Problem Set 1

14. Evaluate $\int_0^1 \int_0^1 \frac{y}{1+xy}\,\mathrm{d}y\,\mathrm{d}x$, *Hint: Change the order of integration.*

Solution

Using Fubini's theorem, we can switch the order of integration and evaluate the iterated integrals:

$$\int_0^1 \int_0^1 \frac{y}{1+xy}\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1 \left[\int_0^1 \frac{y}{1+xy}\,\mathrm{d}x\right]\mathrm{d}y$$
$$= \int_0^1 \Big[\ln(1+xy)\Big]_0^1 \mathrm{d}y$$
$$= \int_0^1 \ln(1+y)\,\mathrm{d}y$$
$$= \Big[(1+y)\ln(1+y) - (1+y)\Big]_0^1 \text{ (Integration by parts.)}$$
$$= 2\ln(2) - 2 - (-1)$$
$$\boxed{= 2\ln(2) - 1}$$

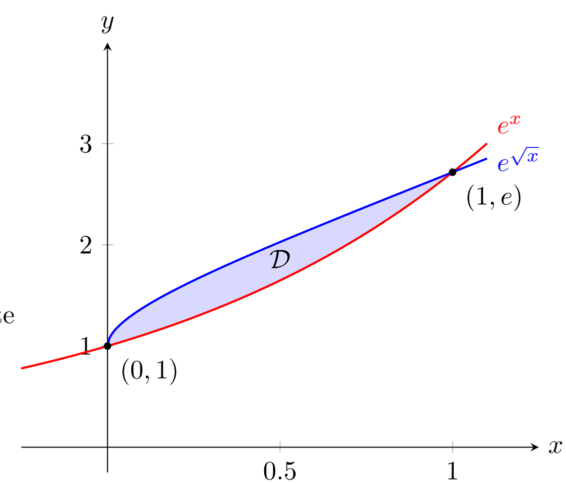
Problem Set 2

6. Compute the integral of $f(x,y) = (\ln y)^{-1}$ over the domain $\mathcal{D}$ bounded by $y = e^x$ and $y = e^{\sqrt{x}}$. *Hint:* Choose the order of integration that enables you to evaluate the integral.

Solution

The function $f(x,y) = (\ln y)^{-1}$ is difficult to integrate with respect to y, so we need to write the region $\mathcal{D}$ as a horizontally simple region. Based on the image, $\mathcal{D}$ can be written as

$$\mathcal{D} = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; 1 \le y \le e, (\ln y)^2 \le x \le \ln y\right\}.$$



Thus, our double integral will be

$$\int_1^e \int_{(\ln y)^2}^{\ln y} \frac{1}{\ln y}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_1^e \left[\frac{x}{\ln y}\right]_{(\ln y)^2}^{\ln y} \mathrm{d}y$$
$$= \int_1^e \frac{\ln y - (\ln y)^2}{\ln y}\,\mathrm{d}y$$

The integrand above is not continuous at $y = 1$, so we evaluate it as an improper integral. Continuing,

$$= \lim_{t\to 1^+} \int_t^e 1 - \ln y\,\mathrm{d}y$$
$$= \lim_{t\to 1^+} \Big[y - y\ln y + y\Big]_t^e \text{ (Integration by parts.)}$$
$$= \lim_{t\to 1^+} \Big[e - e + e - (t - t\ln t + t)\Big]$$
$$= e - (1+1)$$
$$\boxed{= e - 2}$$

8. Find the volume of the region bounded by $y = 1 - x^2$, $z = 1$, $y = 0$ and $z + y = 2$.

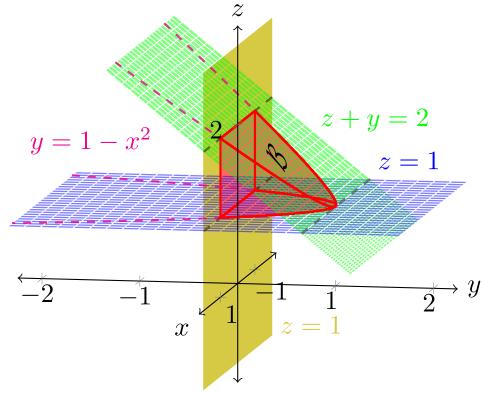
Solution

From the figure, we can see that the bounded region can be described as

$$\mathcal{B} = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; -1 \leq x \leq 1, 0 \leq y \leq 1 - x^2, 1 \leq z \leq 2 - y\right\}.$$

Thus, the volume of the region can be expressed by

$$\iiint_{\mathcal{B}} \mathrm{d}V = \int_{-1}^{1} \int_{0}^{1-x^2} \int_{1}^{2-y} \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}z.$$



The bounded region $\mathcal{B}$ is shaded in red.

Using Fubini's theorem, we can evaluate the triple integral as iterated integrals:

$$\begin{aligned}
&\int_{-1}^{1} \int_{0}^{1-x^2} \int_{1}^{2-y} \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}z \\
&= \int_{-1}^{1} \int_{0}^{1-x^2} \left[\int_{1}^{2-y} \mathrm{d}z\right] \mathrm{d}y \, \mathrm{d}x \\
&= \int_{-1}^{1} \int_{0}^{1-x^2} 2 - y - 1 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_{-1}^{1} \left[\int_{0}^{1-x^2} 1 - y \, \mathrm{d}y\right] \mathrm{d}x \\
&= \int_{-1}^{1} \left[y - \frac{1}{2}y^2\right]_{0}^{1-x^2} \mathrm{d}x \\
&= \int_{-1}^{1} 1 - x^2 - \frac{1}{2}(1 - x^2)^2 \, \mathrm{d}x \\
&= \int_{-1}^{1} 1 - x^2 - \frac{1}{2}(1 - 2x^2 + x^4) \, \mathrm{d}x \\
&= \int_{-1}^{1} 1 - x^2 - \frac{1}{2} + x^2 - \frac{1}{2}x^4) \, \mathrm{d}x \\
&= \int_{-1}^{1} \frac{1}{2} - \frac{1}{2}x^4 \, \mathrm{d}x \\
&= \left[\frac{1}{2}x - \frac{1}{10}x^5\right]_{-1}^{1} \\
&= \frac{1}{2} - \frac{1}{10} - \left(-\frac{1}{2} + \frac{1}{10}\right) \\
&= 1 - \frac{1}{5} \\
&\boxed{= \frac{4}{5}}
\end{aligned}$$