**1** Let $f\colon [a,b] \to \mathbb{R}$ be a Riemann integrable function such that $f \geq 0$ and

$$\int_a^b f(x)\,\mathrm{d}x = 0.$$

Show that if $x \in [a,b]$ is a point of continuity for $f$ then $f(x) = 0$.

**Solution** Suppose $x_0 \in [a,b]$ is a point of continuity, but $f(x_0) > 0$. Then there exists $\delta > 0$ such that $f(x) > 0$ on $[x_0 - \delta, x_0 + \delta]$. Note that $\inf_{x \in [x_0-\delta, x_0+\delta]} f(x) > 0$ also, shrinking $\delta$ if necessary.

Consider the partition $P = \{a = t_0 < t_1 < \cdots < t_i = x_0 - \delta < t_{i+1} = x_0 + \delta < \cdots < t_n = b\}$. Then

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x \geq L(f,P) &= \sum_{j=1}^n m(f, [t_{j-1}, t_j])(t_j - t_{j-1}) \\ &\geq m(f, [x_0 - \delta, x_0 + \delta])(2\delta) \\ &> 0\end{aligned}$$

which is a contradiction. Hence, we must have that $f(x_0) = 0$.

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**2** Let $f\colon [a,b] \to \mathbb{R}$ be a Riemann integrable function such that

$$\int_a^b x^n f(x)\,\mathrm{d}x = 0 \quad \text{for all} \quad n \geq 0.$$

Show that if $x \in [a,b]$ is a point of continuity for $f$ then $f(x) = 0$.

**Solution** Notice that for any polynomial $p(x) = \sum_{i=0}^N a_i x^i$, we have

$$\int_a^b p(x) f(x)\,\mathrm{d}x = \sum_{i=0}^N a_i \int_a^b x^i f(x)\,\mathrm{d}x = 0$$

by assumption.

Let $x_0 \in [a,b]$ be a point of continuity for $f$, but assume, without loss of generality, that $f(x_0) > 0$. Then there exists $\delta > 0$ such that $f(x) > 0$ on $[x_0 - \delta, x_0 + \delta]$.

Let $g\colon [a,b] \to \mathbb{R}$ be a continuous function. Then by Weierstrass, there exists $\{p_n\}_{n \geq 1}$ polynomials such that $p_n \xrightarrow{n\to\infty} g$ uniformly. Then we claim that $p_n f \xrightarrow{n\to\infty} gf$.

Fix $\varepsilon > 0$.

As $f$ is integrable, it is bounded, so there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$.

Moreover, as $p_n \xrightarrow{n\to\infty} g$ uniformly, there exists $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$,

$$d(p_n, g) < \frac{\varepsilon}{M},$$

where $d$ is the uniform metric. Then

$$d(p_n f, gf) = \sup_{x \in [a,b]} |p_n(x) f(x) - g(x) f(x)| \leq M \sup_{x \in [a,b]} |p_n(x) - g(x)| = M d(p_n, g) < \varepsilon,$$

so $p_n f \xrightarrow{n\to\infty} gf$ uniformly.

Additionally, as the convergence is uniform, $gf$ is Riemann integrable and

$$\lim_{n\to\infty} \int_a^b p_n(x) f(x)\,\mathrm{d}x = \int_a^b g(x) f(x)\,\mathrm{d}x \implies \int_a^b g(x) f(x)\,\mathrm{d}x = 0$$

since each $\int_a^b p_n(x)f(x)\,\mathrm{d}x = 0$.

Define $g\colon [a,b] \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} 0 & x \neq [x_0 - \delta, x_0 + \delta] \\ -(x - x_0 - \delta)(x - x_0 + \delta) & \text{otherwise.} \end{cases}$$

Note that $g$ is continuous. Indeed, inside $[x_0 - \delta, x_0 + \delta]$, $g$ is a parabola which opens downwards with roots $\{x_0 - \delta, x_0 + \delta\}$. Moreover, $g(x) > 0$ on $[x_0 - \delta, x_0 + \delta]$.

As shown earlier,

$$\int_a^b g(x)f(x)\,\mathrm{d}x = 0.$$

By construction, $g(x)f(x) \geq 0$ on $[x_0 - \delta, x_0 + \delta]$ and $x_0$ is a point of continuity of $gf$ since it is the product of two functions that are continuous there. By exercise (1), this implies that $f(x_0) = 0$, which is a contradiction.

Hence, we must have that $f(x_0) = 0$ if $x_0$ is a point of continuity for $f$.

In the case that $f(x_0) < 0$, we can simply replace $f$ with $-f$ and use the same argument.

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**3** Let $f, g\colon [a,b] \to \mathbb{R}$ be Riemann integrable functions such that $g$ is monotone. Show that there exists $x_0 \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,\mathrm{d}x = g(a)\int_a^{x_0} f(x)\,\mathrm{d}x + g(b)\int_{x_0}^b f(x)\,\mathrm{d}x.$$

*Hint:* Show that if $g$ is monotonically decreasing on $[a,b]$ with $g(b) = 0$, then

$$g(a)\inf_{x\in[a,b]}\int_a^x f(t)\,\mathrm{d}t \leq \int_a^b f(x)g(x)\,\mathrm{d}x \leq g(a)\sup_{x\in[a,b]}\int_a^x f(t)\,\mathrm{d}t.$$

**Solution** Assume without loss of generality that $g$ is monotonically decreasing with $g(b) = 0$. Then $0 < g(a) = \max_{x\in[a,b]} g(x)$, so

$$\int_a^b f(x)g(x)\,\mathrm{d}x \leq g(a)\int_a^b f(x)\,\mathrm{d}x \leq g(a)\sup_{x\in[a,b]}\int_a^x f(t)\,\mathrm{d}t.$$

If $\int_a^x f(t)\,\mathrm{d}t \geq 0$, then

$$0 = \int_a^a f(t)\,\mathrm{d}t = \inf_{x\in[a,b]}\int_a^x f(t)\,\mathrm{d}t \implies g(a)\inf_{x\in[a,b]}\int_a^x f(t)\,\mathrm{d}t \leq \int_a^b f(x)g(x)\,\mathrm{d}x.$$

If $\int_a^x f(t)\,\mathrm{d}t \leq 0$, note that $\sup_{x\in[a,b]}\bigl(-\int_a^x f(t)\,\mathrm{d}t\bigr) = \inf_{x\in[a,b]}\int_a^x f(t)\,\mathrm{d}t$

$$\int_a^b -f(x)g(x)\,\mathrm{d}x \leq g(a)\sup_{x\in[a,b]}\left(-\int_a^x f(t)\,\mathrm{d}t\right) = -g(a)\inf_{x\in[a,b]}\int_a^x f(t)\,\mathrm{d}t$$
$$\implies g(a)\inf_{x\in[a,b]}\int_a^x f(t)\,\mathrm{d}t \leq \int_a^b f(x)g(x)\,\mathrm{d}x.$$

Thus, for a general $f$, we have

$$\int_a^b -|f(x)|g(x)\,\mathrm{d}x \leq \int_a^b f(x)g(x)\,\mathrm{d}x \leq \int_a^b |f(x)|g(x)\,\mathrm{d}x.$$

Applying the first two cases to the inequality yields

$$g(a)\inf_{x\in[a,b]}\int_a^x f(t)\,\mathrm{d}t \leq \int_a^b f(x)g(x)\,\mathrm{d}x \leq g(a)\sup_{x\in[a,b]}\int_a^x f(t)\,\mathrm{d}t.$$

By a theorem proved in class, $\int_a^x f(t)\,\mathrm{d}t$ is continuous on $[a,b]$ connected and compact. Hence, there exists $x_1 \in [a,b]$ and $x_2 \in [a,b]$ such that

$$\begin{aligned} g(a) \inf_{x\in[a,b]} \int_a^x f(t)\,\mathrm{d}t &= \int_a^{x_1} f(x)g(x)\,\mathrm{d}x \\ g(a) \sup_{x\in[a,b]} \int_a^x f(t)\,\mathrm{d}t &= \int_a^{x_2} f(x)g(x)\,\mathrm{d}x. \end{aligned}$$

Hence, as $\int_a^x f(x)g(x)$ has the Darboux property, there exists $x_0 \in [x_1, x_2]$ (or the other way around if $x_1 > x_2$) such that

$$\int_a^b f(x)g(x)\,\mathrm{d}x = g(a) \int_a^{x_0} f(x)\,\mathrm{d}x.$$

This proves the theorem for a decreasing $g$ with $g(b) = 0$.

If $g$ were increasing, then we can replace $g$ with $-g$ and use the same argument. If $g(b) \neq 0$, then we can apply the argument on $h(x) \coloneqq g(x) - g(b)$. Then as $g$ is decreasing, so is $h$, and $h(b) = 0$. Applying the result gives $x_0 \in [a,b]$ such that

$$\begin{aligned} \int_a^b f(x)(g(x)-g(b))\,\mathrm{d}x &= (g(a)-g(b)) \int_a^{x_0} f(x)\,\mathrm{d}x \\ \implies \int_a^b f(x)g(x)\,\mathrm{d}x &= g(a) \int_a^{x_0} f(x)\,\mathrm{d}x - g(b) \int_a^{x_0} f(x)\,\mathrm{d}x + g(b) \int_a^b f(x)\,\mathrm{d}x \\ &= g(a) \int_a^{x_0} f(x)\,\mathrm{d}x + g(b) \int_{x_0}^b f(x)\,\mathrm{d}x \end{aligned}$$

as desired.

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**4** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function and define $F\colon \mathbb{R} \to \mathbb{R}$ via

$$F(x) = \int_{x-1}^{x+1} f(t)\,\mathrm{d}t.$$

Show that $F$ is differentiable and compute its derivative.

**Solution** For some $c \in \mathbb{R}$, we can write

$$F(x) = \int_c^{x+1} f(t)\,\mathrm{d}t - \int_c^{x-1} f(t)\,\mathrm{d}t.$$

Using the changes of variables $s = t-1$ and $s = t+1$ on each respective integral gives

$$F(x) = \int_c^x f(s+1)\,\mathrm{d}s + \int_c^x f(s-1)\,\mathrm{d}s.$$

As $f$ is continuous on $\mathbb{R}$, we have that $f(s+1)$ and $f(s-1)$ are also continuous on $\mathbb{R}$, so by a theorem, each integral is differentiable and $F'$ is given by

$$F'(x) = f(x+1) - f(x-1).$$

**5** Let $f\colon [a,b] \to \mathbb{R}$ be a twice differentiable function such that $f''$ is Riemann integrable on $[a,b]$.

a. Show that
$$\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{2}[f(a)+f(b)] + \frac{1}{2}\int_a^b f''(x)(x-a)(x-b)\,\mathrm{d}x.$$

b. If additionally $f''$ is continuous, show that there exists $x_0 \in [a,b]$ such that
$$\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{2}[f(a)+f(b)] - \frac{(b-a)^3}{12}f''(x_0).$$

**Solution** a. Applying integration by parts on $\int_a^b f''(x)(x-a)(x-b)\,\mathrm{d}x$ twice, we get
$$\begin{aligned}\int_a^b f''(x)(x-a)(x-b)\,\mathrm{d}x &= f'(b)(b-a)(b-b) - f'(a)(a-a)(b-a) - \int_a^b f'(x)\big[(x-a)+(x-b)\big]\,\mathrm{d}x\\ &= -\big[f(b)(b-a) - f(a)(a-b)\big] + 2\int_a^b f(x)\,\mathrm{d}x\\ &= -(b-a)[f(a)+f(b)] + 2\int_a^b f(x)\,\mathrm{d}x.\end{aligned}$$

Substituting this into the RHS gives
$$\frac{b-a}{2}[f(a)+f(b)] - \frac{b-a}{2}[f(a)+f(b)] + \int_a^b f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x.$$

b. We wish to show that there exists $x_0 \in [a,b]$ such that
$$\int_a^b f''(x)(x-a)(x-b)\,\mathrm{d}x = -\frac{(b-a)^3}{6}f''(x_0).$$

Note that as $f''$ is continuous on $[a,b]$ compact, it attains its minimum $m$ and maximum $M$. Thus,
$$m\int_a^b (x-a)(x-b)\,\mathrm{d}x \le \int_a^b f''(x)(x-a)(x-b)\,\mathrm{d}x \le M\int_a^b (x-a)(x-b)\,\mathrm{d}x.$$

Thus, by the Darboux property of $f''$, there exists $x_0 \in [a,b]$ such that
$$f''(x_0)\int_a^b (x-a)(x-b)\,\mathrm{d}x = \int_a^b f''(x)(x-a)(x-b)\,\mathrm{d}x.$$

Integrating the left side yields
$$\begin{aligned}\int_a^b x^2 - x(a+b) + ab\,\mathrm{d}x &= \frac{1}{3}(b^3-a^3) - \frac{1}{2}(a+b)(b^2-a^2) + ab(b-a)\\ &= \frac{1}{3}(b^3-a^3) - \frac{1}{2}(b-a)(b^2+a^2)\\ &= (b-a)\left(\frac{b^2+ab+a^2}{3} - \frac{b^2+a^2}{2}\right)\\ &= (b-a)\left(\frac{-b^2-a^2-2ab}{6}\right)\\ &= -\frac{(b-a)^3}{6}.\end{aligned}$$

Thus,
$$-\frac{(b-a)^3}{6}f''(x_0) = \int_a^b f''(x)(x-a)(x-b)\,\mathrm{d}x$$
as we wanted.

**6** For $n \geq 1$, let $f_n : [a,b] \to \mathbb{R}$ be a continuous function. Assume that $f_n$ converges pointwise to a continuous function $f : [a,b] \to \mathbb{R}$. Assume that there exists $M > 0$ such that

$$|f_n(x)| \leq M \quad \text{for all} \quad x \in [a,b] \quad \text{and all} \quad n \geq 1.$$

Show that

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x.$$

**Solution** Fix $\varepsilon > 0$.

Consider $D_m^{(k)} \coloneqq \bigcup_{n\geq m}\{x \in [a,b] \mid |f(x) - f_n(x)| > \frac{1}{k}\}$.

Notice that $|f - f_n|$ is continuous since both $f$ and $f_n$ are continuous, so $D_m^{(k)} = \bigcup_{n\geq m}(|f-f_n|)^{-1}\big((\frac{1}{k},\infty)\big) = \bigcup_{n\geq m}(b_{n,m}^{(k)}, a_{n,m}^{(k)})$ is open.

Notice that since for all $x$, $f_n(x) \xrightarrow{n\to\infty} f(x)$, there exists $n_k \in \mathbb{N}$ such that

$$|D_{n_k}^{(k)}| \coloneqq \sum_{n\geq 1} b_{n,n_k}^{(k)} - a_{n,n_k}^{(k)} < \frac{\varepsilon}{2^k}.$$

Then

$$|A| \coloneqq \left|\bigcup_{k\geq 1} D_{n_k}^{(k)}\right| = \sum_{k\geq 1}\frac{\varepsilon}{2^k} < \varepsilon \cdot \frac{\frac{1}{2}}{1-\frac{1}{2}} = \varepsilon,$$

so $A$ has zero content.

Notice that ${}^{\mathrm{c}}A = \bigcap_{k\geq 1} {}^{\mathrm{c}}D_{n_k}^{(k)} = \bigcap_{k\geq 1}\bigcap_{n\geq n_k}\{x \in [a,b] \mid |f(x) - f_n(x)| \leq \frac{1}{k}\}$, which is the set of points on which $f_n \xrightarrow{n\to\infty} f$ uniformly on. Hence, on that set, we can interchange the integral and the limit.

Thus,

$$\begin{aligned}
\lim_{n\to\infty}\int_a^b f_n(x)\,\mathrm{d}x &= \lim_{n\to\infty}\int_A f_n(x)\,\mathrm{d}x + \lim_{n\to\infty}\int_{[a,b]\setminus A} f_n(x)\,\mathrm{d}x \\
&= \lim_{n\to\infty} 0 + \int_{[a,b]\setminus A} \lim_{n\to\infty} f_n(x)\,\mathrm{d}x \\
&= \int_{[a,b]\setminus A} f(x)\,\mathrm{d}x \\
&= \int_{[a,b]\setminus A} f(x)\,\mathrm{d}x + \int_A f(x)\,\mathrm{d}x \\
&= \int_a^b f(x)\,\mathrm{d}x
\end{aligned}$$

as desired.

**7** For $n \geq 1$, let $f_n \colon [0,1] \to \mathbb{R}$ be a continuous function satisfying

$$|f_n(x)| \leq 1 + \frac{n}{1+n^2x^2}$$

and define $F_n \colon [0,1] \to \mathbb{R}$ via

$$F_n(x) = \int_0^x f_n(t)\,\mathrm{d}t.$$

Show that the sequence $\{F_n\}_{n\geq 1}$ admits a subsequence that converges pointwise on $[0,1]$.

**Solution** Consider the interval $[\frac{1}{2}, 1]$. Notice that on this interval, each $f_n$ satisfies

$$|f_n(x)| \leq 1 + \frac{n}{1+n^2\frac{1}{4}} \leq M$$

for some $M > 0$, since $\lim_{n\to\infty} \frac{4n}{4+n^2} = 0$.

Let $\varepsilon > 0$. If we choose $\delta < \frac{\varepsilon}{M}$, then for $x, y \in [\frac{1}{2}, 1]$ with $|x-y| < \delta$,

$$|F_n(x) - F_n(y)| = \left| \int_y^x f_n(t)\,\mathrm{d}t \right| \leq \left| \int_y^x |f_n(t)|\,\mathrm{d}t \right| \leq M|x-y|.$$

Hence, as $M$ is independent of $n$, $\{F_n\}_{n\geq 1}$ is equicontinuous.

Taking $y = 0$ in the above, we get that $\{F_n\}_{n\geq 1}$ is uniformly bounded by $M$. Hence, by Arzelà-Ascoli, $\{F_n\}_{n\geq 1}$ admits a uniformly convergent subsequence which converges on $[\frac{1}{2}, 1]$.

Repeating this process, passing each subsequence along, for the intervals $[\frac{1}{3}, 1], [\frac{1}{4}, 1], \ldots$ and using a diagonal argument, we get $\{F_{k_n}\}_{n\geq 1}$ which converges on $(0, 1]$.

For all $n \geq 1$, $F_{k_n}(0) = 0$, so $\{F_{k_n}\}_{n\geq 1}$ converges pointwise on $[0,1]$.