1 Consider the word *SALESPERSONS*.

a. How many strings can be formed by ordering the letters of *SALESPERSONS*?

b. How many strings can be formed by ordering the letters of *SALESPERSONS* if no two S's are consecutive?

Solution a. There are 12 letters, S is repeated 4 times, and E is repeated 2 times, so there are

$$\frac{12!}{4! \cdot 2!} = 9979200$$

different strings.

b. We start by placing down all letters except for S. Because we have 2 E's, there are $8!/2!$ ways to do this. We may then place the 4 S's in any of the 9 spots in between the letters or at the ends, so there are

$$\frac{8!}{2!}\binom{9}{4} = \frac{8!}{2!} \cdot \frac{9!}{4! \cdot 5!} = 2540160$$

different strings.

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2 You are given three piles of identical red, blue and green balls, where each pile contains at least 10 balls.

a. In how many ways can 10 balls be selected if exactly one red ball and at least one blue ball must be selected?

b. In how many ways can 10 balls be selected if at most one red ball is selected?

Solution Since order doesn't matter, we may assume that we select all the reds at once, then all the blues at once, and then all the greens at once.

a. Because we must have exactly 1 red ball, we just need to pick the blue and green balls. We need at least one blue ball, so we just need to pick 9 balls from the blue and green balls total. This is the same as picking the number of blue balls, since the remaining ones must be green. We can pick 0, 1, . . . , 8, or 9 blue balls, so there are 10 choices here.

b. There are two cases here: choosing 0 red balls or choosing 1 red ball.

If we choose 0 red balls, then we can pick up to 10 blue balls, so there are 11 choices in this case.

If we choose 1 red ball, then from the previous part, there are 10 ways to do this.

By the addition principle, there are $11 + 10 = 21$ ways to pick 10 balls.

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3 Find the number of integer solutions of $x_1 + x_2 + x_3 = 15$ subject to the conditions given:

a. $x_1 = 1$, $x_2 \geq 0$, $x_3 \geq 0$.

b. $0 \leq x_1 \leq 6$, $x_2 \geq 0$, $x_3 \geq 0$.

c. $0 \leq x_1 < 6$, $1 \leq x_2 < 9$, $x_3 \geq 0$.

Solution a. In this case, the problem reduces to counting the number of solutions to $x_2 + x_3 = 14$. There are 15 ways to choose x_2, and afterwards, x_3 is fixed, so there are 15 integer solutions.

b. There are 7 cases. In the $x_1 = k$ case, there are $16 - k$ choices for x_2, which also determines x_3. Summing them up, we get

$$\sum_{k=0}^{6}(16-k) = 16 \cdot 7 - \frac{6(6+1)}{2} = 91$$

solutions.

c. We have 6 choices for x_1. No matter what x_1 is, we have 8 choices for x_2, so in total, there are 48 solutions.

4 Show that the number of solutions in nonnegative integers of the inequality

$$x_1 + x_2 + \cdots + x_n \leq M,$$

where M is a nonnegative integer, is $C(M+n, n)$.

Solution We can think of this as giving each x_i a certain number of 1's. This is analogous to adding balls into n distinct bins, while keeping the total number of balls under or equal to M.

We can have a "trash" bin, where the 1's don't count towards the sum, which we'll call x_{n+1}. Then the problem is distributing M 1's into $n+1$ distinct x_i's. So, we have

$$\binom{M+(n+1)-1}{(n+1)-1} = \binom{M+n}{n}$$

solutions, as required.

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5 What is wrong with the following argument, which supposedly counts the number of partitions of a 10-element set into eight nonempty subsets?

List the elements of the set with blanks between them:

$$x_1\_\_x_2\_\_x_3\_\_x_4\_\_x_5\_\_x_6\_\_x_7\_\_x_8\_\_x_9\_\_x_{10}$$

Every time we fill seven of the nine blanks with seven vertical bars, we obtain a partition of $\{x_1, \ldots, x_{10}\}$ into eight subsets. For example, the partitions $\{x_1\}$, $\{x_2\}$, $\{x_3, x_4\}$, $\{x_5\}$, $\{x_6\}$, $\{x_7, x_8\}$, $\{x_9\}$, $\{x_{10}\}$ would be represented as

$$x_1 \mid x_2 \mid x_3x_4 \mid x_5 \mid x_6 \mid x_7x_8 \mid x_9 \mid x_{10}.$$

Thus the solution to the problem is $C(9, 7)$.

Solution The elements are distinct, so listing them restricts the possible ways to group elements together. For example, we cannot get $\{x_1, x_3\}$ counting this way.

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6 Find the coefficient of the given term when the given expression is expanded:

a. s^6t^6; $(2s-t)^{12}$.

b. a^2x^3; $(a+ax+x)(a+x)^4$.

Solution a. s^6t^6 appears in the following term:

$$\binom{12}{6}(2s)^6(-t)^{12-6} = \frac{12!}{6!\cdot 6!}2^6(-1)^6s^6t^6 = \boxed{59136}\,s^6t^6.$$

b. We have the three expansions to consider:

$$a(a+x)^4,\ ax(a+x)^4,\ \text{and } x(a+x)^4.$$

So, we need to find the coefficients of ax^3, ax^2, and a^2x^2, respectively. ax^2 doesn't show up in the expansion, so the remaining terms show up in:

$$\binom{4}{1}ax^3 = 4ax^3$$
$$\binom{4}{2}a^2x^2 = 6a^2x^2.$$

Thus, the coefficient of a^2x^3 is $4+6=10$.

7 Find the number of terms in the expansion of $(w+x+y+z)^{12}$.

Solution The terms in the expansion are of the form $w^a x^b y^c z^d$, where $a+b+c+d=12$. Indeed, one of w,x,y,z picks up a power from each $w+x+y+z$ factor. Thus, we need to count the number of nonnegative integer solutions to $a+b+c+d=12$.

We can accomplish this by writing out 12 1's, and figuring out how many of them to give to each letter. I.e., we need to split up 12 1's into 4 groups, and there are

$$\binom{12+4-1}{4-1} = \binom{15}{3} = \frac{15!}{12! \cdot 3!} = 455$$

of them, so there are 455 terms.

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8 Use the Binomial Theorem to show that $\displaystyle\sum_{k=0}^{n}(-1)^k C(n,k) = 0$.

Solution By the Binomial Theorem, we have

$$0 = (1-1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k 1^{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k},$$

as desired.

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9 In class we showed that there are 3^n ordered pairs (A,B) satisfying $A \subseteq B \subseteq X$, where X is an n-element set. Give another proof of this result by considering the cases $|A|=0$, $|A|=1$, ..., $|A|=n$, and then using the Binomial Theorem.

Solution If $|A|=k$, we have $\binom{n}{k}$ choices for the elements of A, which gives $n-k$ remaining elements that we could place into B. For any choice of elements of A in this case, we could have $|B \setminus A| = 0,1,\ldots,n-k$, so there are

$$\binom{n-k}{0} + \binom{n-k}{1} + \cdots + \binom{n-k}{n-k} = \sum_{i=0}^{n-k}\binom{n-k}{i}1^i 1^{n-k-i} = 2^{n-k}.$$

So, in this case, there are

$$\binom{n}{k}2^{n-k}$$

choices.

Adding up the possibilities from all the cases, we get

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k} = \sum_{k=0}^{n}\binom{n}{k}2^{n-k}1^k = (2+1)^n = 3^n,$$

as expected.

10 Prove that $\sum_{k=0}^{\lfloor n/2 \rfloor} C(n-k,k) = f_n$ for all $n \geq 1$, where f_n is the n-th Fibonacci number.

Solution Call the sum s_n. Then

$$s_1 = \binom{1}{0} = 1 = f_1 \quad \text{and} \quad s_2 = \binom{2}{0} + \binom{1}{1} = 1 + 1 = 2 = f_2.$$

We will show that $s_n + s_{n+1} = s_{n+2}$. Since $s_1 = f_1$ and $s_2 = f_2$, it follows from the definition of the Fibonacci numbers that $s_n = f_n$ for all $n \geq 1$.

$$\begin{aligned}
s_{n+2} = \sum_{k=0}^{\lfloor (n+2)/2 \rfloor} \binom{n+2-k}{k} &= \binom{n+2}{0} + \sum_{k=1}^{\lfloor (n+2)/2 \rfloor} \left[\binom{n+1-k}{k-1} + \binom{n+1-k}{k} \right] \\
&= \binom{n+1}{0} + \sum_{k=1}^{\lfloor (n+2)/2 \rfloor} \left[\binom{n+1-k}{k-1} + \binom{n+1-k}{k} \right] \\
&= \sum_{k=1}^{\lfloor n/2 \rfloor + 1} \binom{n+1-k}{k-1} + \sum_{k=0}^{\lfloor (n+2)/2 \rfloor} \binom{n+1-k}{k} \\
&= \sum_{k=1}^{\lfloor n/2 \rfloor + 1} \binom{n-(k-1)}{k-1} + \sum_{k=0}^{\lfloor (n+2)/2 \rfloor} \binom{n+1-k}{k} \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} + \sum_{k=0}^{\lfloor (n+2)/2 \rfloor} \binom{n+1-k}{k}.
\end{aligned}$$

If n is odd, then

$$\left\lfloor \frac{n+2}{2} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor.$$

On the other hand, if n is even, then

$$\left\lfloor \frac{n+2}{2} \right\rfloor = \frac{n+2}{2},$$

so the last term in the second sum becomes

$$\binom{n/2}{(n+2)/2} = 0,$$

i.e., we included an extra term when indexing our sum. In either case, we get

$$s_{n+2} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} + \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1-k}{k} = s_n + s_{n+1},$$

as required.

11 Let p be a prime number:

a. Prove that $C(p,i)$ is divisible by p for all i with $1 \le i \le p-1$.

b. Prove *Fermat's Little Theorem*:
For any positive integer a and any prime p, p divides $a^p - a$.

Solution a. By definition,

$$\binom{p}{i} = \frac{p!}{i! \cdot (p-i)!} = \frac{p(p-1)\cdots(p-i)}{(p-i)!}.$$

If $1 \le i \le p-1$, $p - i \neq 0$. Moreover, $(p-i)!$ does not divide p because all of the factors of $(p-i)!$ are strictly smaller than p, and p is prime. Hence, p must divide $\binom{p}{i}$.

b. We will prove this by induction on a.

Base step: $a = 1$

Here, $1^p - 1 = 0$, so p divides it. Hence, the base step holds.

Inductive step:

Assume that the theorem holds for $a = k$. We need to show that it holds for $a = k+1$. Following the hint, we have by the Binomial Theorem that

$$[(a+1)^p - (a+1)] - (a^p - a) = \sum_{i=1}^{p-1} \binom{p}{i} a^i \implies (a+1)^p - (a+1) = \sum_{i=1}^{p-1} \binom{p}{i} a^i + (a^p - a).$$

By part (a), we know that p divides each term in the sum. By the inductive hypothesis, we know that p divides $a^p - a$ also. Hence, p must divide $(a+1)^p - (a+1)$, so the inductive step holds.

By inductino, p divides $a^p - a$ for any positive integer a.

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12 Prove that

$$\left(\frac{m}{m+n}\right)^m \left(\frac{n}{m+n}\right)^n C(m+n, m) < 1$$

for all $m, n \in \mathbb{Z}^+$.

Solution Notice

$$1 = \left(\frac{m}{m+n} + \frac{n}{m+n}\right)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} \left(\frac{m}{m+n}\right)^k \left(\frac{n}{m+n}\right)^{m+n-k}.$$

All the terms are positive since m and n are, so we know that each term is strictly smaller than 1. In particular, when $k = m$, we get

$$\left(\frac{m}{m+n}\right)^m \left(\frac{n}{m+n}\right)^n \binom{m+n}{m} < 1,$$

as required.