38.6 Show that Parseval's equation (14) has the form (17) when the orthonormal sequence $\{\phi_n(x)\}_{n\geq 1}$ is the trigonometric sequence (15).

Solution The Fourier series for a function $f \in L^2([-\pi,\pi])$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx = a_0\sqrt{\frac{\pi}{2}}\left(\frac{1}{\sqrt{2\pi}}\right) + \sum_{n=1}^{\infty} a_n\sqrt{\pi}\left(\frac{\cos nx}{\sqrt{\pi}}\right) + b_n\sqrt{\pi}\left(\frac{\sin nx}{\sqrt{\pi}}\right).$$

Thus, Parseval's equation gives us

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 \, \mathrm{d}x = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

38.7 Obtain the sums

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

by applying Parseval's equation in the preceding to the two Fourier series

$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

and

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}.$$

Solution We'll use the result from problem 6 above.

Parseval's equation gives us

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \, \mathrm{d}x = \frac{2\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2} \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Similarly,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4 \, \mathrm{d}x = \frac{2\pi^4}{5} = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4} \implies 16\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{8\pi^4}{45} \implies \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

38.8 Use the method and results of Problem 7 to obtain the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

from the sine series for x^2.

Solution The sine series for x^2 can be turned into

$$x^2 = \frac{8}{\pi} \sum_{1}^{\infty} -\frac{1}{(2n-1)^3} \sin(2n-1)x + \pi x.$$

Parseval's equation and symmetry gives us

$$\frac{2}{\pi} \int_0^{\pi} (x^2 - \pi x)^2 \, \mathrm{d}x = \frac{64}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} \implies \frac{\pi^4}{15} = \frac{64}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} \implies \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}.$$

Write $s = \sum_{n=1}^{\infty} \frac{1}{n^6}$. Then

$$s = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} + \sum_{n=1}^{\infty} \frac{1}{(2n)^6} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} + \frac{s}{64} \implies \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{63}{64} s.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{64}{63} \frac{\pi^6}{960} = \frac{\pi^6}{945}.$$

38.9 Use the method and results of Problems 7 and 8 to obtain the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450}$$

from the cosine series for x^4.

Solution The cosine series for x^4 is

$$x^4 = \frac{\pi^4}{5} + 8 \sum_{n=1}^{\infty} (-1)^n \frac{\pi^2 n^2 - 6}{n^4} \cos nx.$$

Parseval's equation gives us

$$\begin{aligned} \frac{2\pi^8}{9} = \frac{1}{\pi} \int_{-\pi}^{\pi} x^8 \, \mathrm{d}x &= \frac{4\pi^8}{50} + 64 \sum_{n=1}^{\infty} \left(\frac{\pi^4 n^4}{n^8} - \frac{12\pi^2 n^2}{n^8} + \frac{36}{n^8} \right) \\ &= \frac{4\pi^8}{50} + 64\pi^4 \sum_{n=1}^{\infty} \frac{1}{n^4} - 768\pi^2 \sum_{n=1}^{\infty} \frac{1}{n^6} + 2304 \sum_{n=1}^{\infty} \frac{1}{n^8} \\ &= \frac{4\pi^8}{50} + \frac{64\pi^8}{90} - \frac{768\pi^8}{945} + 2304 \sum_{n=1}^{\infty} \frac{1}{n^8} \\ \implies \sum_{n=1}^{\infty} \frac{1}{n^8} &= \frac{\pi^8}{9450}. \end{aligned}$$

40.1 Find the eigenvalues λ_n and eigenfunctions $y_n(x)$ for the equation $y'' + \lambda y = 0$ in each of the following cases:

d. $y(0) = 0$, $y(L) = 0$ when $L > 0$.

e. $y(-L) = 0$, $y(L) = 0$ when $L > 0$.

Solution d. $\lambda < 0$:

We can write the equation $y'' = -\lambda y$, which has the general solution $y(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$. The initial conditions give us the system

$$\begin{pmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}L} & e^{-\sqrt{-\lambda}L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The matrix is invertible since $L \neq -L$, so we get the trivial solution in this case.

$\lambda = 0$:

The differential equation reduces to $y'' = 0$ here, so $y(x) = ax + b$. $y(0) = 0 \implies b = 0$, and $y(L) = 0 \implies aL = 0 \implies a = 0$, since $L > 0$, so the solution is trivial in this case also.

$\lambda > 0$:

In this case, the general solution is given by $y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$. $y(0) = 0 \implies c_1 = 0$. $y(L) = 0$ gives us

$$c_2 \sin\sqrt{\lambda}L = 0 \implies n\pi = \sqrt{\lambda}L \implies \lambda_n = \frac{n^2\pi^2}{L^2},\ n = 0, 1, \ldots,$$

with eigenfunctions $\sin\dfrac{n\pi x}{L}$.

a. This problem is similar to the previous one. If $\lambda \leq 0$, we get the trivial solution. For $\lambda > 0$, we get the general solution $y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$. The harmonic addition theorem tells us that we can write this in the form $y(x) = c\sin\left(\sqrt{\lambda}x + \phi\right)$. We'll discard c since eigenfunctions differ by a constant multiple.

The initial conditions give us

$$\begin{aligned} y(-L) &= \sin\left(-\sqrt{\lambda}L + \phi\right) = 0 \implies \phi = \sqrt{\lambda}L + \pi n \\ y(-L) &= \sin\left(\sqrt{\lambda}L + \phi\right) = 0 \implies \phi = -\sqrt{\lambda}L + \pi k. \end{aligned}$$

This gives

$$2\phi = \pi(n+k) \quad \text{and} \quad 2\sqrt{\lambda}L = \pi(k-n).$$

If we replace $n+k$ with n, we get

$$\phi_n = \frac{\pi n}{2} \quad \text{and} \quad \lambda_n = \frac{\pi^2 n^2}{4L^2},$$

with eigenfunctions

$$y_n(t) = \sin\left(\frac{\pi n}{2L}x + \frac{\pi n}{2}\right) = \sin\left(\frac{\pi n}{2L}(x+L)\right).$$

40.2 If $y = F(x)$ is an arbitrary function, then $y = F(x+at)$ represents a wave of fixed shape that moves to the left along the x-axis with velocity a. Similarly, if $y = G(x)$ is another arbitrary function, then $y = G(x-at)$ is a wave moving to the right, and the most general one-dimensional wave with velocity a is

$$y(x,t) = F(x+at) + G(x-at). \tag{$*$}$$

a. Show that $(*)$ satisfies the wave equation.

b. It is easy to see that the constant a in equation (8) has the dimensions of velocity. Also, it is intuitively clear that if a stretched string is disturbed, then waves will move in both directions away from the source of the disturbance. These considerations suggest introducing the new variables $\alpha = x+at$ and $\beta = x-at$. Show that with these independent variables, equation (8) becomes

$$\frac{\partial^2 y}{\partial\alpha\partial\beta} = 0,$$

and from this derive $(*)$ by integration.

Solution a. The partial derivatives are given by

$$\frac{\partial^2 y}{\partial x^2} = F''(x+at) + G''(x-at) \quad \text{and} \quad \frac{\partial^2 y}{\partial t^2} = a^2F''(x+at) + a^2G''(x-at).$$

It's easy to see that $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{a^2}\dfrac{\partial^2 y}{\partial t^2}$, so the wave equation is satisfied.

b. By the chain rule,

$$\frac{\partial y}{\partial\alpha} = \frac{\partial y}{\partial x}\frac{\partial x}{\partial\alpha} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial\alpha} = \frac{\partial y}{\partial x} + \frac{1}{a}\frac{\partial y}{\partial t}.$$

Taking the derivative with respect to β, we get

$$\begin{aligned}\frac{\partial^2 y}{\partial\alpha\partial\beta} &= \left[\left(\frac{\partial}{\partial x}\frac{\partial y}{\partial x}\right)\frac{\partial x}{\partial\beta} + \left(\frac{\partial}{\partial t}\frac{\partial y}{\partial x}\right)\frac{\partial t}{\partial\beta}\right] + \frac{1}{a}\left[\left(\frac{\partial}{\partial x}\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial\beta} + \left(\frac{\partial}{\partial t}\frac{\partial y}{\partial t}\right)\frac{\partial t}{\partial\beta}\right] \\ &= \frac{\partial^2 y}{\partial x^2} - \frac{1}{a}\frac{\partial^2 y}{\partial x\partial t} + \frac{1}{a}\frac{\partial^2 y}{\partial x\partial t} - \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2} \\ &= 0,\end{aligned}$$

since

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2} \implies \frac{\partial^2 y}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2} = 0.$$

40.5 Solve the vibrating string problem in the text if the initial shape (12) is given by the function

a. $f(x) = \begin{cases} 2cx/\pi, & \text{if } 0 \le x \le \pi/2 \\ 2c(\pi - x)/\pi & \text{if } \pi/2 \le x \le \pi. \end{cases}$

b. $f(x) = \dfrac{1}{\pi}x(\pi - x).$

c. $f(x) = \begin{cases} x, & \text{if } 0 \le x \le \pi/4, \\ \pi/4, & \text{if } \pi/4 \le x \le 3\pi/4, \\ \pi - x, & \text{if } 3\pi/4 \le x \le \pi. \end{cases}$

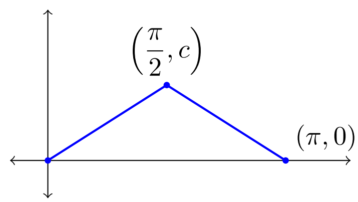
In each case, sketch the initial shape of the string.

Solution a. The Fourier sine coefficients are given by

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx \,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi/2} \frac{2cx}{\pi}\sin nx \,\mathrm{d}x + \frac{2}{\pi}\int_{\pi/2}^{\pi} \frac{2c(\pi - x)}{\pi}\sin nx \,\mathrm{d}x = \frac{8c\sin\frac{\pi n}{2}}{\pi^2 n^2}.$$

Thus, the solution is given by

$$y(x,t) = \frac{8c}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin(2n-1)x\cos(2n-1)at.$$

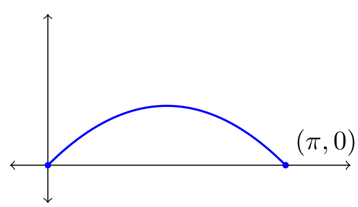


b. The Fourier sine coefficients are given by

$$b_n = \frac{2}{\pi}\int_0^\pi \frac{1}{\pi}x(\pi - x)\sin nx \,\mathrm{d}x = \frac{4(-\cos\pi n + 1)}{\pi^2 n^3}.$$

Notice that all the even terms vanish and the odd terms are $8/\pi^2 n^3$, so the solution is given by

$$y(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin(2n-1)x\cos(2n-1)at.$$



c. The Fourier sine coefficients are given by

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi/4} x\sin nx\,\mathrm{d}x + \frac{2}{\pi}\int_{\pi/4}^{3\pi/4} \frac{\pi}{4}\sin nx\,\mathrm{d}x + \frac{2}{\pi}\int_{3\pi/4}^{\pi} (\pi - x)\sin nx\,\mathrm{d}x \\
&= \frac{2\bigl(\sin\frac{\pi n}{4} + \sin\frac{3\pi n}{4} - \sin \pi n\bigr)}{\pi n^2} \\
&= \frac{2}{\pi n^2}\left(\sin\frac{\pi n}{4} + \sin\frac{3\pi n}{4}\right).
\end{aligned}$$

Thus, the solution is

$$y(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sin\frac{\pi n}{4} + \sin\frac{3\pi n}{4}\right)\sin nx\cos nat.$$