1 a. Let (X, ρ) be a metric space. Prove that there is another metric ρ' on X which (i) generates the same topology as ρ but (ii) satisfies $\rho'(x, y) \leq 1$ for all $x, y \in X$.

b. Let (X_n, ρ_n) be a metric space for each $n \in \mathbb{N}$, assume that $\rho_n(x_n, y_n) \leq 1$ for all $x_n, y_n \in X_n$, and let $X \coloneqq \prod_{n\in\mathbb{N}} X_n$. Define the function

$$\rho(\mathbf{x}, \mathbf{y}) \coloneqq \max\{\rho_n(x_n, y_n)/n \mid n \in \mathbb{N}\} \text{ for } \mathbf{x} = \{x_n\}_n,\ \mathbf{y} = \{y_n\}_n \in X.$$

Prove that (a) ρ is a metric on X and (b) the topology on X generated by ρ is the product of the topologies on the X_n's generated by the ρ_n's.

c. Generalize part (b) to prove the following:

Proposition. *The product topology on a countably infinite product of metrizable spaces is metrizable.*

Solution a. Let $\rho'(x, y) \coloneqq \min\{\rho(x, y), 1\}$. This is a metric:

Since ρ is a metric, ρ' is clearly symmetric, non-negative, and $\rho(x, y) = 0 \iff \rho'(x, y) = 0$, so we just need to show that it satisfies the triangle inequality.

$$\rho'(x, y) = \min\{\rho(x, y), 1\} \leq \min\{\rho(x, z)+\rho(z, y), 1\} \leq \min\{\rho(x, z), 1\}+\min\{\rho(z, y), 1\} = \rho'(x, z)+\rho'(z, y),$$

so ρ' is a metric. Moreover, it's bounded by 1, by definition.

Lastly, we need to show that it generates the same topology as ρ. Since open balls are a base for the metric topology, it suffices to show that an open ρ-ball can be written as a union of open ρ'-balls, and vice versa.

Let x be in an open ρ-ball of radius r centered at x_0.

If $r - \rho(x, x_0) < 1$, then ρ' agrees with ρ, so we can just take the ρ'-ball of radius $r - \rho(x, x_0)$ centered at x, and that is contained in the ρ-ball.

On the other hand, if $r - \rho(x, x_0) \geq 1$, we can just take a ρ'-ball of radius 1.

In either case, we can write a ρ-ball as a union of ρ'-balls.

Now let x be in an open ρ'-ball of radius r centered at x_0. If $r > 1$, then any ρ-ball works is contained in it. Otherwise, we can take a ρ-ball of the same radius. In both cases, a ρ'-ball is a union of ρ-balls.

Hence, the two metrics generate the same topology on X.

b. It's clear that the metric is symmetric and non-negative. We also have

$$\rho(\mathbf{x}, \mathbf{y}) = 0 \iff \frac{\rho_n(x_n, y_n)}{n} = 0\ \forall n \iff x_n = y_n\ \forall n.$$

Thus, we just need to show the triangle inequality. But this follows from the fact that each ρ_n is a metric:

$$\begin{aligned}
\rho(\mathbf{x}, \mathbf{y}) &= \max\left\{\frac{\rho_n(x_n, y_n)}{n} \mid n \in \mathbb{N}\right\} \\
&\leq \max\left\{\frac{\rho_n(x_n, z_n)}{n} + \frac{\rho_n(z_n, y_n)}{n} \mid n \in \mathbb{N}\right\} \\
&\leq \max\left\{\frac{\rho_n(x_n, z_n)}{n} \mid n \in \mathbb{N}\right\} + \max\left\{\frac{\rho_n(z_n, y_n)}{n} \mid n \in \mathbb{N}\right\} \\
&= \rho(\mathbf{x}, \mathbf{z}) + \rho(\mathbf{z}, \mathbf{y}).
\end{aligned}$$

Now, we need to show that the topology on X generated by ρ is the product topology.

It suffices to show this for basic open sets in X. Consider $U = \prod U_n$, where $U_n = X_n$ except for $n = n_1, \ldots, n_m$, and let $\mathbf{x} = \{x_n\} \in U$.

Since X_{n_i} is a metric space, there exists an open ball with $x_{n_i} \in B_{r_{n_i}}(x_{n_i}) \subseteq U_n$.

Take $r = \min_i\{r_{n_i}/n_i\}$. We claim that $B_r(\mathbf{x}) \subseteq U$: Let $\mathbf{y} \in B_r(\mathbf{x})$, so that

$$\frac{\rho_n(x_n, y_n)}{n} < r$$

for every n. Since $U_n = X_n$ except for $n = n_1, \dots, n_m$, we just need to check that $y_{n_i} \in U_{n_i}$ for each i. We have

$$\rho_{n_i}(x_{n_i}, y_{n_i}) < rn_i < r_{n_i} \implies y_{n_i} \in B_{r_{n_i}}(x_{n_i}) \subseteq U_{n_i}$$

for every i, so the generated topology includes the product topology.

Now let $B_r(\mathbf{x})$ be a basic open set generated by ρ, and let $N \in \mathbb{N}$ be so that

$$\frac{1}{N} < r.$$

Pick a basic open set of X as follows: For $1 \leq n \leq N$, let $U_n = B_{r_n}(x_n)$, with $r_n < nr$, for for $n \geq N+1$, let $U_n = X_n$. Then $U \coloneqq \prod U_n$ is a basic open set in X, and if $\mathbf{y} \in U$,

$$\frac{\rho_n(x_n, y_n)}{n} < r \text{ if } 1 \leq n \leq N \quad \text{and} \quad \frac{\rho_n(x_n, y_n)}{n} \leq \frac{1}{N} < r \text{ if } n \geq N+1.$$

Thus, $\rho(\mathbf{x}, \mathbf{y}) < r$, so $\mathbf{y} \in B_r(\mathbf{x})$, so basic open sets in X generate the metric topology.

Since the two topologies generate each other, they must be the same.

c. Suppose $\{(X_n, \rho_n)\}$ are metric spaces and that $X = \prod X_n$ is equipped with the product topology. By part (a), we can replace ρ_n with an equivalent metric ρ_n' which is bounded by 1, for every $n \geq 1$. Thus, by part (b), we can take the given metric ρ, and that generates the product topology on X, so X is metrizable.

4.20 If A is a countable set and X_α is a first (resp. second) countable space for each $\alpha \in A$, then $\prod_{\alpha \in A} X_\alpha$ is first (resp. second) countable.

Solution Each X_α is first countable:

Let $x = \{x_\alpha\}_{\alpha \in A} \in \prod_{\alpha \in A} X_\alpha$, which is non-empty by the axiom of choice, and let $\mathcal{U}_\alpha$ be a countable base for x_α for each $\alpha \in A$.

Since the set of finite sets of a countable set is countable, the set

$$\mathcal{U} \coloneqq \left\{ \bigcap_{\alpha \in A'} \pi_\alpha^{-1}(U_\alpha) \mid A' \subseteq A \text{ finite, } U_\alpha \in \mathcal{E}_\alpha \right\}$$

is countable, and we claim that this is our neighborhood base for x.

Since finite intersections of sets of the form $\pi_\alpha^{-1}(U_\alpha)$ form a base for the product topology, it suffices to show that there exists $V \in \mathcal{U}$ so that V is contained in a basic open set $U \ni x$.

U is of the form $\prod_{\alpha \in A} U_\alpha$, where each $U_\alpha \subseteq X_\alpha$ is open and $U_\alpha = X_\alpha$ for all by finitely many α's. Label these values of α via $\alpha_1, \dots, \alpha_n$. Because each X_α is first countable, for each α_i, there exists $V_{\alpha_i} \in \mathcal{U}_{\alpha_i}$ so that $x_{\alpha_i} \in V_{\alpha_i} \subseteq U_{\alpha_i}$. Then

$$\mathcal{U} \ni V \coloneqq \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(V_{\alpha_i}) \subseteq \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i}) = U,$$

so $\prod_{\alpha \in A} X_\alpha$ is first countable.

Each X_α is second countable:

In this case, for each $\alpha \in A$, there exists a countable collection of open sets $\mathcal{U}_\alpha$, which is a countable base for X_α. Then

$$\mathcal{U} := \left\{ \bigcap_{\alpha \in A'} \pi_\alpha^{-1}(U_\alpha) \mid A' \subseteq A \text{ finite, } U_\alpha \in \mathcal{E}_\alpha \right\}$$

is countable, for the same reason above, and we claim that this is a countable base for $\prod_{\alpha \in A} X_\alpha$.

Let U be an open set in the product. For any $x \in U$, we can find $V_x \in \mathcal{U}$ so that $x \in V_x \subseteq U$. Indeed, there is a basic open set $W \ni x$ contained in U, and by the argument for first countable spaces, we can find $V_x \subseteq W$. Thus,

$$U = \bigcup_{x \in U} V_x,$$

and since $\mathcal{U}$ is countable, it follows that this reduces to an at most countable union of elements of $\mathcal{U}$, so the product is second countable.

4.22 Let X be a topological space, (Y, ρ) a complete metric space, and $\{f_n\}$ a sequence in Y^X such that $\sup_{x \in X} \rho(f_n(x), f_m(x)) \to 0$ as $m, n \to \infty$. There is a unique $f \in Y^X$ such that $\sup_{x \in X} \rho(f_n(x), f(x)) \to 0$ as $n \to \infty$. If each f_n is continuous, so is f.

Solution Notice that for each $x \in X$,

$$\rho(f_n(x), f_m(x)) \leq \sup_{x \in X} \rho(f_n(x), f_m(x)) \xrightarrow{n,m \to \infty} 0,$$

so $\{f_n(x)\}_{n \geq 1}$ is Cauchy. Since Y is complete, there exists $f(x) \in Y$ such that $f_n(x) \xrightarrow{n \to \infty} f(x)$, for every x. This defines a function $f \colon X \to Y$, and f is unique, since convergent sequences in metric spaces have unique limits. By letting $m \to \infty$ in the sup norm, it follows that f_n converges to f.

Suppose each f_n is continuous. We'll now show that f is continuous:

Let $x \in X$ and $\varepsilon > 0$.

Because the sup norm converges to 0, there exists $n_0 \in \mathbb{N}$ so that for $n \geq n_0$,

$$\sup_{z \in X} \rho(f_n(z), f(z)) < \frac{\varepsilon}{3}.$$

Since f_{n_0} is continuous, there exists $x \in U \subseteq X$ open such that $y \in U \implies \rho(f_{n_0}(x), f_{n_0}(y)) < \varepsilon/3$. Then for $y \in U$,

$$\begin{aligned}
\rho(f(x), f(y)) &\leq \rho(f(x), f_{n_0}(x)) + \rho(f_{n_0}(x), f_{n_0}(y)) + \rho(f_{n_0}(y), f(y)) \\
&\leq \sup_{x \in X} \rho(f(x), f_{n_0}(x)) + \rho(f_{n_0}(x), f_{n_0}(y)) + \sup_{y \in X} \rho(f_{n_0}(y), f(y)) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

Thus, $f(U) \subseteq B_\varepsilon(f(x))$, so f is continuous.

4.24 A Hausdorff space X is normal iff X satisfies the conclusion of Urysohn's lemma iff X satisfies the conclusion of the Tietze extension theorem.

Solution We'll label the statements via (a), (b), and (c), in the order that they're presented in the problem.

We know that (a) $\implies$ (b) $\implies$ (c), so we can just show that (c) $\implies$ (a):

Let $\emptyset \neq A, B \subseteq X$ be disjoint closed sets, and define $f\colon A \cup B \to [0,1]$ via $f\big|_A = 0$ and $f\big|_B = 1$. This is continuous:

Notice that $A = B^{\mathrm{c}} \cap (A \cup B)$, so A is open in $A \cup B$. Similarly, B is open in $A \cup B$. Now let $I \subseteq [a,b]$ be an open set. If I contains 0 but not 1, then its preimage is A. Similarly, if $I \cap \{0,1\} = \{1\}$, then its preimage is B. If I contains both, its preimage is $A \cup B$. If I contains neither, then its preimage is empty. In all the cases, $f^{-1}(I)$ is open in X, so f is continuous.

By Tietze's extension theorem, there exists a continuous function $F \in C(X,[0,1])$ so that $F\big|_{A\cup B} = f$. We also see that $F\big|_A = 0$ and $F\big|_B = 1$, so F separates A and B. Consider $U = F^{-1}([0,1/3))$ and $V = F^{-1}((2/3,1])$. $U \cap V = \emptyset$, since no numbers satisfy $2/3 < x < 1/3$, and they are both open in X since F is continuous. Moreover, $A \subseteq U$ and $B \subseteq V$, so U and V separate A and B. Hence, X is normal, and the implication holds, so (a) $\iff$ (b) $\iff$ (c).

4.38 Suppose that $(X,\mathcal{T})$ is a compact Hausdorff space and $\mathcal{T}'$ is another topology on X. If $\mathcal{T}'$ is strictly stronger than $\mathcal{T}$, then $(X,\mathcal{T}')$ is Hausdorff but not compact. If $\mathcal{T}'$ is strictly weaker than $\mathcal{T}$, then $(X,\mathcal{T}')$ is compact but not Hausdorff.

Solution $\mathcal{T}' \supsetneq \mathcal{T}$:

$(X,\mathcal{T}')$ is still Hausdorff since we can still use the open sets from $\mathcal{T} \subseteq \mathcal{T}'$ to separate two distinct points in X.

Suppose $(X,\mathcal{T}')$ is still compact, and let $U \in \mathcal{T}'$. Since $(X,\mathcal{T}')$ is compact Hausdorff, U^{c} is compact. In particular, if $\mathcal{U} \subseteq \mathcal{T}$ is any open cover of U^{c}, $\mathcal{U}$ admits a finite subcover, so U^{c} is compact in $(X,\mathcal{T})$, which is Hausdorff. Hence, U^{c} is closed in $(X,\mathcal{T})$, so $U \in \mathcal{T}$. But this implies that $\mathcal{T}' \subseteq \mathcal{T}$, a contradiction, so $(X,\mathcal{T})'$ cannot be compact.

$\mathcal{T}' \subsetneq \mathcal{T}$:

$(X,\mathcal{T}')$ is still compact because an open cover of X from sets in $\mathcal{T}'$ is an open cover of X from sets in $\mathcal{T}$, which means that the open cover still admits a finite subcover.

Suppose $(X,\mathcal{T}')$ is Hausdorff, and let $U \in \mathcal{T}$. Then U^{c} is closed and hence compact in $(X,\mathcal{T})$. In particular, it is compact in $\mathcal{T}'$ since any open cover in $\mathcal{T}'$ is an open cover in $\mathcal{T}$. Since $(X,\mathcal{T}')$ is Hausdorff, this implies that U^{c} is closed in $(X,\mathcal{T}')$, which means that $U \in \mathcal{T}' \implies \mathcal{T} \subseteq \mathcal{T}'$, but this is impossible. Hence, $(X,\mathcal{T}')$ cannot be Hausdorff.

4.43 For $x \in [0,1)$, let $\sum_1^\infty a_n(x)2^{-n}$ $(a_n(x) = 0$ or $1)$ be the base-2 decimal expansion of x. (If x is a dyadic rational, choose the expansion such that $a_n(x) = 0$ for n large.) Then the sequence $\{a_n\}$ in $\{0,1\}^{[0,1)}$ has no pointwise convergent subsequence. (Hence $\{0,1\}^{[0,1)}$, with the product topology arising from the discrete topology on $\{0,1\}$, is not sequentially compact.)

Solution Suppose otherwise, and that $a_n(x)$ has a pointwise convergent subsequence $a_{k_n}(x)$.

Notice that $k_n \not\equiv n$, since $a_n(2/3)$ does not converge, as it alternates:

$$\frac{2}{3} = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \sum_{n=1}^{\infty} \frac{1}{2^{2n-1}}.$$

This shows that $a_n(2/3) = 0$ if n is even and 1 if n is odd.

Then consider $x \in [0,1)$ with

$$x = \sum_{n=1}^{\infty} b_n 2^{-n},$$

where $b_m = 1$ if $m = k_n$ for n odd, and 0 otherwise. In other words b_{k_n} alternates between 0 and 1, and it is 0 for all the other values of m. Then this series converges, since the sum is bounded by 1, and its value is not 1 since not every b_m is 1.

The expansion for this value of x is unique, since x is not a dyadic rational. But this implies that $a_n(x) = b_n$ for each n, which means that $a_{k_n}(x) = b_{k_n}$ for each n. But b_{k_n} is not convergent, since it alternates between 0 and 1, a contradiction. Hence, no such k_n exists, so $\{a_n\}$ does not converge pointwise.