1 Let (X, d) be a metric space and let A, B be two non-empty subsets of X. Prove that if $A \cap B \neq \emptyset$, then we have the following inequality for the diameters:

$$\delta(A \cup B) \leq \delta(A) + \delta(B).$$

Solution If A or B or unbounded, then there is nothing to prove.

Assume A and B are unbounded. Let $a, b \in A \cup B$. Then there are two cases:

$a, b \in A$ or $a, b \in B$

It follows trivially that $d(a, b) \leq \delta(A) + \delta(B)$ in either case.

$a \in A$ and $b \in B$

Note that this case is this same if a and b are switched since a metric is symmetric.

Let $c \in A \cap B$.

$$d(a, b) \leq d(a, c) + d(c, b)$$

Since $c \in A$, $d(a, c) \leq \delta(A)$, and since $c \in B$, $d(c, b) \leq \delta(B)$. Thus,

$$d(a, b) \leq \delta(A) + \delta(B).$$

Combining the inequalities, we have that $d(a, b) \leq \delta(A) + \delta(B)$ for all $a, b \in A \cup B$. Thus, $\leq \delta(A) + \delta(B)$ is an upper bound for $d(a, b)$ for all $a, b \in A \cup B$, so by definition,

$$\delta(A \cup B) = \sup\{d(a, b) \mid a, b \in A \cup B\} \leq \delta(A) + \delta(B).$$

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2 Let X be a non-empty set and let $d \colon X \times X \to \mathbb{R}$ be the discrete metric on X defined as follows: for any $x, y \in X$,

$$d(x, y) = \begin{cases} 0, & \text{if} \quad x = y \\ 1, & \text{if} \quad x \neq y \end{cases}$$

Find the open and the closed subsets of this metric space.

Solution An open ball with radius $0 < r \leq 1$ centered at $a \in X$ is simply the set $\{a\}$. If $r > 1$, then the open ball is X. In other words, every neighborhood of a point a is either the set $\{a\}$ or the entire set X.

Let A be a non-empty proper subset of X. Then every point of A has an open ball contained in A, so every element of A is in A°. As A is a proper subset of X, any open ball contained in A must have radius $0 < r \leq 1$. It follows that every point in A° is contained in A. Thus, every non-empty proper subset of A is open.

Let $x \in X$. Then x is adherent point of A if $x \in A$. Otherwise, we can take a ball of radius 1, and it will have no intersection with A. Thus, we have that $\bar{A} \subseteq A$. Since $A \subseteq \bar{A}$, we have that $A = \bar{A}$. Thus, every non-empty proper subset of X is closed.

If A is empty or exactly X, then A is both closed and open. So, every subset of X is both open and closed.

3 Let (X, d_1) be a metric space and let $d_2 \colon X \times X \to \mathbb{R}$ be the metric defined as follows: for any $x, y \in X$,

$$d_2(x, y) = \frac{d_1(x, y)}{1 + d_1(x, y)}.$$

Prove that a subset A of X is open with respect to the distance d_1 if and only if it is open with respect to the distance d_2.

Solution We will denote open balls with respect to d_i with a superscript i.

"$\implies$"

We will show that given $r > 0$ and $a \in X$, an open ball of radius r with respect to d_2 is contained in an open ball of radius r with respect to d_1.

First note that if $x \geq 0$,

$$x - \frac{x}{1+x} = \frac{x^2}{1+x} \geq 0 \implies x \geq \frac{x}{1+x}$$

with equality only when $x = 0$. Thus,

$$d_2(x, a) = \frac{d_1(x, a)}{1 + d_1(x, a)} \leq d_1(x, a).$$

So, an open ball with radius r with respect to d_2 is contained in an open ball with radius r with respect to d_1.

Let a be an interior point of A with respect to d_1. Then for some $r > 0$, $B_r^2(a) \subseteq B_r^1(a) \subseteq A$. Thus, a is also an interior point of A with respect to d_2. Thus, all interior points of A with respect to d_1 are also interior points with respect to d_2, so A is open with respect to d_2.

"$\impliedby$"

Note that $d_2(x, y) < 1$ for all $x, y \in X$. So, every open ball with radius $r \geq 1$ is the entire set X.

If $A = X$, then A is clearly open with respect to d_1.

If $A \subset X$, then consider the complement of A. We wish to show that if A^{C} is closed with respect to d_2, then it is also closed with respect to d_1.

Let a be an adherent point of A^{C} with respect to d_2. Then there exists $r > 0$ such that $B_r^2(a) \cap A^{\mathrm{C}} \neq \emptyset$. From the first case, we proved that $B_r^2(a) \subseteq B_r^1(a)$. Thus, $B_r^1(a) \cap A^{\mathrm{C}} \neq 0$. It follows that A^{C} is closed with respect to d_1, which, by definition, means that A is open with respect to d_1.

4 Let $1 \leq p, q \leq \infty$ and consider the two metrics on $\mathbb{R}^n$ given by

$$d_p(x,y) = \left(\sum_{k=1}^{n} |x_k - y_k|^p\right)^{1/p} \quad \text{and} \quad d_q(x,y) = \left(\sum_{k=1}^{n} |x_k - y_k|^q\right)^{1/q},$$

with the obvious modifications if p or q are infinity. Prove that a set $A \subseteq \mathbb{R}^n$ is open with respect to the metric d_p if and only if it is open with respect to the distance d_q.

Solution We will first show that if $p \geq q$, then $d_p(x,0) \leq d_q(x,0)$. Consider $\left(\sum_{k=1}^{n} |a_k|^x\right)^{1/x}$ for $x > 0$. Then its derivative is

$$\left(\sum_{k=1}^{n} |a_k|^x\right)^{1/x} \left(-\frac{1}{x^2} \sum_{k=1}^{n} |a_k|^x + \frac{\sum_{k=1}^{n} |a_k|^x \ln|a_k|}{x \sum_{k=1}^{n} |a_k|^x}\right) = \left(\sum_{k=1}^{n} |a_k|^x\right)^{1/x} \left(\frac{-(\sum_{k=1}^{n} |a_k|^x)^2 + \sum_{k=1}^{n} |a_k|^x \ln|a_k|^x}{x^2 \sum_{k=1}^{n} |a_k|^x}\right)$$

Since $x > \ln x$ for all $x > 0$, the derivative is negative. So, $p \geq q \implies d_p(x,0) \leq d_q(x,0)$.

We will denote open balls with respect to d_i with a superscript i.

Let $1 \leq p, q < \infty$. Assume without loss of generality that $p \geq q$. Then $d_p(x-y, 0) = d_p(x,y) \leq d_q(x-y, 0) = d_q(x,y)$.

"$\implies$"

If A is open with respect to d_p, then A^{C} is closed with respect to d_p. Since $d_p(x,y) \leq d_q(x,y)$, $B_r^p(x) \subseteq B_r^q(x)$ for all $x \in X$ and $r > 0$. Thus, if $x \in A^{\mathrm{C}}$, then $B_r^p(x) \cap A^{\mathrm{C}} \neq \emptyset$. It follows that $B_r^q(x) \cap A^{\mathrm{C}} \neq \emptyset$ as well. Thus, if x is an adherent point of A^{C} with respect to d_p, then it is an adherent point of A^{C} with respect to d_q as well. So, A^{C} is closed with respect to d_q, meaning A is open with respect to d_p.

"$\impliedby$"

Assume A is open with respect to d_q. Let x be an interior point of A with respect to d_q. Then $B_r^p(x) \subseteq B_r^q(x) \subseteq A$, so by definition, A is open with respect to d_p as well.

We can switch the role of p and q, and the result will be the same.

If $p = q = \infty$, then $d_p = d_q$, so there is nothing to prove. If $p = \infty$ while q is finite (or vice versa), then we still have $d_p(x,y) \leq d_q(x,y)$, so we can apply the same argument as the above.

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5 Let (X, d) be a metric space and let A be a non-empty subset of X. Prove that A is open if and only if it can be written as the union of a family of open balls of the form $B_r(x) = \{y \in X \mid d(x,y) < r\}$.

Solution "$\implies$"

Suppose A is open. Then we have that $A = A^\circ$. By definition, for every point $a \in A$, there exists $r_a > 0$ such that $B_{r_a}(a) \subseteq A$. Since $a \in B_{r_a}(a)$,

$$A = \bigcup_{a \in A} \{a\} \subseteq \bigcup_{a \in A} B_{r_a}(a) \subseteq A.$$

Thus, A is a union of open balls.

"$\impliedby$"

Suppose A is a union of open balls. A union (finite or infinite) of open sets is open, so A is open.

6 Fix $r > 0$. Let (X, d) be a metric space and let A be a non-empty subset of X with diameter $\delta(A) < r$. Let $a \in X$ and assume that $A \cap B_r(a) \neq \emptyset$. Then $A \subseteq B_{2r}(a)$.

Solution Let $b \in A$ and $c \in A \cap B_r(a)$. As $b, c \in A$, $d(b, c) < r$. Since $c \in B_r(a)$, we have that $d(a, c) < r$. Then by the triangle inequality, $d(b, a) \leq d(b, c) + d(c, a) < 2r$. By definition, $b \in B_{2r}(a)$. This applies to all $b \in A$, so $A \subseteq B_{2r}(a)$.

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7 Let (X, d) be a metric space and let A, B be two non-empty subsets of X. Prove that

$$A^\circ \cap B^\circ = (A \cap B)^\circ \quad \text{and} \quad (A^\circ)^\circ = A^\circ.$$

Solution $x \in (A \cap B)^\circ \iff \exists r > 0$ s.t. $B_r(x) \subseteq A \cap B \iff B_r(x) \subseteq A$ and $B_r(x) \subseteq B \iff x \in A^\circ$ and $x \in B^\circ$.
$x \in (A^\circ)^\circ \iff \exists r > 0$ s.t. $B_r(x) \subseteq A^\circ \subseteq A \iff x \in A^\circ$.

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8 Let (X, d) be a metric space and let A be a subset of X. Prove that a point $x \in X$ is an adherent point of A if and only if $d(x, A) = 0$.

Solution "$\Longrightarrow$"

Let $x \in X$ be an adherent point of A. Then for all $r > 0$, $A \cap B_r(x) \neq \emptyset$. Thus, for all r, there exists $a \in A$ such that $0 \leq d(x, a) < r$. Since this is true for any $r > 0$, 0 is clearly the greatest lower bound. Thus, $\inf\{d(x, a) \mid a \in A\} = d(x, A) = 0$.

"$\Longleftarrow$"

Let $x \in X$ and assume $d(x, A) = 0$. Let $r > 0$. Then by definition, there exists $a \in A$ such that $d(x, a) < r$. If not, then 0 would not be the greatest lower bound for the distance between x and a. Thus, $B_r(x) \cap A \neq \emptyset$. This holds for all $r > 0$, so by definition, x is an adherent point of A.

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9 Let (X, d) be a metric space and let A be a subset of X. Prove that the diameter of A is equal to the diameter of the closure of A, that is, $\delta(A) = \delta(\bar{A})$.

Solution Let $x, y \in \bar{A}$. Then by definition, for every $r > 0$, we have that $B_{r/2}(x) \cap A \neq \emptyset$ and $B_{r/2}(y) \cap A \neq \emptyset$. Let $a_x \in B_{r/2}(x) \cap A$ and $a_y \in B_{r/2}(y) \cap A$. Then $d(x, y) \leq d(x, a_x) + d(y, a_y) + d(a_x, a_y) < \delta(A) + r$. Thus, $d(x, y) \leq \delta(A)$ for all $x, y \in \bar{A}$. So, $\delta(\bar{A}) \leq \delta(A)$. If $\delta(\bar{A}) = \infty$, then it follows that $\delta(A) = \infty$ also.

We also have that $A \subseteq \bar{A} \implies \delta(A) \leq \delta(\bar{A})$. If $\delta(A) = \infty$, then $\delta(\bar{A}) = \infty$ as well.

Taking the two inequalities, we have $\delta(A) = \delta(\bar{A})$ as desired.

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10 Let (X, d) be a metric space and let A be a subset of X and O be an open subset of X. Prove that

$$O \cap \bar{A} \subseteq \overline{O \cap A} \quad \text{and} \quad \overline{O \cap \bar{A}} = \overline{O \cap A}.$$

Conclude that if $O \cap A = \emptyset$, then $O \cap \bar{A} = \emptyset$.

Solution If $O \cap \bar{A} = \emptyset$, then the proof is trivial. Assume the intersection is non-empty.

Let $a \in O \cap \bar{A}$. Then there exists $R > 0$ such that $B_R(a) \subseteq O$ and $B_R(a) \cap A \neq \emptyset$. Then $B_R(a) \cap O \cap A = B_R(a) \cap A \neq \emptyset$. If $r < R$, then the same reasoning holds. If $r > R$, then $B_R(a) \subseteq B_r(a)$, so $B_r(a) \cap A$ is non-empty.

First note that $\bar{O} \cap \bar{A} \subseteq \overline{\bar{O} \cap \bar{A}} \subseteq \bar{O} \cap \bar{A} \implies \bar{O} \cap \bar{A} = \overline{\bar{O} \cap \bar{A}}$.

$a \in \overline{O \cap \bar{A}} \iff \forall r > 0\ B_r(a) \cap \bar{A} \cap O \neq \emptyset \iff B_r(a) \cap \bar{A} \cap \bar{O} \neq \emptyset \iff a \in \overline{\bar{O} \cap \bar{A}} = \overline{O \cap A}$.

If $O \cap A = \emptyset$, then as $\emptyset$ is closed, $O \cap \bar{A} \subseteq \overline{O \cap A} = O \cap A = \emptyset$. Thus, $O \cap \bar{A} = \emptyset$.

11 Let (X, d) be a metric space and let $a \in X$ and $r > 0$. Prove that the closed ball $K_r(a) = \{x \in X \mid d(x, a) \leq r\}$ is a closed set.

Solution Every element of $K_r(a)$ is an adherent point. We wish to show that these are the only adherent points of $K_r(a)$. Then it follows that $K_r(a) = \overline{K_r(a)}$.

Suppose there exists $x \in \overline{K_r(a)}$ such that $d(x, a) > r$. Then for all $R > 0$, $B_R(x) \cap K_r(a) \neq \emptyset$. Take $R = d(a, x) - r$. Let $y \in B_R(x) \cap K_r(a)$. Then

$$d(a, x) + d(x, y) < r + d(a, x) - r = d(a, x).$$

This is a contradiction since $d(x, y) \geq 0$, so no such x exists. Thus, the only adherent points of $K_r(a)$ must be contained in $K_r(a)$, so $K_r(a)$ is closed.

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12 Let (X, d) be a metric space and let A, B be two subsets of X. Prove that

$$\mathrm{Fr}(A \cup B) \subseteq \mathrm{Fr}(A) \cup \mathrm{Fr}(B).$$

Show also that if $\bar{A} \cap \bar{B} = \emptyset$, then $\mathrm{Fr}(A \cup B) = \mathrm{Fr}(A) \cup \mathrm{Fr}(B)$.

Solution
$$\begin{aligned}
\mathrm{Fr}(A \cup B) &= (\overline{A \cup B}) \cap \overline{(A \cup B)^{\mathrm{C}}} \\
&= (\bar{A} \cup \bar{B}) \cap \overline{A^{\mathrm{C}} \cap B^{\mathrm{C}}} \\
&\subseteq (\bar{A} \cup \bar{B}) \cap (\overline{A^{\mathrm{C}}} \cap \overline{B^{\mathrm{C}}}) \\
&= (\bar{A} \cap \overline{A^{\mathrm{C}}} \cap \overline{B^{\mathrm{C}}}) \cup (\bar{B} \cap \overline{A^{\mathrm{C}}} \cap \overline{B^{\mathrm{C}}}) \\
&\subseteq (\bar{A} \cap \overline{A^{\mathrm{C}}}) \cup (\bar{B} \cap \overline{A^{\mathrm{C}}}) \\
&= \mathrm{Fr}(A) \cup \mathrm{Fr}(B)
\end{aligned}$$

If $\bar{A} \cap \bar{B} = \emptyset$, then $(\bar{A} \cap \bar{B})^{\mathrm{C}} = \bar{A}^{\mathrm{C}} \cup \bar{B}^{\mathrm{C}} = X$. So, $\bar{A} \cap \overline{B^{\mathrm{C}}} = X \cap \bar{A} = \bar{A}$. We also have that $\overline{A^{\mathrm{C}} \cap B^{\mathrm{C}}} = \overline{A^{\mathrm{C}}} \cap \overline{B^{\mathrm{C}}}$. Thus, this turns $\subseteq$ into $=$ in the above, so equality holds.

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13 Let (X, d) be a metric space and let A be a subset of X. Prove that

$$\begin{aligned}
\mathrm{Fr}(\bar{A}) &\subseteq \mathrm{Fr}(A) \\
\mathrm{Fr}(A^\circ) &\subseteq \mathrm{Fr}(A) \\
\bar{A} &= A^\circ \cup \mathrm{Fr}(A).
\end{aligned}$$

Solution $\mathrm{Fr}(\bar{A}) = \bar{\bar{A}} \cap \overline{\bar{A}^{\mathrm{C}}} = A \cap \overline{\mathrm{Ext}(A)} = A \cap \overline{(A^{\mathrm{C}})^\circ} = A \cap \overline{A^{\mathrm{C}}} = \mathrm{Fr}(A)$
$\mathrm{Fr}(A^\circ) = \overline{A^\circ} \cap \overline{(A^\circ)^{\mathrm{C}}} = \bar{A} \cap \overline{\bar{A}^{\mathrm{C}}} = \mathrm{Fr}(A)$
$\mathrm{Fr}(A) = \bar{A} \cap \overline{A^{\mathrm{C}}} = \bar{A} \cap (A^\circ)^{\mathrm{C}}$ Taking the union of both sides with A° yields

$$(\bar{A} \cup A^\circ) \cap ((A^\circ)^{\mathrm{C}} \cup A^\circ) = \bar{A} = A^\circ \cup \mathrm{Fr}(A).$$

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14 Let (X, d) be a metric space and let A be a subset of X. Prove that A is closed if and only if $\mathrm{Fr}(A) \subseteq A$.

Solution "$\Longrightarrow$"

If A is closed, then $A = \bar{A}$. Intersecting both sides with $\overline{A^{\mathrm{C}}}$ yields $A \cap \overline{A^{\mathrm{C}}} = \bar{A} \cap \overline{A^{\mathrm{C}}} = \mathrm{Fr}(A)$. Then $\mathrm{Fr}\, A \subseteq A$.

"$\Longleftarrow$"

If $\mathrm{Fr}(A) \subseteq A$, then $\bar{A} \cap \overline{A^{\mathrm{C}}} = \bar{A} \cap (A^\circ)^{\mathrm{C}} \subseteq A$. Taking the union with A° on both sides yields $\bar{A} \subseteq A$. Since $A \subseteq \bar{A}$ also, it follows that $A = \bar{A} \implies$ A is closed.

15 Let (X, d) be a metric space and let A be a subset of X. Prove that A is open if and only if $\mathrm{Fr}(A) \cap A = \emptyset$.

Solution " $\implies$ "

If A is open, then $A = A^\circ$. Then $\mathrm{Fr}(A) \cap A = (A^\circ)^{\mathrm{C}} \cap \bar{A} \cap A = (A^\circ)^{\mathrm{C}} \cap A = (A^\circ)^{\mathrm{C}} \cap A^\circ = \emptyset$

" $\impliedby$ "

If $\mathrm{Fr}(A) \cap A = \emptyset$, then if we take the union of both sides with A°, we get

$$A^\circ = ((A^\circ)^{\mathrm{C}} \cap \bar{A} \cap A) \cup A^\circ = ((A^\circ)^{\mathrm{C}} \cap A) \cup A^\circ = A$$