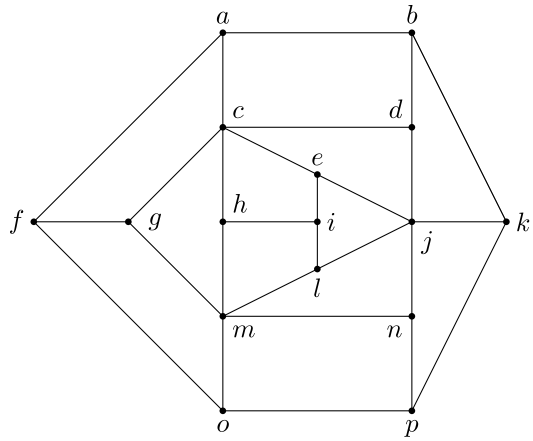
1 Show that the graph does not contain a Hamiltonian cycle:



Solution There are 20 vertices and 31 edges. A Hamiltonian cycle has 20 edges, so it must avoid 11 edges.

Notice that $\delta(c) = 5$, $\delta(m) = 5$, $\delta(j) = 5$, $\delta(p) = \delta(b) = \delta(f) = 3$. In a Hamiltonian cycle, each vertex is incident on precisely 2 edges, so since c, m, j, p, b, f don't have edges between each other, we must discard 3 edges for each of c, m, j, and discard 1 for each of p, b, f, since there are no loops. But this means that such a Hamiltonian cycle cannot use at least $3 + 3 + 3 + 1 + 1 + 1 = 12 > 11$ edges, which is impossible. Thus, the graph does not have a Hamiltonian cycle.

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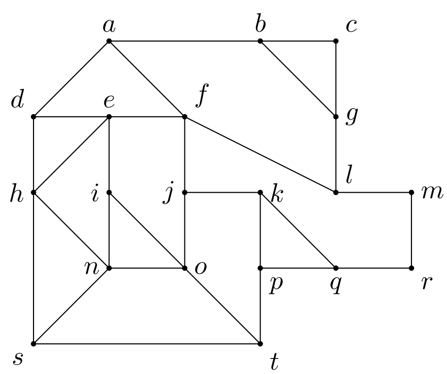
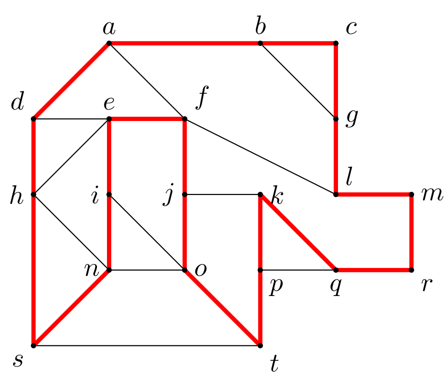
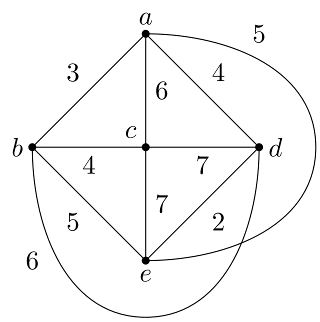
2 Determine whether or not the graph contains a Hamiltonian cycle:



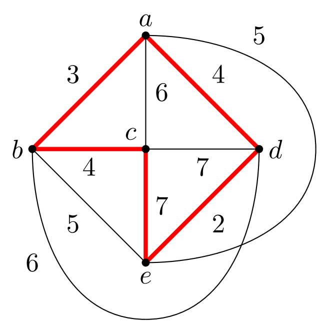
Solution This graph does have a Hamiltonian cycle:



3 Solve the traveling salesman problem for the graph shown (i.e., find a Hamiltonian cycle of smallest possible total weight):

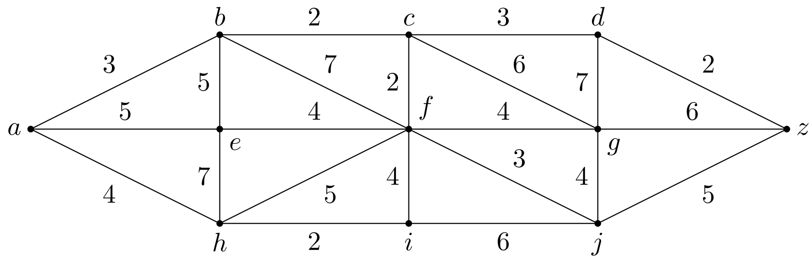


Solution One Hamiltonian cycle is given by following the path (a, b, c, e, d, a), which gives a total weight of 20:



If we replace any of our edges except for (c, e), then we will always replace it with an edge with strictly larger weight, so we cannot replace those. As a result, this forces (c, e) to be in the solution also, so this is the solution to the traveling salesman problem.

4 For each pair of vertices in the weighted graph below, find a shortest path between the vertices, and given an example of such a shortest path:



a. a, z

b. h, d

Solution a. I used a table to keep track of my progress. A boxed number represents a circled vertex.

$$\begin{array}{ccccc} & 3 & \infty & \infty & \\ \boxed{0} & 5 & \infty & \infty & \infty \\ & 4 & \infty & \infty & \end{array} \qquad \begin{array}{ccccc} & \boxed{3} & 5 & \infty & \\ \boxed{0} & 5 & 10 & \infty & \infty \\ & 4 & \infty & \infty & \end{array} \qquad \begin{array}{ccccc} & \boxed{3} & 5 & \infty & \\ \boxed{0} & 5 & 9 & \infty & \infty \\ & \boxed{4} & 6 & \infty & \end{array}$$

$$\begin{array}{ccccc} & \boxed{3} & 5 & \infty & \\ \boxed{0} & \boxed{5} & 9 & \infty & \infty \\ & \boxed{4} & 6 & \infty & \end{array} \qquad \begin{array}{ccccc} & \boxed{3} & \boxed{5} & 8 & \\ \boxed{0} & \boxed{5} & 9 & 11 & \infty \\ & \boxed{4} & 6 & \infty & \end{array} \qquad \begin{array}{ccccc} & \boxed{3} & \boxed{5} & 8 & \\ \boxed{0} & \boxed{5} & 9 & 11 & \infty \\ & \boxed{4} & \boxed{6} & 12 & \end{array}$$

$$\begin{array}{ccccc} & \boxed{3} & \boxed{5} & \boxed{8} & \\ \boxed{0} & \boxed{5} & 9 & 11 & 10 \\ & \boxed{4} & \boxed{6} & 12 & \end{array} \qquad \begin{array}{ccccc} & \boxed{3} & \boxed{5} & \boxed{8} & \\ \boxed{0} & \boxed{5} & \boxed{9} & 11 & 10 \\ & \boxed{4} & \boxed{6} & 12 & \end{array} \qquad \begin{array}{ccccc} & \boxed{3} & \boxed{5} & \boxed{8} & \\ \boxed{0} & \boxed{5} & \boxed{9} & 11 & \boxed{10} \\ & \boxed{4} & \boxed{6} & 12 & \end{array}$$

A shortest path would be (a, b, c, d, z), with length 10.

b.

$$\begin{array}{ccccc} & \infty & \infty & \infty & \\ 4 & 7 & 5 & \infty & \infty \\ & \boxed{0} & 2 & \infty & \end{array} \qquad \begin{array}{ccccc} & \infty & \infty & \infty & \\ 4 & 7 & 5 & \infty & \infty \\ & \boxed{0} & \boxed{2} & 8 & \end{array} \qquad \begin{array}{ccccc} & 7 & \infty & \infty & \\ \boxed{4} & 7 & 5 & \infty & \infty \\ & \boxed{0} & \boxed{2} & 8 & \end{array}$$

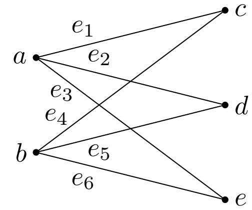
$$\begin{array}{ccccc} & 7 & 7 & \infty & \\ \boxed{4} & 7 & \boxed{5} & 9 & \infty \\ & \boxed{0} & \boxed{2} & 8 & \end{array} \qquad \begin{array}{ccccc} & 7 & \boxed{7} & 10 & \\ \boxed{4} & 7 & \boxed{5} & 9 & \infty \\ & \boxed{0} & \boxed{2} & 8 & \end{array} \qquad \begin{array}{ccccc} & \boxed{7} & \boxed{7} & 10 & \\ \boxed{4} & \boxed{7} & \boxed{5} & 9 & 13 \\ & \boxed{0} & \boxed{2} & \boxed{8} & \end{array}$$

$$\begin{array}{ccccc} & \boxed{7} & \boxed{7} & 10 & \\ \boxed{4} & \boxed{7} & \boxed{5} & \boxed{9} & 13 \\ & \boxed{0} & \boxed{2} & \boxed{8} & \end{array} \qquad \begin{array}{ccccc} & \boxed{7} & \boxed{7} & \boxed{10} & \\ \boxed{4} & \boxed{7} & \boxed{5} & \boxed{9} & 13 \\ & \boxed{0} & \boxed{2} & \boxed{8} & \end{array}$$

The shortest path would be (h, f, c, d), which has length 10.

5 Write the incidence matrix of the graph $K_{2,3}$.

Solution Here is the labeling of $K_{2,3}$ we will use:



This gives the following incidence matrix:

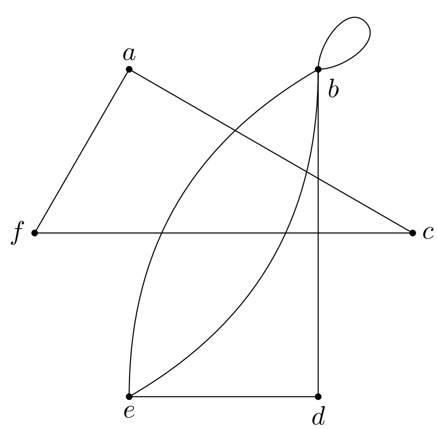
	e_1	e_2	e_3	e_4	e_5	e_6
a	1	1	1	0	0	0
b	0	0	0	1	1	1
c	1	0	0	1	0	0
d	0	1	0	0	1	0
e	0	0	1	0	0	1

---

6 Draw the graph represented by the adjacency matrix:

	a	b	c	d	e	f
a	0	0	1	0	0	1
b	0	2	0	1	2	0
c	1	0	0	0	0	1
d	0	1	0	0	1	0
e	0	2	0	1	0	0
f	1	0	1	0	0	0

Solution The matrix gives the following graph:



7 Suppose that a graph has an adjacency matrix of the form $A = \left(\begin{array}{c|c} W & X \\ \hline Y & Z \end{array}\right)$, where all entries of the sub-matrices X and Y are 0. What must the graph look like?

Solution If our graph has vertices $v_1, \ldots, v_n$, W is a $k \times k$ matrix and Z is a $(n-k) \times (n-k)$ matrix, then our graph may be split up into two subgraphs, with one graph containing $v_1, \ldots, v_k$ and the other containing $v_{k+1}, \ldots, v_n$. Indeed, there are no edges between these two sets of vertices because X and Y are both the 0 matrix, and X and Y describe the edges from $v_1, \ldots, v_k$ to $v_{k+1}, \ldots, v_n$.

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8 What must a graph look like if some row of its incidence matrix consists only of 0's?

Solution Such a graph must have some isolated points: If vertex v has all 0's in its row, it means that v is not the endpoint of any edge, so v must be isolated.

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9 Let A be the adjacency matrix of K_5.

a. Let n be a positive integer. Explain why all the diagonal entries of A^n are equal, and why all off-diagonal entries are equal.

b. Let d_n be the common value of the diagonal elements of A^n and let a_n be the common value of the off-diagonal elements of A^n. Show that $d_{n+1} = 4a_n$, $a_{n+1} = d_n + 3a_n$, and $a_{n+1} = 3a_n + 4a_{n-1}$.

c. Show that $a_n = \frac{1}{5}\left[4^n + (-1)^{n+1}\right]$.

d. Show that $d_n = \frac{4}{5}\left[4^{n-1} + (-1)^n\right]$.

Solution a. We will justify this by induction:

Base step: $n = 1$

This is true by definition of a complete graph: There are no loops, so the diagonal is 0, and every vertex is connected to every vertex by exactly one edge, so the remaining entries are 1.

Inductive step:

Suppose the diagonal entries of A^n are equal and all the off-diagonal entries are equal.

Let I be the identity matrix and let O be the matrix with all ones. Then $A = O - I$, so $A^{n+1} = A^n(O - I) = A^nO - A^n$. Notice that each entry is of the following form:

$$\begin{pmatrix} a & \cdots & a & b & a & \cdots & a \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = a + \cdots + a + b + a + \cdots + a = b + 4a,$$

where b is the term on the diagonal. Thus, A^nO is a matrix will all entries equal.

On the other hand, by assumption, the diagonal entries of A^n are all equal and the non-diagonal entries of A^n are all equal. Hence, $A^nO - A^n$ has the same property, which proves the inductive step.

b. From the calculation above, each entry of A^nO is equal to $d_n + 4a_n$. When we subtract A^n, we subtract d_n from the diagonal of A^nO, which leaves us with diagonal entry $d_{n+1} = 4a_n$.

Similarly, when we subtract A^n, we subtract the non-diagonal entry a_n from a non-diagonal entry of A^nO, so we get $a_{n+1} = d_n + 4a_n - a_n = d_n + 3a_n$. Lastly, substituting $4a_{n-1}$ for d_n, we get $a_{n+1} = 3a_n + 4a_{n-1}$.

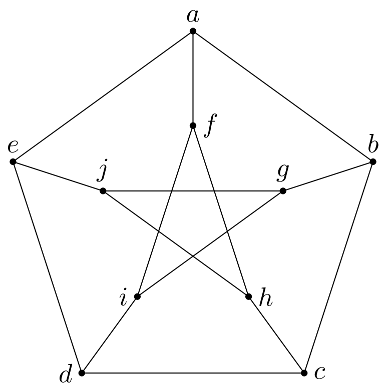
c. The characteristic polynomial of the recurrence relation is $x^2 - 3x - 4 = (x-4)(x+1)$, so the general solution is given by $a_n = a \cdot 4^n + b \cdot (-1)^n$. Our initial conditions are $a_1 = 1$ and $a_2 = 3$, so we can just check that the expression given satisfies these, since it has this form:

$$\frac{1}{5}\left[4^1 + (-1)^2\right] = 1 = a_1 \quad \text{and} \quad \frac{1}{5}\left[4^2 + (-1)^3\right] = 3 = a_2.$$

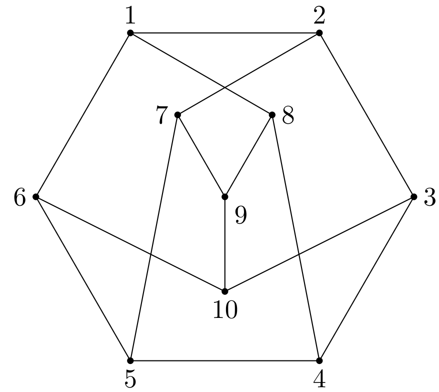
Thus, a_n is indeed given in the problem.

d. By substituting into $d_n = a_{n-1}$, we get $d_n = \frac{4}{5}\left[4^{n-1} + (-1)^n\right]$, as required.

10 Prove that the graphs G_1 and G_2 are isomorphic:



G_1



G_2

Solution We will define the bijection of vertices $f\colon V_1 \to V_2$ as follows:

$$\begin{array}{ccccc} a \mapsto 10 & b \mapsto 3 & c \mapsto 4 & d \mapsto 5 & e \mapsto 6 \\ f \mapsto 9 & g \mapsto 2 & h \mapsto 8 & i \mapsto 7 & j \mapsto 1 \end{array}$$

Then set $g\colon E_1 \to E_2$ as follows:

$$\begin{array}{ccccc} (a,b) \mapsto (10,3) & (b,c) \mapsto (3,4) & (c,d) \mapsto (4,5) & (d,e) \mapsto (5,6) & (e,a) \mapsto (6,10) \\ (a,f) \mapsto (10,9) & (b,g) \mapsto (3,2) & (c,h) \mapsto (4,8) & (d,i) \mapsto (5,7) & (e,j) \mapsto (6,1) \\ (f,h) \mapsto (9,8) & (h,j) \mapsto (8,7) & (j,g) \mapsto (1,2) & (g,i) \mapsto (2,7) & (i,f) \mapsto (7,9). \end{array}$$

g was created just by sending $(p,q) \mapsto (f(p), f(q))$, so these functions define a graph isomorphism.

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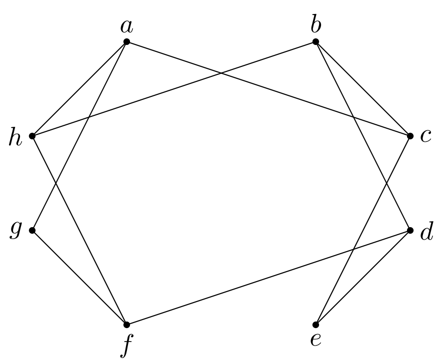
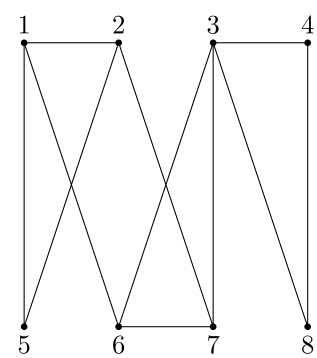
11 Prove that the graphs G_1 and G_2 are not isomorphic:



G_1



G_2

Solution G_1 has exactly 2 elements of degree 2: e and g. On the other hand, G_2 has 3 elements of degree 2: 4, 5, and 8. Hence, the two graphs cannot be isomorphic.