Bass Problems

18.10 Let f_n be a sequence of continuous functions on $\mathbb{R}$ that converges at every point. Prove there exist an interval and a number M such that $\sup_n|f_n|$ is bounded by M on that interval.

Solution Set

$$E_k \coloneqq \bigcap_{n=1}^{\infty}\{x \in \mathbb{R} \mid |f_n(x)| \leq k\}.$$

Since each f_n is continuous, each set in the intersection is closed, so the whole intersectin is closed.

By assumption, for every $x \in \mathbb{R}$, $\{f_n(x)\}$ is bounded, so

$$\mathbb{R} = \bigcup_{k=1}^{\infty} E_k.$$

If all the E_k were nowhere dense, then the Baire category theorem implies that $\mathbb{R}$ is nowhere dense, which is impossible. Thus, at least one E_k has non-empty interior, i.e., it contains an open interval I. Hence, for all $n \geq 1$, $|f_n(x)| \leq k$ on I so $\sup_n|f_n(x)| \leq k$ on I.

Folland Problems

4.76 If X is normal and second countable, there is a countable family $\mathcal{F} \subseteq C(X, I)$ that separates points and closed sets. (Let $\mathcal{B}$ be a countable base for the topology. Consider the set of pairs $(U, V) \in \mathcal{B} \times \mathcal{B}$ such that $\overline{U} \subseteq V$, and use Urysohn's lemma.)

Solution We follow the hint, and use Urysohn's lemma to get a continuous function $f_{U,V} \in C(X, I)$ which separates $\overline{U}$ and V^{c}. There are countably many of these functions, since $\mathcal{B} \times \mathcal{B}$ is countable.

Now let $x \in X$ and $C \subseteq X$ be a closed set which does not contain x. By normality, we can find $V \in \mathcal{B}$ so that $x \in V$, but $C \cap V = \emptyset$. Again by normality, we can find $U, W \in \mathcal{B}$ so that $x \in U$ and $V^{\mathrm{c}} \subseteq W$, but $U \cap W = \emptyset$. Thus,

$$U \subseteq W^{\mathrm{c}} \subseteq V \implies \overline{U} \subseteq V,$$

since W^{c} is closed in X. Thus, $x \in \overline{U}$ and $C \subseteq V^{\mathrm{c}}$, so $f_{U,V}$ separates x and C.

5.3 Complete the proof of Proposition 5.4

Solution We need to show that the limit function $Tx \coloneqq \lim T_n x$ for every x is in $L(\mathcal{X}, \mathcal{Y})$ and that $\|T_n - T\| \to 0$.

We'll first show linearity:

Let $x, y \in \mathcal{X}$. By definition, $T(x + y) = \lim_{n\to\infty} T_n(x + y) = \lim_{n\to\infty}[T_n(x) + T_n(y)]$. Since T_n was Cauchy, both of these limits exist, so we can use linearity of the limit to see that $T(x+y) = T(x)+T(y)$.

Let $\lambda \in \mathbb{R}$. Then $T(\lambda x) = \lim_{n\to\infty} T_n(\lambda x) = \lim_{n\to\infty} \lambda T_n x$. Once again, by linearity of the limit, we have $T(\lambda x) = \lambda T x$, so T is linear.

Now we show that T is bounded:

Since $\{T_n\}$ is Cauchy, it is bounded with respect to the operator norm, so there exists $M > 0$ such that $\|T_n\| \leq M$ for all $n \geq 1$. Then for $\|x\| = 1$,

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq M\|x\| = M,$$

so $\|T\| \leq M$. Thus, $T \in L(\mathcal{X}, \mathcal{Y})$.

Let $\varepsilon > 0$. Then for n, m large, we have $\|T_n - T_m\| \le \varepsilon$. Thus, for $\|x\| = 1$,

$$\|T_n x - Tx\| = \lim_{m\to\infty} \|T_n x - T_m x\| \le \varepsilon.$$

Since x was arbitrary, we have $\|T_n - T\| = \sup_{\|x\|=1} \|(T_n - T)x\| \le \varepsilon$, so $\|T_n - T\| \xrightarrow{n\to\infty} 0$. By the triangle inequality,

$$\Big| \|T_n\| - \|T\| \Big| \le \|T_n - T\| \xrightarrow{n\to\infty} 0,$$

so $\|T\| = \lim_{n\to\infty} \|T_n\|$.

5.8 Let $(X, \mathcal{M})$ be a measurable space, and let $M(X)$ be the space of finite signed measures on $(X, \mathcal{M})$. Then $\|\mu\| = |\mu|(X)$ is a norm on $M(X)$ that makes $M(X)$ into a Banach space.

Solution It's clear that $\|\cdot\|$ is non-negative and symmetric.

Let $\mu, \nu \in M(X)$. Then

$$\|\mu - \nu\| = 0 \iff |\mu - \nu|(X) = 0.$$

Thus, for any $E \in \mathcal{M}$, monotonicity of measures gives

$$0 \le (\mu - \nu)(E) \le |\mu - \nu|(E) \le |\mu - \nu|(X) = 0,$$

so $\mu \equiv \nu$. The other direction is clear, since $\mu \equiv \nu \implies \mu - \nu \equiv 0$.

As for the triangle inequality, we can simply use the triangle inequality on $\mathbb{R}$:

$$\|\mu + \nu\| = |\mu + \nu|(X) = |\mu(X) + \nu(X)| \le |\mu(X)| + |\nu(X)| = \|\mu\| + \|\nu\|.$$

Thus, $\|\cdot\|$ is a norm.

To show that $(X, \mathcal{M})$ is complete, we shall show that absolutely convergent series converges. Let $\{\mu_n\}$ be a sequence of signed measures with $\sum \|\mu_n\| < \infty$.

Let $E \in \mathcal{M}$. Then by monotonicity,

$$\sum_{n=1}^{\infty} |\mu_n|(E) \le \sum_{n=1}^{\infty} |\mu_n|(X) < \infty,$$

so the series converges absolutely in $\mathbb{R}$, which is complete, so $\sum \mu_n(E)$ has a limit $\mu(E) \in \mathbb{R}$, for every measurable E. We now need to show that μ is a finite signed measure.

Taking $E = X$, we see that μ is finite, and taking $E = \emptyset$ in the inequality shows that $\mu(\emptyset) = 0$. We now need to show countable additivity of μ.

Let $\{E_k\}$ be a sequence of disjoint measurable sets in X. Then

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{n=1}^{\infty} \mu_n\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu_n(E_k).$$

Notice that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\mu_n|(E_k) \le \sum_{n=1}^{\infty} |\mu_n|\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{n=1}^{\infty} |\mu_n|(X) < \infty.$$

Hence, the sum is finite, so by Fubini's theorem applied to the counting measure, we may interchange the summation to get

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu_n(E_k) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \mu_n(E_k) = \sum_{k=1}^{\infty} \mu(E_k).$$

Thus, μ is a measure.

5.27 There exist meager subsets of $\mathbb{R}$ whose complements have Lebesgue measure zero.

Solution Fix $k \in \mathbb{Z}$ and consider the interval $I_k := [k, k+1]$.

Let $\varepsilon > 0$, $\{q_n\}$ be an enumeration of $\mathbb{Q} \cap I_k$, and set

$$U_k = \bigcup_{n=1}^{\infty} \left(q_n - \frac{\varepsilon}{2^{n+1}}, q_n + \frac{\varepsilon}{2^{n+1}} \right),$$

which has Lebesgue measure of at most ε, so $1 - \varepsilon \leq m(I_k \setminus U_k) \leq 1$.

Thus, for each $N \geq 1$, we can find a closed set $F_k^{(N)} \subseteq I_k$ with $1 - 1/N \leq m(F_k^{(N)}) \leq 1$. Moreover, its interior in I_k is empty. Otherwise, if $B(x, \delta) \subseteq F_k^{(N)}$, then by density, there exists $q_n \in F_k^{(N)}$, but we assumed that $q_n \in U_n$, a contradiction. Thus, if we set

$$F_k := \bigcup_{N=1}^{\infty} F_k^{(N)},$$

we get that $m(F_k) = 1$. Moreover, F_k has empty interior, or else it contains a rational number inside I_k, which can't happen by construction. Also, its complement in I_k is a Lebesgue null set, by additivity. Furthermore, F_k was closed in I_k, which is also closed, so F_k is closed in $\mathbb{R}$, so F_k is nowhere dense.

Now consider

$$F := \bigcup_{k=1}^{\infty} F_k.$$

This set is meager by definition. Moreover,

$$m(F^{\mathrm{c}}) = m\left(\bigcap_{k=1}^{\infty} F_k^{\mathrm{c}} \right) = m\left(\bigcup_{k=1}^{\infty} (I_k \setminus F_k) \right) = \sum_{k=1}^{\infty} m(I_k \setminus F_k) = \sum_{k=1}^{\infty} 0 = 0,$$

as desired.

5.42 Let E_n be the set of all $f \in C([0,1])$ for which there exists $x_0 \in [0,1]$ (depending on f) such that $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0,1]$.

a. E_n is nowhere dense in $C([0,1])$. (Any real $f \in C([0,1])$ can be uniformly approximated by a piecewise linear function g whose linear pieces, finite in number, have slope of absolute value at least $2n$. If $\|h - g\|_u$ is sufficiently small, then $h \notin E_n$.)

b. The set of nowhere differentiable functions is residual in $C([0,1])$.

Solution a. We first follow the hint:

Let $f \in E_n$ and $\varepsilon > 0$.

Since $[0,1]$ is compact and f is continuous, it is uniformly continuous. Hence, there exists $\delta > 0$ so that

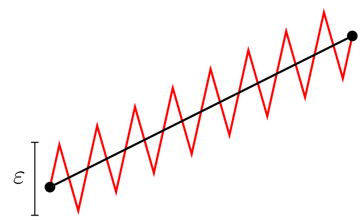
$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}.$$

Now pick $N \in \mathbb{N}$ so that $1/N < \delta$ and partition $[0,1]$ via $x_i = i/N$, where $0 \leq i \leq N$.

Define our piecewise linear function g by letting $g(x_i) = f(x_i)$, and then letting g be linear between the x_i. Then for any $x \in [0,1]$, there exists i so that $x \in [x_i, x_{i+1}]$. Hence,

$$\begin{aligned}
|f(x) - g(x)| &\leq |f(x) - f(x_i)| + |f(x_i) - g(x_i)| + |g(x_i) - g(x)| \\
&= |f(x) - f(x_i)| + |g(x_i) - g(x)| \\
&\leq |f(x) - f(x_i)| + |g(x_i) - g(x_{i+1})| \\
&= |f(x) - f(x_i)| + |f(x_i) - f(x_{i+1})| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Thus, f can be uniformly approximated by piecewise linear functions. We can also approximate linear functions uniformly well by piecewise linear functions with slope of magnitude larger than $2n$ by creating a see-saw:



Thus, f can approximated uniformly by piecewise linear functions whose slope on each piece has magnitude greater than $2n$.

We now show that E_n is closed:

Let $\{f_k\}$ be a sequence in E_n which converges to $f \in C([0,1])$ uniformly. By definition, for each f_k, there exists $x_k \in [0,1]$ so that $|f_k(x) - f_k(x_k)| \leq n|x - x_k|$ for any $x \in [0,1]$. By compactness, $\{x_k\}$ must have a convergent subsequence, which converges to some $x_0 \in [0,1]$.

Now let $x \in [0,1]$ and $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ so that $\|f - f_N\|_u < \varepsilon/4$, $n|x - x_N| \leq n|x - x_0| + \varepsilon/4$, and so $|x_0 - x_N| < \varepsilon/4n$. Then

$$\begin{aligned}
|f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_N)| + |f_N(x_N) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\
&\leq \|f - f_N\|_u + n|x - x_N| + n|x_N - x_0| + \|f - f_N\|_u \\
&< \frac{\varepsilon}{4} + \left(n|x - x_0| + \frac{\varepsilon}{4}\right) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&= n|x - x_0| + \varepsilon.
\end{aligned}$$

Since x and ε were arbitrary, it follows that $|f(x) - f(x_0)| \leq n|x - x_0|$ for any $x \in [0,1]$, so by definition, $f \in E_n$. Hence, E_n is closed.

Lastly, we'll show that E_n is nowhere dense.

Suppose there exist $f \in E_n$ and $\varepsilon > 0$ so that $B(f, \varepsilon) \subseteq E_n$. By the first part of the problem, there is a piecewise linear function g whose pieces have slope larger than $2n$ in magnitude with $\|f - g\|_u < \varepsilon \implies g \in B(f, \varepsilon)$. But for any $x \in [0,1]$ there exists $y \neq x$ in the same piece of g so that

$$\left|\frac{g(x) - g(y)}{x - y}\right| > 2n \implies |g(x) - g(y)| > 2n|x - y| \implies g \notin B(f, \varepsilon).$$

But this is impossible, so E_n must be nowhere dense.

b. Let $\mathcal{F}$ be the set of nowhere differentiable $C([0,1])$ functions. We will show that $\mathcal{F}^c \subseteq \bigcup E_n \coloneqq E$. Since each E_n is nowhere dense, E is meager, which means that $\mathcal{F}^c$ is also meager.

Let f be differentiable at some point $x_0 \in [0,1]$. Then by definition,

$$\lim_{x \to x_0} \left|\frac{f(x) - f(x_0)}{x - x_0}\right|$$

exists, so there exists $\delta > 0$ such that $|x - x_0| < \delta \implies$ the difference quotient is bounded by some $M \in \mathbb{N}$.

If $|x - x_0| \geq \delta$, we have

$$\left|\frac{f(x) - f(x_0)}{x - x_0}\right| \leq \frac{2\|f\|_u}{\delta}.$$

By making M larger if necessary, we thus have

$$\left|\frac{f(x) - f(x_0)}{x - x_0}\right| \leq M \implies |f(x) - f(x_0)| \leq M|x - x_0| \implies f \in E_M.$$

Thus, $\mathcal{F}^c \subseteq E$, and $\mathcal{F}$ is residual.