1 $(A \setminus B) \cap (C \setminus B) = (A \cap C) \setminus B$

Solution We want to show

$$x \in \big((A \setminus B) \cap (C \setminus B)\big) \Rightarrow x \in (A \cap C) \setminus B \tag{1}$$
$$x \in \big((A \setminus B) \cap (C \setminus B)\big) \Leftarrow x \in (A \cap C) \setminus B \tag{2}$$

Proof of (1):

Suppose $x \in \big((A \setminus B) \cap (C \setminus B)\big)$. Then $x \in \big(A \setminus B\big)$ and $x \in \big(C \setminus B\big)$. Thus, $x \in A, x \in C$, and $x \notin B$, so $x \in \big((A \cap C) \setminus B\big)$.

Proof of (2):

Suppose $x \in \big((A \cap C) \setminus B\big)$. Then $x \in A, x \in C$, and $x \notin B$. This means that $x \in \big(A \setminus B\big)$ and $x \in \big(C \setminus B\big)$. Thus, $x \in \big((A \setminus B) \cap (C \setminus B)\big)$. ■

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2 $(A \setminus B) \cap (C \setminus B) = A \setminus (B \cup C)$

Solution Consider $C = B$. Then on the LHS, $(A \setminus B) \cap (B \setminus B) = (A \setminus B) \cap \emptyset = \emptyset$, but on the RHS, we have $A \setminus (B \cup B) = A \setminus B$, which is not necessarily empty. Thus, the identity is not true. ■

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3 If $A \subseteq B$, then $A \cap C \subseteq B \cap C$.

Solution Suppose $x \in (A \cap C)$. Then $x \in A$ and $x \in C$. Since $A \subseteq B$, $x \in B$. Thus, $x \in (B \cap C)$. ■

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4 If $A \subseteq B$, then $A \cup C \subseteq B \cup C$.

Solution Suppose $x \in A \cap C$. Then there are two possibilities:

Case 1: $x \in C$

If $x \in C$, then $x \in (B \cup C)$.

Case 2: $x \in A$

If $x \in A$, then $x \in B$ as well since $A \subseteq B$. Thus, $x \in (B \cup C)$. ■

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5 If $A \cap C = B \cap C$, then $A = B$.

Solution Consider $A = \{1\}, B = \{2\}$, and $C = \{3\}$. Then $A \cap C = B \cap C = \emptyset$, but $A \neq B$. ■

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6 If $A \cup C = B \cup C$, then $A = B$.

Solution Consider $A = \{1\}, B = \{2\}$, and $C = \mathbb{R}$. Then $A \cap C = B \cap C = \mathbb{R}$, but $A \neq B$. ■

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7 $(A \cup B) \cap C = A \cup (B \cap C)$

Solution Take $C = \emptyset$. Then $(A \cup B) \cap C = \emptyset$, but $A \cup (B \cap C) = A$, but A is not necessarily the empty set. Thus, the equality is false. ■

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8 If $A \not\subseteq B$ and $B \not\subseteq C$, then $A \not\subseteq C$.

Solution Consider $A = C = \{1\}$ and $B = \{2\}$. Then $A \not\subseteq B$ and $B \not\subseteq C$, but $A \subseteq C$ since $A = C$. Hence, the equality is not true. ■

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9 If $A \subseteq B$ and $B \subseteq C$, then $A \cup B \subseteq C$.

Solution Suppose $x \in A \cup B$. Then if $x \in A$, $x \in B$ as well. If $x \in B$, then x is obviously in B. If $x \in B$, then $x \in C$ since $B \subseteq C$. Thus, $A \cup B \subseteq C$. ■

10 $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

Solution Let $x \in \big((A \cup B) \cap C\big)$. Then there are two possibilities:

Case 1: $x \in A$

Since x is in $(A \cup B) \cap C$, $x \in C$. So, $x \in (A \cap C)$, and thus, $x \in (A \cap C) \cup (B \cap C)$.

Case 2: $x \in B$

x is also in C, so $x \in (B \cap C)$, so $x \in (A \cap C) \cup (B \cap C)$. ■

****1** Prove that $\sqrt{15} \notin \mathbb{Q}$.

Solution Suppose $\sqrt{15} \in \mathbb{Q}$. Then there exists $a, b \in \mathbb{Z}$ such that

$$\sqrt{15} = \frac{a}{b},$$

where a and b are coprime, and b is nonzero. We can write

$$15 = \frac{a^2}{b^2}$$
$$15b^2 = a^2$$

Notice that a^2 is a multiple of the prime number 5, which means that 5 also divides a. This means we can write a as

$$a = 5c,\ c \in \mathbb{Z}.$$

Making the substitution for a, we get

$$15b^2 = 25c^2$$
$$3b^2 = 5c^2$$

This means that b^2 and therefore b are multiples of 5. However, this is a contradiction, because a and b are coprime. Thus, $\sqrt{15} \notin \mathbb{Z}$. ■