1 Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two bounded sequences. Show that

$$\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Solution There exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$a_n + b_n \leq \sup\{a_n \mid n \geq N\} + \sup\{b_n \mid n \geq N\}$$

As this holds for all $n \geq N$, we conclude that $\sup\{a_n \mid n \geq N\} + \sup\{b_n \mid n \geq N\}$ (which exist since $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are bounded) is an upper bound for $\{a_n + b_n\}_{n\geq 1}$. Thus,

$$\sup\{a_n + b_n \mid n \geq N\} \leq \sup\{a_n \mid n \geq N\} + \sup\{b_n \mid n \geq N\}$$
$$\limsup_{n\to\infty} a_n + b_n \leq \limsup_{n\to\infty} a_n + \limsup_{nto\infty} b_n$$

as desired.

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2 Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two bounded sequences of non-negative numbers. Show that

$$\limsup_{n\to\infty}(a_n b_n) \leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n).$$

Solution By definition, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \leq \sup\{a_n \mid n \geq N\}$ and $b_n \leq \sup\{b_n \mid n \geq N\}$. The supremums exist since $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are bounded. Since $0 < b_n \leq \sup\{b_n \mid n \geq N\}$, we can multiply the inequalities to get

$$a_n b_n \leq (\sup\{a_n \mid n \geq N\})(\sup\{b_n \mid n \geq N\})$$

Thus, $\sup\{a_n \mid n \geq N\}\sup\{b_n \mid n \geq N\}$ is an upper bound for $a_n b_n$, so by definition,

$$\sup\{a_n b_n \mid n \geq N\} \leq \sup\{a_n \mid n \geq N\}\sup\{b_n \mid n \geq N\}$$

Taking $N \to \infty$, we have

$$\limsup_{n\to\infty}(a_n b_n) \leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$$

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3 Show that a sequence $\{a_n\}_{n\geq 1}$ is bounded if and only if $\limsup |a_n| < \infty$.

Solution "$\Longrightarrow$"

Let $\{a_n\}_{n\geq 1}$ be bounded. Then there exists a positive number such that $|a_n| \leq M$ for all $n \geq 1$. By definition, $\sup\{|n| \mid n \geq N\} \leq M$ for all $N \in \mathbb{N}$. Since $|a_n| \geq 0$, $\sup\{|n| \mid n \geq N\} \geq 0$ also for all $N \in \mathbb{N}$. Since $\{\sup\{|n| \mid n \geq N\}\}_{N\in\mathbb{N}}$ is bounded below and decreases monotonically for all N, it must converge to a finite number in the interval $[0, M]$.

"$\Longleftarrow$"

Suppose $\{a_n\}_{n\geq 1}$ is such that $\limsup |a_n| < \infty$. There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\sup\{|a_n| \mid n \geq N\}$ is finite, since $\limsup |a_n|$ is finite, and that $|a_n| \leq \sup\{|a_n| \mid n \geq N\}$. Hence,

$$-\sup\{|a_n| \mid n \geq N\} < a_n < \sup\{|a_n| \mid n \geq N\}$$

For $n < N$, we have finitely many terms of a_n, so $\max\{|a_1|, \ldots, |a_{N-1}|, \sup\{|a_n| \mid n \geq N\}\}$ exists and is an upper bound for $|a_n|$. Thus, a_n is bounded for all $n \geq N$, so $\{a_n\}_{n\geq 1}$ is bounded.

4 Let A denote the set of subsequential limits of a sequence $\{a_n\}_{n\geq 1}$. Suppose that $\{b_n\}_{n\geq 1}$ is a subsequence in $A \cap \mathbb{R}$ such that $\lim_{n\to\infty} b_n$ exists in $\mathbb{R} \cup \{\pm\infty\}$. Show that $\lim_{n\to\infty} b_n$ belongs to A.

Solution $\lim_{n\to\infty} b_n = \infty$

We will show that $\limsup a_n = \sup A = \infty$. As $\lim_{n\to\infty} b_n = \infty$, then for every $M > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $b_n > M$. $b_n \in A$, so there exists $a \in A$ such that $a > M$ for all $M > 0$. It must be that $\sup A = \infty$, as desired.

$\lim_{n\to\infty} b_n = -\infty$

We will show that $\liminf a_n = \inf A = -\infty$. As $\lim_{n\to\infty} b_n = -\infty$, then for every $M < 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $b_n < M$. $b_n \in A$, so there exists $a \in A$ such that $a < M$ for all $M < 0$. It must be that $\inf A = -\infty$, as desired.

$\lim_{n\to\infty} b_n = b \in \mathbb{R}$

Let $\epsilon > 0$. As $\lim_{n\to\infty} b_n = b$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|b_n - b| < \frac{\epsilon}{2}$.

As $b_n \in A$ for all n, there exists a subsequence $\{a_{k_m}\}_{m\geq 1}$ that converges to b_n. So, there exists $M \in \mathbb{N}$ such that for all $m \geq M$, we have $|a_{k_m} - b_n| < \frac{\epsilon}{2}$.

Thus, choose $N_0 = \max\{N, M\}$. Then for all $n, m \geq N_0$,

$$|a_{k_m} - b| \leq |a_{k_m} - b_n| + |b_n - b| < \epsilon$$

Thus, there exists a subsequence of $\{a_n\}_{n\geq 1}$ that converges to b, so $\lim_{n\to\infty} b_n = b \in A$.

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5 Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers that is bounded above. Prove that $L = \limsup a_n$ has the following properties:

a. For every $\epsilon > 0$ there are only finitely many n for which $a_n > L + \epsilon$.

b. For every $\epsilon > 0$ there are infinitely many n for which $a_n > L - \epsilon$.

Solution For all $N \in \mathbb{N}$, $\sup\{a_n \mid n \geq N\}$ is finite since $\{a_n\}_{n\geq 1}$ is bounded. Let $\epsilon > 0$. As $L = \limsup a_n$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$a_n < \sup\{a_n \mid n \geq N\} < L + \epsilon$$

Thus, the only values of n where $a_n > L + \epsilon$ must satisfy $n < N$. There are clearly only finitely many possible values, so it follows that there are finitely many n for which $a_n > L + \epsilon$.

Similarly, for all $n \geq N$, we also have

$$L - \epsilon < \sup\{a_n \mid n \geq N\}$$

Clearly, $L - \epsilon$ is not an upper bound for a_n, so there exists a_{k_1} such that $L - \epsilon < a_{k_1} < \sup\{a_n \mid n \geq N\}$. Similarly, $a_{k_1} < \sup\{a_n \mid n \geq N\}$, so it is not an upper bound. Thus, we can find a_{k_2} such that $L - \epsilon < a_{k_1} < a_{k_2} < \sup\{a_n \mid n \geq N\}$. We proceed by induction.

Inductive step:

Let a_{k_n} be such that $L - \epsilon < a_{k_1} < a_{k_2} < \ldots < a_{k_{n-1}} < a_{k_n} < \sup\{a_n \mid n \geq N\}$. a_{k_n} is not an upper bound for $\sup\{a_n \mid n \geq N\}$, so we can find $a_{k_{n+1}}$ such that $L - \epsilon < a_{k_1} < a_{k_2} < \ldots < a_{k_{n-1}} < a_{k_n} < a_{k_{n+1}} < \sup\{a_n \mid n \geq N\}$.

Since the inductive step holds, then by the principle of mathematical induction, we can find infinitely many a_n such that $a_n > L - \epsilon$. Thus, both (a) and (b) hold.

6 Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Prove that there can be at most one real number L with the following two properties:

a. For every $\epsilon > 0$ there are only finitely many n for which $a_n > L + \epsilon$.

b. For every $\epsilon > 0$ there are infinitely many n for which $a_n > L - \epsilon$.

Solution Suppose there could be more than one real number with those two properties. It suffices to show that there cannot be two real numbers with those properties, as if there were more, we could apply the following argument to any pair of them.

Let L and M be two real numbers that satisfy (a) and (b). Assume without loss of generality that $L < M$. Let $\epsilon > 0$ be such that $L + \epsilon < M - \epsilon$. Then there are finitely many n for which

$$L + \epsilon < a_n$$

and infinitely many n for which

$$L + \epsilon < M - \epsilon < a_n$$

But this is a contradiction, as we cannot have both finitely and infinitely many n that satisfy both inequalities. Thus, there can exist at most one real number that satisfy both.

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7 Let $\{a_n\}_{n\geq 1}$ be a sequence of non-negative numbers. For $n \geq 1$, define

$$s_n = \frac{a_1 + \cdots + a_n}{n}.$$

a. Show that

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} a_n.$$

b. Conclude that if $\lim_{n\to\infty} a_n$ exists, then $\lim_{n\to\infty} s_n$ exists and $\lim_{n\to\infty} = \lim_{n\to\infty} a_n$.

Solution a. It is obvious that $\liminf s_n \leq \limsup s_n$. So, we only need to show that $\liminf a_n \leq \liminf s_n$ and $\limsup s_n \leq \limsup a_n$.

We first prove a lemma: $\limsup(-a_n) = -\liminf(a_n)$.

Let $A = \limsup(-a_n)$. Then for all a_n, we have $-a_n \leq A \implies -A \leq a_n$. Thus, $-A$ is a lower bound for $\{a_n\}_{n\geq 1}$. Suppose there exists M such that $-A < M \leq a_n$. Then $a_n < -M \leq A$. But this is a contradiction, as A is the least upper bound of $\{-a_n\}_{n\geq 1}$. Hence, no such M exists, which means that $-A = -\limsup(-a_n) = \liminf(a_n)$, as desired.

Let $v_N = \sup\{a_n \mid n \geq N\}$ and $N \in \mathbb{N}$. Then for all $n \geq N$,

$$\begin{aligned}
s_n = \frac{a_1 + \cdots + a_n}{n} &= \frac{a_1 + \cdots + a_N}{n} + \frac{a_{N+1} + \cdots + a_n}{n} \\
&\leq \frac{a_1 + \cdots + a_N}{n} + \frac{(n-N)v_N}{n} \\
&< \frac{a_1 + \cdots + a_N}{n} + v_N \\
\sup\{s_n \mid n \geq N\} &\leq \frac{a_1 + \cdots + a_N}{n} + v_N \leq \sup\left\{ \frac{a_1 + \cdots + a_N}{n} \,\middle|\, n \geq N \right\} + \sup\{a_n \mid n \geq N\} \\
&= \frac{M}{N} + \sup\{a_n \mid n \geq N\}
\end{aligned}$$

Where M is some real number (since the supremum of the set will be some number divided by N). Taking $N \to \infty$ yields

$$\limsup s_n \leq \limsup a_n.$$

By the lemma, it follows that

$$\limsup(-s_n) = -\liminf(s_n)$$
$$\limsup(-a_n) = -\liminf(a_n)$$

Applying the same argument as above, but with $-s_n$ and $-a_n$, we get

$$\limsup(-s_n) \le \limsup(-a_n)$$
$$-\liminf(s_n) \le -\liminf(a_n)$$
$$\liminf(s_n) \ge \liminf(a_n)$$

as desired.

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8 Let $\{a_n\}_{n\ge1}$ be a sequence such that $\liminf_{n\to\infty}|a_n| = 0$. Prove that there is a subsequence $\{a_{k_n}\}_{n\ge1}$ such that the series $\sum_{n=1}^{\infty} a_{k_n}$ converges.

Solution For every $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $n \ge N$, we have $|a_n| < \epsilon$. Through induction, we will construct a subsequence $\{|a_{k_n}|\}_{n\ge1}$, which must comprise of non-negative terms, whose terms less than those of $\{\frac{1}{n^2}\}_{n\ge1}$. Then it follows by comparison that the constructed subsequence converges.

Base step:

Let $\epsilon = \frac{1}{1^2}$. Then we can find $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$, we have $|a_{k_n}| < \frac{1}{1^2}$. Take $k_1 = N_1$, so that $|a_{k_1}| < \frac{1}{1^2}$.

Inductive step:

We have k_n so that $|a_{k_n}| < \frac{1}{n^2}$. We will now show that we can find k_{n+1} so that $|a_{k_{n+1}}| < \frac{1}{(n+1)^2}$.

Let $\epsilon = \frac{1}{(n+1)^2}$ whenever $n \ge N_n$. Then we can find $N_{n+1} > N_n \in \mathbb{N}$ such that for all $n \ge N_{n+1}$, we have $|a_{k_{n+1}}| < \frac{1}{(n+1)^2}$. Take $k_{n+1} = N_{n+1}$, so that $|a_{k_{n+1}}| < \frac{1}{(n+1)^2}$.

Thus, by the principle of mathematical induction, we have constructed a subsequence $\{|a_{k_n}|\}_{n\ge1}$ such that $0 \le |a_{k_n}| < \frac{1}{n^2}$ for all n. As the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it follows by comparison that $\sum_{n=1}^{\infty}|a_{k_n}|$ converges $\implies \sum_{n=1}^{\infty} a_{k_n}$ converges. Thus, we have found a subsequence of $\{a_n\}_{n\ge1}$ whose series converges.

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9 Determine which of the following series converge. Justify your answers.

$$(1)\sum_{n\ge1}\frac{n^4}{2^n} \qquad (2)\sum_{n\ge1}\frac{2^n}{n!} \qquad (3)\sum_{n\ge1}(-1)^n \qquad (4)\sum_{n\ge0}\sin\frac{n\pi}{3}$$

Solution (1)

$$\lim_{n\to\infty}\left|\frac{(n+1)^4}{2^{n+1}}\cdot\frac{2^n}{n^4}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^4\cdot\frac{1}{2} = \frac{1}{2} < 1$$

So by the ratio test, the series converges.

(2)

$$\lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right| = \lim_{n\to\infty} 2\cdot\frac{1}{n+1} = 0 < 1$$

So by the ratio test, the series converges.

(3) The subsequential limits of $(-1)^n$ are 1 and -1, so by a theorem, $\lim_{n\to\infty}(-1)^n$ does not exist $\implies$ the series diverges.

(4) The subsequential limits of $\sin\frac{n\pi}{3}$ include at least 0 and $\frac{\sqrt{3}}{2}$, so by a theorem proved in class, $\lim_{n\to\infty}\sin\frac{n\pi}{3}$ does not exist $\implies$ the series diverges.