1 Let $(F, +, \cdot, <)$ be an ordered field with at least two elements and let 1 denote the identity for multiplication. Show that the equation

$$x^2 = 1$$

has exactly two solutions in F.

Solution Notice that $x = 1$ and $x = -1$ are both solutions to the equation. Since F is ordered, we have that $-1 < 0 < 1 \implies 1 \neq -1$ in F. We will now show that these are the only solutions to the equation.

Suppose there exists a such that $a^2 = 1$ with $a \neq 1$ and $a \neq -1$. Then by tracheotomy, there are 4 cases:

$a < -1$:

$a < -1 \implies -1 \cdot a < a \cdot a \implies a^2 > -a > 1.$

$-1 < a < 0$:

$-1 < a \implies a \cdot a < -a < 1 \implies a^2 < 1.$

$0 < a < 1$:

$a \cdot a < 1 \cdot a \implies a^2 < 1.$

$1 < a$:

$1 \cdot a < a \cdot a \implies 1 < a < a^2.$

If $a = 0$, then $a^2 = 0 \neq 1$. Thus, in all cases, $a^2 \neq 1$. Hence, the only possible solutions to $x^2 = 1$ are 1 and -1.

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2 Fix $n \geq 1$. Show that the set of all subsets of $\mathbb{N}$ with n elements is countable.

Solution Let $\mathbf{A} = \{N \subseteq \mathbb{N} \mid |N| = n\}$. We will show that $\mathbf{A} \sim \mathbb{N}$ by constructing two injective functions $f\colon \mathbf{A} \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbf{A}$.

Let $A \in \mathbf{A}$. We define $f(A)$ via the following process:

First order the elements of A so that $A = \{a_1, \ldots, a_n\}$ with $a_1 < \cdots < a_n$. Then $f(A) \coloneqq a_1 0 a_2 0 \ldots 0 a_{n-1} 0 a_n$. I.e., concatenate the numbers and place 0's in between each number. This is clearly an injection.

Let $N \in \mathbb{N}$. Then $g(N) \coloneqq \{N, N+1, \cdots, N + (n-1)\}$. This, clearly, is also an injection.

Hence, by Schröder–Bernstein, $\mathbf{A} \sim \mathbb{N} \iff$ the set of all subsets of $\mathbb{N}$ with n elements is countable.

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3 Let (X, d) be a metric space. Let A be a subset of X and let A' denote the set of accumulation points of A. Show that A' is closed.

Solution Since $A' \subseteq \overline{A'}$, it suffices to show that $\overline{A'} \subseteq A'$.

Let $a \in \overline{A'}$ and fix $r > 0$. Then by definition, $B_r(a) \cap A' \neq \emptyset$. Hence, pick $x \in B_r(a) \cap A'$.

Let $\rho < r - d(a, x)$. As $x \in A'$, $B_\rho(x) \cap A \setminus \{a\} \neq \emptyset$. Hence, choose $y \in B_\rho(x) \cap A \setminus \{a\}$. Then

$$d(a, y) \leq d(a, x) + d(x, y) < d(a, x) + \rho = r.$$

Thus, $y \in B_r(a)$ and $y \neq a$, so $B_r(a) \cap A \setminus \{a\} \neq \emptyset$. We can do this for all $r > 0$, so by definition, $a \in A'$. Thus, $\overline{A'} = A' \iff A'$ is closed.

4 Let A and B be two non-empty sets of real numbers. Assume that A is open. Prove that the set

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\}$$

is open.

Solution Fix $b \in B$. Then $A + b = \{a + b \mid a \in A\}$ is clearly open. Thus,

$$A + B = \bigcup_{b \in B} (A + b)$$

is open, as an infinite union of open sets is open.

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5 Consider the complete metric space (ℓ^∞, d_∞) defined as follows:

$$\ell^\infty = \left\{ \{x_n\}_{n\geq 1} \subseteq \mathbb{R} \mid \sup_{n\geq 1} |x_n| < \infty \right\}$$

and for two points $x = \{x_n\}_{n\geq 1} \in \ell^\infty$ and $y = \{y_n\}_{n\geq 1} \in \ell^\infty$, the distance is given by

$$d_\infty(x, y) = \sup_{n\geq 1} |x_n - y_n|.$$

Let

$$A = \{x \in \ell^\infty \mid \sup_{n\geq 1} n|x_n| \leq 1\}.$$

Show that A is a closed subset of ℓ^∞.

Solution Let $a = \{a_n\}_{n\geq 1} \in \overline{A}$. We will construct a sequence that converges to a.

As $a \in \overline{A}$, $B_1(a) \cap A \neq \emptyset$. Hence, choose $a_1 \in B_1(a) \cap A \neq \emptyset$.

Similarly, $B_{1/2}(a) \cap A \neq \emptyset$. Pick $a_2 \in B_{1/2}(a) \cap A$.

Proceeding inductively, we create a sequence $\{a^{(k)}\}_{k\geq 1}$ so that $d(a, a^{(k)}) = \sup_{n\geq 1} |a_n - a_n^{(k)}| < \frac{1}{k}$. This sequence converges to a trivially by the Archimedean principle. We now show that $a \in A$.

$$\begin{aligned}
|a_n - a_n^{(k)}| &< \frac{1}{k} \\
a_n - a_n^{(k)} &< \frac{1}{k} \\
na_n - na_n^{(k)} &< \frac{n}{k} \\
na_n < \frac{n}{k} + na_n^{(k)} &\leq \frac{n}{k} + 1
\end{aligned}$$

This inequality holds for all k, so taking $k \to \infty$ and holding n fixed, we have

$$na_n \leq 1 \implies \sup_{n\geq 1} |na_n| \leq 1$$

as desired.

6 Let (X, d) be a metric space. For $a, b \in X$ we write $a \sim b$ if there exists a connected subset A of X such that $\{a, b\} \subseteq A$.

a. Show that $\sim$ defines an equivalence relation on X.

b. For $a \in X$, let C_a denote the equivalence class of a, that is,

$$C_a = \{b \in X \mid a \sim b\}.$$

Show that C_a is a connected set. Show that C_a is the largest connected subset of X that contains a.

c. Given $a \in X$, show that C_a is a closed set.

d. Given $a, b \in X$ such that a is not equivalent to b, show that C_a and C_b are separated sets.

Solution a. i. Symmetry: $\{a\} \subseteq X$ is clearly connected and contains a. Thus, $a \sim a$.

ii. Reflexivity: Let $a \sim b \iff \exists A$ connected s.t. $\{a, b\} \subseteq A$. A also contains the set $\{b, a\}$, so $b \sim a$.

iii. Transitivity: Let $a \sim b$ and $b \sim c$. By definition,

$$\exists A \text{ connected s.t. } \{a, b\} \subseteq A$$
$$\exists B \text{ connected s.t. } \{b, c\} \subseteq A$$

Since A and B are connected and $b \in A \cap B \implies A \cap B \neq \emptyset$, then by a theorem, $A \cup B$ is connected. Moreover, $A \cup B$ contains the set $\{a, c\}$. So, $a \sim c$ as desired.

b. Let $x, y \in C_a$. By definition, $a \sim x$ and $a \sim y$. By reflexivity and transitivity, we have that $x \sim y$. Hence, any pair of points in C_a can be joined by a connected subset of C_a, so by a theorem, C_a is connected.

c. By a proposition proved in class, since C_a is connected, $\overline{C_a} \subseteq \overline{C_a}$ is also connected. Thus for all $x \in \overline{C_a}$, $\{a, x\} \subseteq \overline{C_a} \iff a \sim x \iff x \in C_a$. Hence, C_a is closed.

d. Suppose otherwise, and that C_a and C_b are connected. Since both C_a and C_b are closed and connected, $C_a \cap C_b \neq \emptyset$. Let $c \in C_a \cap C_b$. $c \in C_a \iff a \sim c$ and $c \in C_b \iff b \sim c$. Then by transitivity and reflexivity, $a \sim b$, which is a contradiction. Hence, C_a and C_b are separated.

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7 Suppose A and B are compact subsets of a metric space. Show that $A \cap B$ and $A \cup B$ are also compact.

Solution Since A and B are compact, they are both closed. Thus, $A \cap B$ is also closed. Since $A \cap B \subseteq A$ compact, $A \cap B$ is compact by a proposition proved in class.

Let $\{G_n\}_{n \geq 1}$ be an open cover of $A \cup B$.

Then $\{G_n\}_{n \geq 1}$ is an open cover of A. Since A is compact, there exists $n_A \geq 1$ such that $A \subseteq \bigcup_{i=1}^{n_A} G_i$.

Similarly, $\{G_n\}_{n \geq 1}$ is an open cover of B. Since B is compact, there exists $n_B \geq 1$ such that $B \subseteq \bigcup_{i=1}^{n_B} G_i$.

Take $n = \max\{n_A, n_B\}$. Then

$$\left.\begin{aligned} A &\subseteq \bigcup_{i=1}^{n} G_i \\ B &\subseteq \bigcup_{i=1}^{n} G_i \end{aligned}\right\} \implies A \cup B \subseteq \bigcup_{i=1}^{n} G_i$$

Hence, any open cover of $A \cup B$ admits a finite open cover, so $A \cup B$ is compact.