1.6 Show that the intersection of a plane with the unit sphere in $\mathbb{R}^3$ is a circle or a point and conversely every circle or point on the sphere is equal to the intersection of the sphere with a plane. Hint: Rotate the plane and sphere so that the plane is parallel to the $(x, y, 0)$ plane.

Solution "$\implies$"

Let S be the unit sphere, and let P be a plane in $\mathbb{R}^3$. Assume that $S \cap P \neq \emptyset$. Then we rotate the plane and sphere so that we can write the plane in the form $z = c$ for some $c \in [-1, 1]$. Then if $\mathbf{r} = (x, y, z) \in S \cap P$,

$$\begin{aligned} \mathbf{r} \in S &\implies x^2 + y^2 + z^2 = 1 \\ \mathbf{r} \in P &\implies z = c \\ &\implies x^2 + y^2 = 1 - c^2 \end{aligned}$$

Thus, the intersection of a sphere and a plane will be a circle or a point.

"$\impliedby$"

Let C be a circle on the unit sphere S. Then rotate C and S so that C is parallel to the $(x, y, 0)$ plane. E.g., we rotate so that the z-coordinate of all the points of C is equal to $c \in \mathbb{R}$. Hence,

$$C = S \cap \{(x, y, z) \in \mathbb{R} \mid z = c\},$$

so C is the intersection of the sphere with a plane parallel to the $(x, y, 0)$ plane.

Let $\mathbf{r}$ be a point on S. We then rotate S so that $\mathbf{r}$ lies on the positive z-axis. Then

$$\{\mathbf{r}\} = S \cap \{(x, y, z) \in \mathbb{R} \mid z = 1\},$$

so any point on the sphere is equal to the intersection of the sphere and a plane.

1.7 a. Suppose w is a non-zero complex number. Choose z so that $|z| = |w|^{\frac{1}{2}}$ and $\arg z = \frac{1}{2} \arg w$ or $\frac{1}{2} \arg w + \pi$. Show that $z^2 = w$ in both cases, and that these are the only solutions to $z^2 = w$.

b. The quadratic formula gives two solutions to the equation $az^2 + bz + c = 0$, when a, b, c are complex numbers with $a \neq 0$ because completing the square is a purely algebraic manipulation of symbols, and there are two complex roots of every non-zero complex number by part (a). Check the details.

c. If w is a non-zero complex number, find n solutions to $z^n = w$ using polar coordinates.

Solution a. $\arg z = \frac{1}{2} \arg w$,

$z^2 = (|z| e^{i \frac{1}{2} \arg w})^2 = |w| e^{i \arg w} = w$

$\arg z = \frac{1}{2} \arg w + \pi$,

$z^2 = (|z| e^{i \frac{1}{2} \arg w + \pi})^2 = |w| e^{i \arg w} e^{i 2\pi} = w \cdot 1 = w$

Hence, they are both solutions to $z^2 = w$. We will now show that they are the only solutions.

We must have that $|z| = |w|^{\frac{1}{2}}$. Otherwise, $|z^2| \neq |w| \implies z^2 \neq w$. Next, write $\arg z = \frac{1}{2} \arg w + \varphi$, where $\varphi \neq 0, \pi$, and restrict φ to $[0, 2\pi)$. Then

$$z^2 = (|z| e^{i \arg z})^2 = |w| e^{i \arg w} e^{i(2\varphi)} = w e^{i(2\varphi)}.$$

$\varphi \neq 0, \pi \implies 2\varphi \neq 0, 2\pi$. Hence, $e^{i(2\varphi)} \neq 1$, so z cannot be a solution to $z^2 = w$. So the only solutions to $z^2 = w$ are as described above.

b. $0 = az^2 + bz + c \implies 0 = z^2 + \dfrac{b}{a} z + \dfrac{c}{a}$

$$0 = \left(z + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}$$

$$\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

The equation has exactly 2 solutions as shown in part (a), and the solutions are given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

c. $w = |w|e^{i(\arg w + 2\pi k)}$, where $k = 0, 1, \dots, n-1$. Then

$$z^n = w \implies z = |w|^{\frac{1}{n}} e^{i\left(\frac{1}{n}\arg w + 2\pi \frac{k}{n}\right)}$$

Taking the different values of k gives us n solutions to $z^n = w$, as desired.

1.8 Suppose that f is a continuous complex valued function on the real interval $[a, b]$. Let

$$A = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

be the average of f over the interval $[a, b]$.

a. Show that if $|f(x)| \le |A|$ for all $x \in [a, b]$, then $f = A$. Hint: rotate f so that $A > 0$. Then $\frac{1}{b-a}\int_a^b (A - \operatorname{Re} f)\,\mathrm{d}x = 0$, and $A - \operatorname{Re} f$ is continuous and non-negative.

b. Show that if $|A| = \frac{1}{b-a}\int_a^b |f(x)|\,\mathrm{d}x$, then $\arg f$ is constant modulo 2π on $\{z \mid f(z) \neq 0\}$.

Solution a. We can write $A = |A|e^{i\arg A}$. Then we rotate f by $-\arg A$ to get $\tilde{f} = e^{-i\arg A}$. Hence,

$$\frac{1}{b-a}\int_a^b \tilde{f}(x)\,\mathrm{d}x = Ae^{-i\arg A} := \tilde{A} > 0.$$

Then if we take the real part of both sides, we get

$$\begin{aligned} \frac{1}{b-a}\int_a^b \operatorname{Re}\tilde{f}\,\mathrm{d}x = \operatorname{Re}\tilde{A} = \tilde{A} = \frac{1}{b-a}\int_a^b \tilde{A}\,\mathrm{d}x \\ \implies \frac{1}{b-a}\int_a^b \tilde{A} - \operatorname{Re}\tilde{f}\,\mathrm{d}x = 0 \end{aligned}$$

Since $|f(x)| \le |A| \implies |\tilde{f}(x)| \le |\tilde{A}| = \tilde{A}$, we have that $\tilde{A} - \operatorname{Re}\tilde{f} \ge 0$. Hence, for the above integral to be 0, $\tilde{A} - \operatorname{Re}\tilde{f} = 0$. Otherwise, the integral will be non-zero. Thus, $\tilde{A} = \operatorname{Re}\tilde{f}$. Then

$$|\tilde{f}(x)| \le |\tilde{A}| \implies (\operatorname{Re}\tilde{f})^2 + (\operatorname{Im}\tilde{f})^2 = \tilde{A}^2 + (\operatorname{Im}\tilde{f})^2 \le \tilde{A}^2 \implies \operatorname{Im}\tilde{f} = 0.$$

Hence, $\tilde{f} = \tilde{A}$. We undo the rotation by multiplying by $e^{i\arg A}$, which gives us $f = A$, as desired.

b. We rotate in the same manner as above to get $\tilde{A} > 0$ and $\tilde{f}$. Then

$$\frac{1}{b-a}\int_a^b \tilde{f}(x)\,\mathrm{d}x = \tilde{A} = |\tilde{A}| = \frac{1}{b-a}\int_a^b |\tilde{f}(x)|\,\mathrm{d}x \implies \frac{1}{b-a}\int_a^b \tilde{f}(x) - |\tilde{f}(x)|\,\mathrm{d}x = 0 \tag{1}$$

Taking the real part of the abov yields

$$\frac{1}{b-a}\int_a^b \operatorname{Re}\tilde{f}(x) - |\tilde{f}(x)|\,\mathrm{d}x = 0.$$

Since $\operatorname{Re}\tilde{f}(x) \le |\tilde{f}(x)|$, we must have that $\operatorname{Re}\tilde{f}(x) - |\tilde{f}(x)| = 0 \implies \operatorname{Re}\tilde{f}(x) = |\tilde{f}(x)|$ for the above integral to be 0. This implies that

$$(\operatorname{Re}\tilde{f}(x))^2 = (\operatorname{Re}\tilde{f}(x))^2 + (\operatorname{Im}\tilde{f}(x))^2 \implies \operatorname{Im}\tilde{f}(x) \equiv 0.$$

Hence, $\tilde{f}(x)$ lies on the positive real axis, so $\arg\tilde{f} = 2\pi k$, where $k \in \mathbb{Z}$ on $\{x \in \mathbb{R} \mid f(x) \neq 0\}$. Undoing the rotation yields $\arg f = \arg A + 2\pi k$, so $\arg f$ is constant modulo 2π.

1.9 Formally solve the cubic equation $ax^3 + bx^2 + cx + d = 0$, where $x, a, b, c, d \in \mathbb{C}$, $a \neq 0$, by the following reduction process:

a. Set $x = u + t$ and choose the constant t so that the coefficient of u^2 is equal to zero.

b. If the coefficient of u is also zero, then take a cube root to solve. If the coefficient of u is non-zero, set $u = kv$ and choose the constant k so that $v^3 = 3v + r$, for some constant r.

c. Set $v = z + 1/z$ and obtain a quadratic equation for z^3. The map $z + 1/z$ is important for several reasons, including constructing what are called conformal maps.

d. Use the quadratic formula to find two possible values for z^3, and then take a cube root to solve for z.

e. In Section II.2 we will show that the cubic equation has exactly three solutions, counting multiplicity. But the process in this exercise appears to generate more solutions, if we use two solutions to the quadratic and all three cube roots. Moreover there might be more than one valid choice for the constants used to reduce to a simpler equation. Explain.

Solution a. Let $x = u + t$. Then the cubic becomes

$$\begin{aligned} &au^3 + 3au^2t + 3aut^2 + at^3 + bu^2 + 2but + bt^2 + cu + ct + d \\ =\ &au^3 + (3at + b)u^2 + (3at^2 + 2bt + c)u + at^3 + bt^2 + ct + d \end{aligned}$$

For the coefficient of u^2 to be 0, we take $t = -\frac{b}{3a}$.

b. If $t = -\frac{b}{3a}$ is a root of $3at^2 + 2bt + c$, then we have that

$$\begin{aligned} u^3 &= \frac{b^3}{27a^3} - \frac{b^3}{9a^3} + \frac{bc}{3a^2} - \frac{d}{a} \\ u &= \sqrt[3]{|u^3|}e^{i \arg u^3} e^{i2\pi \frac{k}{3}} \end{aligned}$$

where letting $k = 0, 1, 2$ gives us the 3 roots desired.

Otherwise, let $u = kv$. Then the equation becomes

$$ak^3v^3 + \left(3a\frac{b^2}{9a^2} - 2b\frac{b}{3a} + c\right)kv - \frac{b^3}{27a^2} + \frac{b^3}{9a^2} - \frac{bc}{3a} + d = 0$$

$$v^3 + \left(3\frac{b^2}{9a^2} - 2b\frac{b}{3a^2} + \frac{c}{a}\right)\frac{v}{k^2} - \frac{b^3}{27a^3k^3} + \frac{b^3}{9a^3k^3} - \frac{bc}{3a^2k^3} + \frac{d}{ak^3} = 0$$

Taking $k = \frac{1}{\sqrt{3}}\sqrt{\frac{b^2}{3a} - \frac{2b^2}{3a} + c}$ and letting $r = -\left(-\frac{b^3}{27a^3k^3} + \frac{b^3}{9a^3k^3} - \frac{bc}{3a^2k^3} + \frac{d}{ak^3}\right)$ yields $v^3 = 3v + r$ as desired.

c. Set $v = z + \frac{1}{z}$. Then we get

$$\begin{aligned} z^3 + \frac{3}{z} + 3z + \frac{1}{z^3} &= 3z + \frac{3}{z} + r \\ z^6 - rz^3 + 1 &= 0 \end{aligned}$$

d.

$$\begin{aligned} z^3 &= \frac{r \pm \sqrt{r^2 - 4}}{2} \\ z &= \sqrt[3]{|z^3|}e^{i \arg z^3} e^{i2\pi \frac{k}{3}} \end{aligned}$$

and letting $k = 0, 1, 2$ gives us the roots desired.

e. While we have 6 roots, the cubic equation must have 3 roots (counting multiplicity) by the fundamental theorem of algebra. In other words, the equation we ended up with in part (c) must have repetitive roots. I.e., there are roots with multiplicity greater than 1.

In part (b), we could have chose k to be the negative of that expression to reduce the equation. This tells us that for our choice of r, there were two possible ways to reduce the equation in part (b): $v^3 = 3v + r$ and $v^3 = 3v - r$ (note that the v's are different in each equation).