****8** Suppose V and W are two vector spaces. We can make the set

$$V \times W = \{(\alpha, \beta) \mid \alpha \in V,\ \beta \in W\}$$

into a vector space as follows:

$$\begin{aligned}(\alpha_1, \beta_1) + (\alpha_2, \beta_2) &= (\alpha_1 + \alpha_2, \beta_1 + \beta_2)\\ c(\alpha_1, \beta_1) &= (c\alpha_1, c\beta_1)\end{aligned}$$

You can assume the axioms of a vector space hold for $V \times W$.

a. If V and W are finite dimensional, what is the dimension of $V \times W$? Prove your answer.

b. Now suppose W_1 and W_2 are two subspaces of V. We can define a linear map

$$\Phi\colon W_1 \times W_2 \to V$$

by

$$\Phi(\alpha, \beta) = \alpha + \beta.$$

Show that $R(\Phi) = W_1 + W_2$.

c. Show $\dim(N(\Phi)) = \dim(W_1 \cap W_2)$.

Solution a. Let $A = \{\alpha_1, \ldots, \alpha_n\}$ be a basis for V, and let $B = \{\beta_1, \ldots, \beta_m\}$ be a basis for W. Suppose $(\gamma_1, \gamma_2) \in V \times W$. Then by the defined addition, we have $(\gamma_1, \gamma_2) = (\gamma_1, 0) + (0, \gamma_2)$.

Since $\gamma_1 \in V$ and A is a basis for V, we have that $\gamma_1 = c_1\alpha_1 + \cdots + c_n\alpha_n$. Similarly, $\gamma_2 = d_1\beta_1 + \cdots + d_m\beta_m$. Substituting those for γ_1 and γ_2, we have

$$\begin{aligned}(\gamma_1, \gamma_2) &= (\gamma_1, 0) + (0, \gamma_2)\\ &= (c_1\alpha_1 + \cdots + c_n\alpha_n, 0) + (0, d_1\beta_1 + \cdots + d_m\beta_m)\\ &= (c_1\alpha_1, 0) + \cdots + (c_n\alpha_n, 0) + (0, d_1\beta_1) + \cdots + (0, d_m\beta_m)\\ &= c_1(\alpha_1, 0) + \cdots + c_n(\alpha_n, 0) + d_1(0, \beta_1) + \cdots + d_m(0, \beta_m)\end{aligned}$$

We made no assumptions about (γ_1, γ_2), so every vector in $V \times W$ can be written as a linear combination of the vectors $(\alpha_1, 0), \ldots, (\alpha_n, 0), (0, \beta_1), \ldots, (0, \beta_m)$. We will now show that these vectors are linearly independent.

Consider the sum

$$\begin{aligned}c_1(\alpha_1, 0) + \cdots + c_n(\alpha_n, 0) + d_1(0, \beta_1) + \cdots + d_m(0, \beta_m) &= (0, 0)\\ (c_1\alpha_1 + \cdots + c_n\alpha_n, 0) + (0, d_1\beta_1 + \cdots + d_m\beta_m) &= (0, 0)\\ (c_1\alpha_1 + \cdots + c_n\alpha_n, d_1\beta_1 + \cdots + d_m\beta_m) &= (0, 0)\\ \implies c_1\alpha_1 + \cdots + c_n\alpha_n &= 0\\ \implies d_1\beta_1 + \cdots + d_m\beta_m &= 0\end{aligned}$$

But A and B are each linearly independent, so we must have $c_1 = \cdots = c_n = d_1 = \cdots = d_m$. Thus, $(\alpha_1, 0), \ldots, (\alpha_n, 0), (0, \beta_1), \ldots, (0, \beta_m)$ are linearly independent and they span $V \times W$. By definition, the vectors form a basis of $V \times W$. There are $n + m$ vectors, where $n = \dim V$ and $m = \dim W$, so $\dim(V \times W) = n + m = \dim V + \dim W$.

b. Let $\gamma \in W_1 \times W_2$, $A = \{\alpha_1, \ldots, \alpha_n\}$ be a basis for W_1, and $B = \{\beta_1, \ldots, \beta_m\}$ be a basis of W_2. Then

$$(\gamma_1, \gamma_2) = (c_1\alpha_1 + \cdots + c_n\alpha_n, 0) + (0, d_1\beta_1 + \cdots + d_m\beta_m)$$

Then since Φ is linear, we can write the equality as

$$\begin{aligned}\Phi(\gamma_1, \gamma_2) &= \Phi\big((c_1\alpha_1 + \cdots + c_n\alpha_n, 0) + (0, d_1\beta_1 + \cdots + d_m\beta_m)\big)\\ \gamma_1 + \gamma_2 &= \underbrace{c_1\alpha_1 + \cdots + c_n\alpha_n}_{\in W_1} + \underbrace{(d_1\beta_1 + \cdots + d_m\beta_m)}_{\in W_2}\end{aligned}$$

Thus, for any $\gamma \in W_1 \times W_2$, $\Phi(\gamma) \in W_1 + W_2 \iff R(\Phi) \subseteq W_1 + W_2$.

We now wish to show that any vector $\gamma \in W_1 + W_2$ (γ different from the one above) is also an element of $R(\Phi)$. By definition, there exists $\alpha \in W_1$ and $\beta \in W_2$ such that $\alpha + \beta = \gamma$. By the definition of Φ, we can rewrite the left-hand-side as $\Phi(\alpha, \beta) = \gamma$. Thus, $\gamma \in R(\Phi)$, so $W_1 + W_2 \subseteq R(\Phi)$.

Taking the two results together, we have $R(\Phi) \subseteq W_1 + W_2 \subseteq R(\Phi) \implies R(\Phi) = W_1 + W_2$.

c. Let $T : W_1 \cap W_2 \to V \times W,\ \alpha \mapsto T(\alpha) = (\alpha, -\alpha)$.

Note that $\Phi(T(\alpha)) = \alpha - \alpha = 0$ for all $\alpha \in W_1 \cap W_2$. This means that $R(\alpha) \subseteq N(\Phi)$.

Let $(\alpha, \beta) \in N(\Phi)$. Then by definition, $\Phi(\alpha, \beta) = \alpha + \beta = 0 \implies \beta = -\alpha \implies (\alpha, \beta) = (\alpha, -\alpha) \in R(\alpha)$. Thus, $N(\Phi) \subseteq= R(T)$, which, along with the result in the previous paragraph, that $R(T) = N(\Phi)$.

Then consider $T(\alpha) = (\alpha, -\alpha) = 0$. The solution to this equation is $\alpha = -\alpha = 0$, so we have $\dim N(T) = 0$.

Finally, by rank-nullity, we have

$$\dim(W_1 \cap W_2) = \dim R(T) + \dim N(T) = \dim N(\Phi).$$

****9 3.1.13** Let V be a vector space and T a linear transformation from V into V. Prove that the following two statements about T are equivalent.

a. The intersection of the range of T and the null space of T is the zero subspace of V.

b. If $T(T\alpha) = 0$, then $T\alpha = 0$.

Solution $a \implies b$:

Let $R(T) \cap N(T) = \{0\}$. If $T(T\alpha) = 0$, then we must have that $T\alpha \in N(T)$. But by definition, $T\alpha \in R(T)$ also, so $T\alpha \in R(T) \cap N(T) \implies T(\alpha) = 0$.

$b \implies a$:

Suppose that if $T(T\alpha) = 0$, then $T\alpha = 0$. Let $\beta = T\alpha$. By definition, $\beta \in R(T)$. From the hypothesis, if $\beta \in N(T)$ also, then $\beta = 0$. Combining the statements, we have that if $\beta \in R(T) \cap N(T)$, then $\beta = 0$. Thus, $R(T) \cap N(T) = \{0\}$.

Since each statement implies the other, the statements must be equivalent.