21.8 Let X be a topological space and let Y be a metric space. Let $f_n\colon X \to Y$ be a sequence of continuous functions. Let x_n be a sequence of points of X converging to x. Show that if the sequence (f_n) converges uniformly to f, then $(f_n(x_n))$ converges to $f(x)$.

Solution Let $\varepsilon > 0$.

Note that since each f_n is continuous and $f_n \xrightarrow{n\to\infty} f$ uniformly, f is also continuous.

As f_n converges uniformly to f, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|f(x) - f_n(x)| < \varepsilon/2$ for all $x \in X$.

Since f is continuous, $f(x_n) \xrightarrow{n\to\infty} f(x)$. Indeed, since f is continuous, $f^{-1}(B(f(x), \varepsilon/2))$ is open. Since $x_n \xrightarrow{n\to\infty} x$ and $f^{-1}(B(f(x), \varepsilon/2))$ is an open neighborhood of x, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$x_n \in f^{-1}(B(f(x), \varepsilon/2)) \implies f(x_n) \in B(f(x), \varepsilon/2) \iff |f(x_n) - f(x)| < \varepsilon/2.$$

Hence, $f(x_n) \xrightarrow{n\to\infty} f(x)$.

Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$,

$$|f(x) - f_n(x_n)| \leq |f(x) - f(x_n)| + |f(x_n) - f_n(x_n)| < \varepsilon.$$

Thus, $f_n(x_n) \xrightarrow{n\to\infty} f(x)$.

21.9 Let $f_n\colon \mathbb{R} \to \mathbb{R}$ be the function

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1}.$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ be the zero function.

a. Show that $f_n(x) \xrightarrow{n\to\infty} f(x)$ for each $x \in \mathbb{R}$.

b. Show that f_n does not converge uniformly to f.

Solution a. Let $x \in \mathbb{R} - \{0\}$. Then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{n^3[x - (1/n)]^2 + 1} = \lim_{n\to\infty} \frac{1}{n^3} \frac{1}{(x - (1/n))^2 + (1/n^3)} = 0 \cdot \frac{1}{x^2} = 0.$$

For $x = 0$,

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} \frac{1}{n^3(-1/n)^2 + 1} = \lim_{n\to\infty} \frac{1}{n^3(-1/n)^2 + 1} = \lim_{n\to\infty} \frac{1}{1 - n} = 0.$$

b. For all $n \geq 1$,

$$|f_n(1/n) - f(1/n)| = 1,$$

so $\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \geq 1$ for all $n \geq 1$, so the convergence is not uniform.

19.6 Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \ldots$ converges to $\pi_\alpha(\mathbf{x})$ for each α. Is this fact true if one uses the box topology instead of the product topology?

Solution "$\implies$"

Let $\mathbf{x}_n \xrightarrow{n\to\infty} \mathbf{x}$.

Fix α, and let U be an open neighborhood of $\pi_\alpha(\mathbf{x})$. Then since π_α is continuous, $\pi_\alpha^{-1}(U) = U \times \prod_{\beta\neq\alpha} X_\beta$ is an open neighborhood of $\mathbf{x}$. Since $\mathbf{x}_n$ converges to $\mathbf{x}$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\mathbf{x}_n \in \pi_\alpha^{-1}(U) \implies \pi_\alpha(\mathbf{x}_n) \in U.$$

Thus, $\pi_\alpha(\mathbf{x}_n) \xrightarrow{n\to\infty} \pi_\alpha(\mathbf{x})$ for all α.

" $\Longleftarrow$ "

Let $\pi_\alpha(\mathbf{x}_n) \xrightarrow{n\to\infty} \pi_\alpha(\mathbf{x})$ for all α.

Let U be a basic open neighborhood of $\mathbf{x}$, i.e., $U = \prod_\alpha U_\alpha$, with $U_\alpha = X_\alpha$ for all but finitely many α. We index these values of α via $\alpha_1, \ldots, \alpha_k$.

Note that for all $\alpha \notin \{\alpha_1, \ldots, \alpha_k\}$, $\pi_\alpha(\mathbf{x}_n) \in U_\alpha = X_\alpha$ for all n.

Then for each of these α_i, U_α is an open neighborhood of $\pi_{\alpha_i}(\mathbf{x}_n)$ so by convergence, there exists $N_i \in \mathbb{N}$ such that if $n \geq N_i$, $\pi_{\alpha_i}(\mathbf{x}_n) \in U_{\alpha_i}$.

Let $N = \max\{N_1, \ldots, N_k\}$, which is well-defined since k is finite. Then for all $n \geq N$,

$$\pi_\alpha(\mathbf{x}_n) \in U_\alpha \ \forall \alpha \implies \mathbf{x}_n \in U.$$

Hence, $\mathbf{x}_n \xrightarrow{n\to\infty} \mathbf{x}$.

This does not work under the box topology since we relied on the fact that, given an open neighborhood $U = \prod_\alpha U_\alpha$, the values of α such that $U_\alpha = X_\alpha$ were finite to get N.

However, with the box topology, this is not always true. For example, in $\prod_{n\in\mathbb{N}} \mathbb{R}$, we could take the open set $U = \prod_{n\in\mathbb{N}}(-1/n, 1/n)$ and $\mathbf{x}_k = (1/k)_n$. Then $\pi_k(\mathbf{x}_n) \xrightarrow{n\to\infty} 0$, but for any $n \in \mathbb{N}$,

$$\pi_{n+1}(\mathbf{x}_n) = \frac{1}{n} \notin \left(-\frac{1}{n+1}, \frac{1}{n+1}\right),$$

so $\mathbf{x}_n$ does not converge to $\mathbf{x}$.

19.7 Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.

Solution Consider ${}^{c}\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ with the box topology. This is the set of all sequences such that $x_i = 0$ for finitely many values of i.

Let $(x_n) \in {}^{c}\mathbb{R}^\infty$. Then consider the open neighborhood $U = \prod U_n$ defined via

$$U_n = \begin{cases} (-1, 1) & \text{if } x_n = 0 \\ (x_n - |x_n|, x_n + |x_n|) & \text{if } x_n \neq 0. \end{cases}$$

Then $U \subseteq {}^{c}\mathbb{R}^\infty$. Indeed, if $(y_n) \in U$, then the only possible values of i such that $y_i = 0$ is when $x_i = 0$. But there are finitely many of those x_i, so there will be finitely many of those y_i. Hence ${}^{c}\mathbb{R}^\infty$ is open, so $\mathbb{R}^\infty$ is closed, which means that the $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$.

In the product topology, $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$.

Let $(x_n) \in \mathbb{R}^\omega$, and let U be a basic open neighborhood of (x_n) in $\mathbb{R}^\omega$. Then we can write, for a finite $J \subseteq I$ and some $\{\delta_j \mid j \in J\} \subseteq (0, \infty)$,

$$U = \prod_{j\in J}(x_j - \delta_j, x_j + \delta_j) \times \prod_{i\in I-J} \mathbb{R}.$$

$U \cap \mathbb{R}^\infty \neq \emptyset$ since we can take the sequence $(y_n) \in \mathbb{R}^\infty$ such that

$$y_n = \begin{cases} x_n & \text{if } n \in J \\ 0 & \text{if } n \notin J, \end{cases}$$

which is clearly in $\mathbb{R}^\infty$ since J is finite. Moreover, $(y_n) \in U$ since $x_n \in (x_n - \delta_j, x_n + \delta_j)$. Hence, since (x_n) and its open neighborhood was arbitrary, this shows that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$.

17.13 Show that X is Hausdorff if and only if the diagonal $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Solution "$\implies$"

Let X be Hausdorff.

It suffices to show that ${}^{c}\Delta$ is open.

Let $x \times y \in {}^{c}\Delta$, where $x \neq y \in X$. Then since X is Hausdorff, there exist open neighborhoods $U \ni x$ and $V \ni y$ such that $U \cap V = \emptyset$.

Then $W := U \times V$ is an open neighborhood of $x \times y$ with $W \cap \Delta = \emptyset$. If that were not the case, then there exists $z \in X$ such that $z \times z \in W$. But this implies that $z \in U \cap V = \emptyset$, which is a contradiction. Hence, $W \cap \Delta = \emptyset \implies W \subseteq {}^{c}\Delta$, so Δ is closed.

"$\impliedby$"

Let Δ be closed. Then ${}^{c}\Delta$ is open, so for all $x \neq y \in X$, there exists a basic open neighborhood $W = U \times V$ of $x \times y$ such that $W \subseteq {}^{c}\Delta$.

Then $U \cap V = \emptyset$. Otherwise, if there were some $z \in X$ such that $z \in U \cap V$, then $z \times z \in W \cap \Delta$. But this is a contradiction since we assumed $W \subseteq {}^{c}\Delta$, so $U \cap V = \emptyset$.

By definition, $x \times y \in W \implies x \in U$ and $y \in V$ open sets with $U \cap V = \emptyset$, so X is Hausdorff.

9.5 a. Use the choice axiom to show that if $f\colon A \to B$ is surjective, then f has a right inverse $h\colon B \to A$.

b. Show that if $f\colon A \to B$ is injective and A is not empty, then f has a left inverse. Is the axiom of choice needed?

Solution a. Since f is surjective, the set $A_y := f^{-1}(\{y\}) \neq \emptyset$ for all $y \in B$, and each A_y is disjoint since f is a function. Then by the axiom of choice, there exists a set $C \subseteq A$ containing exactly one point from each A_y. Then $A_y \cap C$ is non-empty and is a singleton.

Hence, we can define $h\colon B \to A$ as follows: if $A_y \cap C = \{x\}$, then $h(y) = x$.

Then for any $y \in B$, $h(y) \in f^{-1}(y) \implies f(h(y)) = y$, so h is a right inverse of f.

b. Define $g\colon A \to f(A)$ via $g(x) = f(x)$ for all $x \in A$. Since f is injective, g is injective also. Moreover, g is surjective by definition of g, so g is a bijection. Hence, it has an inverse $g^{-1}\colon f(A) \to A$.

Thus, if $x \in A$, $g^{-1}(f(x)) = g^{-1}(g(x)) = x$, so g^{-1} is a left inverse of f.

The axiom of choice was not needed to show this.

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1 Let I be an indexing set, and let $X = \prod_{i \in I} X_i$ be a product of topological spaces X_i, endowed with the product topology. Show that each projection map π_i for $i \in I$ is open.

Solution Fix $i \in I$, and let U be an open set in X.

Since X has the product topology, $U = \prod_{j \in I} U_j$, with $U_j \subseteq X_j$ open and $U_j = X_j$ for all but finitely many j, which we index via J.

If $i \in J$, then $\pi_i(U) = X_i$, which is open.

If $x \notin J$, then $\pi_i(U) = U_i$, which is open, by definition.

2 Let X and Y be topological spaces, and let $f\colon X \to Y$ be a function. Define

$$\Gamma_f = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y,$$

and given Γ_f the subspace topology of the product topology on $X \times Y$. Show that f is continuous if and only if the map $F\colon X \to \Gamma_f$ given by $F(x) = (x, f(x))$ is a homeomorphism.

Solution First, we'll show that $F^{-1}(U \times V) = U \cap f^{-1}(V)$.

Let $x \in F^{-1}(U \times V)$. Then $F(x) = (x, f(x)) \in U \times V \implies x \in U$ and $f(x) \in V$, so $x \in U \cap f^{-1}(V)$.

Let $x \in U \cap f^{-1}(V)$.

$$\left.\begin{aligned} x \in U \qquad &\implies F(x) = (x, f(x)) \in U \times Y \\ x \in f^{-1}(V) \qquad &\implies F(x) = (x, f(x)) \in X \times V \end{aligned}\right\} \implies F(x) \in (U \times Y) \cap (X \times V) = U \times V,$$

so $x \in F^{-1}(U \times V)$.

Hence, the equality has been proved.

"$\implies$"

Let f be continuous.

$F(x) = F(y) \iff (x, f(x)) = (y, f(y)) \implies x = y$, so F is injective.

F is also a surjection, which is clear because of the definition of Γ_f.

Hence, F is a bijection, so it has an inverse $F^{-1}\colon \Gamma_f \to X$.

We'll now show that F is continuous.

Let W be an open set in Γ_f. Then there exist open sets $U \subseteq X$ and $V \subseteq Y$ such that $W = (U \times V) \cap \Gamma_f$. Then

$$F^{-1}(W) = F^{-1}(U \times V) \cap F^{-1}(\Gamma_f) = (U \cap f^{-1}(V)) \cap X = U,$$

which is open since U and V are open, and f is continuous. Hence F is continuous.

We'll now show that F^{-1} is continuous, which will show that F is a homeomorphism.

Let U be an open set in X. Then

$$(F^{-1})^{-1}(U) = F(U) = \{(x, f(x)) \mid x \in U\} = (U \times Y) \cap \Gamma_f,$$

which is open since $X \times Y$ is open and Γ_f has the subspace topology.

"$\impliedby$"

Let F be a homeomorphism.

Let U be an open set in Y. Then $X \times U$ is an open set in $X \times Y$, so since F is continuous,

$$F^{-1}(X \times U) = X \cap f^{-1}(U) = f^{-1}(U)$$

is open, so f is continuous.