51.1 Show that

$$\mathcal{L}[x\cos ax] = \frac{p^2 - a^2}{(p^2 + a^2)^2},$$

and use this result to find

$$\mathcal{L}^{-1}\left[\frac{1}{(p^2 + a^2)^2}\right].$$

Solution By the differentiation property of Laplace transforms,

$$\mathcal{L}[x\cos ax] = -\mathcal{L}[-x\cos ax] = -\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{p}{p^2 + a^2}\right) = -\frac{p^2 + a^2 - 2p^2}{(p^2 + a^2)^2} = \frac{p^2 - a^2}{(p^2 + a^2)^2}.$$

Then

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{(p^2 + a^2)^2}\right] &= \frac{1}{2a^2}\,\mathcal{L}^{-1}\left[\frac{1}{p^2 + a^2} - \frac{p^2 - a^2}{(p^2 + a^2)^2}\right] \\ &= \frac{1}{2a^2}\left(\frac{1}{a}\sin ax - x\cos ax\right) \\ &\boxed{= \frac{1}{2a^3}\sin ax - \frac{1}{2a^2}x\cos ax.}\end{aligned}$$

51.2 Find each of the following transforms:

a. $\mathcal{L}[x^2 \sin ax]$

b. $\mathcal{L}[x^{3/2}]$

Solution a.

$$\begin{aligned}\mathcal{L}[x^2 \sin ax] &= \mathcal{L}[(-1)^2 x^2 \sin ax] \\ &= \frac{\mathrm{d}^2}{\mathrm{d}p^2}\,\mathcal{L}[\sin ax] \\ &= \frac{\mathrm{d}^2}{\mathrm{d}p^2}\frac{a}{p^2 + a^2} \\ &\boxed{= \frac{2a(3p^2 - a^2)}{(p^2 + a^2)^3}}\end{aligned}$$

b.

$$\begin{aligned}\mathcal{L}[x^{3/2}] &= \mathcal{L}[(-1)^2 x^2 \cdot x^{-1/2}] \\ &= \frac{\mathrm{d}^2}{\mathrm{d}p^2}\sqrt{\frac{\pi}{p}} \\ &\boxed{= \frac{5}{4}\sqrt{\frac{\pi}{p^5}}}\end{aligned}$$

51.3 Solve each of the following differential equations:

a. $xy'' + (3x-1)y' - (4x+9)y = 0$, $y(0) = 0$

b. $xy'' + (2x+3)y' + (x+3)y = 3e^{-x}$, $y(0) = 0$

Solution We'll first deal with the general problem

$$xy'' + (ax+b)y' + (cx+d)y = f(x),\ y(0) = 0.$$

We can rewrite the equation as

$$x(y'' + ay' + cy) + (by' + dy) = f(x).$$

Applying the Laplace transform, letting $Y(p) \coloneqq \mathcal{L}[y]$, and $F(p) \coloneqq \mathcal{L}[f]$, we get

$$\begin{aligned}
\mathcal{L}[x(y'' + ay' + cy)] + \mathcal{L}[by' + dy] &= F \\
-\frac{\mathrm{d}}{\mathrm{d}p}\mathcal{L}[y'' + ay' + cy] + (bpY - by(0) + dY) &= F \\
-\frac{\mathrm{d}}{\mathrm{d}p}(p^2Y - y'(0) + apY + cY) + (bpY + dY) &= F \\
-(2pY + p^2Y' + aY + apY' + cY') + bpY + dY &= F \\
-(p^2 + ap + c)Y' + (bp + d - 2p - a)Y &= F \\
\implies Y' = \frac{(b-2)p + d - a}{p^2 + ap + c}Y - \frac{F}{p^2 + ap + c}&.
\end{aligned}$$

a. In this problem, we have $a = 3$, $b = -1$, $c = -4$, $d = -9$, and $f(x) \equiv 0$, which gives

$$\begin{aligned}
Y' &= \frac{-3p - 9 - 3}{p^2 + 3p - 4}Y \\
\frac{Y'}{Y} &= -3\frac{p+4}{(p+4)(p-1)} \\
\frac{Y'}{Y} &= -\frac{3}{p-1} \\
Y(p) &= C\frac{2!}{(p-1)^3} \\
&\boxed{y(x) = Ce^x x^2.}
\end{aligned}$$

b. For this problem, $a = 2$, $b = 3$, $c = 1$, $d = 3$, and $f(x) = 3e^{-x}$, so

$$\begin{aligned}
Y' &= \frac{p + 3 - 2}{p^2 + 2p + 1}Y - \frac{3}{p^2 + 2p + 1}\frac{1}{p+1} \\
Y' &= \frac{p+1}{(p+1)^2}Y - \frac{3}{(p+1)^3} \\
Y' - \frac{1}{(p+1)}Y &= -\frac{3}{(p+1)^3}.
\end{aligned}$$

This is a first order linear equation in Y, and it has the integrating factor

$$\mu(p) = \exp\left(-\int \frac{1}{p+1}\right) = \frac{1}{p+1}$$

which gives us

$$\begin{aligned}
(Y\mu)' &= -\frac{3}{(p+1)^4} \\
Y\mu &= \frac{1}{(p+1)^3} + C \\
Y(p) &= \frac{1}{(p+1)^2} + \frac{C}{p+1} \\
y(x) &= e^{-x}x + Ce^{-x}.
\end{aligned}$$

Applying the initial condition gives $C = 0$, so the solution to the ODE is $\boxed{y(x) = e^{-x}x.}$

51.4 If $y(x)$ satisfies the differential equation

$$y'' + x^2 y = 0,$$

where $y(0) = y_0$ and $y'(0) = y_0'$, show that its transform $Y(p)$ satisfies the equation

$$Y'' + p^2 Y = py_0 + y_0'.$$

Observe that the second equation is of the same type as the first, so that no progress has been made. The method of Example 3 is advantageous only when the coefficients are first degree polynomials.

Solution If we apply the Laplace transform to the problem, we get

$$\begin{aligned}
p^2 Y - py_0 - y_0' + \frac{\mathrm{d}^2}{\mathrm{d}p^2} Y &= 0 \\
Y'' + p^2 Y &= py_0 + y_0'.
\end{aligned}$$

51.5 If a and b are positive constants, evaluate the following integrals:

a. $\displaystyle\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,\mathrm{d}x$

b. $\displaystyle\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,\mathrm{d}x$

Solution a. Notice that

$$\begin{aligned}
\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,\mathrm{d}x &= -\int_0^\infty \int_a^b e^{-tx}\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_a^b \int_0^\infty e^{-tx}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_a^b \frac{1}{t}\,\mathrm{d}t \\
&\boxed{= \ln\left(\frac{b}{a}\right).}
\end{aligned}$$

b. By the integral property of the Laplace transform,

$$\begin{aligned}
\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,\mathrm{d}x &= \mathcal{L}\left[\frac{\sin bx}{x}\right](a) \\
&= \int_a^\infty \mathcal{L}[\sin bx](p)\,\mathrm{d}p \\
&= \int_a^\infty \frac{b}{p^2+b^2}\,\mathrm{d}p \\
&= \arctan\left(\frac{p}{b}\right)\Big|_a^\infty \\
&\boxed{= \frac{\pi}{2} - \arctan\left(\frac{a}{b}\right)}
\end{aligned}$$

51.8 a. If $f(x)$ is periodic with period a, so that $f(x+a) = f(x)$, show that

$$F(p) = \frac{1}{1-e^{-ap}} \int_0^a e^{-px} f(x)\,\mathrm{d}x.$$

b. Find $F(p)$ if $f(x) = 1$ in the intervals from 0 to 1, 2 to 3, 4 to 5, etc., and $f(x) = 0$ in the remaining intervals.

Solution a. By definition,

$$\begin{aligned}
F(p) = \int_0^\infty e^{-px} f(x)\,\mathrm{d}x &= \int_0^a e^{-px} f(x)\,\mathrm{d}x + \int_a^{2a} e^{-px} f(x)\,\mathrm{d}x + \int_{2a}^{3a} e^{-px} f(x)\,\mathrm{d}x \cdots \\
&= \int_0^a e^{-px} f(x)\,\mathrm{d}x + \int_0^a e^{-p(x-a)} f(x-a)\,\mathrm{d}x + \int_0^a e^{-p(x-2a)} f(x-2a)\,\mathrm{d}x + \cdots \\
&= \int_0^a e^{-px} f(x)\,\mathrm{d}x + e^{ax}\int_0^a e^{-px} f(x)\,\mathrm{d}x + e^{2a}\int_0^a e^{-px} f(x)\,\mathrm{d}x + \cdots \\
&= \left(\sum_{n=0}^\infty e^{-nax}\right)\int_0^a e^{-px} f(x)\,\mathrm{d}x \\
&= \frac{1}{1-e^{-ax}}\int_0^a e^{-px} f(x)\,\mathrm{d}x.
\end{aligned}$$

b. By definition of the Laplace transformation,

$$\begin{aligned}
F(p) &= \int_0^\infty e^{-px} f(x)\,\mathrm{d}x \\
&= \int_0^1 e^{-px}\,\mathrm{d}x + \int_2^3 e^{-px}\,\mathrm{d}x + \int_4^5 e^{-px}\,\mathrm{d}x + \cdots \\
&= \sum_{n=0}^\infty \int_n^{n+1} e^{-px}\,\mathrm{d}x \\
&= \frac{1}{p}\sum_{n=0}^\infty \left(e^{-pnx} - e^{-p(n+1)x}\right).
\end{aligned}$$

The sum telescopes, so the sum is given by

$$\lim_{n\to\infty} 1 - e^{-p(n+1)x} = 1,$$

and the Laplace transform is thus given by

$$\frac{1}{p}.$$

52.1 Find $\mathcal{L}^{-1}[1/(p^2+a^2)^2]$ by convolution.

Solution We can write

$$\mathcal{L}^{-1}\left[\frac{1}{(p^2+a^2)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{p^2+a^2}\cdot\frac{1}{p^2+a^2}\right] = \frac{1}{a^2}\,\mathcal{L}^{-1}\Big[\mathcal{L}[\sin ax]\,\mathcal{L}[\sin ax]\Big],$$

so we can use the convolution formula to get

$$\begin{aligned}
\frac{1}{a}\,\mathcal{L}^{-1}\Big[\mathcal{L}[\sin ax]\,\mathcal{L}[\sin ax]\Big] &= \frac{1}{a}(\sin ax * \sin ax)\\
&= \frac{1}{a^2}\int_0^x \sin a(x-\tau)\sin a\tau\,\mathrm{d}\tau\\
&= \frac{1}{a^2}\int_0^x (\sin ax\cos a\tau - \sin a\tau\cos ax)\sin a\tau\,\mathrm{d}\tau\\
&= \frac{1}{a^2}\int_0^x \sin ax\cos a\tau\sin a\tau - \frac{1}{2}\cos ax + \frac{\cos 2a\tau}{2}\cos ax\,\mathrm{d}\tau\\
&= \frac{1}{a^2}\left[\frac{1}{2a}\sin ax\sin^2 a\tau - \frac{1}{2}\tau\cos ax + \frac{1}{4a}\sin 2a\tau\cos ax\right]_0^x\\
&= \frac{1}{a^2}\left[\frac{1}{2a}\sin^3 ax - \frac{1}{2}x\cos ax + \frac{1}{2a}\sin ax\cos^2 ax\right]\\
&= \frac{1}{a^2}\left[\frac{1}{2a}\sin ax - \frac{1}{2}x\cos ax\right]\\
&= \frac{1}{2a^3}\sin ax - \frac{1}{2a^2}x\cos ax.
\end{aligned}$$

52.2 Solve each of the following integral equations:

a. $y(x) = 1 - \displaystyle\int_0^x (x-t)y(t)\,\mathrm{d}t$

c. $e^{-x} = y(x) + 2\displaystyle\int_0^x \cos(x-t)y(t)\,\mathrm{d}t$

Solution a. Notice that we can write the equation as

$$y(x) = 1 - (t * y(t))(x),$$

so applying the Laplace transform gives

$$\begin{aligned}
\mathcal{L}[y] &= \frac{1}{p} - \mathcal{L}[p]\,\mathcal{L}[y]\\
\mathcal{L}[y] &= \frac{1}{p} - \frac{1}{p^2}\,\mathcal{L}[y]\\
\left(\frac{1}{p^2}+1\right)\mathcal{L}[y] &= \frac{1}{p}\\
\mathcal{L}[y] &= \frac{p}{p^2+1}\\
&\boxed{y(x) = \cos x.}
\end{aligned}$$

c. We can write the equation as

$$e^{-x} = y(x) + 2(\cos t * y(t))(x).$$

Applying the Laplace transformation then gives us

$$\begin{aligned}
\frac{1}{p+1} &= \mathcal{L}[y] + 2\,\mathcal{L}[\cos t]\,\mathcal{L}[y] \\
\frac{1}{p+1} &= \mathcal{L}[y] + \frac{2p}{p^2+1}\,\mathcal{L}[y] \\
\frac{1}{p+1} &= \frac{(p+1)^2}{p^2+1}\,\mathcal{L}[y] \\
\mathcal{L}[y] &= \frac{p^2+1}{(p+1)^3} \\
\mathcal{L}[y] &= \frac{1}{p+1} - \frac{2}{(p+1)^2} + \frac{2}{(p+1)^3} \\
&\boxed{y(x) = e^{-x} - 2e^{-x}x + e^{-x}x^2.}
\end{aligned}$$

52.5 Show that the differential equation

$$y'' + a^2 y = f(x), \qquad y(0) = y'(0) = 0,$$

has

$$y(x) = \frac{1}{a}\int_0^x f(t)\sin a(x-t)\,\mathrm{d}t$$

as its solution.

Solution By Laplace transformation, the differential equation becomes

$$\begin{aligned}
p^2\,\mathcal{L}[y] + a^2\,\mathcal{L}[y] &= \mathcal{L}[f] \\
\mathcal{L}[y] &= \frac{\mathcal{L}[f]}{p^2+a^2} \\
\mathcal{L}[y] &= \frac{1}{a}\,\mathcal{L}[\sin ax]\,\mathcal{L}[f] \\
y(x) &= \frac{1}{a}(\sin at * f)(x) \\
y(x) &= \frac{1}{a}\int_0^x f(t)\sin a(x-t)\,\mathrm{d}t.
\end{aligned}$$

53.1 Show that $f(t)*g(t) = g(t)*f(t)$ directly from the definition, by introducing a new dummy variable $\sigma = t-\tau$. This shows that the operation of forming convolutions is commutative. It is also associative and distributive:

$$f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$$

and

$$\begin{aligned}
f(t) * [g(t) + h(t)] &= f(t) * g(t) + f(t) * h(t) \\
[f(t) + g(t)] * h(t) &= f(t) * h(t) + g(t) * h(t).
\end{aligned}$$

Solution If we use the change of variables $t \mapsto x - t$, we get

$$\begin{aligned}
(f(t) * g(t))(x) &= \int_0^x f(x-t)g(t)\,\mathrm{d}t \\
&= -\int_x^0 f(t)g(t-x)\,\mathrm{d}t \\
&= \int_0^x f(t)g(t-x)\,\mathrm{d}t,
\end{aligned}$$

so the operation commutes.

53.2 Find the convolution of each of the following pairs of functions:

b. e^{at}, e^{bt}, where $a \neq b$

c. t, e^{at}

Solution b.

$$\begin{aligned} e^{at} * e^{bt} &= \int_0^x e^{a(x-t)} e^{bt} \, \mathrm{d}t \\ &= \int_0^x e^{ax} e^{(b-a)t} \, \mathrm{d}t \\ &= \frac{e^{ax}}{b-a}(e^{(b-a)x} - 1) \\ &\boxed{= \frac{e^{bx} - e^{ax}}{b-a}} \end{aligned}$$

c.

$$\begin{aligned} t * e^{at} &= \int_0^x (x-t) e^{at} \, \mathrm{d}t \\ &= \int_0^x x e^{at} - t e^{at} \, \mathrm{d}t \\ &= \frac{x}{a} e^{ax} - \frac{x}{a} - \frac{e^{ax}}{a^2}(ax - 1) - \frac{1}{a^2} \\ &= \frac{x}{a} e^{ax} - \frac{x}{a} - \frac{x}{a} e^{ax} + \frac{e^{ax}}{a^2} - \frac{1}{a^2} \\ &\boxed{= \frac{1}{a^2}(e^{ax} - ax - 1)} \end{aligned}$$

53.4 Use the methods of both Examples 1 and 2 to solve each of the following differential equations:

a. $y'' + 5y' + 6y = 5e^{3t}$, $y(0) = y'(0) = 0$

b. $y'' + y' - 6y = t$, $y(0) = y'(0) = 0$

Solution a. Method of Example 1:

We first consider the equation

$$A'' + 5A' + 6A = u(t), \quad A(0) = A'(0) = 0.$$

We know that

$$\mathcal{L}[A] = \frac{1}{p(p^2 + 5p + 6)} = \frac{1}{p(p+2)(p+3)} = \frac{1/6}{p} - \frac{1/2}{p+2} + \frac{1/3}{p+3},$$

so

$$A(t) = \frac{1}{6} - \frac{1}{2} e^{-2t} + \frac{1}{3} e^{-3t}.$$

By the formula given in the book,

$$\begin{aligned} y(t) &= A(0) f(t) + \int_0^t A'(t-\tau) f(\tau) \, \mathrm{d}\tau \\ &= \int_0^t \left(e^{-2(t-\tau)} - e^{-3(t-\tau)} \right) 5e^{3\tau} \, \mathrm{d}\tau \\ &= \int_0^t 5e^{-2t} e^{5\tau} - 5e^{-3t} e^{6\tau} \, \mathrm{d}\tau \\ &= e^{-2t} e^{5t} - e^{-2t} - \frac{5}{6} e^{-3t} e^{6t} + \frac{5}{6} e^{-3t} \\ &\boxed{= \frac{1}{6} e^{-2t} - e^{-2t} + \frac{5}{6} e^{-3t}.} \end{aligned}$$

Method of Example 2:

We consider the equation $h'' + 5h' + 6h = \delta(t)$. Then the Laplace transformation yields

$$\mathcal{L}[h] = \frac{1}{p+5p+6} = \frac{1}{(p+2)(p+3)} = \frac{1}{p+2} - \frac{1}{p+3} \implies h(t) = e^{-2t} - e^{-3t}.$$

Thus, by a formula given in the book,

$$\begin{aligned} y(t) &= \int_0^t h(t-\tau) f(\tau)\,\mathrm{d}\tau \\ &= \int_0^t (e^{2\tau-2t} - e^{3\tau-3t}) 5e^{3\tau}\,\mathrm{d}\tau \\ &= \int_0^t 5e^{-2t}e^{5\tau} - 5e^{-3t}e^{6\tau}\,\mathrm{d}\tau. \end{aligned}$$

This is the same integral as the above, so we get

$$\boxed{y(t) = \frac{1}{6}e^{-2t} - e^{-2t} + \frac{5}{6}e^{-3t}.}$$

Method of Example 1:

b. We consider $A'' + A - 6A = u(t)$, with $A(0) = A'(0) = 0$. A Laplace transformation gives

$$\mathcal{L}[A] = \frac{1}{p(p^2+p-6)} = \frac{1}{p(p+3)(p-2)} = -\frac{1/6}{p} + \frac{1/15}{p+3} + \frac{1/10}{p-2} \implies A(t) = -\frac{1}{16} + \frac{1}{15}e^{-3t} + \frac{1}{10}e^{2x}.$$

Thus, we get

$$\begin{aligned} y(t) &= \int_0^t A'(t-\tau) f(\tau)\,\mathrm{d}\tau \\ &= \int_0^t \left(-\frac{1}{5}e^{3\tau-3t} + \frac{1}{5}e^{2t-2\tau}\right)\tau\,\mathrm{d}\tau \\ &= \int_0^t -\frac{1}{5}e^{-3t}\tau e^{3\tau} + \frac{1}{5}e^{2t}\tau e^{-2\tau} \\ &\boxed{= \frac{1}{20}e^{2t} - \frac{1}{45}e^{-3t} - \frac{1}{6}t + \frac{1}{36}.} \end{aligned}$$

Method of Example 2:

In this case, we have

$$\mathcal{L}[h] = \frac{1}{p^2+p-6} = \frac{1}{(p+3)(p-2)} = -\frac{1/5}{p+3} + \frac{1/5}{p-2} \implies h(t) = -\frac{1}{5}e^{-3t} + \frac{1}{5}e^{2t}.$$

Thus, by the formula,

$$\begin{aligned} y(t) &= \int_0^t \left(-\frac{1}{5}e^{3\tau-3t} + \frac{1}{5}e^{2t-2\tau}\right)\tau\,\mathrm{d}\tau \\ &= \int_0^t -\frac{1}{5}e^{-3t}\tau e^{3\tau} + \frac{1}{5}e^{2t}\tau e^{-2\tau}\,\mathrm{d}\tau. \end{aligned}$$

We get the same integral as the above, so

$$\boxed{y(t) = \frac{1}{20}e^{2t} - \frac{1}{45}e^{-3t} - \frac{1}{6}t + \frac{1}{36}.}$$