1 In the following problems, let the universe be the set $U = \{1, 2, 3, \ldots, 10\}$. Let $A = \{1, 4, 7, 10\}$, $B = \{1, 2, 3, 4, 5\}$, and $C = \{2, 4, 6, 8\}$. List the elements of each set:

a. $B \cap C$

b. $\overline{A}$

c. $A \cap (B \cup C)$

d. $\overline{A \cap B} \cup C$

Solution a. $B \cap C = \{1, 2, 3, 4, 5\} \cap \{2, 4, 6, 8\} = \{2, 4\}$

b. $\overline{A} = \{2, 3, 5, 6, 8, 9\}$

c. $A \cap (B \cup C) = \{1, 4, 7, 10\} \cap \{1, 2, 3, 4, 5, 6, 8\} = \{1, 4\}$

d. $\overline{A \cap B} \cup C = \overline{\{1, 4\}} \cup \{2, 4, 6, 8\} = \{2, 3, 5, 6, 7, 8, 9, 10\} \cup \{2, 4, 6, 8\} = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$

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2 Let $A = \{1, 2, 3, 4\}$, $C = \{5, 6, 7, 8\}$, and $B = \{n \mid n \in A \text{ and } n + m = 8 \text{ for some } m \in C\}$. Show that A is not a subset of B.

Solution Notice that $4 \in A$ but $4 \notin B$:

$$\{4 + m \mid m \in C\} = \{9, 10, 11, 12\} \not\ni 8,$$

so $A \not\subseteq B$.

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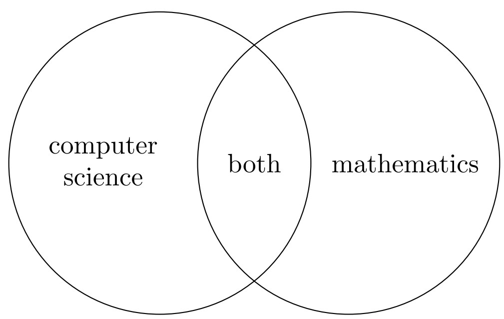
3 In a group of students, each student is taking a mathematics course or a computer science course or both. One-fifth of those taking a mathematics course are also taking a computer science course, and one-eight of those taking a computer science course are also taking a mathematics course. Are more than one-third of the students taking a mathematics course?

Solution Consider the following Venn diagram:



If we let X be the set of all students, C be the set of computer science students, and M be the mathematics students, then

$$\frac{1}{5} = \frac{|C \cap M|}{|M|} \implies \frac{1}{5}|M| = |C \cap M| \quad \text{and}$$
$$\frac{1}{8} = \frac{|C \cap M|}{|C|} \implies |C| = 8|C \cap M|.$$

Hence, by the Venn diagram, we have

$$\begin{aligned} |X| = |C| + |M| - |C \cap M| = 8|C \cap M| + |M| - |C \cap M| &= \frac{7}{5}|M| + |M| \\ &= \frac{12}{5}|M| \\ \implies \frac{5}{12} &= \frac{|M|}{|X|}. \end{aligned}$$

Since $1/3 = 4/12 < 5/12$, more than one-third of the students are taking a mathematics course.

4 Let P denote the set of integers greater than 1. For $i \geq 2$, define $X_i = \{ik \mid k \in P\}$. Describe the set $A = P - \bigcup_{i=2}^{\infty} X_i$.

Solution Notice that $x \in X_i$ if and only if $x = ik$ for some $k \geq 2$, where $i \geq 2$. Hence, $x \in \bigcup_{i=2}^{\infty} X_i$ if there exist $i, k \geq 2$ so that $x = ik$. In other words, the union is the set of all composite integers, so A is the set of integers $n \geq 2$ which are not composite, i.e., A is the set of prime numbers.

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5 Determine whether each set is a function from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$. If it is a function, find its domain and range, draw its arrow diagram, and determine if it is one-to-one, onto or both. If it is both one-to-one and onto, give the description of the inverse function as a set of ordered pairs, draw its arrow diagram, and give the domain and range of the inverse function.

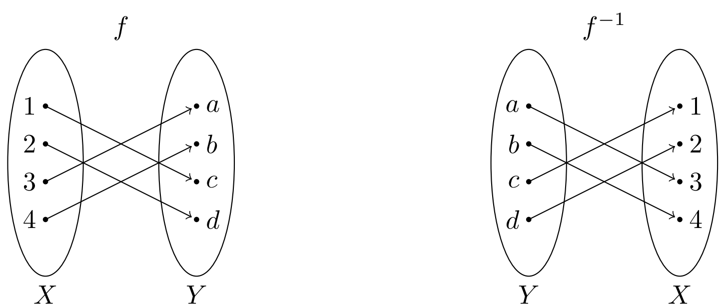
a. $\{(1, c), (2, a), (3, b), (4, c), (2, d)\}$

b. $\{(1, c), (2, d), (3, a), (4, b)\}$

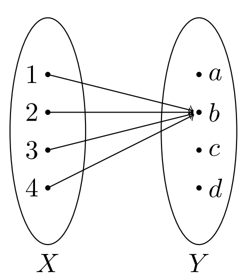
c. $\{(1, b), (2, b), (3, b), (4, b)\}$

Solution a. This set is not a function, since 2 is mapped to two elements: a and d.

b. This set is a function, since each element in X has precisely 1 element in Y.
If we call the function f, then f is a bijection, its inverse is $f^{-1}\colon Y \to X$, $f^{-1} = \{(a, 3), (b, 4), (c, 1), (d, 2)\}$, and its range is X.



c. This set is a function for the same reason as in part (b). This map is neither one-to-one nor surjective.



6 Let f and g be functions from the positive integers to the positive integers defined by the equations

$$f(n) = n^2 \quad \text{and} \quad g(n) = 2^n.$$

Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, and $g \circ f$.

Solution $(f \circ f)(n) = (n^2)^2 = n^4$.
$(g \circ g)(n) = 2^{2^n}$.
$(f \circ g)(n) = (2^n)^2 = 4^n$.
$(g \circ f)(n) = 2^{n^2}$.

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7 Using induction, verify that the equation

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

is true for each positive integer n.

Solution Base step: $n = 1$

$\left(1 \cdot 2 \cdot \frac{1}{2}\right)^2 = 1^2 = 1^3$, so the base case holds.

Inductive step:

Assume that the equation holds for $n = k$. We wish to show that it holds for $n = k + 1$:

$$\begin{aligned}
1^3 + \cdots + k^3 + (k+1)^3 &= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 && \text{(inductive hypothesis)} \\
&= \frac{k^2(k+1)^2 + 4(k+1)(k+1)^2}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \left[\frac{(k+1)((k+1)+1)}{2}\right]^2,
\end{aligned}$$

so the inductive step holds.

By the mathematical principle of induction, the equation holds for each $n \geq 1$.

8 Using induction, verify that $(1+x)^n \geq 1+nx$ for each integer $n \geq 1$ and each real number $x \geq -1$.

Solution Let $x \geq -1$, so that $1+x \geq 0$.

Base step: $n=1$

$(1+x)^1 = 1 + 1 \cdot x$, so the base case holds.

Inductive step:

Suppose the inequality holds for $n=k$. We will show that it holds for $n=k+1$:

$$\begin{aligned} (1+x)^{k+1} &= (1+x)(1+x)^k \\ &\geq (1+x)(1+kx) && (1+x \geq 0 \text{ and the inductive hypothesis}) \\ &= 1+(k+1)x+kx^2 \\ &\geq 1+(k+1)x, && (kx^2 \geq 0) \end{aligned}$$

so the inequality holds for $n=k+1$, so the inductive step holds.

By induction, the inequality holds for all $n \geq 1$ and all $x \geq -1$.

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9 Use induction to prove that if $X_1, X_2, \ldots, X_n$ are finite sets, then

$$|X_1 \times X_2 \times \cdots \times X_n| = |X_1| \cdot |X_2| \cdot \cdots \cdot |X_n|.$$

Solution Base step: $n=1$

$|X_1| = |X_1|$, so the base case holds.

Inductive step:

Now assume that the equality holds for $n=k$. By induction, it suffices to show that $n=k+1$:

Set $Y = X_1 \times \cdots \times X_k$. Then

$$\begin{aligned} |X_1 \times \cdots \times X_k \times X_{k+1}| = |Y \times X_{k+1}| &= |Y| \cdot |X_{k+1}| && (|A \times B| = |A| \cdot |B|) \\ &= |X_1 \times \cdots \times X_k| \cdot |X_{k+1}| \\ &= |X_1| \cdot \cdots \cdot |X_k| \cdot |X_{k+1}|, && (\text{inductive hypothesis}) \end{aligned}$$

so the inductive step holds.

Hence, by induction, the equality holds for all $n \geq 1$.

10 Consider the following pseudocode, which takes as input an array of numbers $a[i], \ldots, a[j]$:

```
val = a[i]
h = i
for k = i + 1 to j do
    if a[k] < val then
        h = h + 1
        swap(a[h], a[k])
    end
end
swap(a[i], a[h])
```

Prove that after the pseudocode terminates that:

a. $a[h] = val$,

b. for all p with $i \leq p < h$, $a[p] < val$,

c. and for all p with $h < p \leq j$, $a[p] \geq val$.

Solution Notice that (a) always holds since $a[i]$ never changes in the algorithm. Hence, in the last line, $a[h]$ takes on the value $a[i] = val$.

We'll prove (b) and (c) by induction on j and leaving i fixed. To do so, we'll prove that the for loop does the following:

(b') For all $i < p \leq h$, $a[p] < val$.

(c') For all $h < p \leq j$, $a[p] \geq val$.

Base step: $j = i$

In this case, the loop never runs. Hence, (b') and (c') hold vacuously, since $h = i = j$, and there are no p with $i < p \leq h$ or $h < p \leq j$.

Inductive step:

Now assume that (b') and (c') hold if we run the loop from $i + 1$ to $j = k$. Then in the next iteration, there are two cases:

$a[k + 1] < val$:

We'll ignore the increment on h and set $h' = h + 1$ in the following analysis.

By (c'), $a[h'] \geq val$. Hence, we swap $a[h']$ with $a[k+1]$. As a result, $a[h'] < val$ now, and $a[k+1] \geq val$.

Thus, for $i < p \leq h$, by the inductive hypothesis (b'), and in this step, we have $a[h'] < val$, so (b') holds: for all $i < p \leq h'$, $a[p] < val$.

Similarly, by the inductive hypothesis, we had for $h < p \leq k$ that $a[p] \geq val$ before the loop. After the loop, $a[h'] < val$, so now $h' < p \leq k$. We also saw that $a[k + 1] \geq val$, so for $h' < p \leq k + 1$, $a[p] \geq val$, so (c') holds.

$a[k + 1] \geq val$:

In this case, nothing changes, so (b') still holds, and (c') clearly holds.

In either case, the inductive step holds, which completes the induction.

In the last line, we swap $a[i]$ and $a[h]$. Thus, because (b') is true for the loop, the swap causes (b) to hold, i.e., $i \leq p < h \implies a[p] < val$. (c') is the same as (c), so (b) and (c) both hold for the algorithm, as required.