2.4.12 $x' - \left(\frac{n}{t}\right)x = e^t e^n, \quad n$ a positive integer

Solution Our integrating factor is

$$u(t) = e^{-\int n/t\,\mathrm{d}t} = e^{-n\ln(t)} = t^{-n}$$

We can neglect the absolute value because we can simply define $u(t)$ as a piecewise function that gives the same result. Using this integrating factor, we get

$$(xt^{-n})' = e^t e^n t^{-n}$$

$$xt^{-n} = \int e^t e^n t^{-n}\,\mathrm{d}t + C$$

$$\boxed{x(t) = e^n t^n \int \frac{e^t}{t^n}\,\mathrm{d}t + Ct^n}$$

$\frac{e^t}{t^n}$ doesn't have an antiderivative in terms of elementary functions, so that is our final answer.

2.4.18 $xy' + 2y = \sin x, \quad y\left(\frac{\pi}{2}\right) = 0$

Solution

$$y' + \frac{2}{x}y = \frac{\sin x}{x}$$

$$(ye^{2\ln|x|})' = e^{2\ln|x|}\frac{\sin x}{x}$$

$$(yx^2)' = x^2\frac{\sin x}{x} = x\sin x$$

$$\int_{\frac{\pi}{2}}^{x} (yt^2)'\,\mathrm{d}t = \int_{\frac{\pi}{2}}^{x} t\sin t\,\mathrm{d}t$$

$$y(x)x^2 - y\left(\frac{\pi}{2}\right)\cdot\left(\frac{\pi}{2}\right)^2 = -x\cos x + \sin x - 1$$

$$y\left(\frac{\pi}{2}\right) = 0 \Rightarrow \boxed{y(x) = -\frac{\cos x}{x} + \frac{\sin x}{x^2} - \frac{1}{x^2}}$$

$y(x)$ doesn't exist only when $x = 0$, so the interval of existence is $(0, \infty)$.

2.4.28 Suppose you have a closed system containing 1000 individuals. A flu epidemic starts. Let N(t) represent the number of infected individuals in the closed system at time t. Assume that the rate at which the number of infected individuals is changing is jointly proportional to the number of infected individuals and to the number of uninfected individuals. Furthermore, suppose that when 100 individuals are infected, the rate at which individuals are becoming infected is 90 individuals per day. If 20 individuals are infected at time $t = 0$, when will 90% of the population be infected? *Hint:* The assumption here is that there are only healthy individuals and sick individuals. Furthermore, the resulting model can be solved using the technique introduced in Exercise 22.

Solution The rate of change of $N(t)$ is proportional to the number of infected individuals and the number of uninfected individuals. Mathematically,

$$\frac{\mathrm{d}N}{\mathrm{d}t} = kN(1000 - N), \quad k \in \mathbb{R}$$

When 100 individuals are infected, we have $N = 100$ and $\frac{\mathrm{d}N}{\mathrm{d}t} = 90$. Solving for k, we get $90 = k \cdot 100 \cdot 900 \Rightarrow$

$k = 1/1000$. Solving for $N(t)$,

$$\begin{aligned}
\frac{\mathrm{d}N}{N(1000-N)} &= k\,\mathrm{d}t \\
\frac{1}{1000}\int \frac{1}{N} + \frac{1}{1000-N}\,\mathrm{d}N &= \int k\,\mathrm{d}t \\
\ln N - \ln(1000-N) &= 1000kt + C_0 \\
\frac{N}{1000-N} &= Ce^{1000kt} \\
N(t) &= \frac{1000Ce^{1000kt}}{1+Ce^{1000kt}} \\
N(0) = 20 \Rightarrow \frac{1000C}{1+C} &= 20 \Rightarrow C = \frac{20}{980} = \frac{1}{49} \\
N(t) &= \frac{\frac{1000}{49}e^t}{1+\frac{1}{49}e^t} = \frac{1000e^t}{49+e^t}
\end{aligned}$$

90% of the population is $N = 900$, so

$$\begin{aligned}
N(t) = \frac{1000e^t}{49+e^t} &= 900 \\
10e^t &= 9\big(49+e^t\big) \\
e^t &= 441 \\
&\boxed{t = \ln 441 \text{ days}}
\end{aligned}$$

2.4.42 Consider anew Newton's law of cooling, where the temperature of a body is modeled by the equation

$$T' = -k(T-A),$$

where T is the temperature of the body and A is the temperature of the surrounding medium (*ambient temperature*). Although this equation is linear and its solution can be found by using an integrating factor, Theorem 4.41 provides a far simpler approach.

(a) Find a solution T_h of the homogeneous equation $T' + kT = 0$.

(b) Find a particular solution T_p of the inhomogeneous equation (4.43). *Note:* This equation is autonomous. See Section 2.9.

(c) Form the general solution $T = T_h + T_p$.

(d) Add a source of constant heat (like a heater in a room) to the model (4.43), as in

$$T' = -k(T-A) + H.$$

Use the technique outlined in parts (a)–(c) to find the general solution of equation (4.44).

Solution

(a) $T_h' = -kT_h$
$T_h = e^{-kt}$

(b) A solution is $T_p = A$, which is an equilibrium solution to the differential equation.

(c) $T(t) = T_p + CT_h, \quad C \in \mathbb{R}$
$T(t) = A + Ce^{-kt}$

(d) Consider the homogeneous linear differential equation equation

$$T_h' = -kT_h$$
$$T_h = e^{-kt}$$

Next, we find a particular solution T_p to the inhomogeneous equation. The equation is autonomous, so we can find an equilibrium solution:

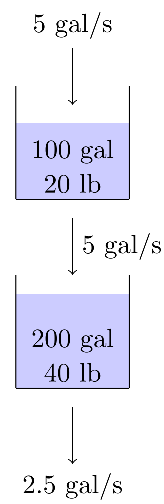
$$0 = -k(T_p - A) + H$$
$$T_p = \frac{H}{k} + A$$

So, the general solution can be formed.

$$T(t) = T_p + CT_h = \boxed{\frac{H}{k} + A + Ce^{-kt}}$$

2.5.12 Consider two tanks, labeled tank A and tank B for reference. Tank A contains 100 gal of solution in which is dissolved 20 lb of salt. Tank B contains 200 gal of solution in which is dissolved 40 lb of salt. Pure water flows into tank A at a rate of 5 gal/s. There is a drain at the bottom of tank A. The solution leaves tank A via this drain at a rate of 5 gal/s and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B at a rate of 2.5 gal/s. What is the salt content in tank B at the precise moment that tank B contains 250 gal of solution?

Solution Let $A(t)$ be the amount of salt of tank A at time t measured in pounds. Let $B(t)$ be the same, but for tank B. Then we have the following differential equations and initial conditions:



$$\frac{\mathrm{d}A}{\mathrm{d}t} = \text{rate in} - \text{rate out} = 0 - 5 \cdot \frac{A}{100} = -\frac{A}{20} \tag{1}$$
$$\frac{\mathrm{d}B}{\mathrm{d}t} = \text{rate in} - \text{rate out} = 5 \cdot \frac{A}{100} - 2.5 \cdot \frac{B}{200 + 2.5t} = \frac{A}{20} - \frac{B}{80 + t} \tag{2}$$
$$A(0) = 20 \tag{3}$$
$$B(0) = 40 \tag{4}$$

(1) is separable, and yields $A(t) = Ce^{-t/20}$. Using the initial condition (3), we get $A(0) = C = 20 \Rightarrow A(t) = 20e^{-t/20}$. Substituting $A(t)$ into (2), we get the linear differential equation

$$\frac{\mathrm{d}B}{\mathrm{d}t} = e^{-t/20} - \frac{B}{80 + t}$$

Using the integrating factor $\mu(t) = e^{\ln(80+t)} = 80 + t$, we get

$$(B\mu)' = \mu\left(e^{-t/20}\right)$$
$$B(80 + t) = \int 80e^{-t/20} + te^{-t/20}\,\mathrm{d}t$$
$$B(t) = \frac{1}{80 + t}\left(-1600e^{-t/20} - 20te^{-t/20} - 400e^{-t/20} + C\right)$$
$$B(t) = \frac{1}{80 + t}\left(-2000e^{-t/20} - 20te^{-t/20} + C\right)$$
$$B(0) = 40 \Rightarrow \frac{-2000 + C}{80} = 40 \Rightarrow C = 5200$$
$$B(t) = \frac{1}{80 + t}\left(-2000e^{-t/20} - 20te^{-t/20} + 5200\right)$$

Solving for when tank B contains 250 gal of solution and substituting into $B(t)$,

$$200 + 2.5t = 250 \Rightarrow t = 20$$
$$B(20) = \frac{1}{100}\left(-2000e^{-1} - 400e^{-1} + 5200\right)$$
$$\boxed{= 52 - \frac{24}{e}\ \text{lb}}$$