**1** Negate the following statements:

a. If pigs had wings, they would fly.

b. If that planes leaves and you are not on it, then you will regret it.

c. For every problem there is a solution that is neat, plausible, and wrong.

**Solution** a. Pigs have wings, and they do not fly.

b. The plane left and you are not on it, but you don't regret it.

c. There exists a problem that does not have a solution that is neat, plausible, and wrong.

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**2** Let X and Y be statements. If we want to DISPROVE the claim that "At least one of X and Y are true", which one of the following do we need to show?

a. At leaset one of X and Y are false.

b. X and Y are both false.

c. Exactly one of X and Y are false.

d. Y is false.

e. X does not imply Y, and Y does not imply X.

f. X is true if and only if Y is false.

g. X is false.

**Solution** We need to know (b).

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**3** Let $P(x)$ be a property about some object $x$ of type X. If we want to DISPROVE the claim that "$P(x)$ is true for all $x$ of type X", which one of the following do we have to do?

a. Show that for every $x$ in X, $P(x)$ is false.

b. Show that for every $x$ in X, there is a $y$ not equal to $x$ for which $P(y)$ is true.

c. Show that $P(x)$ being true does not necessarily imply that $x$ is of type X.

d. Show that there are no objects $x$ of type X.

e. Show that there exists an $x$ of type X for which $P(x)$ is false.

f. Show that there exists an $x$ which is not of type X, but for which $P(x)$ is still true.

g. Assume there exists an $x$ of type X for which $P(x)$ is true, and a derive a contradiction.

**Solution** We need to do (e).

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**4** Let X, Y, Z be statements. Suppose we know that "X is true implies Y is true", and "X is false implies Z is true". If we know that Z is false, then which one of the following can we conclude?

a. X is false.

b. X is true.

c. Y is true.

d. (b) and (c).

e. (a) and (c).

f. (a), (b), and (c).

g. None of the above conclusions can be drawn.

**Solution** We can conclude (d).

**5** Let $P(n,m)$ be a property about two integers $n$ and $m$. If we want to DISPROVE the claim that "There exists an integer $n$ such that $P(n,m)$ is true for all integers $m$", then which one of the following do we need to prove?

a. If $P(n,m)$ is true, then $n$ and $m$ are not ingers.

b. For every integer $n$, there exists an integer $m$ such that $P(n,m)$ is false.

c. For every integer $n$ and every integer $m$, the property $P(n,m)$ is false.

d. For every integer $m$, there exists an integer $n$ such that $P(n,m)$ is false.

e. There exists an integer $n$ such that $P(n,m)$ is false for all integers $m$.

f. There exists integers $n,m$ such that $P(n,m)$ is false.

g. There exists an integer $m$ such that $P(n,m)$ is false for all integers $n$.

**Solution** We need to prove (b).

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**6** Let X and Y be statements. If we know that X implies Y, which one of the following can we conclude?

a. X cannot be false.

b. X is true, and Y is also true.

c. If Y is false, then X is false.

d. Y cannot be false.

e. If X is false, then Y is false.

f. If Y is true, then X is true.

g. At least one of X and Y is true.

**Solution** We know (c).

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**7** Prove the following statement by induction:

$$1+3+5+\cdots+(2n-1)=n^2 \quad \text{for all } n \geq 1.$$

**Solution** Let $P(n)$ denote the statement $1+3+5+\cdots+(2n-1)=n^2$, for $n \geq 1$.

Step 1: (Base step)

$P(1)$: $1 = 1^2 \implies P(1)$ holds.

Step 2: (Inductive step)

Suppose $P(n)$ holds. We wish to show that $P(n+1)$ holds.

$$\begin{aligned} 1+3+5+\cdots+(2n-1) &= n^2 && \mid +(2n+1) \\ 1+3+5+\cdots+(2n-1)+(2n+1) &= n^2+2n+1=(n+1)^2 && P(n+1) \end{aligned}$$

We have shown that $P(n) \implies P(n+1)$. Taking both steps and invoking the principle of mathematical induction, we conclude that $P(n)$ holds for all natural numbers $n \geq 1$. ■

**8** Prove the following statement by induction: The expression

$$4^n + 15n - 1$$

is divisible by 9 for all $n \geq 1$.

**Solution** Let $P(n)$ be the statement that $9 \mid 4^n + 15n - 1$.

Step 1: (Base step)

$P(1)$: $4 + 15 - 1 = 18 = 9 \cdot 2 \implies P(1)$ holds.

Step 2: (Inductive step)

Suppose $P(n)$ holds. Then $4^n + 15n - 1 = 9c$, for some $c \in \mathbb{Z}$. We want to prove that $P(n+1)$ holds.

$$\begin{aligned} & 4^{n+1} + 15(n+1) - 1 \\ = {} & 4 \cdot 4^n + 60n - 4 - 45n + 18 \\ = {} & 4(4^n + 15n - 1) - 45n + 18 \\ = {} & 4 \cdot 9c + 9(-5n + 2) \\ = {} & 9(4c - 5n + 2) \end{aligned}$$

which is divisible by 9. Thus, $P(n+1)$ holds.

If we take both steps, then by the principle of mathematical induction, $P(n)$ holds for all $n \geq 1$. ■

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**9** Decide for which natural numbers the inequality $3^n > n^3$ is true. Prove your claim using mathematical induction.

**Solution** The inequality appears to be true for all $n \geq 4$. I will prove my hypothesis through mathematical induction.

Let $P(n)$ denote the expression $3^n > n^3$.

Step 1: (Base step)

$P(4)$: $3^4 = 81 > 64 = 4^3 \implies P(4)$ holds.

Step 2: (Inductive step)

Suppose $P(n)$ holds. We will show that $P(n+1)$ holds.

$$\begin{aligned} 3^n &> n^3 & \quad | \cdot 3 \\ 3^{n+1} &> (n+1)^3 + 2n^3 - 3n^2 - 3n - 1 \\ 3^{n+1} &> (n+1)^3 + n(2n^2 - 3n - 3) - 1 \\ 3^{n+1} &> (n+1)^3 + n\big[(2n+3)(n-3) + 6\big] - 1 \end{aligned}$$

Since $n \geq 4$, $(2n+3)(n-3) + 6 > 0$. $(2n+3)(n-3) + 6$ is also a natural number, so $n\big[(2n+3)(n-3)+6\big] \geq 4 > 1$. Thus, $n\big[(2n+3)(n-3)+6\big] - 1 > 0$, and it follows that

$$3^{n+1} > (n+1)^3 + n\big[(2n+3)(n-3)+6\big] - 1 > (n+1)^3$$

Collecting steps 1 and 2, $P(n)$ holds for all $n \geq 4$ by the principle of mathematical induction. ■

**10** Prove that there is no rational number whose square is 6.

**Solution** First consider an odd integer $c$, which can be written in the form $2n+1$ for some $n \in \mathbb{Z}$. Then

$$c^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1,$$

meaning $c^2$ is also odd. It follows that if $c^2$ is even, then $c$ must also be even.

Suppose $\exists r \in \mathbb{Q}$ such that $r^2 = 6$. Then we can find $a, b \in \mathbb{Z}$ such that $a$ and $b$ are coprime and $r = \frac{a}{b}$. Then

$$\frac{a^2}{b^2} = 6$$
$$a^2 = 6b^2$$

From the expression, $a^2$ must be even, which means $a$ is also even, so we can find $m \in \mathbb{Z}$ such that $a = 2m$. Substituting this in yields

$$4m^2 = 6b^2 \implies 2m^2 = 3b^2,$$

which means that $b$ is also even. This is a contradiction because $a$ and $b$ must be coprime. Thus, we cannot find $a$ and $b$ as described above, so there is no such $r \in \mathbb{Q}$ such that $r^2 = 6$. ■