1 Let A be a complex $n \times n$ matrix and let us introduce the resolvent of A,

$$R(z) = (A - zI)^{-1}, \quad z \in \mathbb{C} \setminus \operatorname{Spec}(A),$$

where $\operatorname{Spec}(A)$ is the set of eigenvalues of A. Show that the function

$$\mathbb{C} \setminus \operatorname{Spec}(A) \ni z \mapsto \|R(z)\|$$

is subharmonic.

Solution Norms are continuous and linear operators are continuous, so $\|R(z)\|$ is continuous and hence upper semi-continuous.

By definition, we have

$$\|R(z)\| = \sup_{\|v\|=1} \|R(z)v\|.$$

Notice that

$$\|R(z)\| = \sup_{\|u\|=\|v\|=1} \operatorname{Re}\langle R(z)u, v\rangle.$$

In particular, this occurs when v is parallel to $R(z)u$. It now suffices to show that $f_{u,v}(z) \coloneqq \operatorname{Re}\langle R(z)u, v\rangle$ is subharmonic.

This function is actually harmonic, since by Cramer's rule, we have

$$R(z) = \frac{1}{\det(A - zI)} \operatorname{adj}(A - zI).$$

This is a meromorphic function with no poles in $\mathbb{C}\setminus\operatorname{Spec}(A)$, so it is actually analytic. Thus, $f_{u,v}$ is harmonic, so $\|R(z)\|$ is subharmonic.

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2 a. Let f be a holomorphic function in a neighborhood of $\overline{D(0,R)} = \{z \in \mathbb{C} \mid |z| \leq R\}$, for $R > 0$ and let us define

$$M(r) = \sup_{|z|=r} |f(z)|, \quad A(r) = \sup_{|z|=r} \operatorname{Re} f(z), \quad 0 \leq r \leq R.$$

Prove the Borel-Carathéodory inequality,

$$M(r) \leq \frac{2r}{R-r} A(R) + \frac{R+r}{R-r}|f(0)|, \quad 0 \leq r < R.$$

b. Let f be entire holomorphic and assume that there exist positive constants C and N such that

$$\operatorname{Re} f(z) \leq C(1 + |z|)^N, \quad z \in \mathbb{C}.$$

Prove that f is a holomorphic polynomial of degree $\leq N$.

Solution a. We shall first show that

$$f(z) - f(0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{2z}{Re^{i\varphi} - z} (\operatorname{Re} f)(Re^{i\varphi})\, \mathrm{d}\varphi, \quad \text{for } |z| < R.$$

Let $u(z) = \operatorname{Re} f(z)$, and apply the Poisson representation formula to u:

$$u(z) = \frac{1}{2\pi R} \int_{|w|=R} \frac{1 - |z|^2}{|w - z|^2} u(w)\, \mathrm{d}s(w) = \operatorname{Re}\left[\frac{1}{2\pi R} \int_{|w|=R} \frac{w+z}{w-z} u(w)\, \mathrm{d}s(w)\right] \coloneqq \operatorname{Re} \tilde{f}(z).$$

$\tilde{f}$ is analytic as a function of z because we can expand the integrand using geometric series and interchange the sum and integral.

Hence, $f(z) = \tilde{f}(z) + iC$, where $C \in \mathbb{R}$. Indeed, $\mathrm{Re}(f - \tilde{f}) \equiv 0$, so by the Cauchy-Riemann equations, $\mathrm{Im}(f - \tilde{f})$ must be constant. Thus,

$$f(z) - f(0) = \tilde{f}(z) - \tilde{f}(0) = \frac{1}{2\pi R}\int_{|w|=R} \frac{w+z}{w-z} u(w) - u(w)\,\mathrm{d}s(w) = \frac{1}{2\pi R}\int_{|w|=R} \frac{2z}{w-z} u(w)\,\mathrm{d}s(w).$$

By the triangle inequality, we get $|f(z)| \le |f(z) - f(0)| + |f(0)|$. Applying the hint with $|z| = r < R$, we have

$$|f(z) - f(0)| \le \frac{1}{2\pi R}\int_{|w|=R} \left|\frac{2z}{w-z} u(w)\right| \mathrm{d}s(w) \le \frac{2r}{R-r} \sup_{|w|=R} |u(w)| = \frac{2r}{R-r} A(R).$$

Since $(R+r)/(R-r) \ge 1$, we have

$$|f(z)| \le |f(z) - f(0)| + |f(0)| \le \frac{2r}{R-r} A(R) + \frac{R+r}{R-r}|f(0)|,$$

for any $|z| = r < R$. Thus, by definition of the supremum, we get the desired inequality.

b. Fix $R > 0$. By part (a), we have, for any $|z| = r < R$ and $n \ge N$, that

$$\sup_{|z|=r} |f(z)| \le \frac{2r}{R-r} C(1+r)^n + \frac{R+r}{R-r}|f(0)|.$$

By Cauchy's integral formula and using the fact that f is holomorphic in a neighborhood of $\overline{D(0,2R)}$, we have

$$\begin{aligned}
|f^{(n)}(z)| &\le \frac{n!}{4\pi R}\int_{|\zeta|=2R} \left|\frac{\zeta f(\zeta)}{(\zeta - z)^n}\right| \mathrm{d}s(\zeta) \\
&\le \frac{n!}{4\pi R}\frac{2R}{(2R-r)^n} \sup_{|\zeta|=R} |f(\zeta)| \\
&= \frac{n!}{2\pi(2R-r)^n} M(R) \\
&\le \frac{n!}{2\pi(2R-r)^n}\left(\frac{2r}{R-r} C(1+r)^n + \frac{R+r}{R-r}|f(0)|\right) \xrightarrow{R\to\infty} 0.
\end{aligned}$$

Thus, f must be a polynomial of degree $\le N$.

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3 Suppose that u is real and harmonic in $\{0 < |z| < 2\}$ and that

$$u(z) = o\left(\log \frac{1}{|z|}\right), \quad z \to 0.$$

Show that u has a removable singularity at 0.

Solution Let $\tilde{u}$ solve the Dirichlet problem on the unit disk with boundary function $u(z)$.

Notice that $u - \tilde{u} = o(-\log|z|)$, since $\tilde{u}$ is bounded (by continuity) and it is harmonic on the punctured disk. Thus, for any $\varepsilon > 0$, there exists $\delta > 0$ so that

$$0 < |z| < \delta \implies \frac{|u(z) - \tilde{u}(z)|}{-\log|z|} \le \varepsilon \implies |u(z) - \tilde{u}(z)| \le -\varepsilon \log|z|.$$

When $|z| = 1$, $\log|z| = 0$, so $|u(z) - \tilde{u}(z)| = 0$ on $|z| = 1$. By the maximum principle, this equality extends to the entire punctured disk. By continuity, we can extend $u(z) - \tilde{u}(z)$ to the origin with 0, and we see that $u(z) = \tilde{u}(z)$ for all z in the disk, so u has a removable singularity at the origin.

4 Let $\Omega \subseteq \mathbb{C}$ be open, connected and let $f_j \in \mathrm{Hol}(\Omega)$ be a sequence such that $f_j(a)$ converges for some $a \in \Omega$. Assume also that $\operatorname{Re} f_j$ converges locally uniformly in Ω. Show that f_j converges locally uniformly.

Solution From problem 2(a), for $R > 0$ so that $D_R(a) = \{z \mid |z-a| \leq R\} \subseteq \Omega$, we have whenever $|z-a| = r < R$ that

$$\begin{aligned} f_j(z) &= \frac{1}{2\pi R}\int_{|w|=R} \frac{w+(z-a)}{w-(z-a)} \operatorname{Re} f_j(a+w) - \operatorname{Re} f_j(a+w)\, \mathrm{d}s(w) + f_j(a) \\ &= \frac{1}{2\pi R}\int_{|w|=R} \frac{2(z-a)}{w-(z-a)} \operatorname{Re} f_j(a+w)\, \mathrm{d}s(w) + f_j(a) \end{aligned}$$

for any R so that $D_R(a) = \{z \mid |z-a| \leq R\} \subseteq \Omega$. Since $D_R(a)$ is compact, we have uniform convergence of $\operatorname{Re} f_j$, so the RHS converges uniformly to some function f:

$$\begin{aligned} |f_j(z) - f_k(z)| &\leq \frac{1}{2\pi R}\int_{|w|=R} \left|\frac{2(z-a)}{w-(z-a)}\right| |\operatorname{Re} f_j(a+w) - \operatorname{Re} f_k(a+w)|\, \mathrm{d}s(w) + |f_j(a) - f_k(a)| \\ &\leq \frac{2r}{R-r} \sup_{|w|=R} |\operatorname{Re} f_j(a+w) - \operatorname{Re} f_k(a+w)| + |f_j(a) - f_k(a)| \xrightarrow{j,k\to\infty} 0, \end{aligned}$$

uniformly.

Now consider the set

$$E := \{z \in \Omega \mid \exists \omega \text{ compact neighborhood of } z \text{ such that } f_j \text{ converges uniformly on } \overline{\omega} \subseteq \Omega\}.$$

This set is non-empty since we just showed that $a \in E$. It's clear that E is open, so we just need to show that E is closed.

Let z_n be a sequence in E which converges to some $z_0 \in \Omega$. Then there exists $R > 0$ so that $z_0 \in \{z \mid |z - z_n| \leq R\} \subseteq \Omega$ for n large.

Notice that $f_j(z_n) \xrightarrow{j\to\infty} f(z_n)$. Then by the same argument as above, f_j converges uniformly on the whole disk, which implies that the disk is a compact neighborhood of z_0 on which f_j converges uniformly. Thus, $z_0 \in E$, so E is closed.

By connectedness, $E = \Omega$. Thus, for any compact set, we may cover it with finitely many of these compact neighborhoods, which gives locally uniform convergence of f_j.

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5 Let X be a metric space and let $u\colon X \to [-\infty, \infty)$ be an upper semicontinuous function which is bounded above. Show that there exist continuous functions $u_n\colon X \to \mathbb{R}$ such that $u_1 \geq u_2 \geq \cdots \geq u$ on X and $u_n \to u$.

Solution If $u \equiv -\infty$, then it is the limit of the sequence of constant functions $-n$, for $n \geq 1$. Assume from now on that u is not identically ∞.

For $n \geq 1$, define $u_n(x) = \sup_{y\in X}(u(y) - nd(x,y))$. This is an decreasing sequence, since $u(y) - nd(x,y)$ is decreasing.

Notice that

$$u_n^{-1}((a,b)) = \bigcup_{y\in X} \{x \mid a < u(y) - nd(x,y) < b\},$$

which is open since $d(x,y)$ is a continuous function in x for any y, so u_n is continuous.

We now need to show that $u_n \xrightarrow{n\to\infty} u$ pointwise, which is the same as showing that $\inf_n u_n = u$.

Let $x \in X$ and $\varepsilon > 0$. By upper semicontinuity, there exists $\delta > 0$ so that $u(y) \leq u(x) + \varepsilon$ whenever $d(x,y) < \delta$. Let

$$n > \frac{\sup_{z\in X}(u(z)) - u(x)}{\delta} > 0.$$

If $d(x,y) < \delta$, then

$$u(y) - nd(x,y) \leq u(x) + \varepsilon - \frac{\sup_{z\in X}(u(z)) - u(x)}{\delta} d(x,y) < u(x) + \varepsilon.$$

On the other hand, if $d(x,y) \geq \delta$, then

$$u(y) - nd(x,y) \leq u(y) - \frac{\sup_{z\in X}(u(z)) - u(x)}{\delta}\delta \leq u(x).$$

Thus, $u_n(x) \leq u(x) + \varepsilon$, so $u = \inf_n u_n$, as desired.

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6 Let us set $D = \{z \in \mathbb{C} \mid |z| < 1\}$. u_0 to D. Let furthermore $f \in \mathrm{Hol}(D)$ be the unique holomorphic function such that $\mathrm{Re}\, f = u$ and $f(0) = u(0)$. Show that f extends continuously to $\overline{D}$ and compute the boundary value of the imaginary part of f.

Solution Using the Poisson representation formula on u, we have for $|z| < 1$ that

$$u(z) = \mathrm{Re}\left[\frac{1}{2\pi}\int_{|w|=1} \frac{w+z}{w-z} u(w)\,\mathrm{d}s(w)\right] = \mathrm{Re}\,\tilde{f}(z),$$

so that $f(z) = \tilde{f}(z) + iC$. By assumption,

$$f(0) = \tilde{f}(0) + iC = \frac{1}{2\pi}\int_{|w|=1} u(w)\,\mathrm{d}s(w) + iC = u(0) \in \mathbb{R}.$$

The integral must be real-valued, so $C = 0 \implies f = \tilde{f}$.

Fix ζ with $|\zeta| = 1$, and let $0 \leq t < 1$. We will show that $\mathrm{Im}\, f(t\zeta)$ converges to $u_0(\zeta)$.

Taking the imaginary part of f, we get

$$\mathrm{Im}\, f(z) = \frac{1}{2\pi}\int_{|w|=1} \frac{\mathrm{Im}(\overline{z}w)}{|w-z|^2} u_0(w)\,\mathrm{d}s(w).$$

This integral must converge when $|z| = 1$ because the real part of f converges on the boundary, and because $\mathrm{Im}(\overline{z}w)$ is bounded. We now need to show that $\mathrm{Im}\, f(z)$ is continuous on the boundary.

Let $\theta_0 \in \mathbb{R}$, and compute:

$$\left|\mathrm{Im}\, f(e^{i\theta}) - \mathrm{Im}\, f(e^{i\theta_0})\right| = \left|\frac{1}{2\pi}\int_{|w|=1} \frac{\mathrm{Im}(\overline{z}w)}{|w-e^{i\theta}|^2} u_0(w) - \frac{\mathrm{Im}(\overline{z}w)}{|w-e^{i\theta_0}|^2} u_0(w)\,\mathrm{d}s(w)\right|.$$

We may rotate the second integral slightly without changing the value of the integral, so we rotate by $\theta - \theta_0$ degrees:

$$\begin{aligned}
&= \left|\frac{1}{2\pi}\int_{|w|=1} \frac{\mathrm{Im}(\overline{z}w)}{|w-e^{i\theta}|^2} u_0(w) - \frac{\mathrm{Im}(\overline{z}w)}{|w-e^{i\theta_0}|^2} u_0(e^{i(\theta-\theta_0)}w)\,\mathrm{d}s(w)\right| \\
&\leq \frac{1}{2\pi}\int_{|w|=1}\left|\frac{\mathrm{Im}(\overline{z}w)}{|w-e^{i\theta}|^2} u_0(w) - \frac{e^{i(\theta-\theta_0)}\,\mathrm{Im}(\overline{z}w)}{|e^{i(\theta-\theta_0)}w - e^{i\theta}|^2} u_0(e^{i(\theta-\theta_0)}w)\right|\mathrm{d}s(w).
\end{aligned}$$

By simplifying, we may rewrite the integrand as a difference quotient in u_0. By appealing to continuity of u_0, the mean value theorem, and the compactness of $\partial\mathbb{D}$, we will be able to bound the entire integral with a uniform bound, which shows continuity.

7 Let us set $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. Let u be harmonic and bounded in $\mathbb{C}_+$, continuous on $\overline{\mathbb{C}_+}$. Show that u can be represented as a Poisson integral,

$$u(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\operatorname{Im} z}{|z-x|^2} u(x)\, \mathrm{d}x, \quad z \in \mathbb{C}_+.$$

Solution Consider the Möbius transformation

$$F(z) = \frac{z-i}{z+i}.$$

This conformally maps $\mathbb{C}_+$ to the unit disk $\mathbb{D}$. Thus, F extends to be a homeomorphism from $\overline{\mathbb{C}_+}$ to $\overline{\mathbb{D}}$.

Now $\tilde{u} \coloneqq u \circ F^{-1} \colon \mathbb{D} \to \mathbb{C}$ is a harmonic function on the unit disk, by the chain rule, since F^{-1} is analytic, and $\tilde{u}$ is continuous on the boundary. Hence, we can use the Poisson representation formula on $\tilde{u}$:

$$\tilde{u}(w) = \frac{1}{2\pi} \int_{|y|=1} \frac{1-|w|^2}{|y-w|^2} \tilde{u}(y)\, \mathrm{d}s(y),$$

where $w = F(z) \in \mathbb{D}$, for some $z \in \mathbb{C}_+$.

Consider the change of variables $y = F(x)$, which gives

$$\mathrm{d}y = \frac{x+i-(x-i)}{(x+i)^2}\, \mathrm{d}x \implies \mathrm{d}s(y) = \frac{2}{|x+i|^2}\, \mathrm{d}s(x),$$

and because $F(\mathbb{R}) = \partial\mathbb{D}$,

$$\begin{aligned} u(z) = \left(u \circ F^{-1}\right)(w) &= \frac{1}{2\pi} \int_{|y|=1} \frac{1-|w|^2}{|y-w|^2} \tilde{u}(y)\, \mathrm{d}s(y) \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{1-|F(z)|^2}{|F(x)-F(z)|^2} u(x)\, \frac{\mathrm{d}x}{|x+i|^2}. \end{aligned}$$

We now calculate the following quantities:

$$(x+i)(F(x)-F(z)) = x-i-\frac{z-i}{z+i}(x+i) = \frac{2i(x-z)}{z+i} \implies |F(x)-F(z)|^2 = \frac{4|x-z|^2}{|z+i|^2}$$

and

$$|z+i|^2 - |z-i|^2 = |z|^2 + 1 + \overline{z}i - zi - (|z|^2 + 1 - \overline{z}i + zi) = 2i(\overline{z} - z) = 4 \operatorname{Im} z.$$

Thus, the integral becomes

$$u(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\operatorname{Im} z}{|z-x|^2} u(x)\, \mathrm{d}x,$$

as desired.

8 Let $\Omega \subseteq \mathbb{C}$ be open, $\Omega \neq \mathbb{C}$, and let $d(z) = \operatorname{dist}(z, \Omega^c)$ be the distance from z to Ω^c. Show that the function

$$\Omega \ni z \mapsto -\log d(z)$$

is subharmonic.

Solution We first need to show that $u(z) := -\log d(z)$ is upper semicontinuous.

Notice $d(z)$ is continuous. Since Ω^c is closed, $d(z) = 0 \iff z \in \Omega^c$, it follows that $d(z) > 0$, so $u(z)$ is continuous also, so it is upper semicontinuous.

Notice that because $\log x$ is strictly increasing on $(0, \infty)$,

$$d(z) = \inf_{w \in \Omega^c} d(z, w) \implies \log d(z) = \inf_{w \in \Omega^c} \log d(z, w) \implies u(z) = \sup_{w \in \Omega^c} [-\log d(z, w)].$$

This supremum is pointwise finite because $\Omega \neq \mathbb{C} \implies d(z)$ is pointwise finite. Indeed, since Ω^c is closed, the supremum is attained, and the distance between any two points is finite.

Thus, if we can show that $u_w(z) := -\log d(z, w)$ is subharmonic for a fixed $w \in \Omega^c$, it follows by a proposition in class that u is subharmonic.

Notice that

$$u_w(z) = -\log |z - w|.$$

$z - w$ is entire, so u_w is subharmonic, by a proposition in class. Hence, $u = \sup_{w \in \Omega^c} u_w$ is subharmonic.

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9 Let $\Omega \subseteq \mathbb{C}$ be open, $\Omega \neq \mathbb{C}$, and let $f \in \operatorname{Hol}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ be such that $|f| \leq 1$ on $\partial\Omega$ and $|f| \leq M$ in Ω, for some M. Show that $|f| \leq 1$ in Ω.

Solution If Ω is dense in $\mathbb{C}$, then we may analytically extend f to be entire, since its derivative is continuous. Thus, f becomes an entire bounded function, so it is a constant. Since $|f| \leq 1$ on $\partial\Omega$, it follows that $|f| \leq 1$ everywhere.

We may now assume that $\overline{\Omega} \neq \mathbb{C}$, so that there exists $a \in \mathbb{C}$ and $R > 0$ such that $\Omega \cap \{z \mid |z - a| < R\} = \emptyset$.

If Ω is bounded, then we may simply use the maximum principle. From now on, assume that Ω is unbounded.

Let $z_0 \in \Omega$ be in an unbounded component $C \subseteq \Omega$, and consider $h(z) := f^n(z)/(z - a)$. For $z \in \partial\Omega$, we have that

$$|h(z)| \leq \frac{1}{R}.$$

We can also find $R_n > 0$ so that

$$|h(z)| \leq \frac{M^n}{R_n} \leq \frac{1}{R} \quad \text{and} \quad z_0 \in D_{R_n}(a).$$

Now consider $C_n := C \cap D_{R_n}(a)$, which is an open bounded set in Ω. We have that

$$\partial C_n \subseteq \partial C \cap \partial D_{R_n}(a) \subseteq \partial\Omega \cap \partial D_{R_n}(a).$$

Thus, by the maximum principle applied to h, we get

$$|h(z_0)| = \left| \frac{f^n(z_0)}{z_0 - a} \right| \leq \frac{1}{R},$$

for any n. Taking $n \to \infty$, we conclude that $|f(z_0)| \leq 1$ or otherwise, $\lim_{n\to\infty} |f^n(z_0)/(z_0 - a)| = \infty$.