1 Let $\{A_i\}_{i\in I}$ be an infinite family of closed sets with the finite intersection property. Assuming that one member of this family is compact, show that $\bigcap_{i\in I} A_i \neq \emptyset$.

Solution Let $i_0 \in I$ be such that A_{i_0} is compact. Suppose $\bigcap_{i\in I} A_i = \emptyset$. Then $\bigcup_{i\in I} A_i^c = X$. Then as each A_i is closed, $\{A_i^c\}_{i\in I}$ is an open cover for X. In particular, it is an open cover for A_{i_0}. As A_{i_0} is compact, it admits a finite subcover. So, there exists $i_1, i_2, \ldots, i_n$ such that

$$A_{i_0} \subseteq \bigcup_{j=0}^{n} A_{i_j}^c \implies X \setminus A_{i_0} = \bigcap_{j=0}^{n} A_{i_j}.$$

Intersecting both sides with A_{i_0} yields

$$\emptyset = \bigcap_{j=0}^{n} A_{i_j}$$

which is a contradiction. Hence, $\bigcap_{i\in I} A_i \neq \emptyset$.

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2 Let (X, d) be a metric space and let $A \subseteq X$ be a compact subset. Show that

a. For any $y \in X$ there exists $x \in A$ so that $d(y, A) = d(y, x)$.

b. If $B \subseteq X$ and $d(A, B) = 0$ then $A \cap \bar{B} = \emptyset$.

Solution a. Fix $y \in X$. Then define $D \coloneqq d(y, A) = \inf\{d(x, y) \mid x \in A\}$.

Note that $D + 1$ is not a lower bound for $\{d(x, y) \mid x \in A\}$. Hence, there exists $x_1 \in A$ such that $D \leq d(x_1, y) \leq D + 1$.

If $d(x_1, y) = D$, then we're done. Otherwise, $d(x_1, y)$ is not a lower bound for $\{d(x, y) \mid x \in A\}$, so there exists $x_2 \in A$ such that $D \leq d(x_2, y) \leq \min\{d(x_1, y), D + \frac{1}{2}\}$.

Proceeding inductively yields a sequence $\{x_n\}_{n\geq 1} \subseteq A$ with $D \leq d(x_n, y) \leq D + \frac{1}{n}$. For any $m \geq n$, we have that

$$d(x_n, x_m) \leq d(x_n, y) + d(x_m, y) \leq \frac{2}{n}.$$

Then clearly, $\{x_n\}_{n\geq 1}$ is Cauchy. Since A is compact, it is complete, so $x_n \xrightarrow{n\to\infty} x \in A$. Thus, taking $n \to \infty$ in the inequality $D \leq d(x_n, y) \leq D + \frac{1}{n}$ yields

$$d(x, y) = d(y, A)$$

as desired.

b. Suppose $A \cap \bar{B} \neq \emptyset$. Then there exists $x \in A \cap \bar{B}$. As $x \in \bar{B}$, $\exists \{x_n\}_{n\geq 1} \subseteq B$ with $x_n \xrightarrow{n\to\infty} x$. Then for all $\varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon$,

$$0 \leq d(x, x_n) < \varepsilon.$$

Then clearly, $d(A, B) = \inf\{d(a, b) \mid a \in A \text{ and } b \in B\} = 0$. Otherwise, if $d(A, B) > 0$, then we can pick $\varepsilon < d(A, B)$, which gives us infinitely many x_n such that $d(x, x_n) < d(A, B)$.

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3 Let (X, d_X) be a compact metric space.

a. Verify that

$$d_Y(f, g) = \sum_{n\in\mathbb{Z}} 2^{-|n|} d_X(f(n), g(n))$$

defines a metric on $Y = \{f : \mathbb{Z} \to X\}$.

b. Show that Y is compact.

Solution a. $2^{-|n|} > 0$ and $d_X(f(n), g(n)) \geq 0$, so $d_Y(f, g) \geq 0 \; \forall f, g$.
If $f = g$, $d_X(f(n), g(n)) = 0 \implies d_Y(f, g) = 0$. If $d_Y(f, g) = 0$, then we must have that $d_X(f(n), g(n)) = 0$. Otherwise, the sum will be greater than 0. So, $f = g$. Thus, $f = g \iff d_Y(f, g) = 0$.
$d_Y(f, g) = d_Y(g, f)$ trivially, since d_X is a metric.

$$\begin{aligned} d_Y(f, h) + d_Y(h, g) &= \sum_{n\in\mathbb{Z}} 2^{-|n|} d_X(f(n), h(n)) + \sum_{n\in\mathbb{Z}} 2^{-|n|} d_X(h(n), g(n)) \\ &\geq \sum_{n\in\mathbb{Z}} 2^{-|n|} d_X(f(n), g(n)) \\ &= d_Y(f, g) \end{aligned}$$

b. Let $\{f_k\}_{k\geq1} \subseteq Y$. Then as X is compact, the sequence $\{f_k(0)\}_{k\geq1} \subseteq X$ admits a convergent subsequence. Call this subsequence $\{f_k^{(1)}(0)\}_{k\geq1}$ and let its limit as $k \to \infty$ be $f(0)$.
Next, consider $\{f_k^{(1)}\}_{k\geq1} \subseteq Y$. As X is compact, $\{f_k^{(1)}(1)\}_{k\geq1}$ admits a convergent subsequence. Call it $\{f_k^{(2)}\}_{k\geq1}$. We call its limit as $k \to \infty$ $f(1)$. Note that this is the same f as from the previous paragraph, since $f_k^{(2)}(0)$ is a subsequence of the previous convergent subsequence.
Since $\mathbb{Z}$ is countable, we can repeat this process for all $n \in \mathbb{Z}$ through induction, which gives us a subsequence $\{f_{m_k}\}_{k\geq1}$ of $\{f_k\}_{k\geq1}$ with $\lim_{k\to\infty} f_{m_k} = f$, and $f(n)$ is defined for all $n \in \mathbb{Z}$. Hence, every sequence of Y admits a convergent subsequence, so Y is sequentially compact $\implies$ Y is compact.

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4 a. Show that the closed unit ball in ℓ^2, namely,

$$A = \{x \in \ell^2 \mid \sum_{n=1}^{\infty} |x_n|^2 \leq 1\}$$

is not compact in ℓ^2.

b. Define $B \subseteq \ell^2$ by

$$B = \{x \in \ell^2 \mid \sum_{n=1}^{\infty} n|x_n|^2 \leq 1\}.$$

Show that B is compact.

Solution a. Let $\{x^{(n)}\}_{n\geq1} \subseteq A$, where $x_i^{(n)} = \begin{cases} 1 & i = n \\ 0 & i \neq n \end{cases}$. We have $d(x^{(n)}, x^{(m)}) = \sqrt{2} \; \forall m \neq n$. Clearly, $\{x^{(n)}\}_{n\geq1}$ does not admit a convergent subsequence, so A is not compact.

b. Let $\{x^{(n)}\}_{n\geq1} \subseteq B$. As $\sum_{n=1}^{\infty} n|x_n^{(k)}|^2 \leq 1$, we have that $n|x_n|^2 \leq 1 \implies |x_n^{(k)}| \leq \frac{1}{\sqrt{n}}$.
Consider $\{x_1^{(k)}\}_{k\geq1} \subseteq \mathbb{R}$. Thus, it admits a convergent subsequence, which we will denote as $\{x_1^{(1_k)}\}_{k\geq1}$. We will also call its limit x_1. Note that all the terms of this sequence satisfy $|x_1^{(1_k)}| \leq 1$.
Consider $\{x_2^{(1_k)}\}_{k\geq1} \subseteq \mathbb{R}$. Thus, it admits a convergent subsequence, which we will denote as $\{x_2^{(2_k)}\}_{k\geq1}$. We will also call its limit x_2. Note that all the terms of this sequence satisfy $|x_1^{(1_k)}| \leq \frac{1}{\sqrt{2}}$.
We proceed inductively to attain a subsequence $\{x^{(l_k)}\}_{k\geq1}$ with $|x_n^{(l_k)}| \leq \frac{1}{\sqrt{n}}$. Moreover, this subsequence converges to $x = \{x_n\}_{n\geq1}$. We will now show that x belongs to B.

$$\sum_{n=1}^{\infty} n|x_n^{(k)}|^2 \leq 1$$

Taking $k \to \infty$ yields

$$\sum_{n=1}^{\infty} n|x_n|^2 \leq 1 \iff x \in B.$$

Thus, every sequence in B admits a convergent subsequence, so B is compact by Heine–Borel.

5 Let A be a subset of a complete metric space. Assume that for all $\varepsilon > 0$, there exists a compact set A_ε so that

$$\forall x \in A, \quad d(x, A_\varepsilon) < \varepsilon.$$

Show that $\bar{A}$ is compact.

Solution Since $\bar{A}$ is closed and is a subset of a complete metric space, $\bar{A}$ is also complete. All that is left to show is that $\bar{A}$ is totally bounded.

By problem (2), for every $x \in A$, there exists $y \in A_\varepsilon$ such that $d(x, y) = d(x, A_\varepsilon)$.

Let $\varepsilon > 0$. Then as $A_{\varepsilon/3}$ is compact, it is totally bounded, so there exists a finite collection of open balls $\{G_i\}_{i \in I}$ of radius $\frac{\varepsilon}{3}$ which cover $A_{\varepsilon/2}$. Take these balls and triple their radii so that they form the collection $\{H_i\}_{i \in I}$ of radius ε. In particular, if $x \in G_i$, then $B_{2\varepsilon/3}(x) \subseteq H_i$.

Let $x \in \bar{A}$. Then $B_{\varepsilon/3}(x) \cap A \neq \emptyset$, so pick a point y in the intersection. Then as $y \in A$, there exists $z \in A_{\varepsilon/3}$ such that $d(y, z) = d(y, A_{\varepsilon/3}) < \frac{\epsilon}{3}$. By the triangle inequality, $d(x, z) \leq d(x, y) + d(y, z) < \frac{2\varepsilon}{3}$. Hence, there exists $i \in I$ such that $x \in B_{2\varepsilon/3} \subseteq H_i$.

Hence $\{H_i\}_{i \in I}$ is a finite collection of open balls of radius ϵ which cover $\bar{A}$, so $\bar{A}$ is totally bounded. By the Heine–Borel theorem, $\bar{A}$ is compact.

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6 Let (X, d_1) and (Y, d_2) be two compact metric spaces. Show that the space $X \times Y$ endowed with the 'Euclidean' distance

$$d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{d_1^2(x_1, x_2) + d_2^2(y_1, y_2)}$$

is a compact metric space.

Solution By a proposition proved in class, $(X \times Y, d)$ is indeed a metric space. We will now show that it is compact.

Let $\{a_n\}_{n \geq 1} \subseteq X \times Y$ be such that $a_n = (x_n, y_n)$ where $\{x_n\}_{n \geq 1} \subseteq X$ and $\{y_n\}_{n \geq 1} \subseteq Y$. Then as (X, d_1) is compact, $\{x_n\}_{n \geq 1}$ admits a convergent subsequence $\{x_{k_n}\}_{n \geq 1}$ which converges to $x \in X$. Similarly, $\{y_{k_n}\}_{n \geq 1}$ admits a convergent subsequence $\{y_{k_{l_n}}\}_{n \geq 1}$ which converges to $y \in Y$. As a shorthand, we will write l_n instead of k_{l_n}.

We claim that $\{a_{l_n}\}_{n \geq 1}$ is a convergent subsequence of $\{a_n\}_{n \geq 1}$ which converges to (x, y). Fix $\varepsilon > 0$. As $x_{l_n} \xrightarrow{n \to \infty} x$ and $y_{l_n} \xrightarrow{n \to \infty} y$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$,

$$\left.\begin{aligned} d_1(x_n, x) &< \frac{\epsilon}{2} \\ d_2(y_n, y) &< \frac{\epsilon}{2} \end{aligned}\right\} \implies d\big((x_n, y_n), (x, y)\big) < \sqrt{\frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{4}} < \varepsilon$$

Thus, by Heine–Borel, $(X \times Y, d)$ is a compact metric space.

7 Consider the Cantor set

$$K = \left\{ x \in [0,1] \mid x = \sum_{n=1}^{\infty} a_n 3^{-n} \text{ with all } a_n \in \{0,2\} \right\}.$$

For example, $1 \in K$ because it is represented by setting all $a_n = 2$.

a. Show that K is compact.

b. Show that K is uncountable.

c. Show that no connected subset of K contains more than one point.

Solution a. Let $\{x_n\}_{n\geq 1} \subseteq K \subseteq \mathbb{R}$. By Bolzano–Weierstrass, $\{x_n\}_{n\geq 1}$ admits a convergent subsequence. Hence, K is compact by Heine–Borel.

b. Suppose K is countable. Then we can order the elements of K, so that we have

$$\begin{aligned} x_1 &= \sum_{n=1}^{\infty} a_n^{(1)} 3^{-n} \\ x_2 &= \sum_{n=1}^{\infty} a_n^{(2)} 3^{-n} \\ &\vdots \end{aligned}$$

Then consider $x = \sum_{n=1}^{\infty} a_n 3^{-n}$, where

$$a_n = \begin{cases} 0 & \text{if } a_n^{(n)} = 2 \\ 2 & \text{if } a_n^{(n)} = 0 \end{cases}$$

This is a different set of coefficients from the others as it does not match the n-th term of the sequence of coefficients of x_n for all n. Hence, $x \neq x_n$ for all $n \geq 1$, but $x \in K$, which is a contradiction. So, K is uncountable.

c. Let A be a connected subset of K. Assume that $\{x, y\} \subseteq A$ where $y > x$. Then since the only connected subsets of $\mathbb{R}$ are intervals, we must have that $a \in A$ for all a such that $x < a < y$. In particular, $a = \frac{3x+y}{4}$ must be contained in the set. Then since x and y are convergent series, we have that

$$\frac{3x+y}{4} = \sum_{n=1}^{\infty} \left(\frac{3x_n + y_n}{4} \right) 3^{-n}$$

where $x = \sum_{n=1}^{\infty} x_n 3^{-n}$, $y = \sum_{n=1}^{\infty} y_n 3^{-n}$, and all $x_n, y_n \in \{0,2\}$. Since $y > x$, then there exists $i \geq 1$ such that $\frac{3x_n+y_n}{4} = \frac{0+2}{4} = \frac{1}{2}$, so a cannot be in K. Hence, the only connected subsets of K contain only one point.