2 The Cantor set $\mathcal{C}$ can also be described in terms of ternary expansions.

a. Every number in $[0,1]$ has a ternary expansion

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}, \quad \text{where } a_k = 0, 1, \text{ or } 2.$$

Note that this decomposition is not unique since, for example, $1/3 = \sum_{k=2}^{\infty} 2/3^k$.
Prove that $x \in \mathcal{C}$ if and only if x has a representation as above where every a_k is either 0 or 2.

b. The **Cantor-Lebesgue function** is defined on $\mathcal{C}$ by

$$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k} \quad \text{if } x = \sum_{k=1}^{\infty} a_k 3^{-k}, \text{ where } b_k = a_k/2.$$

In this definition, we choose the expansion of x in which $a_k = 0$ or 2.
Show that F is well defined and continuous on $\mathcal{C}$, and moreover $F(0) = 0$ as well as $F(1) = 1$.

c. Prove that $F\colon \mathcal{C} \to [0,1]$ is surjective, that is, for every $y \in [0,1]$ there exists $x \in \mathcal{C}$ such that $F(x) = y$.

d. One can also extend F to be a continuous function on $[0,1]$ as follows. Note that if (a,b) is an open interval of the complement of $\mathcal{C}$, then $F(a) = F(b)$. Hence we may define F to have the constant value $F(a)$ in that interval.

Solution a. We'll first prove the following lemma: In each step of the construction of the Cantor set, each open middle third we remove from a closed interval is of the form $(a, a + 1 \cdot 3^{-k})$, where

$$a = \sum_{i=1}^{n-1} a_i 3^{-i} + 1 \cdot 3^{-n}$$

where each $a_i = 2$. This also means that the endpoints of our "full" closed intervals have the same form as a, since the resulting endpoints are the endpoints of $(a, a + 1 \cdot 3^{-n})$.

Base step: $n = 1$

In this step, we remove the open interval $(1/3, 2/3)$ from $[0,1]$.
The ternary expansions of $1/3$ and $2/3$ are

$$\frac{1}{3} = 1 \cdot 3^{-1} \quad \text{and} \quad \frac{2}{3} = \frac{1}{3} + 1 \cdot 3^{-1}.$$

so the base step holds for this claim.

Inductive step:

Assume that the claim is true for the first n steps.
Pick one of the closed intervals of length $1/3^n$. The left endpoint can be written in the form

$$a = \sum_{i=1}^{n} a_i 3^{-i} + 1 \cdot 3^{-(n+1)}$$

by the inductive hypothesis. The right endpoint is then $a + 1 \cdot 3^{-n}$.
In the $(n+1)$-th step, we remove the middle third of this interval, which has length $1/3^{n+1}$, meaning we remove the interval

$$\begin{aligned}
&(a + 1 \cdot 3^{-(n+1)}, a + 2 \cdot 3^{-(n+1)}) \\
&= \left(\sum_{i=1}^{n} a_i 3^{-i} + 1 \cdot 3^{-(n+1)} + 1 \cdot 3^{-(n+1)}, \sum_{i=1}^{n} a_i 3^{-i} + 1 \cdot 3^{-(n+1)} + 2 \cdot 3^{-(n+1)} \right) \\
&= \left(\sum_{i=1}^{n} a_i 3^{-i} + 2 \cdot 3^{-(n+1)}, \sum_{i=1}^{n} a_i 3^{-i} + 2 \cdot 3^{-(n+1)} + 1 \cdot 3^{-(n+1)} \right)
\end{aligned}$$

as desired. Hence, the inductive step holds.

By induction, the claim holds for all steps of the construction of the Cantor set. We're now ready to prove the original problem.

" $\implies$ "

We will prove by induction that in each step of the construction of the Cantor set, removing the middle open third of each closed interval in C_k removes all numbers in $[0, 1]$ with $a_k = 1$, except for the endpoints, which have two ternary expansions; one with infinitely many 2's.

Base step: $k = 1$

We can write any number x in $(1/3, 2/3)$ in the form

$$x = 1 \cdot 3^{-1} + \varepsilon_x$$

where $0 < \varepsilon_x < 1 \cdot 3^{-1}$. Indeed, the length of the open interval is $1/3$, so ε_x is at most $1/3$. Since $\varepsilon_x \in [0, 1]$, it has a ternary expansion

$$\varepsilon_x = \sum_{k=1}^{\infty} e_k 3^{-k}$$

where $e_k \in \{0, 1, 2\}$. In particular, we must have $e_1 = 0$. Otherwise, since every term in the sum is non-negative, we would have $\varepsilon_x \geq 1 \cdot 3^{-1}$, which is a contradiction. Thus,

$$\varepsilon_x = \sum_{k=2}^{\infty} e_k 3^{-k},$$

which means that

$$x = 1 \cdot 3^{-1} + \sum_{k=2}^{\infty} e_k 3^{-k}$$

which shows that all the numbers removed in the middle interval has 1 as its 1st coefficient. So, the base step holds.

Inductive step:

Assume that the claim holds for $k = 1, \ldots, n$. I.e., in the n-th step, we removed all numbers from $[0, 1]$ with $a_1 = \cdots = a_n = 1$. We wish to show that the same holds with the $(n+1)$-th step.

We first pick a "full" closed interval. By the lemma, the interval we remove has the form $B = (a, a + 1 \cdot 3^{-(n+1)})$, where a has the ternary expansion

$$a = \sum_{i=1}^{n} a_i 3^{-i} + 1 \cdot 3^{-(n+1)}$$

with $a_1, \ldots, a_n = 2$.

Let $x \in B$. Then we can write

$$x = \sum_{k=1}^{\infty} x_k 3^{-k}$$

where $x_i = a_i = 2$ for $1 \leq i \leq n$. Indeed, if there were some $1 \leq j \leq n$ where $x_j \neq a_j$, then x will not be contained in B since the length of B is $1 \cdot 3^{-(n+1)}$ and x will differ from a by at least $1 \cdot 3^{-j} > 1 \cdot 3^{-(n+1)}$. We will show that $x_{n+1} = 1$.

We can write x as

$$x = a + \varepsilon_x$$

where $0 < \varepsilon_x < 1 \cdot 3^{-(n+1)}$. Since $\varepsilon_x \in [0, 1]$, it has a ternary expansion

$$\varepsilon_x = \sum_{i=1}^{\infty} e_i 3^{-i}.$$

In particular, we must have that $e_1 = \cdots = e_{n+1} = 0$. If $e_i \neq 0$ for some $1 \leq i \leq n+1$, then $\varepsilon_x > 1 \cdot 3^{-i} \geq 1 \cdot 3^{-(n+1)}$ since all terms are non-negative, which is a contradiction. Hence,

$$x = a + \varepsilon_x = \sum_{i=1}^{n} a_i 3^{-i} + 1 \cdot 3^{-(n+1)} + \sum_{i=n+2}^{\infty} e_i 3^{-i},$$

which means $x_{n+1} = 1$, so all $x \in B$ will satisfy $x_{n+1} = 1$, as desired. Thus, the inductive step holds.

By induction, if $x \in \mathcal{C}$, then if

$$x = \sum_{i=1}^{\infty} x_i 3^{-i}$$

then $x_i \in \{0, 2\}$ for all $i \geq 1$, as desired.

"$\Longleftarrow$"

Let

$$x = \sum_{k=1}^{\infty} x_k 3^{-k}$$

with each $x_k \in \{0, 2\}$.

We will show that

$$x \in \bigcap_{n=0}^{\infty} C_n$$

where C_n is defined as in the textbook in the construction of the Cantor set.

We will prove by induction that $x \in C_n$ for all $n \geq 0$.

Base step: $n = 0$

x is at most 1, which occurs when $x_k = 2$ for all $k \geq 1$. Moreoever, x is at least 0, which occurs when $x_k = 0$ for all $k \geq 1$. Hence, $x \in [0, 1] = C_0$.

Inductive step:

Assume that $x \in C_n$. We will show that $x \in C_{n+1}$.

We can write C_n as a union of closed intervals of length $1/3^n$. x must be in one of these closed intervals. In the construction of the Cantor set, by the lemma, we remove a set of the form $(a, a + 1 \cdot 3^{-(n+1)})$, where

$$a = \sum_{i=1}^{n} a_i 3^{-i} + 1 \cdot 3^{-(n+1)}$$

and $x_i = a_i = 2$ for all $1 \leq i \leq n$, which we proved in the other implication.

Suppose $x \in (a, a + 1 \cdot 3^{-(n+1)})$. Then since the first n digits match,

$$\sum_{i=1}^{n} a_i 3^{-i} + 1 \cdot 3^{-(n+1)} < \sum_{k=1}^{\infty} x_k 3^{-k} < \sum_{i=1}^{n} a_i 3^{-i} + 2 \cdot 3^{-(n+1)}$$

$$1 \cdot 3^{-(n+1)} < \sum_{k=n+1}^{\infty} x_k 3^{-k} < 2 \cdot 3^{-(n+1)},$$

but this implies that $x_{n+1} = 1$. Indeed, if $x_{n+1} = 0$, then $\sum_{k=n+1}^{\infty} x_k 3^{-k}$ is at most

$$\sum_{k=n+2}^{\infty} 2 \cdot 3^{-k} = 2 \cdot \frac{3^{-(n+2)}}{1 - \frac{1}{3}} = 1 \cdot 3^{-(n+1)},$$

which is a contradiction. Additionally, if $x_{n+1} = 2$, then $\sum_{k=n+1}^{\infty} x_k 3^{-k}$ is at least $2 \cdot 3^{-(n+1)}$, since all the terms are positive, which is also a contradiction.

In all cases, we have a contradiction, which means that $x \notin (a, a + 1 \cdot 3^{-(n+1)})$. Thus, the inductive step holds.

By induction, $x \in C_n$ for all $n \geq 0$, so, $x \in \mathcal{C}$.

b. We'll first show that $F(0) = 0$ and $F(1) = 1$.
The ternary expansion of 0 has $a_k = 0$ for all $k \geq 1$. Then

$$F(x) = \sum_{k=1}^{\infty} \frac{0}{2^{k+1}} = 0.$$

The ternary expansion of 1 has $a_k = 2$ for all $k \geq 1$. Indeed,

$$\sum_{k=1}^{\infty} 2 \cdot 3^{-k} = 2 \cdot \frac{1/3}{1 - 1/3} = 1.$$

Hence,

$$F(1) = \sum_{k=1}^{\infty} \frac{2}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

F is well defined since each term in the sum is non-negative, meaning the sequence of partial sums is increasing. Moreover, $F(x)$ is bounded above by $F(1) = 1$, since $F(x)$ is maximized when $b_k = 1$ for all $k \geq 1$.
Moreover, the ternary expansion with 0's and 2's is unique for a particular $x \in \mathcal{C}$. Indeed, suppose we have two $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n \geq 1}$ such that

$$a = \sum_{n=1}^{\infty} a_n 3^{-n} = \sum_{n=1}^{\infty} b_n 3^{-n} = b.$$

Assume without loss of generality that the first differing digit is the N-th one, and that we have $a_N < b_N$. We can switch a_N and b_N around otherwise in the following argument.
The expansion with a_n is maximized if $a_n = 2$ for all $n \geq N+1$, and we get

$$a = \sum_{n=1}^{N-1} a_n 3^{-n} + \sum_{n=N+1}^{\infty} 2 \cdot 3^{-n} = \sum_{n=1}^{N-1} a_n 3^{-n} + 1 \cdot 3^{-N} < b$$

which is a contradiction. Hence, the expansion is unique if we restrict the digits to 0's and 2's, so each $x \in \mathcal{C}$ has a unique image under F.
We'll now show that F is continuous on $\mathcal{C}$.
Let $x \in \mathcal{C}$, $x = \sum_{k=1}^{\infty} a_k 3^{-k}$. Consider a sequence $\{x_n\}_{n \geq 1} \subseteq [0,1]$ with $x_n \xrightarrow{n\to\infty} x$. For each x_n, consider its expansion

$$x_n = \sum_{k=1}^{\infty} a_k^{(n)} 3^{-k}$$

where $a_k^{(n)} \in \{0,2\}$. Notice that we must have $\lim_{n\to\infty} a_k^{(n)} \in \{0,2\}$ since $\{0,2\}$ is compact and the limit exists. Moreover, we must have $a_k^{(n)} \xrightarrow{n\to\infty} a_k$ for all $k \geq 1$. Otherwise, if for some k they are not equal, then $\lim_{n\to\infty} x_n$ will differ from x by at least

$$2 \cdot 3^{-k} - \sum_{i=k+1}^{\infty} 2 \cdot 3^{-i} = 1 \cdot 3^{-k}$$

which occurs when $a_i = \lim_{n\to\infty} a_i^{(n)}$ for $1 \leq i \leq k-1$, $a_k = 2$, $\lim_{n\to\infty} a_k^{(n)} = 0$, and $\lim_{n\to\infty} a_j^{(n)} = 2$ for all $j > k$.
Note that as $\{a_k^{(n)}\}_{n\geq 1} \subseteq \{0,2\}$ converges to a_k for all $k \geq 1$, there exists some $n_k \in \mathbb{N}$ such that for all $n \geq n_k$, $a_k^{(n)} = a_k$.

Fix $\varepsilon > 0$. Then by the Archimedean property, there exists $N \in \mathbb{N}$ such that $1/2^N < \varepsilon$. Then for $n \geq \max\{n_1, n_2, \dots, n_N\}$,

$$|F(x_n) - F(x)| = \left| \sum_{k=1}^{\infty} \frac{a_k^{(n)} - a_k}{2^{k+1}} \right| \leq \sum_{k=1}^{\infty} \frac{|a_k^{(n)} - a_k|}{2^{k+1}} = \sum_{k=N+1}^{\infty} \frac{|a_k^{(n)} - a_k|}{2^{k+1}} \leq \sum_{k=N+1}^{\infty} \frac{2}{2^{k+1}} = \frac{1}{2^N} < \varepsilon.$$

Hence, $F(x_n) \xrightarrow{n\to\infty} F(x)$. As $\{x_n\}_{n\geq 1}$ was arbitrary, F is continuous on $\mathcal{C}$.

c. Let $y \in [0,1]$. We will construct a sequence $\{b_k\}_{k\geq 1} \subseteq \{0,1\}$, which gives us some $x \in \mathcal{C}$, so that $F(x) = y$. We will construct it by splitting the unit interval into halves, then quarters, than eighths, etc. Define $\{c_n\}_{n\geq 1} \subseteq [0,1]$ as follows:

$$c_n := \sum_{k=1}^{n} \frac{b_k}{2^k}$$

If $y \geq 1/2$, then choose $b_1 = 1$. Otherwise, choose $b_1 = 0$. Then $|y - c_1| < 1/2$.
If $c_1 + 1/4 \geq y$, then choose $b_2 = 1$. Otherwise, choose $b_2 = 0$. Then $|y - c_2| < 1/4$.
If $c_2 + 1/8 \geq y$, then choose $b_3 = 1$. Otherwise, choose $b_3 = 0$. Then $|y - c_3| < 1/8$.
Proceeding inductively yields $\{b_k\}_{k\geq 1}$ so that each $b_k \in \{0,1\}$ and

$$|y - c_n| < \frac{1}{n}$$

for all $n \geq 1$. Hence,

$$\lim_{n\to\infty} c_n = \sum_{k=1}^{\infty} \frac{b_k}{2^k} = y.$$

Moreover, defining $a_k = 2b_k$ gives us

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}$$

with $a_k \in \{0,2\}$ for all $k \geq 1$. Hence, $x \in \mathcal{C}$ and $F(x) = y$.

d. Note that $\mathcal{C}$ is closed since it is an arbitrary intersection of closed sets. Hence, its complement in $[0,1]$ is open, and can be written as a union of open intervals.

Also note that the complement of $\mathcal{C}$ is the union of all the open intervals removed in each step of the construction of the Cantor set. Hence, by the lemma proved in part (a), we can write one of these open intervals as $(a, a + 1 \cdot 3^{-n})$, where

$$a = \sum_{i=1}^{n-1} a_i 3^{-i} + 1 \cdot 3^{-n}$$

where $a_i = 2$ for all $1 \leq i \leq n-1$.
Notice that we can rewrite a as

$$a = \sum_{i=1}^{n-1} a_i 3^{-i} + \sum_{i=n+1}^{\infty} 2 \cdot 3^{-i}.$$

Then

$$F(a) = \sum_{k=1}^{\infty} \frac{1}{2^k} - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

and

$$F(b) = \sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2} \cdot \frac{1 - 1/2^n}{1 - 1/2} = \frac{2^n - 1}{2^n}.$$

Hence, we define $F(x) = (2^n - 1)/2^n$ for the values of x removed in the n-th step of the construction of the Cantor set.

We'll now show continuity for F on this extension to $[0,1]$.

F is continuous on each connected subset of the complement of $\mathcal{C}$ since it is constant on that region. It suffices to check the endpoints of each of these intervals for continuity.

Let a be as above. Let $\{x_n\}_{n\geq 1} \subseteq [0,1]$ be such that $x_n \xrightarrow{n\to\infty} a$. $F(x_n) = F(a) = (2^n - 1)/2^n$ for all x_n such that $x_n \in [0,1] - \mathcal{C}$.

Hence, since F is continuous on $\mathcal{C}$, $F(x_n) \xrightarrow{n\to\infty} F(a)$, so the extension of F is continuous on $[0,1]$.

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4 Construct a closed set $\hat{\mathcal{C}}$ so that at the k-th stage of the construction one removes 2^{k-1} centrally situated open intervals each of length ℓ_k, with

$$\ell_1 + 2\ell_2 + \cdots + 2^{k-1}\ell_k < 1.$$

a. If ℓ_j are chosen small enough, then $\sum_{k=1}^{\infty} 2^{k-1}\ell_k < 1$. In this case, show that $m(\hat{\mathcal{C}}) > 0$, and in fact, $m(\hat{\mathcal{C}}) = 1 - \sum_{k=1}^{\infty} 2^{k-1}\ell_k$.

b. Show that if $x \in \hat{\mathcal{C}}$, then there exists a sequence of points $\{x_n\}_{n\geq 1}$ such that $x_n \notin \hat{\mathcal{C}}$, yet $x_n \to x$ and $x_n \in I_n$, where I_n is a sub-interval in the complement of $\hat{\mathcal{C}}$ with $|I_n| \to 0$.

c. Prove as a consequence that $\hat{\mathcal{C}}$ is perfect, and contains no open interval.

d. Show also that $\hat{\mathcal{C}}$ is uncountable.

Solution a. Suppose we chose $\{\ell_k\}_{k\geq 1}$ with $\sum_{k=1}^{\infty} 2^{k-1}\ell_k < 1$.

Notice that at the open intervals we remove at the k-th stage, $E_i^{(k)}$, satisfy $m(E_i^{(k)}) = \ell_k$.

Moreover, all the intervals that we remove are pairwise separated by at least one non-empty closed interval, so the distance between them is positive. By the sub-additivity property of the outer measure,

$$\hat{\mathcal{C}}^{\mathrm{c}} = \bigcup_{i,k} E_i^{(k)} \implies m(\hat{\mathcal{C}}^{\mathrm{c}}) = \sum_{i,k} \left| E_i^{(k)} \right| = \sum_{k=1}^{\infty} 2^{k-1}\ell_k$$

since at each step, we remove 2^{k-1} intervals of length ℓ_k. Thus,

$$1 = m(\hat{\mathcal{C}}) + m(\hat{\mathcal{C}}^{\mathrm{c}}) \implies m(\hat{\mathcal{C}}) = 1 - \sum_{k=1}^{\infty} 2^{k-1}\ell_k$$

as desired.

b. Let $x \in \hat{\mathcal{C}}$.

In the k-th stage of the construction of $\hat{\mathcal{C}}$, take x_k to be in the middle of the closed interval which contains x. This gives us $\{x_k\}_{k\geq 1} \subseteq [0,1]$ such that $|x_k - x| < d_k$, where d_k is the length of the remaining closed intervals after the k-th step. Notice that $d_k \xrightarrow{k\to\infty} 0$. Indeed, d_k is given by

$$d_k = \frac{1}{2^k}\left(1 - \sum_{i=1}^{k} 2^{i-1}\ell_i\right).$$

This is obtained by noticing that at each iteration, we remove 2^{i-1} intervals of length ℓ_i from the unit interval, so the total length of all the remaining intervals at each step is $1 - \sum_{i=1}^{k} 2^{i-1}\ell_i$. There are 2^k equally long intervals, so each interval is the quotient of the two numbers.

As $k \to \infty$, the numerator of d_k approaches a positive number, while $1/2^k$ approaches 0, so $d_k \xrightarrow{k\to\infty} 0$. Hence $x_k \xrightarrow{k\to\infty} x$. Moreover, $|I_n| = d_k \xrightarrow{k\to\infty} 0$.

c. Note that $\hat{\mathcal{C}}$ is closed, as it is an arbitrary intersection of closed intervals, similar to the Cantor set.

Let $x \in \hat{\mathcal{C}}$. Then there exists a sequence of the closed sub-intervals $\{I_k\}_{k\geq 1}$ generated at each step such that $x \in I_k$ for all $k \geq 1$. The endpoints a_k, b_k of each I_k are always included in $\hat{\mathcal{C}}$, which means that $|a_k - x| < |I_k| \xrightarrow{k\to\infty} 0$. Hence, x is not isolated.

As x was arbitrary, $\hat{\mathcal{C}}$ is perfect.

$\hat{\mathcal{C}}$ contains no open interval. Suppose there exists $(a,b) \subseteq \hat{\mathcal{C}}$ with $a < b$. As $|I_k| \xrightarrow{k\to\infty} 0$, this means that there exists $k_{a,b}$ such that each closed sub-interval past the $k_{a,b}$-th stage of the construction satisfies $|I_k| < (b-a)/2$. But this means that (a,b) cannot be in $\hat{\mathcal{C}}$ because it is too long, which is a contradiction.

d. Suppose $\hat{\mathcal{C}}$ is countable. Then there exists an enumeration $\hat{\mathcal{C}} = \{c_1, c_2, \ldots\}$.

In the first stage of the construction of $\hat{\mathcal{C}}$, we split the unit interval into a left and right interval. If c_1 is in the left interval, then pick x_1 in the right interval. Otherwise, pick x_1 in the left interval.

In the second stage, we split the interval containing x_1 into a left and right interval. If c_2 is in the left interval, then pick x_2 to be in the right interval. Otherwise, pick x_2 to be in the left interval.

Proceeding inductively, we obtain a sequence $\{x_n\}_{n\geq 1} \subseteq [0,1]$ which converges to some $x \in \hat{\mathcal{C}}$, since $|I_k| \xrightarrow{k\to\infty} 0$ and $\hat{\mathcal{C}}$ is closed, such that for all $n \geq 1$, $|x - c_n| > \ell_n$. But this is a contradiction, because this means there does not exist n_0 such that $x = c_{n_0}$. Hence, $\hat{\mathcal{C}}$ is uncountable.

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5 Suppose E is a given set, and $\mathcal{O}_n$ is the open set:

$$\mathcal{O}_n = \{x \mid d(x, E) < 1/n\}.$$

Show:

a. If E is compact, then $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$.

b. However, the conclusion in (a) may be false for E closed and unbounded; or E open and bounded.

Solution a. Note that $\{m(\mathcal{O}_n)\}_{n\geq 1}$ is a decreasing sequence of real numbers. Indeed, since $\mathcal{O}_{n+1} \subseteq \mathcal{O}_n$, we're taking the infimum on a subset of $\mathcal{O}_n$, so $m(\mathcal{O}_{n+1}) \leq m(\mathcal{O}_n)$. Moreover, the sequence is bounded below by 0 by the definition of the outer measure. Hence, $\lim_{n\to\infty}|\mathcal{O}_n|$ exists in $\mathbb{R}$.

Notice that

$$\mathcal{O}_n = \bigcap_{i=1}^{n} \mathcal{O}_i.$$

We claim that $E = \bigcap_{i=1}^{\infty} \mathcal{O}_i$.

Let $x \in E$. Since E is closed, $d(x, E) = 0$, so $x \in \bigcap_{i=1}^{n} \mathcal{O}_i \implies E \subseteq \bigcap_{i=1}^{\infty} \mathcal{O}_i$.

Let $x \in \bigcap_{i=1}^{\infty} \mathcal{O}_i$. Suppose $x \notin E$. Since E is closed, $d(x, E) > \delta$ for some $\delta > 0$. By the Archimedean principle, there exists $N \in \mathbb{N}$ such that $1/N < \delta \implies d(x, E) > 1/N$. But this means that $x \notin \mathcal{O}_N$, which is a contradiction. Hence, $x \in E$, so $E = \bigcap_{i=1}^{\infty} \mathcal{O}_i$.

As each $\mathcal{O}_n$ is open, it is measurable, so we must have that $\lim_{n\to\infty} m(\mathcal{O}_n) = m(E)$.

b. Consider $E = \mathbb{N}$, which is closed and unbounded. It has measure 0 since it is countable.

However, $\mathcal{O}_n = \bigcup_{i=1}^{\infty}(i - 1/n, i + 1/n)$, which means $|\mathcal{O}_n| = 2/n$. Then $m(\mathcal{O}_n) = \sum_{i=1}^{\infty} 2/n = \infty$ so $\lim_{n\to\infty} m(\mathcal{O}_n) = \infty \neq 0$.

Consider $E = [0,1] \setminus \hat{\mathcal{C}}$, which is bounded and open since $\hat{\mathcal{C}}$ is closed. As we showed in the previous problem, the measure of E is less than 1.

However, as we proved in part (b) in the previous problem, for all $x \in \hat{\mathcal{C}}$ and $n \geq 1$, there exists $y \in [0,1] \setminus \hat{\mathcal{C}}$ such that $d(x, y) < 1/n$. Hence, $[0,1] \subseteq \mathcal{O}_n$. But this implies that $m(\mathcal{O}_n) \geq 1$ for all $n \geq 1$, which means that $\lim_{n\to\infty} m(\mathcal{O}_n) \geq 1$.

6 Using translations and dilations, prove the following: Let B be a ball in $\mathbb{R}^d$ of radius r. Then $m(B) = v_d r^d$, where $v_d = m(B_1)$, and B_1 is the unit ball, $B_1 = \{x \in \mathbb{R}^d \mid |x| < 1\}$.

Solution Using the notation in exercise 7, if B is a ball of radius r centered at the origin, then $B = rB_1$, so by the next exercise, $m(B) = r \cdots r m(B_1) = v_d r^d$.

Shifting B preserves measure, so the result holds for any ball in $\mathbb{R}^d$ of radius r.

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7 If $\delta = (\delta_1, \ldots, \delta_d)$ is a d-tuple of positive numbers $\delta_i > 0$, and E is a subset of $\mathbb{R}^d$, we define δE by

$$\delta E = \{(\delta_1 x_1, \ldots, \delta_d x_d) \mid \text{where } (x_1, \ldots, x_d) \in E\}.$$

Prove that δE is measurable whenever E is measurable, and

$$m(\delta E) = \delta_1 \cdots \delta_d m(E).$$

Solution It suffices to show when we dilate one component of an element in E by a factor of δ, that scale $m_*(E)$ by δ. We can then repeat the argument for all other components.

Note that dilation in the i-th component by a factor of $\delta > 0$ is a bijective mapping. Indeed, its inverse is mapping is multiplying the i-th component by $1/\delta$.

Consider a rectangle $R = [a_1, b_1] \times \cdots \times [a_d, b_d]$. If we dilate the i-th component by a factor of δ, then we get $R' = [a_1, b_1] \times \cdots \times [\delta a_i, \delta b_i] \times \cdots [a_d, b_d]$. Then $|R'| = \delta |R|$.

Let $\varepsilon > 0$. Then there exists a sequence of cubes $\{Q_n\}_{n\geq 1}$ such that $E \subseteq \bigcup_{n\geq 1} Q_n$ and $\sum_{n\geq 1} |Q_n| \leq m_*(E) + \varepsilon$.

Then dilating the i-th component of E by δ gives a new set E'. Moreover, if $x \in Q_n$, then $x' \in Q'_n$ also. Since dilation is bijective, this means that Q'_n covers E'. Thus,

$$m_*(E') \leq \sum_{n\geq 1} |Q'_n| = \sum_{n\geq 1} \delta |Q'_n| \leq \delta m_*(E) + \delta\varepsilon$$

As ε was arbitrary, $m_*(E') \leq \delta m_*(E)$.

Let $\{Q'_n\}_{n\geq 1}$ be a sequence of cubes which cover E' and have total area less than or equal to $m_*(E') + \varepsilon$. If we dilate these cubes by $1/\delta$, then we get cubes $\{Q_n\}_{n\geq 1}$ which cover E. Hence,

$$m_*(E) \leq \frac{1}{\delta} \sum_{n\geq 1} |Q'_n| \leq \frac{1}{\delta}(m_*(E') + \varepsilon).$$

As ε was arbitrary and $\delta > 0$, we have $\delta m_*(E) \leq m_*(E')$.

Thus, $m_*(E') = \delta m_*(E)$. Repeating the argument for each component of E, we have that

$$m_*(\delta E) = \delta_1 \cdots \delta_d m_*(E)$$

as desired.

If E is measurable, then for all $\varepsilon > 0$, there exists O_ε such that $m_*(O - E) \leq \varepsilon/\delta$. By what we proved,

$$m_*(\delta O - \delta E) = \delta m_*(O - E) \leq \varepsilon$$

so δE is measurable also.