8 Suppose L is linear transformation of $\mathbb{R}^d$. Show that if E is a measurable subset of $\mathbb{R}^d$, then so is $L(E)$, by proceeding as follows:

a. Note that if E is compact, so is $L(E)$. Hence if E is an F_σ set, so is $L(E)$.

b. Because L automatically satisfies the inequality

$$|L(x) - L(x')| \leq M|x - x'|$$

for some M, we can see that L maps any cube of side length ℓ into a cube of side length $c_d M\ell$, with $c_d = 2\sqrt{d}$. Now if $m(E) = 0$, there is a collection of cubes $\{Q_j\}$ such that $E \subseteq \bigcup_j Q_j$, and $\sum_j m(Q_j) < \varepsilon$. Thus $m_*(L(E)) \leq c'\varepsilon$, and hence $m(L(E)) = 0$. Finally, use Corollary 3.5.

Solution a. Let $E \subseteq \mathbb{R}^d$ be compact. We'll show that $L(E)$ is closed and bounded also.
First note that L is continuous. Indeed, for any $\varepsilon > 0$, we can pick $\delta < \varepsilon/\|L\|$. Then for any $x \in \mathbb{R}^d$,

$$y \in B_\delta(x) \implies \|L(y) - L(x)\| \leq \|L\|\|y - x\| < \varepsilon.$$

Let $\{y_n\}_{n\geq 1}$ be a sequence in $L(E)$. Then there exists $\{x_n\}_{n\geq 1} \subseteq E$ such that $L(x_n) = y_n$.
Since E is compact, $\{x_n\}_{n\geq 1}$ admits a convergent subsequence $\{x_{k_n}\}_{n\geq 1}$ with limit $x \in E$. Moreover, since L is continuous,

$$x_{k_n} \xrightarrow{n\to\infty} x \in E \implies L(x_{k_n}) \xrightarrow{n\to\infty} L(x) \in L(E).$$

Hence, $L(E)$ is sequentially compact $\implies$ $L(E)$ is compact.
As a result, if E is F_σ, then $L(E)$ is also F_σ. This is because we can write E as a union of compact sets, since if $E = \bigcup_j E_j$ for some closed sets E_j,

$$E = \bigcup_{n=1}^{\infty} \bigcup_j \left(\overline{B_n(0)} \cap E_j\right),$$

which is a countable union of compact sets. $\overline{B_n(0)} \cap E_j$ is compact since the closed ball is bounded and intersections of closed sets are closed.
Thus,

$$L(E) = L\left(\bigcup_{n=1}^{\infty} \bigcup_j \left(\overline{B_n(0)} \cap E_j\right)\right) = \bigcup_{n=1}^{\infty} \bigcup_j L\left(\overline{B_n(0)} \cap E_j\right)$$

and each $L\left(\overline{B_n(0)} \cap E_j\right)$ is compact by the first part of the problem. Hence $L(E)$ is F_σ.

b. We showed the inequality in part (a). Just replace y with x' and let $M = \|L\|$.
Let x and x' be any two points on a cube. Then consider the two opposite corners of the cube, which has the maximal distance $\ell\sqrt{d}$ between them. Then by the inequality,

$$\|L(x) - L(x')\| \leq M\|x - x'\| \leq M\sqrt{d}\ell,$$

so the image of a cube under L fits in a cube of side length $c_d M\ell$.
If $m(E) = 0$, then there exist cubes $\{Q_j\}$ with $E \subseteq \bigcup_j Q_j$ and $\sum_j m(Q_j) < \varepsilon$. Note that $L(E) \subseteq L(Q_j)$ also, and by the above, $m(L(Q_j)) \leq (c_d M\ell)^2 = (c_d M)^2 m(Q_j) := c' m(Q_j)$, so

$$m_*(L(E)) \leq \sum_j m_*(L(Q_j)) \leq c' \sum_j m(Q_j) < c'\varepsilon.$$

Since ε was arbitrary, $m_*(L(E)) = 0$, so $L(E)$ is measurable.
Let $E \subseteq \mathbb{R}^d$ be measurable. By Corollary 3.5, E differs from an F_σ set F with $m(E - F) = m(F - E) = 0$. Hence,

$$m(L(E - F)) = m(L(E) - L(F)) = 0.$$

But by (a), $L(F)$ is F_σ, so $L(E)$ differs from an F_σ set by a set of measure zero, so $L(E)$ is measurable.

9 Give an example of an open set $\mathcal{O}$ with the following property: the boundary of the closure of $\mathcal{O}$ has positive Lebesgue measure.

[Hint: Consider the set obtained obtained by taking the union of open intervals which are deleted at the odd steps in the construction of a Cantor-like set.]

Solution We will construct a nowhere dense set E with positive measure. Then if we define $U = [0,1] \setminus E$, then $\partial U = E$, which has positive measure.

We first start with the unit interval $[0,1]$ and remove the middle $1/4$-th.

Next, from each closed interval, we remove the middle $1/4^2$-th, so we remove a total length of $2/4^2$.

Continuing by induction, by the k-th step, we have removed a total length of

$$\sum_{n=1}^{k} \frac{2^{n-1}}{4^n}$$

from $[0,1]$. Thus, from the resulting set we construct, we have removed

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{4^n} = \frac{1/4}{1-1/2} = \frac{1}{2}.$$

Moreover, E is nowhere dense by construction, since every interval is cut in the middle at some point in the construction.

Hence, $m(\partial U) = m([0,1] \setminus E) = 1 - m(E) = 1/2 > 0$.

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11 Let A be the subset of $[0,1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $m(A)$.

Solution We'll show that $m_*(A) = 0$, so by a theorem, A is measurable and $m(A) = 0$ by definition.

Consider the first step of the construction of A, where we remove the interval $(4/10, 5/10)$ from $[0,1]$, similar to the construction of the Cantor set. In this stage, the resulting closed intervals have total length $9/10$.

In the next step, we divide each interval into 10 equal pieces, and remove the 5-th open interval. The resulting length of the intervals have length $(9/10)^2$.

Continuing by induction, after the k-th step, we have the set E_k, which comprises of 2^k closed intervals with total length $(9/10)^k$. Moreover, $E_{k+1} \subseteq E_k$ for all $k \geq 0$ and $A = \bigcap_{k \geq 0} E_k$. By a corollary in the book,

$$m_*(A) = m_*\left(\bigcap_{k \geq 0} E_k\right) = \lim_{k \to \infty} m_*(E_k) = \lim_{k \to \infty} \left(\frac{9}{10}\right)^k = 0.$$

Thus, A is measurable and its Lebesgue measure is 0.

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12 Theorem 1.3 states that every open set in $\mathbb{R}$ is the disjoint union of open intervals. The analogue in $\mathbb{R}^d$, $d \geq 2$, is generally false. Prove the following:

a. An open disc in $\mathbb{R}^2$ is not the disjoint union of open rectangles.

[Hint: What happens to the boundary of any of these rectangles?]

b. An open connected set Ω is the disjoint union of open rectangles if and only if Ω is itself an open rectangle.

Solution a. An open disc cannot be written as the disjoint union of open rectangles because the boundary of the closure of each rectangle will not be included in their union. Since each open rectangle is disjoint from the others, no other rectangle will contain the boundary of another. Otherwise, the two rectangles will intersect. Hence, an open disc cannot be written as the disjoint union of open rectangles.

b. "$\Longleftarrow$"

Ω is an open rectangle, so this is clear since Ω is a union with just itself. Moreover, open rectangles are connected since we can connect any two points in the rectangle with a line segment.

"$\Longrightarrow$"

Let Ω be an open connected set which is the disjoint union of open rectangles.

Suppose Ω was not an open rectangle. Then we need at least two disjoint open rectangles in the union to form Ω. Otherwise, Ω would simply be an open rectangle.

Let $\Omega = \bigcup_j R_j$, where R_j is an open rectangle and $R_j \cap R_i = \emptyset$ for all $i \neq j$. But since each R_j is open, this means that R_i and R_j are separated for all $i \neq j$. This implies that Ω is disconnected, which is a contradiction. Hence, Ω must be an open rectangle.

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1 Given an irrational x, one can show (using the pigeon-hole principle, for example) that there exists infinitely many fractions p/q, with relatively prime integers p and q such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}.$$

However, prove that the set of those $x \in \mathbb{R}$ such that there exist infinitely many fractions p/q, with relatively prime integers p and q such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^3} \quad \text{or} \leq 1/q^{2+\varepsilon},$$

is a set of measure zero.

[Hint: Use the Borel-Cantelli lemma.]

Solution Fix $\varepsilon > 0$.

Choose $E_k(n) = \left\{ x \in [k-1, k] \mid \left| x - \left(k - 1 + \frac{p}{n} \right) \right| \leq \frac{1}{n^{2+\varepsilon}},\ p = 0, \ldots, n-1 \right\}$. This is closed (and therefore measurable) since it is a union of n closed balls of radius $1/n^3$. Each closed ball is the set of numbers that is within $1/n^{2+\varepsilon}$ of a rational number between $[k-1, k]$.

Note that

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} m(E_k(n)) \leq \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{2n}{n^{2+\varepsilon}} = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{2}{n^{1+\varepsilon}} < \infty \quad (p\text{-series with } p > 1),$$

so $\limsup_{n \to \infty} E_k(n) = \{x \in [k-1, k] \mid \text{there are infinitely many } n \text{ such that } x \in E_k(n)\} \coloneqq E_k$ is measurable and has measure 0 by the Borel-Cantelli lemma.

Hence, $\bigcup_{k \in \mathbb{Z}} E_k = \left\{ x \in \mathbb{R} \mid \left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}} \right\} \coloneqq E$ with p and q relatively prime, which is the set we're interested in. By countable sub-additivity,

$$m(E) = m\left(\bigcup_{k \in \mathbb{Z}} E_k \right) \leq \sum_{k \in \mathbb{Z}} m(E_k) = 0$$

as desired.