1 Show that any tree with two or more vertices has a vertex of degree 1.

Solution Pick any root for the tree, and let v be a vertex with maximal depth. Then $\delta(v) = 1$. Otherwise, suppose it is adjacent to two edges a and b. Since v can only have one parent, one of these vertices must be a child, but this child would have a larger depth than v, which is impossible.

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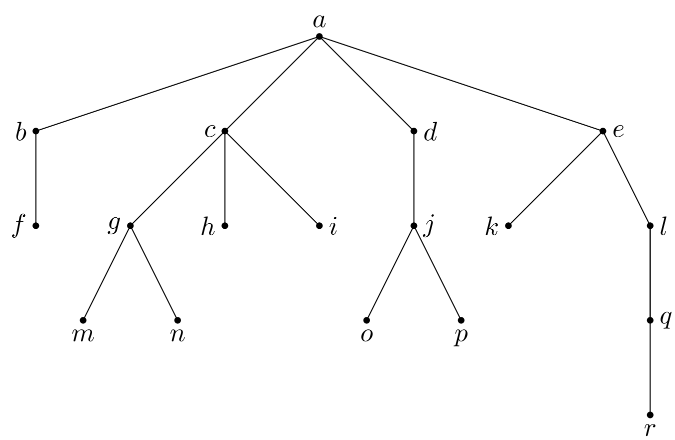
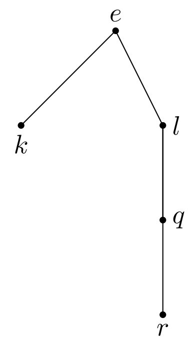
2 Answer the questions for the following tree:



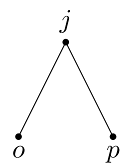
a. Find the parents of c and h.
b. Find the children of d and e.
c. Find the ancestors of c and j.
d. Find the descendants of c and e.
e. Find the terminal vertices.
f. Find the siblings of f and h.
g. Find the internal vertices.
h. Draw the subtree rooted at e.
i. Draw the subtree rooted at j.

Solution a. The parent of c is a, and the parent of h is c.
b. The child of d is j and the children of e are k and l.
c. The ancestor of c is a, and the ancestors of j are d and a.
d. The descendants of c are g, h, i, m, and n, and the descendants of e are k, l, q, and r.
e. The terminal vertices are f, m, n, h, i, o, p, k, and r.
f. f has no siblings, and the siblings of h are g and i.
g. The internal vertices are a, b, c, d, e, g, j, and l.

h.



i.



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3 Draw a graph having the given properties or explain why no such graph exists:

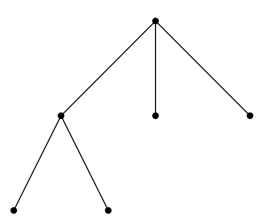
a. Acyclic; four edges, six vertices.

b. Tree; six vertices having degrees 1, 1, 1, 1, 3, 3.

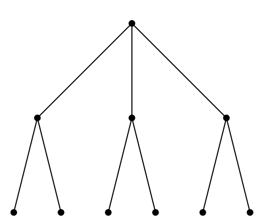
c. Tree; four internal vertices, six terminal vertices.

Solution a.

b.



c.



4 A *forest* is a simple graph with no cycles.

a. Explain why a forest is a union of trees.

b. If a forest F consists of m trees and n vertices, how many edges does F have?

Solution a. Consider the connected components of a forest. Since the forest has no cycles, the connected components also have no cycles, which means that the connected subgraphs are trees. Hence, since the forest is a union of these connected subgraphs, it is a union of trees.

b. Suppose that the k-th tree has n_k vertices, so that $\sum_k n_k = n$, where k ranges from 1 to m. Then the k-th tree has $n_k - 1$ edges, and so the total number of edges is

$$\sum_{k=1}^{m}(n_k - 1) = \sum_{k=1}^{m} n_k - \sum_{k=1}^{m} 1 = n - m.$$

---

5 If $P_1 = (v_0, \ldots, v_n)$ and $P_2 = (w_0, \ldots, w_m)$ are two distinct simple paths from $a = v_0 = w_0$ to $b = v_n = w_m$ in a simple graph G, is

$$(v_0, \ldots, v_{n-1}, v_n = w_m, w_{m-1}, \ldots, w_0)$$

necessarily a cycle? Explain.

Solution No, since P_1 and P_2 may possibly share an edge. For example, say $v_1 = w_1$. Then $(a, v_1) = (w_1, a)$ is a repeated edge in the concatenation.

---

6 Prove that T is a tree if and only if T is connected and when an edge is added between any two vertices, exactly one cycle is created.

Solution "$\Longrightarrow$"

Let T be a tree. By definition, T is connected.

Pick two vertices v and w. By definition of a tree, there is a simple path P from v to w. If we add the edge (v, w), then P concatenated with (v, w) is a cycle.

This is the only cycle: suppose (v, w) is an edge on two distinct cycles C_1 and C_2. Then v and w must both be part of these cycles, so $C_1 \setminus \{(v, w)\}$ and $C_2 \setminus \{(v, w)\}$ are both distinct simple paths from v to w. But this is impossible, as there can be exactly one simple path from v to w in a tree.

Hence, adding an edge between any two vertices creates precisely one cycle.

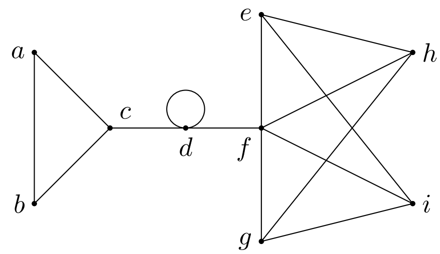
"$\Longleftarrow$"

We need to show that T has no cycles, and since T is connected, it suffices to show that T has exactly one simple path between any two vertices.

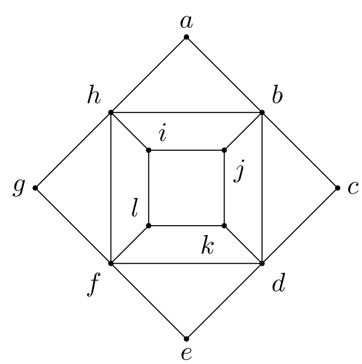
Pick two vertices v and w. Now suppose T has two distinct simple paths between v and w, which we call P_1 and P_2. Then if we add an edge (v, w) to T, then two cycles are created: $P_1 \cup \{(v, w)\}$ and $P_2 \cup \{(v, w)\}$, which is impossible. Thus, T cannot have two distinct simple paths between any two vertices, so T is a tree.

7 Find a spanning tree for each graph.

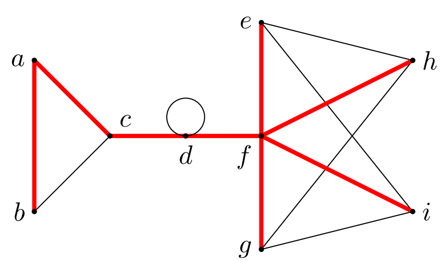
a.



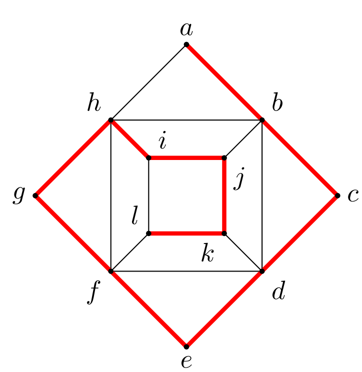
b.



Solution a.



b.



8 Under what conditions is an edge in a connected graph G contained in every spanning tree of G?

Solution The edge must be part of every spanning tree of G if and only if removing the edge causes G to be disconnected.

Indeed, if a spanning subgraph does not contain that edge, it cannot be connected, so it cannot be a tree. Conversely, say (v, w) is in every spanning tree of G, but removing (v, w) keeps G connected. Then there is a different simple path P from v to w, which means that given a spanning tree, we can replace (v, w) with P, and delete edges until we get a new spanning tree. But this new spanning tree does not contain (v, w), which is a contradiction.

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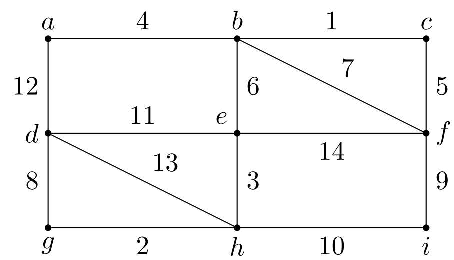
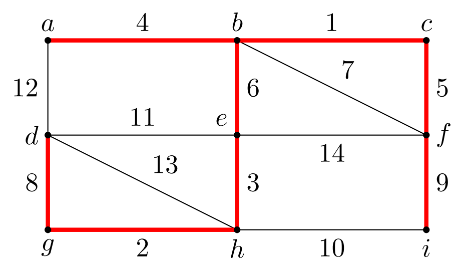
9 Find the minimal spanning tree in the following graph using Prim's algorithm:



Solution We start at the vertex a. Then we get the following minimal spanning tree:



This tree has total weight 38.

10 a. Let T and T' be two spanning trees of a connected graph G. Suppose that an edge x is in T but not T'. Show that there is an edge y in T' but not T such that $(T \setminus \{x\}) \cup \{y\}$ and $(T' \setminus \{y\}) \cup \{x\}$ are spanning trees of G.

b. Show that if all weights in a connected graph G are distinct, G has a unique minimal spanning tree.

Solution a. Suppose $x = (a, b)$. Since T is a tree, $T \setminus \{x\}$ is disconnected, since there are no other simple paths in T from a to b. Let V_a be the vertices in the connected component of a, and let V_b be the vertices in the connected component of b.

Since T' is a tree, there is exactly one simple path P from a to b in T'. This path starts in V_a and ends in V_b, so there must be an edge y which connects the two sets of vertices.

Hence, $(T \setminus \{x\}) \cup \{y\}$ must be a spanning tree. It is certainly connected, and if it had a cycle, that cycle must contain y as an edge. However, this means that we can remove y and the subgraph $T \setminus \{x\}$ will still be connected, which is impossible. Lastly, $(T \setminus \{x\}) \cup \{y\}$ has the same number of edges as T, which means it has the same number of vertices, i.e., it contains all vertices. Hence, it is a spanning tree.

On the other hand, $P \cup \{x\}$ is the unique simple cycle in $T' \cup \{x\}$ containing a and b, by problem (6). Hence, y is part of this cycle, so removing y breaks the only cycle in $T' \cup \{x\}$, which means that $(T' \setminus \{y\}) \cup \{x\}$ has no cycles. It is also connected: removing y and adding x replaces a path with another path between the two same vertices. Thus, $(T' \setminus \{y\}) \cup \{x\}$ is a spanning tree, by the same argument as before.

b. Let T and T' be two minimal spanning trees with total weight W, and from all the edges between T and T', let x be the edge with minimal weight w. Such an x is unique, since all the weights are distinct, so x is either in T or T'. Without loss of generality, assume that x is in T.

By (a), there exists an edge y in T' but not T so that $(T \setminus \{x\}) \cup \{y\}$ and $(T' \setminus \{y\}) \cup \{x\}$ are also spanning trees. If we denote the weight of an edge e by $w(e)$, then the total weight of $(T' \setminus \{y\}) \cup \{x\}$ is $W - w(y) + w(x) = W - (w(y) - w(x))$. $w(y) > w(x)$, which means that the total weight of this spanning tree is strictly smaller than W. But this is impossible, as we assumed that W was the minimal weight. Thus, G must have a minimal spanning tree.