1.4.3 Prove that

$$\left|\frac{a-b}{1-\overline{a}b}\right| = 1$$

if either $|a| = 1$ or $|b| = 1$. What exception must be made if $|a| = |b| = 1$?

Solution Assume without loss of generality that $|a| = 1$. The following argument can be done for $|b| = 1$.

Note that it suffices to show that the square of the left-hand side is 1:

$$\begin{aligned}\left|\frac{a-b}{1-\overline{a}b}\right|^2 &= \frac{a-b}{1-\overline{a}b} \cdot \frac{\overline{a}-\overline{b}}{1-a\overline{b}} \\ &= \frac{|a|^2+|b|^2-a\overline{b}-\overline{a}b}{1+|ab|^2-a\overline{b}-\overline{a}b} \\ &= \frac{1+|b|^2-a\overline{b}-\overline{a}b}{1+|b|^2-a\overline{b}-\overline{a}b} \\ &= 1.\end{aligned}$$

In the instance that $|a| = |b| = 1$, we must have that $a \neq b$. Otherwise, if $a = b$, the fraction becomes $0/0$.

1.4.4 Find the conditions under which the equation $az + b\overline{z} + c = 0$ in one complex unknown has exactly one solution, and compute that solution.

Solution Write $z = x + iy$. Then

$$\begin{aligned}a(x+iy) + b(x-iy) + c &= 0 \\ x(a+b) + iy(a-b) &= -c.\end{aligned}$$

We need $a \neq b$ and $a \neq -b$, or else x or y vanishes from the equation, and we get infinitely many equations. In this case, the general solution is

$$z = -\frac{\operatorname{Re} c}{a+b} - i\frac{\operatorname{Im} c}{a-b}.$$

1.4.5 Prove Lagrange's identity in the complex form

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 = \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{1 \leq i < j \leq n} |a_i\overline{b_j} - a_j\overline{b_i}|^2.$$

Solution We will prove this by induction.

Base step:

When $n = 1$, the double-indexed sum has no terms, so the right-hand side becomes

$$|a_1|^2|b_1|^2 = |a_1 b_1|^2,$$

so the base case holds.

Inductive step:

Assume that Lagrange's identity holds for $n = k$. We wish to show that this implies that it holds for $n = k + 1$.

$$\sum_{i=1}^{k+1}|a_i|^2\sum_{i=1}^{k+1}|b_i|^2 - \sum_{1\le i<j\le k+1}|a_i\overline{b_j} - a_j\overline{b_i}|^2$$

$$= \left(|a_{k+1}|^2 + \sum_{i=1}^{k}|a_i|^2\right)\left(|b_{k+1}|^2 + \sum_{i=1}^{k}|b_i|^2\right) - \left(\sum_{i=1}^{k}|a_i\overline{b_{k+1}} - a_{k+1}\overline{b_i}|^2 + \sum_{1\le i<j\le k}|a_i\overline{b_j} - a_j\overline{b_i}|^2\right)$$

$$= |a_{k+1}|^2|b_{k+1}|^2 + |a_{k+1}|^2\sum_{i=1}^{k}|b_i|^2 + |b_{k+1}|^2\sum_{i=1}^{k}|a_i|^2 + \left(\sum_{i=1}^{k}|a_i|^2\sum_{i=1}^{k}|b_i|^2 - \sum_{1\le i<j\le k}|a_i\overline{b_j} - a_j\overline{b_i}|^2\right) - \sum_{i=1}^{k}|a_i\overline{b_{k+1}} - a_{k+1}\overline{b_i}|^2$$

$$= |a_{k+1}b_{k+1}|^2 + |a_{k+1}|^2\sum_{i=1}^{k}|b_i|^2 + |b_{k+1}|^2\sum_{i=1}^{k}|a_i|^2 + \left|\sum_{i=1}^{k}a_ib_i\right|^2 - \sum_{i=1}^{k}|a_i\overline{b_{k+1}} - a_{k+1}\overline{b_i}|^2.$$

Note that

$$|a_i\overline{b_{k+1}} - a_{k+1}\overline{b_i}|^2 = |a_ib_{k+1}|^2 + |a_{k+1}b_i|^2 - a_ia_{k+1}\overline{b_{k+1}b_i} - \overline{a_ia_{k+1}}b_{k+1}b_i.$$

Next,

$$\begin{aligned}\left|\sum_{i=1}^{k+1}a_ib_i\right|^2 &= \left|a_{k+1}b_{k+1} + \sum_{i=1}^{k}a_ib_i\right|^2\\ &= |a_{k+1}b_{k+1}|^2 + \overline{a_{k+1}b_{k+1}}\sum_{i=1}^{k}a_ib_i + a_{k+1}b_{k+1}\sum_{i=1}^{k}\overline{a_ib_i} + \left|\sum_{i=1}^{k}a_ib_i\right|^2,\end{aligned}$$

so substitution yields

$$\left|\sum_{i=1}^{k+1}a_ib_i\right|^2 + |a_{k+1}|^2\sum_{i=1}^{k}|b_i|^2 + |b_{k+1}|^2\sum_{i=1}^{k}|a_i|^2 - \overline{a_{k+1}b_{k+1}}\sum_{i=1}^{k}a_ib_i - a_{k+1}b_{k+1}\sum_{i=1}^{k}\overline{a_ib_i} - \sum_{i=1}^{k}|a_i\overline{b_{k+1}} - a_{k+1}\overline{b_i}|^2.$$

It suffices to prove that

$$\sum_{i=1}^{k}|a_i\overline{b_{k+1}} - a_{k+1}\overline{b_i}|^2 = |b_{k+1}|^2\sum_{i=1}^{k}|a_i|^2 + |a_{k+1}|^2\sum_{i=1}^{k}|b_i|^2 - \overline{a_{k+1}b_{k+1}}\sum_{i=1}^{k}a_ib_i - a_{k+1}b_{k+1}\sum_{i=1}^{k}\overline{a_ib_i},$$

so that the remaining terms cancel out.

$$\begin{aligned}\sum_{i=1}^{k}|a_i\overline{b_{k+1}} - a_{k+1}\overline{b_i}|^2 &= \sum_{i=1}^{k}(|a_ib_{k+1}|^2 + |a_{k+1}b_i|^2 - a_ib_i\overline{a_{k+1}b_{k+1}} - \overline{a_ib_i}a_{k+1}b_{k+1})\\ &= |b_{k+1}|^2\sum_{i=1}^{k}|a_i|^2 + |a_{k+1}|^2\sum_{i=1}^{k}|b_i|^2 - \overline{a_{k+1}b_{k+1}}\sum_{i=1}^{k}a_ib_i - a_{k+1}b_{k+1}\sum_{i=1}^{k}\overline{a_ib_i}.\end{aligned}$$

Thus, we finally have that

$$\left|\sum_{i=1}^{k+1}a_ib_i\right|^2 = \sum_{i=1}^{k+1}|a_i|^2\sum_{i=1}^{k+1}|b_i|^2 - \sum_{1\le i<j\le k+1}|a_i\overline{b_j} - a_j\overline{b_i}|^2,$$

so the inductive step holds.

By induction, Lagrange's inequality holds.

1.5.1 Prove that

$$\left|\frac{a-b}{1-\overline{a}b}\right| < 1$$

if $|a| < 1$ and $|b| < 1$.

Solution Let $|a| < 1$ and $|b| < 1$. Note that

$$\begin{aligned} |a-b|^2 &= |a|^2 - a\overline{b} - \overline{a}b + |b|^2 \\ |1-\overline{a}b|^2 &= 1 - a\overline{b} - \overline{a}b + |ab|^2. \end{aligned}$$

Subtracting the two equations yields

$$|a-b|^2 - |1-\overline{a}b|^2 = |a|^2 + |b|^2 - 1 - |ab|^2 < 1 + 1 - 1 - 1 = 0,$$

so

$$|a-b|^2 < |1-\overline{a}b|^2,$$

which is equivalent to what we wanted to show.

1.5.3 If $|a_i| < 1$, $\lambda_i \geq 0$ for $i = 1, \ldots, n$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, show that

$$|\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n| < 1.$$

Solution We will prove the following claim by induction: If $\lambda_1 + \lambda_2 + \cdots + \lambda_n = c > 0$ and the other conditions hold, then

$$|\lambda_1 a_1 + \cdots + \lambda_n a_n| < c.$$

Base step:

When $n = 1$, we get $|\lambda_1 a_1| = \lambda_1 |a_1| < c|a_1| < c$, as desired.

Inductive step:

Assume that the claim holds for $n = k$. We wish to show that it holds for $n = k + 1$.

Note that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = c - \lambda_{n+1}$. Then by the triangle inequality,

$$\begin{aligned} |\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n + \lambda_{n+1} a_{n+1}| &\leq |\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n| + |\lambda_{n+1} a_{n+1}| \\ &< c - \lambda_{n+1} + \lambda_{n+1} a_{n+1} \\ &< c - \lambda_{n+1} + \lambda_{n+1} \\ &= c, \end{aligned}$$

so the inductive step holds.

By induction, the claim holds. Let $c = 1$, and we have solved the original problem.

1.5.4 Show that there are complex numbers z satisfying

$$|z - a| + |z + a| = 2|c|$$

if and only if $|a| \leq |c|$. If this condition is fulfilled, what are the smallest and largest values of $|z|$?.

Solution "$\Longrightarrow$"

Suppose there exists z such that $|z - a| + |z + a| = 2|c|$. Notice by the triangle inequality that

$$|2a| = |(a + z) + |a - z|| \leq |z + a| + |z - a| = 2|c|.$$

It follows immediately that $|a| \leq |c|$.

"$\Longleftarrow$"

Suppose $|a| \leq |c| \implies |c|/|a| \geq 1$. Then $z = \dfrac{|c|}{|a|}a$ is a solution to the equation, if $a \neq 0$:

$$\left|\frac{|c|}{|a|}a - a\right| + \left|\frac{|c|}{|a|}a + a\right| = \left(\frac{|c|}{|a|} - 1\right)|a| + \left(\frac{|c|}{|a|} + 1\right)|a| = 2\frac{|c|}{|a|}|a| = 2|c|.$$

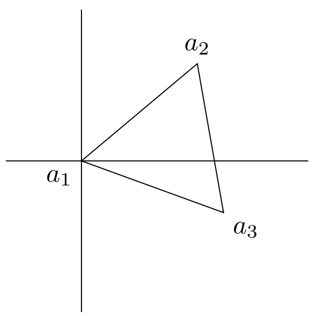
In the case that $a = 0$, then it is easy to see that $z = c$ is a solution to the equation.

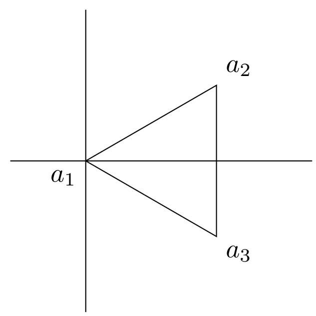
2.1.2 Prove that the points a_1, a_2, a_3 are vertices of an equilateral triangle if and only if $a_1^2 + a_2^2 + a_3^2 = a_1a_2 + a_2a_3 + a_3a_1$.

Solution We can shift the points by $-a_1$ so that a_1 is at the origin and preserve the geometry of the problem. So, without loss of generality, assume $a_1 = 0$.

"$\Longrightarrow$"

Let a_1, a_2, a_3 be vertices of an equilateral triangle. We can rotate the points so that $a_2 = \overline{a_3}$.





Indeed, we can rotate by the correct angle θ via the map $z \mapsto e^{i\theta}z$ to get $a_2 \mapsto e^{i\theta}a_2$ and $a_3 \mapsto e^{i\theta}a_3$. So, we can assume without loss of generality that $a_2 = \overline{a_3}$.

Since the points form an equilateral triangle,

$$|a_2|^2 = |a_3|^2 = |a_2 - a_3|^2.$$

Thus,

$$-a_2a_3 = -a_2\overline{a_3} = -|a_2|^2 = -|a_2 - a_3|^2 = (a_2 - a_3)(\overline{a_2} - \overline{a_3}) = a_2^2 + a_3^2 - 2a_2a_3$$
$$\implies a_2^2 + a_3^2 = a_2a_3.$$

“ $\Longleftarrow$ ”

By assumption, we know $a_2^2 + a_3^2 = a_2 a_3$. Then

$$\begin{aligned} a_2^2 + a_3^2 &= a_2 a_3 \\ a_2^2 - a_2 a_3 + a_3^2 &= 0 \\ a_2 &= \frac{a_3 \pm \sqrt{a_3^2 - 4a_3^2}}{2} \\ a_2 &= a_3\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right). \end{aligned}$$

Then

$$|a_2| = |a_3|\left|\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right| = |a_3|\left|\frac{1}{4} + \frac{3}{4}\right| = |a_3|.$$

Lastly,

$$|a_2 - a_3| = |a_3|\left|-\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right| = |a_3|.$$

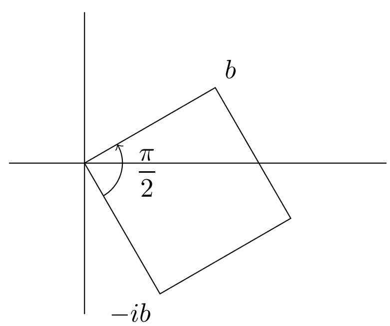
Thus,

$$|a_2| = |a_3| = |a_2 - a_3|,$$

so the points lie on an equilateral triangle.

2.1.3 Suppose that a and b are two vertices of a square. The the two other vertices in all possible cases.

Solution Assume without loss of generality that $a = 0$, which we can do since we can shift the points while preserving the geometry of the problem, i.e., angles and lengths are preserved.



Since b is a vertex of a square, we can get another vertex by rotation by $\frac{\pi}{2}$ or $-\frac{\pi}{2}$, which is

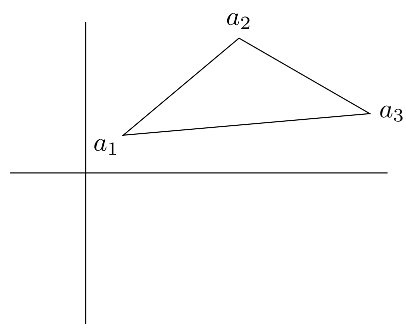
$$e^{i\pi/2}b = ib \quad \text{or} \quad e^{-i\pi/2}b = -ib.$$

So, the other vertices are given by the sums:

$$0, b, ib, b + ib \quad \text{or} \quad 0, b, -ib, b - ib.$$

2.1.4 Find the center and the radius of the circle which circumscribes the triangle with vertices a_1, a_2, a_3. Express the result in symmetric form.

Solution By geometry, the center of the circle which circumscribes the points is the intersection of the bisectors of the sides.



Two of the bisectors are given by

$$\begin{aligned} b_1(t) &= \frac{a_1 + a_3}{2} + ti(a_3 - a_1) \\ b_2(t) &= \frac{a_1 + a_2}{2} - ti(a_2 - a_1). \end{aligned}$$

We wish to find where they intersect:

$$\begin{aligned} b_1(t) &= b_2(s) \\ \frac{a_1 + a_3}{2} + ti(a_3 - a_1) &= \frac{a_1 + a_2}{2} - si(a_2 - a_1) \\ t(a_3 - a_1) + s(a_2 - a_1) &= \frac{a_2 - a_3}{2i}. \end{aligned}$$

If we write $a_j = x_j + jy_j$, we get the system

$$\begin{pmatrix} x_3 - x_1 & x_2 - x_1 \\ y_3 - y_1 & y_2 - y_1 \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(y_2 - y_3) \\ -\frac{1}{2}(x_2 - x_3) \end{pmatrix}.$$

By Cramer's rule, we get

$$t = \frac{1}{2} \frac{(y_2 - y_3)(y_2 - y_1) + (x_2 - x_3)(x_2 - x_1)}{(x_3 - x_1)(y_2 - y_1) - (y_3 - y_1)(x_2 - x_1)},$$

so the center is given by

$$c = \frac{a_1 + a_3}{2} + i\frac{a_3 - a_1}{2} \frac{(y_2 - y_3)(y_2 - y_1) + (x_2 - x_3)(x_2 - x_1)}{(x_3 - x_1)(y_2 - y_1) - (y_3 - y_1)(x_2 - x_1)}.$$

The radius is then given by

$$\begin{aligned} |c - a_1| &= \left| \frac{a_3 - a_1}{2} + i\frac{a_3 - a_1}{2} \frac{(y_2 - y_3)(y_2 - y_1) + (x_2 - x_3)(x_2 - x_1)}{(x_3 - x_1)(y_2 - y_1) - (y_3 - y_1)(x_2 - x_1)} \right| \\ &= \left| \frac{a_3 - a_1}{2} \right| \left| 1 + i\frac{(y_2 - y_3)(y_2 - y_1) + (x_2 - x_3)(x_2 - x_1)}{(x_3 - x_1)(y_2 - y_1) - (y_3 - y_1)(x_2 - x_1)} \right|, \end{aligned}$$

as desired.

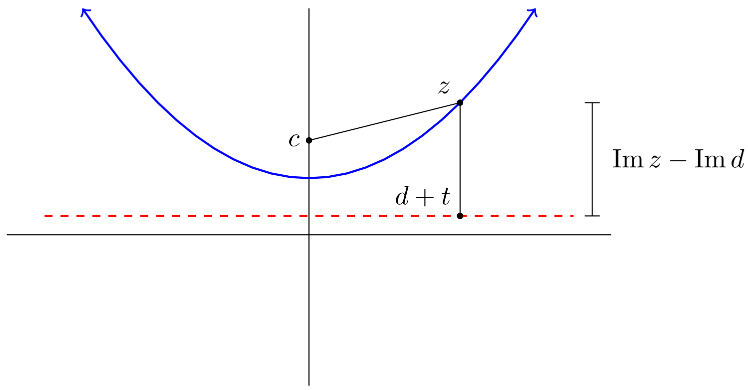
2.3.2 Write the equation of an ellipse, hyperbola, parabola in complex form.

Solution An ellipse is defined as follows: Given two foci c and $-c$, a point z is on the ellipse if the sum of its distance to c and $-c$ is $d \in \mathbb{R}$, which gives the equation

$$|z - c| + |z + c| = d.$$

For a hyperbola, given two foci, we want the magnitude of the distances to be constant:

$$\big||z - c| - |z + c|\big| = d.$$

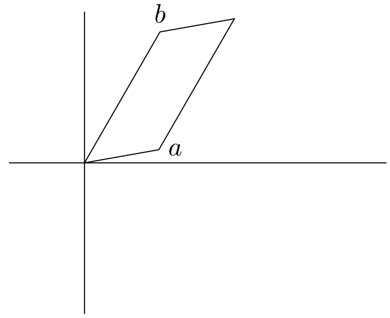


Lastly, for a parabola that opens upwards, we need a focus c and a line called the directrix parallel to the real axis, so it can be expressed as $d + t$, for $t \in \mathbb{R}$. A point of the parabola is equidistance from the focus and the directrix. So, the equation is given by

$$|z - c| = |\operatorname{Im} z - \operatorname{Im} d|.$$

2.3.3 Prove that the diagonals of a parallelogram bisect each other and that the diagonals of a rhombus are orthogonal.

Solution By shifting like in the other problems, we can assume without loss of generality that one of the vertices is at the origin.



The diagonals are then $d_1(t) = a + (b-a)t$ and $d_2(t) = (a+b)t$, for $0 \le t \le 1$, respectively. To show that they bisect each other, it suffices to show that they intersect at $t = 1/2$.

$$d_1\left(\frac{1}{2}\right) = a + \frac{b-a}{2} = \frac{a+b}{2} = d_2\left(\frac{1}{2}\right),$$

so the diagonals bisect each other.

The diagonals for a rhombus are the same, except we also know that $|a| = |b|$. The direction of diagonal 1 is given by $b-a$ and for diagonal 2, $a+b$. If we write $a = x+iy$ and $b = \alpha + i\beta$, we can associate the numbers with the following vectors:

$$\begin{aligned} b + a &\longleftrightarrow \begin{pmatrix} \alpha + x \\ \beta + y \end{pmatrix} \\ b - a &\longleftrightarrow \begin{pmatrix} \alpha - x \\ \beta - y \end{pmatrix}. \end{aligned}$$

Then their inner product is

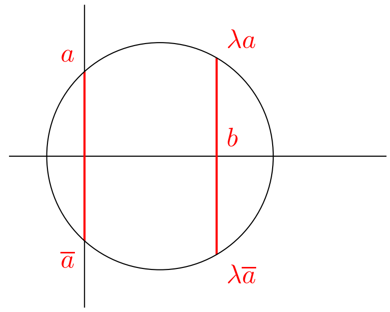
$$\begin{pmatrix} \alpha + x \\ \beta + y \end{pmatrix} \cdot \begin{pmatrix} \alpha - x \\ \beta - y \end{pmatrix} = \alpha^2 - x^2 + \beta^2 - y^2 = |y|^2 - |x|^2 = 0,$$

so the diagonals are orthogonal.

2.3.4 Prove analytically that the midpoints of parallel chords to a circle lie on a diameter perpendicular to the chords.

Solution We first fix two chords on a circle, and shift and rotate the circle so that the first chord lies directly on the imaginary axis and the origin is the midpoint of that chord.

The real axis contains the diameter by symmetry, since a and $\overline{a}$ are equidistant to the real line.



In the picture above, b is the midpoint of the second chord, and since the two chords are parallel, one of the endpoints of the chord is $b + \lambda a$ for some $\lambda \in \mathbb{R}$. By symmetry, the other endpoint must be $b + \lambda\overline{a}$, since the chord is parallel to the imaginary axis.

Since b is the midpoint of λa and $\lambda\overline{a} = -\lambda a$, we have that $\operatorname{Im} b = 0$ and b is a real number. Hence, the line passing through 0 and b is the real axis, so 0 and b lie on a diameter perpendicular to the chords.

2.3.5 Show that all circles that pass through a and $1/\overline{a}$ intersect the circle $|z| = 1$ at right angles.

Solution We can rotate a so that a is real, which means $1/\overline{a} = 1/a$ will be real also. So, assume without loss of generality that a is real.

A circle passing through a and $1/a$ must be equidistant to both those points, i.e., if the center of the circle has real part h,

$$h = \frac{1}{2}\left(a + \frac{1}{a}\right) = \frac{a^2+1}{2a}.$$

We can let the imaginary part of the center vary as $t \in \mathbb{R}$. Since a is on the circle, we must have that the radius is

$$r^2 = (a-h)^2 + t^2 = \left(\frac{a^2-1}{2a}\right)^2 + t^2.$$

Let $e^{i\theta}$ be on the unit circle and on the circle described above. Then the vector lying between $e^{i\theta}$ and the center of the circle is

$$e^{i\theta} - \left[\frac{a^2+1}{2a} + it\right].$$

It suffices to show that $e^{i\theta}$ and the vector above are orthogonal, so that the tangents at those points are also orthogonal, which is because the radius to a point on a circle is orthogonal to the tangent at that point.

We know that $e^{i\theta}$ lies on the circle passing through a, so it must satisfy

$$\begin{aligned}
\left|e^{i\theta} - \left[\frac{a^2+1}{2a} + it\right]\right|^2 &= R^2 \\
\left(\cos\theta - \frac{a^2+1}{2a}\right)^2 + (\sin\theta - t)^2 &= \left(\frac{a^2-1}{2a}\right)^2 + t^2 \\
\cos^2\theta - \frac{a^2+1}{a}\cos\theta + \frac{a^4+2a^2+1}{4a^2} + \sin^2\theta - 2t\sin\theta + t^2 &= \frac{a^4-2a^2+1}{4a^2} + t^2 \\
1 - \frac{a^2+1}{a}\cos\theta - 2t\sin\theta + 1 &= 0 \\
2 - \frac{a^2+1}{a}\cos\theta - 2t\sin\theta &= 0 \\
\frac{a^2+1}{a}\cos\theta + 2t\sin\theta &= 2.
\end{aligned}$$

Then the inner product between $e^{i\theta}$ and $e^{i\theta} - \left[\frac{a^2+1}{2a} + it\right]$ is given by

$$\begin{aligned}
\cos\theta\left(\cos\theta - \frac{a^2+1}{2a}\right) + \sin\theta(\sin\theta - t) &= \cos^2\theta - \frac{a^2+1}{2a}\cos\theta + \sin^2\theta - t\sin\theta \\
&= 1 - \frac{1}{2}\left(\frac{a^2+1}{a}\cos\theta + 2t\sin\theta\right) \\
&= 1 - \frac{1}{2}\cdot 2 \\
&= 0,
\end{aligned}$$

so the two circles intersect at right angles.

2.4.1 Show that z and z' correspond to diametrically opposite points on the Riemann sphere if and only if $z\overline{z'} = -1$.

Solution Let Z, Z' represent the stereographic projections of z, z', respectively.

"$\implies$"

Let $Z = -Z'$. If we write $Z = (x_1, x_2, x_3)$, then the projections onto $\mathbb{C}$ are given by

$$z = \frac{x_1 + ix_2}{1 - x_3} \quad \text{and} \quad z' = \frac{-x_1 - ix_2}{1 + x_3}.$$

Since $x_1^2 + x_2^2 + x_3^2 = 1$, we have

$$\begin{aligned}
z\overline{z'} &= \frac{x_1 + ix_2}{1 - x_3}\,\frac{-x_1 + ix_2}{1 + x_3} \\
&= \frac{-x_1^2 - x_2^2}{x_1^2 + x_2^2} \\
&= -1,
\end{aligned}$$

as desired.

"$\impliedby$"

Let $z\overline{z'} = -1$. Note that $\overline{z\overline{z'}} = \overline{z}z' = \overline{-1} = -1$ and that $|zz'|^2 = z\overline{z'}\overline{z}z' = 1$.

Then the chordal distance between the two points is given by

$$\begin{aligned} d^2(z,z') &= \frac{4|z-z'|^2}{(1+|z|^2)(1+|z'|^2)} \\ &= \frac{4(|z|^2+|z'|^2-z\overline{z'}-\overline{z}z')}{1+|z|^2+|z'|^2+|zz'|^2} \\ &= \frac{4(2+|z|^2+|z'|^2)}{2+|z|^2+|z'|^2} \\ &= 4, \end{aligned}$$

so the chordal distance between the two points is 2. Hence, they must be diametrically opposite points on the Riemann sphere, since the sphere has diameter 2.

2.4.2 A cube has its vertices on the sphere S and its edges parallel to the coordinate axes. Find the stereographic projections of the vertices.

Solution Note that a diagonal on the cube makes an angle of $\pi/4$ with every axis. Hence, the coordinates of the vertex in the first octant are given by

$$\left(\sin\frac{\pi}{4}\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\sin\frac{\pi}{4}, \cos\frac{\pi}{4}\right) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right).$$

Flipping the signs on any coordinate gives the remaining vertices, so we will consider

$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right),$$

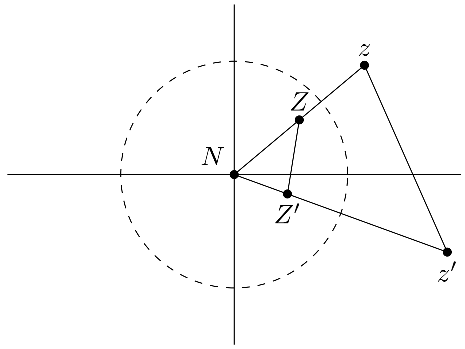
but it should be understood that the coordinates may not be all the same sign. Then the stereographic projection is given by

$$z = \frac{\pm\frac{1}{2} \pm i\frac{1}{2}}{1 \pm \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{2}\frac{\pm 1 \pm i}{2\sqrt{2} \pm 2}$$

gives us all the vertices.

2.4.4 Let Z, Z' denote the stereographic projections of z, z', and let N be the north pole. Show that the triangles NZZ' and Nzz' are similar, and use this to derive (28).

Solution Note that N, Z, z lie on the same line, by the definition of the stereographic projection. Similarly, N, Z, z' also lie on the same (but different from the first) line.



The bird's-eye view of the Riemann sphere and the relevant points shows that that $\angle ZNZ' = \angle zNz'$, so by the SAS similarity theorem, $\triangle NZz \sim \triangle NZ'z'$ if we can show that two of the sides correspond to each other.

The claim is that $NZ \sim Nz'$ and $Nz \sim NZ'$. We can show that by showing that

$$\frac{NZ}{Nz'} = \frac{NZ'}{Nz}.$$

We'll write $Z = (x_1, x_2, x_3)$ and $Z' = (x_1', x_2', x_3')$.

$$\begin{aligned}
\frac{NZ}{Nz'} &= \frac{\|N - Z\|}{\left\| N - \left(\dfrac{x_1'}{1 - x_3'}, \dfrac{x_2'}{1 - x_3'}, 0 \right) \right\|} \\
&= \frac{x_1^2 + x_2^2 + (1 - x_3)^2}{\left(\dfrac{x_1'}{1 - x_3'} \right)^2 + \left(\dfrac{x_1'}{1 - x_3'} \right)^2 + 1} \\
&= (1 - x_3)^2 \frac{|z|^2 + 1}{|z'|^2 + 1}
\end{aligned}$$

We know that

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1},$$

so substituting in, we get

$$\left(\frac{NZ}{Nz'} \right)^2 = \left(\frac{2}{|z|^2 + 1} \right)^2 \frac{|z|^2 + 1}{|z'|^2 + 1} = \frac{4}{(1 + |z|^2)(1 + |z'|^2)}.$$

A similar calculation yields

$$\left(\frac{NZ'}{Nz} \right)^2 = \left(\frac{2}{|z'|^2 + 1} \right)^2 \frac{|z'|^2 + 1}{|z|^2 + 1} = \frac{4}{(1 + |z'|^2)(1 + |z|^2)} = \left(\frac{NZ}{Nz'} \right)^2,$$

so we have shown that the triangles are similar with the ratio

$$\frac{2}{\sqrt{(1 + |z'|^2)(1 + |z|^2)}}.$$

Hence, the chordal distance is given by

$$\|Z - Z'\| = |z - z'| \cdot \frac{2}{\sqrt{(1 + |z'|^2)(1 + |z|^2)}} = \frac{2|z - z'|}{\sqrt{(1 + |z'|^2)(1 + |z|^2)}},$$

as desired.

2.4.5 Find the radius of the spherical image of the circle in the plane whose center is a and radius R.

Solution We can write a point on the circle as $a + Re^{i\theta}$, for $0 \le \theta < 2\pi$.

By rotating a, we can rotate the circle so that its center lies on the real axis, so without loss of generality, assume that a is a real number.

Then the farthest points on the projected circle must be $a + R$ and $a - R$. Their chordal distance is given by

$$d(a + R, a - R) = \frac{2|2R|}{\sqrt{(1 + |a + R|^2)(1 + |a - R|^2)}} = \frac{4R}{\sqrt{(1 + |a + R|^2)(1 + |a - R|^2)}}$$