4 If a day is sunny the probability that the next day will be rainy is $\frac{1}{2}$. If exactly k consecutive days have been raining, the probability that the following day will be sunny is $\frac{1}{k+1}$. If today is sunny, what is the probability that the next n days will all be rainy?

Solution If today was sunny, then the probability the 1st day after today is rainy is $\frac{1}{2}$. Then the probability the 2nd day after today is rainy will be $1 - \frac{1}{1+1} = \frac{1}{2}$. The 3rd will be $1 - \frac{1}{2+1} = \frac{2}{3}$, and so on. So, the probability that the next n days will be rainy is

$$\begin{aligned} &\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{n-2}{n-1}\right)\left(\frac{n-1}{n}\right) \\ = &\frac{1}{2n} \end{aligned}$$

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6 You are running a day care center and today you are looking after n babies. At the end of the day, n parents arrive to collect their respective children. You are lazy and terrible at running a day care center so you give every parent a random baby.

a. What is the probability that no parent ends up with the correct baby?

b. What is the probability that exactly k parents end up with the correct baby?

c. What do these probabilities tend to as n becomes very large?

Solution a. Each baby is distinguishable, so there are a total of $n!$ ways to return each baby to a parent. It will be easier to count the number of ways that at least 1 baby is returned to the correct parent first.

Let B_i be the event that the i-th baby is returned to the correct parents. Then the event A where at least 1 baby is returned is

$$A = B_1 \cup B_2 \cup \cdots \cup B_n.$$

To return baby i correctly, there is 1 choice for baby i. Then we need to arrange the last $n-i$ babies in any way, so we will have $(n-i)!$ ways to return baby i.

Then applying the inclusion-exclusion principle, we get

$$\begin{aligned} |A| &= |B_1 \cup B_2 \cup B_3 \cup B_4 \cup \cdots \cup B_n| \\ &= \sum_{i=1}^{n} |B_i| - \sum_{1\le i<j\le n} |B_i \cap B_j| + \sum_{1\le i<j<k\le n} |B_i \cap B_j \cap B_k| - \cdots + (-1)^{n-1}\left|\bigcap_{i=1}^{n} B_i\right| \end{aligned}$$

The first sum is equal to the number of ways to return 1 specific baby correctly. The second sum is equal to the number of ways to return 2 specific babies correctly, and so on. This is because for the m-th sum, we return m babies correctly, and return the remaining $n-m$ babies any way we like. Each sum will have $\binom{n}{m}$ terms because we need to pick m sets from n total to union. Thus, the sum is equal to

$$\begin{aligned} &\binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! + \cdots + (-1)^{n-1}\binom{n}{n}(n-n)! \\ = &\frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} - \cdots + (-1)^{n-1} \end{aligned}$$

Since each outcome is equally likely, the probability will be

$$\begin{aligned} &\frac{1}{n!}\left(\frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} - \cdots + (-1)^{n-1}\right) \\ = &\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^{n-1}}{n!} \end{aligned}$$

b. The problem is similar to the above, but first, we return k babies correctly. There will be $\binom{n}{k}$ ways to do so, and then we have to make sure that the remaining $n-k$ babies are returned to the wrong parent. Thus, the number of ways for this to happen is

$$\begin{aligned}
&\binom{n}{k}\left(\frac{(n-k)!}{1!}-\frac{(n-k)!}{2!}+\frac{(n-k)!}{3!}-\cdots+(-1)^{n-k-1}\right)\\
=&\frac{n!}{k!(n-k)!}\left(\frac{(n-k)!}{1!}-\frac{(n-k)!}{2!}+\frac{(n-k)!}{3!}-\cdots+(-1)^{n-k-1}\right)\\
=&\frac{n!}{k!}\left(\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\cdots+\frac{(-1)^{n-k-1}}{(n-k)!}\right)
\end{aligned}$$

Each outcome is equally likely, and there are $n!$ ways to return the babies, so the probability will be

$$\frac{1}{k!}\left(\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\cdots+\frac{(-1)^{n-k-1}}{(n-k)!}\right)$$

c. $$\lim_{n\to\infty}\left(\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\cdots+\frac{(-1)^{n-1}}{n!}\right)=e^{-1}$$

$$\lim_{n\to\infty}\left[\frac{1}{k!}\left(\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\cdots+\frac{(-1)^{n-k-1}}{(n-k)!}\right)\right]=\frac{e^{-1}}{k!}$$