****13 3.4.12** Let V be an n-dimensional vector space over the field F, and let $\mathfrak{B} = \{\alpha_1, \dots, \alpha_n\}$ be an ordered basis for V.

a. According to Theorem 1, there is a unique linear operator T on V such that

$$T\alpha_j = \alpha_{j+1}, \qquad j = 1, \dots, n-1, \qquad T\alpha_n = 0.$$

What is the matrix A of T in the ordered basis $\mathfrak{B}$?

b. Prove that $T^n = 0$ but $T^{n-1} \neq 0$.

c. Let S be any linear operator on V such that $S^n = 0$ but $S^{n-1} \neq 0$. Prove that there is an ordered basis $\mathfrak{B}'$ for V such that the matrix S in the ordered basis $\mathfrak{B}'$ for V such that the matrix of S in the ordered basis $\mathfrak{B}'$ is the matrix A of part (a).

d. Prove that if M and N are $n \times n$ matrices over F such that $M^n = N^n = 0$ but $M^{n-1} \neq 0 \neq N^{n-1}$, then M and N are similar.

Solution a. We have

$$\begin{aligned}
T\alpha_1 &= 0\alpha_1 + 1\alpha_2 + 0\alpha_3 + \cdots + 0\alpha_n \\
T\alpha_2 &= 0\alpha_1 + 0\alpha_2 + 1\alpha_3 + \cdots + 0\alpha_n \\
&\vdots \\
T\alpha_{n-1} &= 0\alpha_1 + 0\alpha_2 + 0\alpha_3 + \cdots + 1\alpha_n \\
T\alpha_n &= 0\alpha_1 + 0\alpha_2 + 0\alpha_3 + \cdots + 0\alpha_n
\end{aligned}$$

So

$$[T]_{\mathfrak{B}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

b. Let

$$U = \begin{pmatrix} | & | & | & | \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ | & | & | & | \end{pmatrix},$$

which is the matrix that takes coordinates from the $\mathfrak{B}$ basis to the standard basis. So, U^{-1} does the opposite. Then

$$\begin{aligned}
[T]_{\mathfrak{B}} &= U^{-1}TU \\
[T]_{\mathfrak{B}}^n &= U^{-1}T^nU \\
&= U^{-1}T^n \begin{pmatrix} | & | & | & | \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ | & | & | & | \end{pmatrix} \\
&= U^{-1} \begin{pmatrix} | & | & | & | \\ T^n\alpha_1 & T^n\alpha_2 & \cdots & T^n\alpha_n \\ | & | & | & | \end{pmatrix} \\
&= U^{-1} \begin{pmatrix} | & | & | & | \\ T\alpha_n & T^2\alpha_n & \cdots & T^n\alpha_n \\ | & | & | & | \end{pmatrix} \\
&= U^{-1}0 \\
&= 0
\end{aligned}$$

Thus,

$$T^n = U[T]_{\mathfrak{B}}^n U^{-1} = U0U^{-1} = 0$$

as desired. Similarly

$$\begin{aligned}
[T]_{\mathfrak{B}}^{n-1} &= U^{-1}T^{n-1}U \\
&= U^{-1}T^n \begin{pmatrix} | & | & | & | \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ | & | & | & | \end{pmatrix} \\
&= U^{-1} \begin{pmatrix} | & | & | & | \\ T^{n-1}\alpha_1 & T^{n-1}\alpha_2 & \cdots & T^{n-1}\alpha_n \\ | & | & | & | \end{pmatrix} \\
&= U^{-1} \begin{pmatrix} | & | & | & | \\ \alpha_n & T\alpha_n & \cdots & T^n\alpha_n \\ | & | & | & | \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \neq 0
\end{aligned}$$

Thus, as U and U^{-1} are invertible $\implies$ $U\alpha$ and $U^{-1}\alpha$ are 0 only if $\alpha = 0$, we have

$$T^{n-1} = U[T]_{\mathfrak{B}}U^{-1} \neq 0$$

c. We take $\alpha \in V$ such that $S^{n-1}\alpha \neq 0$. Then the set $\mathfrak{B}' = \{\alpha, S\alpha, \ldots, S^{n-1}\alpha\}$ is a basis of $\mathbb{R}^n$. To show this, consider the sum

$$\begin{aligned}
c_0\alpha + c_1S\alpha + \cdots + c_{n-1}S^{n-1}\alpha &= 0 \\
c_0S^{n-1}\alpha + c_1S^n\alpha + \cdots + c_{n-1}S^{2n-2}\alpha &= 0 \\
c_0S^{n-1}\alpha &= 0
\end{aligned}$$

Since $S^{n-1}\alpha \neq 0$ by assumption, $c_0 = 0$. Repeating a similar argument yields

$$\begin{aligned}
c_1S^{n-1}\alpha = 0 &\implies c_1 = 0 \\
c_2S^{n-1}\alpha = 0 &\implies c_2 = 0 \\
&\vdots \\
c_{n-1}S^{n-1}\alpha = 0 &\implies c_{n-1} = 0
\end{aligned}$$

Thus, $\mathfrak{B}'$ contains n linearly independent vectors, so it is a basis of $\mathbb{R}^n$. Thus, we let

$$U = \begin{pmatrix} | & | & | & | \\ \alpha & S\alpha & \cdots & S^{n-1}\alpha \\ | & | & | & | \end{pmatrix}$$

which has the same properties as the U from part (b). Thus,

$$
\begin{aligned}
[S]_{\mathfrak{B}'} &= U^{-1}SU \\
&= U^{-1}S\begin{pmatrix} | & | & | & | \\ \alpha & S\alpha & \cdots & S^{n-1}\alpha \\ | & | & | & | \end{pmatrix} \\
&= U^{-1}\begin{pmatrix} | & | & | & | \\ S\alpha & S^2\alpha & \cdots & S^{n}\alpha \\ | & | & | & | \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}
\end{aligned}
$$

as desired.

d. By part (c), M and N are both similar to the matrix of part (a), which we will call T. Thus, there exist invertible U and V such that

$$
\begin{aligned}
T = UMU^{-1} &= VNV^{-1} \\
M &= (U^{-1}V)N(V^{-1}U) \\
&= (U^{-1}V)N(U^{-1}V)^{-1}
\end{aligned}
$$

Hence, M and N are similar.

****14 3.4.9** Let V be a finite-dimensional vector space over the field F and let S and T be linear operators on V. We ask: When do there exist ordered bases $\mathfrak{B}$ and $\mathfrak{B}'$ for V such that $[S]_{\mathfrak{B}} = [T]_{\mathfrak{B}'}$? Prove that such bases exist if and only if there is an invertible linear operator U on V such that $T = USU^{-1}$. (*Outline of proof:* If $[S]_{\mathfrak{B}} = [T]_{\mathfrak{B}'}$, let U be the operator which carries $\mathfrak{B}$ onto $\mathfrak{B}'$ and show that $S = UTU^{-1}$. Conversely, if $T = USU^{-1}$ for some invertible U, let $\mathfrak{B}$ be any ordered basis for V and let $\mathfrak{B}'$ be its image under U. Then show that $[S]_{\mathfrak{B}} = [T]_{\mathfrak{B}'}$.)

Solution "$\implies$"

Suppose there exist bases $\mathfrak{B}$ and $\mathfrak{B}'$ such that $[S]_{\mathfrak{B}} = [T]_{\mathfrak{B}'}$. Let P be the invertible matrix that transforms $\mathfrak{B}$ coordinates to standard coordinates, and let Q be the invertible matrix that transforms standard coordinates to $\mathfrak{B}'$ coordinates. If we let $U = PQ$, then

$$
\begin{aligned}
S &= P[S]_{\mathfrak{B}}P^{-1} \\
&= P[T]_{\mathfrak{B}'}P^{-1} \\
&= P(QTQ^{-1})P^{-1} \\
&= UTU^{-1}
\end{aligned}
$$

Hence, the invertible linear operator U exists.

"$\impliedby$"

Suppose there exists an invertible linear operator U on V such that $T = USU^{-1}$. We wish to show that for some bases $\mathfrak{B}$ and $\mathfrak{B}'$, we have $[S]_{\mathfrak{B}} = [T]_{\mathfrak{B}'}$. Let P be the invertible matrix that transforms standard coordinates to $\mathfrak{B}$ coordinates, and let Q be the invertible matrix that transforms standard coordinates to

$\mathfrak{B}'$ coordinates. Then if we let $U = Q^{-1}P$, which is invertible as Q and P are invertible, we get

$$\begin{aligned}
[T]_{\mathfrak{B}'} &= QTQ^{-1} \\
&= QUSU^{-1}Q^{-1} \\
&= QQ^{-1}PSP^{-1}QQ^{-1} \\
&= PSP^{-1} \\
&= [S]_{\mathfrak{B}}
\end{aligned}$$

as desired.