37.2 In the case of the orthonormal sequence (5), verify in detail that the Fourier coefficients (8) are slightly different from the ordinary Fourier coefficients, but that the Fourier series (6) is exactly the same as the ordinary Fourier series.

Solution For an orthonormal sequence, the coefficients are simply given by the inner product. So, if our orthonormal sequence is $\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \dots\right\}$, then the coefficients are given by

$$a_0 = \int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2\pi}}\,\mathrm{d}x, \quad a_n = \int_{-\pi}^{\pi} \frac{f(x)\cos nx}{\sqrt{\pi}}\,\mathrm{d}x, \quad b_n = \int_{-\pi}^{\pi} \frac{f(x)\sin nx}{\sqrt{\pi}}\,\mathrm{d}x.$$

The corresponding series is then given by

$$\begin{aligned} f(x) &\sim \frac{a_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} a_n \frac{\cos nx}{\sqrt{\pi}} + \sum_{n=1}^{\infty} b_n \frac{\sin nx}{\sqrt{\pi}} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2\pi}}\,\mathrm{d}x + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} \frac{f(x)\cos nx}{\sqrt{\pi}}\,\mathrm{d}x\right)\frac{\cos nx}{\sqrt{\pi}} + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} \frac{f(x)\sin nx}{\sqrt{\pi}}\,\mathrm{d}x\right)\frac{\sin nx}{\sqrt{\pi}} \\ &= \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x\right) + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x\right)\cos nx + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x\right)\sin nx, \end{aligned}$$

which is the same series as before.

37.3 Prove the Schwarz inequality.

Solution In the instance that $g \equiv 0$, $\langle f, g\rangle = 0$, so the inequality holds trivially. Assume from now on that g is not identically 0.

Let $\lambda \in \mathbb{R}$, and consider $\|f + \lambda g\|^2$, which is non-negative, since norms are non-negative. Then by linearity of the inner product,

$$\begin{aligned} \|f - \lambda g\|^2 &= \langle f - \lambda g, f - \lambda g\rangle \\ &= \|f\|^2 + \lambda^2\|g\|^2 - 2\lambda\langle f, g\rangle. \end{aligned}$$

If we take λ to be $\langle f, g\rangle/\|g\|^2$, we get

$$0 \le \|f - \lambda g\|^2 = \|f\|^2 + \frac{\langle f, g\rangle^2}{\|g\|^4}\|g\|^2 - 2\frac{\langle f, g\rangle^2}{\|g\|^2} = \|f\|^2 - \frac{\langle f, g\rangle^2}{\|g\|^2}.$$

Finally, by rearranging terms, we get

$$\langle f, g\rangle^2 \le \|f\|^2\|g\|^2 \implies |\langle f, g\rangle| \le \|f\|\|g\|,$$

which proves the inequality.

37.4 A well-known theorem of elementary geometry states that the sum of the squares of the sides of a parallelogram equals the sum of the squares of its diagonals. Prove that this called *parallelogram law* is true for the norm in R:

$$2\|f\|^2 + 2\|g\|^2 = \|f + g\|^2 + \|f - g\|^2.$$

Solution If we expand the right-hand side:

$$\|f + g\|^2 + \|f - g\|^2 = \|f\|^2 + \|g\|^2 + 2\langle f, g\rangle + \|f\|^2 + \|g\|^2 - 2\langle f, g\rangle = 2\|f\|^2 + 2\|g\|^2,$$

as desired.

37.5 Prove the *Pythagorean theorem* and its converse in R: f is orthogonal to g if and only if $\|f-g\|^2 = \|f\|^2+\|g\|^2$.

Solution "$\Longrightarrow$"

Let f and g be orthogonal. By definition, this means $\langle f, g\rangle = 0$. Thus, if we expand the distance between them,

$$\|f-g\|^2 = \|f\|^2 + \|g\|^2 - 2\langle f, g\rangle = \|f\|^2 + \|g\|^2,$$

as desired.

"$\Longleftarrow$"

By the same expansion, we see that

$$\|f-g\|^2 = \|f\|^2 + \|g\|^2 - 2\langle f, g\rangle = \|f\|^2 + \|g\|^2,$$

by assumption. Thus, if we subtract $\|f\|^2 + \|g\|^2$ from both sides, we get

$$-2\langle f, g\rangle = 0 \implies \langle f, g\rangle = 0,$$

so by definition, f and g are orthogonal.

37.6 Show that a null function is zero at each point of continuity, so that a continuous null function is identically zero.

Solution Let f be a null function. Suppose f is continuous at x_0, but $f(x_0) \neq 0$. Since f is continuous, $|f|^2$ is continuous also, since it is a composition of continuous functions. Thus, by continuity of f at x_0, there exists $\delta > 0$ such that for $x \in (x_0 - \delta, x_0 + \delta)$, we have

$$\left||f(x)|^2 - |f(x_0)|^2\right| < \frac{|f(x_0)|^2}{2} \implies |f(x)|^2 > -\frac{|f(x_0)|^2}{2} + |f(x_0)|^2 = \frac{|f(x_0)|^2}{2}.$$

Since $|f|^2$ is positive, integrating over a smaller interval makes the integral smaller, so

$$0 = \|f\|^2 = \int_a^b |f(x)|^2 \, \mathrm{d}x \geq \int_{x_0-\delta}^{x_0+\delta} |f(x)|^2 \, \mathrm{d}x > \int_{x_0-\delta}^{x_0+\delta} \frac{|f(x_0)|^2}{2} \, \mathrm{d}x = \delta|f(x_0)|^2 > 0,$$

which is a contradiction. Thus, f must be 0 at each point of continuity, so if f is continuous on its domain, then it must be identically zero.

38.1 Consider the sequence of functions $f_n(x)$, $n = 1, 2, 3, \ldots$, defined on the interval $[0, 1]$ by

$$f_n(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1/n \\ \sqrt{n}, & \text{if } 1/n < x < 2/n, \\ 0, & \text{if } 2/n \leq x \leq 1. \end{cases}$$

a. Show that the sequence $\{f_n(x)\}$ converges pointwise to the zero function on the interval $[0, 1]$.
b. Show that the sequence $\{f_n(x)\}$ does *not* converge in the mean to the zero function on the interval $[0, 1]$.

Solution a. Let $x \in [0, 1]$. By the Archimedean principle of the real numbers, there exists $N \in \mathbb{N}$ such that $1/N > x$. Thus, for $n \geq N$, we have

$$\frac{1}{n} \leq \frac{1}{N} < x \implies f_n(x) = 0,$$

by definition. Thus, for all but finitely many n, $f_n(x) = 0$, so $f_n(x) \xrightarrow{n\to\infty} 0$ for every x.

b. Notice that for all $n \in \mathbb{N}$,

$$\|f_n - 0\|^2 = \int_0^1 |f_n(x)|^2 \, \mathrm{d}x = \int_{1/n}^{2/n} n \, \mathrm{d}x = 1,$$

which doesn't converge to 0 as $n \to \infty$. Thus, f_n does not converge in the mean to the zero function.

38.2 Consider the following sequence of closed intervals of $[0,1]$:

$$\left[0,\frac{1}{2}\right],\ \left[\frac{1}{2},1\right],\ \left[0,\frac{1}{4}\right],\ \left[\frac{1}{4},\frac{1}{2}\right],\ \left[\frac{3}{4},1\right],\ \left[0,\frac{1}{8}\right],\ \left[\frac{1}{8},\frac{1}{4}\right],\ \ldots,$$

and denote the n-th subinterval by I_n. Now define a sequence of functions $f_n(x)$ on $[0,1]$ by

$$f_n(x) = \begin{cases} 1 & \text{for } x \text{ in } I_n \\ 0 & \text{for } x \text{ not in } I_n. \end{cases}$$

a. Show that the sequence $\{f_n(x)\}$ converges in the mean to the zero function on the interval $[0,1]$.

b. Show that the sequence $\{f_n(x)\}$ does *not* converge pointwise at any point of the interval $[0,1]$.

Solution a. Let $\varepsilon > 0$. Notice that there exists $N \in \mathbb{N}$ such that $1/2^N < \varepsilon$.
Also note that there exists $n_0 \in \mathbb{N}$ such that,

$$\|f_{n_0}\|^2 = \int_0^1 |f_{n_0}(x)|^2 \,\mathrm{d}x = \frac{1}{2^N}.$$

By the definition of the intervals, it is easy to see that $\{\|f_n\|^2\}$ is a decreasing sequence, so for all $n \geq n_0$,

$$\|f_{n_0}\|^2 \leq \frac{1}{2^N} < \varepsilon.$$

As ε was arbitrary, f_n converges in the mean to the zero function.

b. Let $x \in [0,1]$. Then there exists a subsequence I_{k_n} such that $x \in I_{k_n}$ for all $n \geq 1$. Thus, for all $n \geq 1$, $f_{k_n}(x) = 1$. Hence, f does not converge pointwise anywhere in $[0,1]$. Indeed, if $\varepsilon < 1/2$, then for any $n \geq 1$, $k_n \geq n$ but $|f_{k_n}(x) - 0| = 1 > \varepsilon$.

38.4 The function $f(x) = 1$ is to be approximated on $[0,\pi]$ by

$$p(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + b_5 \sin 5x$$

in such a way that $\int_0^\pi [1 - p(x)]^2 \,\mathrm{d}x$ is minimized. What values should the coefficients b_k have?

Solution First notice that

$$\begin{aligned}
\int_0^\pi \sin nx \sin mx \,\mathrm{d}x &= \frac{1}{2}\int_0^\pi \cos(n-m)x - \cos(n+m)x \,\mathrm{d}x \\
&= \frac{1}{2}\left[\frac{1}{n-m}\sin(n-m)x - \frac{1}{n+m}\sin(n+m)x\right]_0^\pi \\
&= 0,
\end{aligned}$$

if $n \neq m$. If $n = m$,

$$\int_0^\pi \sin^2 nx \,\mathrm{d}x = \int_0^\pi \frac{1}{2} - \frac{\cos 2nx}{2} \,\mathrm{d}x = \frac{\pi}{2} - \frac{\sin 2nx}{4}\bigg|_0^\pi = \frac{\pi}{2},$$

since $n \in \mathbb{Z}$.

Thus, we can write p as

$$p(x) = \left(b_1\sqrt{\frac{\pi}{2}}\right)\frac{\sin x}{\sqrt{\pi/2}} + \left(b_2\sqrt{\frac{\pi}{2}}\right)\frac{\sin 2x}{\sqrt{\pi/2}} + \cdots + \left(b_5\sqrt{\frac{\pi}{2}}\right)\frac{\sin 5x}{\sqrt{\pi/2}},$$

which approximates 1 with an orthonormal sequence. By a theorem proved in class, the coefficients are given by

$$b_k\sqrt{\frac{\pi}{2}} = \left\langle 1, \frac{\sin kx}{\sqrt{\pi/2}}\right\rangle = \int_0^\pi \frac{\sin kx}{\sqrt{\pi/2}} \,\mathrm{d}x = -\frac{1}{k}\sqrt{\frac{2}{\pi}}\cos kx\bigg|_0^\pi = \frac{(-1)^k - 1}{k}\sqrt{\frac{2}{\pi}} \implies b_k = \frac{2 - 2(-1)^k}{k\pi}.$$

38.5 The function $f(x) = x$ is to be approximated on $[0, \pi]$ by

$$p(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x$$

in such a way that $\int_0^\pi [x - p(x)]^2 \, \mathrm{d}x$ is minimized. What values should the coefficients b_k have?

Solution We can use the same argument as in the previous problem to see that

$$b_k = \frac{2}{\pi} \langle x, \sin kx \rangle = \frac{2}{\pi} \int_0^\pi x \sin kx \, \mathrm{d}x = \frac{2(-1)^{k-1}}{k}$$