25.6 A space X is said to be weakly locally connected at x if for every neighborhood U of x, there is a connected subspace of X contained in U that contains a neighborhood of x. Show that if X is weakly locally connected at each of its points, then X is locally connected. [*Hint:* Show that the components of open sets are open.]

Solution Let X be weakly locally connected at every point.

Pick $x \in X$ and an open neighborhood $U \subseteq X$ of x. By definition, there exists a connected subspace A of X such that $A \subseteq U$ and A contains an open neighborhood V of x.

Let $y \in A$. Note that U is also an open neighborhood of y, so since X is weakly locally connected at y, there exists $V_y \subseteq A$ open containing y. Then

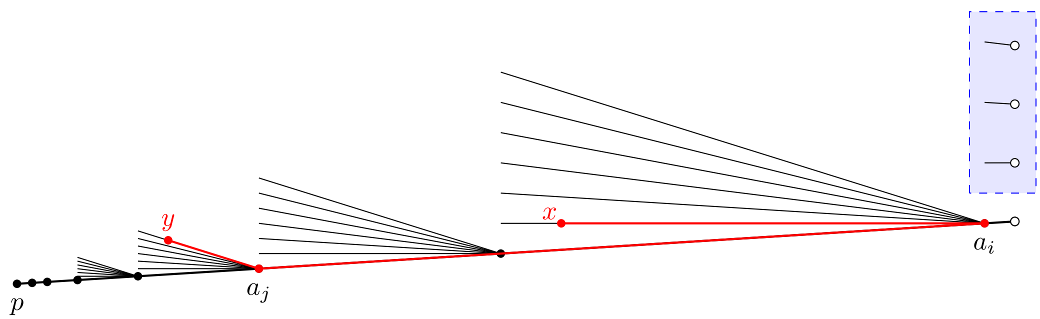
$$A = \bigcup_{y \in A} V_y.$$

Hence, A is open in X, since each V_y was open in X. So A is an open connected neighborhood of x, which means X is locally connected at x. Since x and U were arbitrary, it follows that X is locally connected.

25.7 Consider the "infinite broom" X pictured in Figure 25.1. Show that X is not locally connected at p, but is weakly locally connected at p. [*Hint:* Any connected neighborhood of p must contain all the points a_i.]

Solution We'll first show that X is weakly locally connected at p.

Let $U \subseteq X$ be an open neighborhood of p. Then U must contain at least one a_i, by definition of X. Then the broom from p to a_i is connected. Indeed, it is path connected; given any two points, we can trace it back to an a_i, move to another a_j, and up into the other point.



An open neighborhood U of p. From a_i to the left, we have a connected subspace of X.

Because of self-similarity, the connected subspace contains an open neighborhood of p also.

However, X is not locally connected at p. Indeed, any open set U that is not all of X, such as the above image, contains a largest a_i. Then in our open neighborhood, we have parts of spokes from a_{i-1}, such as in the image. So, we can take the intersection of the blue open rectangle in $\mathbb{R}^2$ in the above picture and the rest of our open neighborhood U, which gives us two disjoint open sets in X whose disjoint union is U.

Thus, any open connected neighborhood of p must be X, so X is not locally connected at p.

25.10 Let X be a space. Let us define $x \sim y$ if there is no separation $X = A \cup B$ of X into disjoint open sets such that $x \in A$ and $y \in B$.

a. Show this relation is an equivalence relation. The equivalence classes are called the quasicomponents of X.

b. Show that each component of X lies in a quasicomponent of X, and that the components and quasicomponents of X are the same if X is locally connected.

c. Let K denote the set $\{1/n \mid n \in \mathbb{Z}_+\}$ and let $-$K denote the set $\{-1/n \mid n \in \mathbb{Z}_+\}$. Determine the components, path components, and quasicomponents of the following subspaces of $\mathbb{R}^2$:

$$A = (K \times [0,1]) \cup \{0 \times 0\} \cup \{0 \times 1\}.$$
$$B = A \cup ([0,1] \times \{0\}).$$
$$C = (K \times [0,1]) \cup (-K \times [-1,0]) \cup ([0,1] \times -K) \cup ([-1,0] \times K).$$

Solution a. Reflexivity: If $x = y$, every open neighborhood of x trivially contains y and vice versa, so $x \sim x$.

Symmetry: By commutativity of unions, i.e., $A \cup B = B \cup A$, we have $y \sim x$.

Transitivity: Let $x \sim y$ and $y \sim z$. Suppose there were a separation $X = A \cup B$ with $x \in A$ and $z \in B$. Then either $y \in A$ or $y \in B$. Then this implies that $y \not\sim z$ (respectively, $y \not\sim x$), which is a contradiction. Hence, we must have $x \sim z$.

Hence, $\sim$ is an equivalence relation on X.

b. Let U be a connected component of X. Then for every $x, y \in U$, we have $x \sim y$. Otherwise, we would be able to separate the connected subspace U by two disjoint open sets which don't contain both x and y. Thus, U is "quasiconnected," so it is contained in a quasicomponent of X.

Let X be locally connected.

First note that connected components are open and closed in X. Indeed, if U is a connected component of X, then U is open since X is locally connected.

Consider $X - U$, and let $y \in X - U$. Then since X is locally connected, there exists V open in X which contains a connected neighborhood W in X of y.

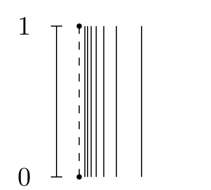
If $W \cap U \neq \emptyset$, then $W \cup U$ is connected since W and U are connected, which implies $W \subseteq U$. But we assumed y to be outside U. Hence, $W \subseteq X - U$, so U is open and closed.

Let U be a quasicomponent of X. Suppose U were disconnected. Then consider the separation $U = A \cup B$, and let $x \in A$, $y \in B$.

A is a neighborhood of x, so since X is locally connected, there exists an open component $V \subseteq A$ in X such that $x \in V$ but $y \notin V$.

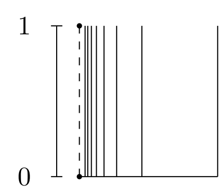
Since V is open and closed, $V \cup {}^{c}V$ is a separation of X with $x \in V$ and $y \in {}^{c}V$, which is a contradiction. Thus, a U is connected, so we have that quasicomponents and components are the same.

c. The components and path components are the sets $\{(0,0)\}, \{(0,1)\}$, and the vertical lines.
The quasicomponents are the vertical lines, and the set $\{(0,0),(0,1)\}$. $\{(0,0),(0,1)\}$ is quasiconnected because any two disjoint open sets which separate the two points would separate a vertical line, which are connected.



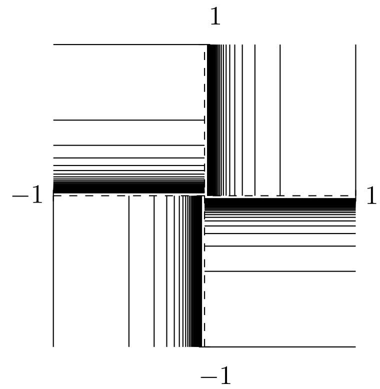
A

B is connected since any attempt to separate the point $(0,1)$ and the rest of B would separate a connected line.
The path connected components (and also regular components) are the sets $\{(0,0)\}$, and $B - \{(0,1)\}$.
The quasicomponent is all of B, since B connected is contained in a quasicomponent.



B

C is connected, since the lines intersect the closure of adjacent sets on the lines $x = 0$ and $y = 0$.
The path components of C are the vertical and horizontal lines. The "squares" are not path connected because any path would be cut at the lines $x = 0$ and $y = 0$.
The quasicomponent is C, since C itself is connected.



C

26.11 Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of X that is simply ordered by proper inclusion. Then

$$Y = \bigcap_{A \in \mathcal{A}} A$$

is connected.

[*Hint:* If $C \cup D$ is a separation of Y, choose disjoint open sets U and V of X containing C and D, respectively, and show that

$$\bigcap_{A \in \mathcal{A}} (A - (U \cup V))$$

is not empty.]

Solution Let A and B be any two sets in $\mathcal{A}$. Then either $A \subsetneq B$ or $B \subsetneq A$. So, their intersection $A \cap B$ is either A or B, so $A \cap B$ is connected and closed.

Let $C \cup D$ be a separation of Y. Then take U and V disjoint and open sets in X containing C and D, respectively.

Note that if $A \in \mathcal{A}$, then $A - (U \cup V) = A \cap {}^{\mathrm{c}}(U \cup V)$ is closed, since A and ${}^{\mathrm{c}}(U \cup V)$ are both closed.

Since $\mathcal{A}$ is simply ordered by proper inclusion, so is the collection $\{A - (U \cup V)\}$.

Note that if $A - (U \cup V) = \emptyset$ for some A, then A is separated by U and V, which is a contradiction, since we assumed A to be connected.

Thus, since any finite intersection of the $A - (U \cup V)$ gives another set of the same form, the collection $\{A - (U \cup V)\}$ has the finite intersection property. Since X is compact,

$$\bigcap_{A \in \mathcal{A}} (A - (U \cup V)) \neq \emptyset.$$

Hence

$$\emptyset \neq \bigcap_{A \in \mathcal{A}} (A - (U \cup V)) = \left(\bigcap_{A \in \mathcal{A}} A \right) - (U \cup V) = Y - (U \cup V),$$

but as assumed $U \cup V$ to cover Y, which is a contradiction. Hence, Y must be connected.

28.4 A space X is said to be countably compact if every countable open covering of X contains a finite subcollection that covers X. Show that for a T_1 space X, countable compactness is equivalent to limit point compactness.

[*Hint:* If no finite subcollection of U_n covers X, choose $x_n \notin U_1 \cup \cdots \cup U_n$, for each n.]

Solution Let X be a T_1 space.

"$\implies$"

Let X be countably compact, and suppose X was also not limit point compact.

Let A be a countably infinite set with no limit point, which we can do since X is infinite. Note that since A has no limit point in X, $\overline{A} = A \cup A' = A$, so A is closed.

By assumption, A does not have a limit point. Hence, for every $a_n \in A$, there exists an open neighborhood $U_n \subseteq X$ of a_n such that $U_n \cap (A - \{a_n\}) = \emptyset$.

Then $\{U_n \mid n \geq 1\} \cup \{{}^{\mathrm{c}}A\}$ forms a countable open cover of X, so there exists $N \in \mathbb{N}$ such that

$$X = {}^{\mathrm{c}}A \cup \bigcup_{n=1}^{N} U_n.$$

But for $k > N$, $a_k \notin {}^{\mathrm{c}}A$ and $a_k \notin \bigcup_{n=1}^{N} U_n$, since by construction,

$$A \cap \left(\bigcup_{n=1}^{N} U_n \right) = \{a_1, \dots, a_N\}.$$

But this is a contradiction, since we assumed that we had a cover of X. Hence, A must have a limit point, so X is limit point compact.

"$\impliedby$"

Let X be limit point compact.

Suppose $(U_n)_{n\geq 1}$ was a countable open covering of X with no finite subcover.

For each $n \in \mathbb{N}$, pick $x_n \notin U_1 \cup \cdots \cup U_n$, which we can do, since $(U_n)_{n\geq 1}$ does not have a finite subcover of X.

Define $A = \{x_1, x_2, \dots\}$ and let x be a limit point of A, which exists by assumption. Since $(U_n)_{n\geq 1}$ covers X, there exists N such that $x \in U_N$. Then by construction, $U_N \cap A$ contains at most $\{x_1, \dots, x_{N-1}\}$.

Since X is T_1, there exist open neighborhoods $V_1, \dots, V_{N-1} \subseteq X$ such that $x \in V_1, \dots, V_{N-1}$, but $x_i \notin V_i$ for all $1 \leq i \leq N-1$.

Hence, $W := U \cap \bigcap_{i=1}^{N-1} V_i$ is an open neighborhood of x, but $W \cap (A - \{x\}) = \emptyset$. This a contradiction. Hence, U_n must admit a finite subcover, so X is countably compact.

28.6 Let (X, d) be a metric space. If $f\colon X \to X$ satisfies the condition

$$d(f(x), f(y)) = d(x, y)$$

for all $x, y \in X$, then f is called an isometry of X. Show that if f is an isometry and X is compact, then f is bijective and hence a homeomorphism.

[*Hint:* If $a \notin f(X)$, choose ε so that the ε-neighborhood of a is disjoint from $f(X)$. Set $x_1 = a$, and $x_{n+1} = f(x_n)$ in general. Show that $d(x_n, x_m) \geq \varepsilon$ for $n \neq m$.]

Solution Let f be an isometry and X be compact. Note that X is Hausdorff, also, since it is a metric space. Moreover, in a metric space, X compact $\iff$ X sequentially compact.

f is continuous: we can simply take $\delta = \varepsilon$ for any $\varepsilon > 0$. Hence, $f(X)$ is compact. Since X is Hausdorff, it follows that $f(X)$ is closed.

We'll first show f is injective:

Let $x \neq y \in X$. Then $d(x, y) > 0 \iff d(f(x), f(y)) > 0 \iff f(x) \neq f(y)$, so f is injective.

Now we'll show that f is surjective:

Suppose there exists $a \in X$ such that $a \notin f(X)$. Since X is compact Hausdorff and $f(X)$ is closed, there exists $\varepsilon > 0$ such that $B(a, \varepsilon) \cap f(X) = \emptyset$.

Let $x_1 = a$, and define recursively $x_{n+1} = f(x_n)$. Notice that for all $n \geq 2$, $x_n \in f(X)$, so we have that $d(x_1, x_n) \geq \varepsilon$. Then for $n < m$,

$$d(x_n, x_m) = d(f(x_{n-1}), f(x_{m-1})) = d(x_{n-1}, x_{m-1}) = \cdots = d(x_1, x_{m-n}) \geq \varepsilon.$$

But this means that $(x_n)_{n \geq 1}$ does not admit a convergent subsequence, which is a contradiction. Hence, f is surjective.

f is bijective and continuous, so all that's left to do is to show that it's an open map:

Pick a basic open set $B(x, r)$ for some $x \in X$ and $r > 0$. Then

$$\begin{aligned} f(B(x,r)) &= f(\{y \in X \mid d(x,y) < r\}) \\ &= \{f(y) \in X \mid d(x,y) < r\} \\ &= \{f(y) \in X \mid d(f(x), f(y)) < r\} \\ &= B(f(y), r), \end{aligned}$$

so f is open. Thus, f is a homeomorphism.

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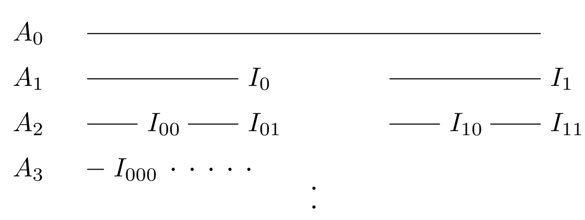
1 Show that the Cantor set defined in Munkres 27.6 is homeomorphic to the product space $\prod_{n=1}^{\infty}\{0,1\}$, where each space $\{0,1\}$ has the discrete topology.

Solution Let $X = \prod_{n=1}^{\infty}\{0,1\}$.

For the first stage A_1, define the "left" interval and the "right" interval as I_0 and I_1, respectively.

Consider the next intervals in A_2, given by $I_0 \cap A_2$ and $I_1 \cap A_2$. $I_i \cap A_2$ gives us two intervals. Define the "left" interval and the "right" interval via I_i0 and I_i1, respectively.

Continuing this process, we get connected intervals $I_{i_1 i_2 \cdots i_n}$.



Note that given a sequence $(i_n)_{n\geq 1} \subseteq \{0,1\}$, $\bigcap_{n=1}^{\infty} I_{i_1\cdots i_n}$ contains a single point. Indeed, the diameter of $I_{i_1\cdots i_n}$ is, by construction, $1/3^n \xrightarrow{n\to\infty} 0$, and $I_{i_1\cdots i_n} \supseteq I_{i_1\cdots i_{n+1}}$ for all n. By a theorem in analysis, the intersection contains exactly one point.

Hence, we can define $f\colon X \to C$ as follows: If $x = (i_n)_{n\geq 1}$, where each $i_n \in \{0,1\}$,

$$\{f(x)\} = \bigcap_{n=1}^{\infty} I_{i_1\cdots i_n}.$$

Then f is injective:

If $x \neq y \in X$, then they must differ at at least one coordinate, so they will end up in different disjoint intervals, so $f(x) \neq f(y)$.

f is surjective:

Given a point y in C, we are able to construct each $I_{i_1\cdots i_n}$ by checking if y belongs to that interval. Continuing inductively, we find $(i_n)_{n\geq 1} \subseteq X$ such that

$$\{y\} = \bigcap_{n=1}^{\infty} I_{i_1\cdots i_n} = \{f((i_n)_{n\geq 1})\},$$

so f is surjective.

f is continuous:

A basic open set in C is as follows: Fix $i_1, \ldots, i_n$, and let

$$U = \{x \in C \mid x \in \bigcap_{k=1}^{n} I_{i_1\cdots i_k}\}.$$

Then

$$f^{-1}(U) = \prod_{k=1}^{n}\{i_k\} \times \prod_{k=n+1}^{\infty} \{0,1\},$$

which is a basic open set in X, so f is continuous.

f is open:

Let U be a basic open set in X. Then we can write U as

$$\prod_{k=1}^{n}\{i_k\} \times \prod_{k=n+1}^{\infty} \{0,1\}.$$

Then

$$f(U) = \{x \in C \mid x \in \bigcap_{k=1}^{n} I_{i_1\cdots i_k}\},$$

which is open.

Thus, f is a homeomorphism.

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2 Let $f\colon X \to Y$ be a surjective continuous map of compact Hausdorff spaces. Show that if U is open in X, then so is its subset $f^{-1}(B)$, where $B = \{y \in Y \mid f^{-1}(y) \subseteq U\}$.

Solution Let U be open in X, and let $B = \{y \in Y \mid f^{-1}(y) \subseteq U\}$.

Let F be a closed set. Since X is compact, F is also compact, so $f(F)$ is compact in Y Hausdorff, which means $f(F)$ is closed. Hence, f is a closed map.

Note that ${}^{\mathrm{c}}B = f({}^{\mathrm{c}}U)$.

If $x \in {}^{\mathrm{c}}B$, then $f^{-1}(x) \in {}^{\mathrm{c}}U \implies x \in f({}^{\mathrm{c}}U)$, so ${}^{\mathrm{c}}B \subseteq f({}^{\mathrm{c}}U)$.

If $x \in f({}^{\mathrm{c}}U)$, then $f^{-1}(x) \in {}^{\mathrm{c}}U$. By definition, $f^{-1}(x) \in {}^{\mathrm{c}}B$, so ${}^{\mathrm{c}}B = f({}^{\mathrm{c}}U)$.

Thus, $B = {}^{\mathrm{c}}f({}^{\mathrm{c}}U)$. Note that since U is open, ${}^{\mathrm{c}}U$ is closed, so $f({}^{\mathrm{c}}U)$ is closed. Finally, this means that ${}^{\mathrm{c}}f({}^{\mathrm{c}}U) = B$ is open. Since f is continuous, $f^{-1}(B)$ is open as well.