1 Show that $L^2([0,1]) \subseteq L^1([0,1])$ (for example, using the Cauchy-Schwarz inequality). Give an example to show that this inclusion is not an equality. Show that neither of $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$ contains the other.

Solution Let $f \in L^2([0,1])$. By definition, $\|f\|_2 < \infty$, so

$$\|f\|_1 = \int_0^1 |f| = \int_0^1 |f \cdot 1| = \langle f, 1\rangle_2 \leq \|f\|_2 \|1\|_2 = \|f\|_2 \int_0^1 1^2 = \|f\|_2 < \infty,$$

so $f \in L^1([0,1])$ also. Thus, $L^2([0,1]) \subseteq L^1([0,1])$.

This inclusion is not an equality: Consider the function $1/\sqrt{x}$.

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}} = 2\sqrt{x}\Big|_0^1 = 2 < \infty \iff \frac{1}{\sqrt{x}} \in L^1([0,1])$$
$$\int_0^1 \frac{\mathrm{d}x}{x} = \lim_{t \to 0^+} 2\log x\Big|_t^1 = \infty \iff \frac{1}{\sqrt{x}} \notin L^2([0,1]).$$

To show that $L^1(\mathbb{R})$ does not contain $L^2(\mathbb{R})$, consider the function $f(x) = \frac{\chi_{x>1}}{x}$. Then

$$\int_{\mathbb{R}} f = \int_1^\infty \frac{1}{x} = \lim_{t \to \infty} 2\log x\Big|_1^\infty = \infty \iff f \notin L^1([0,1])$$
$$\int_{\mathbb{R}} f^2 = \int_1^\infty \frac{1}{x^2} = \lim_{t \to \infty} -\frac{1}{x}\Big|_1^\infty = 1 < \infty \iff f \in L^2([0,1]).$$

To show that $L^2(\mathbb{R})$ does not contain $L^1(\mathbb{R})$, we can use the function $f(x) = \frac{\chi_{[0,1]}}{\sqrt{x}}$, and use the same argument as in the first part of the problem to see that $f \in L^1([0,1])$, but $f \notin L^2([0,1])$.

---

2 Show that the space of smooth (C^∞) functions with compact support is dense in $L^2(\mathbb{R})$. (Hint: Reduce to showing that step functions can be approximated in the L^2 norm by smooth functions with compact support. Reduce further to the case of the characteristic function of an interval. Graph the function

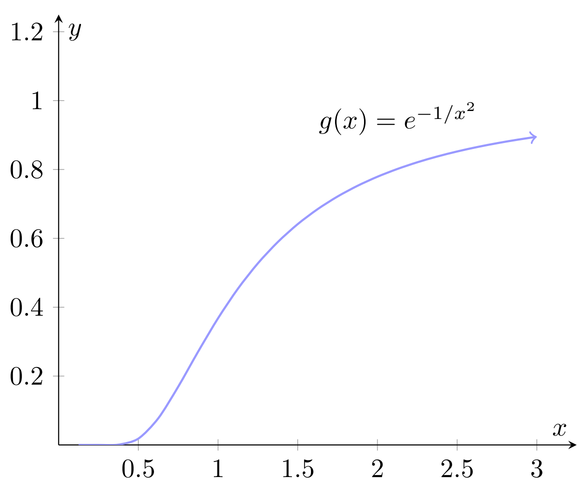
$$g(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0, \end{cases}$$

which is known to be smooth. Use g to construct the approximating functions you need.)

Solution Note that all functions in $L^2(\mathbb{R})$ are measurable. Hence, we can approximate $L^2(\mathbb{R})$ functions by step functions, so it suffices to approximate step functions by smooth functions with compact support.

Since step functions are finite linear combinations of characteristic functions, it suffices to approximate characteristic functions by smooth functions with compact support.

We can make the difference between a measurable set and open intervals containing it arbitrarily small, so it suffices to approximate characteristic functions of open interval via smooth functions with compact support.



Notice that $1 - g(x)$ is integrable since $1 - g(x) \leq 1/x^2$. Indeed, notice $e^x \geq 1 + x$ for all x, because

$$(e^x - (1+x))' = e^x - 1 = 0 \iff x = 0 \quad \text{and} \quad (e^x - (1+x))'' = e^x > 0 \; \forall x,$$

so $x = 0$ is the global minimizer for $e^x - (1+x)$, which gives $e^0 - 1 = 0$, which proves the claim. Thus, $e^x > x \implies e^{-1/x^2} \geq 1 - 1/x^2$, so

$$0 \leq 1 - e^{-1/x^2} \leq \frac{1}{x^2}.$$

Thus, by comparison, $1 - g(x)$ is integrable.

Assume that we are approximating the characteristic of interval $[-1,1]$. We can get any other interval via translations and scalings.

Thus, consider the functions

$$g_n(x) = \begin{cases} 1 - e^{-1/n(x-1)^2} & \text{if } x > 1 \\ 1 & \text{if } -1 \leq x \leq 1 \\ 1 - e^{-1/n(x+1)^2} & \text{if } x < 1, \end{cases}$$

which is smooth since g_n is smooth. Moreover, $g_n(x) = 1$ on $[-1,1]$, so it suffices to show that on $|x| > 1$, g_n vanishes quickly at $n \to \infty$.

Note that each g_n is integrable, since we showed that $1 - g(x)$ is integrable. Moreover, $g_n(x) \xrightarrow{n \to \infty} \chi_{[-1,1]}$ for all x, since

$$\lim_{x \to \infty} 1 - e^{-1/x^2} = 1 - e^0 = 0.$$

Notice that the sequence $|\chi_{[-1,1]} - g_n(x)|^2$ is monotonically decreasing and converges pointwise to 0 since $g_n(x)$ converges to $\chi_{[-1,1]}$. It vanishes on $[-1,1]$, and on $|x| > 1$, we essentially have the functions of the form $1 - e^{-1/x^2}$, which is monotonically decreasing.

Thus, by the monotone convergence theorem,

$$\|\chi_{[-1,1]} - g_n\|_2 = \int_0^1 |\chi_{[-1,1]} - g_n|^2 \xrightarrow{n \to \infty} \int_0^1 0 = 0,$$

so smooth functions are dense in $L^2([0,1])$.

---

3 Show that the functions x^n for $n \geq 0$ span $L^2([-1,1])$. (Hint: Reduce to the propeties of Fourier series in $L^2([0,1])$, which you can easily translate to $L^2([-1,1])$.) By the Gram-Schmidt process, these functions determine an orthonormal basis $e_0(x), e_1(x), \ldots$ for $L^2([-1,1])$ (which are polynomials, clearly). Compute $e_0, \ldots, e_3$ explicitly. (For example, $e_0 = 1/\sqrt{2}$.)

Solution It is known that Fourier series are dense in $L^2([0,1])$. By translations and scalings, it follows that they are dense in $L^2([-1,1])$ also, so it suffices to show that we can approximate $e^{2\pi i n x}$ via polynomials.

$e^{2\pi i n x}$ is analytic on $[-1,1]$, and its power series is given by

$$e^{2\pi i n x} = \sum_{k=0}^{\infty} \frac{(2\pi i n x)^k}{k!},$$

which converges uniformly on $[-1,1]$, since the radius of convergence of the series is ∞. Thus, for any $\varepsilon > 0$, there exists $N > 0$ such that for all $n \geq N$,

$$\left| e^{2\pi i n x} - \sum_{k=0}^{n} \frac{(2\pi i n x)^k}{k!} \right| < \frac{\varepsilon}{\sqrt{2}}$$

for all $x \in [-1,1]$. Then

$$\left\| e^{2\pi i n x} - \sum_{k=0}^{n} \frac{(2\pi i n x)^k}{k!} \right\|_2 = \sqrt{\int_{-1}^{1} \left| e^{2\pi i n x} - \sum_{k=0}^{n} \frac{(2\pi i n x)^k}{k!} \right|^2 \mathrm{d}x} \le \sqrt{\int_{-1}^{1} \frac{\varepsilon^2}{2} \,\mathrm{d}x} = \varepsilon.$$

Note that the partial sum is a finite linear combination of polynomials, so $\{x^n \mid n \ge 0\}$ is dense in $L^2([-1,1])$, so it spans $L^2([-1,1])$.

By the Gram-Schmidt process, we can construct an orthonormal basis as follows.

Notice that if $n+m$ is odd, then $\langle x^n, x^m \rangle = \int_{-1}^{1} x^{n+m} \,\mathrm{d}x = 0$.

$$\begin{aligned}
e_0 &= \frac{1}{\|1\|_2} = \frac{1}{\sqrt{\int_{-1}^{1} \mathrm{d}x}} = \frac{1}{\sqrt{2}} \\
u_1 &= x - \langle x, e_0 \rangle e_0 = x - \frac{1}{\sqrt{2}} \int_{-1}^{1} \frac{x}{\sqrt{2}} \,\mathrm{d}x = x \\
e_1 &= \frac{u_1}{\|u_1\|_2} = \frac{x}{\sqrt{\int_{-1}^{1} x^2 \,\mathrm{d}x}} = \sqrt{\frac{3}{2}} x \\
u_2 &= x^2 - \big(\langle x^2, e_0 \rangle e_0 + \langle x^2, e_1 \rangle e_1\big) = x^2 - \frac{1}{2} \int_{-1}^{1} x^2 \,\mathrm{d}x = x^2 - \frac{1}{3} \\
e_2 &= \frac{u_2}{\|u_2\|_2} = \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{-1}^{1} x^2 - \frac{1}{3} \,\mathrm{d}x}} = \frac{x^2 - \frac{1}{3}}{} \\
u_3 &= x^3 - \big(\langle x^3, e_0 \rangle e_0 + \langle x^3, e_1 \rangle e_1 + \langle x^3, e_2 \rangle e_2\big)
\end{aligned}$$

---

4 Show that the "Haar functions"

$$e_0^0(x) = 1 \text{ for } 0 \le x \le 1$$

$$e_n^k(x) = \begin{cases} 2^{n/2} & \text{for } \dfrac{k-1}{2^n} \le x \le \dfrac{k-\frac{1}{2}}{2^n} \\ -2^{n/2} & \text{for } \dfrac{k-\frac{1}{2}}{2^n} \le x \le \dfrac{k}{2^n} \\ 0 & \text{otherwise,} \end{cases}$$

defined for $1 \le k \le 2^n$ and $n \ge 1$ (and $k = n = 0$), form an orthonormal basis for $L^2([0,1])$. Graph the first few functions. (Hint: To show that these functions span, show that $\int_0^x f = 0$ for every x of the form $k/2^n$ with $1 \le k \le 2^n$ and $n \ge 1$, and deduce that $\int_E f = 0$ for every measurable subset E of $[0,1]$.)

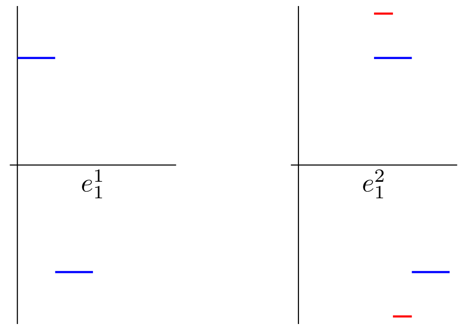
Solution We first show that the Haar functions are orthonormal.

Notice that $e_n^k(x)$ are odd about $x = 1/2$, except when $k = n = 0$, so

$$\langle e_0^0, e_n^k \rangle = \int_0^1 e_n^k(x) \,\mathrm{d}x = 0.$$

Notice that for a fixed n, the support of $e_n^k(x)$ is almost disjoint from the other k. Hence, $\langle e_n^k, e_n^{k'} \rangle = 0$ for all $1 \le k < k' \le 2^n$.

For two e_n^k and $e_{n'}^{k'}$ with $n \ne n'$ and $k \ne k'$, their supports are also almost disjoint, or they overlap like in the graph for e_1^2, which simply scales the function, so by symmetry their inner product is still 0. Thus, the set is orthogonal.



Lastly,

$$\int_0^1 |e_n^k(x)|^2 \,\mathrm{d}x = \left(\frac{k-\frac{1}{2}}{2^n} - \frac{k-1}{2^n}\right)2^n - \left(\frac{k}{2} + \frac{k-\frac{1}{2}}{2^n}\right)2^n = \frac{2\cdot 2^n}{2^{n+1}} = 1,$$

and $\int_0^1 e_0^0 = 1 \cdot 1 = 1$, so the set is orthonormal.

Let $f \in L^2([0,1])$ such that $\langle f, e_n^k \rangle = 0$ for all $n \geq 1$ and $1 \leq k \leq 2^n$. Note that by problem (1), $f \in L^1([0,1])$ also, so $\|f\|_1 < \infty$.

Notice that $\chi_{[k/2^n,(k+1)/2^n]}$ is the limit a.e. of Haar functions. Indeed, consider the finite linear combination of the Haar functions

$$g_N(x) = \sum_{m=0}^{N} \frac{1}{2^{n/2}} e_{(n+m)}^{(m+1)k} = \begin{cases} 1 & \text{if } 0 \leq x \leq \dfrac{(N+1)k - \frac{1}{2}}{2^{n+N}} := \ell_N \\ -M & \text{if } \dfrac{(N+1)k-\frac{1}{2}}{2^{n+N}} \leq x \leq \dfrac{(N+1)k}{2^{n+N}} := m_N, \end{cases}$$

where

$$M = \sum_{m=0}^{N} \frac{1}{2^{(n+m)/2}} \leq \sum_{m=0}^{N} \frac{1}{2^{m/2}} = \frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1}$$

for all $N \geq 1$. The length of the interval on which the sum is $-M$ is

$$\frac{(N+1)k}{2^{n+N}} - \frac{(N+1)k - \frac{1}{2}}{2^{n+N}} = \frac{1}{2^{n+N+1}},$$

so we can make this arbitrarily small, say $\varepsilon > 0$ small. Hence the partial sums converge everywhere to $\chi_{[k/2^n,(k+1)/2^n]}$ except at the single point $(k+1)/2^n$.

$$\begin{aligned}
\left|\langle f, \chi_{[k/2^n,(k+1)/2^n]}\rangle\right| &= \left|\langle f, \chi_{[k/2^n,(k+1)/2^n]} - g_N\rangle + \langle f, g_N\rangle\right| \\
&\leq \left|\sum_{n=0}^{N} \frac{1}{2^{n/2}} \int_0^1 f(x)(\chi_{[k/2^n,(k+1)/2^n]} - g_N)\,\mathrm{d}x + 0\right| \\
&\leq \sum_{n=0}^{N} \frac{1}{2^{n/2}} \int_0^1 |f(x)(\chi_{[k/2^n,(k+1)/2^n]} - g_N)|\,\mathrm{d}x \\
&\leq \sum_{n=0}^{N} \frac{1}{2^{n/2}} (1+M)\varepsilon \int_0^1 |f(x)|\,\mathrm{d}x \\
&= \sum_{n=0}^{N} \frac{1}{2^{n/2}} (1+M)\|f\|_1 \varepsilon \\
&\leq \frac{\sqrt{2}}{\sqrt{2}-1}(1+M)\|f\|_1\varepsilon.
\end{aligned}$$

Since ε was arbitrary and M is bounded, it follows that $\langle f, \chi_{[k/2^n,(k+1)/2^n]}\rangle = 0$ for every n, k.

Notice that numbers of the form $k/2^n$ are dense in $[0, 1]$. Indeed, let $(a, b) \subseteq [0, 1]$. Then there exists N such that $2^N|b - a| \geq 1$, by the Archimedean principle. Then we can fit a natural number k in the interval $(2^N a, 2^N b)$ since its length is bigger than 1. Hence,

$$2^N a < k < 2^N b \implies a < \frac{k}{2^N} < b,$$

which proves density.

Thus, given any open interval (a, b), we can approximate a from above and b from below via these numbers, which give us a countable collection of open sets of the form $(k/2^n, (k + 1)/2^n)$ whose union is (a, b).

If E is a measurable set, then we can approximate it with a G_δ set G which differs from E be a set of measure 0. We can write each open set of G as a countable union of sets of the form $(k/2^n, (k + 1)/2^n)$, which gives us

$$\int_E f = \int_G f = 0$$

for any measurable $E \subseteq \mathbb{R}$. This implies that $f = 0 \in L^2([0, 1])$, which shows that the Haar functions are dense in $L^2([0, 1])$.