17.4.41 Prove a famous result of Archimedes: The surface area of the portion of the sphere of radius R between two horizontal planes $z = a$ and $z = b$ is equal to the surface area of the corresponding portion of the circumscribed cylinder (Figure 22).

Solution The portion of the sphere of interest can be expressed as

$$x^2 + y^2 + z^2 = R^2, \text{ where } a \leq z \leq b$$

A parameterization G for this surface makes use of spherical coordinates, and is

$$G(\theta, \phi) = R \langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi \rangle$$

where

$$\theta \in [0, 2\pi], \phi \in \left[\arccos\frac{b}{R}, \arccos\frac{a}{R}\right].$$

Now we need to find the normal vector. The partial derivatives of G are

$$\begin{aligned} G_\theta &= R \langle -\sin\phi \sin\theta, \sin\phi\cos\theta, 0 \rangle \\ G_\phi &= R \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi \rangle. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{N}(\theta, \phi) &= G_\theta \times G_\phi \\ &= R^2 \left\langle -\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\sin\phi\cos\phi\sin^2\theta - \sin\phi\cos\phi\cos^2\theta \right\rangle \\ &= R^2 \left\langle -\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\sin\phi\cos\phi \right\rangle \\ &= R^2 \sin\phi \langle -\sin\phi\cos\theta, -\sin\phi\sin\theta, -\cos\phi \rangle \\ \|\mathbf{N}(\theta, \phi)\| &= \left|R^2\sin\phi\right| \sqrt{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi} \\ &= \left|R^2\sin\phi\right| \sqrt{\sin^2\phi + \cos^2\phi} \\ &= \left|R^2\sin\phi\right|, \text{ but } 0 \leq \phi \leq \pi, \text{ so} \\ &= R^2 \sin\phi \end{aligned}$$

Thus, the surface area is

$$\begin{aligned} &\iint_S \mathrm{d}S \\ &= \int_0^{2\pi} \int_{\arccos\frac{b}{R}}^{\arccos\frac{a}{R}} \|\mathbf{N}(\theta, \phi)\| \,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \int_{\arccos\frac{b}{R}}^{\arccos\frac{a}{R}} R^2 \sin\phi \,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= 2\pi R^2 \Big[-\cos\phi\Big]_{\arccos\frac{b}{R}}^{\arccos\frac{a}{R}} \\ &= 2\pi R^2 \left(\frac{b}{R} - \frac{a}{R}\right) \\ &= 2\pi R (b - a). \end{aligned}$$

The surface area of the circumscribed cylinder is $2\pi Rh$, and in this scenario, it is easy to see that $h = b - a$. So, the surface area of the circumscribed cylinder is $2\pi R (b - a)$, which matches the surface area above. ■

17.5.22 Let $\mathcal{S}$ be the ellipsoid $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$. Calculate the flux of $\mathbf{F} = z\mathbf{i}$ over the portion of $\mathcal{S}$ where $x, y, z \leq 0$ with upward-pointing normal. *Hint:* Parametrize $\mathcal{S}$ using a modified form of spherical coordinates θ, ϕ.

Solution A parameterization of $\mathcal{S}$ would be

$$G(\theta, \phi) = \langle 4\sin\phi\cos\theta, 3\sin\phi\sin\theta, 2\cos\phi\rangle .$$

Calculating the normal vector,

$$\begin{aligned}
G_\theta &= \langle -4\sin\phi\sin\theta, 3\sin\phi\cos\theta, 0\rangle \\
G_\phi &= \langle 4\cos\phi\cos\theta, 3\cos\phi\sin\theta, -2\sin\phi\rangle \\
G_\theta \times G_\phi &= \left\langle -6\sin^2\phi\cos\theta, 8\sin^2\phi\cos\theta, -12\sin\phi\cos\phi\sin^2\theta - 12\sin\phi\cos\phi\cos^2\theta\right\rangle \\
&= \left\langle -6\sin^2\phi\cos\theta, 8\sin^2\phi\cos\theta, -12\sin\phi\cos\phi\right\rangle
\end{aligned}$$

The region of the ellipsoid described with respect to the parameterization, $\mathcal{D}$, is $\pi \leq \theta \leq \frac{3\pi}{2}$ and $\frac{\pi}{2} \leq \phi \leq \pi$. On this region, $\sin\phi \geq 0$ and $\cos\phi \leq 0$, meaning $-12\sin\phi\cos\phi \geq 0$, so we can conclude that the cross product points upwards, and so our normal is

$$\mathbf{N}(\theta, \phi) = \left\langle -6\sin^2\phi\cos\theta, 8\sin^2\phi\cos\theta, -12\sin\phi\cos\phi\right\rangle .$$

Thus, we can begin calculating the surface integral:

$$\begin{aligned}
&\iint_{\mathcal{S}} \mathbf{F}\cdot \mathrm{d}\mathcal{S} \\
=& \iint_{\mathcal{D}} \mathbf{F}(G(\theta,\phi)) \cdot \mathbf{N}(\theta,\phi)\,\mathrm{d}A_{\theta\phi} \\
=& \int_{\pi/2}^{\pi}\int_{\pi}^{3\pi/2} \langle 2\cos\phi, 0, 0\rangle \cdot \left\langle -6\sin^2\phi\cos\theta, 8\sin^2\phi\cos\theta, -12\sin\phi\cos\phi\right\rangle \,\mathrm{d}\theta\,\mathrm{d}\phi \\
=& \int_{\pi/2}^{\pi}\int_{\pi}^{3\pi/2} -12\sin^2\phi\cos\phi\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\
=& -12\left(\int_{\pi/2}^{\pi} \sin^2\phi\cos\phi\,\mathrm{d}\phi\right)\left(\int_{\pi}^{3\pi/2}\cos\theta\,\mathrm{d}\theta\right) \\
=& -12\left(\left[\frac{1}{3}\sin^3\phi\right]_{\pi/2}^{\pi}\right)\left(\left[\sin\theta\right]_{\pi}^{3\pi/2}\right) \\
=& -12\left(-\frac{1}{3}\right)(-1) \\
&\boxed{= -4}
\end{aligned}$$