1 If the set A has n elements and the set B has m elements, show that there are m^n many functions from A to B.

Solution As A is finite with n elements, we can list its elements as $A = \{a_1, \ldots, a_n\}$. Similarly, we can list the elements of B as $B = \{b_1, \ldots, b_m\}$.

Consider a function $f : A \to B$. As f is a function, each element $a_i \in A$ must have an image $f(a_i) \in B$. As there are m elements in B, there are m^n different ways to map elements from A to B. Thus, there are m^n different functions from A to B.

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2 Fix $n \geq 1$. Show that if $A_1, A_2, \ldots, A_n$ are countable, then $A_1 \times A_2 \times \cdots \times A_n$ is countable.

Solution We will prove this by induction.

Base step:

A_1 is countable, as given.

Inductive step:

Suppose $A_1 \times A_2 \times \cdots \times A_n$ is countable. Then there exists a bijection $f : A_1 \times A_2 \times \cdots \times A_n \to \mathbb{N}$. Since A_{n+1} is countable, there exists a bijection $g : A_{n+1} \to \mathbb{N}$. By a proposition proved in class, $\mathbb{N} \times \mathbb{N}$ is countable. Let $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection. Then $h(f, g)$ is a bijection from $A_1 \times A_2 \times \cdots \times A_n \times A_{n+1}$ to $\mathbb{N}$. Thus, $A_1 \times A_2 \times \cdots \times A_n \times A_{n+1}$ is countable.

By the principle of mathematical induction, $A_1 \times A_2 \times \cdots \times A_n$ is countable for all $n \geq 1$.

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3 If $A \sim B$, show that $\mathcal{P}(A) \sim \mathcal{P}(B)$.

Solution As $A \sim B$, there exists a bijection $f : A \to B$. If $a \in \mathcal{P}(A)$, we define $F : \mathcal{P}(A) \to \mathcal{P}(B)$ as follows:

$$F(a) = \begin{cases} \emptyset & a = \emptyset \\ f(a) & \text{otherwise} \end{cases}$$

E.g., if $a = \{x_1, x_2, \ldots, x_n\}$, then $F(a) = \{f(x_1), f(x_2), \ldots, f(x_n)\}$. We claim that F is bijective.

F is injective:

Let $a_1, a_2 \in \mathcal{P}(A)$ such that they are not the empty set and $F(a_1) = F(a_2)$. Then $f(a_1) = f(a_2)$. Since f is bijective, we have that $a_1 = a_2$. If a_1 is the empty set, then $f(a_1) = \emptyset = f(a_2) \implies a_1 = a_2 = \emptyset$ by construction.

Thus, F is injective.

F is surjective:

If $b = \emptyset$, then $F(\emptyset) = b$.

Let $b \in \mathcal{P}(B)$ such that $b \neq \emptyset$. Then its preimage is $f^{-1}(b)$, which exists in $\mathcal{P}(A)$ since each element of b has a preimage in A. Thus, every element of $\mathcal{P}(B)$ has a preimage in $\mathcal{P}(A) \iff F$ surjective.

F is a bijection between $\mathcal{P}(A)$ and $\mathcal{P}(B)$, so by definition, $\mathcal{P}(A) \sim \mathcal{P}(B)$.

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4 Prove that $\mathcal{P}(\mathbb{N})$ is equivalent with the set of functions

$$2^{\mathbb{N}} = \{f : \mathbb{N} \to \{0, 1\} \mid f \text{ is a function}\}.$$

In particular, the cardinality of $\mathcal{P}(\mathbb{N})$ is $2^{\aleph_0}$.

Solution Let $N \in \mathcal{P}(\mathbb{N})$. Let $f : \mathcal{P}(N) \to 2^{\mathbb{N}}$. Note that the image of N under f is a sequence of 1's and 0's. We define $f(N) = \{N_n\}_{n \geq 1}$ as follows: if $i \in N$, then $N_i = 1$. Otherwise, $N_i = 0$. We claim that F is bijective.

F is injective:

Let $N_1, N_2 \in \mathcal{P}(\mathbb{N})$ such that $f(N_1) = f(N_2) \iff (N_1)_n = (N_2)_n$ with the sequences defined as above for all n. Then by the definition of F, N_1 and N_2 must contain the same elements as each other, which means that $N_1 = N_2$. Thus, F is injective.

F is surjective:

Let $\{a_n\}_{n\geq 1} \in 2^{\mathbb{N}}$. The sequence clearly has a preimage in $\mathcal{P}(N)$, which is given by $\{i \in \mathbb{N} \mid a_i = 1\}$. Thus, F is surjective.

Hence, F is a bijection from $\mathcal{P}(N)$ to $2^{\mathbb{N}}$, so by definition, $\mathcal{P}(\mathbb{N}) \sim 2^{\mathbb{N}}$. By definition, their cardinalities are the same, i.e., $|\mathcal{P}(\mathbb{N})| = |2^{\mathbb{N}}| = 2^{\aleph_0}$.

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5 Show that $\mathbb{N}^{\mathbb{N}} \sim 2^{\mathbb{N}}$, that is, the set of sequences with values in $\mathbb{N}$ is equivalent with the set of sequences with values in $\{0, 1\}$.

Solution Let $\{N_n\}_{n\geq 1} \in \mathbb{N}^{\mathbb{N}}$. Let $f : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$. We define $f(\{N_n\}_{n\geq 1}) = \{N'_n\}_{n\geq 1}$ by induction:

Base step:

Put the first $N_1 - 1$ elements as 0, unless $N_1 = 1$, and then let the N_1-th element be 1.

Inductive step:

Suppose we have gone through the first n elements of $\{N_n\}_{n\geq 1}$. Then we have defined the first $\sum_{i=1}^{n} N_i$ elements of $\{N'_n\}_{n\geq 1}$. Then define the next $N_{n+1} - 1$ elements to be 0, and define the element after that to be 1.

Thus, we have defined $f(\{N_n\}_{n\geq 1})$ through induction.

To illustrate what f does, consider the sequence $\{4, 1, 5, 6, \cdots\}$. Then the first elements of its image under f is given by

$$\underbrace{0, 0, 0}_{\text{3 zeroes}}, 1, \underbrace{\quad}_{\text{0 zeroes}} 1, \underbrace{0, 0, 0, 0}_{\text{4 zeroes}}, 1, \underbrace{0, 0, 0, 0, 0}_{\text{5 zeroes}}, 1, \cdots$$

f is clearly injective:

Given a sequence in $\mathbb{N}^{\mathbb{N}}$, its image encodes the original sequence, so if two sequences have the same image under f, the sequences in $\mathbb{N}^{\mathbb{N}}$ must be the same. Thus, f is injective.

Next, we define $g : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, $\{a_n\}_{n\geq 1} \mapsto \{a_n + 1\}_{n\geq 1}$. g is clearly an injective function.

Since f and g are both injective, it follows by Schr oder–Bernstein that there exists a bijection between the two functions. Thus, $\mathbb{N}^{\mathbb{N}} \sim 2^{\mathbb{N}}$.

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6 Show that the cardinality of $\mathbb{R}$ is $2^{\aleph_0}$. You may use the fact that the interval $(0, 1)$ has cardinality $2^{\aleph_0}$.

Solution Define $f : (0, 1) \to \mathbb{R}$ by $x \mapsto \tan(\frac{\pi}{2} + \pi x)$. We will show that f is bijective.

Consider $f'(x) = \dfrac{\pi}{\cos^2(\frac{\pi}{2} + \pi x)}$. f' exists on the interval $(0, 1)$ and is clearly positive. Thus, $f(x)$ is strictly increasing, so if $f(x_1) > f(x_2) \implies x_1 > x_2$ for all $x_1, x_2 \in (0, 1)$.

As $\lim_{x\to 0} f(x) = -\infty$ and $\lim_{x\to 1} f(x) = \infty$, f is unbounded on $(0, 1)$. Since f is continuous on that interval, f must attain all values of $\mathbb{R}$. Thus, f is surjective.

Hence f is a bijection from $(0, 1)$ to $\mathbb{R}$, so $|\mathbb{R}| = |(0, 1)| = 2^{\aleph_0}$.

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7 Prove that the set of irrational numbers has the cardinality of $\mathbb{R}$.

Solution We start by showing that $\mathbb{R} \setminus \mathbb{Q}$ has a cardinality greater than $\aleph_0$. Suppose otherwise, and that $\mathbb{R} \setminus \mathbb{Q}$ is at most countable. Then there are two cases:

$\mathbb{R} \setminus \mathbb{Q}$ is finite:

$\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$ as proven in class, so the set cannot be finite.

$\mathbb{R} \setminus \mathbb{Q}$ is countable:

Consider $\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} \cup \mathbb{Q} \cup \cdots$. By a theorem, a countable union of countable sets is countable, but this is a contradiction, as this implies that $\mathbb{R}$ is countable. Thus, $\mathbb{R} \setminus \mathbb{Q}$ is not countable.

Hence, $\mathbb{R} \setminus \mathbb{Q}$ is not at most countable, so it must be uncountable.

Let $f : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$, where $x \mapsto x$. This is clearly an injective function.

Let $g : \mathbb{R} \to \mathbb{R} \setminus \mathbb{Q}$.

By a theorem proven in class, since $(-1, 1)$ is infinite, it admits a countable subset, which we will denote as $A = \{a_1, a_2, \ldots\}$. As $\mathbb{Q}$ is countable also, there exists a bijection $h : \mathbb{Q} \to A$.

We then define $g(x)$ as follows:

$$g(x) = \begin{cases} x + \operatorname{sign}(x) & x \text{ irrational} \\ h(x) & x \text{ rational} \end{cases}$$

We claim that g is injective:

Let $x_1, x_2 \in \mathbb{R}$ such that $g(x_1) = g(x_2)$.

We cannot have one number be rational and the other be irrational as the rational number lies in the interval $(-1, 1)$ and the irrational number lies in the interval $(-\infty, -1) \cup (1, \infty)$. So, x_1 and x_2 must fall in one of the following cases.

x_1 and x_2 are both rational:

Then $g(x_1) = g(x_2) \implies h(x_1) = h(x_2)$. Since h is bijective, it follows that $x_1 = x_2$.

x_1 and x_2 are both irrational:

Over the irrational numbers, g is strictly increasing, so we must have that $x_1 = x_2$.

In all cases, we have $g(x_1) = g(x_2) \implies x_1 = x_2$, so g is injective.

We have shown that there exist injections from $\mathbb{R}$ to $\mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ to $\mathbb{R}$, so by Schröder–Bernstein, $\mathbb{R} \sim \mathbb{R} \setminus \mathbb{Q} \implies$ there exists a bijection between the two sets, so their cardinalities must be the same.