1 Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Show that for any $c \in (a,b)$ that is not a point of maximum or minimum for f' there exist $x_1, x_2 \in (a,b)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Solution Consider $h\colon [a,b] \to \mathbb{R}$, $h(x) \coloneqq f(x) - f'(c)x$. Note that since h is the sum of two differentiable functions on (a,b), it is differentiable on that same interval. Similarly, it is continuous on $[a,b]$.

If h is injective, then as h is continuous, it is strictly monotone. Assume without loss of generality that h is strictly increasing. Then

$$h'(x) \geq 0 \implies f'(x) - f'(c) \geq 0 \implies f'(x) \geq f'(c),$$

but this is a contradiction as this means that c is a point of minimum for f'.

We can apply the same argument, but with inequality signs switched, and get the same contradiction. Hence, h must not be injective.

Since h is not injective, there exists $x_1, x_2 \in (a,b)$ such that $x_1 < x_2$ and $h(x_1) = h(x_2)$. Then

$$\begin{aligned} 0 &= h(x_2) - h(x_1) \\ 0 &= f(x_2) - f(x_2) + f'(c)(x_2 - x_1) \\ f'(c) &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \end{aligned}$$

as desired.

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2 Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Assume that f' is strictly increasing. Show that for any $c \in (a,b)$ such that $f'(c) = 0$ there exist $x_1, x_2 \in [a,b]$, $x_1 < c < x_2$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Solution As f' is strictly increasing, there exists $y_1 \in (a,c)$ such that $f'(y_1) < 0$. Moreover, there exists $y_2 \in (c,b)$ such that $f'(y_2) > 0$.

Thus, on (y_1, c), f is strictly decreasing, and on (c, y_2), f is strictly increasing. Let $M = \frac{1}{2}\min\{f(y_1), f(y_2)\}$. As f is continuous and $[a,b]$ is connected, f has the Darboux property. Thus, since $f(c) < M < f(x_1)$, there exists $x_1 \in (y_1, c)$ such that $f(x_1) = M$. Similarly, there exists $f(x_2) = M$ where $x_2 \in (c, y_2)$. Hence,

$$\frac{f(x_2) - f(x_2)}{x_2 - x_1} = 0 = f'(c)$$

as desired.

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3 Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function on the closed interval $[0,1]$ and differentiable on the open interval $(0,1)$. Assume that $f(0) = 0$ and f' is an increasing function on $(0,1)$. Show that

$$g(x) = \frac{f(x)}{x}$$

is an increasing function on $(0,1)$.

Solution f is continuous on $[0,1]$ and differentiable on $(0,1)$, so applying the mean value theorem on the interval $(0,x)$ gives us $c \in (0,x)$

$$\frac{f(x) - f(0)}{x - 0} = f'(c) \implies f(x) = f'(c)x \leq f'(x)x$$

The last inequality holds since f' is an increasing function on $(0,1)$. Hence,

$$g'(x) = \frac{f'(x)x - f(x)}{x^2} \geq 0$$

since $f'(x)x - f(x) \geq 0$ and $x^2 > 0$ on $(0,1)$. Hence, g is an increasing function $(0,1)$.

4 Assume $f\colon [a,b] \to \mathbb{R}$ is a continuous function on the closed interval $[a,b]$ and differentiable on the open interval (a,b) with $f(a) = f(b) = 0$. Prove that for every $\lambda \in \mathbb{R}$ there exists $x_0 \in (a,b)$ such that $f'(x_0) = \lambda f(x_0)$.

Solution Fix $\lambda \in \mathbb{R}$.

Let $h_\lambda\colon [a,b] \to \mathbb{R}$, $h(x) = e^{-\lambda x} f(x)$. h_λ is continuous since it is a product of two continuous functions. Moreover, it is differentiable since $e^{-\lambda x}$ and $f(x)$ are differentiable on (a,b).

Notice that $h'_\lambda(x) = -\lambda e^{-\lambda x} f(x) + e^{-\lambda x} f'(x)$. Applying the mean value theorem on the interval $[a,b]$ gives us that there exists $x_0 \in (a,b)$ such that

$$h'_\lambda(x_0) = \frac{h_\lambda(b) - h_\lambda(a)}{b-a} \implies -\lambda e^{-\lambda x_0} f(x_0) + e^{-\lambda x_0} f'(x_0) = 0 \implies f'(x_0) = \lambda f(x_0)$$

as desired. (Note that we could divide by $e^{-\lambda x_0}$ since it is non-zero.)

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5 Assume $f\colon (1,\infty) \to \mathbb{R}$ is differentiable. If

$$\lim_{x\to\infty} f(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} f'(x) = c,$$

prove that $c = 0$.

Solution Consider $g\colon (1,\infty) \to \mathbb{R}$, where $g(x) = \frac{f(x)}{x}$.

Note that

$$\lim_{x\to\infty} \frac{f(x)}{x} = \left(\lim_{x\to\infty} f(x)\right)\left(\lim_{x\to\infty} \frac{1}{x}\right) = 1 \cdot 0 = 0$$

Hence, as $x \xrightarrow{x\to\infty} \infty$ and $x > 0$ for all $x \in (1,\infty)$, we can apply L'Hôpital's rule.

$$0 = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{f'(x)}{1} = c$$

Thus, by uniqueness of limits, $c = 0$.