1 Show that if $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable, then f is bounded on $[a,b]$.

Solution As f is Riemann integrable, $|f|$ is integrable also, so let

$$\int_a^b |f(x)| \, \mathrm{d}x = r$$

for some $r \in \mathbb{R}$.

Fix $\varepsilon > 0$. Then as there exists a partition $P = \{a = t_0 < \cdots < t_n = b\}$ of $[a,b]$ whose associated Riemann sum S satisfies

$$|S - r| < \varepsilon \implies r - \varepsilon < S < r + \varepsilon$$

Let $\delta = \min_{1 \le k \le n}\{t_k - t_{k-1}\}$.

Suppose f were unbounded. Then for all $M > 0$, there exists $x_0 \in [a,b]$ such that $|f(x_0)| > \frac{M}{\delta}$. Note that x_0 lies in $[t_{l-1}, t_l]$ for some l.

Choose $x_l = x_0$. Then

$$\begin{aligned} S &= \sum_{k=1}^{n} |f(x_k)|(t_k - t_{k-1}) \\ &> M + \sum_{k \neq l} |f(x_k)|(t_k - t_{k-1}) \\ &\geq M \end{aligned}$$

Hence, if we choose $M = r + \varepsilon$, we have $S \geq r + \varepsilon$, which is a contradiction. Thus, f must be bounded.

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2 Let $f\colon [a,b] \to \mathbb{R}$ be integrable and let $g\colon [a,b] \to \mathbb{R}$ be a function such that $f(x) = g(x)$ except for finitely many x in $[a,b]$. Show that g is integrable on $[a,b]$ and that

$$\int_a^b f(x) \, \mathrm{d}x = \int_a^b g(x) \, \mathrm{d}x.$$

Solution It suffices to show that $f - g$ is integrable, that f and g differ by a single $x_0 \in [a,b]$, and that

$$\int_a^b (f-g)(x) \, \mathrm{d}x = 0 \implies \int_a^b f(x) \, \mathrm{d}x = \int_a^b g(x) \, \mathrm{d}x$$

Indeed, if $f - g$ is integrable, then $-(f-g) + f$ is also integrable, since f is integrable. Moreover, we can extend the proof to an arbitrary finite number of differences through induction.

As $f(x) = g(x)$ except at x_0, we have that $h\colon [a,b] \to \mathbb{R}$, $h(x) = f(x) - g(x) = 0$ except for at x_0. Assume without loss of generality that $h(x_0) > 0$.

Let $\varepsilon > 0$. Let $P = \{a = t_0 < \cdots < t_n = b\}$ be a partition of $[a,b]$ with mesh $P = \delta$, with $\delta > 0$ to be chosen later.

Note that there exists l such that $x_0 \in [t_{l-1}, t_l]$ and possibly in either $[t_{l-2}, t_{l-1}]$ or $[t_l, t_{l+1}]$ also.

We will show that

$$U(h,P) - L(h,P) < \varepsilon$$

Notice that each $[t_{k-1}, t_k]$ contains a point where $h(x) = 0$, so $L(h, [t_{k-1}, t_k]) = 0$ for all $k \implies L(h) = 0$.

Thus,

$$\begin{aligned} U(h,P) - L(h,P) &= U(h,P) \\ &= \sum_{k=1}^{n} M(h, [t_{k-1}, t_k])(t_k - t_{k-1}) \\ &< h(x_0)(t_{l-2} - t_{l-1}) + h(x_0)(t_l - t_{l-1}) + h(x_0)(t_{l+1} - t_l) \\ &< 3h(x_0)\delta \end{aligned}$$

If we pick $\delta = \frac{\varepsilon}{3h(x_0)}$, then we get the desired result. Hence, $f - g$ is integrable. In particular, $f - g$ is Darboux integrable. Thus,

$$\int_a^b (f-g)(x)\,\mathrm{d}x = L(h) = 0 \implies \int_a^b f(x)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x.$$

We can apply the same argument if $f(x_0) < 0$, but $U(f - g) = 0$ and we would need to choose $\delta = -\frac{\varepsilon}{3h(x_0)}$ instead to get the desired result.

By induction, we can extend our argument for an arbitrary finite number of points where $f(x) \neq g(x)$.

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3 Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function and let $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$.

a. Show that if P is a partition of $[a,b]$, then

$$U(f^2, P) - L(f^2, P) \leq 2M[U(f,P) - L(f,P)].$$

b. Deduce that if f is integrable on $[a,b]$, then f^2 is also integrable on $[a,b]$.

c. Prove that if f and g are two integrable functions on $[a,b]$, then the product fg is integrable on $[a,b]$.

Solution a. Notice that for $S \subseteq [a,b]$, we have

$$\begin{aligned} M(f^2,S) - m(f^2,S) &= \sup_{x,y\in S} \{f^2(x) - f^2(y)\} \\ &= \sup_{x,y\in S} \big\{(f(x)+f(y))(f(x)-f(y))\big\} \\ &\leq 2M \sup_{x,y\in S} \big\{f(x)-f(y)\big\} \\ &= 2M[M(f,S) - m(f,S)] \end{aligned}$$

Thus, $U(f^2, P) - L(f^2, P) \leq 2M[U(f,P) - L(f,P)]$ as desired.

b. Fix $\varepsilon > 0$.

As f is integrable, there exists δ such that if P is a partition of $[a,b]$ with mesh $P < \delta$, then

$$\begin{aligned} &U(f,P) - L(f,P) < \frac{\varepsilon}{4M} \\ \implies &U(f^2,P) - L(f^2,P) \leq 2M[U(f,P) - L(f,P)] < \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

as desired. Hence, f^2 is also integrable on $[a,b]$.

c. By a theorem proved in class and by part (b), the following functions are integrable:

$$-\frac{f^2}{2}, -\frac{g^2}{2}, \frac{(f+g)^2}{2}.$$

Thus,

$$\frac{(f+g)^2}{2} - \frac{f^2}{2} - \frac{g^2}{2} = fg$$

is integrable, as desired.

4 Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a,b]$. Assume that

$$\int_a^b f(x)\,\mathrm{d}x = 0.$$

Show that $f(x) = 0$ for all $x \in [a,b]$.

Solution Note that on any subinterval $[c,d]$ with $a \leq c < d \leq b$, we have $0 \leq f(x)$, so

$$0 \leq \int_c^d f(x)\,\mathrm{d}x$$

which means that we must have $\int_c^d f(x)\,\mathrm{d}x = 0$. Otherwise, if this were not the case, $\int_a^b f(x)\,\mathrm{d}x \neq 0$. Indeed, as f is continuous on $[a,b]$, it is piecewise continuous on $[a,c]$, $[c,d]$, and $[d,b]$, so

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^d f(x)\,\mathrm{d}x + \int_d^b f(x)\,\mathrm{d}x > 0$$

which is a contradiction.

Suppose otherwise, and that there exists $x_0 \in [a,b]$ such that $f(x_0) > 0$. As f is continuous, there exists $\delta > 0$ such that $f(x) > 0$ on $A \coloneqq [x_0 - \delta, x_0 + \delta]$. As f is integrable on $[a,b]$, it is integrable on A also. Note that as f is continuous on A compact, there exists $x_a \in A$ such that

$$0 < f(x_a) = \inf_{x \in A}\{f(x)\} \leq f(x) \implies 0 < 2\delta f(x_a) \leq \int_{x_0-\delta}^{x_0+\delta} f(x)\,\mathrm{d}x$$

which contradicts the above observation. Hence, $f \equiv 0$ on $[a,b]$.

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5 Let $f, g\colon [a,b] \to \mathbb{R}$ be two Riemann integrable functions such that the set $\{x \in [a,b] \mid f(x) = g(x)\}$ is dense in $[a,b]$. Show that

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x.$$

Solution Note that for $h\colon [a,b] \to \mathbb{R}$, $h(x) = f(x) - g(x)$, the set $E \coloneqq \{x \in [a,b] \mid h(x) = 0\}$ is dense in $[a,b]$. Note that since f and g are integrable, so is h.

We wish to show that

$$\int_a^b h(x)\,\mathrm{d}x = 0.$$

Let $\varepsilon > 0$.

As h is integrable, $|h|$ is also integrable, so $U(|h|) = L(|h|)$.

Let $S \subseteq [a,b]$ such that S an interval containing more than 1 point. Then as E is dense in $[a,b]$, $\emptyset \neq E \cap \mathring{S} \subseteq E \cap S$. Thus, there exists $x_0 \in S$ such that $|h(x_0)| = \inf_{x \in S}\{h(x)\} = 0$.

So, for any partition $P = \{a = t_0 < \cdots < t_n = b\}$ of $[a,b]$,

$$m(|h|, [t_{k-1}, t_k]) = \inf_{x \in [t_{k-1}, t_k]}\{|h(x)|\} = 0$$

for all k. Thus,

$$L(|h|) = U(|h|) = \int_a^b |h(x)|\,\mathrm{d}x = 0.$$

Hence,

$$\left|\int_a^b h(x)\,\mathrm{d}x\right| \leq \int_a^b |h(x)|\,\mathrm{d}x = 0 \implies \int_a^b h(x)\,\mathrm{d}x = 0 \implies \int_a^b f(x)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x$$

as desired.

6 Suppose $f\colon [1,\infty) \to \mathbb{R}$ is Riemann integrable on $[1,a]$ for all $a > 1$. If

$$\lim_{a\to\infty} \int_1^a f(x)\,\mathrm{d}x$$

exists and is finite, we say that the integral $\int_1^\infty f(x)\,\mathrm{d}x$ converges and we write

$$\int_1^\infty f(x)\,\mathrm{d}x = \lim_{a\to\infty} \int_1^a f(x)\,\mathrm{d}x.$$

Now assume $f\colon [1,\infty) \to \mathbb{R}$ is non-negative and decreasing. Show that

$$\int_1^\infty f(x)\,\mathrm{d}x \text{ converges if and only if } \sum_{n\geq 1} f(n) \text{ converges.}$$

Solution As f is non-negative,

$$F(x) := \int_1^x f(t)\,\mathrm{d}t$$

is an increasing function. Indeed, if $x < y$, then

$$F(y) - F(x) = \int_1^y f(t)\,\mathrm{d}t - \int_1^x f(t)\,\mathrm{d}t = \int_x^y f(t)\,\mathrm{d}t \geq 0.$$

As f is decreasing, we have that given $n \in \mathbb{N}$, $f(n+1) \leq f(x) \leq f(n)$ for all $x \in [n, n+1]$. Thus,

$$f(n+1) = \int_n^{n+1} f(n+1)\,\mathrm{d}x \leq \int_n^{n+1} f(x)\,\mathrm{d}x \leq \int_n^{n+1} f(n)\,\mathrm{d}x = f(n)$$

for all $n \geq 1$.

Summing the inequality yields

$$\sum_{k=1}^n f(k+1) \leq \sum_{k=1}^n \int_k^{k+1} f(x)\,\mathrm{d}x \leq \sum_{k=1}^n f(k)$$

$$\sum_{k=2}^n f(k) \leq \int_1^n f(x)\,\mathrm{d}x \leq \sum_{k=1}^n f(k).$$

Hence, for $x \geq 1$, there exists $n \geq 2$ such that

$$\int_1^{n-1} f(t)\,\mathrm{d}t \leq \int_1^x f(t)\,\mathrm{d}t \leq \int_1^n f(t)\,\mathrm{d}t \leq \sum_{k=1}^n f(k) \tag{1}$$

and $m \geq 1$ such that

$$\sum_{k=2}^m f(k) \leq \int_1^m f(t)\,\mathrm{d}t \leq \int_1^x f(t)\,\mathrm{d}t \leq \int_1^{m+1} f(t)\,\mathrm{d}t \tag{2}$$

Thus, if $\sum_{n\geq 1} f(n)$ converges, then by (1), as $x \to \infty$, $n \to \infty$ also so we have

$$\int_1^\infty f(t)\,\mathrm{d}t \leq \sum_{k=1}^\infty f(k) < \infty.$$

If the sum diverges, then by (2), the integral must diverge.

Similarly, if the integral converges, then as $x \to \infty$, we have $m \to \infty$ so by (2),

$$\sum_{k=1}^\infty f(k) \leq f(1) + \int_1^\infty f(t)\,\mathrm{d}t < \infty.$$

If the integral diverges, then by (1), then the sum must diverge also. Hence,

$$\int_1^\infty f(x)\,\mathrm{d}x \text{ converges } \iff \sum_{n\geq 1} f(n) \text{ converges.}$$