****5 2.3.9** Let V be a vector space over a subfield F of the complex numbers. Suppose α, β, and γ are linearly independent vectors in V. Prove that $(\alpha+\beta)$, $(\beta+\gamma)$, and $(\gamma+\alpha)$ are linearly independent.

Solution We wish to show that if

$$c_1(\alpha+\beta)+c_2(\beta+\gamma)+c_3(\gamma+\alpha)=0$$

then $c_1=c_2=c_3=0$.

$$\begin{aligned} c_1(\alpha+\beta)+c_2(\beta+\gamma)+c_3(\gamma+\alpha)&=0\\ c_1\alpha+c_1\beta+c_2\beta+c_2\gamma+c_3\gamma+c_3\alpha&=0\\ (c_1+c_3)\alpha+(c_1+c_2)\beta+(c_2+c_3)\gamma&=0 \end{aligned}$$

Since α, β, and γ are linearly independent, $c_1+c_3=c_1+c_2=c_2+c_3=0$. Then $c_1+c_3=c_1+c_2 \implies c_2=c_3$, so $c_2+c_3=2c_2=0 \implies c_2=c_3=0$. Thus, $c_1+c_2=c_2+c_3 \implies c_1=0$. Thus, $c_1=c_2=c_3=0$, which means $(\alpha+\beta)$, $(\beta+\gamma)$, and $(\gamma+\alpha)$ are linearly independent.

****6** V is a vector space. Suppose $S\subset V$ and that

$$S=S_1\cup S_2$$

and that

$$S_1\cap S_2=\emptyset$$

and that S is linearly independent. Prove:

$$\operatorname{span} S_1\cap\operatorname{span} S_2=\{0\}.$$

You can assume S is finite.

Solution Let $S_1=\{\alpha_1,\ldots,\alpha_n\}$ and $S_2=\{\beta_1,\ldots,\beta_m\}$. Since $S_1\cap S_2=\emptyset$, $S=\{\alpha_1,\ldots,\alpha_n,\beta,\ldots,\beta_m\}$. Suppose $\gamma\in\operatorname{span} S_1\cap\operatorname{span} S_2$. Then for $c_1,\ldots,c_n$ and $d_1,\ldots,d_m$, we have

$$\begin{aligned} \gamma=c_1\alpha_1+\cdots+c_n\alpha_n&=d_1\beta_1+\cdots+d_m\beta_m\\ \implies c_1\alpha_1+\cdots+c_n\alpha_n+(-d_1)\beta_1+\cdots+(-d_m)\beta_m&=0 \end{aligned}$$

Since S is linearly independent, we must have by definition $c_1=\cdots=c_n=d_1=\cdots=d_m=0$. Thus, if $\gamma\in\operatorname{span} S_1\cap\operatorname{span} S_2$, γ must be equal to 0. Hence, $\operatorname{span} S_1\cap\operatorname{span} S_2=\{0\}$.

****7 2.3.14** Let V be the set of real numbers. Regard V as a vector space over the field of *rational* numbers, with the usual operations. Prove that this vector space is *not* finite-dimensional.

Solution Let $\alpha\in\mathbb{R}$ be non-algebraic; that is, given $n\in\mathbb{Z}$, there does not exist $c_0,\ldots,c_{n-1}\in\mathbb{Q}$ such that

$$\alpha^n+c_{n-1}\alpha^{n-1}+\cdots+c_1\alpha+c_0=0$$

Consider the set $A(n)=\{1,\alpha,\alpha^2,\ldots,\alpha^n\}$. We will prove by induction that $A(n)$ is linearly independent for $n\geq 0$ over $\mathbb{Q}$. (In this proof, all spans refer to spans over the field $\mathbb{Q}$.)

Base step:

$A(0)=\{1\}$ is obviously linearly independent.

Inductive step:

Suppose $A(n)$ is linearly independent. Then we wish to show that $A(n+1)$ is also linearly independent. We can do this by showing that $\alpha^{n+1}\notin\operatorname{span} A(n)$. Suppose otherwise, and that $\alpha^{n+1}\in\operatorname{span} A(n)$. Then there are $c_0,\ldots,c_n\in\mathbb{Q}$ such that

$$\begin{aligned} c_n\alpha^n+\cdots+c_0&=\alpha^{n+1}\\ \alpha^{n+1}+(-c_n)\alpha^n+\cdots+(-c_0)&=0, \end{aligned}$$

but α is non-algebraic. Thus, there are no values in $c_0,\ldots,c_n\in\mathbb{Q}$ such that the equation holds. Hence, α^{n+1} is not in the span of $A(n)$, so $A(n+1)$ must be linearly independent.

Since both steps hold, we use the principle of induction to conclude that $A(n)$ is linearly independent for all natural numbers $n\geq 0$. There is no upper bound on n, so meaning we can find infinitely many vectors in $\mathbb{R}$ that are linearly independent over $\mathbb{Q}$. Hence, $\dim\mathbb{R}=\infty$ over the field of rational numbers.