22.2 a. Let $p\colon X \to Y$ be a continuous map. Show that if there is a continuous map $f\colon Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

b. If $A \subseteq X$, a retraction of X onto A is a continuous map $r\colon X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

Solution a. Let p and f be the functions described in the problem.

Then $p \circ f\colon Y \to Y$ is bijective, since it is the identity map, i.e., for all $y \in Y$, $(p \circ f)(y) = y$ and $(p \circ f)^{-1}(y) = y$.

Since p is continuous, for any open set $V \subseteq Y$, $p^{-1}(V)$ is open. It suffices to show that if $p^{-1}(V)$ is open, then V is open as well.

Let $p^{-1}(V)$ be open. Then since f is continuous, $f^{-1}(p^{-1}(V))$ is open. But $f^{-1} \circ p^{-1} = (p \circ f)^{-1}$, which is the identity map on Y, so $f^{-1}(p^{-1}(V)) = V$ is open.

Thus, V is open in Y if and only if $p^{-1}(V)$ is open in X, so p is a quotient map.

b. Let r be a retraction. Note that r is onto, by definition.

Since r is continuous, then for any open $V \subseteq A$, $r^{-1}(V)$ is open, so we just need to check that $r^{-1}(V)$ open $\implies V$ is open.

Let $r^{-1}(V)$ be open in X. Since r is onto, for every $y \in V$, there exists $x \in A$ such that $r(x) = y$. Since r is a retraction, it follows that $y = x$. Hence, $V = r(r^{-1}(V)) = r^{-1}(V)$ is open.

23.2 Let $\{A_n\}$ be a sequence of connected subspaces of X such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\bigcup A_n$ is connected.

Solution Suppose $A \coloneqq \bigcup A_n$ is disconnected. Then there exist disjoint, non-empty, open sets U and V such that $A = U \amalg V$.

Since A_1 is connected, either $A_1 \subseteq U$ or $A_1 \subseteq V$. Assume without loss of generality that $A_1 \subseteq U$. The argument is the same for $A_1 \subseteq V$, but with U and V switched.

We'll finish the proof by induction.

Base step:

By assumption, $\emptyset \neq A_1 \cap A_2 \subseteq A_1 \subseteq U$. Since A_2 is connected, $A_2 \subseteq U$ also. Hence, the base step holds.

Inductive step:

Assume that $A_n \subseteq U$. Then since $A_n \cap A_{n+1} \neq \emptyset$ and $A_n \subseteq U$, $A_n \cap A_{n+1} \subseteq U$ also. But A_{n+1} is connected, so $A_{n+1} \subseteq U$. Thus, the inductive step holds.

By induction, we conclude that $A_n \subseteq U$ for all $n \geq 1$, so $\bigcup_n A_n \subseteq U$. But $A = \bigcup_n A_n$, which implies that $A \subseteq U \implies V = \emptyset$. This is a contradiction. Hence, A must be connected.

23.12 Let $Y \subseteq X$; let X and Y be connected. Show that if A and B form a separation of $X - Y$, then $Y \cup A$ and $Y \cup B$ are connected.

Solution If $Y = \emptyset$, then $X - Y = X$ is connected, so no such separation exists. Hence, $Y \neq \emptyset$.

Suppose $Y \cup A$ were disconnected. Then there exist non-empty, disjoint, and open sets U and V in $Y \cup A$ such that $Y \cup A = U \cup V$. Note that A is open and closed in $X - Y$, since $B = (X - Y) - A$ is also open. Hence, there exist sets D open in X and F closed in X such that $D \cap (X - Y) = A$ and $F \cap (X - Y) = A$.

Since Y is connected, either $Y \subseteq U$ or $Y \subseteq V$. Assume without loss of generality that $Y \subseteq U$. We can switch U and V in the following argument to get the same result.

Also, since $Y \subseteq U$, it follows that $V \subseteq A$, since $U \cup V = Y \cup A$. Moreover, V is closed and open in $Y \cup A$, also, so V is open and closed in A.

This implies that V is an open subset of D. Indeed, note that $V \cap Y = \emptyset$, so we get

$$V \subseteq A = D \cap (X - Y) = D - Y \implies V \subseteq D.$$

Then the only set in D such that $S \cap D = V$ is $S = V$ itself, so V is an open subset of D.

By a similar argument, V is a closed subset of F.

D was open in X and F was closed in X, so using the same argument as the above once again, we get that V is closed and open in X. But the only closed and open sets in X are $\emptyset$ and X, which is a contradiction. Thus, $Y \cup A$ must be connected.

We can use the same argument as the above, but with A and B swapped, to get the same result.

For a topological space X, consider the set

$$\Sigma(X) = \{0\} \amalg (X \times (0,1)) \amalg \{1\}$$

endowed with the quotient topology from the map $\pi \colon X \times [0,1] \to \Sigma(X)$ given by

$$\pi(x,y) = \begin{cases} 0 & \text{if } y = 0 \\ (x,y) & \text{if } 0 < y < 1 \\ 1 & \text{if } y = 1. \end{cases}$$

1 Show that if X is Hausdorff, then so is $\Sigma(X)$.

Solution Let X be Hausdorff.

Note that basic open sets of $X \times [0,1]$ are sets of the form $U \times (a,b)$, where U is an open set in X and (a,b) is an open set in $[0,1]$.

Also, π is a bijection if we restrict it to $X \times (0,1)$ to $X \times (0,1)$. So, open sets on $X \times (0,1)$ are still open under π since it is just itself.

Let $a \neq b \in \Sigma(X)$. Then there are four cases to consider:

$a = 0$ and $b = (x,y)$:

Consider the open neighborhood of 0, $U = \{0\} \cup (X \times (0,\varepsilon))$. This is open since

$$\pi^{-1}(U) = \pi^{-1}(\{0\}) \cup \pi^{-1}(X \times (0,\varepsilon)) = (X \times \{0\}) \cup (X \times (0,\varepsilon)) = X \times [0,\varepsilon),$$

and this is open in $X \times [0,1]$, since X is open in X and $[0,\varepsilon)$ is open in $[0,1]$, and $\Sigma(X)$ has the quotient topology.

Let V be an open neighborhood of (x,y) in $\Sigma(X)$ in the form of $W \times (c,d)$, where W is an open neighborhood of x in X. Take $\varepsilon < d$. Then $[0,y) \cap (c,d) = \emptyset$, so U and V are disjoint neighborhoods of a and b respectively.

$a = 1$ and $b = (x, y)$:

This case is similar to the above set. We can take $U = \{1\} \cup (X \times (\varepsilon, 1))$ for some $\varepsilon > 0$. Then its inverse image under π is $X \times (\varepsilon, 1]$. Then we can use a similar argument to the above to find an open neighborhood of b which is disjoint to U.

$a = 0$ and $b = 1$:

We can take $X \times [0, 1/4)$ and $X \times (3/4, 1]$ to be open neighborhoods of a and b respectively. These are clearly disjoint.

$a = (x_1, y_1)$ and $b = (x_2, y_2)$:

Note that as π is bijective when restricted as $\pi\big|_{X\times(0,1)}$, it is a bijection. Hence, we can treat a and b as if they were elements of $X \times (0, 1)$ in our domain space.

X is Hausdorff, so there exist disjoint open sets U_1 and V_1 containing x_1 and x_2, respectively.

y_1 and y_2 are in the open set $(0, 1)$, so there exist open sets U_2 and V_2 containing y_1 and y_2, respectively.

Hence, since $U_1 \cap V_1 = \emptyset$, we have that $U_1 \times U_2$ and $V_1 \times V_2$ are disjoint open neighborhoods of a and b, respectively.

In all cases, we can find disjoint open neighborhoods of a and b. Hence, $\Sigma(X)$ is Hausdorff.

2 Show that if X and Y are homeomorphic, then so are $\Sigma(X)$ and $\Sigma(Y)$.

Solution Since X and Y are homeomorphic, there exists a bicontinuous function $f\colon X \to Y$.

We define $F\colon \Sigma(X) \to \Sigma(Y)$ as follows:

$$F(a) = \begin{cases} 0 & \text{if } a = 0 \\ (f(x), y) & \text{if } a = (x, y) \\ 1 & \text{if } a = 1. \end{cases}$$

This is a homeomorphism between $\Sigma(X)$ and $\Sigma(Y)$.

F is bijective:

If $a = 0$ and $b \neq 0$, then $F(a) = 0 \neq b$. A similar argument holds for $a = 1$ and $b \neq 1$.

Let $a = (x_1, y_1)$ and $b = (x_2, y_2)$. Then

$$F(a) = F(b) \implies (f(x_1), y_1) = (f(x_2), y_2) \implies f(x_1) = f(x_2) \text{ and } y_1 = y_2.$$

Since f is a bijection, it follows that $x_1 = x_2$, so $a = b$. Hence, F is injective.

F is also a surjection. Clearly, $\{0, 1\} \subseteq F(\Sigma(X))$. For $(x, y) \in \Sigma(Y)$, since f is a bijection and $x \in X$, $F(f^{-1}(x), y) = (f(f^{-1}(x)), y) = (x, y)$, so F is a surjection.

Hence, F is a bijection.

F is continuous:

Let U be an open set in $\Sigma(Y)$. Then using the open neighborhoods of 0 and 1 from problem (1), we get, for some $\varepsilon > 0$ and $\delta > 0$,

$$F^{-1}(U) = \underbrace{(X \times [0, \varepsilon))}_{\text{omit this if } 0 \notin U} \cup \underbrace{(X \times (\delta, 1])}_{\text{omit this if } 1 \notin U} \cup \left\{(f^{-1}(x), y) \in \Sigma(X) \mid (x, y) \in U\right\}.$$

The right-most set is open. Indeed, let $(f^{-1}(x), y)$ be in that set. Note that since U is open, there exists an open neighborhood of $V \times W \subseteq U$ containing (x, y), where $V \subseteq X$ and $W \subseteq [0, 1]$ are open sets. Then

$$(f^{-1}(x), y) \in \underbrace{f^{-1}(V)}_{\text{open}} \times W \subseteq \left\{(f^{-1}(x), y) \in \Sigma(X) \mid (x, y) \in U\right\},$$

so it's open. Hence, F is continuous.

F^{-1} is continuous:

Since $F = (F^{-1})^{-1}$, it suffices to show that F is an open mapping.

Let $U \subseteq \Sigma(X)$ be an open set. Then for some $\varepsilon > 0$ and $\delta > 0$,

$$F(U) = \underbrace{(X \times [0, \varepsilon))}_{\text{omit this if } 0 \notin U} \cup \underbrace{(X \times (\delta, 1])}_{\text{omit this if } 1 \notin U} \cup \{(f(x), y) \in \Sigma(X) \mid (x, y) \in U\}.$$

Using the same argument as in showing F is continuous but with f and f^{-1} replaced, we get that F^{-1} is continuous also, since f is bicontinuous.

Thus, F is a homeomorphism between $\Sigma(X)$ and $\Sigma(Y)$, so the two sets are homeomorphic.

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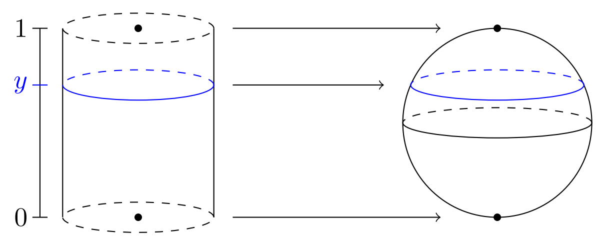
3 If we take $X = S^1$, to what familiar object of geometry is $\Sigma(X)$ homeomorphic?

Solution If $X = S^1$, then $\Sigma(X)$ is homeomorphic to a sphere in $\mathbb{R}^3$. We can "squeeze" the cylinder close to 0 and 1 to get the circular cross-section.



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4 Show that a product $\prod_{i \in I} X_i$ of connected topological spaces is connected under the product topology.

Solution Suppose $X \coloneqq \prod_{i \in I} X_i$ were disconnected. Then there exist open, disjoint, and non-empty sets U and V such that $X = U \cup V$.

If two elements differ at a single i, then they are in the same connected subspace of X. Indeed, the map π_i^{-1} is continuous, since π_i is an open mapping, so it preserves the connectedness of X_i.

Hence, by induction, if two elements differ at finitely many i, they must be in the same connected subspace of X.

By connectedness, the set of all points which differ at finitely many i is connected, so it must lie in either U or V.

U must contain a basic open neighborhood, say $W = \prod_{i \in I} W_i$, where $W_i = X_i$ for all but finitely many i, which we index via $i_1, \ldots, i_n$.

Fix one (x_i) in W. Then for each point in X, (x_i) can differ from that point at, in the worst case, n different indices: $i_1, i_2, \ldots, i_n$.

Hence, by connectedness, all points of X must be in the same connected subspace as (x_i), which implies that $X \subseteq U$ or $X \subseteq V$. In either case, we get a contradiction. Hence X is connected.

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5 Show that any infinite set with the finite complement topology is connected.

Solution Let X be an infinite set with the finite complement topology.

Suppose X were disconnected. Then there exist open, disjoint, and non-empty sets U and V such that $X = U \cup V$.

Since X has the finite complement topology, ${}^{\mathrm{c}}U$ and ${}^{\mathrm{c}}V$ are finite. But

$$X = {}^{\mathrm{c}}\emptyset = {}^{\mathrm{c}}(U \cap V) = {}^{\mathrm{c}}U \cup {}^{\mathrm{c}}V,$$

which implies that X is finite. But X is infinite, which is a contradiction. Thus, X must be connected.