****21 8.2.16** Let V be an inner product space and W a finite-dimensional subspace of V. There are (in general) many projections which have W as their range. One of these, the orthogonal projection on W, has the property that $\|E\alpha\| \leq \|\alpha\|$ for every α in V. Prove that if E is a projection with range W, such that $\|E\alpha\| \leq \|\alpha\|$ for all α in V, then E is the orthogonal projection on W.

Solution Let E be a projection with range W such that $\|E\alpha\| \leq \|\alpha\|$ for all $\alpha \in V$. Assume E is not the orthogonal projection onto W. Then there exists $\beta \in W$ and $\gamma \in \ker E$ such that $<\beta, \gamma> \neq 0$. If this were not true, then $<\beta, \gamma> = 0$ for all $\beta \in W$ and $\gamma \in \ker E$, which means $\ker E = W^{\perp} \implies E$ is the orthogonal projection. Define $\alpha = \lambda\beta + \gamma$. Note that $E\alpha = \lambda\beta$. Then

$$\begin{aligned}
\|\alpha\|^2 &= \|\lambda\beta + \gamma\|^2 \\
&= \lambda^2\|\beta\|^2 + \lambda<\beta, \gamma> + \overline{\lambda}<\gamma, \beta> + \|\gamma\|^2 \\
&= \|E\alpha\|^2 + \|\gamma\|^2 + \lambda<\beta, \gamma> + \overline{\lambda<\beta, \gamma>}
\end{aligned}$$

This is true for any λ. Take $\lambda = -\frac{\|\gamma\|^2}{<\beta,\gamma>}$. Then

$$\begin{aligned}
\|\alpha\|^2 &= \|E\alpha\|^2 + \|\gamma\|^2 - \|\gamma\|^2 - \overline{\|\gamma\|^2} \\
&= \|E\alpha\|^2 - \|\gamma\|^2
\end{aligned}$$

This implies that $\|E\alpha\| > \|\alpha\|$, which is a contradiction. Hence E must be the orthogonal projection.

****22 8.2.11** Let V be a finite-dimensional inner product space, and let $\{\alpha_1, \ldots, \alpha_n\}$ be an orthonormal basis for V. Show that for any vector α, β in V,

$$<\alpha, \beta> = \sum_{k=1}^{n} <\alpha, \alpha_k>\overline{<\beta, \alpha_k>}.$$

Solution We can write $\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n$ and $\beta = b_1\alpha_1 + \cdots + b_n\alpha_n$. Note that since $\{\alpha_1, \ldots, \alpha_n\}$ is orthonormal, $a_i = <\alpha, \alpha_i>$ and $b_i = <\beta, \alpha_i>$. Then

$$\sum_{k=1}^{n} <\alpha, \alpha_k>\overline{<\beta, \alpha_k>} = \sum_{k=1}^{n} a_k\overline{b_k}.$$

Moreover,

$$\begin{aligned}
<\alpha, \beta> &= <a_1\alpha_1 + \cdots + a_n\alpha_n, b_1\alpha_1 + \cdots + b_n\alpha_n> \\
&= \sum_{k=1}^{n} a_k<\alpha_k, b_1\alpha_1 + \cdots + b_n\alpha_n> \\
&= \sum_{k=1}^{n} a_k\left(\overline{b_1}<\alpha_k, \alpha_1> + \cdots + \overline{b_n}<\alpha_k, \alpha_n>\right) \\
&= \sum_{k=1}^{n} a_k\overline{b_k}<\alpha_k, \alpha_k> \\
&= \sum_{k=1}^{n} a_k\overline{b_k}
\end{aligned}$$

Thus,

$$<\alpha, \beta> = \sum_{k=1}^{n} a_k\overline{b_k} = \sum_{k=1}^{n} <\alpha, \alpha_k>\overline{<\beta, \alpha_k>}$$

as desired.