6.5 Suppose $0 < p < q < \infty$. Then $L^p \nsubseteq L^q$ iff X contains sets of arbitrarily small positive measure, and $L^q \nsubseteq L^p$ iff X contains sets of arbitrarily large finite measure. What about the case $q = \infty$?

Solution $L^p \nsubseteq L^q$ iff X contains sets of arbitrarily small positive measure:

"$\Longrightarrow$"

Assume that there exists $\varepsilon > 0$ so that $\varepsilon \leq \mu(E)$ for all non-null measurable sets $E \subseteq X$.

Now let $f = \sum_{n=1}^N a_n \chi_{E_n}$ be an integrable simple function in L^p. By Minkowski's inequality,

$$\|f\|_q \leq \sum_{n=1}^N \|a_n \chi_{E_n}\|_q = \sum_{n=1}^N |a_n| \mu(E_n)^{1/q} = \sum_{n=1}^N |a_n| \mu(E_n)^{\frac{1}{p}} \mu(E_n)^{\frac{1}{q}-\frac{1}{p}} \leq \frac{1}{\varepsilon^{(q-p)/qp}} \|f\|_p.$$

Hence, simple integrable functions in L^p are all contained in L^q, which is a Banach space, hence closed. By density of these functions, this means that $L^p \subseteq L^q$.

"$\Longleftarrow$"

Let $\{E_n\}$ be a disjoint sequence of measurable sets so that $0 < \mu(E_n) < 2^{-n}$, which exists by assumption. Then set $a_n = \mu(E_n)^{-1/q}$ and $f = a_n \chi_{E_n}$. By the monotone convergence theorem,

$$\|f\|_p^p = \int \left| \sum_{n=1}^\infty a_n \chi_{E_n} \right|^p = \int \sum_{n=1}^\infty a_n^p \chi_{E_n} = \sum_{n=1}^\infty \int a_n^p \chi_{E_n} = \sum_{n=1}^\infty \mu(E_n)^{1-(p/q)} < \sum_{n=1}^\infty 2^{-n(q-p)/q} < \infty,$$

since $(q-p)/q > 0$. On the other hand, the same calculation yields

$$\|f\|_q^q = \sum_{n=1}^\infty \mu(E_n)^{1-(q/q)} = \sum_{n=1}^\infty 1 = \infty.$$

Hence, $L^p \nsubseteq L^q$.

$L^q \nsubseteq L^p$ iff X contains sets of arbitrarily large finite measure:

"$\Longrightarrow$"

Assume that there exists $M > 0$ so that $\mu(E) \leq M$ for all measurable $E \subseteq X$.

Let $f = \sum_{n=1}^N a_n \chi_{E_n}$ be an integrable simple function in L^q. Then by the triangle inequality,

$$\|f\|_p \leq \sum_{n=1}^N \|a_n \chi_{E_n}\| \leq \sum_{n=1}^N |a_n| \mu(E_n)^{1/p} = \sum_{n=1}^N |a_n| \mu(E_n)^{\frac{1}{q}} \mu(E_n)^{\frac{1}{p}-\frac{1}{q}} \leq M^{(q-p)/pq} \|f\|_p$$

As before, a dense subset of L^q is contained in the complete space L^p, so $L^q \subseteq L^p$.

"$\Longleftarrow$"

Now let $\{E_n\}$ is a sequence of disjoint measurable subsets of X so that $2^n \leq \mu(E_n) < \infty$, which exists by assumption. Set $a_n = \mu(E_n)^{-1/p}$, and let $f = \sum a_n \chi_{E_n}$. Then by the similar calculation as the first part,

$$\|f\|_q^q = \sum_{n=1}^\infty a_n^q \mu(E_n) = \sum_{n=1}^\infty \mu(E_n)^{-q/p} \mu(E_n) \leq \sum_{n=1}^\infty \frac{1}{2^{n(q/p-1)}} < \infty,$$

because $q/p - 1 > 0$. However

$$\|f\|_p^p = \sum_{n=1}^\infty \mu(E_n)^{-p/p} \mu(E_n) = \sum_{n=1}^\infty 1 = \infty,$$

so $f \in L^q$ but not L^p.

6.19 Define $\varphi_n \in (\ell^\infty)^*$ by $\varphi_n(f) = n^{-1}\sum_{j=1}^n f(j)$. Then the sequence $\{\varphi_n\}$ has a weak* cluster point φ, and φ is an element of $(\ell^\infty)^*$ that does not arise from an element of ℓ^1.

Solution Notice that if $\|f\| \leq 1$, then

$$|\varphi_n(f)| = \left| n^{-1}\sum_{j=1}^n f(j) \right| \leq n^{-1} \cdot n\|f\|_\infty = 1,$$

for all $n \geq 1$. Thus, $\varphi_n \in B^*$, so by Banach–Alaoglu, φ_n admits a convergent subsequence $\varphi_{n_k} \to \varphi$ weakly*, so φ is a weak* cluster point of the sequence. Because K is a Banach space, $(\ell^\infty)^*$ is one also, so $\varphi \in (\ell^\infty)^*$. Now suppose φ arose from an element $g \in \ell^1$. Then by definition of weak convergence, for all $f \in \ell^\infty$,

$$\varphi_{n_k}(f) \xrightarrow{k\to\infty} \sum_j g(j)f(j).$$

Now consider the basis sequence $e_m \in \ell^\infty$, where $e_m(k) = 0$ unless $k = m$, where $e_m(m) = 1$. Then

$$\varphi_{n_k}(e_m) = \frac{1}{n_k} \xrightarrow{k\to\infty} 0 = \sum_j g(j)e_m(j) = g(j)$$

for all $j \geq 1$. But if we test g against the constant 1 sequence,

$$\varphi_{n_k}(1) = 1 \xrightarrow{k\to\infty} 1 = \sum_j 1 \cdot g(j) = 0,$$

which is impossible. Hence, φ does not arise from an element of ℓ^1.

6.20 Suppose $\sup_n\|f_n\|_p < \infty$ and $f_n \to f$ a.e.

a. If $1 < p < \infty$, then $f_n \to f$ weakly in L^p.

b. The result of (a) is false in general for $p = 1$. It is, however, true for $p = \infty$ if μ is σ-finite and weak convergence is replaced by weak* convergence.

Solution a. Let $g \in L^q$ and $\varepsilon > 0$.

We follow the hint: By density of L^q-integrable simple functions, there exists $\varphi \in L^q$ so that $\|g - \varphi\|_q < \varepsilon/2$. Now let $\delta > 0$, which will be chosen later, and let $\mu(E) < \delta$. We have, by Minkowski's inequality, that

$$\|g\chi_E\|_q \leq \|(g - \varphi)\chi_E\|_q + \|\varphi\chi_E\|_q.$$

We simply need to investigate the L^q norm of φ on E. If we write

$$\varphi = \sum_{n=1}^N a_n\chi_{E_n}, \quad \text{then} \quad \varphi\chi_E = \sum_{n=1}^N a_n\chi_{E_n\cap E},$$

which gives

$$\|\varphi\chi_E\|_q \leq \sum_{n=1}^N \int |a_n|^q\chi_{E_n\cap E} \leq \sum_{n=1}^N |a_n|^q\mu(E) < \left(\sum_{n=1}^N |a_n|^q\right)\delta \xrightarrow{\delta\to 0} 0.$$

Hence, we may find δ which makes the last term smaller than $\varepsilon/2$, independently of E. Thus,

$$\|g\chi_E\|_q \leq \|(g - \varphi)\chi_E\|_q + \|\varphi\chi_E\|_q < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Now for the next part, for each $n \geq 1$, consider the measurable set $E_n := \{x \mid |g(x)|^q \geq 1/n\}$. This satisfies $E_1 \subseteq E_2 \subseteq \cdots$, and each of these have finite measure, or else $g \notin L^q$. Then

$$\int_{\bigcup E} |g|^q = \int_{\{|g|^q > 0\}} |g|^q = \int |g|^q.$$

Notice that $|g - g\chi_{E_n}|^q \le |g|^q \in L^1$, so by the dominated convergence theorem,

$$\int_{X \setminus E_n} |g|^q = \int |g - g\chi_{E_n}|^q \xrightarrow{n\to\infty} 0.$$

This proves the second part: there exists $n \ge 1$ so that $\mu(E_n) < \infty$ and $\int_{X\setminus E_n} |g|^q < \varepsilon$.

For the last part, we may simply invoke Egorov's theorem on $A := E_n$, which gives us a set $B \subseteq A$ so that $f_n \to f$ uniformly on B and $\mu(A \setminus B) < \delta$.

Finally, let $M = \sup_n \|f_n\|$. By Hölder's inequality,

$$\begin{aligned}
\int |f_n g - fg| &= \int_B |f_n - f||g| + \int_{A\setminus B} |f_n - f||g| + \int_{X\setminus A} |f_n - f||g| \\
&\le \|(f_n - f)\chi_B\|_p \|g\|_q + \|(f_n - f)\chi_{A\setminus B}\|_p \|g\chi_{A\setminus B}\|_q + \|(f_n - f)\chi_{X\setminus A}\|_p \|g\chi_{X\setminus A}\|_q \\
&\le \|(f_n - f)\chi_B\|_p \|g\|_q + 2M\|g\chi_{A\setminus B}\|_q + 2M\|g\chi_{X\setminus A}\|_q.
\end{aligned}$$

The first term can be made smaller than $\varepsilon/3$, because $f_n \to f$ uniformly on B by picking n large enough. Next, the second term can be made smaller than $\varepsilon/3$ also by picking $\delta(A, \varepsilon, n)$ small enough, so that $\mu(A \setminus B)$ is small. Lastly, the final term may be made smaller than $\varepsilon/3$ again by picking A large enough, shrinking $\delta(A, \varepsilon, n)$ if necessary. Hence,

$$\int |f_n g - fg| < \varepsilon$$

so $f_n \to f$ weakly in L^p.

b. Consider $f_n = \chi_{[n,n+1]}$. $f_n \in L^1$, since the measure of $[n, n+1]$ is 1 for all $n \ge 1$. Moreover, $f_n \to 0$ pointwise. But if we let $g \equiv 1 \in L^\infty$, then

$$\int f_n g = 1 \xrightarrow{n\to\infty} 1 \ne 0 = \int 0 \cdot g,$$

so the conclusion of (a) fails.

Now, assume that $p = \infty$ and μ is a σ-finite measure, which means that the dual of L^1 is L^∞. Let $f_n \to f$ a.e. and $M := \sup_n \|f_n\|_\infty < \infty$.

Since $(L^1)^* = L^\infty$, it suffices to show that $\int f_n g \to \int fg$ for all $g \in L^1$.

Let $g \in L^1$. Notice that $|f_k - f||g| \le 2M|g| \in L^1$. Hence, because $|f_n - f| \to 0$ pointwise, dominated convergence gives us

$$\int |f_n g - fg| = \int |f_n - f||g| \xrightarrow{n\to\infty} 0,$$

so $f_k \to f$ weakly in L^∞.

6.22 Let $X = [0,1]$, with Lebesgue measure.

a. Let $f_n(x) = \cos 2\pi nx$. Then $f_n \to 0$ weakly in L^2, but $f_n \not\to 0$ a.e. or in measure.

b. Let $f_n(x) = n\chi_{(0,1/n)}$. Then $f_n \to 0$ a.e. and in measure, but $f_n \not\to 0$ weakly in L^p for any p.

Solution a. It suffices to show that $\int \varphi f_n \to 0$ for every integrable simple function, since they're dense. Hence, it suffices to show it for step functions. Because the Lebesgue measurable sets differ from a G_δ set by a null set, it further suffices to show this for step functions on an interval. Hence, let $E = (a,b) \subseteq [0,1]$. Then

$$\int_0^1 \chi_E \cos 2\pi nx \, \mathrm{d}x = \int_a^b \cos 2\pi nx \, \mathrm{d}x = \frac{1}{2\pi n}(\sin 2\pi nb - \sin 2\pi na) \xrightarrow{n\to\infty} 0.$$

Hence, $f_n \to 0$ weakly in L^2.

Now suppose $f_n \to 0$ a.e. Notice that $|f_n(x) - f_m(x)| \le 2 \in L^2([0,1])$. Then by the dominated convergence theorem,

$$\int_0^1 (f_{n+1}(x) - f_n(x))^2 \, \mathrm{d}x \xrightarrow{n\to\infty} 0.$$

But by a calculation, (e.g., via WolframAlpha)

$$\int_0^1 (\cos 2\pi(n+1)x - \cos 2\pi nx)^2 \, \mathrm{d}x = 1,$$

which is absurd. So, f_n admits no convergent subsequence.

Moreover, f_n cannot converge in measure, or else f_n has a pointwise convergent subsequence, which contradicts the previous part.

b. The set on which f_n and 0 differ is $(0,1/n)$, whose measure is $1/n$, and this tends to 0, so f_n converges in measure. f_n also converges pointwise everywhere except at the origin.

f_n does not converge to 0 weakly in any L^p, since $1 \in L^p([0,1])$, but

$$\int_0^1 1 \cdot f_n \, \mathrm{d}x = 1 \xrightarrow{n\to\infty} 1 \ne 0.$$

6.26 Complete the proof of Theorem 6.18 for the case $p = 1$.

Solution Assume

$$\int |K(x,y)| \, \mathrm{d}\mu(x) \le C,$$

for some $C > 0$. We need to show that for $f \in L^1(\nu)$,

$$Tf(x) = \int K(x,y) f(y) \, \mathrm{d}\nu(y)$$

converges absolutely for a.e. $x \in X$, that $Tf \in L^1(\mu)$, and that $\|Tf\|_1 \le C\|f\|_1$.

By Tonelli's theorem,

$$\begin{aligned}
\|Tf\|_1 &= \iint |K(x,y)f(y)| \, \mathrm{d}\nu(y) \, \mathrm{d}\mu(x) \\
&= \int \left(\int |K(x,y)| \, \mathrm{d}\mu(x) \right) |f(y)| \, \mathrm{d}\nu(y) \\
&\le C \int |f(y)| \, \mathrm{d}\nu(y) \\
&= C\|f\|_1.
\end{aligned}$$

This also shows that the integral converges absolutely for a.e. x.

6.36 If $f \in \text{weak}\, L^p$ and $\mu(\{x \mid f(x) \neq 0\}) < \infty$, then $f \in L^q$ for all $q < p$. On the other hand, if $f \in (\text{weak}\, L^p) \cap L^\infty$, then $f \in L^q$ for all $q > p$.

Solution Let f be as in the problem, and let $M = \mu(\{x \mid f(x) \neq 0\})$. By definition,

$$[f]_p = \left(\sup_{\alpha>0} \alpha^p \lambda_f(\alpha)\right)^{1/p} < \infty.$$

Also, notice that M is an upper bound for $\lambda_f(\alpha)$, by definition of λ_f. Then for $q < p$,

$$\begin{aligned}
\|f\|_q^q &= \int |f|^q \,\mathrm{d}\mu \\
&= q \int_0^\infty \alpha^{q-1} \lambda_f(\alpha)\,\mathrm{d}\alpha \\
&= q \int_0^\infty \alpha^p \alpha^{q-p-1} \lambda_f(\alpha)\,\mathrm{d}\alpha \\
&= q\left(\int_0^1 \alpha^p \alpha^{q-p-1} \lambda_f(\alpha)\,\mathrm{d}\alpha + \int_1^\infty \alpha^p \alpha^{q-p-1} \lambda_f(\alpha)\,\mathrm{d}\alpha\right) \\
&\leq q\left(\int_0^1 \lambda_f(\alpha)\,\mathrm{d}\alpha + \int_1^\infty \alpha^{q-p-1} (\alpha^p \lambda_f(\alpha))\,\mathrm{d}\alpha\right) \\
&\leq q\left(M + [f]_p^p \int_1^\infty \alpha^{q-p-1}\,\mathrm{d}\alpha\right) \\
&< \infty.
\end{aligned}$$

Indeed, the integral converges because $q - p - 1 < -1$.

On the other hand, let $f \in (\text{weak}\, L^p) \cap L^\infty$, and let $q > p$. Since $f \in L^\infty$, there exists $\alpha_0 > 0$ so that $\lambda_f(\alpha) = 0$ for all $\alpha > \alpha_0$. Then by the same calculation as above,

$$\begin{aligned}
\|f\|_q^q &= q\left(\int_0^1 \alpha^p \alpha^{q-p-1} \lambda_f(\alpha)\,\mathrm{d}\alpha + \int_1^\infty \alpha^p \alpha^{q-p-1} \lambda_f(\alpha)\,\mathrm{d}\alpha\right) \\
&= q\left([f]_p^p \int_0^1 \alpha^{q-p-1}\,\mathrm{d}\alpha + \int_1^{\alpha_0} \alpha^p \alpha^{q-p-1} \lambda_f(\alpha)\,\mathrm{d}\alpha\right) \\
&< \infty.
\end{aligned}$$

The left integral convergences because $q - p - 1 > -1$, and the right integral converges because α^{q-1} is continuous on $[1, \alpha_0]$, hence integrable. Thus, $f \in L^q$.