33.1 Find the Fourier series for the function defined by

$$f(x) = \pi, \qquad -\pi \le x \le \frac{\pi}{2}$$
$$f(x) = 0, \qquad \frac{\pi}{2} < x \le \pi$$

Solution We first find the cosine coefficients:

By inspection, $a_0 = 3\pi/2$. Then

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi/2} \pi\cos nx \,\mathrm{d}x \\ &= \frac{\sin nx}{n}\Big|_{-\pi}^{\pi/2} \\ &= \frac{1}{n}\sin\frac{n\pi}{2} \end{aligned}$$

We then find the sine coefficients:

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi/2} \pi\sin nx \,\mathrm{d}x \\ &= -\frac{\cos nx}{n}\Big|_{-\pi}^{\pi/2} \\ &= -\frac{1}{n}\cos\frac{n\pi}{2} + \frac{1}{n}\cos n\pi. \end{aligned}$$

This gives the series

$$\begin{aligned} f(x) &\sim \frac{3\pi}{4} + \sum_1^\infty \frac{1}{n}\sin\frac{n\pi}{2}\cos nx + \sum_1^\infty -\frac{1}{n}\cos\frac{n\pi}{2}\sin nx + \sum_1^\infty \frac{1}{n}\cos n\pi \sin nx \\ &= \frac{3\pi}{4} + \sum_1^\infty \frac{(-1)^{n-1}}{2n-1}\cos(2n-1)x + \sum_1^\infty \frac{(-1)^{n-1}}{2n}\sin(2nx) + \sum_1^\infty \frac{(-1)^n}{n}\sin(nx). \end{aligned}$$

33.2 Find the Fourier series for the function defined by

$$f(x) = \begin{cases} 0, & \text{if } -\pi \le x < 0 \\ 1, & \text{if } 0 \le x \le \frac{\pi}{2} \\ 0, & \text{if } \frac{\pi}{2} < x \le \pi. \end{cases}$$

Solution By inspection $a_0 = 1/2$.

The cosine coefficients are given by

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{0}^{\pi/2} \cos nx \,\mathrm{d}x \\ &= \frac{1}{n\pi}\sin nx\Big|_{0}^{\pi/2} \\ &= \frac{1}{n\pi}\sin\frac{n\pi}{2}. \end{aligned}$$

Then the sine coefficients are

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{0}^{\pi/2} \sin nx\,\mathrm{d}x \\
&= -\frac{1}{n\pi}\cos nx\Big|_0^{\pi/2} \\
&= -\frac{1}{n\pi}\cos\frac{n\pi}{2} + \frac{1}{n\pi}.
\end{aligned}$$

So, the Fourier series is given by

$$\begin{aligned}
f(x) &\sim \frac{1}{4} + \sum_{1}^{\infty}\frac{1}{n\pi}\sin\frac{n\pi}{2}\cos nx - \sum_{1}^{\infty}\frac{1}{n\pi}\cos\frac{n\pi}{2}\sin nx + \sum_{1}^{\infty}\frac{1}{n\pi}\sin nx \\
&= \frac{1}{4} + \frac{1}{\pi}\sum_{1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos(2n-1)x - \frac{1}{\pi}\sum_{1}^{\infty}\frac{(-1)^n}{2n}\sin 2nx + \sum_{1}^{\infty}\frac{1}{n\pi}\sin nx.
\end{aligned}$$

33.3 Find the Fourier series for the function defined by

$$\begin{aligned}
f(x) &= 0, \qquad -\pi \le x < 0 \\
f(x) &= \sin x, \quad\ 0 \le x \le \pi
\end{aligned}$$

Solution Notice that $a_0 = 2/\pi$.

$$a_n = \frac{1}{\pi}\int_0^{\pi}\sin x\cos nx\,\mathrm{d}x = \frac{1}{\pi}\frac{\cos\pi n + 1}{1-n^2}.$$

If $n = 1$, $a_1 = 0$. So, for the cosine terms, the coefficient is 0 if n is odd.

$$b_n = \frac{1}{\pi}\int_0^{\pi}\sin x\sin nx\,\mathrm{d}x = -\frac{1}{\pi}\frac{\sin\pi n}{n^2-1}.$$

For $n = 1$, $b_n = 1/2$. Otherwise, the coefficients are all 0, so the series is given by

$$\frac{1}{\pi} + \frac{1}{2}\sin x + \frac{1}{\pi}\sum_{2}^{\infty}\frac{(-1)^n + 1}{1-n^2}\cos nx.$$

33.4 Solve Problem 3 with $\sin x$ replaced by $\cos x$.

Solution $a_0 = 0$, $a_1 = 1/2$, $b_1 = 0$, and

$$a_n = \frac{1}{\pi}\int_0^{\pi}\cos x\cos nx\,\mathrm{d}x = -\frac{1}{\pi}\frac{n\sin\pi n}{n^2-1} = 0.$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}\cos x\sin nx\,\mathrm{d}x = \frac{1}{\pi}\frac{n(\cos\pi n + 1)}{n^2-1}.$$

So, the series is given by

$$\frac{1}{2}\cos x + \frac{1}{\pi}\sum_{2}^{\infty}\frac{n((-1)^n+1)}{n^2-1}\sin nx.$$

35.2 Show that any function $f(x)$ defined on a symmetrically placed interval can be written as the sum of an even function and an odd function. *Hint*:

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)].$$

Solution The hint is easy to verify by using the distribution law. It suffices to show that one is even and one is odd.

$$\frac{1}{2}[f(-x) + f(-(-x))] = \frac{1}{2}[f(-x) + f(x)] = \frac{1}{2}[f(x) + f(-x)]$$
$$\frac{1}{2}[f(-x) - f(-(-x))] = \frac{1}{2}[f(-x) - f(x)] = -\frac{1}{2}[f(x) - f(-x)],$$

so the first term is even and the second term is odd.

35.4 Show that the sine series of the constant function $f(x) = \pi/4$ is

$$\frac{\pi}{4} = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots, \qquad 0 < x < \pi.$$

What sum is obtained by putting $x = \pi/2$? What is the cosine series of this function?

Solution We first calculate the sine series:

$$b_n = \frac{2}{\pi}\int_0^\pi \frac{\pi}{4}\sin nx \, \mathrm{d}x = \frac{1 - \cos \pi n}{2n}.$$

This is 0 whenever n is even and $1/n$ when n is odd, so

$$\frac{\pi}{4} \sim \sin x + \frac{\sin 3x}{3} + \cdots.$$

Substituting $x = \pi/2$ gives

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_0^\infty \frac{(-1)^n}{2n+1}.$$

We now find the cosine series. The first coefficient is given by $a_0 = \pi/2$. Then

$$a_n = \frac{1}{\pi}\int_{-\pi}^\pi \frac{\pi}{4}\cos nx \, \mathrm{d}x = \frac{\sin \pi n}{2n} = 0.$$

So, the cosine series is given by $\pi/4$.

35.6 Find the sine and cosine series for $\sin x$.

Solution The sine coefficients are given by

$$b_n = \frac{2}{\pi}\int_0^\pi \sin x \sin nx \, \mathrm{d}x = -\frac{2\sin \pi n}{\pi(n^2 - 1)}.$$

So, we need to check the $n = 1$ case separately:

$$b_1 = \frac{2}{\pi}\int_0^\pi \sin^2 x \, \mathrm{d}x = 1,$$

which means that the sine series is given by $\sin x$.

As for the cosine series,

$$a_0 = \frac{2}{\pi}\int_0^\pi \sin x \, \mathrm{d}x = \frac{4}{\pi},$$

and

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \,\mathrm{d}x = \frac{2}{\pi}\frac{\cos \pi n + 1}{1 - n^2}.$$

Again, we need to check the $n = 1$ case, which gives

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \,\mathrm{d}x = 0.$$

So, a_n is $2/(1-n^2)$ whenever n is even and 0 whenever n is odd, which means that the sine series is

$$\frac{2}{\pi} + \frac{4}{\pi}\sum_1^{\infty} \frac{\cos(2nx)}{1-(2n)^2}.$$

35.9 If $f(x) = x$ for $0 \le x \le \pi/2$ and $f(x) = \pi - x$ for $\pi/2 < x \le \pi$, show that the cosine series for this function is

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_1^{\infty} \frac{\cos 2(2n-1)x}{(2n-1)^2}.$$

Sketch the graph of the sum of this series on the interval $-5\pi \le x \le 5\pi$.

Solution

$$a_0 = \frac{2}{\pi}\int_0^{\pi/2} x \,\mathrm{d}x + \frac{2}{\pi}\int_{\pi/2}^{\pi} \pi - x \,\mathrm{d}x = \frac{2}{\pi}\cdot\frac{\pi^2}{4} = \frac{\pi}{2}.$$

Then

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi/2} x\cos nx \,\mathrm{d}x + \frac{2}{\pi}\int_{\pi/2}^{\pi} \pi\cos nx - x\cos nx \,\mathrm{d}x \\ &= \frac{2}{\pi}\left(\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{2}{n^2}\right) - \frac{2}{n}\sin\frac{n\pi}{2} - \frac{2}{\pi}\left(\frac{1}{n^2}\cos n\pi - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right) \\ &= \frac{2}{n^2\pi}\sin^2\frac{\pi n}{4}\cos\frac{\pi n}{2}. \end{aligned}$$

When n is even, the coefficient is given by $\left(2\sin^2\frac{\pi n}{4}\right)/\pi n^2$ with alternating signs, and when n is odd, the coefficient is 0.

This gives the series

$$\frac{\pi}{4} + \frac{2}{\pi}\sum_1^{\infty}(-1)^n\frac{\cos 2nx}{n^2}\sin^2\frac{\pi n}{2} = \frac{\pi}{4} - \frac{2}{\pi}\sum_1^{\infty}\frac{\cos 2(2n-1)x}{(2n-1)^2},$$

as desired. Here is the graph of the given solution with 175 terms:

5"/series.png.bb"

35.10 a. Show that the cosine series for x^2 is

$$x^2 = \frac{\pi^2}{3} + 4\sum_1^\infty (-1)^n \frac{\cos nx}{n^2}, \qquad -\pi \le x \le \pi.$$

b. Find the sine series for x^2 and use this expansion together with formula (7) to obtain the sum

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}.$$

c. Denote by s the sum of the reciprocals of the cubes of the odd numbers,

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \cdots = s,$$

and show that then

$$\sum_1^\infty \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots = \frac{8}{7}s.$$

Solution a. By inspection, $a_0 = 2\pi^2/3$. Then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \,\mathrm{d}x = \frac{2(\pi^2 n^2 - 2)\sin \pi n + 4\pi n \cos \pi n}{\pi n^3} = \frac{4\cos \pi n}{n^2}.$$

This gives

$$\frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}\cos nx.$$

b. For the sine series,

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx \,\mathrm{d}x = \frac{2}{\pi}\frac{(2-\pi^2 n^2)\cos \pi n + 2\pi n \sin \pi n - 2}{n^3} = \frac{4(\cos \pi n - 1)}{\pi n^3} - \frac{2\pi \cos \pi n}{n}.$$

The first term is 0 whenever n is even and $-8/\pi n^3$ otherwise. The second term alternates. This gives the series

$$x^2 \sim \frac{8}{\pi}\sum_1^\infty -\frac{1}{n^3}\sin nx - 2\pi \sum_1^\infty \frac{(-1)^n}{n}\sin nx.$$

Substituting the expansion for x, we see

$$x^2 = \frac{8}{\pi}\sum_1^\infty -\frac{1}{n^3}\sin nx + \pi x.$$

Substituting $x = \pi/2$ and rearranging, we get

$$\sum_1^\infty \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}$$

c. Notice that for the even cubes,

$$\sum_1^\infty \frac{1}{(2n)^3} = \frac{1}{8}\sum_1^\infty \frac{1}{n^3}.$$

So,

$$\sum_1^\infty \frac{1}{n^3} - \frac{1}{8}\sum_1^\infty \frac{1}{n^3} = s \implies \sum_1^\infty \frac{1}{n^3} = \frac{8}{7}s.$$

35.11 a. Show that the cosine series for x^3 is

$$x^3 = \frac{\pi^3}{4} + 6\pi \sum_1^\infty (-1)^n \frac{\cos nx}{n^2} + \frac{24}{\pi}\sum_1^\infty \frac{\cos(2n-1)x}{(2n-1)^4},$$

for $0 \le x \le \pi$.

b. Use the series in (a) to obtain, in this order, the sums

$$\sum_1^\infty \frac{1}{(2n-1)^4} = \frac{\pi^4}{96} \quad \text{and} \quad \sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Solution a. The first coefficient is given by $a_0 = \pi^3/2$. Then the coefficients are given by

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi x^3 \cos nx \,\mathrm{d}x \\ &= \frac{2}{\pi}\frac{\pi n(\pi^2 n^2 - 6)\sin \pi n + 3(\pi^2 n^2 - 2)\cos \pi n + 6}{n^4} \\ &= \frac{2}{\pi}\frac{6(1-\cos \pi n)}{n^4} + \frac{2}{\pi}\frac{3\pi^2 \cos \pi n}{n^2} \\ &= \frac{12(1-\cos \pi n)}{n^4} + \frac{6\pi \cos \pi n}{n^2}. \end{aligned}$$

In the first term, we get 0 whenever n is even, and we get $24/n^4$ otherwise, so we get

$$x^3 = \frac{\pi^3}{4} + 6\pi \sum_1^\infty (-1)^n \frac{\cos nx}{n^2} + \frac{24}{\pi}\sum_1^\infty \frac{\cos(2n-1)x}{(2n-1)^4}$$

as desired.

b. Notice that if we look at the even terms,

$$\sum_1^\infty \frac{1}{(2n)^4} = \frac{1}{16}\sum_1^\infty \frac{1}{n^4},$$

so if we remove the even terms,

$$\sum_1^\infty \frac{1}{n^4} - \sum_1^\infty \frac{1}{(2n)^4} = \frac{15}{16}\sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{96} \implies \sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{90}.$$

36.2 Find the Fourier series for the functions defined by

a. $f(x) = 1 + x$, $-1 \le x < 0$ and $f(x) = 1 - x$, $0 \le x \le 1$

b. $f(x) = |x|$, $-2 \le x \le 2$.

Solution a. $a_0 = 1$ and

$$a_n = \int_{-1}^0 (1+x)\cos \pi nx \,\mathrm{d}x + \int_0^1 (1-x)\cos \pi nx \,\mathrm{d}x = \frac{2(1-\cos \pi n)}{\pi^2 n^2}.$$

The function is even, so it has no sine terms. Thus,

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi^2}\sum_1^\infty \frac{(1-(-1)^n)}{n^2}\cos \pi nx.$$

b. $a_0 = 2$, since $L = 2$ in this problem. Then

$$a_n = \frac{1}{2}\int_{-2}^2 |x| \cos \frac{\pi nx}{2}\,\mathrm{d}x = \frac{1}{2}\int_{-2}^0 -x\cos\frac{\pi nx}{2}\,\mathrm{d}x + \frac{1}{2}\int_0^2 x\cos\frac{\pi nx}{2}\,\mathrm{d}x = \frac{4(\cos \pi n - 1)}{\pi^2 n^2}.$$

This coefficient is 0 whenever n is odd, and is $-8/\pi^2 n^2$ otherwise.
The sine coefficients are 0 since f is even, which gives the series

$$|x| \sim 1 - \frac{8}{\pi^2}\sum_1^\infty \frac{1}{(2n-1)^2}\cos\frac{\pi(2n-1)x}{2}.$$

36.3 Show that

$$\frac{1}{2}L - x = \frac{L}{\pi}\sum_{1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{L}, \quad 0 < x < L.$$

Solution We'll calculate its sine series.

$$b_n = \frac{2}{L}\int_0^L \left(\frac{1}{2}L - x\right)\sin\frac{\pi n x}{L}\,\mathrm{d}x = \frac{2}{L}\frac{L^2\pi n(\cos\pi n + 1)}{2\pi^2 n^2} = \frac{L(\cos\pi n + 1)}{\pi n}.$$

This is 0 whenever n is odd, so only the even terms remain with coefficient $2L/\pi n$. This gives us

$$\frac{1}{2}L - x \sim \frac{2L}{\pi}\sum_{1}^{\infty}\frac{1}{2n}\sin\frac{2\pi n x}{L} = \frac{L}{\pi}\sum_{1}^{\infty}\frac{1}{n}\sin\frac{2\pi n x}{L}.$$

36.7 Find the cosine series for the function defined by

$$f(x) = \frac{1}{4} - x,\ 0 \le x < \frac{1}{2} \quad \text{and} \quad f(x) = x - \frac{3}{4},\ \frac{1}{2} \le x \le 1.$$

Solution $a_0 = 0$. Then

$$a_n = 2\int_0^{1/2}\left(\frac{1}{4} - x\right)\cos\pi n x\,\mathrm{d}x + \int_{1/2}^{1}\left(x - \frac{3}{4}\right)\cos\pi n x\,\mathrm{d}x = \frac{-8\cos\frac{\pi n}{2} + 4\cos\pi n + 4}{2\pi^2 n^2}.$$

So the series is given by

$$f(x) \sim \frac{1}{2\pi^2}\sum_{1}^{\infty}\frac{-8\cos\frac{\pi n}{2} + 4(-1)^n + 4}{n^2}\cos\pi n x.$$