5.9 Let $C^k([0,1])$ be the space of functions on $[0,1]$ possessing continuous derivatives up to order k on $[0,1]$, including one-sided derivatives at the endpoints.

a. If $f \in C([0,1])$, then $f \in C^k([0,1])$ iff f is k times continuously differentiable on $(0,1)$ and $\lim_{x\searrow 0} f^{(j)}(x)$ and $\lim_{x\nearrow 1} f^{(j)}(x)$ exist for $j \le k$. (The mean value theorem is useful.)

b. $\|f\| = \sum_0^k \|f^{(j)}\|_u$ is a norm on $C^k([0,1])$ that makes $C^k([0,1])$ into a Banach space. (Use induction on k. The essential point is that if $\{f_n\} \subseteq C^1([0,1])$, $f_n \to f$ uniformly and $f_n' \to g$ uniformly, then $f \in C^1([0,1])$ and $f' = g$. The easy way to prove this is to show that $f(x) - f(0) = \int_0^x g(t)\,\mathrm{d}t$.)

Solution a. "$\implies$"

This direction is clear from definition.

"$\impliedby$"

Let $1 \le j \le k$. The limits exist for $j = 0$ by continuity.

To show differentiability at the endpoints, we just need to show that

$$\lim_{\delta\to 0^+} \frac{f^{(j-1)}(\delta) - f^{(j-1)}(0)}{\delta} \quad \text{and} \quad \lim_{\delta\to 1^-} \frac{f^{(j-1)}(1) - f^{(j-1)}(\delta)}{1-\delta}$$

exist.

By the mean value theorem, given $\delta \in (0,1)$, there exists $\zeta \in (0,\delta)$ so that

$$\frac{f^{(j-1)}(\delta) - f^{(j-1)}(0)}{\delta} = f^{(j)}(\zeta).$$

If we let $\delta \to 0^+$, we get $\zeta \to 0^+$ and by assumption,

$$\lim_{\delta\to 0^+} \frac{f^{(j-1)}(\delta) - f^{(j-1)}(0)}{\delta} = \lim_{\zeta\to 0^+} f^{(j)}(\zeta)$$

exists. Similarly, we get $\eta \in (\delta, 1)$ so that

$$\frac{f^{(j-1)}(1) - f^{(j-1)}(\delta)}{1-\delta} = f^{(j)}(\eta).$$

Letting $\delta \to 1^-$, we get $\eta \to 1^-$, so by assumption,

$$\lim_{\delta\to 1^-} \frac{f^{(j-1)}(1) - f^{(j-1)}(\delta)}{1-\delta} = \lim_{\zeta\to 1^-} f^{(j)}(\eta)$$

exists also.

b. We proceed by induction.

Base step:

We already know that $C([0,1])$ is a Banach space with the uniform norm.

Inductive step:

Suppose that $C^k([0,1])$ is complete with the given metric. We wish to show that $C^{k+1}([0,1])$ is also complete. We follow the hint:

Assume that $f_n \xrightarrow{n\to\infty} f$ and $f_n' \xrightarrow{n\to\infty} g$ uniformly. Then

$$f_n(x) - f_n(0) = \int_0^x f_n'(t)\,\mathrm{d}t.$$

By uniform convergence, we get

$$f(x) - f(0) = \int_0^x g(t)\,\mathrm{d}t.$$

Notice that since each f_n' was continuous, uniform convergence tells us that g is also continuous, and so $\int_0^x g(t)\,\mathrm{d}t$ is continuous. Thus, it follows that f is differentiable, and that

$$f'(x) = g(x),$$

which also shows that f is C^1.

Now suppose that $\{f_n\} \subseteq C^{k+1}([0,1])$ is Cauchy, so it is also Cauchy in $C^k([0,1])$, so by the inductive hypothesis, we know that the limit function f exists and is C^k. Also, by definition of the norm, we know that $\{f_n^{(k+1)}\}$ is Cauchy in $C([0,1])$. By completeness, it converges to some continuous function g, and by the hint, we know that $(f^{(k)})' = g$, so $f^{(k)} \in C^1([0,1])$, so $f \in C^{k+1}([0,1])$, so $C^{k+1}([0,1])$ with the given norm is a Banach space.

5.15 Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $T \in L(\mathcal{X}, \mathcal{Y})$. Let $\mathcal{N}(T) = \{x \in \mathcal{X} \mid Tx = 0\}$.

a. $\mathcal{N}(T)$ is a closed subspace of $\mathcal{X}$.

b. There is a unique $S \in L(\mathcal{X}/\mathcal{N}(T), \mathcal{Y})$ such that $T = S \circ \pi$ where $\pi \colon \mathcal{X} \to \mathcal{X}/\mathcal{M}$. Moreover, $\|S\| = \|T\|$.

Solution a. Let $\{x_n\} \subseteq \mathcal{N}(T)$ be a convergent sequence in $\mathcal{X}$, and let its limit be x. By continuity of T,

$$Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} 0 = 0 \implies x \in \mathcal{N}(T),$$

so $\mathcal{N}(T)$ is closed in $\mathcal{X}$.

b. Let S be such a function. Then for any $x \in \mathcal{X}$, we have

$$T(x) = S(x + \mathcal{M}),$$

so such an S is unique.

We will show that such an S is well-defined and that $S \in L(\mathcal{X}/\mathcal{N}(T), \mathcal{Y})$.

Well-definedness:

Let $x, x' \in \mathcal{X}$ so that $x - x' \in \mathcal{M}$. Then

$$T(x) = S(x + \mathcal{M}) = S(x - (x - x') + \mathcal{M}) = S(x' + \mathcal{M}) = T(x'),$$

so S is well-defined.

Linearity:

Let $x, y \in \mathcal{X}$. Then

$$S(x + \mathcal{M} + y + \mathcal{M}) = S(x + y + \mathcal{M}) = T(x + y) = T(x) + T(y) = S(x + \mathcal{M}) + S(y + \mathcal{M}).$$

Now let $\lambda \in K$. Then

$$S(\lambda x + \mathcal{M}) = T(\lambda x) = \lambda T(x) = \lambda S(\lambda x + \mathcal{M}),$$

so S is linear.

Boundedness:

By exercise 12, $\|\pi\| = 1$, so $\|T\| \leq \|S\|\|\pi\| = \|S\|$.

On the other hand,

$$\|S(x + \mathcal{M})\| = \|Tx\| = \|T(x + y)\| \leq \|T\|\|x + y\|,$$

where $y \in \mathcal{M}$. If we take the infimum over $\mathcal{M}$, $\|x + y\|$ becomes $\|x + \mathcal{M}\|$, by definition, so

$$\|S(x + \mathcal{M})\| \leq \|T\|\|x + \mathcal{M}\|,$$

so $\|S\| \leq \|T\|$, as desired.

Thus, S is a well-defined linear functional which is in $L(\mathcal{X}/\mathcal{N}(T), \mathcal{Y})$.

5.20 If $\mathcal{M}$ is a finite-dimensional subspace of a normed vector space $\mathcal{X}$, there is a closed subspace $\mathcal{N}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N} = \mathcal{X}$.

Solution Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for $\mathcal{M}$.

Let T be the natural isomorphism between $\mathcal{M}$ and K^n. It's clear that T and T^{-1} are bounded, since all the coordinates will be bounded by 1, if $\|x\| = 1$. Now consider the coordinate projections on K^n, $\pi_i \colon K^n \to K$. These functions give us the linear functionals $\pi_i \circ T$ on $\mathcal{M}$, which is like a "coordinate projection to K."

By Hahn-Banach, for any $x \in \mathcal{X} \setminus \mathcal{M}$, there exists a linear functional $f_i \in \mathcal{X}$ so that $f_i(x) \neq 0$ which extends the "projection" to all of $\mathcal{X}$.

Now consider the set $\mathcal{N} := \bigcap_{i=1}^n \ker f_i$. This is a closed subspace since kernels are closed subspaces, and because intersections of closed sets are closed.

First, $\mathcal{M} \cap \mathcal{N} = \{0\}$. If not, then there exists $x \in \mathcal{M} \cap \mathcal{N}$ and $1 \leq i \leq n$ so that $(\pi_i \circ T)x \neq 0$. But since $x \in \mathcal{N} \subseteq \ker f_i$, this implies that $f_i(x) = 0$, but this cannot happen since $f_i\big|_{\mathcal{M}} = \pi_i \circ T$. Hence, their intersection is trivial.

Lastly, let $x \in \mathcal{X}$. Notice that

$$\sum_{i=1}^n f_i(x) v_i \in \mathcal{M} \quad \text{and} \quad x - \sum_{i=1}^n f_i(x) v_i \in \mathcal{N}.$$

The first one is true, since the v_i give us a basis of $\mathcal{M}$. As for the second, notice that

$$f_j\left(x - \sum_{i=1}^n f_i(x) v_i\right) = f_j(x) - f_j(x) = 0 \implies x - \sum_{i=1}^n f_i(x) v_i \in \ker f_j.$$

This works for all j, so it's in all the $\ker f_j$, i.e., it's in $\mathcal{N}$. This shows that $\mathcal{M} + \mathcal{N}$, as needed.

5.21 If $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces, define $\alpha \colon \mathcal{X}^* \times \mathcal{Y}^* \to (\mathcal{X} \times \mathcal{Y})^*$ by $\alpha(f, g)(x, y) = f(x) + g(y)$. Then α is an isomorphism which is isometric if we use the norm $\|(x, y)\| = \max(\|x\|, \|y\|)$ on $\mathcal{X} \times \mathcal{Y}$, the corresponding operator norm on $(\mathcal{X} \times \mathcal{Y})^*$, and the norm $\|(f, g)\| = \|f\| + \|g\|$ on $\mathcal{X}^* \times \mathcal{Y}^*$.

Solution We'll first show that α is an isomorphism. It's clear that it's a homomorphism, since the linear functionals form a vector space. So, we just need to show that it's a bijection.

Suppose that $\alpha(f, g)(x, y) = \alpha(f', g')(x, y)$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Then $(f - f')(x) + (g - g')(y) \equiv 0$. In particular, if we set $x = 0$, then $g = g'$ and similarly, we get that $f = f'$, so α is injective.

For surjectivity, let $h(x, y) \in (\mathcal{X} \times \mathcal{Y})^*$. Notice that by linearity, we can decompose h into $h(x, 0) + h(0, y) := h_x(x) + h_y(y)$, so $\alpha(h_x, h_y) = h$, which shows surjectivity.

Lastly, we need to show that α^{-1} is bounded. This is easy to see, as if we had h with $\|h\| = 1$, then

$$\|\alpha^{-1}(h)\| = \|(h(x, 0), h(0, y))\| = \|h(x, 0)\| + \|h(0, y)\| \leq 2 \implies \|\alpha^{-1}\| \leq 2.$$

Hence, α is an isomorphism.

We now need to show that it's an isometry. Let $(f, g) \in \mathcal{X}^* \times \mathcal{Y}^*$.

By the triangle inequality, $\|\alpha(f, g)\| = \sup\limits_{\|x\|=1 \text{ or } \|y\|=1} |f(x) + g(y)| \leq \|f\| + \|g\| = \|(f, g)\|$.

Conversely, let $\varepsilon > 0$. Then there exist $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ with $\|x\| = \|y\| = 1$ so that $|f(x)| \geq \|f\| - \varepsilon/2$ and $|g(y)| \geq \|g\| - \varepsilon/2$. Then, by multiplying by a rotation $e^{i\theta}$ if necessary, we may assume that $f(x)$ and $g(x)$ have the same direction (this is mainly necessary in the complex case), which gives

$$\|(f, g)\| - \varepsilon = \|f\| + \|g\| - \varepsilon \leq |f(x)| + |g(y)| = |f(x) + g(y)| = |\alpha(f, g)(x, y)| \leq \|\alpha(f, g)\|.$$

Letting $\varepsilon \to 0$, we get that $\|\alpha(f, g)\| = \|(f, g)\|$, so α is an isometry.

5.22 Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $T \in L(\mathcal{X}, \mathcal{Y})$.

a. Define $T^\dagger \colon \mathcal{Y}^* \to \mathcal{X}^*$ by $T^\dagger f = f \circ T$. Then $T^\dagger \in L(\mathcal{Y}^*, \mathcal{X}^*)$ and $\|T^\dagger\| = \|T\|$.

b. Applying the construction in (a) twice, one obtains $T^{\dagger\dagger} \in L(\mathcal{X}^{**}, \mathcal{Y}^{**})$. If $\mathcal{X}$ and $\mathcal{Y}$ are identified with their natural images $\hat{\mathcal{X}}$ and $\hat{\mathcal{Y}}$ in $\mathcal{X}^{**}$ and $\mathcal{Y}^{**}$, then $T^{\dagger\dagger}\big|_{\mathcal{X}} = T$.

c. $T^\dagger$ is injective iff the range of T is dense in $\mathcal{Y}$.

d. If the range of $T^\dagger$ is dense in $\mathcal{X}^*$, then T is injective; the converse is true if $\mathcal{X}$ is reflexive.

Solution a. Since $\mathcal{X}^*$ and $\mathcal{Y}^*$ are vector spaces, it follows that $T^\dagger$ is linear, so we just need to show that it is bounded. By definition, it is clear that if $\|f\|_{\mathcal{Y}^*} = 1$, then

$$\|T^\dagger f\|_{\mathcal{X}^*} = \|f \circ T\|_{\mathcal{X}^*} \leq \|f\|_{\mathcal{Y}^*}\|T\|_{\text{op}} = \|T\|_{\text{op}}$$

so $\|T^\dagger\|_{\text{op}} \leq \|T\|_{\text{op}}$.

Conversely, by definition, there exists $x_n \in \mathcal{X}$ with $\|x_n\| = 1$ for each n so that $\|Tx_n\| \xrightarrow{n\to\infty} \|T\|$.

By an application of Hahn-Banach, for each $n \geq 1$, there exists $f_n \in \mathcal{Y}^*$ so that $\|f_n\| = 1$ and $f_n(Tx_n) = \|Tx_n\|$. Thus,

$$\|T^\dagger\|_{\text{op}} \geq \|T^\dagger f_n\| \geq \|(f_n \circ T)x_n\| = \|Tx_n\| \xrightarrow{n\to\infty} \|T\|_{\text{op}},$$

so $\|T^\dagger\|_{\text{op}} = \|T\|_{\text{op}}$.

b. For $x \in \mathcal{X}$, we identify it with $\hat{x} \in \mathcal{X}^{**}$, where $\hat{x}(f) = f(x)$. Then

$$T^{\dagger\dagger}\hat{x} = \hat{x} \circ T^\dagger = T^\dagger\hat{x} = \hat{x} \circ T = Tx,$$

for any $x \in \mathcal{X}$, so $T^{\dagger\dagger}\big|_{\mathcal{X}} = T$.

c. "$\Longrightarrow$"

Let $T^\dagger$ be injective, and suppose that $T(\mathcal{X})$ is not dense in $\mathcal{Y}$. Then there exists $x \in \mathcal{Y}$ with positive distance δ to $T(\mathcal{X})$. By Hahn-Banach, there exists $f \in \mathcal{X}^*$ so that $\|f\| = 1$, $f(x) = \delta$, and $f\big|_{T(\mathcal{X})} = 0$. Thus, $T^\dagger f = 0$ and $f \neq 0$, but this cannot happen, since $T^\dagger$ is injective. Thus, $T(\mathcal{X})$ must be dense in $\mathcal{Y}$.

"$\Longleftarrow$"

Let $T(\mathcal{X})$ be dense in $\mathcal{Y}$, and assume that $T^\dagger$ is not injective, so that there exists a non-zero $f \in \mathcal{Y}^*$ with $T^\dagger f = 0$.

By definition, this means that $f \circ T = 0$, so $T(\mathcal{X}) \subseteq \ker f$. But $T(\mathcal{X})$ is dense in $\mathcal{Y}$ and f is a continuous linear functional, which means that $Y \subseteq \ker f$, but this means that $f = 0$, a contradiction. Thus, T must be injective.

d. "$\Longrightarrow$"

Let $T^\dagger(\mathcal{Y}^*)$ be dense in $\mathcal{X}^*$, and suppose that T is not injective, i.e., there exists a non-zero $x \in \ker T$. Hahn-Banach gives us a functional $f \in \mathcal{X}^*$ so that $\|f\| = 1$ and $f(x) = \|x\| > 0$.

Since $T^\dagger(\mathcal{Y}^*)$ is dense in $\mathcal{X}$, there exists $g_n \in \mathcal{Y}^*$ so that $T^\dagger(g_n) \xrightarrow{n\to\infty} f$. But

$$0 = g_n(0) = g_n(Tx) = T^\dagger(g_n)(x) \xrightarrow{n\to\infty} f(x) = \|x\| > 0,$$

which is impossible, so T must be injective.

"$\Longleftarrow$"

Assume that $\mathcal{X}$ is reflexive.

Now let T be injective, and assume that the range of $T^\dagger$ is not dense in $\mathcal{X}^*$. So, there exists a non-zero $f \in \mathcal{X}^*$ with positive distance (with respect to the operator norm) from $T^\dagger(\mathcal{Y}^*)$. By Hahn-Banach, there exists $\hat{x} \in \mathcal{X}^{**} = \mathcal{X}$ so that $\|\hat{x}\| = 1$ and $\hat{x}\big|_{T^\dagger(\mathcal{Y}^*)} = 0$.

Since $\mathcal{X}$ is reflexive, we may identify $\hat{x}$ with $x \in \mathcal{X}$. Since $y \mapsto \hat{y}$ is an isometry, we have that $\|x\| = \|\hat{x}\| > 0$ and by injectivity, $Tx \neq 0$. Again, by Hahn-Banach, there exists $g \in \mathcal{Y}^*$ so that $g(Tx) \neq 0$. But this means that

$$0 = \hat{x}(T^\dagger g) = \hat{x}(g \circ T) = g(Tx) \neq 0,$$

a contradiction. So the image of $T^\dagger$ must be dense in $\mathcal{X}$.

5.25 If $\mathcal{X}$ is a Banach space and $\mathcal{X}^*$ is separable, then $\mathcal{X}$ is separable. (Let $\{f_n\}_1^\infty$ be a countable dense subset of $\mathcal{X}^*$. For each n choose $x_n \in \mathcal{X}$ with $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$. Then the linear combinations of $\{x_n\}_1^\infty$ are dense in $\mathcal{X}$.

Solution We follow the hint, and let $\{f_n\}_1^\infty$ be a countable dense subspace in $\mathcal{X}^*$. Then for each $n \geq 1$, there exists $x_n \in \mathcal{X}$ so that $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$, by definition of the operator norm. Now consider

$$\left\{ \sum_{j \in N} q_j x_j \mid N \subseteq \mathbb{N} \text{ finite}, \ q_j = a_j + ib_j, \text{ where } a_j, b_j \in \mathbb{Q} \right\},$$

which is countable, since the collection of finite subsets of $\mathbb{N}$ is countable and because $\mathbb{Q} \times \mathbb{Q}$ is countable.

We claim that this, which we call $\mathcal{M}$, is dense in $\mathcal{X}$.

Suppose otherwise, and that there exists $x \in \mathcal{X}$ with positive distance $\delta > 0$ from $\mathcal{M}$. Then by Hahn-Banach, there exists $f \in \mathcal{X}^*$ so that $f(x) = \delta$ and $\|f\| = 1$, and $f\big|_{\mathcal{M}} = 0$.

Since $\{f_n\}$ was dense in $\mathcal{X}^*$ and $\mathcal{X}$ is complete, there exists a sequence of linear combinations f_{n_k} so that $f_{n_k} \xrightarrow{k \to \infty} f$. But by assumption, for every k,

$$|f(x_{n_k}) - f_k(x_{n_k})| = |f_k(x_{n_k})| \geq \frac{\|f_{n_k}\|}{2} \implies \|f - f_{n_k}\| \geq \frac{\|f_{n_k}\|}{2}.$$

Thus, since $\|f - f_k\| \xrightarrow{k \to \infty} 0$, we see that $\|f_{n_k}\| \xrightarrow{k \to \infty} 0 \implies \|f\| = 0$, a contradiction. Thus, the linear combinations of $\{x_n\}$ must be dense in $\mathcal{X}$.