1.3 Prove the parallelogram equality:

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$

In geometric terms, the equality says that the sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of the lengths of the sides. It is perhaps easier to prove it using the complex notation of this chapter than to prove it using high school geometry.

Solution First note that if $a = a_1 + ia_2$ and $b = b_1 + ib_2$,

$$|a+b|^2 = (a_1+b_1)^2 + (a_2+b_2)^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2a_1b_1 + 2a_2b_2 = |a|^2 + |b|^2 + 2(a_1b_1 + a_2b_2).$$

Then expanding out the left side of the inequality yields

$$\big(|z|^2 + |w|^2 + 2(z_1w_1 + z_2w_2)\big) + \big(|z|^2 + |w|^2 - 2(z_1w_1 + z_2w_2)\big) = 2(|z|^2 + |w|^2)$$

as desired.

1.5 Check the formula for the chordal distance between two points on the unit sphere given in §3. The chordal distance is bounded by 2, by the triangle inequality. Verify analytically that the formula for this distance given in the text is bounded by 2 using the Cauchy–Schwarz inequality, and also by directly multiplying it out using complex notation and one of the estimates at the start of §2.

Solution Let $z = a + bi$, $w = c + di$. Then

$$z^* = \left(\frac{2a}{|z|^2+1}, \frac{2b}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$$
$$w^* = \left(\frac{2c}{|w|^2+1}, \frac{2d}{|w|^2+1}, \frac{|w|^2-1}{|w|^2+1}\right)$$

Then the distance between them is given by

$$\begin{aligned}
|z^*-w^*|^2 &= \left(\frac{2a}{|z|^2+1} - \frac{2c}{|w|^2+1}\right)^2 + \left(\frac{2b}{|z|^2+1} - \frac{2d}{|w|^2+1}\right)^2 + \left(\frac{|z|^2-1}{|z|^2+1} - \frac{|w|^2-1}{|w|^2+1}\right)^2 \\
&= \frac{4|z|^2 + |z|^4 - 2|z|^2 + 1}{(|z|^2+1)^2} + \frac{4|w|^2 + |w|^4 - 2|w|^2 + 1}{(|w|^2+1)^2} - \frac{8ac + 8bd}{(|z|^2+1)(|w|^2+1)} - \frac{(|z|^2-1)(|w|^2-1)}{(|z|^2+1)(|w|^2+1)} \\
&= 2 - \frac{8ac+8bd}{(|z|^2+1)(|w|^2+1)} - 2\frac{|z|^2|w|^2 - |z|^2 - |w|^2 + 1}{(|z|^2+1)(|w|^2+1)} \\
&= \frac{2|z|^2|w|^2 + 2|z|^2 + 2|w|^2 + 2 - 8(ac+bd) - 2|z|^2|w|^2 + 2|z|^2 + 2|w|^2 - 2}{(|z|^2+1)(|w|^2+1)} \\
&= \frac{4(|z|^2 + |w|^2 - 2(ac+bd))}{(|z|^2+1)(|w|^2+1)} \\
&= \frac{4|z-w|}{(|z|^2+1)(|w|^2+1)} \\
\implies |z^*-w^*| &= \frac{2|z-w|}{\sqrt{|z|^2+1}\sqrt{|w|^2+1}}
\end{aligned}$$

Next, we wish to verify

$$|z^*-w^*| = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} \leq 2.$$

By Cauchy–Schwarz,

$$|z-w|^2 = |1\cdot z - w \cdot 1|^2 \leq (1^2 + |w|^2)(|\bar{z}|^2 + 1^2) = (1+|z|^2)(1+|w|^2)$$
$$\implies |z-w| \leq \sqrt{1+|z|^2}\sqrt{1+|w|^2}$$
$$\implies |z^*-w^*| = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} \leq 2$$

First note that $|x| \leq 1 + |x|^2$: if $|x| \leq 1$, then $|x|^2 \leq 1 \implies 1 + |x|^2 \geq 1$. If $|x| > 1$, then $|x|^2 > |x| \implies 1 + |x|^2 > |x|$.

Hence,

$$\begin{aligned}
|z - w|^2 &= (z - w)(\bar{z} - \bar{w}) \\
&= z\bar{z} + w\bar{w} - \bar{z}w - \bar{w}z \\
&\leq |z|^2 + |w|^2 + |z||w| \qquad \left(\frac{1}{2}\operatorname{Im}\bar{z}w \leq |z||w|\right) \\
&\leq |z|^2 + |w|^2 + 1 + |z|^2|w|^2 \\
&= (1 + |z|^2)(1 + |w|^2) \\
\implies |z - w| &\leq \sqrt{1 + |z|^2}\sqrt{1 + |w|^2} \\
\implies |z^* - w^*| &= \frac{2|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}} \leq 2
\end{aligned}$$

1.12 Stereographic projection combined with rigid motions of the sphere can be used to describe some transformations of the plane.

a. Map a point $z \in \mathbb{C}$ to $\mathbb{S}^2$, apply a rotation of the unit sphere, then map the resulting point back to the plane. For a fixed rotation, find this map of the extended plane to itself as an explicit function of z. Two cases are worth working out first: rotation about the x_3 axis and rotation about the x_1 axis.

b. Another map can be obtained by mapping a point $z \to \mathbb{C}$ to $\mathbb{S}^2$, then translating the sphere so that the origin is sent to (x_0, y_0, z_0), then projecting back to the plane. The projection to the plane is given by drawing a line through the (translated) north pole and a point on the (translated) sphere and finding the intersection with the plane $\{(x, y, 0)\}$. For a fixed translation, find this map as an explicit function of z. In this case it is worth working out a vertical translation and a translation in the plane separately. Then view an arbitrary translation as a composition of these two maps. Partial answer: the maps in (a) and (b) are of the form $(az + b)/(cz + d)$ with $ad - bc \neq 0$.

Solution a. Let $z \in \mathbb{C}$, where $z = |z|e^{i\theta} = |z|\cos\theta + i|z|\sin\theta$. Then its stereographic projection is

$$z^* = \left(\frac{2|z|}{|z|^2 + 1}\cos\theta, \frac{2|z|}{|z|^2 + 1}\sin\theta, \frac{|z|^2 - 1}{|z|^2 + 1}\right).$$

Given a point (x_1, x_2, x_3) on the sphere, its corresponding complex number is

$$z = \frac{x_1}{1 - x_3} + i\frac{x_2}{1 - x_3}$$

Next consider the matrix associated with a counter-clockwise rotation of φ about the x_3 axis:

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying this matrix to z^* yields

$$\begin{aligned}
z^\dagger &= \left(\frac{2|z|}{|z|^2 + 1}(\cos\theta\cos\varphi - \sin\theta\sin\varphi), \frac{2|z|}{|z|^2 + 1}(\sin\theta\cos\varphi + \cos\theta\sin\varphi), \frac{|z|^2 - 1}{|z|^2 + 1}\right) \\
&= \left(\frac{2|z|}{|z|^2 + 1}\cos(\theta + \varphi), \frac{2|z|}{|z|^2 + 1}\sin(\theta + \varphi), \frac{|z|^2 - 1}{|z|^2 + 1}\right),
\end{aligned}$$

which, when projected back onto $\mathbb{C}$, obviously gives $z' = |z|e^{i(\theta + \varphi)}$, so this is a rotation of the plane about the x_3 axis.

Now consider the matrix associated with a counter-clockwise rotation of φ about the x_1 axis when looking from the positive x_1 axis:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}.$$

Applying this to z^* yields

$$\begin{aligned} z^\dagger &= \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}\cos\varphi - \frac{|z|^2-1}{|z|^2+1}\sin\varphi, \frac{2y}{|z|^2+1}\sin\varphi + \frac{|z|^2-1}{|z|^2+1}\cos\varphi\right) \\ &= \frac{1}{|z|^2+1}\left(2x, 2y\cos\varphi - (|z|^2-1)\sin\varphi, 2y\sin\varphi + (|z|^2-1)\cos\varphi\right) \end{aligned}$$

So, its projection back onto the x_1x_2-plane is

$$z = \frac{2x + i(2y\cos\varphi - (|z|^2-1)\sin\varphi)}{(|z|^2+1) - 2y\sin\varphi - (|z|^2-1)\cos\varphi}$$

Similarly, for a rotation about the x_2 axis through an angle φ, we get

$$z = \frac{2x\cos\varphi + (|z|^2-1)\sin\varphi + i2y}{2x\sin\varphi - (|z|^2-1)\cos\varphi + 1}.$$

We can get any arbitrary rotation we want by composing the three results together.

b. Let $z = x + iy$. Then its stereographic projection is

$$z^* = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right).$$

Shifting the sphere by (x_0, y_0, z_0) yields a North pole of $(x_0, y_0, 1+z_0)$ and

$$z^\dagger = \left(\frac{2x}{|z|^2+1} + x_0, \frac{2y}{|z|^2+1} + y_0, \frac{|z|^2-1}{|z|^2+1} + z_0\right).$$

Thus, the line joining the new North pole and $z^\dagger$ is

$$\begin{aligned} &(x_0, y_0, 1+z_0) + t\left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} - 1\right) \\ &= (x_0, y_0, 1+z_0) + t\left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, -\frac{2}{|z|^2+1}\right) \end{aligned}$$

which intersects the x_1x_2-plane when

$$t = \frac{(1+z_0)(|z|^2+1)}{2}$$

so we get the new complex number

$$\begin{aligned} z &= x_0 + x(1+z_0) + iy_0 + iy(1+z_0) \\ &= x + iy + z_0(x+iy) + x_0 + iy_0 \\ &= (1+z_0)z + (x_0 + iy_0) \end{aligned}$$

as desired. In this transformation, we have $a = 1+z_0$, $b = x_0+iy_0$, $c = 0$, and $d = 1$. $ad-bc = 1+z_0 \neq 0$ as long as $z_0 \neq -1$.

2.2a If p is a polynomial with real coefficients, prove that p can be factored into a product of linear and quadratic factors, each of which has real coefficients, such that the quadratic factors are non-zero on $\mathbb{R}$. Most Engineering problems involving polynomials only need polynomials with real coefficients.

Solution Let $p = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$. By the fundamental theorem of algebra, p has a root, which we call z_0. We claim that $\bar{z_0}$ is also a root.

Write $z_0 = |z_0| e^{i\theta}$. Then $\bar{z_0} = |z_0| e^{-i\theta}$. Hence,

$$\begin{aligned} p(z_0) &= a_n |z_0|^n e^{in\theta} + a_{n-1} |z_0|^{n-1} e^{i(n-1)\theta} + \cdots + a_1 |z_0| e^{i\theta} + a_0 = 0 \\ p(\bar{z_0}) &= a_n |z_0|^n e^{-in\theta} + a_{n-1} |z_0|^{n-1} e^{-i(n-1)\theta} + \cdots + a_1 |z_0| e^{-i\theta} + a_0 \\ p(z_0) + p(\bar{z_0}) &= \sum_{k=0}^{n} a_i |z_0|^i (e^{ik\theta} + e^{-ik\theta}) = 2 \sum_{k=0}^{n} a_i |z_0|^i \cos k\theta = 2 \operatorname{Re} p(z_0) = 0 \end{aligned}$$

Hence $p(\bar{z_0}) = 0$ also.

By the fundamental theorem of algebra, p has n roots, counting multiplicity, so we can write

$$p = a_n \prod_{k=1}^{n} (z - z_k),$$

where z_i is a root. We order $z_1, \ldots, z_n$ as follows: $z_1, \ldots, z_k$ are all of the real roots, and for $k < j < n$, $\bar{z_j} = z_{j+1}$. In other words, put the real roots first, and put all conjugates next to each other. Then

$$\begin{aligned} p &= a_n \left(\prod_{l=1}^{k} (z - z_l) \right) (z - z_{k+1})(z - \overline{z_{k+1}}) \cdots (z - z_{n-1})(z - \overline{z_{n-1}}) \\ &= a_n \left(\prod_{l=1}^{k} (z - z_l) \right) \left(z^2 - z\overline{z_{k+1}} - z(z_{k+1} + \overline{z_{k+1}}) + z_{k+1}\overline{z_{k+1}}\right) \cdots \left(z^2 - z\overline{z_{n-1}} - z(z_{n-1} + \overline{z_{n-1}}) + z_{n-1}\overline{z_{n-1}}\right) \end{aligned}$$

The linear factors obviously have real coefficients. Since $z + \bar{z} = 2 \operatorname{Re} z \in \mathbb{R}$ and $z\bar{z} = |z|^2 \in \mathbb{R}$, all the coefficients of the quadratics are also real.

2.3 For what values of z is

$$\sum_{n=0}^{\infty} \left(\frac{z}{1+z} \right)^n$$

convergent? Draw a picture of the region.

Solution The series converges when $\left| \dfrac{z}{1+z} \right| < 1$. If $z = x + iy$,

$$|z| < |1 + z| \implies x^2 + y^2 < (x+1)^2 + y^2 \implies x > -\frac{1}{2}.$$

So, the series converges when $\operatorname{Re} z > \frac{1}{2}$.