4.3 Prove the following version of Weierstrass's theorem: Suppose $\{U_n\}$ is an increasing sequence of regions and suppose f_n is defined and analytic on U_n. If the sequence f_n converges uniformly on compact subsets of $U := \bigcup_{n\geq 1} U_n$ to a function f, then f is analytic on U and the sequence f_n' converges to f' uniformly on compact subsets of U (even though perhaps none of the f_n are defined on all of U).

Solution Let $z_0 \in U$. Then there exists $N \in \mathbb{N}$ such that $z_0 \in U_n$ for all $n \geq N$.

For $n \geq N$, f_n is analytic on U_n, since $\{U_n\}$ is an increasing sequence of regions, so we can write

$$f_n(x) = \sum_{m\geq 0} a_m^{(n)} (z - z_0)^m.$$

Since f_n converges to f uniformly on a compact subset containing z_0,

$$f(x) = \lim_{n\to\infty} \sum_{m\geq 0} a_m^{(n)} (z - z_0)^m = \sum_{m\geq 0} \left(\lim_{n\to\infty} a_m^{(n)} \right) (z - z_0)^m$$

so f is analytic on U.

Since f_n converges uniformly to f on compact subsets, f_n' converges to f'. Moreover, as f_n' is analytic, the convergence is uniform.

4.4 a. Use Cauchy's estimate to prove Liouville's theorem.

b. Use Cauchy's estimate (2.3) to compute a lower bound on the radius of convergence of the power series representation of a holomorphic function.

Solution a. Let f be a bounded entire function. Thus, there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. By Cauchy's estimate, on $|z - z_0| \leq r$, $r > 0$, we have

$$|f'(z_0)| \leq \frac{\sup|f|}{r} \leq \frac{M}{r}$$

Since f is entire, this estimate holds for all $r > 0$. Hence, take $r \to \infty$ to get that

$$|f'(z_0)| = 0 \ \forall z_0 \in \mathbb{C}.$$

Hence, f is constant on $\mathbb{C}$.

b. Let f be a holomorphic function. Then we can write

$$f(z) = \sum_{n\geq 0} a_n (z - z_0)^n$$

for $|z - z_0| < r$ for some $r > 0$. By Cauchy's estimate,

$$|a_n| = \left| \frac{f^{(n)}(z_0)}{n!} \right| \leq \frac{\sup_{|z-z_0|<r}|f|}{r^n}.$$

Then by the root test, we can estimate the radius of convergence

$$R = \limsup |a_n|^{-1/n} \geq \frac{r}{\sup_{|z-z_0|<r}|f|}.$$

4.5 At the end of the proof of Runge's Theorem 3.4, we stated that an inequality remains true for all z in a disk containing z_0 and for all refinements of the partition. Supply the details. *Hint:* Using Cauchy's integral formula, prove that $|f(z) - f(z_0)| \le C|z - z_0|$ for all z with $|z - z_0| < \frac{1}{2}\operatorname{dist}(z_0, \Gamma)$, where C depends only on $\sup|f|$, $\ell(\Gamma)$, and $\operatorname{dist}(z_0, \Gamma)$. Do the same for the difference of the Riemann sums at z and z_0 (with the same partition).

Solution By Cauchy's integral formula,

$$\begin{aligned}
|f(z) - f(z_0)| &= \frac{1}{2\pi}\left|\oint_\Gamma \frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - z_0}\,\mathrm{d}\zeta\right| \\
&\le \frac{1}{2\pi}\oint_\Gamma \left|\frac{f(\zeta)(z - z_0)}{(\zeta - z)(\zeta - z_0)}\right|\mathrm{d}\zeta \\
&\le \frac{1}{2\pi}\cdot \ell(\Gamma)\cdot |z - z_0| \cdot \sup_\Gamma\left|\frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)}\right| \\
&\le \frac{1}{2\pi}\cdot \ell(\Gamma)\cdot |z - z_0| \cdot \sup_\Gamma|f(\zeta)| \cdot \frac{4}{\operatorname{dist}(z_0, \Gamma)} \\
&= \frac{2\ell(\Gamma)\sup|f|}{\pi \operatorname{dist}(z_0, \Gamma)}|z - z_0|
\end{aligned}$$

Hence, f is Lipschitz continuous, so it is uniformly continuous. We can perform a similar calculation for the difference of the Riemann sums to get that the difference is uniformly continuous also. Let $R(z)$ be the difference in the Riemann sums, where we replace z with z_0.

The result follows from the triangle inequality.

4.6 Show that Theorem 3.2 and Corollary 3.3 hold for a convex sets S with piecewise continuously differentiable boundary. A set S is convex if for all $z, w \in S$ and $0 < t < 1$, we have $tz + (1 - t)w \in S$. *Hint:* To prove the analog of Proposition 3.1, use Corollary 2.9 if there is an appropriate disk B or show that you can replace the integral around ∂S with an integral around a small square S_0 centered at a by writing $\partial S - \partial S_0$ as the sum of four curves chosen so that Corollary 2.9 applies to each.

Solution Let S_0 be a closed square oriented counter-clockwise which lies in S whose four corners touch ∂S. We can do so since S is convex, so vertical and horizontal lines are contained in S. Then the part of ∂S between each pair of adjacent corners is a closed C^1 curve. We call them C_1, C_2, C_3, C_4, oriented counter-clockwise.

Then $\partial S_0 + C_1 + \cdots + C_4 = \partial S$. Hence, as f is analytic, we can write

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{\partial S} \frac{f(\zeta) - f(z)}{\zeta - z}\,\mathrm{d}\zeta &= \frac{1}{2\pi i}\oint_{\partial S} f'(z + t(\zeta - z))\,\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\left(\oint_{\partial S_0} f'(z + t(\zeta - z))\,\mathrm{d}\zeta + \sum_{j=1}^{4}\oint_{\partial C_j} f'(z + t(\zeta - z))\,\mathrm{d}\zeta\right) \\
&= 0
\end{aligned}$$

since ∂S_0 and each C_j is a closed C^1 curve and f is analytic. Hence, Theorem 3.2 holds.

Applying the result on $g(\zeta) \coloneqq f(\zeta)(\zeta - z)$ yields Corollary 3.3.

4.7 Suppose Ω is a region which is symmetric about $\mathbb{R}$. Set $\Omega^+ = \Omega \cap \mathbb{H}$ and $\Omega^- = \Omega \cap (\mathbb{C} \setminus \overline{\mathbb{H}})$. If f is analytic on Ω^+, continuous on $\Omega^+ \cup (\Omega \cap \mathbb{R})$ and $\operatorname{Im} f = 0$ on $\Omega \cap \mathbb{R}$ then the function defined by

$$F(z) = \begin{cases} f(z) & \text{for } z \in \Omega \setminus \Omega^- \\ \overline{f(\overline{z})} & \text{for } z \in \Omega^- \end{cases}$$

is analytic on Ω. *Hint:* Divide a rectangle with sides parallel to the axes and intersecting $\mathbb{R}$ into two rectangles. Then use Exercise 6 for rectangles.

Solution Let $R \subseteq \Omega$ be a rectangle with its sides parallel to the axes oriented counter-clockwise. Define R^+ to be the part of R which lies above and including $\mathbb{R}$. Similarly, define R^- to be part of R which lies below and including $\mathbb{R}$. Then $R = R^+ + R^-$.

Consider D, which is a rectangle oriented counter-clockwise who shares its bottom side with R, with width w and with height h. Note that $R^+ - D$ is a closed rectangle which lies entirely in Ω^+. Then

$$\begin{aligned} \oint_{\partial R^+} F(z)\,\mathrm{d}z &= \oint_{\partial R^+ \partial - D} f(z)\,\mathrm{d}z + \oint_{\partial D} f(z)\,\mathrm{d}z \\ &= \oint_D f(z)\,\mathrm{d}z \end{aligned}$$

Taking $h \to 0$, D becomes a line segment which is traversed twice in opposite directions, so the integral vanishes. Hence, $\oint_{R^+} F(z)\,\mathrm{d}z = 0$.

In Ω^-, notice that $\bar{z} \in \Omega^+$. As f is continuous on Ω^+, we have

$$\overline{f(\bar{z})} \xrightarrow{\operatorname{Im} z \to 0} \overline{f(\operatorname{Re} z)} = f(\operatorname{Re} z),$$

so F is continuous on Ω^- also.

Hence, by symmetry,

$$\oint_{\partial R^-} F(z)\,\mathrm{d}z = 0$$

also.

Thus, F is continuous on Ω and

$$\oint_{\partial R} F(z)\,\mathrm{d}z = 0$$

for all rectangles with its sides parallel to the axes, so by Morera's theorem, F is analytic on Ω.

4.9 Show that there is a constant $C < \infty$ so that if f is analytic on $\mathbb{D}$, then

$$|f'(z)| \leq C \int_{\mathbb{D}} |f(x+iy)|\,\mathrm{d}x\,\mathrm{d}y$$

for all $|z| \leq 1/2$.

Solution Note that since f is analytic, applying Cauchy's estimate yields

$$|f'(z)| = \frac{1}{2\pi}\left|\oint_{C_r} \frac{f(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta\right| \leq \frac{1}{2\pi}\oint_{C_r}\left|\frac{f(\zeta)}{(\zeta - z)^2}\right|\mathrm{d}\zeta \leq \frac{1}{2\pi}\oint_{C_r}\left|\frac{f(\zeta)}{\frac{1}{4}}\right|\mathrm{d}\zeta$$

Integrating from $r = 0$ to $r = 1$ yields

$$|f'(z)| \leq \frac{2}{\pi}\iint_{\mathbb{D}} |f(x+iy)| \implies C = \frac{2}{\pi}.$$

5.1 Suppose γ is a closed curve and $a \notin \gamma$. Suppose R_a is a ray (half-line) from a to ∞ such that $\gamma \cap R_a$ consists of finitely many points. We say that the curve $\gamma(t)$ crosses R_a counter-clockwise (clockwise) at $\gamma(t_1) \in R_a$ if $\arg(\gamma(t) - a)$ is increasing (respectively decreasing) as a function of t in a neighborhood of t_1. Prove that $n(\gamma, a)$ equals the number of counter-clockwise crossings of R_a minus the number of clockwise crossings. The set $\gamma \setminus R_a$ consists of finitely many pieces, but not all endpoints are necessarily "crossings".

Solution It suffices to show that if a closed curve crosses R_a counter-clockwise once at a point x. Then by induction, we can bring the result to an arbitrary number of points. Then to get clockwise crossings, we can simply replace γ with $-\gamma$ to get the same result.

If γ crosses R_a counter-clockwise once at a point x, then it must make a loop about a without crossing R_a again, which is true by our assumption and by the fact that γ is a closed (continuous) curve. Then by Cauchy's theorem,

$$\int_\gamma \frac{\mathrm{d}\zeta}{\zeta - a} = 1$$

since a lies within the interior of the region with boundary γ.

Hence, by induction, if we had N counter-clockwise crossings, then the winding number is N. Indeed, we can add and subtract curves so that we get a cycle which goes around a, which differ from γ by only closed curves.

For clockwise crossings, we can replace γ with $-\gamma$ to turn them into counter-clockwise ones. Then if we have M clockwise crossings,

$$\int_{-\gamma} \frac{\mathrm{d}\zeta}{\zeta - a} = M \implies n(\gamma, a) = -M$$

Thus, if γ has N counter-clockwise crossings and M clockwise crossings, we can add and subtract curves to γ so that we have a closed curve γ_1 with only counter-clockwise crossings and another closed curve γ_2 with only clockwise crossings. Then difference between γ and the curves γ_1 and γ_2 will simply be a closed loop, and an integral of an analytic function over a closed loop is 0. Hence,

$$n(\gamma, a) = \oint_\gamma \frac{\mathrm{d}\zeta}{\zeta - a} = \oint_{\gamma_1} \frac{\mathrm{d}\zeta}{\zeta - a} + \oint_{\gamma_2} \frac{\mathrm{d}\zeta}{\zeta - a} = N - M$$

as desired.