5.29 Let $\mathcal{Y} = L^1(\mu)$ where μ is the counting measure on $\mathbb{N}$, and let $\mathcal{X} = \{f \in \mathcal{Y} \mid \sum_1^\infty n|f(n)| < \infty\}$, equipped with the L^1 norm.

a. $\mathcal{X}$ is a proper dense subspace of $\mathcal{Y}$; hence $\mathcal{X}$ is not complete.

b. Define $T\colon \mathcal{X} \to \mathcal{Y}$ by $Tf(n) = nf(n)$. Then T is closed but not bounded.

c. Let $S = T^{-1}$. Then $S\colon \mathcal{Y} \to \mathcal{X}$ is bounded and surjective but not open.

Solution a. It is easy to see that $\mathcal{X}$ is a subspace. $0 \in \mathcal{X}$. It is closed under addition by using the triangle inequality and splitting the sum, and it is clearly closed under scaling, so $\mathcal{Y}$ is a subspace of $\mathcal{X}$.

Let $f \in \mathcal{Y}$, and let $\varepsilon > 0$.

Since f is integrable, there exists $N \in \mathbb{N}$ so that

$$\sum_{n=N_1}^{\infty} |f(n)| < \frac{\varepsilon}{2}.$$

Then for $1 \le n \le N-1$, set $g(n) = f(n)$, and for $n \ge N$, set $g(n) = f(n)$ for $n \le N-1$ and 0 for $n \ge N$. Then

$$\|f - g\| = \sum_{n=1}^{\infty} |f(n) - g(n)| = \sum_{n=N}^{\infty} |f(n)| < \varepsilon.$$

Moreover, $ng(n)$ is summable since it has finite support. Thus, $\mathcal{X}$ is dense in $\mathcal{Y}$.

It is also not dense, since $f(n) = 1/n^2$ is in $L^1(\mu)$, but $nf(n)$ is not, since it diverges. Hence, $\mathcal{X}$ is a proper dense subspace of $\mathcal{Y}$.

b. Let $\{Tf_k(n)\} \subseteq T(\mathcal{X})$ be a sequence which converges to $f(n)$ in $\mathcal{Y}$. By definition

$$\sum_{n=1}^{\infty} |nf_k(n) - f(n)| \xrightarrow{k\to\infty} 0.$$

This forces $|nf_k(n) - f(n)| \xrightarrow{k\to\infty} 0$ for each $n \ge 1$. Thus, $f_k(n)$ must converge to $f(n)/n$, which is in $\mathcal{X}$ since $f(n)$ is summable. Hence, $f(n) = Tf(n)/n$, so T is closed.

Now consider $\chi_{\{m\}}$. It's clearly an element of $\mathcal{X}$ with norm 1, but

$$\|T\chi_{\{m\}}\| = \sum_{n=1}^{\infty} \chi_{\{m\}} = m.$$

This works for any m, which shows that T is unbounded.

c. Notice that $\|T^{-1}f\| \le \|f\|$, since each term is smaller, for any $f \in L^1(\mu)$. In particular, it works for $\|f\| = 1$, which shows that $\|T^{-1}\|$ is bounded by 1.

If $f(n) \in \mathcal{X}$, then by definition, $nf(n) \in L^1(\mu)$ and $T^{-1}nf(n) = f(n)$, so T^{-1} is surjective.

T^{-1} being open is equivalent to T be continuous, which implies that it is bounded, which is impossible by part (b). Thus, T^{-1} is not open.

5.37 Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. If $T\colon \mathcal{X} \to \mathcal{Y}$ is a linear map such that $f \circ T \in \mathcal{X}^*$ for every $f \in \mathcal{Y}^*$, then T is bounded.

Solution By the closed graph theorem, it's enough to show that $\Gamma(T)$ is closed.

Let $\{(x_n, Tx_n)\} \subseteq \Gamma(T)$ be a sequence which converges to $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

Suppose $y \neq Tx$. Since $f \circ T \in \mathcal{X}^*$, it is continuous, so $(f \circ T)(x_n) \xrightarrow{n\to\infty} (f \circ T)(x)$ for any $f \in \mathcal{Y}$. Similarly, f is continuous, so $f(Tx_n) \xrightarrow{n\to\infty} f(y)$.

By Hahn-Banach, $\mathcal{Y}^*$ separates points, so there exists $f \in \mathcal{Y}$ so that $f(y) \neq f(Tx)$. But this contradicts the above, so $y = Tx$. Thus, $\Gamma(T)$ is closed, so T is bounded.

5.38 Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let $\{T_n\}$ be a sequence in $L(\mathcal{X}, \mathcal{Y})$ such that $\lim T_n x$ exists for every $x \in X$. Let $Tx = \lim T_n x$; then $T \in L(\mathcal{X}, \mathcal{Y})$.

Solution We'll first show linearity: Let $x, y \in \mathcal{X}$ and $\lambda \in K$. Then because the limit of a sum of limits that exist is the sum of the limits,

$$T_n(x + y) = T_n(x) + T_n(y) \implies T(x + y) = T(x) + T(y).$$

Similarly, $T_n(\lambda x) = \lambda T_n x \implies T(\lambda x) = \lambda T x$, so T is linear. We now need to show that T is bounded.

Notice that $\sup_n \|T_n x\| < \infty$ for all $x \in \mathcal{X}$, since the limit exists for each x. By the uniform boundedness principle, $\sup_n \|T_n\| < \infty$, so $\|T\| < \infty$. Hence, T is bounded.

5.45 The space $C^\infty(\mathbb{R})$ of all infinitely differentiable functions on $\mathbb{R}$ has a Fréchet space topology with respect to which $f_n \to f$ iff $f_n^{(k)} \to f^{(k)}$ uniformly on compact sets for all $k \geq 0$.

Solution For each $j \geq 1$, consider the compact set $K_j := \overline{B(0, j)}$, and for each $k \geq 0$, consider the seminorms

$$\rho_{(j,k)}(f) := \sup_{x \in K_j} \left| f^{(k)}(x) \right|.$$

This is a seminorm since it is a norm on $C(\mathbb{R})$, and there are countably many since $\mathbb{N}^2$ is countable.

With this topology, $C^\infty(\mathbb{R})$ is complete, as seen on a previous homework assignment. Indeed, if f_n' converges uniformly to g, then f_n converges to a function f with $f' = g$, i.e., the limit function is in $C^1(\mathbb{R})$. By induction, this tells us that the limit function is in $C^\infty(\mathbb{R})$.

The space is also Hausdorff, since $p_{(j,1)}$ will separate different functions for j sufficiently large.

"$\implies$"

Suppose $f_n \to f$ in the topology generated by these seminorms.

Let K be a compact set in $\mathbb{R}$. Then there exists $j \geq 1$ so that $K \subseteq K_j$. By definition,

$$\sup_{x \in K} \left| f^{(k)}(x) - f_n^{(k)}(x) \right| \leq \sup_{x \in K_j} \left| f^{(k)}(x) - f_n^{(k)}(x) \right| = \rho_{(j,k)}(f - f_n) \xrightarrow{n\to\infty} 0,$$

for any $k \geq 0$, so $f_n^{(k)}$ converges to $f^{(k)}$ locally uniformly.

"$\impliedby$"

Suppose $f_n^{(k)} \to f^{(k)}$ locally uniformly. In particular, the sequences converges uniformly on each K_j, so

$$\rho_{(j,k)}(f - f_n) = \sup_{x \in K_j} \left| f^{(k)}(x) - f_n^{(k)}(x) \right| \xrightarrow{n\to\infty} 0$$

for every $j \geq 1$, so $f_n \to f$ in the topology generated by the seminorms.

Thus, $C^\infty(\mathbb{R})$ is a Fréchet space.

5.51 A vector subspace of a normed vector space $\mathcal{X}$ is norm-closed iff it is weakly closed.

Solution Let $\mathcal{M}$ be a vector subspace of $\mathcal{X}$.

"$\implies$"

For any $x \in \mathcal{X} \setminus \mathcal{M}$, because $\mathcal{M}$ is norm-closed, Hahn-Banach gives us a linear functional $f_x \in \mathcal{X}^*$ so that $f_x\big|_{\mathcal{M}} = 0$ and $f_x(x) \neq 0$. Notice that $\ker f_x$ is weakly closed, since $\ker f_x = f_x^{-1}(\{0\})$. Lastly,

$$\mathcal{M} = \bigcap_{x \in \mathcal{X} \setminus \mathcal{M}} \ker f_x.$$

"$\subseteq$" is clear from construction of each f_x. Conversely, if x is in the right-hand side, then $x \in \mathcal{M}$, or else $f_x(x) \neq 0$. Thus, $\mathcal{M}$ is a weakly closed subspace of $\mathcal{X}$.

"$\impliedby$"

Let $\{x_n\} \subseteq \mathcal{M}$ converge to x in $\mathcal{X}$ in norm, and let f be a linear functional on $\mathcal{X}$. Then

$$|f(x - x_n)| \leq \|f\| \|x - x_n\| \xrightarrow{n \to \infty} 0$$

by assumption. This is holds for any f, so $x_n \to x$ weakly, so $x \in \mathcal{M}$.

5.53 Suppose that $\mathcal{X}$ is a Banach space and $\{T_n\}$, $\{S_n\}$ are sequences in $L(\mathcal{X}, \mathcal{X})$ such that $T_n \to T$ strongly and $S_n \to S$ strongly.

a. If $\{x_n\} \subseteq \mathcal{X}$ and $\|x_n - x\| \to 0$, then $\|T_n x_n - Tx\| \to 0$.

b. $T_n S_n \to TS$ strongly.

Solution a. By Exercise 47(a), we have $M \coloneqq \sup_n \|T_n\| < \infty$. Thus,

$$\begin{aligned} \|T_n x_n - Tx\| &\leq \|T_n x_n - T_n x\| + \|T_n x - Tx\| \\ &\leq \|T_n\| \|x_n - x\| + \|T\| \|x_n - x\| \\ &\leq 2M \|x_n - x\| \xrightarrow{n \to \infty} 0 \end{aligned}$$

as required.

b. Let $x \in \mathcal{X}$. By definition, we know that $\|S_n x - Sx\| \xrightarrow{n \to \infty} 0$. Thus, if we set $y_n = S x_n$ and $y = Sx$, part (a) gives

$$\|T_n S_n x - TSx\| = \|T_n y_n - Ty\| \xrightarrow{n \to \infty} 0,$$

so $T_n S_n \to TS$ strongly.