1 a. Show how Theorem 3.1 of the notes to Lecture 1 is a special case of Zorn's lemma.

b. Let A be nonempty and let X_α be nonempty for every $\alpha \in A$. Let $\mathcal{Y}$ be the collection of all pairs $(B, \{x_\alpha\}_{\alpha\in B})$ where $B \subseteq A$ and $x_\alpha \in X_\alpha$ for all $\alpha \in B$. Define a relation $\preceq$ on $\mathcal{Y}$ by:

$$(B, \{x_\alpha\}_{\alpha\in B}) \preceq (C, \{y_\alpha\}_{\alpha\in C}) \iff B \subseteq C \text{ and } x_\alpha = y_\alpha \ \forall \alpha \in B.$$

Prove that $\mathcal{Y}$ is nonempty and that $\preceq$ is a partial order on $\mathcal{Y}$.

c. Prove AC using Zorn's lemma.

Solution a. Zorn's lemma tells us that given a poset such that every chain has an *upper bound*, the poset must have a maximal element.

If we consider the poset $(\mathcal{F}, \subseteq)$ ($\mathcal{F}$ is as in the theorem statement). Condition (i) and (ii) demand that each chain has a *maximal element*, namely, $\cup\mathcal{C}$, which is an upper bound for the chain, since a chain is totally ordered. In this case, $\mathcal{F}$ has a maximal element.

b. Notice that $(\emptyset, \emptyset) \in \mathcal{Y}$, so $\mathcal{Y}$ is non-empty.

Reflexivity:

It's clear that $B \subseteq B$ and $x_\alpha = x_\alpha$ for all α, so reflexivity holds.

Transitivity:

Let

$$(B, \{x_\alpha\}_{\alpha\in B}) \preceq (C, \{y_\alpha\}_{\alpha\in C}) \quad \text{and} \quad (C, \{y_\alpha\}_{\alpha\in C}) \preceq (D, \{z_\alpha\}_{\alpha\in D}).$$

For all $\alpha \in B \subseteq C$, $x_\alpha = y_\alpha = z_\alpha$, by definition and transitivity of equality. Since $\subseteq$ is a partial ordering on $\mathcal{P}(A)$, we get $B \subseteq C \subseteq D$ by reflexivity. Thus,

$$(B, \{x_\alpha\}_{\alpha\in B}) \preceq (D, \{z_\alpha\}_{\alpha\in D})$$

so transitivity holds.

Antisymmetry:

Let

$$(B, \{x_\alpha\}_{\alpha\in B}) \preceq (C, \{y_\alpha\}_{\alpha\in C}) \quad \text{and} \quad (C, \{y_\alpha\}_{\alpha\in C}) \preceq (B, \{x_\alpha\}_{\alpha\in B}).$$

By definition, we have $B \subseteq C \subseteq B$, so $B = C$. It follows that since $x_\alpha = y_\alpha$ for all $\alpha \in B = C$, the two pairs are the same, so antisymmetry holds.

Thus, $(\mathcal{Y}, \preceq)$ is a poset.

c. Let $\mathcal{C} \subseteq \mathcal{Y}$ be a chain and write $\mathcal{C} = \{(B_1, \{x_\alpha^{(1)}\}), (B_2, \{x_\alpha^{(2)}\}), \ldots\}$. We claim that

$$(B, \{x_\alpha\}), \quad \text{where} \quad B = \bigcup_{i=1}^\infty B_i$$

and $x_\alpha = x_\alpha^{(i)}$ if $\alpha \in B_i$. This is well-defined since if $\alpha \in B_i$ and $\alpha \in B_j$, then because chains are totally ordered, we have $B_i \subseteq B_j$ or $B_j \subseteq B_i$. Either way, by definition of the partial ordering, $x_\alpha^{(i)} = x_\alpha^{(j)}$.

Given any $(B_i, \{x_\alpha^{(i)}\}) \in \mathcal{C}$, it's clear that $B_i \subseteq B$. Moreover, if $\alpha \in B_i$, then by definition, $x_\alpha = x_\alpha^{(i)}$, so $(B_i, \{x_\alpha^{(i)}\}) \preceq (B, \{x_\alpha\})$, so $\mathcal{C}$ has an upper bound.

By Zorn's lemma, $\mathcal{Y}$ has a maximal element, and this element must be of the form $(B, \{x_\alpha\}_{\alpha\in B})$. We claim that $B = A$.

If not, then there exists $\alpha_0 \in A \setminus B$. X_{α_0} is non-empty by assumption, so pick an element $x_{\alpha_0} \in X_{\alpha_0}$. But

$$(B, \{x_\alpha\}_{\alpha\in B}) \preceq (B \cup \{\alpha_0\}, \{x_\alpha\}_{\alpha\in B} \cup \{x_{\alpha_0}\}),$$

without equality. This is a contradiction, as we assumed that B was a maximal element, so $B = A$.

Thus, there exists a function $f\colon A \to \cup X_\alpha$, namely $f(\alpha) = x_\alpha$ for all $\alpha \in A$, so the axiom of choice is proved.

2 For $A \subseteq \mathbb{R}$ and $f\colon A \to \mathbb{R}$, consider the **additivity equation**:

$$f(x+y) = f(x) + f(y) \quad \text{whenever } x, y, x+y \text{ all lie in } A. \tag{1}$$

a. Prove that, if $A = \mathbb{R}$ and f satisfies (1), then f also satisfies

$$f(ax) = af(x) \quad \text{whenever } a \in \mathbb{Q} \text{ and } x \in \mathbb{R}.$$

b. Prove that, if $A = \mathbb{R}$, f satisfies (1), and f is continuous, then f has the form $f(x) = \lambda x$ for some fixed $\lambda \in \mathbb{R}$.

c. Let $u, v \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$, meaning that

$$\text{if } a, b \in \mathbb{Q} \text{ and } au + bv = 0 \text{ then } a = b = 0.$$

Let $A := \{au + bv \mid a, b \in \mathbb{Q}\}$. Prove that there exists a function $f\colon A \to \mathbb{R}$ which satisfies (1) but is *not* of the form $f(x) = \lambda x$ for any fixed $\lambda \in \mathbb{R}$.

d. Prove that there is a function $f\colon \mathbb{R} \to \mathbb{R}$ which satisfies (1) but is not of the form $f(x) = \lambda x$ for any $\lambda \in \mathbb{R}$.

Solution a. First notice that (1) holds whenever $a \in \mathbb{N}$, by induction.
Let $x \in \mathbb{R}$, $a/b \in \mathbb{Q}$, with $a, b \in \mathbb{N}$. Then $(a/b)x \in \mathbb{R}$, and

$$af(x) = f(ax) = f\Big(b \cdot \frac{a}{b}x\Big) = bf\Big(\frac{a}{b}x\Big) \implies f\Big(\frac{a}{b}x\Big) = \frac{a}{b}f(x),$$

so the equation holds for rational coefficients.

b. Let us first consider a function $f\big|_{\mathbb{Q}}$ satisfying the hypotheses of the problem. Let $\lambda = f\big|_{\mathbb{Q}}(1)$. Then for any $r \in \mathbb{Q}$,

$$f(r) = f(r \cdot 1) = rf(1) = \lambda r,$$

so $f\big|_{\mathbb{Q}}$ has the desired form.
Let f be a continuous extension of $f\big|_{\mathbb{Q}}$, and pick $x \in \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a sequence $\{r_n\}n \geq 1 \subseteq \mathbb{Q}$ with $r_n \xrightarrow{n \to \infty} x$. Since f is continuous,

$$f(x) = \lim_{n \to \infty} f(r_n) = \lim_{n \to \infty} \lambda r_n = \lambda x,$$

by linearity of the limit. Hence, f has the desired form.

c. Let $f(u) = v$ and $f(v) = u$, and extend f so that it satisfies that additivity equation, i.e., for any $a, b \in \mathbb{Q}$, set

$$f(au + bv) = af(u) + bf(v).$$

Assume that there exists $\lambda \in \mathbb{R}$ so that $f(x) = \lambda x$. Then

$$v = f(u) = \lambda u \implies v - \lambda u = 0.$$

Since u and v are linearly independent and $1 \neq 0$, we have that $\lambda = 0$, which implies that f is identically 0. But this cannot be the case, since this implies that $0 = f(u) = v$, which means that u and v cannot be linearly independent, a contradiction. Hence, f does not have the form given in the question.

d. We will regard $\mathbb{R}$ as a vector space over $\mathbb{Q}$.
Consider the set $S = \{(A, f) \mid A \subseteq \mathbb{R},\ f\colon A \to \mathbb{R} \text{ and satisfies (1), but is not of the form } \lambda x\}$ with the partial ordering

$$(A, f) \preceq (B, g) \iff A \subseteq B \text{ and } g\big|_A = f.$$

S is non-empty, by part (c).
Let $\mathcal{C} \subseteq S$ be a chain, and write $\mathcal{C} = \{(A_1, f_1), \ldots\}$. Take $A = \cup A_i$. We claim that (A, f) is an upper bound, where

$$f\colon A \to \mathbb{R} \quad \text{and} \quad f\big|_{A_i} = f_i \ \forall i.$$

f is well-defined: if $x \in A_i$ and $x \in A_j$, then because $\mathcal{C}$ is a chain, $A_i \subseteq A_j$ or $A_j \subseteq A_i$. Either way, $f_j(x) = f_j\big|_{A_i}(x) = f_i\big|_{A_j}(x) = f_i(x)$.

Let $(A_i, f_i) \in \mathcal{C}$. It's clear that $A_i \subseteq A$ by definition, and furthermore, $f\big|_{A_i} = f_i$, by definition also. Hence, $(A_i, f_i) \preceq (A, f)$. This shows that f does not have the form λx, since f_i does not have that form, so each chain has an upper bound.

By Zorn's lemma, S has a maximal element (A, f). We claim that $A = \mathbb{R}$. If not, then there is $x \in \mathbb{R} \backslash \operatorname{span} A$, and we can extend f to f' by setting $f'(x)$ to be anything, and letting $f'(x+y) = f'(x)+f(y)$, for $y \in A$. Then $f'\big|_A = f$ does not have the form λx, which means that f' does not have the same form. But this implies that $(A, f) \preceq (A \cup \operatorname{span}\{x\}, f')$ without equality, a contradiction. So $A = \mathbb{R}$, and this completes the proof.

---

3 Consider the set $\mathcal{P}(\mathbb{N})$ of all subsets of $\mathbb{N} = \{1, 2, \ldots\}$ as a partially ordered set under set inclusion.

a. Given an example of an infinite subfamily $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ which has all four of the following properties:

   i. it is down-closed (meaning that $A \subseteq B \in \mathcal{F} \implies A \in \mathcal{F}$),

   ii. it is finite-chain-closed (meaning that if $\{C_1, \ldots, C_m\} \subseteq \mathcal{F}$ is a chain, then $C_1 \cup \cdots \cup C_m \in \mathcal{F}$),

   iii. it is not chain-closed, and

   iv. it has no maximal element.

b. Let $\sigma \colon \mathbb{N} \to \mathbb{N}$ be a permutation, and let $\mathcal{C}_\sigma$ be the family

$$\{\emptyset, \{\sigma(1)\}, \{\sigma(1), \sigma(2)\}, \{\sigma(1), \sigma(2), \sigma(3)\}, \ldots, \mathbb{N}\}.$$

   Prove that $\mathcal{C}_\sigma$ is a maximal chain in $\mathcal{P}(\mathbb{N})$.

c. Prove that $\mathcal{P}(\mathbb{N})$ contains a chain $\mathcal{C}$ with the property that the set

$$\{C \in \mathcal{C} \mid A \subseteq C \subseteq B\}$$

   is uncountable whenever $A, B \in \mathcal{C}$ and $A \subsetneq B$.

Solution a. Let $\mathcal{F} = \{N \subseteq \mathbb{N} \mid N \text{ is finite}\}$.

(i) is clear.

(ii) follows from the fact that finite unions of finite sets are finite.

$\mathcal{F}$ satisfies (iii) since

$$\mathbb{N} = \bigcup_{n \in \mathbb{N}} \{1, \ldots, n\} \notin \mathcal{F}.$$

Lastly, it satisfies (iv) because given any element N in $\mathcal{F}$, we can add a new element n to it so that $N \subsetneq N \cup \{n\}$ and $N \cup \{n\}$ is finite.

b. It's clear that it's a chain, since each succeeding element contains the previous.

Suppose it were not a maximal chain, so that there exists $N \in \mathcal{P}(\mathbb{N}) \setminus C_\sigma$ such that $C_\sigma \cup \{N\}$ is a chain. If N is infinite, it must omit some $\sigma(i)$, or else it is $\mathbb{N}$. But in this case, N cannot be compared with $\{\sigma(1), \ldots, \sigma(i)\}$, a contradiction, so N must be finite.

By definition, there exists $\{\sigma(1), \ldots, \sigma(n)\} \subseteq N \subseteq \{\sigma(1), \ldots, \sigma(m)\}$, with $0 \leq n < m$, so N contains $\sigma(1), \ldots, \sigma(n)$.

But this means that N must contain $\sigma(n+1), \ldots, \sigma(m)$. Otherwise, suppose N does not contain $\sigma(i)$, but contains $\sigma(j)$, with $n+1 \leq i < j \leq m$. This must be the case or else $N \in C_\sigma$. But in this case also, N cannot be compared with $\{\sigma(1), \ldots, \sigma(i)\}$, a contradiction. Hence, no such N exists, so C_σ is maximal.

c. Notice that $\mathbb{Q} \cap (0,1)$ is countably infinite, so there exists a bijection $\varphi \colon \mathbb{N} \to \mathbb{Q} \cap (0,1)$. Hence, we can consider $\mathcal{P}(\mathbb{Q} \cap (0,1))$ instead of $\mathcal{P}(\mathbb{N})$.

For any real number $x \in (0,1)$, consider $\mathcal{Q}_r = \{x \in \mathbb{Q} \mid x < r\}$, and let $\mathcal{C} = \{\mathcal{Q}_r \mid r \in \mathbb{R}\}$. This is a chain, since $\mathcal{Q}_r \subseteq \mathcal{Q}_s$ whenever $r \leq s \in \mathbb{R}$, which is totally ordered. Moreover, any interval (r, s) is uncountable, so there are uncountably many t so that $\mathcal{Q}_r \subseteq \mathcal{Q}_t \subseteq \mathcal{Q}_s$.

4.8 If X is an infinite set with the cofinite topology and $\{x_j\}$ is a sequence of distinct points in X, then $x_j \to x$ for every $x \in X$.

Solution Let $\{x_j\}$ be as in the problem, and let $x \in X$ be arbitrary.

Let U be an open neighborhood of x, i.e., $U^{\mathrm{c}} = \{y_1, \ldots, y_n\}$ for some $n \in \mathbb{N}$. Since $\{x_j\}$ has infinitely many distinct points, there exists $k \in \mathbb{N}$ so that $x_j \notin U^{\mathrm{c}} \iff x_j \in U$ whenever $j \geq k$.

Since U arbitrary, it follows that $x_j \xrightarrow{j \to \infty} x$. Similarly, since x was arbitrary, it follows that x_j converges to every x in X.

4.13 If X is a topological space, U is open in X, and A is dense in X, then $\overline{U} = \overline{U \cap A}$.

Solution "$\subseteq$"

Let $x \in \overline{U}$, and let V be an open neighborhood of x.

Since A is dense, $V \cap A \neq \emptyset$. Since $x \in \overline{U}$, we have that $U \cap V \neq \emptyset$. Hence, $V \cap (U \cap A) \neq \emptyset$. Since V was arbitrary, $x \in \overline{U \cap A}$.

"$\supseteq$"

Let $x \in \overline{U \cap A}$, and let V be an open neighborhood of x.

By definition, $V \cap (U \cap A) \neq \emptyset$. In particular, $V \cap U \neq \emptyset$, so as V was arbitrary, $x \in \overline{U}$.

4.15 If X is a topological space, $A \subseteq X$ is closed, and $g \in C(A)$ satisfies $g = 0$ on ∂A, then the extension of g to X defined by $g(x) = 0$ for $x \in A^{\mathrm{c}}$ is continuous.

Solution Let $g \colon X \to \mathbb{C}$ be as described in the problem.

Notice that $\partial A = \partial(A^{\mathrm{c}})$. We'll first show that $(A^\circ)^{\mathrm{c}} = \overline{A^{\mathrm{c}}}$.

Let $x \in (A^\circ)^{\mathrm{c}}$. By definition, for all open neighborhoods $U \ni x$, $U \cap A^{\mathrm{c}} \neq \emptyset$. Otherwise, $U \subseteq A \implies x \in A^\circ$.

Now let $x \in \overline{A^{\mathrm{c}}}$. By definition, for any open neighborhood $U \ni x$, $U \cap A^{\mathrm{c}} \neq \emptyset$, so $U \subsetneq A^\circ$ for any U, so $x \in (A^\circ)^{\mathrm{c}}$, and equality is proved.

Then
$$\partial A = \overline{A} \setminus A^\circ = A \cap (A^\circ)^{\mathrm{c}} = A \cap \overline{A^{\mathrm{c}}} = \overline{A^{\mathrm{c}}} \setminus A^{\mathrm{c}} = \overline{A^{\mathrm{c}}} \setminus (A^{\mathrm{c}})^\circ = \partial(A^{\mathrm{c}}).$$

Also note that this shows that $\overline{A^{\mathrm{c}}} = \partial A \cup A^{\mathrm{c}}$.

Let C be closed in $\mathbb{C}$. If C does not contain 0, then

$$g^{-1}(C) = \left(g\big|_A\right)^{-1}(C)$$

is closed, since g doesn't take on any non-zero values on A^{c} and because $g \in C(A)$.

If C contains 0, then
$$g^{-1}(C) = \left(g\big|_A\right)^{-1}(C) \cup A^{\mathrm{c}} \cup \partial A = \left(g\big|_A\right)^{-1}(C) \cup \overline{A^{\mathrm{c}}}.$$

Indeed, let $x \in g^{-1}(C)$. If $x \in A$, then $x \in \left(g\big|_A\right)^{-1}(C)$. Otherwise, $x \in A^{\mathrm{c}}$.

On the other hand, let $x \in \left(g\big|_A\right)^{-1}(C) \cup A^{\mathrm{c}}$. If $x \in A^{\mathrm{c}}$, then $g(x) = 0 \in C \implies x \in g^{-1}(C)$. Otherwise, $x \in A$, so $x \in \left(g\big|_A\right)^{-1}(C) \subseteq g^{-1}(C)$, so the two sets are equal.

Since $g \in C(A)$, $\left(g\big|_A\right)^{-1}(C)$ is closed and by definition, $\overline{A^{\mathrm{c}}}$ is closed. Since finite unions of finite sets are closed, it follows that $g^{-1}(C)$ is closed. Hence, g is continuous.