2 Suppose $\{K_\delta\}$ is a family of kernels that satisfies:

(i) $|K_\delta(x)| \leq A\delta^{-d}$ for all $\delta > 0$.

(ii) $|K_\delta(x)| \leq A\delta/|x|^{d+1}$ for all $\delta > 0$.

(iii) $\int_{-\infty}^{\infty} K_\delta(x)\,\mathrm{d}x = 0$ for all $\delta > 0$.

Thus K_δ satisfies conditions (i) and (ii) of approximations to the identity, but the average value of K_δ is 0 instead of 1. Show that if f is integrable on $\mathbb{R}^d$, then

$$(f * K_\delta)(x) \to 0 \quad \text{for a.e. } x, \text{ as } \delta \to 0.$$

Solution Let f be integrable on $\mathbb{R}^d$. By definition,

$$\begin{aligned}
|(f * K_\delta)(x)| &= \left|(f * K_\delta)(x) - f(x)\int_{\mathbb{R}^d} K_\delta\right| \\
&= \left|\int_{\mathbb{R}^d} f(x-y)K_\delta(y)\,\mathrm{d}y - \int_{\mathbb{R}^d} f(x)K_\delta(y)\,\mathrm{d}y\right| \\
&\leq \int_{\mathbb{R}^d} |f(x-y) - f(x)||K_\delta(y)|\,\mathrm{d}y \\
&= \int_{|y|\leq\delta} |f(x-y) - f(x)||K_\delta(y)|\,\mathrm{d}y + \sum_{n=0}^{\infty}\int_{2^n\delta\leq|y|\leq 2^{n+1}\delta} |f(x-y) - f(x)||K_\delta(y)|\,\mathrm{d}y.
\end{aligned}$$

By property (i), we have

$$\int_{|y|\leq\delta} |f(x-y) - f(x)||K_\delta(y)|\,\mathrm{d}y \leq \frac{A}{\delta^d}\int_{|y|\leq\delta} |f(x-y) - f(x)|\,\mathrm{d}y = A\mathcal{A}(\delta),$$

where $\mathcal{A}$ is as defined in the book.

By property (ii), we have for each $n \geq 0$,

$$\begin{aligned}
\int_{2^n\delta\leq|y|\leq 2^{n+1}\delta} |f(x-y) - f(x)||K_\delta(y)|\,\mathrm{d}y &\leq \frac{A\delta}{(2^{n+1}\delta)^{d+1}}\int_{|y|\leq 2^{n+1}\delta} |f(x-y) - f(x)|\,\mathrm{d}y \\
&= \frac{A}{2^{n+1}(2^{n+1}\delta)^d}\int_{|y|\leq 2^{n+1}\delta} |f(x-y) - f(x)|\,\mathrm{d}y \\
&= A'2^{-n}\mathcal{A}(2^{n+1}\delta).
\end{aligned}$$

Let $\varepsilon > 0$

We pick $N \in \mathbb{N}$ large enough so that $\sum_{n\geq N} 2^{-n} < \varepsilon$.

By Lemma 2.2, we can make δ small enough so that for each $i = 0, \ldots, N-1$, we have

$$\mathcal{A}(2^n\delta) < \frac{\varepsilon}{N}.$$

By the same lemma, we also have that $\mathcal{A}(\delta) \leq M$ for some $M > 0$ for all δ.

Hence, putting the estimates together, we have

$$\begin{aligned}
\int_{|y|\leq\delta} |f(x-y) - f(x)||K_\delta(y)|\,\mathrm{d}y &\leq \sum_{n=0}^{N-1} A'2^{-n}\mathcal{A}(2^{n+1}\delta) + \sum_{n=N}^{\infty} A'2^{-n}\mathcal{A}(2^{n+1}\delta) \\
&\leq \frac{A\varepsilon}{N} + (N-1)\frac{A'\varepsilon}{N} + A'M\varepsilon \\
&\leq C\varepsilon,
\end{aligned}$$

for some C.

Since ε was arbitrary, it follows from $|(f * K_\delta)(x)| \leq C\varepsilon$ that $(f * K_\delta)(x) \xrightarrow{\delta\to 0} 0$.

3 Suppose 0 is a point of (Lebesgue) density of the set $E \subseteq \mathbb{R}$. Show that for each of the individual conditions below there is an infinite sequence of points $x_n \in E$, with $x_n \neq 0$, and $x_n \to 0$ as $n \to \infty$.

a. The sequence also satisfies $-x_n \in E$ for all n.

b. In addition, $2x_n$ belongs to E for all n.

Generalize.

Solution Let t be a non-zero real number such that $|t| \leq 1$. We will show that there is a sequence $\{x_n\}_{n\geq 1} \subseteq E \setminus \{0\}$ converging to 0 such that $x_n/t \in E$ for all $n \geq 1$, also. This will generalize both (a) and (b), by taking $t = -1$ and $t = 1/2$, respectively.

Let $r > 0$. Consider the sets $(-r, r) \cap E$ and $(-r, r) \cap tE := \{tx \mid x \in E\}$. Note that both of these sets are measurable since E and tE are measurable, and since measurable sets are closed under finite intersections. Then

$$\begin{aligned} m((-r,r) \cap tE) &= m\left(t\left(-\frac{r}{|t|}, \frac{r}{|t|}\right) \cap tE\right) \\ &= m\left(t\left[\left(-\frac{r}{|t|}, \frac{r}{|t|}\right) \cap E\right]\right) \\ &= |t| m\left(\left(-\frac{r}{|t|}, \frac{r}{|t|}\right) \cap E\right) \\ &\geq |t| m((-r,r) \cap E). \end{aligned}$$

The last inequality holds since $|t| \leq 1$, so

$$(-r, r) \subseteq \left(-\frac{r}{|t|}, \frac{r}{|t|}\right).$$

This gives us

$$m((-r,r) \cap tE) + m((-r,r) \cap E) \geq (1 + |t|) m((-r,r) \cap E).$$

By definition,

$$\lim_{r \to 0} \frac{m((-r,r) \cap E)}{2r} = 1,$$

so for any $\varepsilon > 0$, there exists $R > 0$ such whenever $0 < r < R$, we have

$$\left| \frac{m((-r,r) \cap E)}{2r} - 1 \right| < \varepsilon \implies \frac{m((-r,r) \cap E)}{2r} > 1 - \varepsilon.$$

In particular, since $1/(1 + |t|) < 1$, we can choose ε so that

$$\frac{m((-r,r) \cap E)}{2r} > \frac{1}{1 + |t|}.$$

Combining with the previous inequality, we get that

$$m((-r,r) \cap tE) + m((-r,r) \cap E) \geq (1 + |t|) m((-r,r) \cap E) > 2r = m((-r,r)).$$

Thus, the intersection of $(-r, r) \cap E$ and $(-r, r) \cap tE$ must have positive measure. Otherwise, discarding the intersection from both sets, we can invoke the subset property and additivity of Lebesgue measure to conclude that

$$m((-r,r) \cap tE) + m((-r,r) \cap E) = m(((-r,r) \cap tE) \cup ((-r,r) \cap E)) \leq 2r,$$

which is a contradiction, so their intersection has positive measure. In particular, it is non-empty for all r.

Thus, we can define x_n as follows:

Let $0 < x_1 \in (-r_1, r_1) \cap E \cap tE$ for any $r_1 > 0$. Then $x_1 \in tE \implies x_1/t \in E$.

Take $r_2 = |x_1|/2$. Then pick $0 < x_2 \in (-r_2, r_2) \cap E \cap tE \neq \emptyset$.

Proceeding inductively, we construct a sequence with $|x_n| \leq |x_{n-1}|/2 \implies |x_n| \leq |x_1|/2^{n-1}$ for all n with $x_n/t \in E$ for all $n \geq 1$. The sequence clearly converges to 0 as $n \to \infty$, since $1/2^{n-1} \xrightarrow{n\to\infty} 0$, as desired.

4 Prove that if f is integrable on $\mathbb{R}^d$, and f is not identically zero, then

$$f^*(x) \geq \frac{c}{|x|^d}, \quad \text{for some } c > 0 \text{ and all } |x| \geq 1.$$

Conclude that f^* is not integrable on $\mathbb{R}^d$. Then, show that the weak type estimate

$$m(\{x \mid f^*(x) > \alpha\}) \leq \frac{c}{\alpha}$$

for all $\alpha > 0$, whenever $\int |f| = 1$, is best possible in the following sense: if f is supported in the unit ball with $\int |f| = 1$, then

$$m(\{x \mid f^*(x) > \alpha\}) \geq \frac{c'}{\alpha}$$

for some $c' > 0$ and all sufficiently small α.

[Hint: For the first part, use the fact that $\int_B |f| > 0$ for some ball B.]

Solution Let f be integrable on $\mathbb{R}^d$ and not identically 0.

Since f is not identically 0, there exists a ball $B \subseteq \mathbb{R}^d$ centered at the origin with finite radius such that $\int_B |f| > 0$, which implies, by definition, that $f^*(x) > 0$ for all $x \in \mathbb{R}^d$, since we can make any ball around x large enough to include all of B. We can also scale B so that its radius is $R \geq 1$.

$x \in B$

In this case, we have, by definition,

$$f^*(x) \geq \frac{1}{v_d R^d} \int_B |f| \geq \frac{1}{v_d R^d |x|^d} \int_B |f| \coloneqq \frac{c_0}{|x|^d}.$$

$x \notin B$

We can take the ball $D \coloneqq B_{|x|+R}(0)$, which satisfies $x \in D$ and $B \subseteq D$. Notice that $|x| + R \leq |x| + R|x|$, since $|x| \geq 1$. Then

$$f^*(x) \geq \frac{1}{m(D)} \int_D |f| \geq \frac{1}{v_d(|x|+R)^d} \int_B |f| \geq \frac{1}{v_d(|x|+R|x|)^d} \int_B |f| = \frac{1}{v_d|x|^d(1+R)^d} \int_B |f| \coloneqq \frac{c_1}{|x|^d}.$$

Take $c = \min\{c_0, c_1\}$ to get that $f^*(x) \geq c/|x|^d$ whenever $|x| \geq 1$, as desired.

Hence, f^* is not integrable on $\mathbb{R}^d$ since $1/|x|^d$ is not integrable on $\{x \in \mathbb{R}^d \mid |x| \geq 1\}$.

Let f be supported in the unit ball B_1 with $\int |f| = 1$. In particular, f satisfies the conditions of the first part of this problem.

For $|x| \geq 1$, we have, from the first part of the problem, that $f^*(x) \geq c/|x|^d$, for some $c > 0$ and $|x| \geq 1$.

Notice that given any $\alpha > 0$, we have

$$\left\{x \in \mathbb{R}^d \mid 1 \leq |x| < \left(\frac{c}{\alpha}\right)^{1/d}\right\} = \left\{x \in \mathbb{R}^d \mid f^*(x) \geq \frac{c}{|x|^d} > \alpha\right\} \subseteq \{x \in \mathbb{R}^d \mid f^*(x) > \alpha\}.$$

Hence, by the subset property of Lebesgue measure,

$$v_d\left(\frac{c}{\alpha} - 1\right) = v_d\frac{c}{\alpha} - v_d(1^d) = m\left(\left\{x \in \mathbb{R}^d \mid 1 \leq |x| < \left(\frac{c}{\alpha}\right)^{1/d}\right\}\right) \leq m\big(\{x \in \mathbb{R}^d \mid f^*(x) > \alpha\}\big).$$

If $\alpha < c/2$, then

$$m\big(\{x \in \mathbb{R}^d \mid f^*(x) > \alpha\}\big) \geq v_d\left(\frac{c-\alpha}{\alpha}\right) \geq \frac{v_d c}{2\alpha} \coloneqq \frac{c'}{\alpha},$$

where $c' = v_d c/2$.

1 Show that properties (i) and (ii) in the definition of a Hilbert space imply property (iii): the Cauchy-Schwarz inequality $|\langle f, g\rangle| \le \|f\| \cdot \|g\|$ and the triangle inequality $\|f+g\| \le \|f\| + \|g\|$.

[Hint: For the first inequality, consider $\langle f+\lambda g, f+\lambda g\rangle$ as a positive quadratic function of λ. For the second, write $\|f+g\|^2$ as $\langle f+g, f+g\rangle$.]

Solution Cauchy-Schwarz inequality:

Let $\lambda \in \mathbb{C}$.

Then since $\langle f+\lambda g, f+\lambda g\rangle = \|f+\lambda g\|^2 \ge 0$,

$$\begin{aligned}
0 &\le \langle f+\lambda g, f+\lambda g\rangle \\
&= \langle f, f+\lambda g\rangle + \lambda\langle g, f+\lambda g\rangle \\
&= \overline{\langle f+\lambda g, f\rangle} + \lambda\overline{\langle f+\lambda g, g\rangle} \\
&= \overline{\langle f, f\rangle + \lambda\langle g, f\rangle} + \lambda\left(\overline{\langle f, g\rangle + \lambda\langle g, g\rangle}\right) \\
&= \|f\|^2 + \lambda(\langle f, g\rangle + \overline{\langle f, g\rangle}) + \lambda^2\|g\|^2
\end{aligned}$$

Take $\lambda = -\langle f, g\rangle/\|g\|^2$. Then

$$\begin{aligned}
0 &\le \|f\|^2 + \lambda(\langle f, g\rangle + \overline{\langle f, g\rangle}) + \lambda^2\|g\|^2 \\
&= \|f\|^2 + \frac{-\langle f, g\rangle^2}{\|g\|^2} - \frac{|\langle f, g\rangle|^2}{\|g\|^2} + \frac{\langle f, g\rangle^2}{\|g\|^2} \\
|\langle f, g\rangle|^2 &\le \|f\|^2\|g\|^2 \\
|\langle f, g\rangle| &\le \|f\|\|g\|,
\end{aligned}$$

so Cauchy-Schwarz holds.

Triangle inequality:

Taking $\lambda = 1$ in the calculation above, we find that

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2 + \langle f, g\rangle + \overline{\langle f, g\rangle}.$$

$\langle f, g\rangle + \overline{\langle f, g\rangle} = \operatorname{Re}\langle f, g\rangle$, so using the triangle inequality on $\mathbb{R}$ with the usual metric, we get

$$\|f+g\|^2 \le \|f\|^2 + \|g\|^2 + 2|\langle f, g\rangle| \le \|f\|^2 + \|g\|^2 + 2\|f\|\|g\| = (\|f\| + \|g\|)^2.$$

Taking square roots on both sides preserves inequality since $\sqrt{x}$ is an increasing function on $[0, \infty)$, so we get

$$\|f+g\| \le \|f\| + \|g\|$$

as desired.

2 In the case of equality in the Cauchy-Schwarz inequality we have the following. If $\|\langle f, g\rangle\| = \|f\|\|g\|$ and $g \neq 0$, then $f = cg$ for some scalar c.

[Hint: Assume $\|f\| = \|g\| = 1$ and $\langle f, g\rangle = 1$. Then $f - g$ and g are orthogonal, while $f = f - g + g$. Thus, $\|f\|^2 = \|f - g\|^2 + \|g\|^2$.]

Solution Note that if $\|f\| = 0$, then by definition, $f = 0$, so $f = 0g$. Assume from now on that f is not identically 0.

Assume, without loss of generality, that $\|f\| = \|g\| = 1$. Given any f and g, we can normalize them by dividing by $\|f\|$ and $\|g\|$, respectively. Then by the result, we have that

$$\frac{f}{\|f\|} = c\frac{g}{\|g\|} \implies f = c\frac{\|f\|}{\|g\|}g,$$

so the result holds for a general f and g.

Assume from now on that f and g are normalized.

Notice that $\langle f - g, g\rangle = \langle f, g\rangle - \langle g, g\rangle = 1 - 1 = 0$, so $f - g$ and g are orthogonal. Hence, by the calculation in Exercise 1,

$$\|f\|^2 = \|f - g + g\|^2 = \|f - g\|^2 + \|g\|^2.$$

But $\|f\|^2 = \|g\|^2$, so we get that $\|f - g\|^2 = 0 \iff f = g$, as desired.

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4 Prove from the definition that $\ell^2(\mathbb{Z})$ is complete and separable.

Solution $\ell^2(\mathbb{Z})$ is complete:

Let $\{x^{(n)}\}_{n\geq 1}$ be a Cauchy sequence in $\ell^2(\mathbb{Z})$. By definition, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$\|x^{(n)} - x^{(m)}\| = \left(\sum_{k=-\infty}^{\infty} |x_k^{(n)} - x_k^{(m)}|^2\right)^{1/2} < \sqrt{\varepsilon} \implies |x_k^{(n)} - x_k^{(m)}| < \varepsilon \ \forall k.$$

Thus, for every k, the sequence $\{x_k^{(n)}\}_{n\geq 1}$ is Cauchy in $\mathbb{R}$, which is complete. Hence, it converges to some $x_k \in \mathbb{R}$.

We'll now show that $\{x^{(n)}\}_{n\geq 1}$ converges to $x := \{x_k\}_{k=-\infty}^{\infty}$ in the $\ell^2(\mathbb{Z})$ norm.

Fix $M \in \mathbb{N}$. Since $\{x^{(n)}\}_{n\geq 1}$ is Cauchy in $\ell^2(\mathbb{Z})$, for $n, m \geq N$,

$$\sum_{k=-M}^{M} |x_k^{(n)} - x_k^{(m)}|^2 < \varepsilon$$

Since the sum is finite, we can take $m \to \infty$ to get

$$\sum_{k=-M}^{M} |x_k^{(n)} - x_k|^2 \leq \varepsilon.$$

Then the sum as a sequence of M is monotonically increasing, since each $|x_k^{(n)} - x_k|^2 \geq 0$, and the sum is bounded by ε for all M, so by the monotone convergence theorem,

$$\sum_{k=-M}^{M} |x_k^{(n)} - x_k|^2 \xrightarrow{M\to\infty} \sum_{k=-\infty}^{\infty} |x_k^{(n)} - x_k|^2 \leq \varepsilon,$$

so the sequence converges to x in the $\ell^2(\mathbb{Z})$ norm.

We'll now show that $x \in \ell^2(\mathbb{Z})$.

For $n \geq N$, we have, by the triangle inequality, that

$$\|x\| \leq \|x^{(n)} - x\| + \|x^{(n)}\|.$$

Since $x^{(n)} \xrightarrow{n\to\infty} x$ in the $\ell^2(\mathbb{Z})$ norm, $\|x^{(n)} - x\| \leq \varepsilon < \infty$. Since $x^{(n)} \in \ell^2(\mathbb{Z})$, $\|x^{(n)}\| < \infty$. Hence, their sum is finite, so $\|x\| < \infty \implies x \in \ell^2(\mathbb{Z})$.

Thus, $\ell^2(\mathbb{Z})$ is complete.

$\ell^2(\mathbb{Z})$ is separable:

Consider $e^{(i)}$, where $e_j^{(i)} = 1$ if $i = j$ and 0 if $i \neq j$. Then $\{e^{(i)}\}_{i\in\mathbb{Z}}$ is a countable collection since $\mathbb{Z}$ is countable.

Fix $x \in \ell^2(\mathbb{Z})$.

Since

$$\sum_{k=-\infty}^{\infty} |x_k|^2 < \infty,$$

for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{k=-\infty}^{\infty} |x_k|^2 - \sum_{k=-N}^{N} |x_k|^2 < \varepsilon^2.$$

Then consider the linear combination

$$x^{(N)} := \sum_{k=-N}^{N} x_k e^{(k)},$$

and note that $x_k^{(N)} = x_k$ for all $-N \leq k \leq N$.

Thus,

$$\|x - x^{(N)}\|^2 = \sum_{k=-\infty}^{\infty} |x_k - x_k^{(N)}|^2 = \sum_{k=-\infty}^{\infty} |x_k|^2 - \sum_{k=-N}^{N} |x_k|^2 < \varepsilon^2 \implies \|x - x^{(N)}\| < \varepsilon,$$

since all terms are 0 except for $|k| > N$.

Hence, as ε was arbitrary, it follows that the set of linear combinations of $\{e^{(i)} \mid i \in \mathbb{Z}\}$ is dense in $\ell^2(\mathbb{Z})$, so $\ell^2(\mathbb{Z})$ is separable.