1 Let $\Omega \subseteq \mathbb{C}$ be open and let $f \in L^1_{\text{loc}}(\Omega)$ be such that

$$\int f(z) \frac{\partial \varphi}{\partial \overline{z}} \, L(\mathrm{d}z) = 0,$$

for all $\varphi \in C_0^1(\Omega)$. Show that there exists $g \in \text{Hol}(\Omega)$ such that $f = g$ a.e.

Solution We follow the hint: let $\omega \subseteq \Omega$ be relatively compact in Ω, and consider

$$g(z) = -\frac{1}{\pi} \int \frac{\partial \varphi}{\partial \overline{\zeta}}(\zeta) f(\zeta) \frac{L(\mathrm{d}\zeta)}{\zeta - z}, \quad z \in \omega,$$

where $\varphi \in C_0^1(\Omega)$ and is 1 in a neighborhood of $\overline{\omega}$.

Notice that the integrand is in $L^1(\overline{\omega})$, since $f \in L^1_{\text{loc}}(\Omega)$, φ is continuous, and because $(\zeta - z)^{-1}$ is also locally L^1. Hence, we may pass derivatives through the integral sign.

The only term may depend on $\overline{z}$ is $(\zeta - z)^{-1}$, but the entire integrand vanishes if ζ is close to z since $\partial_{\overline{z}} \varphi = 0$ in a neighborhood of $\overline{\omega}$, so the entire integrand vanishes. When ζ is far from z, $(\zeta - z)^{-1}$ is locally holomorphic, so the derivative vanishes. Hence, $\partial_{\overline{z}} g = 0$ and hence $g \in \text{Hol}(\omega)$.

It suffices to show that for any cutoff $\psi \in C_0^1(\omega)$ that

$$\int f(z)\psi(z) \, L(\mathrm{d}z) = \int g(z)\psi(z) \, L(\mathrm{d}z).$$

We then get that f and g agree almost everywhere on compact sets in ω, so we get agreement almost everywhere on ω.

Since g is integrable, we may apply Fubini's theorem:

$$\begin{aligned}
\int g(z)\psi(z) \, L(\mathrm{d}z) &= \int \left(-\frac{1}{\pi} \int \frac{\partial \varphi}{\partial \overline{\zeta}}(\zeta) f(\zeta) \frac{L(\mathrm{d}\zeta)}{\zeta - z} \right) \psi(z) \, L(\mathrm{d}z) \\
&= -\frac{1}{\pi} \int \left(\int \psi(z) \frac{L(\mathrm{d}z)}{\zeta - z} \right) f(\zeta) \frac{\partial \varphi}{\partial \overline{\zeta}}(\zeta) \, L(\mathrm{d}\zeta)
\end{aligned}$$

Since ψ is $C_0^1(\omega)$, there exists a $u \in C_0^1(\Omega)$ so that $\partial u / \partial \overline{\zeta} = -\psi$. Indeed, we can solve the inhomogeneous Cauchy–Riemann equation, and multiply by a cutoff function in a neighborhood of the support of ψ. Thus, applying the Cauchy integral formula on ω to u, u vanishes on $\partial \omega$, so we are left with

$$u(\zeta) = -\frac{1}{\pi} \int \frac{\partial u}{\partial \overline{z}}(z) \frac{L(\mathrm{d}z)}{z - \zeta} = -\frac{1}{\pi} \int \psi(z) \, \frac{L(\mathrm{d}\zeta)}{\zeta - z}.$$

Thus, we can rewrite the integral as

$$\int g(z)\psi(z) \, L(\mathrm{d}z) = \int u(\zeta) f(\zeta) \frac{\partial \varphi}{\partial \overline{\zeta}}(\zeta) \, L(\mathrm{d}\zeta).$$

Adding and subtracting

$$\frac{\partial u}{\partial \overline{\zeta}}(\zeta) f(\zeta) \varphi(\zeta)$$

and invoking the product rule, we get

$$\begin{aligned}
\int g(z)\psi(z) \, L(\mathrm{d}z) &= \int f(\zeta) \left[u(\zeta) \frac{\partial \varphi}{\partial \overline{\zeta}}(\zeta) - \frac{\partial u}{\partial \overline{\zeta}}(\zeta) \varphi(\zeta) \right] L(\mathrm{d}\zeta) + \int \frac{\partial u}{\partial \overline{\zeta}}(\zeta) f(\zeta) \varphi(\zeta) \, L(\mathrm{d}\zeta) \\
&= \int f(\zeta) \frac{\partial}{\partial \overline{\zeta}} [u(\zeta)\varphi(\zeta)] \, L(\mathrm{d}\zeta) + \int \frac{\partial u}{\partial \overline{\zeta}}(\zeta) f(\zeta) \varphi(\zeta) \, L(\mathrm{d}\zeta).
\end{aligned}$$

The first integral vanishes, since $u(\zeta)\varphi(\zeta) \in C_0^1(\Omega)$, by definition of f. On the other hand, recall that $\partial u/\partial\overline{\zeta} = \psi$, and that the integral is over ω, so that $\varphi = 1$ there. Hence

$$\int g(z)\psi(z)\, L(\mathrm{d}z) = \int f(\zeta)\psi(\zeta)\, L(\mathrm{d}\zeta),$$

as wanted. Thus, we make take a compact exhaustion, and construct g on each compact set. Taking a limit and invoking the dominated convergence theorem gives us g which agrees with f a.e. on all of Ω.

---

2 Let $\Omega \subseteq \mathbb{C}$ be open and assume that $f, g \in \mathrm{Hol}(\Omega)$ have no common zeros in Ω. Show that there exist $a, b \in \mathrm{Hol}(\Omega)$ such that

$$af + bg = 1.$$

Solution Assume that f and g do not vanish identically. If, without loss of generality, f vanishes identically, then g must not vanish anywhere by assumption. Then $b := 1/g$ is holomorphic in Ω, which completes the problem in this case.

Consider the meromorphic function $(fg)^{-1}$, and let A be the zeros of f, without counting multiplicities. This set has no limit points, or else f must vanish identically.

For each $w \in C$, let $c_{w,1}, \dots, c_{w,N_w}$ be so that

$$\frac{1}{fg} - \sum_{j=1}^{N_w} \frac{c_{w,j}}{(z-w)^j}$$

is holomorphic in a neighborhood of w. Indeed, the Laurent expansion of $(fg)^{-1}$ must have finitely many terms of the form c_j/w^j, or else $(fg)^{-1}$ has a pole of infinite order, which means that f or g has a zero of infinite order, which is impossible, since neither of them vanish identically.

By Mittag–Leffler, there exists $a_1 \in \mathrm{Hol}(\Omega \setminus A)$ so that for every $w \in A$,

$$a_1 - \sum_{j=1}^{N_w} \frac{c_{w,j}}{(z-w)^j}$$

is holomorphic in a neighborhood of w. Moreover, a_1 has poles precisely where f has zeros. By the same argument, we find b_1 so that b_1 has poles precisely where g has zeros.

Then

$$a_2 := \frac{1}{fg} - a_1 - b_1$$

is holomorphic in Ω. Thus,

$$\frac{1}{fg} = (a_1 + a_2) + b_1.$$

Set $a = b_1 g$ and $b = (a_1 + a_2)f$ so that a and b are analytic. a is holomorphic because the zeros of g kill off the poles of b_1, and similarly for b. Then

$$af + bg = b_1 fg + (a_1 + a_2)fg = fg \cdot [(a_1 + a_2) + b_1] = fg \cdot \frac{1}{fg} = 1,$$

as required.

3 Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and let $f \in \mathrm{Hol}(D)$ be bounded, not vanishing identically. Show that the zeros $\alpha_1, \alpha_2, \ldots$ of f, counted with multiplicities, satisfy the Blaschke condition,

$$\sum_{j=1}^{\infty}(1 - |\alpha_j|) < \infty. \tag{1}$$

Conversely, assume that $\{\alpha_n\}$ is a sequence in D such that $\alpha_n \neq 0$ and (1) holds. Show that there exists a function $B \in \mathrm{Hol}(D)$ such that $|B(z)| \leq 1$ and B has precisely the zeros α_n.

Solution Let f be as given. Reorder the zeros so that $|\alpha_1| \leq |\alpha_2| \leq \cdots$.

We may write $f(z) = z^N g(z)$, where g does not vanish at 0. We may then apply Jensen's formula to g for $0 < r < 1$:

$$\frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\varphi})|\,\mathrm{d}\varphi = \log \frac{|g(0)|r^{n(r)}}{\prod_{j=1}^{n(r)} \alpha_j}.$$

Because g is bounded by, say, $M > 0$, we have

$$\log \frac{|g(0)|r^{n(r)}}{\prod_{j=1}^{n(r)}|\alpha_j|} \leq M.$$

Exponentiating, we get

$$|g(0)|\frac{r^{n(r)}}{\prod_{j=1}^{n(r)}|\alpha_j|} \leq e^M \implies |g(0)|r^{n(r)}e^{-M} \leq \prod_{j=1}^{n(r)}|\alpha_j|.$$

In particular, this is true for any $k \leq n(r)$ since $0 < r < 1$ and all the zeros have magnitude $\leq r$, so we have

$$|g(0)|r^k e^{-M} \leq \prod_{j=1}^{k}|\alpha_j|.$$

Sending $r \to 1$ and then $k \to \infty$, we get

$$0 < |g(0)|e^{-M} \leq \prod_{j=1}^{\infty}|\alpha_j| < 1,$$

which then implies (because $0 < |\alpha_j| < 1$)

$$\sum_{j=1}^{\infty}(1 - |\alpha_j|) < \infty.$$

Thus, adding the finitely remaining term from the zeros from the z^N factor, we see that the sum also holds for the zeros of f.

On the other hand, consider the infinite Blaschke product

$$B(z) := \prod_{j=1}^{\infty} \frac{\overline{\alpha_j}}{|\alpha_j|}\frac{z - \alpha_j}{\overline{\alpha_j}z - 1}.$$

Then for each $j \geq 1$, notice that

$$\begin{aligned}\left|1 - \frac{\overline{\alpha_j}}{|\alpha_j|}\frac{z - \alpha_j}{\overline{\alpha_j}z - 1}\right| &= \left|\frac{|\alpha_j|\overline{\alpha_j}z - |\alpha_j| - \overline{\alpha_j}z + |\alpha_j|^2}{|\alpha_j|(\overline{\alpha_j}z - 1)}\right| \\ &= \left|\frac{(|\alpha_j| - 1)(\overline{\alpha_j}z + |\alpha_j|)}{|\alpha_j|(\overline{\alpha_j}z - 1)}\right| \\ &\leq (1 - |\alpha_j|)\frac{1 + |z|}{1 - |\alpha_j||z|}\end{aligned}$$

By the maximum principle, the factor is bounded above by 1 (set $|z| = 1$) so the whole product is bounded above by 1. Moreover, for $|z| \leq r < 1$, we get

$$\leq (1 - |\alpha_j|)\frac{1+r}{1-r},$$

for all $|z| < 1$.

By a proposition in class, the product thus converges uniformly on the disc $|z| \leq r$, since

$$\sum_{j=1}^{\infty} \sup_{z \in D} \left| 1 - \frac{\overline{\alpha_j}}{|\alpha_j|} \frac{z - \alpha_j}{\overline{\alpha_j} z - 1} \right| \leq \frac{1+r}{1-r} \sum_{j=1}^{\infty} (1 - |\alpha_j|) < \infty$$

Uniform convergence on compact sets follows since it is contained in one of these discs, and it has the zeros we need.

---

4 Suppose that α_j is a sequence in $\mathbb{C}$ such that

$$\sum \frac{1}{|a_j|^{p+1}} < \infty.$$

Show that

$$f(z) = \prod_j E_p\left(\frac{z}{\alpha_j}\right)$$

defines an entire function such that $|f(z)| \leq \exp(C|z|^{p+1})$.

Solution From a previous discussion in class, we already know that the canonical product converges locally uniformly, so we just need to show that the function is of acceptable growth.

We need to control $E_p(z)$. First notice that

$$\log|E_p(z)| = \log\left[|1 - w| \exp\left(\sum_{j=1}^{p-1} \frac{z^j}{j}\right) \exp\left(\frac{z^p}{p}\right)\right] \leq \log|E_{p-1}(z)| + \frac{|z|^p}{p}.$$

Continuing by induction, we see that

$$\log|E_p(z)| \leq \log|E_0(z)| + \sum_{k=1}^{p} \frac{|z|^k}{k} \leq C|z|^{p+1}.$$

Taking the logarithm of the product, we see that

$$\left|\log\left[\prod_j E_p\left(\frac{z}{\alpha_j}\right)\right]\right| = \prod_j \left|\log E_p\left(\frac{z}{\alpha_j}\right)\right| \leq C|z|^{p+1} \sum_j \frac{1}{|\alpha_j|^{p+1}} = C|z|^{p+1},$$

since $\sum 1/|\alpha_j|^{p+1}$ converges. We simply absorb the sum into the constant C. Exponentiating, we finally get

$$\left|\prod_j E_p\left(\frac{z}{\alpha_j}\right)\right| \leq \exp\bigl(C|z|^{p+1}\bigr),$$

as required.

5 Let f be entire of finite order ρ, with the Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Show that

$$\rho = \limsup_{n\to\infty} \frac{n\log n}{\log(1/|a_n|)}. \tag{2}$$

Solution Let $\varepsilon > 0$, which gives us $C = C_\varepsilon > 0$ so that $|f(z)| \le Ce^{|z|^{\rho+\varepsilon}}$.

Let C_r be a circle of radius $r > 0$ centered at the origin. Cauchy's estimates gives us

$$|a_n| \le \frac{1}{r^n} \sup_{|\zeta|=r} |f(\zeta)| \le \frac{C}{r^n} e^{r^{\rho+\varepsilon}}.$$

Taking logarithms, we get

$$\log|a_n| \le \log C - n\log r + r^{\rho+\varepsilon}.$$

We now optimize the right-hand side: taking the derivative and setting it to 0 gives us

$$-\frac{n}{r} + (\rho+\varepsilon)r^{\rho+\varepsilon-1} = 0 \implies r = \left(\frac{n}{\rho+\varepsilon}\right)^{1/(\rho+\varepsilon)}.$$

Substituting, we get

$$\log|a_n| \le \log C - \frac{n}{\rho+\varepsilon}\log n + n\frac{\log(\rho+\varepsilon)}{\rho+\varepsilon} + \frac{n}{\rho+\varepsilon}.$$

For n large, $|a_n| < 1$, so $\log(1/|a_n|) > 0$. Multiplying through by $\rho+\varepsilon$ and rearranging, we get

$$(\rho+\varepsilon)\log|a_n| \le (\rho+\varepsilon)\log C - n\log n + n\log(\rho+\varepsilon) + n$$
$$\frac{n\log n}{(\rho+\varepsilon)\log(1/|a_n|)} \le \frac{\log C}{\log(1/|a_n|)} + \frac{\log(\rho+\varepsilon)}{\rho+\varepsilon}\frac{n}{\log(1/|a_n|)} + \frac{1}{\rho+\varepsilon}\frac{n}{\log(1/|a_n|)} + 1.$$

Because f is entire, the root test tells us that $|a_n|^{1/n} \xrightarrow{n\to\infty} 0$. Thus,

$$\frac{1}{n}\log|a_n| \xrightarrow{n\to\infty} -\infty \implies \frac{n}{\log(1/|a_n|)} \xrightarrow{n\to\infty} 0.$$

Hence, taking $n \to \infty$ in the inequality, we get

$$\limsup_{n\to\infty} \frac{n\log n}{(\rho+\varepsilon)\log(1/|a_n|)} \le 1 \implies \limsup_{n\to\infty} \frac{n\log n}{\log(1/|a_n|)} \le \rho+\varepsilon.$$

Taking $\varepsilon \to 0^+$, we get

$$\limsup_{n\to\infty} \frac{n\log n}{\log(1/|a_n|)} \le \rho.$$

As for the other direction, let μ be the given limit, and let $\varepsilon > 0$. For large $n \ge 1$, we have by definition that

$$\frac{n\log n}{\log(1/|a_n|)} \le \mu+\varepsilon \implies \log n^n \le \log\left(\frac{1}{|a_n|^{\mu+\varepsilon}}\right) \implies |a_n| \le n^{-n/(\mu+\varepsilon)}.$$

Now, for $|z| = r > 0$, we need to bound the following series:

$$\sum_{n=1}^{\infty} n^{-n/(\mu+\varepsilon)} r^n.$$

Consider the function $x^{-x/(\mu+\varepsilon)}r^x$. Taking the logarithm optimizing with respect to x, we get

$$-\frac{1}{\mu+\varepsilon}\log x - \frac{1}{\mu+\varepsilon} + \log r = 0 \implies \log x + 1 = \log r^{\mu+\varepsilon} \implies x = e^{-1}r^{\mu+\varepsilon}.$$

The coefficient of r decays rapidly, so the series is essentially bounded by its largest term, which grows at most like

$$Cr^{r^{\mu+\varepsilon}} = C\exp(|z|^{\mu+\varepsilon}\log|z|).$$

Hence, f is bounded by this function for large $|z|$ and the order of f is ρ, so $\rho \le \mu+\varepsilon$ by definition. Since ε was arbitrary, it follows that $\rho \le \mu$, hence $\rho = \mu$, as required.

6 Let $\Omega \subseteq \mathbb{C}$ be a bounded open set and let $H(\Omega)$ be the subspace of $L^2(\Omega)$ consisting of holomorphic functions (the Bergman space).

a. Show that $H(\Omega)$ is a closed subspace of $L^2(\Omega)$.

b. Let $\{f_n\}$ be a sequence in $H(\Omega)$, such that $\|f_n\|_{L^2(\Omega)} \leq C$. Show that there exists a subsequence which converges locally uniformly in Ω.

c. Show that for each $z \in \Omega$, we have

$$f(z) = \int_\Omega K(z,\zeta) f(\zeta)\, L(\mathrm{d}\zeta), \quad f \in H(\Omega),$$

where $\Omega \ni \zeta \mapsto \overline{K(z,\zeta)} \in H(\Omega)$. (The function K is called the Bergman kernel.)

d. Compute the Bergman kernel for the unit disc.

Solution a. Let $f, g \in H(\Omega)$ and $\lambda \in \mathbb{C}$. $f + \lambda g$ is holomorphic, since sums and linear combinations of holomorphic functions are holomorphic. We just need to show that $H(\Omega)$ is closed.

Let $\{f_n\} \subseteq H(\Omega)$ be a Cauchy sequence in $L^2(\Omega)$. Let $B \coloneqq B(w, 2R) \subseteq \Omega$ with closure contained in Ω also.

By applying the mean value property, multiplying both sides by $0 < r \leq R$, and integrating, we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{i\theta})\,\mathrm{d}\theta \\ \implies f(z) &= \frac{1}{\pi R^2}\int_0^R \int_0^{2\pi} r f(z + re^{i\theta})\,\mathrm{d}\theta\,\mathrm{d}r \\ &= \frac{1}{\pi R^2}\iint_B f(\zeta)\, L(\mathrm{d}\zeta). \end{aligned}$$

Applying this to $f_n - f_m$, we get for $z \in B$ that

$$|f_n(z) - f_m(z)| \leq \frac{1}{\pi R^2}\iint_B |f_n(\zeta) - f_m(\zeta)|\, L(\mathrm{d}\zeta).$$

By Cauchy–Schwarz (the integrand is multiplied against 1), we get

$$|f_n(z) - f_m(z)| \leq R\sqrt{\pi}\|f_n - f_m\|_{L^2(\Omega)} \xrightarrow{n,m\to\infty} 0.$$

Thus, $\{f_n\}$ is uniformly Cauchy on B. By covering compact sets with finitely many open balls, we have local uniform convergence, and this limit will be holomorphic.

b. By Arzelà–Ascoli, it's enough to show that the functions are equicontinuous. Let B be an open ball of radius $R > 0$, and let $z \in B \coloneqq B(w, 2R) \subseteq \Omega$ with $\overline{B} \subseteq \Omega$.

By applying Green's theorem to Cauchy's integral formula, we get

$$f(z) = \frac{1}{\pi}\iint_{|\zeta - z| \leq R} \frac{f(\zeta)}{(1 - \overline{\zeta} z)^2}\, L(\mathrm{d}\zeta).$$

We can differentiate under the integral sign, since $(1 - \overline{\zeta}z)^{-2}$ is L^1, since its only pole occurs when $\zeta = z/|z|^2$, and we can avoid this by shrinking R if necessary. This gives

$$f'(z) = \frac{2}{\pi}\iint_{|\zeta - z| \leq R} \frac{\overline{\zeta} f(\zeta)}{(1 - \overline{\zeta} z)^3}\, L(\mathrm{d}\zeta).$$

This can be bounded uniformly by $M > 0$ for each f_n, since they are uniformly bounded by C. Thus, by estimating

$$|f(z) - f(w)| \leq \int_w^z |f'(\zeta)|\,\mathrm{d}\zeta \leq M|z - w|,$$

we can achieve equicontinuity. Thus, by Arzelà–Ascoli, the sequence admits a locally uniform convergent subsequence.

c. Consider $\hat{z} \in (H(\Omega))^*$ given by $\hat{z}(f) = f(z)$. This is a bounded linear functional, since f is continuous. Thus, by Riesz representation, there exists $\overline{K(z,\zeta)} \in H(\Omega)$ so that

$$\hat{z}(f) = \langle f, \overline{K} \rangle = \int_\Omega K(z,\zeta) f(\zeta)\, L(\mathrm{d}\zeta),$$

as required.

d. We gave the kernel in part (b):

$$K(z,\zeta) = \frac{1}{\pi} \frac{1}{(1-\overline{\zeta} z)^2}.$$

We'll now give the full derivation. Starting with the Cauchy integral formula, we get by a previous application of Green's theorem in class that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \\ &= \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\overline{\zeta} f(\zeta)}{1-\overline{\zeta} z}\, \mathrm{d}\zeta \\ &= \frac{1}{2\pi i} \cdot 2i \cdot \int_{|\zeta|\le 1} \frac{\partial}{\partial \overline{\zeta}} \frac{\overline{\zeta} f(\zeta)}{1-\overline{\zeta} z}\, L(\mathrm{d}\zeta) \\ &= \frac{1}{\pi} \int_{|\zeta|\le 1} \frac{f(\zeta)(1-\overline{\zeta} z) + \overline{\zeta} f(\zeta) z}{(1-\overline{\zeta} z)^2}\, L(\mathrm{d}\zeta) \\ &= \frac{1}{\pi} \int_{|\zeta|\le 1} \frac{f(\zeta)}{(1-\overline{\zeta} z)^2}\, L(\mathrm{d}\zeta), \end{aligned}$$

as required.