1 Suppose that f and g are entire functions with the property that

$$f^n + g^n = 1.$$

If $n = 2$, prove that there is an entire function h such that $f = \cos h$ and $g = \sin h$. Prove that if $n > 2$, then f and g are constant.

Solution Suppose $n = 2$, so that $f^2 + g^2 = 1$. Notice that we can write this as

$$(f + ig)(f - ig) = 1.$$

Thus, $f + ig$ may not vanish, or else the product would not equal 1. Hence, it has a well-defined logarithm, i.e., there exists $h \in \mathrm{Hol}(\mathbb{C})$ so that

$$\cos h + i \sin h = e^{ih} = f + ig.$$

Notice that by our factorization,

$$e^{-ih} = \frac{1}{f + ig} = f - ig.$$

Then

$$\cos h = \frac{e^{ih} + e^{-ih}}{2} = \frac{2f}{2} = f \quad \text{and}$$
$$\sin h = \frac{e^{ih} - e^{-ih}}{2i} = \frac{2gi}{2i} = g,$$

as required.

Now suppose $n \geq 3$, and suppose that

$$f^n + g^n = 1 \implies \left(\frac{f}{g}\right)^n + 1 = \frac{1}{g^n}.$$

Because g is entire, the $1/g$ never vanishes. Thus, the left-hand side does not have solutions to

$$\left(\frac{f}{g}\right)^n = -1.$$

First, notice that if f and g are both polynomial, then they are necessarily constant. Otherwise, $f^n + g^n$ is at least an n-th order polynomial, which cannot be identically equal to 1. So, we may assume from now on that at least one of f and g is non-polynomial.

This means that f/g is a non-rational meromorphic function whose range omits the n-th roots of -1, of which there are $n \geq 3$. But by Picard's theorem, f/g may only omit 2 values, a contradiction. Thus, f/g must be rational, so f and g are polynomials, hence constant.

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2 Let

$$\Phi = \{f \in \mathrm{Hol}(D) \mid \mathrm{Re}\, f > 0,\ f(0) = 1\}.$$

Here, $D = \{|z| < 1\}$. Show that Φ is a normal family. What happens if one removes the assumption $f(0) = 1$?

Solution Consider the family of functions $e^{-\Phi} := \{e^{-f} \mid f \in \Phi\}$. Because $\mathrm{Re}\, f > 0$, we have $\left|e^{-f}\right| = e^{\mathrm{Re}(-f)} \leq 1$. Hence, the family is uniformly bounded, so it is normal, so there is a sequence $\{e^{-f_n}\} \subseteq e^{-\Phi}$ which either converges to some function $g \in \mathrm{Hol}(D)$ locally uniformly or to ∞ locally uniformly.

Because $f_n(0) = 1$ for every $n \geq 1$, $e^{-f_n(0)} = e^{-1}$, so the sequence does not tend to ∞, so $f_n \to g$ in $\mathrm{Hol}(D)$. Moreover, $g(0) = e^{-1} > 0$, so by Hurwitz, g vanishes nowhere, since each e^{-f} vanishes nowhere. Thus, it has a well-defined logarithm. By continuity of the logarithm,

$$\lim_{n \to \infty} f_n = \log g \in \mathrm{Hol}(D),$$

locally uniformly.

If we remove the assumption that $f(0) = 1$, then we may get locally uniform convergence to ∞. For example, consider the sequence of constants $f_n(z) = n$, where $n \in \mathbb{N}$. In this case, however, Φ is still a normal family in the classical sense.

3 Let $f \in \mathrm{Hol}(\mathrm{Im}\, z > 0)$ be such that $\mathrm{Im}\, f \geq 0$ and

$$|f(z)| \leq \frac{C}{\mathrm{Im}\, z},$$

for some $C > 0$. Show that

$$\int_{-\infty}^{\infty} \mathrm{Im}\, f(x+iy)\, \mathrm{d}x \leq C\pi, \quad y > 0,$$

and that there exists a unique positive bounded measure μ on $\mathbb{R}$ such that

$$f(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{t-z}, \quad \mathrm{Im}\, z > 0.$$

Solution We start with a preliminary calculation:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\, \mathrm{d}x = \frac{1}{a} \arctan\left(\frac{x}{a}\right)\Big|_{-\infty}^{\infty} = \frac{\pi}{a}.$$

Now, for $z = x+iy$, where $y > 0$, we have that $B(z, \delta) \subseteq \{\mathrm{Im}\, z > 0\}$ for $\delta = y/2$. Thus, we have the Poisson representation formula and the following estimates:

$$\begin{aligned}
\mathrm{Im}\, f(x+iy) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta^2 - |x+iy|^2}{(x-\delta\cos\theta)^2 + (y-\delta\sin\theta)^2}\, \mathrm{Im}\, f(x+iy+\delta e^{i\theta})\, \mathrm{d}\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta^2}{(x-\delta\cos\theta)^2 + (y-\delta\sin\theta)^2}\, \frac{C}{y+\delta\sin\theta}\, \mathrm{d}\theta.
\end{aligned}$$

Then if we integrate by x and apply Tonelli's theorem (the integrand is non-negative), we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{Im}\, f(x+iy)\, \mathrm{d}x &= \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{\delta^2}{(x-\delta\cos\theta)^2 + (y-\delta\sin\theta)^2}\, \frac{C}{y+\delta\sin\theta}\, \mathrm{d}x\, \mathrm{d}\theta \\
(x \mapsto x+\delta\cos\theta) &= \frac{1}{2\pi} \int_0^{2\pi} \left(\int_{-\infty}^{\infty} \frac{1}{x^2 + (y-\delta\sin\theta)^2}\, \mathrm{d}x \right) \frac{C\delta^2}{y+\delta\sin\theta}\, \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{\pi}{y-\delta\sin\theta}\, \frac{C\delta^2}{y+\delta\sin\theta}\, \mathrm{d}\theta \\
&= \frac{1}{2} \int_0^{2\pi} \frac{C\delta^2}{y^2-\delta^2\sin\theta}\, \mathrm{d}\theta \\
&\leq \pi \frac{C\delta^2}{y^2-\delta^2} \\
&= \pi \frac{C(y/2)^2}{y^2-(y/2)^2} \\
&= \frac{C\pi}{3} \\
&\leq C\pi,
\end{aligned}$$

which proves the inequality.

Now, from the first homework assignment, we have the following Poisson kernel representation for a harmonic function u on the upper-half plane:

$$u(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Im} z}{|z-x|^2} u(x)\,\mathrm{d}x$$

Notice that

$$\frac{\operatorname{Im} z}{|z-x|^2} = \frac{1}{2i}\left(\frac{1}{x-z} - \frac{1}{x-\overline{z}}\right) = \operatorname{Im}\left(\frac{1}{x-z}\right).$$

Now applying the formula to $u = \operatorname{Im} f_n(z) := \operatorname{Im} f(z + in^{-1})$ (which is harmonic in a neighborhood of $\{\operatorname{Im} z \geq 0\}$) yields

$$\operatorname{Im} f_n(z) = \operatorname{Im}\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Im} f(t + in^{-1})}{t-z}\,\mathrm{d}t\right).$$

The integral defines a holomorphic function, since we can pass the derivative under the integral sign. Indeed, the denominator stays away from 0 since $\operatorname{Im} z > 0$, so it is C^1 on the domain of f_n.

Now consider the positive linear functional on $L^1(\mathbb{R})$ given by

$$\mu_n(g) := \frac{1}{\pi}\int_{-\infty}^{\infty} g(t) \operatorname{Im} f_n(t)\,\mathrm{d}t.$$

We consider $L^1(\mathbb{R}) \subseteq M(\mathbb{R}) = C_0(\mathbb{R})^*$, where $C_0(\mathbb{R})$ is the set of continuous functions vanishing at infinity. This embedding is an isometry, so we also have

$$\|\mu_n\| = \|\operatorname{Im} f_n\|_{L^1(\mathbb{R})} \leq C\pi$$

for all $n \geq 1$. Hence, by Banach–Alagolu, there exists a convergent subsequence μ_{n_k} which converges weakly* to a bounded measure $\mu \in M(\mathbb{R})$.

This μ is unique because it is obtained via the limit, and $\mu \geq 0$ since each $\mu_n \geq 0$. Thus, because $(t-z)^{-1}$ is a continuous function in t which vanishes at infinity, we have the following weak* convergence:

$$\mu_{n_k}\left(\frac{1}{t-z}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Im} f_{n_k}(t)}{t-z}\,\mathrm{d}t \xrightarrow{k\to\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu(t)}{t-z} = \mu\left(\frac{1}{t-z}\right).$$

Thus,

$$\operatorname{Im} f(z) = \lim_{k\to\infty} \operatorname{Im} f_{n_k}(z) = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu(t)}{t-z}\right).$$

By the Cauchy–Riemann equations, f and the integral only differ by a real constant a. But notice that for $y > 0$,

$$\left|\frac{1}{t-iy}\right| \in C_0(\mathbb{R}) \subseteq L^1(\mu),$$

so by dominated convergence,

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu(t)}{t-iy} \xrightarrow{y\to\infty} 0.$$

On the other hand,

$$|f(iy)| \leq \frac{C}{y} \xrightarrow{y\to\infty} 0.$$

As a result, taking $y \to \infty$, we get

$$f(iy) = a + \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu(t)}{t-iy} \implies 0 = a + 0 \implies a = 0.$$

Hence,

$$f(z) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu(t)}{t-z},$$

as desired.