1 Let $f\colon [0,\infty) \to [0,\infty)$ be continuous with $f(0) = 0$. Show that if

$$f(t) \le 1 + \frac{1}{10} f(t)^2 \qquad \text{for all } t \in [0,\infty)\,,$$

then f is uniformly bounded throughout $[0,\infty)$.

Solution Notice that

$$f(t) \le 1 + \frac{1}{10} f(t)^2 \iff \Big(f(t) - (5+\sqrt{15})\Big)\Big(f(t) - (5-\sqrt{15})\Big) \ge 0 \quad \forall t \in [0,\infty)$$

So, we must have either $f(t) \le 5 - \sqrt{15} > 0$ or $f(t) \ge 5 + \sqrt{15} > 0$.

Since f is continuous and $[0,\infty)$ is connected, $f([0,\infty))$ must also be connected. In particular, since it is a subset of $\mathbb{R}$, $f([0,\infty))$ must be an interval. The interval must also contain 0 since $f(0) = 0$, so we must have $0 \le f(t) \le 5 - \sqrt{15}$ for all $t \in [0,\infty)$. Hence, f is uniformly bounded on $[0,\infty)$.

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2 Show that the function

$$H(x,y) = x^2 + y^2 + |x-y|^{-1}$$

achieves its global minimum somewhere on the set $\big\{(x,y) \in \mathbb{R}^2 \mid x \ne y\big\}$.

Solution Consider the change in variables $u = x - y$ and $v = x + y$. Then we can rewrite H as $H(u,v) = u^2 + v^2 + \frac{1}{|u|}$. Then the problem is equivalent to showing that $H(u,v)$ achieves its global minimum on the set

$$\big\{(u,v) \in \mathbb{R}^2 \mid u \ne 0\big\}.$$

One value of $H(u,v)$ is $H(1,1) = 3$.

Fix v. Notice that $f(u) \coloneqq u^2 + \frac{1}{|u|} \le H$, and that $\lim_{u\to 0} f(u) = \infty$. Then by definition, there exists $\delta_0 > 0$ such that $f(u) > 3$ whenever $|u| < \delta_0$. We can discard these values of u on the domain of H since they give values of H greater than $H(1,1)$, so they cannot be global minima. So, we have reduced the set to $\big\{(u,v) \in \mathbb{R}^2 \mid |u| \ge \delta_0\big\}$.

Similarly, $\lim_{|u|\to\infty} f(u) = \infty$. So, there exists $\delta_1 > 0$ such that $f(u) > 3$ whenever $|u| > \delta_1$. We can discard these values also, so our set is reduced to $\big\{(u,v) \in \mathbb{R}^2 \mid \delta_0 \le |u| \le \delta_1\big\}$.

Next, fix a value of u so that $\delta_0 \le |u| \le \delta_1$. Then consider $g(v) \coloneqq v^2 \le H(u,v)$. Then as $\lim_{|v|\to\infty} g(v) = \infty$, there exists $\delta_2 > 0$ such that $g(v) > 3$ whenever $|u| > \delta_2$. We can discard these values of v as candidates for global minima since they are greater than $H(1,1) = 3$. So, our set is reduced to

$$\Omega = \big\{(u,v) \in \mathbb{R}^2 \mid \delta_0 \le |u| \le \delta_1,\ |v| \le \delta_2\big\}.$$

Finally, this remaining set is compact, since it is a closed and bounded subset of $\mathbb{R}^2$. As H is the sum of functions continuous on their domains, H itself is continuous, so $H(\Omega)$ is compact. Hence, there exists some (u_0, v_0) such that

$$H(u_0, v_0) = \inf_{(u,v)\in\Omega} H(u,v) = \inf_{(u,v)\in\mathbb{R}^2 \mid u\ne 0} H(u,v)$$

so it indeed achieves it global minimum somewhere on the set $\big\{(u,v) \in \mathbb{R}^2 \mid u \ne 0\big\}$.

3 Let $a, b \in \mathbb{R}$ with $a < b$ and let $f\colon [a,b] \to [a,b]$ be continuous. Show that there exists $x_0 \in [a,b]$ such that $f(x_0) = x_0$.

Solution Consider the function $g\colon [a,b] \to \mathbb{R}$ with $g(x) = f(x) - x$, which is continuous on $[a,b]$ since both $f(x)$ and x are.

Suppose there is no such $x_0 \in [a,b]$ such that $f(x_0) = x_0$. Then there is no $x \in [a,b]$ such that $g(x) = 0$. Since $[a,b]$ is connected, g has the Darboux property. Hence, $g([a,b])$ must contain all positive or all negative numbers. Otherwise, the Darboux property implies that there exists some value such that $g(x_0) = 0 \iff f(x_0) = x_0$, which is a contradiction.

Assume that $g(x) > 0$ for all $x \in [a,b]$. Then $g(b) = f(b) - b > 0 \implies f(b) > b$. But the image of f must be a subset of $[a,b]$, which is a contradiction. So, $g(x) < 0$ for all $x \in [a,b]$.

But if $g(x) < 0$ for all $x \in [a,b]$, $g(a) = f(a) - a < 0 \implies f(a) < a \implies f(a) \notin [a,b]$, which is a contradiction.

Thus, there must exist some $x_0 \in [a,b]$ such that $g(x_0) = 0 \iff f(x_0) = x_0$.

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4 Let $f, g\colon [-1,1] \to [-1,1]$ be defined as follows:

$$f(x) = \begin{cases} \frac{x-1}{2}, & x \in [-1,0] \\ x - \frac{1}{2}\sin(\frac{\pi}{x}), & x \in \left(0, \frac{1}{2}\right] \\ x, & x \in [\frac{1}{2}, 1] \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1-x}{2}, & x \in [-1,0] \\ -x - \frac{1}{2}\sin(\frac{\pi}{x}), & x \in \left(0, \frac{1}{2}\right] \\ -x, & x \in [\frac{1}{2}, 1] \end{cases}$$

Let A and B denote the graphs of f and g, respectively, that is,

$$A = \{(x, f(x)) \mid x \in [-1,1]\} \quad \text{and} \quad B = \{(x, g(x)) \mid x \in [-1,1]\}.$$

a. Show that $A \cap B = \emptyset$ and $\{(-1,-1), (1,1)\} \subseteq A$ and $\{(-1,1), (1,-1)\} \subseteq B$.

b. Prove that A and B are connected subsets of $[-1,1] \times [-1,1]$.

Solution a. Consider the function $h\colon [-1,1] \to [-1,1]$ with $h(x) = f(x) - g(x)$. Then

$$h(x) = \begin{cases} x - 1 & x \in [-1,0] \\ 2x & x \in \left(0, \frac{1}{2}\right] \\ 2x & x \in [\frac{1}{2}, 1] \end{cases} = \begin{cases} x - 1 & x \in [-1,0] \\ 2x & x \in (0,1] \end{cases}$$

Note that $A \cap B = \emptyset \iff \nexists x \in [-1,1]\ f(x) = g(x) \iff \nexists x \in [-1,1]\ h(x) = 0$.

$$\begin{aligned} x - 1 = 0 &\implies x = 1 \notin [-1,0] \\ 2x = 0 &\implies x = 0 \notin (0,1] \end{aligned}$$

Hence, there are no solutions to $h(x) = 0 \implies A \cap B = \emptyset$.

$$\begin{aligned} f(-1) = \frac{-1-1}{2} = -1 &\implies (-1,1) \in A \\ f(1) = 1 &\implies (1,1) \in A \\ g(-1) = \frac{1-(-1)}{2} = 1 &\implies (-1,1) \in B \\ g(1) = -1 &\implies (1,-1) \in B \end{aligned}$$

b. Consider an arbitrary continuous function $f\colon [a,b] \to \mathbb{R}$ and the function $g\colon [a,b] \to \mathbb{R}^2$ with $g(x) = (x, f(x))$. We claim that g is also continuous.

Let $\varepsilon > 0$ and $c \in [a,b]$. Then there exists $\delta_c > 0$ such that for all $x \in [a,b]$ such that $|x - c| < \delta$, we have $|f(x) - f(c)| < \frac{\varepsilon}{2}$.

Choose $\delta = \min\{\delta_c, \frac{\varepsilon}{2}\}$. Then if $|x - c| < \delta$,

$$d(g(x), g(c)) = \sqrt{(x-c)^2 + (f(x) - f(c))^2} < \varepsilon.$$

Hence, g is continuous on $[a,b]$.

A is connected:

Let $F\colon [-1,1] \to \mathbb{R}^2$ with $F(x) = (x, f(x))$. Then $F([-1,1]) = A$.

F is continuous on $[-1,0]$ connected, so $F([-1,0])$ is also connected. The same argument applies to the intervals $\left(0, \frac{1}{2}\right]$ and $[\frac{1}{2}, 1]$.

Note that $(0, -\frac{1}{2}) \in F([-1,0])$. Also, if $x_n = \frac{2}{1+4n}$, we have that $(x_n, \sin(\frac{\pi}{x_n})) \xrightarrow{n\to\infty} (0, -\frac{1}{2}) \implies F([-1,0]) \cap \overline{F(\left(0, \frac{1}{2}\right])} \neq \emptyset$, so they are not separated. Since $F([-1,0])$ and $F(\left(0, \frac{1}{2}\right])$ are both connected, we have that $F([-1,0]) \cup F(\left(0, \frac{1}{2}\right])$ is connected.

Lastly, note that $F(\frac{1}{2})$ is contained in both $F([-1,0]) \cup F(\left(0, \frac{1}{2}\right])$ and $F([\frac{1}{2}, 1])$. Since both sets are connected and they intersect at a point, their union is connected. Hence, $F([-1,1]) = A$ is connected.

B is connected

A similar argument to the above holds for this case.

Lastly, it is clear that A and B are subsets of $[-1,1] \times [-1,1]$ because f and g are functions from $[-1,1]$ to $[-1,1]$. Thus, A and B are both connected subsets of $[-1,1] \times [-1,1]$.

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5 Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ with } (p,q) = 1 \end{cases}$$

Prove that f is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous on $\mathbb{Q}$.

Solution Let $\{x_n\}_{n\geq 1} \subseteq \mathbb{Q}$ that converges to $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. We don't need to handle the case where $\{x_n\}_{n\geq 1} \subseteq \mathbb{R} \setminus \mathbb{Q}$ since the sequence of their images obviously converge to $f(x_0) = 0$.

As x_n is rational, we can find $\{a_n\}_{n\geq 1} \subseteq, \{b_n\}_{n\geq 1} \subseteq \mathbb{Z}$ such that $x_n = \frac{a_n}{b_n}$ with $\gcd(a_n, b_n) = 1$. Then $\lim_{n\to\infty} b_n = \infty$. If not, then b_n must be bounded, which means that the limit of x_n as $n \to \infty$ will not be irrational. Hence,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{1}{b_n} = 0.$$

Thus, the sequence of the images of rational numbers converges to $f(x_0)$.

So, any sequence $\{x_n\}_{n\geq 1}$ which converges to x_0 will have $f(x_n) \xrightarrow{n\to\infty} f(x_0)$.

Let $x_0 \in \mathbb{Q}$. Then $f(x_0) = \frac{1}{b_0} \neq 0$. Let $\{x_n\}_{n\geq 1} \subseteq \mathbb{R} \setminus \mathbb{Q}$ be such that $x_n \xrightarrow{n\to\infty}$. Such a sequence exists since $\mathbb{Q}$ is dense in $\mathbb{R}$. But $f(x_n) \xrightarrow{n\to\infty} = 0 \neq f(x_0)$. Hence, f cannot be continuous on $\mathbb{Q}$.

6 Let (X, d) be a metric space and let $f, g \colon X \to \mathbb{R}$ be two continuous functions.

a. Prove that the set $\{x \in X \mid f(x) < g(x)\}$ is open.

b. Prove that if the set $\{x \in X \mid f(x) \leq g(x)\}$ is dense in X, then $f(x) \leq g(x)$ for all $x \in X$.

Solution a. Consider the function $h \colon X \to \mathbb{R}$ with $h(x) = f(x) - g(x)$, so $h(x) < 0 \iff f(x) - g(x) < 0 \iff f(x) < g(x)$. h is also continuous on X since it is the difference of two continuous functions.

Let $a \in \{x \in X \mid f(x) < g(x)\}$, so $h(a) < 0$. Then as h is continuous, there exists $\delta > 0$ such that if $|x - a| < \delta$, we have

$$|h(x) - h(a)| < -h(a) \implies h(x) < h(a) < 0$$

Thus, the ball $B_{h(a)}(x) \subseteq \{x \in X \mid f(x) < g(x)\}$, so by definition, the set is open.

b. Consider the complement of the set, which is $\{x \in X \mid f(x) > g(x)\}$. By (a), we have that this set is open, so $\{x \in X \mid f(x) \leq g(x)\}$ is closed, by definition. Since it is dense,

$$\{x \in X \mid f(x) \leq g(x)\} = \overline{\{x \in X \mid f(x) \leq g(x)\}} = X$$

Thus, $f(x) \leq g(x)$ for all $x \in X$.

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7 Let $a, b \in \mathbb{R}$ with $a < b$. Show that a function $f \colon (a, b) \to \mathbb{R}$ is uniformly continuous on (a, b) if and only if it can be extended to a continuous function $\tilde{f}$ on $[a, b]$.

Solution "$\implies$"

Suppose f is uniformly continuous. As (a, b) is connected and f is continuous, $f((a, b))$ must also be connected, so it must be an interval.

Consider the sequence $\{a_n\}_{n \geq 1} \subseteq (a, b)$ such that $a_n \xrightarrow{n \to \infty} a$ and $\{f(a_n)\}_{n \geq 1}$.

Fix $\varepsilon > 0$. As f is uniformly continuous on (a, b), there exists $\delta_\varepsilon > 0$ such that for all $x, y \in (a, b)$ with $|x - y| < \delta_\varepsilon$, we have $|f(x) - f(y)| < \varepsilon$.

Since $a_n \xrightarrow{n \to \infty} a$, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n, m \geq N$, we have $|a_n - a_m| < \delta_\varepsilon$. Thus, since $a_n, a_m \in (a, b)$,

$$|f(a_n) - f(a_m)| < \varepsilon$$

so since $\mathbb{R}$ is complete, $\{f(a_n)\}_{n \geq 1}$ converges in $\mathbb{R}$. Let its unique limit be A. Moreover, $A \in \overline{f((a, b))}$ since a sequence in $f((a, b))$ converges to it.

We can use a similar argument with a sequence $\{b_n\}_{n \geq 1}$ that converges to b in order to define a value $B \in \overline{f((a, b))}$.

Define $\tilde{f} \colon [a, b] \to \mathbb{R}$ as follows:

$$\tilde{f}(x) = \begin{cases} A & x = a \\ f(x) & x \in (a, b) \\ B & x = b \end{cases}$$

Then $\tilde{f}$ is continuous on $[a, b]$ by construction. Indeed, f is continuous on (a, b), so $\tilde{f}$ is continuous on that interval also. $\tilde{f}$ is continuous at $x = a$ since as show above, any sequence $\{a_n\}_{n \geq 1}$ with $a_n \xrightarrow{n \to \infty} a \implies f(a_n) \xrightarrow{n \to \infty} A = f(a)$. The same argument applies at $x = b$, so $\tilde{f}$ is continuous on $[a, b]$.

"$\impliedby$"

Suppose there is an extension $\tilde{f}$ for f which is continuous on $[a, b]$. Since $[a, b]$ is compact, $\tilde{f}$ is uniformly continuous, by a proposition proved in class. Hence, it is also uniformly continuous on (a, b), so f is uniformly continuous on (a, b).