3.10 a. Suppose p is a non-constant polynomial with all its zeroes in the upper half-plane $\mathbb{H} = \{z \mid \operatorname{Im} z > 0\}$. Prove that all of the zeroes of p' are contained in $\mathbb{H}$. *Hint:* Look at the partial fraction expansion of p'/p.

b. Use (a) to prove that if p is a polynomial then the zeroes of p' are contained in the (closed) convex hull of the zeroes of p. (The closed convex hull is the intersection of all half-planes containing the zeroes.)

Solution a. By the fundamental theorem of algebra, we can write $p(z) = a\prod_{j=1}^{n}(z - z_j)$, where z_j is a root of p and $a \in \mathbb{C}$. By assumption, $\operatorname{Im} z_j > 0$ for all j.

Note that by the product rule,

$$p'(z) = \sum_{j=1}^{n} \frac{a\prod_{j=1}^{n}(z - z_j)}{z - z_j} = \sum_{j=1}^{n} \frac{p(z)}{z - z_j}$$

Hence,

$$\begin{aligned}\frac{p'(z)}{p(z)} &= \frac{\sum_{j=1}^{n} \frac{p(z)}{z-z_j}}{p(z)} \\ &= \sum_{j=1}^{n} \frac{1}{z - z_j}\end{aligned}$$

Assume that there exists $z_k \in \mathbb{C}$ such that $p'(z_k) = 0$ and $\operatorname{Im} z_k \leq 0$. Note that this means that z_k cannot be a root of p. Then

$$\frac{p'(z_k)}{p(z_k)} = \sum_{j=1}^{n} \frac{1}{z_k - z_j} = \sum_{j=1}^{n} \frac{1}{(\operatorname{Re} z_k - \operatorname{Re} z_j) + i(\operatorname{Im} z_k - \operatorname{Im} z_j)},$$

but $\operatorname{Im} z_k - \operatorname{Im} z_j < 0$ for all j, so $\frac{p'(z_k)}{p(z_k)} \neq 0 \implies p'(z_k) \neq 0$. Hence, z_k cannot be a root of p', so all roots of p' must belong to $\mathbb{H}$.

b. Let p be a polynomial with roots $z_1, \ldots, z_n$. Suppose all the roots of p are contained in a half-plane. Then we can rotate p by multiplying it by a number of the form $e^{i\theta}$ so that all the roots of p lie in $\mathbb{H}$, as described in (a). Then by part (a), the roots of the (rotated) p' also lie in $\mathbb{H}$, so undoing the rotation gives us that the roots of p' lie in the same half-plane as p. We can repeat this for all half-planes which contain the roots of p to attain our result.

3.11 Suppose f is analytic in $\mathbb{D}$ and $|f(z)| \leq 1$ in $\mathbb{D}$ and $f(0) = 1/2$. Prove that $|f(1/3)| \geq 1/5$.

Solution Consider the transformation $T_a\colon \mathbb{D} \to \mathbb{D}$, $T_a(z) = \frac{z-a}{1-\bar{a}z}$. Then consider $\varphi\colon \mathbb{D} \to \mathbb{D}$, $\varphi = T_{1/2} \circ f$. Since $|f(z)| \leq 1$ and $f(0) = 1/2$, we have that $|\varphi(z)| \leq 1$ and $\varphi(0) = 0$. Then by the Schwarz lemma,

$$\left|\varphi\left(\frac{1}{3}\right)\right| \leq \frac{1}{3} \implies \left|\frac{f(\frac{1}{3}) - \frac{1}{2}}{1 - \frac{1}{2}f(\frac{1}{3})}\right| \leq \frac{1}{3} \implies \left|f\left(\frac{1}{3}\right)\right| \geq \frac{1}{5}$$

as desired.

4.1 a. A finite union of boundaries of squares, oriented in the usual counter-clockwise direction is a cycle, by definition. Prove that if a subarc of their union is traced twice, in opposite directions, then after removal of the common edge, the union is still a cycle.

b. The boundary of a finite union of squares is a cycle, oriented so that the region lies on the left.

Solution a. Let R be a finite union of boundaries of squares. Let S_1 and S_2 be two squares such that $\partial S_1 \cap \partial S_2$ is a side which is oriented in different directions in S_1 and S_2. If we remove it, then we get another closed shape which is oriented counter-clockwise. The polygon is also closed, so if we repeat this process for all the squares in R, we still get a finite union of boundaries of squares, so the union is still a cycle.

b. Let ∂S_1 be the boundary of a square oriented counter-clockwise. Then the claim is obviously true, so the base case holds.

Let ∂S_n be the boundary of a finite union of squares. By assumption the region is to the left in this boundary. Consider $\partial S_{n+1} = \partial(S_n \cup S)$, where S is a square oriented counter-clockwise. After removing subarcs that are traced twice in opposite directions, we still end up with a cycle, by (a). Since the region is to the left in S and in S_n, this is true of S_{n+1}. Hence, the inductive step holds.

By induction, the boundary of a finite union of squares is a cycle, and a counter-clockwise orientation means the region will always be on the left.

4.2 a. Let U be an open set in $\mathbb{C}$. A **polygonal curve** is a curve consisting of a finite union of line segments. Define an equivalence relation on the points of U by: $a \sim b$ if and only if there is a polygonal curve contained in U with edges parallel to the axes and with endpoints a and b. Show that each equivalence class is open and closed in U and connected and that there are at most countably many equivalence classes. The equivalence classes are called the **components** of U. For open sets in $\mathbb{C}^*$, allow polygonal curves to contain a half-line and obtain a similar result.

b. Let K be a compact set. Define an equivalence on the points of K by: $a \sim b$ if and only if there is a connected subset of K containing both a and b. Prove that the equivalence classes are connected and closed. The equivalence classes are called the **(closed) components** of K. There can be uncountably many (closed) components. In both parts (a) and (b) the components are the maximal connected subsets.

Solution a. Let $z_0 \in U$ and $C(z_0) \subseteq U$ be the equivalence class of z_0 which lies in U.

We first show that $C(z_0)$ is open in U.

Let $z \in C(z_0)$. Since $z \in U$, there exists $r > 0$ such that $B_r(z) \subseteq U$. Pick $w \in B_r(z)$. Then the curve $\gamma\colon [0,2] \to \mathbb{C}$,

$$\gamma(t) = \begin{cases} \operatorname{Re} z + i\Big[(1-t)\operatorname{Im} z + t \operatorname{Im} w\Big] & 0 \le t \le 1 \\ (2-t)\operatorname{Re} z + (1-t)\operatorname{Re} w + i \operatorname{Im} w & 1 \le t \le 2 \end{cases}$$

Note that $\gamma(t) \subseteq B_r(z) \subseteq U$. $\gamma(0) = z$ and $\gamma(2) = w$. Hence, by definition, $z \sim w \implies w \in C(z_0) \implies B_r(z) \subseteq C(z_0)$, so $C(z_0)$ is open in U.

We now show that $C(z_0)$ is closed in U.

Let $\{z_n\}_{n \ge 1} \subseteq C(z_0)$ which converges to $z \in U$. As U is open, there exists $r > 0$ such that $B_r(z) \subseteq U$. Since $z_n \xrightarrow{n \to \infty}$, there exists z_N such that $z_N \in B_r(z) \subseteq U$. Using the same argument as above, we find that $z_N \sim z \implies z \in C(z_N) = C(z_0)$. Hence, $C(z_0)$ is closed in U.

$C(z_0)$ is also obviously connected, since every pair of points in $C(z_0)$ can be connected by a polygonal curve. Hence, $C(z_0)$ is open and closed in U, and is connected.

We now show that there are at most countably many equivalence classes.

Suppose there were uncountably many equivalence classes. Let K be a compact subset of U which contains uncountably many equivalence classes. Let $r > 0$ and consider the uncountable union

$$\bigcup_{z \in K} B_r(z) \supseteq K.$$

Note that this is an open cover of K, so as K is compact, there exists $z_1, \ldots, z_n$ in K such that

$$K \subseteq \bigcup_{i=1}^{n} B_r(z_i).$$

But $B_r(z_i) \subseteq C(z_i)$, meaning K contains a finite number of equivalence classes, which is a contradiction. Hence, there are countably many equivalence classes.

b. Let $z_0 \in K$. Then let $z, w \in C(z_0)$. By definition, there exists a connected subset of K that contains a and b, so $C(z_0)$ is connected (by 131).

Let $z \in \overline{C(z_0)}$. Then for all $r > 0$, $B_r(z) \cap C(z_0) \neq \emptyset$. Pick $w \in B_r(z) \cap C(z_0)$. Then as $B_r(z)$ and $C(z_0)$ are both connected and their intersection is non-empty, $B_r(z) \cap C(z_0)$ is connected. Hence, $z \sim w \sim z_0 \implies z \in C(z_0)$, so $C(z_0)$ is closed, as desired.