1 Prove that a polynomial of degree n is uniformly continuous on $\mathbb{R}$ if and only if $n = 0$ or $n = 1$.

Solution "$\Longleftarrow$"

Suppose $p(x)$ is a polynomial of degree $n = 0$. Then $p(x) \equiv a$ for some $a \in \mathbb{R}$. Then for all $x, y \in \mathbb{R}$, $|p(x) - p(y)| = 0$, so $p(x)$ is uniformly continuous on $\mathbb{R}$.

Suppose $p(x)$ is a polynomial of degree $n = 1$. Then $p(x) = a_0 + a_1 x$ for some $a_0, a_1 \in \mathbb{R}$ with $a_1 \neq 0$. Fix $\varepsilon > 0$ and choose $\delta = \frac{\varepsilon}{|a_1|}$. Then for all $x, y \in \mathbb{R}$ such that $|x - y| < \delta$, we have

$$|p(x) - p(y)| = |a_0 + a_1 x - (a_0 + a_1 y)| = |a_1||x - y| < |a_1| \frac{\varepsilon}{|a_1|} = \varepsilon.$$

Hence, by definition, $p(x)$ is uniformly continuous on $\mathbb{R}$.

"$\Longrightarrow$"

Let $p(x)$ be a polynomial of degree $n \geq 2$. Then we can write $p(x) = \sum_{i=0}^n a_i x^i$, with $a_n \neq 0$.

Fix $\varepsilon > 0$ and let $\delta > 0$. Then for all $|x - y| < \delta \implies x - \delta < y < x + \delta$,

$$\begin{aligned}
|p(x) - p(y)| &= \left| \sum_{i=0}^n a_i x^i - \sum_{i=0}^n a_i y^i \right| \\
&= \left| \sum_{i=1}^n a_i (x^i - y^i) \right| \\
&= \delta \left| \sum_{i=1}^n a_i \sum_{k=0}^{i-1} x^{i-1-k} y^k \right| \\
&\geq \delta \left| \sum_{i=1}^n a_i \sum_{k=0}^{i-1} x^{i-1-k} (x - \delta)^k \right| \xrightarrow{x \to \infty} \infty
\end{aligned}$$

Thus, for all $\delta > 0$, we can find x sufficiently large enough so that $|p(x) - p(y)| \geq \varepsilon$. Hence, polynomials of degree 2 or higher are not uniformly continuous.

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2 Let $f \colon [0, 1] \to [0, 1]$ be a continuous function such that $f(0) = 0$ and $f(1) = 1$. Consider the sequence of functions $f_n \colon [0, 1] \to [0, 1]$ defined as follows:

$$f_1 = f \quad \text{and} \quad f_{n+1} = f \circ f_n \text{ for } n \geq 1$$

Prove that if $\{f_n\}_{n \geq 1}$ converges uniformly, then $f(x) = x$ for all $x \in [0, 1]$.

Solution Note that if $f_n \xrightarrow{n \to \infty} F$, then since f is continuous and f_n converges pointwise to F, we have that $f_{n+1} = f \circ f_n \xrightarrow{n \to \infty} f \circ F$. Thus, by uniqueness of limits, $F(x) = f(F(x))$, so $F(x)$ is a fixed point of f for all $x \in [0, 1]$.

$F(x)$ is not constant, since $F(0) = 0 \neq 1 = F(1)$. Moreover, as f_n converges uniformly to F, F is continuous on $[0, 1]$. Since $[0, 1]$ is connected, F has the Darboux property on this interval.

Hence, for all $y \in [0, 1]$, we there exists $x \in [0, 1]$ such that $f(F(x)) = F(x) \implies f(y) = y$, as desired.

3 Let

$$\mathcal{F} = \big\{ f \in C(\mathbb{R}) \mid \lim_{|x|\to\infty} f(x) = 0 \big\}.$$

Show that $\mathcal{F}$ is closed in $C(\mathbb{R})$.

Solution Let $f \in \mathcal{F}$. We claim that $\{f_n\}_{n\geq 1} \subseteq \mathcal{F}$ with $f_n \colon \mathbb{R} \to \mathbb{R}$, with $f_n = \big(1 - \frac{1}{n}\big) f$ belongs to $\mathcal{F}$ and converges to f with respect to the uniform metric.

f_n is continuous since it is the product of two continuous functions, $1 - \frac{1}{n}$ and f. Moreover, f_n is bounded above. Indeed, since $f \in C(\mathbb{R})$, there exists M such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$, which means that $|f_n(x)| \leq M\big(1 - \frac{1}{n}\big)$ for all $x \in \mathbb{R}$, so $f_n \in C(\mathbb{R})$.

Lastly,

$$\lim_{|x|\to\infty} f_n(x) = \lim_{|x|\to\infty} \left(1 - \frac{1}{n}\right) f(x) = \left(1 - \frac{1}{n}\right)\left(\lim_{|x|\to\infty} f(x)\right) = 0$$

and thus, $\{f_n\}_{n\geq 1} \subseteq \mathcal{F}$.

Let $\varepsilon > 0$. If $n_\varepsilon > \frac{1}{\varepsilon}$, then for all $n \geq n_\varepsilon$,

$$\begin{aligned} d(f, f_n) &= \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \\ &= \sup_{x\in\mathbb{R}} \left| f(x) - \left(1 - \frac{1}{n}\right) f(x) \right| \\ &= \sup_{x\in\mathbb{R}} \left| \frac{1}{n} \right| \\ &= \frac{1}{n} \leq \frac{1}{n_\varepsilon} < \varepsilon \end{aligned}$$

so $f_n \xrightarrow{n\to\infty} f$. Hence, $f \in \overline{\mathcal{F}} \implies \mathcal{F} = \overline{\mathcal{F}}$.

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4 Let $f \colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x^2}$. Find

a. an open set $D \subseteq \mathbb{R}$ such that $f(D)$ is not open;

b. a closed set $F \subseteq \mathbb{R}$ such that $f(F)$ is not closed;

c. a set $A \subseteq \mathbb{R}$ such that $f(\bar{A}) \neq \overline{f(A)}$.

Solution Note that f is continuous since it is the composition of two continuous functions: e^x and $-x^2$. Moreover, since $f'(x) = -2xe^{-x^2}$, f is strictly increasing on $(-\infty, 0)$ and strictly decreasing on $(0, \infty)$. So, $f(0) = 1$ is the maximum value of f. Lastly, $\lim_{|x|\to\infty} e^{-x^2} = 0$, and $\mathbb{R}$ is connected, so by the continuity of f, the image of f is the interval $(0, 1]$.

We claim that $\mathbb{R}$ satisfies all three criteria. Note that $\mathbb{R}$ is both closed and open.

a. If $D = \mathbb{R}$, then $f(D) = (0, 1]$, which is not open.

b. If $F = \mathbb{R}$, then $f(F) = (0, 1]$, which is not closed.

c. If $A = \mathbb{R}$, note that $A = \bar{A}$ and $f(A) = (0, 1] \implies \overline{f(A)} = [0, 1]$. But $f(\bar{A}) = (0, 1] \neq [0, 1] = \overline{f(A)}$.

5 Let $\{F_n\}_{n\geq 1}$ be a sequence of closed sets such that $F_n \subseteq F_{n+1}$ for all $n \geq 1$. Set $F = \bigcup_{n\geq 1} F_n$ and $F_0 = \emptyset$. For $n \geq 1$ we define

$$A_n = \big[(F_n \setminus F_{n-1}) \setminus \operatorname{Int}(F_n \setminus F_{n-1})\big] \cup \big[\operatorname{Int}(F_n \setminus F_{n-1}) \cap \mathbb{Q}\big].$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 2^{-n} & \text{if } x \in A_n \\ 0 & \text{if } x \notin \bigcup_{n\geq 1} A_n. \end{cases}$$

Show that f is discontinuous on F and continuous on $\mathbb{R} \setminus F$.

Solution We first show f is discontinuous on F.

Fix $n \geq 1$. Then $F_n = \bigcup_{i=1}^n F_i \setminus F_{i-1}$. This is true for $n = 1$ since clearly $F_1 = (F_1 \setminus \emptyset) \cup \emptyset$. Moreover, if we assume it is true for n, then

$$F_{n+1} = (F_{n+1} \setminus F_n) \cup F_n = (F_{n+1} \setminus F_n) \cup \bigcup_{i=1}^{n} F_i \setminus F_{i-1} = \bigcup_{i=1}^{n+1} F_i \setminus F_{i-1}$$

so by induction, it holds for all n.

Let $x \in F$. Then there exists $n \geq 1$ such that $x \in F_n \setminus F_{n-1}$.

Case 1: $x \in \operatorname{Int}(F_n \setminus F_{n-1}) \cap \mathbb{Q}$

Then $x \in A_n \implies f(x) = 2^{-n}$.

As $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$ and $\operatorname{Int}(F_n \setminus F_{n-1})$ is open, there exists $\{x_k\}_{k\geq 1} \subseteq \operatorname{Int}(F_n \setminus F_{n-1}) \cap (\mathbb{R} \setminus \mathbb{Q})$ with $x_k \xrightarrow{k\to\infty} x$. But each $x_k \notin A_i$ for all $i \geq 1$. Hence, $f(x_k) = 0$ for all $k \geq 1$, so

$$f(x_k) \xrightarrow{k\to\infty} 0 \neq 2^{-n} = f(x).$$

So, in this case, f is not continuous at x.

Case 2: $x \in (F_n \setminus F_{n-1}) \setminus \operatorname{Int}(F_n \setminus F_{n-1})$

Then $x \in A_n \implies f(x) = 2^{-n}$.

If $\operatorname{Int}(F_n \setminus F_{n-1}) = \emptyset$, then since the only connected subsets of $\mathbb{R}$ are intervals, x must be isolated. But then we can't have $\lim_{y\to x} f(y) = 2^{-n}$ since there are no other points close to x whose image under f is 2^{-n}.

Otherwise, there exists $\{x_k\}_{k\geq 1} \subseteq \operatorname{Int}(F_n \setminus F_{n-1}) \cap (\mathbb{R} \setminus \mathbb{Q})$ such that $x_k \xrightarrow{k\to\infty} x$. But then

$$f(x_k) = 0 \xrightarrow{k\to\infty} 0 \neq f(x) = 2^{-n}.$$

so in this case, f is not continuous at x.

Case 3: $x \in \operatorname{Int}(F_n \setminus F_{n-1}) \cap (\mathbb{R} \setminus \mathbb{Q})$

Then $x \notin A_i$ for all $i \geq 1$, so $f(x) = 0$. As $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\operatorname{Int}(F_n \setminus F_{n-1})$ is open, there exists $\{x_k\}_{k\geq 1} \subseteq \operatorname{Int}(F_n \setminus F_{n-1}) \cap \mathbb{Q}$ with $x_k \xrightarrow{k\to\infty} x$. But then $x_k \in A_n$, so

$$f(x_k) = 2^{-n} \implies f(x_k) \xrightarrow{k\to\infty} 2^{-n} \neq 0 = f(x).$$

So in this case, f is not continuous at x.

Thus, in all cases, f is discontinuous at $x \in F$.

We next show that f is continuous on $\mathbb{R} \setminus Q$.

For $x \in \mathbb{R} \setminus F$, then $x \notin A_n$ for all $n \geq 1$. Indeed, $x \notin F = \bigcup_{n\geq 1} F_n \setminus F_{n-1}$, so for all $n \geq 1$, $x \notin F_n \setminus F_{n-1} \implies x \notin A_n$. Thus, for all $x \in \mathbb{R} \setminus F$, $f(x) \equiv 0$, so f is continuous on $\mathbb{R} \setminus F$.

6 Let (X, d) be a metric space with at least two points and let $\mathcal{A} \subseteq C(X)$ be an algebra that is dense in the metric space $C(X)$.

a. Show that $\mathcal{A}$ separates points on X.

b. Show that $\mathcal{A}$ vanishes at no point in X.

See pages 161–162 in Rudin's textbook for the definitions.

Solution a. Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Consider f continuous and bounded such that $f(x_1) > f(x_2)$. If $f \in \mathcal{A}$, then we are done. Otherwise, as $\mathcal{A}$ is dense in $C(X)$, there exists $\{f_n\}_{n\geq 1} \subseteq \mathcal{A}$ such that $f_n \xrightarrow{n\to\infty} f$ with respect to the uniform metric.
By definition, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\begin{aligned} |f(x) - f_n(x)| \leq \sup_{x\in X} |f(x) - f_n(x)| &< \frac{f(x_1) - f(x_2)}{2} \\ \implies f(x) + \frac{f(x_2) - f(x_1)}{2} < f_n(x) < f(x) &+ \frac{f(x_1) - f(x_2)}{2} \end{aligned}$$

for all $x \in X$. Evaluating the inequality at x_1 and x_2 yields

$$\begin{aligned} \frac{f(x_2) + f(x_1)}{2} &< f_n(x_1) \\ f_n(x_2) &< \frac{f(x_1) + f(x_2)}{2} \\ \implies f_n(x_2) &< f_n(x_1). \end{aligned}$$

Hence, $\mathcal{A}$ separates points on X.

b. Let $x_0 \in X$. Then consider $f(x) \equiv 1$, which is continuous and bounded. It clearly doesn't vanish at x_0. If $f \in \mathcal{A}$, then we're done. Otherwise, as $\mathcal{A}$ is dense, there exists $\{f_n\}_{n\geq 1} \subseteq \mathcal{A}$ such that $f_n \xrightarrow{n\to\infty} f$ uniformly.
By definition, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$f(x) - f_n(x) \leq \sup_{x\in X} |f(x) - f_n(x)| < \frac{1}{2} \implies f_n(x_0) > \frac{1}{2}.$$

Hence, $\mathcal{A}$ vanishes at no point in X.

7 a. Show that given any continuous function $f\colon [0,1]\times[0,1]\to\mathbb{R}$ and any $\varepsilon>0$ there exist $n\in\mathbb{N}$ and functions $g_1,\dots,g_n,h_1,\dots,h_n\in C([0,1])$ such that

$$\left|f(x,y)-\sum_{k=1}^{n} g_k(x)h_k(y)\right|<\varepsilon \quad \text{for all} \quad (x,y)\in[0,1]\times[0,1].$$

b. If $f(x,y)=f(y,x)$ for all $(x,y)\in[0,1]\times[0,1]$, can this be done with $g_k=h_k$ for each $1\le k\le n$? Justify your answer!

Solution a. Consider the compact metric space $([0,1]\times[0,1],d_2)$, where d_2 is the Euclidean metric. Consider the set $\mathcal{A}=\{P(x,y)\mid P \text{ is a polynomial}\}\subseteq C([0,1]\times[0,1])$. We claim that this is an algebra on $C([0,1]\times[0,1])$ that separates points and vanishes nowhere.

It is clearly an algebra. If $f,g\in\mathcal{A}$, then we can write

$$\begin{aligned} f(x,y)&=\sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}x^i y^j\\ g(x,y)&=\sum_{i=0}^{p}\sum_{j=0}^{q} b_{ij}x^i y^j. \end{aligned}$$

fg will be a polynomial of degree $n+m+p+q$, $f+g$ will be a polynomial of degree $\max\{n+m,p+q\}$, and cf will clearly be a polynomial, for $c\in\mathbb{R}$.

We now show that it separates points. Let $(x_0,y_0),(x_1,y_1)\in\mathbb{R}^2$ such that $x\neq x_1$ or $y\neq y_1$. Then $f(x,y)=a(x-x_0)+b(y-y_0)$, where $a,b\neq 0$. separates them. Indeed, $f(x_0,y_0)=0$, and $f(x_1,y_1)\neq 0$ since at least one of $(x-x_0)$ and $(y-y_0)$ is non-zero.

Lastly, it clearly vanishes at no point—take the constant polynomial $f(x,y)\equiv 1$.

By Stone–Weierstrass, $\mathcal{A}$ is dense in $C([0,1]\times[0,1])$.

Since $[0,1]\times[0,1]$ compact, f is continuous and bounded on that interval, so $f\in C([0,1]\times[0,1])$. So, there exists $\{P_n\}_{n\ge 1}$ such that $P_n\xrightarrow{n\to\infty} f$ with respect to the uniform metric (i.e., P_n converges to f uniformly). Thus, for all $\varepsilon>0$ and $(x,y)\in[0,1]\times[0,1]$, there exists $n_\varepsilon\in\mathbb{N}$ such that for all $n\ge n_\varepsilon$,

$$|f(x,y)-P_n|<\varepsilon$$

Pick $N\ge n_\varepsilon$. Since P_N is a polynomial, we can write it in the form

$$P_N(x,y)=\sum_{i=0}^{n} A_i x^{p(i)}y^{q(i)}$$

for some $n\in\mathbb{N}$, $A_i\in\mathbb{R}$, and $p,q\colon\mathbb{N}\to\mathbb{N}$. Thus, taking $g_k(x)=A_kx^{(k)}$ and $h_k(y)=y^{q(k)}$ gives

$$\left|f(x,y)-\sum_{k=0}^{n} g_k(x)h_k(y)\right|=\left|f(x,y)-\sum_{k=0}^{n} A_p x^{p(k)}y^{q(k)}\right|<\varepsilon$$

as desired.

b. No. Consider $f(x,y)=f(y,x)\equiv -1$. Then suppose there exist $g_1,\dots,g_k$ that satisfy (a). If $x=y$, we have

$$\left|f(x,x)-\sum_{k=1}^{n} g_k^2(x)\right|\ge|f(x,x)|=1$$

which does not satisfy what we want.