4.2.3 Map the complement of the arc $|z| = 1$, $y \geq 0$ on the outside of the unit circle so that the points at ∞ correspond to each other.

Solution First consider the map

$$f_1(z) = \alpha \frac{z+1}{z-1},$$

where α is chosen so that $f_1(i) \in \mathbb{R}$. This maps the upper-half unit circle to the non-negative real line.

The consider $f_2(z) = \sqrt{z}$, defined so that its imaginary part is positive. This is holomorphic since the complement of the non-negative real axis is simply connected and doesn't contain 0, and this maps the complement of the non-negative real axis to the upper-half plane. Then take the mapping

$$f_3(z) = \frac{z+i}{z-i},$$

which maps the upper-half plane to the outside of the unit disk.

The last step is to ensure that ∞ is mapped to ∞. Suppose that after these mappings, ∞ lands on the point β. Then take the map

$$f_4(z) = \frac{z - \frac{1}{\overline{\beta}}}{1 - \frac{z}{\beta}},$$

which is an automorphism on the disk, i.e., it fixes the unit disk and the outside of the unit disk. Moreover, this maps β to ∞, so we are done if we use $f_4 \circ f_3 \circ f_2 \circ f_1$.

4.2.4 Map the outside of the parabola $y^2 = 2px$ on the disk $|w| < 1$ so that $z = 0$ and $z = -p/2$ correspond to $w = 1$ and $w = 0$.

Solution The map

$$\varphi(w) = \frac{p}{2} - w^2$$

maps the line $\operatorname{Re} w = \sqrt{p/2}$ to the given parabola. Then its inverse is

$$f_1(z) = \sqrt{\frac{p}{2} - z},$$

where we define the square root so that we get the branch with the positive real part, and this maps the outside of the parabola to the half-plane $\{z \mid \operatorname{Re} z > \sqrt{p/2}\}$. We shift it with $f_2(z) = z - \sqrt{p/2}$ so that we get the right-half plane. Notice that

$$(f_2 \circ f_1)(0) = 0 \quad \text{and} \quad (f_2 \circ f_1)\left(-\frac{p}{2}\right) = \sqrt{p} - \sqrt{\frac{p}{2}}.$$

Finally, take the Möbius transformation

$$f_3(z) = -\frac{z - (\sqrt{p} - \sqrt{p/2})}{z + (\sqrt{p} - \sqrt{p/2})},$$

which maps the right-half plane to the unit disk. It sends $\sqrt{p} - \sqrt{p/2}$ to 0 and 0 to 1, as desired. Thus, our map is $f_3 \circ f_2 \circ f_1$.

4.2.5 Map the inside of the right-hand branch of the hyperbola $x^2 - y^2 = a^2$ on the disk $|w| < 1$ so that the focus corresponds to $w = 0$ and the vertex to $w = -1$.

Solution Consider the map

$$w = f_1(z) = z^2.$$

If we write $w = u + iv$ and $z = x + iy$, we get

$$u = x^2 - y^2 \quad \text{and} \quad v = 2y\sqrt{a^2 + y^2}.$$

In particular, the hyperbola is sent to $u = a^2$, and $2y\sqrt{a^2 + y^2}$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$. Moreover, the focus $\sqrt{2}a$ is sent to $2a^2$, which means that f_1 sends the hyperbola to the line $u = a^2$ and the inside of the hyperbola to the region $\operatorname{Re} w > a^2$.

We shift with $f_2(z) = z - a^2$, and take the Möbius transformation

$$f_3(z) = \frac{z - a^2}{z + a^2},$$

which maps the right-half plane to the unit disk.

Lastly, we need to check that our points are mapped properly:

$$(f_3 \circ f_2 \circ f_1)(a) = f_3(0) = -1$$
$$(f_3 \circ f_2 \circ f_1)(a) = f_3(\sqrt{2}a) = f_3(a^2) = 0,$$

so we're done.

4.2.7 Map the outside of the ellipse $(x/a)^2 + (y/b)^2 = 1$ onto $|w| < 1$ with preservation of symmetries.

Solution Take $c(z + 1/z)$. If we write $z = e^{i\theta}$, we get that the result $x + iy$ can be written

$$x = c\left(r + \frac{1}{r}\right)\cos\theta \quad \text{and} \quad y = c\left(r - \frac{1}{r}\right)\sin\theta \implies \frac{x^2}{c^2(r + r^{-1})^2} + \frac{x^2}{c^2(r - r^{-1})^2} = 1.$$

If

$$c = \sqrt{\frac{a^2 - b^2}{2}} \quad \text{and} \quad r = \frac{1}{\sqrt{2}}\sqrt{\left|\frac{a + b}{a - b}\right|},$$

then the ellipse is the image of the circle of that radius. The map takes 0 to ∞, so the interior of that circle is mapped to the outside of the ellipse. Thus, the inverse, which we'll call $f_1(z)$, takes the outside of the ellipse into the circle of radius r. Take a dilation $f_2(z) = 1/r$, and $f_2 \circ f_1$ is the map we want.

4.2.8 Map the part of the z-plane to the left of the right-hand branch of the hyperbola $x^2 - y^2 = 1$ on a half plane.

Solution Notice that $z = w^2$ maps $\{z \mid \operatorname{Re} z > 1\}$ to the right-side of the hyperbola. Thus, it takes $\{z \mid \operatorname{Re} z < 1\}$ to the left side. Call its inverse f_1.

Next, shift the image of f_1 via $f_2(z) = z - 1$ so that we get the right-half plane. $f_2 \circ f_1$ is the desired mapping.

6.4.5 Show that the mean-value formula remains valid for $u = \log|1+z|$, $z_0 = 0$, $r = 1$, and use this fact to compute

$$\int_0^\pi \log \sin\theta \,\mathrm{d}\theta.$$

Solution By a theorem, we know that

$$\int_{|z|=r} \log|1+z| \,\mathrm{d}z = \alpha \log r + \beta,$$

for some constants α and β.

For $0 < r < 1$, $u(z)$ is harmonic, so the mean-value property holds on any closed ball contained in the unit disk, i.e., for any $0 < r < 1$,

$$\alpha \log r + \beta = \frac{1}{2\pi} \int_{|z|=r} \log|1+z| \,\mathrm{d}z = \log 1 = 0.$$

Thus, $\alpha = \beta = 0$, so as $r \to 1$, we see that the mean value property holds for $r = 1$.

Notice that for $z \in \{z \mid |z+1| = 1\}$, we can write $z = -1 + e^{i\theta}$ which gives

$$|z|^2 = 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta = 2 - 2\cos\theta.$$

Thus,

$$\operatorname{Re} \log z = \log|z| = \log\sqrt{2 - 2\cos\theta},$$

and using the double angle identity gives

$$\begin{aligned}
\int_0^\pi \log \sin\theta \,\mathrm{d}\theta &= \int_0^\pi \log \frac{1}{2}\sqrt{2 - 2\cos 2\theta} \,\mathrm{d}\theta \\
&= -\pi \log 2 + \int_0^\pi \log\sqrt{2 - 2\cos 2\theta} \,\mathrm{d}\theta \\
&= -\pi \log 2 + \frac{1}{2}\int_0^{2\pi} \log\sqrt{2 - 2\cos\theta} \,\mathrm{d}\theta \\
&= -\pi \log 2 + \operatorname{Re}\left[\frac{1}{2}\int_0^{2\pi} \log|1+z| \,\mathrm{d}\theta\right] \\
&= -\pi \log 2 + \operatorname{Re} \log 1 \\
&= -\pi \log 2.
\end{aligned}$$

6.4.6 If $f(z)$ is analytic in the whole plane and if $z^{-1} \operatorname{Re} f(z) \to 0$ when $z \to \infty$, show that f is a constant.

Solution Let $u = \operatorname{Re} f$ so that $u(z)/z \xrightarrow{z\to\infty} 0$, and pick an appropriate C so that

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{\mathrm{d}\zeta}{\zeta} + iC.$$

Let $0 < r < R$. Then for $|z| = r$, Harnack's principle gives us

$$|f(z)| \leq \frac{R+r}{R-r} u(0) + C \xrightarrow{R\to\infty} u(0) + C,$$

so by Liouville's theorem, f is constant.

6.5.3 If $f(z)$ is analytic in $|z| \leq 1$ and satisfies $|f| = 1$ on $|z| = 1$, show that $f(z)$ is rational.

Solution Consider

$$B(z) = \prod \frac{z - z_j}{1 - \overline{z_j} z},$$

which has modulus 1 when $|z| = 1$ and has the same zeroes as f. It follows that B/f and f/B are analytic on the open disk and continuous on the closed disk. By applying the maximum modulus principle on both functions, we see that $|f/B| = 1$, which implies that f is a constant multiple of B, i.e., f is rational, as desired.

6.5.4 Use

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{\mathrm{d}\zeta}{\zeta} + iC$$

to derive a formula for $f'(z)$ in terms of $u(z)$.

Solution Let z_0 with $|z_0| < R$. Then

$$f(z) - f(z_0) = \frac{1}{2\pi i} \int_{|\zeta|=R} \left(\frac{\zeta + z}{\zeta - z} - \frac{\zeta - z_0}{\zeta - z_0} \right) \frac{u(\zeta)}{\zeta} \, \mathrm{d}\zeta = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{2(z - z_0)}{(\zeta - z)(\zeta - z_0)} u(\zeta) \, \mathrm{d}\zeta.$$

Thus,

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{2u(\zeta)}{(\zeta - z)(\zeta - z_0)} \, \mathrm{d}\zeta \xrightarrow{z \to z_0} = \frac{1}{\pi i} \int_{|\zeta|=R} \frac{u(\zeta)}{(\zeta - z_0)^2} \, \mathrm{d}\zeta.$$

5.5.3 If $f(z)$ is analytic in the whole plane, show that the family formed by all functions $f(kz)$ with constant k is normal in the annulus $r_1 < |z| < r_2$ if and only if f is a polynomial.

Solution "$\Longrightarrow$"

Let $\mathfrak{F} = \{f(kz) \mid k \in \mathbb{C}\}$ be normal in the given annulus, which we call A.

Let k_n be a sequence of complex numbers. We can regard $f(k_n z)$ as $f(z)$ restricted to $|k_n| r_1 < |z| < |k_n| r_2$.

Let $a_n = n$, and use normality of $\mathfrak{F}$ to get a uniform limit function g. Either $g \equiv \infty$ or g is analytic on the whole plane.

If $g \equiv \infty$, then for all $M > 0$, we can find n large enough so that $M \leq |f(k_n z)|$, which shows that $f(k_n z) \xrightarrow{z \to \infty} \infty$. This shows that f has a non-essential singularity at infinity, so by a previous exercise, f is a polynomial.

In the other case, since f was entire, so is g, which means $g(B)$ is bounded. It follows that $f(n_k z)$ is bounded also by a number M. Since f is analytic,

$$|f^{(n)}(n_k z)| \leq \frac{M}{r^n} n! \xrightarrow{k \to \infty} 0,$$

which shows that f is constant in this case.

"$\Longleftarrow$"

Let f be polynomial.

Consider $f(k_n z)$. If $k_n \xrightarrow{n \to \infty} \infty$, then $f(k_n z) \xrightarrow{n \to \infty} \infty$ uniformly, since polynomials have non-essential singularities at infinity. In the other case, k_n admits a convergent subsequence k_{n_j}, which means that $f(k_{n_j} z)$ is a uniformly convergent subsequence.

3.2.1 If E is a compact set in a region Ω, prove that there exists a constant M, depending only on E and Ω, such that every positive harmonic function $u(z)$ in Ω satisfies $u(z_2) \leq Mu(z_1)$ for any two points $z_1, z_2 \in E$.

Solution Since Ω is connected in $\mathbb{C}$, it is path connected. So, without loss of generality, we can assume E is connected by connecting the components of a compact set together. The result is still compact.

Let $0 < R < \frac{1}{2}d(E, \partial\Omega)$. Then for any point $z \in E$, $B(z, R) \subseteq \overline{B(z, 2R)} \subseteq \Omega$. By compactness, there exist $w_1, \ldots, w_n \in E$ so that

$$E \subseteq \bigcup_{j=1}^{n} B(w_j, R)$$

with $w_j \in B(w_k, R)$ for some k for all j. Then for $\zeta_1 \in B(w_k, R)$ and $\zeta_2 \in B(w_j, R)$, Harnack's inequality gives us

$$\frac{1}{3}u(\zeta_1) = \frac{2R - R}{2R + R}u(\zeta_1) \leq u(w_k) \leq \frac{2R + R}{2R - R}u(w_j) = 3u(w_j) \leq 3\frac{2R + R}{2R - R}u(\zeta_2) = 3^2u(\zeta_2).$$

Given $z_1, z_2 \in E$, we can find a chain of these balls which connect the two z_1 and z_2, and with the center of every ball contained in some other ball. Any chain will have at most n balls, so we get the estimate

$$u(z_2) \leq 3^{n+1}u(z_1),$$

as desired.