9.1.4 $A = \begin{pmatrix} -4 & 1 \\ -2 & 1 \end{pmatrix}$

Solution $p(\lambda) = \lambda^2 + 3\lambda - 2 = 0$

$$\lambda_{1/2} = \frac{-3 \pm \sqrt{9+8}}{2} = \boxed{-\frac{3}{2} \pm \frac{\sqrt{17}}{2}}$$

9.1.14 The characteristic polynomial of

$$A = \begin{pmatrix} 5 & 4 \\ -8 & -7 \end{pmatrix}$$

is $p(\lambda) = \lambda^2 + 2\lambda - 3$. Use hand calculations to show that the matrix A satisfies the equation $p(A) = 0$ (i.e., show that $A^2 + 2A - 3I$ equals the zero matrix, where I is the 2×2 identity matrix). This result is known as the Cayley-Hamilton theorem.

Solution
$$\begin{aligned} p(A) &= A^2 + 2A - 3I \\ &= \begin{pmatrix} 5 & 4 \\ -8 & -7 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ -8 & -7 \end{pmatrix} + 2\begin{pmatrix} 5 & 4 \\ -8 & -7 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -7 & -8 \\ 16 & 17 \end{pmatrix} + \begin{pmatrix} 10 & 8 \\ -16 & -14 \end{pmatrix} - \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ \square \end{aligned}$$

9.1.20 $\mathbf{y}' = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}\mathbf{y}$

Solution $p(\lambda) = \lambda^2 - 1 = 0$

$\lambda_{1/2} = \pm 1$

$$E_1 = \ker(A - I) = \ker\begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} = \operatorname{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$E_{-1} = \ker(A + I) = \ker\begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix} = \operatorname{span}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

A fundamental set of solutions is given by

$$y_1(t) = e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } y_2(t) = e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

9.1.48 Use Definition 1.4 to show that if $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of A associated to the eigenvalue λ, then $a\mathbf{v} + b\mathbf{w}$ is also an eigenvector associated to λ for any scalars a and b.

Solution Since $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of A associated to the eigenvalue λ,

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ A\mathbf{w} &= \lambda\mathbf{w} \end{aligned}$$

By the linearity of matrix multiplication,

$$\begin{aligned} A(a\mathbf{v} + b\mathbf{w}) &= A(a\mathbf{v}) + A(b\mathbf{w}) \\ &= aA(\mathbf{v}) + bA(\mathbf{w}) \\ &= a\lambda\mathbf{v} + b\lambda\mathbf{w} \\ &= \lambda(a\mathbf{v} + b\mathbf{w}) \Rightarrow a\mathbf{v} + b\mathbf{w} \text{ is an eigenvector of } A \text{ associated to the eigenvalue } \lambda. \ \square \end{aligned}$$

9.2.8 $\mathbf{y}' = \begin{pmatrix} -1 & 6 \\ -3 & 8 \end{pmatrix} \mathbf{y}, \quad \mathbf{y}(0) = (1, -2)^T$

Solution $p(\lambda) = \lambda^2 - 7\lambda + 10 = 0$

$$\lambda_1 = 5, \quad \lambda_2 = 2$$

$$E_5 = \ker \begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$E_2 = \ker \begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\mathbf{y}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\mathbf{y}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow c_1 = -5, \quad c_2 = 3$$

$$\boxed{\mathbf{y}(t) = -5e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

9.2.24 $\mathbf{y}' = \begin{pmatrix} -1 & 1 \\ -5 & -5 \end{pmatrix} \mathbf{y}, \quad \mathbf{y}(0) = (1, -5)^T$

Solution $p(\lambda) = \lambda^2 + 6\lambda + 10 = 0$

$$\begin{aligned} \lambda_{1/2} &= \frac{-6 \pm \sqrt{36 - 40}}{2} \\ &= -3 \pm i \end{aligned}$$

$$E_{-3+i} = \ker \begin{pmatrix} 2 - i & 1 \\ -5 & -2 - i \end{pmatrix} \overset{(\text{row } 1)\cdot(2+i)}{=} \ker \begin{pmatrix} 5 & 2 + i \\ -5 & -2 - i \end{pmatrix} = \operatorname{span} \begin{pmatrix} 2 + i \\ 5 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 2 + i \\ -5 \end{pmatrix}$$

$$\begin{aligned} \mathbf{z}(t) &= e^{(-3+i)t} \begin{pmatrix} 2 + i \\ -5 \end{pmatrix} \\ &= e^{-3t}(\cos t + i \sin t) \left[\begin{pmatrix} 2 \\ -5 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} i \right] \\ &= e^{-3t} \left[\begin{pmatrix} 2 \\ -5 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t \right] + i e^{-3t} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 2 \\ -5 \end{pmatrix} \sin t \right] \end{aligned}$$

$$\mathbf{y}(t) = c_1 e^{-3t} \left[\begin{pmatrix} 2 \\ -5 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t \right] + c_2 e^{-3t} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 2 \\ -5 \end{pmatrix} \sin t \right]$$

$$\mathbf{y}(0) = \begin{pmatrix} 1 \\ -5 \end{pmatrix} \Rightarrow c_1 = 1, \quad c_2 = -1$$

$$\boxed{\mathbf{y}(t) = e^{-3t} \left[\begin{pmatrix} 2 \\ -5 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t \right] - e^{-3t} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 2 \\ -5 \end{pmatrix} \sin t \right]}$$

9.2.36 $\mathbf{y}' = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{y}, \quad \mathbf{y}(0) = (0, -3)^T$

Solution $p(\lambda) = \lambda^2 + 4\lambda + 4 = 0$

$$\lambda = -2$$

Choose $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then

$$\begin{aligned} \mathbf{v}_1 &= (A + 2I)\mathbf{v}_2 \\ &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

$$\mathbf{y}(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$$

$$\mathbf{y}(0) = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \Rightarrow c_1 = 0, \quad c_2 = -3$$

$$\mathbf{y}(t) = -3e^{-2t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$$