1 Study the convergence of the series

$$(1)\sum_{n\geq 2}\frac{1}{[n+(-1)^n]^2}\qquad (2)\sum_{n\geq 1}(\sqrt{n+1}-\sqrt{n})\qquad (3)\sum_{n\geq 1}\frac{n!}{n^n}.$$

Solution (1) Note that

$$0<\frac{1}{[n+(-1)^n]^2}\leq\frac{1}{(n-1)^2}$$

Since $\sum\frac{1}{(n-1)^2}$ converges, then by comparison, the series $\sum\frac{1}{[n+(-1)^n]^2}$ also converges.

(2) The sum telescopes, so the partial sum is given by

$$\begin{aligned}s_n &= (\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+\cdots+(\sqrt{n}-\sqrt{n-1})+(\sqrt{n+1}-\sqrt{n})\\ &= -\sqrt{1}+\sqrt{n+1}\end{aligned}$$

Then $\lim_{n\to\infty}s_n=\infty$, so by definition, $\sum(\sqrt{n+1}-\sqrt{n})$ diverges.

(3) We will apply the ratio test.

$$\begin{aligned}\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} &= \lim_{n\to\infty}\frac{(n+1)}{(n+1)}\cdot\left(\frac{n}{n+1}\right)^n\\ &= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{-1}\\ &= \frac{1}{e}<1\end{aligned}$$

Thus, by the ratio test, the series converges.

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2 Study the convergence of the series

$$(1)\sum_{n\geq 2}\frac{n^{\ln n}}{(\ln n)^n}\qquad (2)\sum_{n\geq 2}\frac{1}{(\ln n)^{\ln n}}\qquad (3)\sum_{n\geq 1}\frac{(-1)^n n!}{2^n}.$$

Solution (1) We start by proving a relevant limit:

$$\begin{aligned}\lim_{n\to\infty}n^{\frac{\ln n}{n}} &= \lim_{n\to\infty}e^{\frac{(\ln n)^2}{n}}\\ &= \lim_{n\to\infty}e^{\left(\frac{\frac{1}{2}\ln\sqrt{n}}{\sqrt{n}}\right)^2}\\ &= \lim_{n\to\infty}e^{\frac{1}{4}(\ln\sqrt{n}^{\frac{1}{\sqrt{n}}})^2}\\ &= e^{\frac{1}{4}(\ln 1)^2}\\ &= 1\end{aligned}$$

Then

$$\lim_{n\to\infty}\left(\frac{n^{\ln n}}{(\ln n)^n}\right)^{\frac{1}{n}}=\lim_{n\to\infty}\frac{n^{\frac{\ln n}{n}}}{\ln n}=1\cdot 0=0<1$$

Thus, by the root test, the series converges.

(2) As $\ln n$ increases for all n, $\frac{1}{(\ln n)^{\ln n}}$ is clearly decreasing. Hence, we can apply the dyadic test. The series $\sum \frac{1}{(\ln n)^{\ln n}}$ converges iff $\sum \frac{2^n}{(n \ln 2)^{n \ln 2}}$ converges. We apply the ratio test on the second series:

$$\begin{aligned}\lim_{n\to\infty} \frac{2^{n+1}}{\left[(n+1)^{n+1}(\ln 2)^{n+1}\right]^{\ln 2}} \cdot \frac{\left[n^n(\ln 2)^n\right]^{\ln 2}}{2^n} &= \lim_{n\to\infty} 2\left[\frac{1}{n+1}\left(\frac{n}{n+1}\right)^n \frac{1}{\ln 2}\right]^{\ln 2} \\ &= 2\left[0 \cdot \frac{1}{e} \cdot \frac{1}{\ln 2}\right]^{\ln 2} \\ &= 0 < 1\end{aligned}$$

Thus, by the root test, $\sum \frac{2^n}{(n \ln 2)^{n \ln 2}}$, so $\sum \frac{1}{(\ln n)^{\ln n}}$ converges also.

(3)

$$\lim_{n\to\infty} \left|(-1)^{n+1}\frac{(n+1)!}{2^{n+1}} \cdot (-1)^n \frac{2^n}{n!}\right| = \lim_{n\to\infty} \frac{n+1}{2} = \infty > 1$$

By the ratio test, the series diverges.

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3 a. Given an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.

b. Show that if $\sum a_n$ is absolutely convergent, then the series $\sum a_n^2$ also converges.

c. Given an example of a convergent series $\sum a_n$, for which $\sum a_n^2$ diverges.

Solution a. $\sum \frac{1}{n}$ diverges, but $\sum \frac{1}{n^2}$ converges.

b. If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ converges, and $\lim_{n\to\infty} |a_n| = 0$. Thus, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|a_n| < 1 \implies 0 \leq a_n^2 < |a_n|$$

By comparison, the sum $\sum_{n \geq N} a_n^2$ converges as $\sum |a_n|$ converges. As $\sum_{1 \leq n < N} a_n^2$ is a finite sum, it follows that $\sum_{n \geq 1} a_n^2$ converges.

c. By the Leibniz criterion, $\sum \frac{(-1)^n}{\sqrt{n}}$ converges, but $\sum \frac{1}{n}$ diverges.

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4 Prove that

$$\sum_{n \geq 1} \frac{1}{n(n+1)} = 1.$$

Solution Note that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Then the partial sum of the series is

$$\begin{aligned}s_n &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}\end{aligned}$$

Thus,

$$\sum_{n \geq 1} \frac{1}{n(n+1)} = \lim_{n\to\infty} s_n = 1.$$

5 a. Prove that

$$\sum_{n\geq 1} \frac{n-1}{2^{n+1}} = \frac{1}{2}.$$

b. Use part (1) to calculate

$$\sum_{n\geq 1} \frac{n}{2^n}.$$

Hint: Note that $\frac{n-1}{2^{n+1}} = \frac{n}{2^n} - \frac{n+1}{2^{n+1}}$.

Solution a. The partial sum of the series is

$$\begin{aligned} s_n &= \left(\frac{1}{2} - \frac{2}{2^2}\right) + \left(\frac{2}{2^2} - \frac{3}{2^3}\right) + \cdots + \left(\frac{n-1}{2^{n-1}} - \frac{n}{2^n}\right) + \left(\frac{n}{2^n} - \frac{n+1}{2^{n+1}}\right) \\ &= \frac{1}{2} - \frac{n+1}{2^{n+1}} \end{aligned}$$

Thus,

$$\sum_{n\geq 1} \frac{n-1}{2^{n+1}} = \lim_{n\to\infty} s_n = \frac{1}{2}.$$

b. We can write the original series as $\frac{1}{2}\sum(\frac{n}{2^n} - \frac{1}{2^n})$. As both $\sum \frac{n}{2^n}$ and $\sum \frac{1}{2^n}$ are convergent,

$$\begin{aligned} \frac{1}{2}\sum_{n\geq 1}\left(\frac{n}{2^n} - \frac{1}{2^n}\right) &= \frac{1}{2}\left(\sum_{n\geq 1}\frac{n}{2^n} - \sum_{n\geq 1}\frac{1}{2^n}\right) \\ &= \frac{1}{2}\left(\sum_{n\geq 1}\frac{n}{2^n} - 1\right) = \frac{1}{2} \\ \implies \sum_{n\geq 1}\frac{n}{2^n} &= 2 \end{aligned}$$

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6 Let $\{a_n\}_{n\geq 1}$ be a sequence of non-negative numbers such that $\sum_{n\geq 1} a_n$ diverges. For $n \geq 1$, let $s_n = a_1 + \cdots + a_n$.

a. Prove that the series

$$\sum_{n\geq 1} \frac{a_n}{a_n + 1}$$

diverges.

b. Prove that for all $N \geq 1$ and all $n \geq 1$,

$$\sum_{k=1}^{n} \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+n}}.$$

Deduce that the series $\sum \frac{a_n}{s_n}$ diverges.

c. Prove that for all $n \geq 2$,

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}.$$

Deduce that the series $\sum \frac{a_n}{s_n^2}$ converges.

Solution a. As $\sum a_n$ diverges, we have three cases: $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} a_n > 0$, or the limit does not exist. In the last two cases, the series must diverge. So, we need to handle the first case.
As $\lim_{n\to\infty} a_n = 0$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n| = a_n < 1$. Then

$$0 < a_n + 1 < 2 \implies 0 < \frac{1}{2} < \frac{1}{a_n+1} \implies 0 < \frac{a_n}{2} < \frac{a_n}{a_n+1}$$

Since $\sum a_n$ diverges, $\sum \frac{a_n}{2}$ diverges also, so by comparison, $\sum \frac{a_n}{a_n+1}$ diverges.

b. Note that s_n is monotonically increasing as $a_n \geq 0$.

$$
\begin{aligned}
1 - \frac{s_N}{s_{N+n}} &= \frac{s_{N+n} - s_N}{s_{N+n}} \\
&= \frac{a_{N+1} + a_{N+2} + \cdots + a_{N+n}}{s_{N+n}} \\
&\leq \frac{a_{N+1}}{s_{N+n}} + \frac{a_{N+2}}{s_{N+n}} + \cdots + \frac{a_{N+n-1}}{s_{N+n}} + \frac{a_{N+n}}{s_{N+n}} \\
&\leq \frac{a_{N+1}}{s_{N+1}} + \frac{a_{N+2}}{s_{N+2}} + \cdots + \frac{a_{N+n-1}}{s_{N+n-1}} + \frac{a_{N+n}}{s_{N+n}} = \sum_{k=1}^{n} \frac{a_{N+k}}{s_{N+k}}
\end{aligned}
$$

The series diverges by the Cauchy criterion for series, as

$$
\left| \sum_{n=N}^{N+k} \frac{a_n}{s_n} \right| = \sum_{n=N}^{N+k} \frac{a_n}{s_n} \geq 1 - \frac{s_N}{s_{N+k}}
$$

for every N and k in $\mathbb{N}$.

c. As s_n is monotonically increasing, $s_{n-1} \leq s_n \implies s_n s_{n-1} \leq s_n^2$. Thus,

$$
0 \leq \frac{a_n}{s_n^2} \leq \frac{a_n}{s_n s_{n-1}} = \frac{s_n - s_{n-1}}{s_n s_{n-1}} = \frac{1}{s_{n-1}} - \frac{1}{s_n}
$$

The partial sum of $\sum\left(\frac{1}{s_{n-1}} - \frac{1}{s_n}\right)$ is given by

$$
\frac{1}{s_1} - \frac{1}{s_n} < \frac{1}{s_1}
$$

Thus, the partial sums are increasing and bounded above, so the sequence of partial sums must converge, which means that $\sum \frac{a_n}{s_n^2}$ must converge also by comparison.

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7 Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence of non-negative numbers such that $\sum_{n\geq 1} a_n < \infty$. Show that

$$
\lim_{n\to\infty} n a_n = 0.
$$

Solution Let s_n be the partial sums of the series. Then

$$
\begin{aligned}
\lim_{n\to\infty} s_{2n-1} - s_{n-1} = \lim_{n\to\infty} a_n + \cdots + a_{2n-1} = \lim_{n\to\infty} n a_{2n-1} = 0 &\implies \lim_{n\to\infty} (2n-1) a_{2n-1} = 0 \\
\lim_{n\to\infty} s_{2n} - s_n = \lim_{n\to\infty} a_{n+1} + \cdots + a_{2n} = \lim_{n\to\infty} n a_{2n} = 0 &\implies \lim_{n\to\infty} 2n a_{2n} = 0
\end{aligned}
$$

Let $\epsilon > 0$. Then as the above limits exist, there exists $N \in \mathbb{N}$ such that for all $k \geq \frac{N+1}{2}$, we have

$$
\begin{aligned}
|(2k-1) a_{2k-1}| &< \epsilon \\
|2k a_{2k}| &< \epsilon
\end{aligned}
$$

Thus, as any $n \in \mathbb{N}$ can be written as $2k-1$ or $2k$ for $k \in \mathbb{N}$, we have that for all $n \geq N$,

$$
|n a_n| < \epsilon
$$

so by definition, $\lim_{n\to\infty} n a_n = 0$.