13.6.3 Let G be a finite group with subgroups H and K. Recall that $HK := \{hk \mid h \in H,\ k \in K\}$. Show that $|HK| = |H||K|/|H \cap K|$.

Solution Notice that if $hk = h'k'$ for $h, h' \in H$ and $k, k' \in K$, we get

$$h^{-1}h' = k(k')^{-1} \in H \cap K.$$

This means that every element in HK gets counted $|H \cap K|$ times.

Every product between elements of H and K is given by $|H||K|$, counting duplicates. Thus, if we remove duplicates, we see that

$$|HK| = |H||K|/|H \cap K|,$$

as desired.

13.6.4 Let N and H be two normal subgroups of G such that $G = HN$. Show that there is an isomorphism $G/(H \cap N) \simeq G/H \times G/N$.

Solution Define $\varphi\colon G \to G/H \times G/N$ as follows: if $h \in H$ and $n \in N$,

$$\varphi(hn) = (hnH, hnN) = (nH, hN),$$

by normality.

This is clearly surjective, and we also have that

$$\varphi(g) = (eH, eN) \iff gH = eH \text{ and } gN = eN \iff g \in H \text{ and } g \in N \iff g \in H \cap N.$$

Thus, $\ker \varphi = H \cap N$, so by the first isomorphism theorem,

$$G/(H \cap N) \overset{\varphi}{\simeq} G/H \times G/N.$$

13.6.5** Let G be a finite group with normal subgroups H and K of relatively prime order. Show that the group HK is cyclic if H and K are cyclic and abelian if H and K are abelian.

Solution Notice that $H \cap K = \{e\}$. Indeed, $H \cap K$ is a subgroup of H and K, so its order must divide both $|H|$ and $|K|$ by Lagrange, which are relatively prime. We'll show that if $h \in H$ and $k \in K$, that $hk = kh$.

Consider $h^{-1}k^{-1}hk$. Since K is normal, there exists $k' \in K$ such that

$$h^{-1}k^{-1}hk = h^{-1}k^{-1}k'h \in K,$$

by normality of K.

Similarly, there exists $h' \in H$ such that

$$h^{-1}k^{-1}hk = k^{-1}h'hk \in H.$$

Thus, $h^{-1}k^{-1}hk \in H \cap K \implies h^{-1}k^{-1}hk = e \implies hk = kh$, as desired.

Let H and K both be abelian. Then for $h, h' \in H$ and $k, k' \in K$

$$(hk)(h'k') = hkk'h' = hk'kh' = k'hkh' = k'hh'k = (h'k')(hk),$$

by repeatedly applying the lemma we proved using the fact that H and K are abelian, so HK is abelian.

Let H and K be cyclic, and thus abelian, and let their generators be h and k, respectively, with order m and n, respectively.

Notice that for $h \in H$ and $k \in K$, we have $h^a k^b = k^b h^a$, by the lemma. Thus, let $h^a k^b, h^c k^d \in HK$. Then

$$(h^a k^b)(h^c k^d) = h^{a+c}k^{b+d}.$$

Thus, $HK \simeq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. By the Chinese remainder theorem, $HK \simeq \mathbb{Z}/mn\mathbb{Z}$, which is cyclic.

13.6.6 Let G be a group with subgroups H and N. We say that G is the *(internal) semidirect product* of H and N, denoted by $N \rtimes H$ if

(i) $N \triangleright G$

(ii) $G = NH$

(iii) $N \cap H = \{1\}$.

Let $G = N \rtimes H$. Define $\varphi \colon H \to \mathrm{Aut}(N)$ by $\varphi_x(n) = xnx^{-1}$. Show φ is a group homomorphism and $\psi \colon N \rtimes H \to N \rtimes_\varphi H$ induced by $h \mapsto (e_N, h)$ and $n \mapsto (n, e_H)$ is a group isomorphism, where $N \rtimes_\varphi H$ is the (external) semidirect product of Exercise 11.9(14).

Solution Let $h_1, h_2 \in H$. Then

$$\varphi(h_1h_2)(n) = h_1h_2n(h_1h_2)^{-1} = h_1(h_2nh_2^{-1})h_1^{-1} = \varphi(h_1) \circ \varphi(h_2),$$

so φ is a group homomorphism.

We'll first show that ψ is a homomorphism. Let $n_1, n_2 \in N$ and $h_1, h_2 \in H$. By normality of N, there exists $n_2' \in N$ such that $h_1n_2 = n_2'h_1 \iff n_2' = h_1n_2h_1^{-1}$. Then

$$\psi(n_1h_1n_2h_2) = (n_1n_2', h_1h_2) = (n_1(h_1n_2h_1^{-1}), h_1h_2) = (n_1\varphi_{(h_1)}(n_2), h_1h_2) = (n_1, n_2)\cdot(h_1, h_2) = \psi(n_1h_1)\cdot\psi(n_2h_2),$$

so ψ is a homomorphism.

The inverse of ψ is

$$\psi^{-1}(n, h) = nh,$$

so ψ is an isomorphism.

13.6.8** Let G be a group. Recall the *commutator* G' of G is the subgroup generated by the *commutators* of elements of G, i.e., elements of the form $[x, y] \coloneqq xyx^{-1}y^{-1}$. Show

a. G/G' is abelian. G/G' is called the *abelianization* of G and denoted by G^{ab}.

b. If $N \triangleright G$ and G/N is abelian then $G' \subseteq N$. In particular, the abelianization $G^{ab} = G/G'$ of G is the maximal abelian quotient of G.

c. If $G' \subseteq H \subseteq G$ then $H \triangleright G$.

Solution a. We first check that G' is normal. Let $g \in G$ and $g' \in G'$. Then

$$gg'g^{-1} = g'(g')^{-1}gg'g^{-1} = g'[(g')^{-1}, g] \in G'.$$

So G' is normal, which means that G/G' is a group.
Let $g_1G', g_2G' \in G/G'$. Since $g_2^{-1}g_1^{-1}g_2g_1 \in G'$,

$$g_1G'g_2G' = g_1g_2G' = g_1g_2(g_2^{-1}g_1^{-1}g_2g_1)G' = g_2g_1G' = g_2G'g_1G',$$

so G' is abelian.

b. Since G/N is abelian, for $xyx^{-1}y^{-1} \in G'$,

$$xyx^{-1}y^{-1}N = xNyNx^{-1}Ny^{-1}N = N,$$

so $xyx^{-1}y^{-1} \in N$. Thus, $G' \subseteq N$.

c. Let $g \in G$ and $h \in H$, so

$$ghg^{-1} = ghg^{-1}h^{-1}h = [g, h]h \in H.$$

Thus, $gHg^{-1} \subseteq H$. Similarly,

$$h = g(g^{-1}hgh^{-1}h)g^{-1} = g[g^{-1}, h]g^{-1} \in gHg^{-1},$$

so $H \triangleleft G$.

14.12.2 Let G be an abelian group. Show that $G(p) \coloneqq \{x \in G \mid x^{p^r} = e \text{ for some } r\}$ is a subgroup of G.

Solution $e \in G(p)$, so unity holds.

For $x \in G(p)$, $x^{p^r - 1}$ is its inverse. It's clear that their product is e, so we just need to show that it's in $G(p)$:

$$(x^{p^r-1})^{p^r} = x^{p^r p^r - p^r} = (x^{p^r})^{p^r} = e.$$

If $x, y \in G(p)$, we have $x^{p^r} = y^{p^s} = e$, which means that

$$(xy)^{p^{r+s}} = (x^{p^r})^{p^s}(y^{p^s})^{p^r} = e,$$

so $xy \in G(p)$.

Thus $G(p)$ is a subgroup of G.

14.12.3 Show that $p(\mathbb{Z}/p^k\mathbb{Z}) \simeq \mathbb{Z}/p^{k-1}\mathbb{Z}$ for all primes p and positive integers k.

Solution Consider $\varphi\colon \mathbb{Z} \to p(\mathbb{Z}/p\mathbb{Z})$, $\varphi(n) = [pn]_p$. Then $\varphi(n) = e \iff pn = mp^{k-1}$ for some $m \in \mathbb{Z}$, so $\ker\varphi = p^{k-1}\mathbb{Z}$. Thus, by the first isomorphism theorem,

$$\mathbb{Z}/p^{k-1}\mathbb{Z} \simeq p(\mathbb{Z}/p^k\mathbb{Z}).$$

---

1 Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Label $(0,0)$ as 1, $(1,0)$ as 2, $(0,1)$ as 3, $(1,1)$ as 4.

Define the action

$$T_g(g') = g + g'.$$

Using the labels, you can get 4 permutations of $\{1,2,3,4\}$. What are the permutations?

Solution By inspection,

$$\begin{aligned} T'_{(0,0)} &\sim \mathrm{id} \\ T'_{(1,0)} &\sim \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix} \\ T'_{(0,1)} &\sim \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix} \\ T'_{(1,1)} &\sim \begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix}. \end{aligned}$$