1 Suppose that six distinct integers are selected from the set $\{1,2,3,4,5,6,7,8,9,10\}$. Prove that at least two of the six have a sum of 11.

Solution Let $x_1,\dots,x_6$ be six distinct integers from that set, and consider $11-x_1,\dots,11-x_6$, which lie in the same set. Then $x_1,\dots,x_6,11-x_1,\dots,x_6$ are 12 numbers in a set of 10 numbers, so by the pigeonhole principle, adding $11-x_1,\dots,11-x_6$ must have added 2 duplicates to our list. Thus, there exist i_1,i_2,j_1,j_2 all distinct so that

$$x_{i_1}=11-x_{j_1} \implies x_{i_1}+x_{j_1}=11 \quad \text{and} \quad x_{i_2}=11-x_{j_2} \implies x_{i_2}+x_{j_2}=11,$$

as required.

---

2 Prove that for each positive integer n, there exists a positive integer which, when expressed in decimal, has at most n digits, all of which are either 0 or 1 and is a multiple of n.

Solution Let $n\geq 1$, and consider the set $\{1,11,\dots,11\dots1\}$ with n elements, which we label $\{x_1,\dots,x_n\}$. Now consider their remainder when divided by n: $\{r_1,\dots,r_n\}$. There are $n-1$ possible distinct remainders: $1,2,\dots,n-1$, so by the pigeonhole principle, there exist distinct $i<j$ so that $r_i=r_j$.

Thus, $x_j-x_i>0$ is divisible by n, since the remainder of the difference is the difference of the remainders by Euclidean division. Moreover,

$$x_j-x_i=\underbrace{11\dots1}_{j-i}\underbrace{00\dots0}_{i},$$

which is what we wanted.

---

3 The goal of this question is to prove the following: *Given a group of* 10 *persons, there are at least two such that the sum or difference of their ages is divisible by* 16.

Let $a_1,\dots,a_{10}$ be the ages, and let $r_i=a_i \mod 16$ (the remainder of a_i when divided by 16). Let

$$s_i=\begin{cases} r_i & \text{if } r_i\leq 8 \\ 16-r_i & \text{if } r_i>8. \end{cases}$$

a. Show that $s_1,\dots,s_{10}$ range in value from 0 to 8.

b. Explain why $s_j=s_k$ for some $j\neq k$.

c. Suppose that $s_j=s_k$ for some $j\neq k$. Explain why if $s_j=r_j$ and $s_k=r_k$, or $s_j=16-r_j$ and $s_k=16-r_k$, then 16 divides a_j-a_k.

d. Show that if the conditions of part (c) fail, then 16 divides a_j+a_k.

Solution a. If $0\leq r_i\leq 8$, then $0\leq s_i=r_i\leq 8$. On the other hand, if $r_i>8$, then $8<r_i\leq 15$, so $-15\leq -r_i<-8 \implies 1\leq 16-r_i<8$, so s_i must range between 0 and 8.

b. There are 10 s_i which all lie in a set of 9 numbers, so by the pigeonhole principle, there must be $i\neq j$ so that $s_j=s_k$.

c. By Euclidean division, we know that the remainder of the difference is the difference of the remainders. Thus, if $r_j=r_k$, then $a_j-a_k=r_j-r_k \mod 16$, which is 0. Similarly, if $16-r_j=16-r_k$, then $r_j=r_k$, and the same reasoning holds.

d. If the conditions fail to hold, then it must be that $s_j=r_j$ but $s_k=16-r_k$ (or vice versa). Then

$$r_j=16-r_k \implies r_j+r_k=16,$$

so

$$a_j+a_k=r_j+r_k \mod 16=0,$$

so 16 divides a_j+a_k in this case.

4 Let $X = \{1, 2, \ldots, n\}$ and let $f \colon X \to X$ be a one-to-one and onto function from X to itself. For any positive integer k, let $f^k = f \circ f \circ \cdots \circ f$ be the k-fold composition of f with itself.

a. Prove that there are distinct positive integers i and j such that $f^i(x) = f^j(x)$ for all $x \in X$.

b. Prove that there is some positive integer k for which $f^k(x) = x$ for all $x \in X$.

Solution a. There are $n!$ bijections $g \colon X \to X$. Indeed, we have n choices to map 1 to, $n-1$ choices to map 2 to, etc., so by the multiplication principle, we get $n!$ possible ways to map X to itself bijectively. In particular, there are finitely many bijections.

Thus, there must be $i < j$ so that $f^i(x) = f^j(x)$ for all x. Otherwise, f^k would be distinct from $f^1, \ldots, f^k$ for all $k \geq 1$, which implies that there are infinitely many bijections.

b. Since f is a bijection, f^{-1} exists. Notice that $f^{-k} = (f^k)^{-1}$ if $k \geq 1$.

Let $i < j$ be as in part (a), and apply f^{-i} to both sides:

$$f^i(x) = f^j(x) \ \forall x \implies f^{j-i}(x) = x \ \forall x,$$

as required.

---

5 Let S_n denote the number of n-bit strings which do not contain the substring 010.

a. Compute S_1, S_2, S_3, and S_4.

From now on, assume that $n \geq 4$.

b. Show that the number of n-bit strings that start with 1 and do not contain the substring 010 is exactly S_{n-1}.

c. For any integer $1 \leq k \leq n-3$, show that the number of n-bit strings that start with exactly k 0's and do not contain the substring 010 is exactly $S_{n-(k+2)}$.

d. Show that there are exactly three n-bit strings that start with at least $n-2$ 0's and don't contain the substring 010.

e. By combining parts (b) – (d), derive the recurrence relation

$$S_n = S_{n-1} + S_{n-3} + S_{n-4} + S_{n-5} + \cdots + S_2 + S_1 + 3$$

for any $n \geq 4$.

f. Replace n by $n-1$ in the equation from part (e) to get a formula for S_{n-1}. Subtract this from the formula for S_n to derive the recurrence relation

$$S_n = 2S_{n-1} - S_{n-2} + S_{n-3}$$

for $n \geq 5$.

Solution a. $S_1 = 2$, $S_2 = 2^2$, $S_3 = 2^3 - 1 = 7$, and $S_4 = 2^4 - 4 = 12$ (0100, 0101, 0010, and 1010 are the only bad strings).

b. If an n-bit string starts with 1, then 010 can only appear in the remaining $n-1$ bits. Thus, there are S_{n-1} strings of this form.

c. The $(k+1)$-th bit must be 1, since there are exactly k 0's. The next bit must also be a 1, or else we get a 010. Thus, the $(k+2)$-th and next bits can be whatever they want, and the problem reduces to the number of $(n-(k+2))$-bit strings don't have a 010 in it. Thus, the number of strings is $S_{n-(k+2)}$.

d. We just need to look at the last 2 bits. The possibilities are 00, 01, 10, and 11, but 10 isn't allowed, since the bit right before it is 0, so there are exactly 3 strings here.

e. If the string starts with 1, then there are no new restrictions for the next $n-1$ bytes, so we have S_{n-1} strings in this case.

If the string starts with k 0's with $1 \leq k \leq n-3$, then from (c), we know that there are $S_{n-(k+2)}$ possibilities.

If $k = n-2$, then there are exactly 3 possibilities. Thus,

$$S_n = S_{n-1} + \sum_{k=1}^{n-3} S_{n-(k+2)} + 3 = S_{n-1} + S_{n-3} + S_{n-4} + \cdots + S_2 + S_1 + 3,$$

as desired.

f. We have

$$S_{n-1} = S_{n-2} + S_{n-4} + \cdots + S_2 + S_1 + 3.$$

Subtracting, we get

$$S_n - S_{n-1} = S_{n-1} - S_{n-2} + S_{n-3} \implies S_n = 2S_{n-1} - S_{n-2} + S_{n-3},$$

as required.

---

6 The sequence $g_1, g_2, \ldots$ is defined by the recurrence relation $g_n = g_{n-1} + g_{n-2} + 1$ for $n \geq 3$ and the initial conditions $g_1 = 1$ and $g_2 = 3$. By using mathematical induction or otherwise, show that $g_n = 2f_{n+1} - 1$ for all $n \geq 1$, where $f_1, f_2, \ldots$ is the Fibonacci sequence.

Solution Set $h_n = 2f_{n+1} - 1$. By definition of the Fibonacci sequence, $f_2 = 1$ and $f_3 = 2$, so $h_1 = 1$ and $h_2 = 3$, which means that h_n agrees with the initial conditions on g_n.

Next, we have

$$h_n = 2f_{n+1} - 1 = 2(f_n + f_{n-1}) - 1 = (2f_n - 1) + (2f_{n-1} - 1) + 1 = h_{n-1} + h_{n-2} + 1,$$

so h_n satisfies the same recurrence relation as g_n. Thus, $h_n = g_n$ for all $n \geq 1$, so $g_n = 2f_{n+1} - 1$.

---

7 Consider the formula

$$u_n = \begin{cases} u_{3n+1} & \text{if n is odd and greater than 1} \\ u_{n/2} & \text{if n is even and greater than 1} \end{cases}$$

and the initial condition $u_1 = 1$. Explain why this formula is *not* a recurrence relation. A long standing open problem in number theory states that for every positive integer n, u_n is well-defined and equal to 1. Compute u_n for $n = 2, 3, \ldots, 7$.

Solution This is not a recurrence relation because u_n depends on large indices; namely, u_{3n+1}, so u_n is not a recurrence relation.

$u_2 = u_1 = 1$.

It's easy to see that $u_{2^n} = u_{2^{n-1}} = \cdots = u_2 = u_1 = 1$. Then $u_5 = u_{16} = 1$, so $u_{10} = u_5 = 1$. Thus, $u_3 = u_{10} = 1$.

Next, $u_6 = u_3 = 1$.

Lastly, $u_7 = u_2 = 1$, so $u_1 = u_2 = \cdots = u_7 = 1$.

8 Solve the given recurrence relations for the given initial conditions:

a. $a_n = 6a_{n-1} - 8a_{n-2}$; $a_0 = 1$, $a_1 = 0$.

b. $L_n = L_{n-1} + L_{n-2}$; $L_1 = 1$, $L_2 = 3$.

c. $9a_n = 6a_{n-1} - a_{n-2}$; $a_0 = 6$, $a_1 = 5$.

d. $S_n = \dfrac{S_{n-1} + S_{n-2}}{2}$; $S_1 = 0$, $S_2 = 1$.

Solution a. The characteristic polynomial is

$$x^2 - 6x + 8 = (x-4)(x-2),$$

so we'll be looking for solutions in the form $a_n = b \cdot 2^n + c \cdot 4^n$:

$$\begin{aligned} b + c &= 1 \\ 2b + 4c &= 0. \end{aligned}$$

By inspection, $b = 2$ and $c = -1$, so our solution is $a_n = 2^{n+1} - 2^{2n}$.

b. Here, we get the polynomial $x^2 - x - 1 = 0$, and the roots are

$$\varphi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \psi = \frac{1-\sqrt{5}}{2}.$$

So, we need to look for solutions of the form $L_n = b\varphi^n + c\psi^n$. We'll shift the indices by 1 for convenience. Then the initial conditions give us

$$\begin{aligned} b + c &= 1 \\ b\varphi + c\psi &= 3. \end{aligned}$$

Thus,

$$b = \frac{3+\psi}{\varphi+\psi} \quad \text{and} \quad c = \frac{\psi - 3}{\varphi + \psi},$$

and that gives the solution

$$L_{n+1} = \left(\frac{3+\psi}{\varphi+\psi}\right)\varphi^n + \left(\frac{\psi-3}{\varphi+\psi}\right)\psi^n \implies L_n = \left(\frac{3+\psi}{\varphi+\psi}\right)\varphi^{n-1} + \left(\frac{\psi-3}{\varphi+\psi}\right)\psi^{n-1}.$$

c. The polynomial here is $9x^2 - 6ax + 1 = (3x-1)^2$, so we need to look for solutions in the form

$$a_n = b\left(\frac{1}{3}\right)^n + cn\left(\frac{1}{3}\right)^n.$$

By inspection, $b = 6$ and $c = 9$.

d. Here, $2x^2 - x - 1 = (x-1)(2x+1)$, so we will look for solutions of the form $S_n = a + b(-1/2)^n$. By inspection, $a = 2/3$ and $b = -2/3$, which gives us our solution:

$$S_n = \frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^{n-1}.$$

---

9 Solve the recurrence relation $c_n = 2 + \sum_{i=1}^{n-1} c_i$ for $n \geq 2$ for the inital condition $c_1 = 1$.

Solution Notice that

$$c_{n+1} - c_n = 2 + \sum_{i=1}^{n} c_i - 2 - \sum_{i=1}^{n-1} c_i = c_n \implies c_{n+1} = 2c_n,$$

so $c_n = a \cdot 2^n$. Notice that $c_2 = 3$, so $a = 3/4$, and hence $c_n = 3 \cdot 2^{n-2}$ for $n \geq 2$. This works:

$$2 + \sum_{i=1}^{n-1} c_i = 2 + 1 + \sum_{i=2}^{n-1} 3 \cdot 2^{i-2} = 3 + 3 \cdot \frac{1 - 2^{n-2}}{1-2} = 3 - 3 + 3 \cdot 2^{n-2} = c_n,$$

as required.