****10 3.2.9** Let T be a linear operator on the finite-dimensional space V. Suppose there is a linear operator U on V such that $TU = I$. Prove that T is invertible and $U = T^{-1}$. Give an example which shows that this is false when V is not finite dimensional. (*Hint:* Let $T = D$, the differentiation operator on the space of polynomial functions.)

Solution Since $\dim V = \dim V < \infty$, T is onto $\iff$ T is one-to-one. The operator $TU = I$ is obviously an isomorphism on V. We will prove that this implies that T is also one-to-one.

Since $TU = I$, then for all $\alpha, \beta \in V$, $TU\alpha = \beta \implies \alpha = \beta$. Let $\alpha_1, \alpha_2 \in V$ be such that $U\alpha_1 = U\alpha_2 \implies TU\alpha_1 \implies TU\alpha_2$. Since TU is injective, $\alpha_1 = \alpha_2 \implies U$ is injective. Since U is a map from V onto itself, U is also surjective.

Suppose T is not one-to-one. Then there exists distinct $\alpha_1, \alpha_2 \in V$ different from above such that $T\alpha_1 = T\alpha_2$. Since U is an isomorphism and $\alpha_1 \neq \alpha_2$, there exists distinct $\beta_1, \beta_2 \in V$ such that $U\beta_1 = \alpha_1$ and $U\beta_2 = \alpha_2$. Thus, we have $T\alpha_1 = TU\beta_1 = TU\beta_2 = T\alpha_2$. But TU is isomorphic, so $\beta_1 = \beta_2 \implies U\beta_1 = \alpha_1 = \alpha_2 = U\beta_2$. We said that $\alpha_1 \neq \alpha_2$, so we have a contradiction. Thus, T is one-to-one and therefore also onto, so its inverse T^{-1} exists. Thus,

$$T^{-1}TU = T^{-1}I \implies IU = T^{-1} \implies U = T^{-1}$$

as desired.

An example of where this is false in infinite dimensional vector spaces is the differentiation operator D on the space of polynomials. Consider the integration operator U and the coefficient vector $(a_0, a_1, a_2, \ldots,)^T$. Then

$$DU(a_0, a_1, a_2 \ldots,)^T = D\left(C, a_0, \frac{1}{2}a_1, \frac{1}{3}a_2, \ldots,\right)^T = (a_0, a_1, a_2, \ldots)^T \implies DU = I.$$

However, $N(T) \neq \{0\}$, as $D(1, 0, 0, \ldots,)^T = D(2, 0, 0, \ldots,)^T = (0, 0, 0, \ldots,)^T$. In other words, T is not invertible.

****11 3.2.11** Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that $\operatorname{rank}(T^2) = \operatorname{rank} T$. Prove that the range and null space of T are disjoint, i.e., have only the zero vector in common.

Solution By rank-nullity, $\dim V = \operatorname{rank} T + \operatorname{null} T = \operatorname{rank} T^2 + \operatorname{null} T^2 \implies \operatorname{null} T = \operatorname{null} T^2$.

Suppose the range and null space of T are not disjoint. That is, $\dim(\operatorname{im} T \cap \ker T) > 0$. Then we can find $\alpha_1, \ldots, \alpha_k \in V$ such that $T\alpha_1, \ldots, T\alpha_k$ is a basis of the intersection of the range and null space. Then we can find vectors $\alpha_{k+1}, \ldots, \alpha_n \in \operatorname{im} T \setminus \ker T$ such that $T\alpha_1, \ldots, T\alpha_n$ spans $\operatorname{im} T$. The image of these vectors under T must span of $\operatorname{im} T^2$ because the second application of T is a map from $\operatorname{im} T$ to V.

Applying T to these vectors yields $T^2\alpha_1, \ldots, T^2\alpha_k, T^2\alpha_{k+1}, \ldots, T^2\alpha_n$, which reduces to $T^2\alpha_{k+1}, \ldots, T^2\alpha_n$. These vectors span $\operatorname{im} T^2$, but that means $\operatorname{rank} T^2 = n - k \neq n = \operatorname{rank} T$, which is a contradiction. Thus, we must have that the range and null space of T are disjoint.

****12 3.4.10** We have seen that the linear operator T on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1, 0)$ is represented in the standard ordered basis by the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This operator satisfies $T^2 = T$. Prove that if S is a linear operator on $\mathbb{R}^2$ such that $S^2 = S$, then $S = 0$, or $S = I$, or there is an ordered basis $\mathfrak{B}$ for $\mathbb{R}^2$ such that $[S]_{\mathfrak{B}} = A$ (above).

Solution There are three cases:

$\operatorname{rank} S = 0$:

If $\operatorname{rank} S = 0$, then $\operatorname{im} S$ has 0 linearly independent vectors; i.e., $T\alpha = 0$ for all $\alpha \in \mathbb{R}^2$. So, we must have $S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which obviously satisfies $S^2 = S$.

rank $S = 1$:

If $\operatorname{rank} S = 1$, then $\operatorname{null} S = 1$. Let $\alpha, \beta \in \mathbb{R}^2$ be such that $S\alpha$ and β span $\operatorname{im} S$ and $\ker S$, respectively. Since $S^2 = S$, $S^2\alpha = S\alpha$. Then $\mathfrak{B} = \{S\alpha, \beta\}$ is a basis for $\mathbb{R}^2$, since $\mathbb{R}^2$ has dimension 2 and the vectors are linearly independent.

We will show that under the basis $\mathfrak{B}$, S can be written as the matrix A. Let $\mathfrak{E} = \{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$.

Let P be the matrix that switches coordinates from $\mathfrak{B}$ and $\mathfrak{E}$. Then P^{-1} switches coordinates from $\mathfrak{E}$ to $\mathfrak{B}$. Note that $Pe_1 = S\alpha$, $Pe_2 = \beta$, $P^{-1}S\alpha = e_1$ and $P^{-1}\beta = e_2$. Then

$$\begin{aligned}
[S]_{\mathfrak{B}} &= P^{-1}SP \\
&= P^{-1}S \begin{pmatrix} | & | \\ S\alpha & \beta \\ | & | \end{pmatrix} \\
&= P^{-1} \begin{pmatrix} | & | \\ S^2\alpha & S\beta \\ | & | \end{pmatrix} \\
&= P^{-1} \begin{pmatrix} | & | \\ S\alpha & 0 \\ | & | \end{pmatrix} \\
&= \begin{pmatrix} | & | \\ P^{-1}S\alpha & P0 \\ | & | \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = A
\end{aligned}$$

as desired.

rank $S = 2$:

Let α, β be the first and second column of S, respectively. $\operatorname{rank} S = 2$, α and β must be linearly independent, so they span $\mathbb{R}^2$. Since $S^2 = S$, we have $S\alpha = \alpha$ and $S\beta = \beta$.

Since α and β span $\mathbb{R}^2$, for all $\gamma \in \mathbb{R}^2$, there exists $c_1, c_2 \in \mathbb{R}$ such that $\gamma = c_1\alpha + c_2\beta$. Then

$$S\gamma = c_1 S\alpha + c_2 S\beta = c_1\alpha + c_2\beta = \gamma$$

This holds for all $\gamma \in \mathbb{R}^2$, so it must be that $S = I$.

Thus, in all cases, S must be either 0 or I, or there exists a basis $\mathfrak{B}$ such that $[S]_{\mathfrak{B}} = A$.