1.12a Stereographic projection combined with rigid motions of the sphere can be used to describe some transformations of the plane.

Map a point $z \in \mathbb{C}$ to $\mathbb{S}^2$, apply a rotation of the unit sphere, then map the resulting point back to the plane. For a fixed rotation, find this map of the extended plane to itself as an explicit function of z. Two cases are worth working out first: rotation about the x_3 axis and rotation about the x_1 axis.

Solution Let $z \in \mathbb{C}$, where $z = |z|e^{\theta i =}|z|\cos\theta + i|z|\sin\theta$. Then its stereographic projection is

$$z^* = \left(\frac{2|z|}{|z|^2+1}\cos\theta, \frac{2|z|}{|z|^2+1}\sin\theta, \frac{|z|^2-1}{|z|^2+1}\right).$$

Given a point (x_1, x_2, x_3) on the sphere, its corresponding complex number is

$$z = \frac{x_1}{1-x_3} + i\frac{x_2}{1-x_3}$$

Next consider the matrix associated with a counter-clockwise rotation of φ about the x_3 axis:

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying this matrix to z^* yields

$$\begin{aligned} z^\dagger &= \left(\frac{2|z|}{|z|^2+1}(\cos\theta\cos\varphi - \sin\theta\sin\varphi), \frac{2|z|}{|z|^2+1}(\sin\theta\cos\varphi + \cos\theta\sin\varphi), \frac{|z|^2-1}{|z|^2+1}\right) \\ &= \left(\frac{2|z|}{|z|^2+1}\cos(\theta+\varphi), \frac{2|z|}{|z|^2+1}\sin(\theta+\varphi), \frac{|z|^2-1}{|z|^2+1}\right), \end{aligned}$$

which, when projected back onto $\mathbb{C}$, obviously gives $z' = |z|e^{(\theta+\varphi)i}$, so this is a rotation of the plane about the x_3 axis.

A rotation of the point (a, b, c) about the x_1 axis can be expressed as $(a, be^{i\varphi}, ce^{i\varphi})$. Define $w = b + ic$ and $w' = we^{i\varphi}$.

Then

$$z = \frac{a+ib}{1-c} \implies \frac{z-1}{z+1} = \frac{a+ib-1+c}{a+ib+1-c} = \frac{a-1+w}{a+1-\overline{w}} = \frac{a-1+w}{a+1-\frac{|w|^2}{w}} = \frac{a-1+w}{a+1-\frac{1-a^2}{w}} = \frac{w}{a+1}.$$

Applying the same algebra to z' yields

$$\frac{z'-1}{z'+1} = \frac{w'}{a+1} = \frac{z-1}{z+1}e^{i\varphi} \implies z' = \frac{z(1+e^{i\varphi})+1-e^{i\varphi}}{z(1-e^{i\varphi})+1+e^{i\varphi}}$$

The same calculation yields an identical formula for a rotation about the x_2 axis. Note that

$$ad - bc = (1+e^{i\varphi})^2 - (1-e^{i\varphi})^2 = 2 + 2e^{i2\varphi} > 0$$

Any proper rotation can be expressed as a composition of these three rotations. Composing these transformations will yield an expression of the form

$$\frac{az+b}{cz+d}$$

since all three rotations are of this form. Each rotation is invertible, so we also have $ad - bc \neq 0$.

2.5 a. Prove that f has a power series expansion about z_0 with radius of convergence $R > 0$ if and only if $g(z) = \frac{f(z)-f(z_0)}{z-z_0}$ has a power series expansion about z_0, with the same radius of convergence. (How must you define $g(z_0)$ in terms of the coefficients of the series for f to make this a true statement?)

b. It follows from (a) that if f has a power series expansion at z_0 with radius of convergence R and if $|z - z_0| \leq r < R$ then there is a constant C so that $f(z) - f(z_0) \leq C|z - z_0|$. Use the same idea to show that if $f(z) = \sum a_n(z - z_0)^n$ then

$$\left| f(z) - \sum_{n=0}^{k} a_n(z - z_0)^n \right| \leq D_k|z - z_0|^{k+1},$$

where D_k is a constant and $|z - z_0| \leq r < R$.

c. Use the proof of the root test to give an explicit estimate of D_k (for large k and therefore an estimate of the rate of convergence of the series of f if $|z - z_0| \leq r < R$).

Solution a. "$\implies$"

Suppose $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ with radius of convergence $R > 0$. Then

$$\begin{aligned} g(z) &= \frac{f(z) - f(z_0)}{z - z_0} \\ &= \frac{\sum_{n=0}^{\infty} a_n(z - z_0)^n - a_0}{z - z_0} \\ &= \frac{\sum_{n=1}^{\infty} a_n(z - z_0)^n}{z - z_0} \\ &= \sum_{n=1}^{\infty} a_n(z - z_0)^{n-1} \\ &= \sum_{n=0}^{\infty} a_{n+1}(z - z_0)^n \end{aligned}$$

So, we define $g(z_0) = a_1$. Clearly, $g(z)$ has the same radius of convergence as f as $f(z_0)$ and $\frac{1}{z-z_0}$ are constants, so they don't affect the convergence of the power series of g.

"$\impliedby$"

Suppose $g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$, which has a radius of convergence of $R > 0$. Then

$$\begin{aligned} \frac{f(z) - f(z_0)}{z - z_0} &= \sum_{n=0}^{\infty} a_n(z - z_0)^n \\ f(z) &= f(z_0) + \sum_{n=0}^{\infty} a_n(z - z_0)^{n+1} \\ &= \sum_{n=0}^{\infty} b_n(z - z_0)^n \end{aligned}$$

where $b_0 = f(z_0)$ and $b_n = a_n$ for all $n \geq 1$. $f(z_0)$ and $(z - z_0)$ are constants, so they don't affect the convergence of the series. Hence, the power series for f also has radius of convergence of R.

b. Assume f has a power series expansion $\sum a_n(z - z_0)^n$ about $z = z_0$ with radius of convergence $R > 0$. Note that

$$\left| f(z) - \sum_{n=0}^{k} a_n(z - z_0)^n \right| = \left| \sum_{n=k+1}^{\infty} a_n(z - z_0)^n \right|$$

will have the same radius of convergence as f, since we only subtract off finitely many terms. Hence, if $|z - z_0| \leq r < R$, then by the root test, the series on the right side converges uniformly. So, there exists D_k such that

$$\left| \sum_{n=k+1}^{\infty} a_n(z - z_0)^n \right| \leq D_k|z - z_0|^{k+1}$$

on $\{z \in \mathbb{C} \mid |z - z_0| \le r\}$.

c. Choose r_1 such that $|z - z_0| \le r < r_1 < R$. Then as $r_1 < R = \liminf_{n\to\infty} |a_n|^{-\frac{1}{n}}$, there exists $N \in \mathbb{N}$ such that for all $k \ge N$, we have $r_1 < |a_n|^{-\frac{1}{n}}$. Hence, for all $k \ge N$, we have

$$|a_k||z - z_0|^{k+1} \le \frac{r^{k+1}}{r_1^k} \implies \left|\sum_{n=k+1}^{\infty} a_n (z - z_0)^n\right| \le \sum_{n=0}^{\infty} r\left(\frac{r}{r_1}\right)^n = \frac{rr_1}{r_1 - r} \le \frac{Rr}{r_1 - r}$$

Taking $r_1 = \frac{R+r}{2}$ gives us

$$\left|f(z) - \sum_{n=0}^{k} a_n (z - z_0)^n\right| \le \frac{2Rr}{R - r}$$

The inequality in (b) is satisfied when

$$\frac{2Rr}{R - r} \le D_k R^{k+1} \implies D_k \ge \frac{2R^{-k} r}{R - r}$$

2.6 Define $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Show

a. This series converges for all $z \in \mathbb{C}$.

b. $e^z e^w = e^{z+w}$.

c. Define $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$. Using the series for e^z show that you obtain the same series expansions for $\sin\theta$ and $\cos\theta$ that you learned in calculus. Check that $\cos^2\theta + \sin^2\theta = 1$ by multiplying out the definitions, so that $e^{i\theta}$ is a point on the unit circle corresponding to the cartesian coordinate $(\cos\theta, \sin\theta)$.

d. $|e^z| = e^{\operatorname{Re} z}$ and $\arg e^z = \operatorname{Im} z$. If z is a non-zero complex number then $z = re^{it}$, where $r = |z|$ and $t = \arg z$. Moreover, $z^n = r^n e^{int}$.

e. $e^z = 1$ only when $z = 2\pi k i$ for some integer k.

Solution a. In this problem, we have $z_0 = 0$ and $a_n = \frac{1}{n!}$. We clearly have $|n!|^{\frac{1}{n}} \xrightarrow{n\to\infty} \infty$, so we have $R = \liminf_{n\to\infty} |a_n|^{-\frac{1}{n}} = \infty$. So, the series converges everywhere by the root test.

b. Consider $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ and $e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$. Then

$$\begin{aligned}
e^{z+w} &= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} \binom{n}{k} z^{n-k} w^k}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^{n-k} w^k}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n-k} w^k}{(n-k)!k!} \\
&= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) \qquad \text{(Cauchy Product)} \\
&= e^z e^w
\end{aligned}$$

c. First note that

$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\frac{\theta}{1!} - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \\
e^{-i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 - i\frac{\theta}{1!} - \frac{\theta^2}{2!} + i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots
\end{aligned}$$

Then

$$\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}) = \frac{1}{2}\left(2 - 2\frac{\theta^2}{2!} + 2\frac{\theta^4}{4!} + \cdots\right) = \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!}$$
$$\sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) = 2\left(2i\frac{\theta}{1!} - 2i\frac{\theta^3}{3!} + \cdots\right) = \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!}$$

Next,

$$\begin{aligned}\cos^2\theta &= \frac{1}{4}(e^{i2\theta}+e^{-i2\theta}+2)\\ \sin^2\theta &= -\frac{1}{4}(e^{i2\theta}+e^{-i2\theta}-2)\\ \implies \cos^2\theta+\sin^2\theta &= \frac{1}{4}(4) = 1\end{aligned}$$

d. Note that $e^{i\theta} = \cos\theta + i\sin\theta$. Write $z = x+iy$. Then

$$e^{x+iy} = e^x e^{iy}$$

We get a complex number in polar coordinates. Its radius is e^x and its angle is y. Hence, $|e^z| = e^{\operatorname{Re} z}$ and $\arg e^z = \operatorname{Im} z$.

e. $e^z = 1 \implies e^{x+iy} = e^x(\cos y + i\sin y) = 1 \implies \cos y = 1$ and $\sin y = 0 \implies y = 2\pi k$ for some $k\in\mathbb{Z}$.

2.7 Using the definitions in Exercise 6, prove

a. $\frac{\mathrm{d}}{\mathrm{d}z}e^z = e^z$.

b. Use (a) and the chain rule to compute the indefinite integral

$$\int e^{nt}\cos mt\,\mathrm{d}t.$$

Hint: Use $\operatorname{Re}\int e^{(n+im)t}\,\mathrm{d}t$, which results in a lot less work than the standard calculus trick of integrating by parts twice.

c. Use (a), the chain rule, and the fundamental theorem of calculus to prove $\int_0^{2\pi} e^{int}\,\mathrm{d}t = 0$, if n is a non-zero integer.

Solution a.
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}e^z &= \lim_{w\to z}\frac{e^w - e^z}{w-z}\\ &= \lim_{w\to z}\frac{\sum_{n=0}^{\infty}\frac{w^n}{n!} - \sum_{n=0}^{\infty}\frac{z^n}{n!}}{w-z}\\ &= \lim_{w\to z}\frac{\sum_{n=0}^{\infty}\frac{w^n-z^n}{n!}}{w-z}\\ &= \lim_{w\to z}\frac{\sum_{n=0}^{\infty}\frac{(w-z)(w^{n-1}+w^{n-2}z+\cdots+wz^{n-2}+z^{n-1})}{n!}}{w-z}\\ &= \lim_{w\to z}\sum_{n=1}^{\infty}\frac{w^{n-1}+w^{n-2}z+\cdots+wz^{n-2}+z^{n-1}}{n!}\\ &= \sum_{n=1}^{\infty}\frac{(n-1)z^{n-1}}{n!}\\ &= \sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!}\\ &= \sum_{n=0}^{\infty}\frac{z^n}{n!} = e^z\end{aligned}$$

b. Notice that $e^{nt}\cos mt = \operatorname{Re} e^{(n+im)t}$. Thus,

$$\begin{aligned}\int e^{nt}\cos mt\,\mathrm{d}t &= \operatorname{Re}\int e^{(n+im)t}\,\mathrm{d}t\\ &= \operatorname{Re}\left(\frac{1}{n+im}e^{(n+im)t} + C + iD\right)\\ &= \frac{n}{n^2+m^2}e^{nt}\cos mt + \frac{m}{n^2+m^2}e^{nt}\sin mt + C\end{aligned}$$

c. $\displaystyle\int_0^{2\pi} e^{int}\,\mathrm{d}t = \frac{1}{in}e^{i2\pi n} - \frac{1}{in} = \frac{1}{in}(\cos 2\pi n + i\sin 2\pi n - 1) = 0$

2.9 a. Suppose p and q are polynomials with no common zero and suppose $q(z_0) \neq 0$. Let d denote the distance from z_0 to the nearest zero of q. Then the rational function $r = p/q$ has a power series expansion which converges in $\{z \mid |z - z_0| < d\}$ and no larger disk. Hint: Use the partial fraction expansion, Exercise 4, Theorem 5.3, and (3.2).

b. Find the series expansion and radius of convergence of

$$\frac{z+2i}{(z-6)^2(z^2+6z+10)}$$

about the point 1. Hint: set $z = 1 + w$, then expand in powers of w.

Solution a. Consider the partial fraction expansion of $r(z)$. Note that $r(z)$ has no poles on the disk $\{z \mid |z - z_0| < d\}$. Also, the terms in the partial fraction expansion are analytic since they are a product of an analytic function (a polynomial) with the composition of two analytic functions ($\frac{1}{z}$ and a polynomial). Hence, $r(z)$ is analytic on that disk, so it has a power series expansion which converges on that disk.

Moreover, that disk is the largest disk on which the power series of $r(z)$ converges. This is because any larger disk will contain a zero of $q(z)$, and at least one partial fraction will no longer have a power series that converges.

b. The partial fraction expansion is given by

$$\frac{z+2i}{(z-6)^2(z^2+6z+10)} = \frac{1}{6724}\frac{93+147i}{z+3-i} - \frac{1}{6724}\frac{67+111i}{z+3+i} - \frac{1}{3362}\frac{13+18i}{z-6} + \frac{1}{41}\frac{3+i}{(z-6)^2}$$

The power series expansion can be attained by using the geometric series. Moreover, the smallest distance between 1 and a zero of the denominator is $\sqrt{17}$, so the radius of convergence is $\sqrt{17}$, by (a).

2.11 Suppose $\sum_{j=0}^{\infty}|a_j|^2 < \infty$. Show $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is analytic in $\{z \mid |z| < 1\}$. Compute (and prove your answer):

$$\lim_{r\uparrow 1}\int_0^{2\pi}\frac{|f(re^{i\theta})|^2}{2\pi}\,\mathrm{d}\theta.$$

Solution As $\sum_{j=0}^{\infty}|a_j|^2 < \infty$, we have that $\lim_{j\to\infty}|a_j|^2 = 0 \implies \lim_{j\to\infty}|a_j| = 0$. Then there exists $N \in \mathbb{N}$ such that for all $j \geq N$, we have that $|a_j| < 1$. Then

$$\sum_{j=N}^{\infty}|a_j z^j| \leq \sum_{j=N}^{\infty}|z^j| < \infty$$

since $|z| < 1$. Hence, by the Weierstrass M-test, the series $\sum a_j z^j$ converges absolutely, so $f(z)$ is analytic on $\{z \mid |z| < 1\}$.

Notice that

$$
\begin{aligned}
|f(re^{i\theta})|^2 &= f(re^{i\theta})\overline{f(re^{i\theta})} \\
&= \left(\sum_{j=0}^{\infty} a_j r^j e^{ij\theta}\right)\overline{\left(\sum_{j=0}^{\infty} a_j r^j e^{ij\theta}\right)} \\
&= \left(\sum_{j=0}^{\infty} a_j r^j e^{ij\theta}\right)\left(\sum_{j=0}^{\infty} \overline{a_j} r^j e^{-ij\theta}\right) \\
&= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j} a_k\overline{a_{j-k}} r^k r^{j-k} e^{ik\theta} e^{-i(j-k)\theta}\right) \\
&= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j} a_k\overline{a_{j-k}} r^j e^{-i(2k-j)\theta}\right)
\end{aligned}
$$

For $0 < r < 1$, the power series expansion of $f(z)$ is uniformly convergent, so the integral of the sum is the sum of the integrals. Hence,

$$
\frac{1}{2\pi}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\left(\int_0^{2\pi} a_k\overline{a_{j-k}} r^j e^{-i(2k-j)\theta}\right)\mathrm{d}\theta = \sum_{j=0}^{\infty}|a_j|^2 r^j
$$

Hence,

$$
\lim_{r\uparrow 1}\int_0^{2\pi}\frac{|f(re^{i\theta})|^2}{2\pi}\,\mathrm{d}\theta = \sum_{j=0}^{\infty}|a_j|^2
$$

2.12 Suppose f has a power series expansion at 0 which converges in all of $\mathbb{C}$. Suppose also that $\int_{\mathbb{C}}|f(x+iy)|\,\mathrm{d}x\,\mathrm{d}y < \infty$. Prove $f \equiv 0$. Hint: Use polar coordinates to prove $f(0) = 0$.

Solution By definition, f is analytic on $\mathbb{C}$, so it has a power series $f(z) = \sum a_n z^n$.

Let $z = re^{i\theta}$. Then by uniform convergence, the integral becomes

$$
\begin{aligned}
\int_{\mathbb{C}}\left|\sum_{n=0}^{\infty} a_n r^n e^{in\theta}\right|\mathrm{d}x\,\mathrm{d}y &\geq \left|\int_0^{\infty}\int_0^{2\pi}\sum_{n=0}^{\infty} a_n r^n e^{in\theta} r\,\mathrm{d}\theta\,\mathrm{d}r\right| \\
&= \left|\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{2\pi} a_n r^n e^{in\theta} r\,\mathrm{d}\theta\,\mathrm{d}r\right| \\
&= \lim_{R\to\infty}\left|\frac{a_0 R^2}{2}\right|
\end{aligned}
$$

For the inequality to hold, we must have $a_0 = 0$ since the left side is finite.

Then consider $f(z) = z\sum_{n=0}^{\infty} a_{n+1} z^n := zF(z)$. The power series expansion of $F(z)$ converges everywhere since the power series expansion of $f(z)$ does, by the root test.

The integral $\int_{|z|\leq 1}|F(z)|\,\mathrm{d}x\,\mathrm{d}y$ is convergent since $\{z \mid |z| \leq 1\}$ is compact and $F(z)$ is analytic $\implies$ it is continuous. So, we need to consider the integral on $|z| > 1$.

For $|z| > 1$, we have that $|F(z)| \leq |f(z)| \implies \int_{|z|>1}|F(x+iy)|\,\mathrm{d}x\,\mathrm{d}y \leq \int_{|z|>1}|f(x+iy)|\,\mathrm{d}x\,\mathrm{d}y < \infty$.

Hence, $\int_{\mathbb{C}}|F(z)|\,\mathrm{d}x\,\mathrm{d}y < \infty$. We can then aplpy the same argument as above in order to show that $a_1 = 0$.

Proceeding inductively yields that $a_n = 0$ for all $n \geq 0$, so $f \equiv 0$.

2.13 Suppose f is analytic in a connected open set U such that for each $z \in U$, there exists an n (depending upon z) such that $f^{(n)}(z) = 0$. Prove that f is a polynomial.

Solution Consider the set $E_n = \{z \in U \mid f^{(n)}(z) = 0\}$. We claim that E_n is closed and nowhere dense.

Assume the zeroes of $f^{(n)}$ are isolated. If they weren't, then $f^{(n)} \equiv 0$ and is clearly a polynomial. Then clearly, E_n is closed. Since the zeroes are isolated, the interior of E_n is empty, so E_n is nowhere dense.

As $\mathbb{C}$ is complete, it has the Baire property, and so the union of countably many closed, nowhere dense sets is also nowhere dense. So, $\bigcup_{n=0}^{\infty} E_n$ is nowhere dense in U. But $\bigcup_{n=0}^{\infty} E_n = U$ since it includes all z. This is a contradiction as U must be dense in itself, as $\overline{U} = U$. Hence, there is some n such that $f^{(n)} \equiv 0$, so $f^{(m)} \equiv 0$ for all $m \geq n$. Hence, f is a polynomial.