****18 6.2.13** Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$ which are continuous, i.e., the space of continuous real-valued functions on the real line. Let T be the linear operator on V defined by

$$(Tf)(x) = \int_0^x f(t)\,\mathrm{d}t.$$

Prove that T has no characteristic values.

Solution Suppose f is an eigenvector of T. Then there exists $\lambda \in \mathbb{R}$ such that $(Tf)(x) = \lambda f(x)$. I.e.,

$$\begin{aligned}
\int_0^x f(t)\,\mathrm{d}t &= \lambda f(x) \\
f(x) &= \lambda f'(x) \\
f(x) &= Ce^{\frac{x}{\lambda}} \\
\int_0^x f(t)\,\mathrm{d}t = \int_0^x Ce^{\frac{t}{\lambda}}\,\mathrm{d}t &= \lambda Ce^{\frac{x}{\lambda}} \\
C\lambda e^{\frac{x}{\lambda}} - C\lambda &= C\lambda e^{\frac{x}{\lambda}} \\
C\lambda &= 0
\end{aligned}$$

This implies that $\lambda = 0$ or $C = 0$. In the first case, we get that $f(x) = 0$, and in the second case, $f(x)$ is undefined. Thus, T has no characteristic values.

****19** Suppose V has dimension n and that $T : V \to V$. Suppose $T^2 = 0$ and let k be the dimension of $R(T)$. Show that there is a basis $\mathfrak{B}$ so that $[T]_{\mathfrak{B}}$ is all zeros except for k ones.

Solution Note that if $\alpha \in R(T)$, $T\alpha = 0$ since $T^2 = 0$, so $R(T) \subseteq N(T)$. By rank-nullity, we have that the dimension of $N(T)$ is $n-2k$. Let $\alpha_1 = \beta_1, \dots, \alpha_k = \beta_k$ be linearly independent vectors such that $\alpha_{k+1} = T\beta_1, \dots, \alpha_{2k} = T\beta_k$ are a basis of $R(T)$. Then let $\alpha_{2k+1}, \dots, \alpha_n$ be so that $\alpha_{k+1}, \dots, \alpha_n$ is a basis of $N(T)$. We will show that $\mathfrak{B} = \{\alpha_1, \dots, \alpha_n\}$ is indeed linearly independent, which means that it spans V. Consider

$$\begin{aligned}
c_1\alpha_1 + \cdots + c_k\alpha_k + c_{k+1}\alpha_{k+1} + \cdots + c_{2k}\alpha_{2k} + c_{2k+1}\alpha_{2k+1} + \cdots + c_n\alpha_n = 0 \\
c_1\alpha_1 + \cdots + c_k\alpha_k + c_{k+1}T\alpha_1 + \cdots + c_{2k}T\alpha_k + c_{2k+1}\alpha_{2k+1} + \cdots + c_n\alpha_n = 0
\end{aligned}$$

If we apply T to the left-side, we get that $c_1\alpha_1 + \cdots + c_k\alpha_k = 0$. Since $\alpha_1, \dots, \alpha_k$ were linearly independent, it follows that $c_1 = \cdots = c_k = 0$. Then we are left with

$$c_{k+1}T\alpha_1 + \cdots + c_{2k}T\alpha_k + c_{2k+1}\alpha_{2k+1} + \cdots + c_n\alpha_n = 0.$$

By construction, $T\alpha_1, \dots, T\alpha_k, \alpha_{2k+1}, \dots, \alpha_n$ are a basis of $N(T)$, so it follows that $c_{k+1} = \cdots = c_n = 0$. Thus, the set $\mathfrak{B} = \{\alpha_1, \dots, \alpha_n\}$ is linearly independent. Then let

$$U = \begin{pmatrix} | & & | \\ \alpha_1 & \cdots & \alpha_n \\ | & & | \end{pmatrix}$$

U takes $\mathfrak{B}$ coordinates to standard coordinates, and U^{-1} does the opposite. Thus,

$$\begin{aligned}
[T]_{\mathfrak{B}} &= U^{-1}TU \\
&= U^{-1}T \begin{pmatrix} | & & | \\ \alpha_1 & \cdots & \alpha_n \\ | & & | \end{pmatrix} \\
&= U^{-1} \begin{pmatrix} | & & | \\ T\alpha_1 & \cdots & T\alpha_n \\ | & & | \end{pmatrix} \\
&= U^{-1} \begin{pmatrix} | & & | & | & & | \\ \alpha_{k+1} & \cdots & \alpha_k & 0 & \cdots & 0 \\ | & & | & | & & | \end{pmatrix} \\
&= \begin{pmatrix} | & & | & | & & | \\ U^{-1}\alpha_{k+1} & \cdots & U^{-1}\alpha_k & 0 & \cdots & 0 \\ | & & | & | & & | \end{pmatrix}
\end{aligned}$$

Since $\alpha_{k+1}, \ldots, \alpha_k$ are the basis vectors, their image under U^{-1} will have all components be zero except for a single one. There are k of these vectors, so the resulting matrix will have all entries be zero except for k ones.

****20** The field of scalars for this problem is $\mathbb{C}$. Suppose $T : V \to V$. Suppose that every eigenvalue of T is zero and that the dimension of $N(T) = 1$. Show that $N(T) \subseteq R(T)$ if $R(T) \neq \{0\}$. Hint: What can you say about the restriction of T to $R(T)$?

Solution Since all eigenvalues are 0, the characteristic polynomial is $\lambda^n = 0$, where $n = \dim V$. By Cayley–Hamilton, $A^n = 0$. Let $\alpha \in R(T)$. Since $A^{n-1}\alpha = 0$, then for some $1 \leq k \leq n-1$, $A^k\alpha = 0$ and $A^m\alpha \neq 0$ if $m < k$. Then $A^{k-1}\alpha \in R(T)$ and it also spans $N(T)$, since the nullity of T is 1. It follows that $N(T) \subseteq R(T)$.