\mathrm{d}x + 3x\mu\,\mathrm{d}y = 2x^a y^{b+1}\,\mathrm{d}x + 3x^{a+1}y^b\,\mathrm{d}y = 0$ is exact. So,

$$\begin{aligned}\frac{\partial\left(2x^a y^{b+1}\right)}{\partial y} &= \frac{\partial\left(3x^{a+1}y^b\right)}{\partial x}\\ 2(b+1)x^a y^b &= 3(a+1)x^a y^b\\ \Rightarrow a &= \frac{1}{3}(2b-1)\end{aligned}$$

A particular integrating factor with $a = 1$ and $b = 2$ is $\mu = xy^2$. Substituting,

$$\begin{aligned}2xy^3\,\mathrm{d}x + 3x^2y^2\,\mathrm{d}y &= 0\\ F(x,y) &= \int 2xy^3\,\mathrm{d}x = x^2y^3 + \phi(y)\\ \frac{\partial F}{\partial y} &= 3x^2y^3 + \phi'(y) = 3x^2y^2\\ &\Rightarrow \phi = 0\end{aligned}$$

Thus, the implicitly defined solution to the differential equation is

$$\boxed{F(x,y) = x^2y^3 = C}$$

2.6.32 $\left(x^2 - xy + y^2\right)\mathrm{d}x + 4xy\,\mathrm{d}y = 0$

Solution

$$\begin{aligned}P(x,y) &= x^2 - xy + y^2\\ P(tx,ty) &= t^2x^2 - t^2xy + t^2y^2 = t^2\left(x^2 - xy + y^2\right) = t^2P(x,y)\\ Q(x,y) &= 4xy\\ Q(tx,ty) &= 4t^2xy = t^2(4xy) = t^2Q(x,y)\end{aligned}$$

The degree of P and Q is 2.

2.6.36

$$(x+y)\,\mathrm{d}x + (y-x)\,\mathrm{d}y = 0$$

Solution By inspection, the equation is homogeneous with degree 2. Letting $y = xv$, we get

$$\begin{aligned}\mathrm{d}y &= x\,\mathrm{d}v + v\,\mathrm{d}x\\ (x + xv)\,\mathrm{d}x + (xv - x)(x\,\mathrm{d}v + v\,\mathrm{d}x) &= 0\\ (1+v)\,\mathrm{d}x + (v-1)(x\,\mathrm{d}v + v\,\mathrm{d}x) &= 0\\ \left(v^2+1\right)\mathrm{d}x + x(v-1)\,\mathrm{d}v &= 0\end{aligned}$$

Dividing through by $x\left(v^2+1\right)$,

$$\frac{1}{x}\,\mathrm{d}x + \frac{v-1}{v^2+1}\,\mathrm{d}v = 0$$

Solving for the solution implicitly,

$$\begin{aligned}F(x,v) &= \int \frac{1}{x}\,\mathrm{d}x + \int \frac{v}{v^2+1} - \frac{1}{v^2+1}\,\mathrm{d}v\\ &\ln|x| + \frac{1}{2}\ln\left|v^2+1\right| - \arctan v = C\end{aligned}$$

$$\boxed{F(x,y) = \ln|x| + \frac{1}{2}\ln\left(\frac{y^2}{x^2} + 1\right) - \arctan\frac{y}{x} = C}$$

2.6.44 $x\,\mathrm{d}x + y\,\mathrm{d}y = y^2\big(x^2+y^2\big)\,\mathrm{d}y$
Hint: Consider $\mathrm{d}\big(\ln\big(x^2+y^2\big)\big)$.

Solution The total differential for $\ln\big(x^2+y^2\big)$ is

$$\mathrm{d}\big(\ln\big(x^2+y^2\big)\big) = \frac{2x}{x^2+y^2}\,\mathrm{d}x + \frac{2y}{x^2+y^2}\,\mathrm{d}y.$$

If we divide our original equation through by x^2+y^2 and multiply by 2, we get

$$\begin{aligned}\frac{2x}{x^2+y^2}\,\mathrm{d}x + \frac{2y}{x^2+y^2}\,\mathrm{d}y &= 2y^2\,\mathrm{d}y\\ \mathrm{d}\big(\ln\big(x^2+y^2\big)\big) &= 2y^2\,\mathrm{d}y\\ \int \mathrm{d}\big(\ln\big(x^2+y^2\big)\big) &= \int 2y^2\,\mathrm{d}y\\ \boxed{\ln\big(x^2+y^2\big) = \frac{2}{3}y^3 + C}\end{aligned}$$