16.6.22 Let $G(u, v) = (u - uv, uv)$.

(a) Show that the image of the horizontal line of the horizontal line $v = c$ is $y = \dfrac{c}{1-c}x$ if $c \neq 1$, and is the y-axis if $c = 1$.

(b) Determine the images of the vertical lines in the uv-plane.

(c) Compute the Jacobian of G.

(d) Observe that by the formula for the area of a triangle, the region $\mathcal{D}$ in Figure 14 has area $\frac{1}{2}(b^2 - a^2)$. Compute this area again, using the Change of Variables Formula applied to G.

(e) Calculate $\displaystyle\iint_D xy \, \mathrm{d}x \, \mathrm{d}y$.

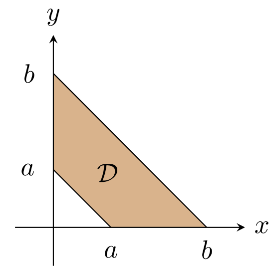

Figure 14

Solution $G(u, v) = (u - uv, uv)$

(a) With $v = c$ and $c \neq 1$, we have $x = u - cu \Rightarrow u = \dfrac{1}{1-c}x$ and $y = uc$. Substituting in our expression for u, we get $y = \dfrac{c}{1-c}x$, as desired.

In the case where $v = c = 1$, we get $x = u - u = 0$, which is the y-axis.

(b) Any vertical line in the uv-plane can be expressed as $u = c$, where $c \in \mathbb{R}$. Applying G to this line yields $x = c - cv \Rightarrow cv = c - x$ and $y = cv$. Substituting in our expression for cv, we arrive at $y = c - x$.

(c)
$$
\begin{aligned}
J(G) &= \det \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} \\
&= \det \begin{pmatrix} 1-v & -u \\ v & u \end{pmatrix} \\
&= (1-v)(u) - (-u)(v) \\
&= u - vu + uv \\
&\boxed{= u}
\end{aligned}
$$

(d) Notice that the lines bounding $\mathcal{D}$ are expressed by $y = a - x$, $y = b - x$, $x = 0$, and $y = 0$. Using the results from part (a) and (b), $y = a - x$ and $y = b - x$ in the uv-plane are $u = a$ and $u = b$, respectively. $x = 0$ and $y = 0$ in the uv-plane are $v = 1$ and $v = 0$, respectively. Then the preimage of $\mathcal{D}$, $\mathcal{D}_0$, in the uv-plane can be expressed by $a \leq u \leq b$, $0 \leq v \leq 1$. Thus, the area of $\mathcal{D}$ when computed from the uv-plane is

$$
\begin{aligned}
&\iint_{\mathcal{D}_0} J(G) \, \mathrm{d}A \\
&= \int_0^1 \int_a^b u \, \mathrm{d}u \, \mathrm{d}v \\
&= \int_0^1 \frac{1}{2}(b^2 - a^2) \, \mathrm{d}u \\
&\boxed{= \frac{1}{2}\left(b^2 - a^2\right)}
\end{aligned}
$$

(e)
$$\begin{aligned}
&\iint_{\mathcal{D}} xy \,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{\mathcal{D}_0} (u-uv)(uv)u \,\mathrm{d}u\,\mathrm{d}v \\
&= \int_0^1 \int_a^b u^3(v-v^2)\,\mathrm{d}u\,\mathrm{d}v \\
&= \left(\int_0^1 v - v^2 \,\mathrm{d}v\right)\left(\int_a^b u^3 \,\mathrm{d}u\right) \text{(Separation of Variables)} \\
&= \left(\frac{1}{2}-\frac{1}{3}\right)\left(\frac{1}{4}\left(b^4-a^4\right)\right) \\
&\boxed{= \frac{1}{24}\left(b^4-a^4\right)}
\end{aligned}$$

16.4.53 Calculate the integral of
$$f(x,y,z) = z(x^2+y^2+z^2)^{-3/2}$$
over the part of the ball $x^2+y^2+z^2 \leq 16$ defined by $z \geq 2$.

Solution We want to find $\iiint_D f(x,y,z)\,\mathrm{d}V$. $x^2+y^2+z^2 \leq 16$ is a sphere, so it will be helpful to switch to cylindrical coordinates. For cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, and the Jacobian is r. Let $\mathcal{E}_0$ be such that its image through this change of coordinates is the region of interest, $\mathcal{E}$. In cylindrical coordinates, the ball becomes $r^2+z^2 \leq 16$, and this intersects the plane $z = 2$ when $r = \sqrt{12}$. So, $\mathcal{E}_0$ can be expressed as

$$\begin{aligned}
\mathcal{E}_0 &= \left\{(r,\theta,z) \in \mathbb{R}^3 \;\middle|\; (r,\theta) \in \mathcal{D}_0, 2 \leq z \leq \sqrt{16-r^2}\right\}, \text{ where} \\
\mathcal{D}_0 &= \left\{(r,\theta) \in \mathbb{R}^2 \;\middle|\; 0 \leq r \leq \sqrt{12}, 0 \leq \theta \leq 2\pi\right\}
\end{aligned}$$

Therefore,
$$\begin{aligned}
&\iiint_{\mathcal{E}} f(x,y,z)\,\mathrm{d}V \\
&= \iiint_{\mathcal{E}_0} rf(r\cos\theta, r\sin\theta, z)\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \iiint_{\mathcal{E}_0} rz(r^2+z^2)^{-3/2}\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{12}}\int_2^{\sqrt{16-r^2}} rz(r^2+z^2)^{-3/2}\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{12}} \left[-r(r^2+z^2)^{-1/2}\right]_2^{\sqrt{16-r^2}}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{12}} -\frac{r}{4} + r(r^2+4)^{-1/2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \left[-\frac{r^2}{8} + (r^2+4)^{1/2}\right]_0^{\sqrt{12}}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} -\frac{12}{8} + (12+4)^{1/2} - 4^{1/2}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{1}{2}\,\mathrm{d}\theta \\
&\boxed{= \pi}
\end{aligned}$$