10 If $f \in \mathbb{Q}[t]$ and K is a splitting field of f over $\mathbb{Q}$, determine $[K : \mathbb{Q}]$ if f is:

a. $t^4 + 1$

b. $t^6 + 1$

c. $t^4 - 2$

d. $t^6 - 2$

e. $t^6 + t^3 + 1$

Solution a. $t^4 + 1$ has roots $e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4}$, which we denote as $\zeta_1, \zeta_2, \zeta_3, \zeta_4$. Notice that

$$\zeta_1 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2},\ \zeta_2 = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2},\ \ldots.$$

If $\alpha = \sqrt{2}$ and $\beta = \sqrt{2}i$, we claim that $K = \mathbb{Q}(\zeta_1, \ldots, \zeta_4) = \mathbb{Q}(\alpha, \beta)$.
It's clear that $\mathbb{Q}(\zeta_1, \ldots, \zeta_4) \subseteq \mathbb{Q}(\alpha, \beta)$. On the other hand,

$$\alpha = \zeta_1 - \zeta_2 \quad \text{and} \quad \beta = \zeta_1 + \zeta_2,$$

so $\mathbb{Q}(\alpha, \beta) \subseteq \mathbb{Q}(\zeta_1, \ldots, \zeta_4)$.
Next, notice that $\{1, \beta\}$ is linearly independent in $\mathbb{Q}(\alpha, \beta)/\mathbb{Q}(\alpha)$: if we have nonzero $a, b \in \mathbb{Q}(\alpha)$ such that

$$a + b\beta = 0 \implies \sqrt{2} = i\frac{a}{b}.$$

But $a, b \in \mathbb{R}$, so this is impossible. Hence, $\{1, \beta\}$ is a basis for $\mathbb{Q}(\alpha, \beta)/\mathbb{Q}(\alpha)$. Also, it's clear that $\{1, \alpha\}$ is a basis for $\mathbb{Q}(\alpha)/\mathbb{Q}$, so

$$[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] = [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)][Q(\alpha) : \mathbb{Q}] = 2 \cdot 2 = 4.$$

b. The roots of f are $e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}$, which we'll label as $\zeta_1, \ldots, \zeta_6$.
We have

$$\zeta_1 = \frac{\sqrt{3}}{2} + \frac{i}{2},\ \zeta_2 = i,\ \zeta_3 = -\frac{\sqrt{3}}{2} + \frac{i}{2},\ \zeta_4 = -\frac{\sqrt{3}}{2} - \frac{i}{2},\ \zeta_5 = -i,\ \zeta_6 = \frac{\sqrt{3}}{2} - \frac{i}{2}.$$

We claim that $\mathbb{Q}(\zeta_1, \ldots, \zeta_6) = \mathbb{Q}(\sqrt{3}, i)$.
It's clear that $\mathbb{Q}(\zeta_1, \ldots, \zeta_6) \subseteq \mathbb{Q}(\sqrt{3}, i)$ over $\mathbb{Q}$. On the other hand,

$$\sqrt{3} = \zeta_1 - \zeta_3 \quad \text{and} \quad i = \zeta_1 + \zeta_3,$$

so because we're working with vector spaces, $\mathbb{Q}(\sqrt{3}, i) = \mathbb{Q}(\zeta_1, \ldots, \zeta_6)$.
It's clear that 1 and i are linearly independent over $\mathbb{Q}(\sqrt{3})$, since $1, \sqrt{3} \in \mathbb{R}$ but $i \in \mathbb{C}$. Hence, $\{1, i\}$ is a basis for $\mathbb{Q}(\sqrt{3}, i)/\mathbb{Q}(\sqrt{3})$. Thus, because $\{1, \sqrt{3}\}$ is a basis for $\mathbb{Q}(\sqrt{3})/\mathbb{Q}$

$$\Big[\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}\Big] = \Big[\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}(\sqrt{3})\Big]\Big[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}\Big] = 2 \cdot 2 = 4.$$

c. Let $\alpha = \sqrt[4]{2}$, and ζ_i be as in (a). Then the roots of f are

$$\alpha,\ \alpha\zeta_1,\ \alpha\zeta_2,\ \alpha\zeta_3,\ \alpha\zeta_4.$$

Hence, the splitting field of f is $\mathbb{Q}(\alpha, \zeta_1, \ldots, \zeta_4)$. From (a), we know that $[\mathbb{Q}(\zeta_1, \ldots, \zeta_4) : \mathbb{Q}] = 2$. Moreover, $t^4 - 2$ is irreducible by Eisenstein, so $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$.
We claim that $\{1, \sqrt{2}i\}$ is linearly independent over $\mathbb{Q}(\alpha)$. If there were $a, b \in \mathbb{Q}(\alpha)$ so that $a + b\sqrt{2}i = 0$, then

$$\sqrt{2}i = -\frac{a}{b} \in \mathbb{Q}(\alpha) \subseteq \mathbb{R},$$

which is impossible. Thus, $\{1, \sqrt{2}i\}$ is a basis for $\mathbb{Q}(\zeta_1, \ldots, \zeta_4)/\mathbb{Q}(\alpha)$, since $\sqrt{2} \in \mathbb{Q}(\alpha)$ and so,

$$[\mathbb{Q}(\alpha, \zeta_1, \ldots, \zeta_4) : \mathbb{Q}] = [\mathbb{Q}(\alpha, \zeta_1, \ldots, \zeta_4) : \mathbb{Q}(\alpha)][\mathbb{Q}(\alpha) : \mathbb{Q}] = 2 \cdot 4 = 8.$$

d. Let $\alpha = \sqrt[6]{2}$ and ζ_i be as in (b). By the same argument as the above, the splitting field of f is $\mathbb{Q}(\alpha, \zeta_1, \ldots, \zeta_6)$, and

$$[\mathbb{Q}(\alpha, \zeta_1, \ldots, \zeta_6) : \mathbb{Q}] = [\mathbb{Q}(\alpha, \zeta_1, \ldots, \zeta_6) : \mathbb{Q}(\alpha)][\mathbb{Q}(\alpha) : \mathbb{Q}] = 2 \cdot 6 = 12.$$

e. Notice that f is irreducible. Indeed, we can write $f = (t^3)^2 + t^3 + 1$, so f is a quadratic in $\mathbb{Q}(t^3)$. Thus, it must factor into linear terms in t^3. By the rational root theorem, the only possible roots are 1 and -1, but substituting them for t^3 does not give 0, so f is irreducible.

Next, note that $(t^3 - 1)f = t^9 - 1$, so the roots of f must be $\zeta, \zeta^2, \zeta^4, \zeta^5, \zeta^7, \zeta^8$, where $\zeta = e^{2\pi i/9}$. Indeed, $1, \zeta^3, \zeta^6$ are all roots of $t^3 - 1$.

Thus, the splitting field of f is $\mathbb{Q}(\zeta)$. Because f was irreducible, f is the minimal polynomial for ζ, so $[\mathbb{Q}(\zeta) : \mathbb{Q}] = 6$.

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11 Find the splitting fields K for $f \in \mathbb{Q}[t]$ and $[K : \mathbb{Q}]$ if f is:

a. $t^4 - 5t^2 + 6$

b. $t^6 - 1$

c. $t^6 - 8$

Solution a. f factors as $f = (t^2 - 3)(t^2 - 2)$. So, the splitting field is $\mathbb{Q}(\sqrt{3}, \sqrt{2})$.

It's clear that $\{1, \sqrt{2}\}$ is a basis for $\mathbb{Q}(\sqrt{2})$.

$\sqrt{2}$ and $\sqrt{3}$ are linearly independent because they are distinct primes, so 1 and $\sqrt{3}$ are linearly independent over $\mathbb{Q}(\sqrt{2})$. Hence, $\{1, \sqrt{3}\}$ is a basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}(\sqrt{2})$, so

$$\left[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}\right] = \left[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})\right]\left[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}\right] = 2 \cdot 2 = 4.$$

b. f factors as $(t-1)(t^5 + t^4 + t^3 + t^2 + t + 1)$, so the roots of f are $1, \zeta, \zeta^2, \ldots, \zeta^5$, where $\zeta = e^{2\pi i/6}$. Since $1 \in \mathbb{Q}$, the splitting field of f is $\mathbb{Q}(\zeta)$. Next, we have

$$\zeta = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \ \zeta^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \ \ldots.$$

Let $\beta = \sqrt{3}i$. We claim that $\mathbb{Q}(\zeta) = \mathbb{Q}(\beta)$. It's clear that $\mathbb{Q}(\zeta) \subseteq \mathbb{Q}(\beta)$. However,

$$\beta = \zeta + \zeta^2 \quad \text{and} \quad 1 = \zeta - \zeta^2,$$

so $\mathbb{Q}(\beta) \subseteq \mathbb{Q}(\zeta)$. Hence, $\mathbb{Q}(\beta) = \mathbb{Q}(\zeta)$, and $[\mathbb{Q}(\beta) : \mathbb{Q}] = 2$, since $\{1, \beta\}$ is a basis.

c. Let $\alpha = \sqrt[6]{8} = \sqrt{2}$, and let $\zeta = e^{2\pi i/6}$. Notice that $\zeta, \alpha\zeta, \ldots, \alpha\zeta^5$ are distinct roots of f, so the splitting field of f is $\mathbb{Q}(\zeta, \alpha)$.

From the previous part, if $\beta = \sqrt{3}i$, then $\mathbb{Q}(\beta) = \mathbb{Q}(\zeta)$. A basis of $\mathbb{Q}(\alpha)$ is $\{1, \alpha\}$, since α is irrational and $\alpha^2 = 2$.

Notice that $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] \leq 4$, since $\{1, \alpha, \beta, \alpha\beta\}$ spans the set. On the other hand,

$$[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] = [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)][\mathbb{Q}(\alpha) : \mathbb{Q}] = 2[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)].$$

But $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)] > 1$, since 1 and β are linearly independent. Hence, it must be that $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] = 4$.

12 Let $F = \mathbb{Z}/p\mathbb{Z}$. Then show:

a. There exists $f \in F[t]$ with $\deg f = 2$ and f irreducible.

b. Use the f in (a) to construct a field with p^2 elements.

c. If $f_1, f_2 \in F[t]$ have $\deg f_i = 2$ and f_i irreducible for $i = 1, 2$, show that their splitting fields are isomorphic.

Solution a. For $p = 2$, $f(t) = t^2 + t + 1$ works: $f(0) = 1$ and $f(1) = 1$, so f has no roots in F.

For $p > 2$, notice that there are p^2 polynomials of degree 2 with leading coefficient 1, but there are

$$\binom{p}{2} = \frac{p(p+1)}{2}$$

reducible ones. Hence there F has an irreducible polynomial with degree 2.

b. $K = F[t]/(f)$ does the trick. The elements of K are

$$K = \{a + bt \mid a, b \in F\},$$

so $|K| = p^2$. Moreover, (f) is a maximal ideal, so K is a field, as required.

c. Notice that K is the splitting field of f. Hence, $K_1 = F[t]/(f_1)$ and $K_2 = F[t]/(f_2)$ are the splitting fields of f_1 and f_2, respectively. Then the homomorphism given by $K_1 \ni \overline{1}_1, \overline{t}_1 \mapsto \overline{1}_2, \overline{t}_2 \in K_2$ is an isomorphism.

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13 Let K/F and $f \in F[t]$. Show the following:

a. If $\varphi\colon K \to K$ is an F-automorphism, then φ takes roots of f in K to roots of f in K.

b. If $F \subseteq \mathbb{R}$ and $\alpha = a + ib$ is a root of f with $a, b \in \mathbb{R}$, then $\overline{\alpha} = a - ib$ is also a root of f.

c. Let $F = \mathbb{Q}$. If $m \in \mathbb{Z}$ is not a square and $a + b\sqrt{m} \in \mathbb{C}$ is a root of f with $a, b \in \mathbb{Q}$, then $a - b\sqrt{m}$ is also a root of f in $\mathbb{C}$.

Solution a. Let $\alpha \in K$ be a root of $f(t) = a_n t^n + \cdots + a_1 t + a_0$, where $a_n \in F$. Then

$$f(\alpha) = a_n \alpha^n + \cdots + a_1 \alpha + a_0 = 0.$$

Applying φ,

$$\begin{aligned} 0 = \varphi(a_n \alpha^n + \cdots + a_1 \alpha + a_0) &= \varphi(a_n)\varphi(\alpha)^n + \cdots + \varphi(a_1)\varphi(\alpha) + \varphi(a_0) \\ &= a_n \varphi(\alpha)^n + \cdots + a_1 \varphi(\alpha) + a_0, \end{aligned}$$

so $\varphi(\alpha)$ is a root of f.

b. Set $\varphi\colon \mathbb{C} \to \mathbb{C}$, $\varphi(a + ib) = a - ib$. Then φ is an $\mathbb{R}$-automorphism, and by (a), $a - ib$ is a root of f.

c. The map $\mathbb{R} \ni a + b\sqrt{m} \mapsto a - b\sqrt{m} \in \mathbb{R}$ is clearly a $\mathbb{Q}$-automorphism. By (a), $a - b\sqrt{m}$ must then be a root of f.

14 Prove any field automorphism $\varphi\colon \mathbb{R} \to \mathbb{R}$ is the identity automorphism.

Solution φ must satisfy $\varphi(0) = 0$ and $\varphi(1) = 1$. Then $\varphi(n) = n$ for all $n \in \mathbb{Z}$. But if $a, b \in \mathbb{Z}$ with $b \neq 0$, we then have

$$a = \varphi(a) = \varphi\Big(\frac{a}{b} \cdot b\Big) = \varphi\Big(\frac{a}{b}\Big)\varphi(b) = \varphi\Big(\frac{a}{b}\Big)b \implies = \varphi\Big(\frac{a}{b}\Big) = \frac{a}{b}.$$

So, φ is the identity on $\mathbb{Q}$.

Next, if $0 < x \in \mathbb{R}$, then $\sqrt{x} \in \mathbb{R}$. Hence,

$$\varphi(x) = \varphi((\sqrt{x})^2) = \varphi(\sqrt{x})^2 > 0.$$

Thus, if $x < y \in \mathbb{R}$, then $0 < y - x$, so $\varphi(y - x) > 0 \implies \varphi(y) > \varphi(x)$, so φ is monotone.

Finally, assume that φ is not the identity on $\mathbb{R}$, so that there exists $x \in \mathbb{R}$ with $\varphi(x) \neq x$. Assume without loss of generality that $\varphi(x) > x$. Then pick $r \in \mathbb{Q}$ so that $x < r < \varphi(x)$. Since φ is monotone,

$$\varphi(x) < \varphi(r) = r,$$

but we assumed that $r < \varphi(x)$, a contradiction. Thus, φ must be the identity automorphism.

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15 Let $p_1, \dots, p_n$ be n distinct prime numbers. Let $f = (t^2 - p_1) \cdots (t^2 - p_n) \in \mathbb{Q}[t]$. Show that $K = \mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_n})$ is a splitting field of f over $\mathbb{Q}$ and $[K : \mathbb{Q}] = 2^n$. Formulate a generalization of the statement for which your proof still works.

Solution $-\sqrt{p_1}, \sqrt{p_1}, \dots, -\sqrt{p_n}, \sqrt{p_n}$ are all distinct roots of f, and f is a polynomial of degree $2n$, so they must be all the roots of f. Hence, K must be a splitting field of f.

We will show that 1 and $\sqrt{p_n}$ are linearly independent in $\mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_n})/\mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_{n-1}})$ by induction on n.

Base step: $n = 1$

Since p_1 is prime, $\sqrt{p_1}$ must be irrational, and it follows immediately that 1 and $\sqrt{p_1}$ are linearly independent.

Inductive step:

Suppose the claim holds for n primes. We wish to show it holds for $n + 1$ primes.

Let $a, b \in \mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_n})$ be such that $a + b\sqrt{p_{n+1}} = 0$. Then

$$a + b\sqrt{p_{n+1}} = 0 \implies a^2 = b^2 p_{n+1}.$$

Since p_{n+1} is rational, this is an equation in $\mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_n})/\mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_{n-1}})$, if we factor the terms appropriately. But by the inductive hypothesis, all the coefficients must be 0, and hence $a = b = 0$. Thus, 1 and $\sqrt{p_{n+1}}$ are linearly independent, as required.

Because each p_k is prime, $t^2 - p_k$ is irreducible by Eisenstein, so it is the minimal polynomial of p_k over $\mathbb{Q}$.

Notice that for each $k \geq 1$, $t^2 - p_k$ is still the minimal polynomial for p_k over $\mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_{k-1}})$, since $t^2 - p_k = (t - \sqrt{p_k})(t + \sqrt{p_k})$, and $\sqrt{p_k} \notin \mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_{k-1}})$.

Thus, $\big[\mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_k}) : \mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_{k-1}})\big] = 2$ for all $1 \leq k \leq n$, so if $K_k = \mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_k})$ with $K_0 = \mathbb{Q}$,

$$[\mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_n}) : \mathbb{Q}] = \prod_{1 \leq k \leq n} [K_n : K_{n-1}] = 2^n,$$

as required.

Here is a generalization of the statement:

Let $p_1, \dots, p_n$ be n distinct prime numbers. Let $f = (t^{k_1} - p_1) \cdots (t^{k_n} - p_n) \in \mathbb{Q}(t)$. Then

$$K = \mathbb{Q}(\sqrt[k_1]{p_1}, \dots, \sqrt[k_n]{p_n})$$

is a splitting field for f over $\mathbb{Q}$ and

$$[K : \mathbb{Q}] = k_1 k_2 \cdots k_n.$$

16 Find a splitting field of $f \in F[t]$ if $F = \mathbb{Z}/p\mathbb{Z}$ and $f = t^{p^e} - t$, $e > 0$.

Solution If $p = 2$, then f splits in F. So, assume that p is odd from now on.

Let S be the set of roots of f. Then $|S| \le p^e$, since $\deg f = p^e$. Notice that S contains F: $F^\times$ is a multiplicative group with $p-1$ elements, so

$$n^{p^e} - n = n^{p^{e-1}} \cdot (n^{p-1})^{p^{e-1}} - n = n^{p^{e-1}} - n = \cdots = n^{p^{e-e}} - n = n - n = 0,$$

so $n \in S$.

We claim that S is a field:

It's clear that 0 and 1 are roots. If $\alpha \in S$, then $-\alpha \in S$ also, since p is odd, which means p^e, so f is odd. For α^{-1}, notice that $\alpha^{p^e} = \alpha$, so that

$$\alpha^{-p^e} - \alpha^{-1} = \left(\alpha^{p^e}\right)^{-1} - \alpha^{-1} = \alpha^{-1} - \alpha^{-1} = 0.$$

We just need to show that the roots are closed under addition and multiplication.

Let $\alpha, \beta \in S$. Then because F has characteristic p,

$$\begin{aligned}(\alpha+\beta)^{p^e} - (\alpha+\beta) = [(\alpha+\beta)^p]^{p^{e-1}} - (\alpha+\beta) &= (\alpha^p + \beta^p)^{p^{e-1}} - (\alpha+\beta) \\ &\vdots \\ &= \left(\alpha^{p^e} + \beta^{p^e}\right)^{p^{e-e}} - (\alpha+\beta) \\ &= 0,\end{aligned}$$

so $\alpha + \beta \in S$. On the other hand, we know $\alpha^{p^e} = \alpha$ and $\beta^{p^e} = \beta$, so

$$(\alpha\beta)^{p^e} - \alpha\beta = \alpha^{p^e}\beta^{p^e} - \alpha\beta = \alpha\beta - \alpha\beta = 0.$$

Thus, $\alpha\beta \in S$, so S is a field, and f splits over S.

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17 Let F be a field of characteristic $p > 0$. Show that $f = t^4 + 1 \in F[t]$ is not irreducible. Let K be a splitting field of f over F. Determine which finite field F must contain so that $K = F$.

Solution We can embed $\mathbb{Z}/p\mathbb{Z}$ into F with the homomorphism $n \mapsto 1 + 1 + \cdots + 1$, n times. Since F is a field, p must be prime.

If $p = 2$, then 1 is a root of f. F just needs to contain $\mathbb{Z}/2\mathbb{Z}$ so that $F = K$.

If $p > 2$, then notice that $p^2 - 1 = (p-1)(p+1)$. Thus, these are two consecutive even numbers, so one must be divisible by 2 and the other by 4, so $8 \mid p^2 - 1$.

Now consider the multiplicative group of $K = \mathbb{Z}/p^2\mathbb{Z}$, which has $p^2 - 1$ elements. $K^\times$ is cyclic, and $8 \mid p^2 - 1$, so K has an element α of order 8, so α^4 has order 2, i.e., α^4 is a root of

$$t^2 - 1 = (t-1)(t+1).$$

Hence, $\alpha^4 = -1$, since -1 is the only root with order 2.

Next, notice that $[K : F] = 2$. Indeed, let $\beta \in K \setminus F$. Then $\{1, \beta\}$ is linearly independent over F. Moreover, we have p choices for each coefficient of 1 and β, which means that there are p^2 elements in its span, i.e., $\{1, \beta\}$ is a basis for K/F.

Thus, α is a root of f. If f were irreducible, then f is the minimal polynomial of α over F, which implies that $[F(\alpha) : F] = 4$. But this is impossible, since $F(\alpha) \subseteq K$:

$$2 = [K : F] = [K : F(\alpha)][F(\alpha) : F] \ge 4.$$

Hence, f must be reducible. In this case, F must contain $\mathbb{Z}/p^2\mathbb{Z}$.

18 Let $f = t^6 - 3 \in F[t]$. Construct a splitting field K of f over F and determine $[K : F]$ for each of the cases: $F = \mathbb{Q}, \mathbb{Z}/5\mathbb{Z}$, or $\mathbb{Z}/7\mathbb{Z}$. Do the same thing if f is replaced by $g = t^6 + 3 \in F[t]$.

Solution Let $f = t^6 - 3$.

$F = \mathbb{Q}$:

Let $\alpha = \sqrt[6]{3}$ and $\beta = \sqrt{3}i$. Then by the same argument in problem 11(c), the splitting field of f is given by $\mathbb{Q}(\alpha, \beta)$ and $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)] = 2$. On the other hand, $\{1, \alpha, \dots, \alpha^5\}$ is a basis for $\mathbb{Q}(\alpha)/\mathbb{Q}$, so

$$[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] = [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)][\mathbb{Q}(\alpha) : \mathbb{Q}] = 2 \cdot 6 = 12.$$

Now let $f = t^6 + 3$.

$F = \mathbb{Q}$:

Let $\alpha = \sqrt[6]{3}$ and let $\zeta_1, \dots, \zeta_6$ be the 6-th roots of -1. As in problem 10(b), $\mathbb{Q}(\zeta_1, \dots, \zeta_6) = \mathbb{Q}(\sqrt{3}, i)$. The roots of f are $\alpha\zeta_1,\ \alpha\zeta_2,\ \dots,\ \alpha\zeta_6$, so the splitting field of f is $\mathbb{Q}(\alpha, \sqrt{3}, i)$.

From before, $\left[\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}\right] = 2$. On the other hand, we claim that $\left[\mathbb{Q}(\alpha, \sqrt{3}, i) : \mathbb{Q}(\sqrt{3}, i)\right] = 3$. Indeed, we have

$$\alpha^4 = \sqrt{3}\alpha,$$

so $\{1, \alpha, \alpha^2\}$ is a basis over $\mathbb{Q}(\sqrt{3}, i)$. Thus,

$$\left[\mathbb{Q}(\alpha, \sqrt{3}, i) : \mathbb{Q}\right] = \left[\mathbb{Q}(\alpha, \sqrt{3}, i) : \mathbb{Q}(\sqrt{3}), i\right]\left[\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}\right] = 3 \cdot 2 = 6.$$