8.2.8 Suppose that $f \in L^p(\mathbb{R})$. If there exists $h \in L^p(\mathbb{R})$ such that

$$\lim_{y \to 0} \left\| y^{-1}(\tau_{-y} f - f) - h \right\|_p = 0,$$

we call h the **(strong)** L^p **derivative** of f. If $f \in L^p(\mathbb{R}^n)$, L^p partial derivatives of f are defined similarly. Suppose that p and q are conjugate exponents, $f \in L^p$, $g \in L^q$, and the L^p derivative $\partial_j f$ exists. Then $\partial_j(f * g)$ exists (in the ordinary sense) and equals $(\partial_j f) * g$.

Solution By fixing every coordinate except x_j, we may assume that $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$. So, from now on, we will write f' for $\partial_j f$.

To begin, we define $\Delta_y(f, h) := y^{-1}(\tau_{-y} f - f) - h$, so that by assumption, $\|\Delta_y(f, f')\|_p \xrightarrow{y \to 0} 0$. Hence, to show that $(f * g)'$ exists in the usual sense, we need to show that

$$\lim_{y \to 0} |\Delta_y(f * g, f' * g)| = 0.$$

By Proposition 8.6, $\tau_{-y}(f * g) = (\tau_{-y} f) * g$, so

$$\begin{aligned}
\Delta_y(f * g, f' * g)(x) &= \frac{1}{y}\Big(((\tau_{-y} f) * g)(x) - (f * g)(x)\Big) - (f' * g)(x) \\
&= \int_{\mathbb{R}} \left(\frac{1}{y}\Big(f(x + y - z) - f(x - z)\Big) - f'(x - z)\right) g(z)\, \mathrm{d}z \\
&= \int_{\mathbb{R}} \Delta_y(f, f')(x - z) g(z)\, \mathrm{d}z.
\end{aligned}$$

Notice that $\Delta_y(f, f') \in L^p(\mathbb{R})$ for all $y \neq 0$:

$$\left\|\Delta_y(f, f')\right\|_p \leq 2y^{-1}\|f\|_p + \|f'\|_p < \infty.$$

Hence, by Proposition 8.8, $\|\Delta_y(f * g, f' * g)\|_u \leq \|\Delta_y(f, f')\|_p \|g\|_q \xrightarrow{y \to 0} 0$, i.e., $\Delta_y(f * g, f' * g)$ converges uniformly to 0, which completes the proof.

8.2.9 If $f \in L^p(\mathbb{R})$, the L^p derivative of f (call it h) exists iff f is absolutely continuous on every bounded interval (perhaps after modification on a null set) and its pointwise derivative f' is in L^p, in which case $h = f'$ a.e..

Solution "$\implies$"

Suppose the L^p derivative h of f exists. We follow the hint and let $g \in C_c$ be with unit mass. Such functions exist: take ψ as in (8.1) in Folland, and set $g = \psi / \int \psi$. Then g is in L^1 and satisfies the assumptions of Theorem 8.15, since exponentials decay much faster than functions of the form $1/(1 + |x|)^{n+\varepsilon}$.

By Theorem 8.14, $f * g_t \xrightarrow{t \to 0} f$ in L^p, and by Exercise 8.2.8, $(f * g_t)' = h * g_t \xrightarrow{t \to 0} h$ in L^p also. Moreover, if $-\infty < a < b < \infty$, notice that if $g \in L^p([a, b])$, then

$$\int_a^b |g(x)|\, \mathrm{d}x \leq \|g\|_p (b - a)^{1/p} < \infty,$$

i.e., $L^p([a, b]) \subseteq L^1([a, b])$, so convergence in L^p implies convergene in L^1. Thus, if a and b are Lebesgue points of f,

$$(f * g_t)(b) - (f * g_t)(a) = \int_a^b (f * g_t)'(x)\, \mathrm{d}x = \int_a^b (h * g_t)(x)\, \mathrm{d}x.$$

Taking $t \to 0$, we have by Theorem 8.15 that $(f * g_t)(a) \xrightarrow{t\to 0} f(a)$, $(f * g_t)(b) \xrightarrow{t\to 0} f(b)$, since they are Lebesgue points, and by convergence in L^1, we also get

$$\int_a^b (h * g_t)(x)\,\mathrm{d}x \xrightarrow{t\to 0} \int_a^b h(x)\,\mathrm{d}x,$$

so we have

$$f(b) - f(a) = \int_a^b h(x)\,\mathrm{d}x.$$

Now, let $\delta > 0$ and assume we have any finite interval I. Then I is contained in an interval $[a, b]$, where a and b are Lebesgue points of f, since almost every point is a Lebesgue point. Hence, if we partition $[a, b]$ at Lebesgue points into pairwise disjoint subintervals (a_k, b_k) with $\sum_k |b_k - a_k| < \delta$, we have by Hölder's inequality that

$$\sum_k |f(b_k) - f(a_k)| \le \sum_k \int_{a_k}^{b_k} |h(x)|\,\mathrm{d}x = \int_{\bigcup_k (a_k,b_k)} |h(x)|\,\mathrm{d}x \le \|h\|_p \delta^{(p-1)/p}.$$

This tends to 0 as $\delta \to 0$, which proves absolute continuity of f on bounded intervals. By Theorem 3.35, it follows that f' exists almost everywhere and that $f' = h$ almost everywhere, by the argument below.

"$\Longleftarrow$"

Suppose f is absolutely continuous on every bounded interval and its pointwise derivative f' is in L^p. We follow the hint and write

$$y^{-1}(\tau_{-y} f(x) - f(x)) - f'(x) = \frac{f(x+y) - f(x)}{y} - f'(x) = \frac{1}{y}\int_0^y f'(x+t) - f'(x)\,\mathrm{d}t,$$

which follows from applying Lebesgue's fundamental theorem of calculus for absolutely continuous functions (Theorem 3.35) to f' on $[0, y]$. Then by Minkowski's inequality,

$$\begin{aligned}
\|y^{-1}(\tau_{-y} f - f) - f'\|_p &= \left(\int_{\mathbb{R}} \left|\frac{1}{y}\int_0^y f'(x+t) - f'(x)\,\mathrm{d}t\right|^p \mathrm{d}x\right)^{1/p} \\
&\le \frac{1}{y}\int_0^y \left(\int_{\mathbb{R}} |f'(x+t) - f'(x)|^p\,\mathrm{d}x\right)^{1/p} \mathrm{d}t \\
&= \frac{1}{y}\int_0^y \|\tau_{-t} f' - f'\|_p\,\mathrm{d}t
\end{aligned}$$

Then by Proposition 8.5, $\|\tau_{-t} f' - f'\|_p \xrightarrow{t\to 0} 0$. Hence, if $\varepsilon > 0$, then there exists $\delta > 0$ so that if $|t| < \delta$, then $\|\tau_{-t} f' - f'\|_p < \varepsilon$. Thus, if $|y| < \delta$,

$$\|y^{-1}(\tau_{-y} f - f) - f'\|_p \le \frac{1}{y}\int_0^y \|\tau_{-t} f' - f'\|_p\,\mathrm{d}t \le \frac{1}{y}\int_0^y \varepsilon\,\mathrm{d}t = \varepsilon.$$

Hence, $\lim_{y\to 0} \|y^{-1}(\tau_{-y} f - f) - f'\|_p = 0$. Thus, by definition, the L^p derivative of f exists and is equal to h a.e.

$$\|f' - h\|_p \le \|y^{-1}(\tau_{-y} f - f) - f'\|_p + \|y^{-1}(\tau_{-y} f - f) - h\|_p \xrightarrow{y\to 0} 0.$$

8.3.14 If $f \in C^1([a,b])$ and $f(a) = f(b) = 0$, then

$$\int_a^b |f(x)|^2 \,\mathrm{d}x \le \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'(x)|^2 \,\mathrm{d}x.$$

Solution We first assume that $a = 0$ and $b = 1/2$, and we will justify this at the end. Then we need to show that

$$\int_0^{1/2} |f(x)|^2 \,\mathrm{d}x \le \frac{1}{4\pi^2} \int_0^{1/2} |f'(x)|^2 \,\mathrm{d}x.$$

We extend f to be $C^1([-1/2, 1/2])$ by setting $f(-x) = -f(x)$, for $x \in [-1/2, 0]$. It's clear that f is C^1 everywhere but the origin, so we need to show that f is also C^1 at the origin.

f continuous at 0: $f(0) = 0$, so by assumption, $\lim_{x\to 0^+} f(x) = 0$ and hence from our extension, $\lim_{x\to 0^-} f(x) = 0$ also. Next, By the mean value theorem applied to the interval $(-x, 0)$, we get for some $\xi \in (-x, 0)$ that

$$f'(\xi_x) = \frac{0 - f(-x)}{0 - (-x)} = \frac{-f(-x)}{x} = \frac{f(x)}{x}.$$

But the right-hand side is just the mean value theorem applied to f on the interval $(0, x)$, so for some $\xi'_x \in (0, x)$, we have

$$f'(\xi_x) = f'(\xi'_x).$$

Since f was C^1, we know that $\lim_{x\to 0^-} f'(x)$ and $\lim_{x\to 0^+} f'(x)$ exist, and because $\xi_x \to 0^-$, $\xi'_x \to 0^+$ as $x \to 0^+$, it follows that $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x)$, by uniqueness of limits. Thus, $f \in C^1([-1/2, 1/2])$.

Next, we extend f to be periodic on all of $\mathbb{R}$, and by the same argument as above with 0 replaced by 1/2, it follows that $f \in C^1(\mathbb{T})$.

For $k \neq 0$, integration by parts yields

$$\begin{aligned}\langle f, e^{2\pi i k x}\rangle = \int_{\mathbb{T}} f(x) e^{-2\pi i k x} \,\mathrm{d}x = \left[-\frac{f(x)e^{-2\pi i k x}}{2\pi i k}\right]_{-1/2}^{1/2} + \frac{1}{2\pi i k}\int_{\mathbb{T}} f'(x) e^{-2\pi i k x} \,\mathrm{d}x &= \frac{1}{2\pi i k}\int_{\mathbb{T}} f'(x) e^{-2\pi i k x} \,\mathrm{d}x \\ &= \frac{1}{2\pi i k}\langle f', e^{-2\pi i k x}\rangle.\end{aligned}$$

Indeed, $f(1/2) = f(-1/2) = 0$, so the first term vanishes. If $k = 0$, then

$$\int f(x) \,\mathrm{d}x = 0,$$

because we extended f to an odd function on $[-1/2, 1/2]$. By Parseval's identity, we get

$$\|f\|_2^2 = \sum_{k\in\mathbb{Z}} |\langle f, E_k\rangle|^2 = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{4\pi^2 k^2} |\langle f', E_k\rangle|^2 \le \frac{1}{4\pi^2} \sum_{k\in\mathbb{Z}} |\langle f', E_k\rangle|^2 = \frac{1}{4\pi^2}\|f'\|_2^2.$$

Indeed, $\langle f', 1\rangle = 0$, by the fundamental theorem of calculus. Lastly, notice that $|f|$ and $|f'|$ are even functions, so

$$\int_0^{1/2} |f(x)|^2 \,\mathrm{d}x = \frac{\|f\|_2^2}{2} \quad \text{and} \quad \int_0^{1/2} |f'(x)|^2 \,\mathrm{d}x = \frac{\|f'\|_2^2}{2},$$

so by dividing by 2 on both sides of our inequality, we have

$$\int_0^{1/2} |f(x)|^2 \,\mathrm{d}x \le \frac{1}{4\pi^2} \int_0^{1/2} |f'(x)|^2 \,\mathrm{d}x,$$

as required.

To justify setting $a = 0$ and $b = 1/2$, we apply the linear change in coordinates

$$u = \frac{1}{2}\left(\frac{x-a}{b-a}\right) \implies \mathrm{d}u = \frac{1}{2}\left(\frac{\mathrm{d}x}{b-a}\right),$$

and since f is C^1, we may apply the usual change of variables to get

$$\int_a^b |f(x)|^2 \,\mathrm{d}x = \int_0^{1/2} 2(b-a)|f(x(u))|^2 \,\mathrm{d}u.$$

If we set $g(u) = f(x(u))$, then $g'(u) = 2(b-a)f'(x(u))$, and by the above, we have

$$\begin{aligned}
\int_a^b |f(x)|^2 \,\mathrm{d}x = \int_0^{1/2} 2(b-a)|f(x(u))|^2 \,\mathrm{d}u &= 2(b-a)\int_0^{1/2} |g(u)|^2 \,\mathrm{d}u \\
&\le 2(b-a)\int_0^{1/2} |g'(u)|^2 \,\mathrm{d}u \\
&\le \frac{2(b-a)}{4\pi^2}\int_0^{1/2} 4(b-a)^2|f'(x(u))|^2 \,\mathrm{d}u.
\end{aligned}$$

Undoing the change of variables, we get

$$\int_a^b |f(x)|^2 \,\mathrm{d}x \le \frac{1}{4\pi^2}\int_a^b 4(b-a)^2|f'(x)|^2 \,\mathrm{d}x = \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'(x)|^2 \,\mathrm{d}x,$$

which concludes the proof.