3 Recall Problem 5 from Problem Set 2: A burglar found your keychain with n keys and he is trying to find one of the two keys which open your house door (whenever he tries a wrong key he discards it). Find the expectation and the variance of the number of tries he has to do to find a key which opens the door. Simplify your answers, they should be closed expressions, not summations. *Hint: you will need to know some formulas for the sum of the first n integers, etc, you can look these up and simply state them.*

Solution Let A_i denote the event that the burglar opens the door on the i-th try. We wish to find $P(A_k) = P(A_1^\complement \cap A_2^\complement \cap \ldots \cap A_{k-1}^\complement \cap A_k)$. By the multiplication rule, we have

$$\begin{aligned}
P(A_k) &= P(A_1^\complement \cap A_2^\complement \cap \ldots \cap A_{k-1}^\complement \cap A_k) \\
&= \left(\frac{n-2}{n}\right)\left(\frac{(n-1)-2}{n-1}\right)\left(\frac{(n-2)-2}{n-2}\right)\left(\frac{(n-3)-2}{n-3}\right)\cdots\left(\frac{(n-(k-2))-2}{n-(k-2)}\right)\left(\frac{2}{n-(k-1)}\right) \\
&= \left(\frac{n-2}{n}\right)\left(\frac{n-3}{n-1}\right)\left(\frac{n-4}{n-2}\right)\left(\frac{n-5}{n-3}\right)\cdots\left(\frac{n-k}{n-k+2}\right)\left(\frac{2}{n-k+1}\right) \\
&= \frac{2(n-k)}{n(n-1)}
\end{aligned}$$

Let the random variable X denote the number of tries it takes before finding a correct key. Then our probability mass function is $p_X(k) = \frac{2(n-k)}{n(n-1)}$, with $k \in \{1, 2, 3, \ldots, n-1\}$. Thus,

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=1}^{n-1} k p_X(k) \\
&= \sum_{k=1}^{n-1} k \frac{2(n-k)}{n(n-1)} \\
&= \frac{2}{n(n-1)} \sum_{k=1}^{n-1} k(n-k) \\
&= \frac{2}{n(n-1)} \left[n\sum_{k=1}^{n-1} k - \sum_{k=1}^{n-1} k^2 \right] \\
&= \frac{2}{n(n-1)} \left[n\frac{(n-1)n}{2} - \frac{(n-1)n(2n-1)}{6} \right] \\
&= n - \frac{2n-1}{3} = \boxed{\frac{n+1}{3}}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(X^2) &= \sum_{k=1}^{n-1} k^2 p_X(k) \\
&= \sum_{k=1}^{n-1} k^2 \frac{2(n-k)}{n(n-1)} \\
&= \frac{2}{n(n-1)} \sum_{k=1}^{n-1} k^2(n-k) \\
&= \frac{2}{n(n-1)} \left[n\sum_{k=1}^{n-1} k^2 - \sum_{k=1}^{n-1} k^3 \right] \\
&= \frac{2}{n(n-1)} \left[n\frac{(n-1)n(2n-1)}{6} - \frac{(n-1)^2 n^2}{4} \right] \\
&= n\frac{2n-1}{3} - \frac{(n-1)n}{2} = \frac{n(n+1)}{6}
\end{aligned}$$

Thus, $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \dfrac{n(n+1)}{6} - \left(\dfrac{n+1}{3}\right)^2 = \boxed{\dfrac{(n+1)(n-2)}{18}}$.

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4 Every hour, a clock tower rings either once or twice. Once with probability $\frac{1}{3}$ and twice with probability $\frac{2}{3}$. If you're listening the clock for n hours (you hear the bell ringing on n occasions) let X be the number of times you heard the bell ring. Find the expectation and the variance of X.

Solution

$$\begin{aligned}
p_X(n) &= \left(\frac{1}{3}\right)^n \\
p_X(n+1) &= \binom{n}{1}\left(\frac{1}{3}\right)^{n-1}\left(\frac{2}{3}\right)^1 \\
p_X(n+2) &= \binom{n}{2}\left(\frac{1}{3}\right)^{n-2}\left(\frac{2}{3}\right)^2 \\
p_X(n+k) &= \binom{n}{k}\left(\frac{1}{3}\right)^{n-k}\left(\frac{2}{3}\right)^k
\end{aligned}$$

with $k \in \{0, 1, \ldots, n\}$. We have a binomial distribution, but with a linear transformation applied (a translation). If we let Y be a binomial random variable with $p = \frac{2}{3}$ and range $\{0, 1, \ldots, n\}$, then $X = Y + n$. Thus,

$$\mathbb{E}(X) = \mathbb{E}(Y + n) = \mathbb{E}(Y) + n = n\frac{2}{3} + n = \boxed{\frac{5n}{3}}$$

$$\operatorname{Var}(X) = \operatorname{Var} Y + n = \operatorname{Var} Y = n \cdot \frac{2}{3} \cdot \frac{1}{3} = \boxed{\frac{2n}{9}}$$