**2.1.2** If $V$ is a vector space over the field $F$, verify that

$$(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) = [\alpha_2 + (\alpha_3 + \alpha_1)] + \alpha_4$$

for all vectors $\alpha_1$, $\alpha_2$, $\alpha_3$, and $\alpha_4$ in $V$.

**Solution**

$$\begin{aligned}(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) &= \alpha_1 + (\alpha_2 + \alpha_3) + \alpha_4 \\ &= [\alpha_1 + (\alpha_2 + \alpha_3)] + \alpha_4\end{aligned}$$

---

**2.1.4** Let $V$ be the set of all pairs $(x, y)$ of real numbers, and let $F$ be the field of real numbers. Define

$$\begin{aligned}(x, y) + (x_1, y_1) &= (x + x_1, y + y_1) \\ c(x, y) &= (cx, y).\end{aligned}$$

Is $V$, with these operations, a vector space over the field of real numbers?

**Solution** Yes, $V$ satisfies the axioms of a vector space over $F$ with the given operations.

Addition:

(a) $(x, y) + (x_1 + y_1) = (x + x_1, y + y_1) = (x_1 + x, y_1 + y) = (x_1, y_1) + (x, y)$

(b) $$\begin{aligned}(x, y) + ((x_1, y_1) + (x_2, y_2)) &= (x + (x_1 + x_2), y + (y_1, y_2)) \\ &= ((x + x_1) + x_2, (y + y_1) + y_2) \\ &= ((x, y) + (x_1, y_1)) + (x_2, y_2)\end{aligned}$$

(c) $\mathbf{0} = (0, 0)$ is the identity.

(d) For a given tuple $(x, y)$, $-(x, y) = (-x, -y)$ is its inverse.

Multiplication

(a) $1(x, y) = (1x, y) = (x, y)$

(b) $(c_1c_2)(x, y) = (c_1c_2x, y) = (c_1(c_2x), y) = c_1(c_2(x, y))$

(c) $$\begin{aligned}c((x, y) + (x_1, y_1)) &= c(x + x_1, y + y_1) \\ &= (c(x + x_1), y + y_1) \\ &= (cx + cx_1, y + y_1) \\ &= (cx, y) + (cx_1, y_1) \\ &= c(x, y) + c(x_1, y_1)\end{aligned}$$

(d) $(c + d)(x, y) = ((c + d)x, y) = (cx + dx, y) = (cx, y) + (dx, y) = c(x, y) + d(x, y)$

---

**2.1.5** On $\mathbb{R}^n$, define two operations

$$\begin{aligned}\alpha \oplus \beta &= \alpha - \beta \\ c \cdot \alpha &= -c\alpha.\end{aligned}$$

The operations on the right are the usual ones. Which of the axioms for a vector space are satisfied by $(\mathbb{R}^n, \oplus, \cdot)$?

**Solution** The vector space satisfies (3c), (3d), (4c), and (4d).

**2.1.6** Let $V$ be the set of all complex-valued functions $f$ on the real line such that (for all $t$ in $\mathbb{R}$)

$$f(-t) = \overline{f(t)}.$$

The bar denotes complex conjugation. Show that $V$, with the operations

$$\begin{aligned}(f+g)(t) &= f(t) + g(t)\\ (cf)(t) &= cf(t)\end{aligned}$$

is a vector space over the field of *real* numbers. Give an example of a function in $V$ which is not real-valued.

**Solution** We have the typical vector addition and scalar multiplication, so they will satisfy the axioms (3a) through (4d). We only need to show that $V$ with the given operations over $\mathbb{R}$ are closed under addition and scalar multiplication.

$$(f+g)(-t) = f(-t) + g(-t) = \overline{f(t)} + \overline{g(t)} = \overline{f(t) + g(t)} = \overline{(f+g)(t)}$$
$$(cf)(-t) = cf(-t) = c\overline{f(t)} = \overline{cf(t)} = \overline{(cf)(t)}$$

Thus, over $F$, $V$ is closed under vector addition and scalar multiplication.

---

**2.1.7** Let $V$ be the set of pairs $(x, y)$ of real numbers and let $F$ be the field of real numbers. Define

$$\begin{aligned}(x, y) + (x_1, y_1) &= (x + x_1, 0)\\ c(x, y) &= (cx, 0)\end{aligned}$$

Is $V$, with these operations, a vector space?

**Solution** No, there is no additive identity. We will prove by contradiction: Suppose we have $(x, y) \in V$ such that $y_1 \neq 0$, and that the identity is $\mathbf{0} = (x_1, y_1)$. Then

$$\begin{aligned}(x, y) + \mathbf{0} &= (x, y) + (x_1, y_1)\\ &= (x + x_1, 0)\end{aligned}$$

We have a contradiction since $y$ is non-zero, so there is no identity vector in $V$. Thus $V$ over the real numbers with the given operations do not form a vector space.

---

**2.2.2** Let $V$ be the (real) vector space of all functions $f$ from $\mathbb{R}$ onto $\mathbb{R}$. Which of the following sets of functions are subspaces of $V$?

a. all $f$ such that $f(x^2) = f(x)^2$;
b. all $f$ such that $f(0) = f(1)$;
c. all $f$ such that $f(3) = 1 + f(-5)$;
d. all $f$ such that $f(-1) = 0$;
e. all $f$ which are continuous.

**Solution** a. This set is not closed under addition. Let $f$ and $g$ be in the set described. Then

$$(f+g)(x^2) = f(x^2) + g(x^2) \neq (f+g)(x)^2 = f(x)^2 + 2f(x)g(x) + g(x)^2$$

Thus the set is not a subspace of $V$.

b. This set is a subspace. $f(x) = 0$ is in the set, so it is non-empty. Let $f$ and $g$ be in the set. Then

$$(f+g)(0) = f(0) + g(0) = f(1) + g(1) = (f+g)1 \implies \text{the set is closed under addition}$$

$$(cf)(0) = cf(0) = cf(1) = (cf)(1) \implies \text{the set is closed under scalar multiplication}$$

c. The set is not closed under scalar multiplication. Let $f$ be in the set described. Then

$$(cf)(3) = cf(3) = c(1 + f(-5)) = c + cf(-5) \neq 1 + cf(-5) = 1 + (cf)(-5)$$

Thus the set is not a subspace of $V$.

d. This set is a subspace. $f(x) = 0$ satisfies $f(-1) = 0$. Let $f$ and $g$ be in the set as described. Then

$$(f+g)(-1) = f(-1) + g(-1) = 0 + 0 = 0 \implies \text{the set is closed under addition}$$

$$(cf)(-1) = cf(-1) = c \cdot 0 = 0 \implies \text{the set is closed under scalar multiplication}$$

Thus the set is a subspace of $V$.

e. This set is a subspace. $f(x) = 0$ is continuous, so the set is non-empty. Let $f$ and $g$ be continuous functions. Then

$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = f(a)+g(a) = (f+g)(a) \implies f+g \text{ is continuous}$$

$$\lim_{x\to a}(cf)(x) = \lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = cf(a) = (cf)(a) \implies cf \text{ is continuous}$$

Hence, the set is also closed under both addition and scalar multiplication, so the set is a subspace of $V$.

---

**2.2.4** Let $W$ be the set of all $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ which satisfy

$$\begin{aligned} 2x_1 - x_2 + \tfrac{4}{3}x_3 - x_4 \phantom{- x_5} &= 0 \\ x_1 \phantom{- x_2} + \tfrac{2}{3}x_3 \phantom{- x_4} - x_5 &= 0 \\ 9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 &= 0. \end{aligned}$$

Find a finite set of vectors which spans $W$.

**Solution**

$$\begin{pmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 \\ 9 & -3 & 6 & -3 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & \frac{2}{3} & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

This gives the equations

$$\begin{aligned} x_1 &= -\frac{2}{3}x_3 + x_5 \\ x_2 &= 0 \\ x_4 &= 2x_5 \end{aligned}$$

Thus a tuple that satisfies the equation will be in the form of

$$\begin{aligned} \left(-\frac{2}{3}x_3 + x_5, 0, x_3, 2x_5, x_5\right) &= \left(-\frac{2}{3}x_3, 0, x_3, 0, 0\right) + \left(x_5, 0, 0, 2x_5, x_5\right) \\ &= x_3\left(-\frac{2}{3}, 0, 1, 0, 0\right) + x_5\left(1, 0, 0, 2, 1\right) \end{aligned}$$

If we let $x_3$ and $x_5$ be free variables, then every solution of the system is a linear combination of the set

$$\left\{\left(-\frac{2}{3}, 0, 1, 0, 0\right), \left(1, 0, 0, 2, 1\right)\right\}.$$

**2.2.5** Let $F$ be a field and let $n$ be a positive integer ($n \geq 2$). Let $V$ be the vector space of all $n \times n$ matrices over $F$. Which of the following sets of matrices $A$ in $V$ are subspaces of $V$?

a. all invertible $A$;

b. all non-invertible $A$;

c. all $A$ such that $AB = BA$, where $B$ is some fixed matrix in $V$;

d. all $A$ such that $A^2 = A$.

**Solution** a. This is not a subspace because it is not closed under scalar multiplication. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is invertible, but $0 \cdot A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is not.

b. This is not a subspace because it is not closed under addition. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are both singular, but $A + B = I_2$ is invertible.

c. This is a subspace. The 0 matrix is contained in the set, so it is non-empty. Let $C$ and $D$ be matrices from the set. Then

$$(C + D)B = CB + DB = BC + BD = B(C + D) \implies \text{the set is closed under addition}$$

$$(cC)B = cCB = cBC = B(cC) \implies \text{the set is closed under scalar multiplication}$$

Thus the set forms a subspace of $V$.

d. The set is not closed under scalar multiplication. Let $A$ be from the set and $c \in F$. Then

$$(cA)^2 = c^2 A^2 = c^2 A \neq cA$$

Hence the set does not form a subspace of $V$.

---

**2.2.6** a. Prove that the only subspaces of $\mathbb{R}^1$ are $\mathbb{R}^1$ and the zero subspace.

b. Prove that a subspace of $\mathbb{R}^2$ is $\mathbb{R}^2$ or the zero subspace, or consists of all scalar multiples of some fixed vector in $\mathbb{R}^2$. (The last type of subspace is, intuitively, a straight line through the origin.)

c. Can you describe the subspaces of $\mathbb{R}^3$?

**Solution** a. $\mathbb{R}^1$ and $\{0\}$ are both obviously subspaces of $\mathbb{R}^1$, so we only need to prove that any other subset of $R$ are not subspaces.

Suppose $V$ is a subset of $\mathbb{R}^1$. Then we can express $V$ as an interval $(a, b)$, $(a, b]$, $[a, b)$, or $[a, b]$, where $b > a$. In any case, the number $a + (b - a)/2$ is contained in $V$, but $a + 3(b - a)/2$ will not be contained in the set. Thus, any set other than $\mathbb{R}^1$ and $\{0\}$ is not closed under scalar multiplication, and cannot be a subspace of $\mathbb{R}^1$.

b. $\mathbb{R}^2$ and $\{0\}$ are both obviously subspaces of $\mathbb{R}^2$, so we need to show that any subset of $\mathbb{R}^2$ other than the scalar multiples of a fixed vector in $\mathbb{R}^2$ is not a subspace.

Let $V$ be a subset of $\mathbb{R}^2$ that is not $\mathbb{R}^2$, $\{0\}$, or the set of scalar multiples of a fixed vector in $\mathbb{R}^2$. Suppose $V \subseteq \mathbb{R}^2$ is a subspace. Since $V$ the set of scalar multiples of a vector, then there must exist two vectors $\alpha = (a_1, a_2)$ and $\beta = (b_1, b_2)$ such that $\alpha$ is not a scalar multiple of $\beta$ and vice versa. We will show that there exists $c, d \in \mathbb{R}$ such that $c\alpha + d\beta = \gamma \ \forall \gamma$, which means $\gamma \in V \implies V = \mathbb{R}^2$.

$$ca_1 + db_1 = x \implies d = -\frac{a_1}{b_1}c + \frac{1}{b_1}x$$

$$ca_2 + db_2 = y \implies d = -\frac{a_2}{b_2}c + \frac{1}{b_2}y$$

Since $\alpha$ and $\beta$ are not scalar multiples of each other, $a_1/b_1 \neq a_2/b_2$, which means the two lines described above in the $cd$-plane are not parallel and thus must intersect. Thus, there exists $c$ and $d$ as described above, which implies that $V = \mathbb{R}^2$, which is a contradiction. Hence, if $V$ is a subspace of $\mathbb{R}^2$, then it can only be as described in the problem.

c. Geometrically, the subspaces of $\mathbb{R}^3$ will be planes, lines, or the zero subspace.

---

**Problem** Let $F \subseteq \mathbb{C}$ be a field and let $S$ be a non-empty set. Let $V$ be an $F$ vector space. Let $W$ be the space of all functions from $S$ to $V$. We can add two elements $f$ and $g$ of $W$:

$$(f+g)(s) = f(s) + g(s).$$

Scalar multiplication by an element $c \in F$ is defined:

$$cf(s) = c(f(s)).$$

Show $W$ is an $F$ vector space with these operations. You should check the axioms on page 28 until you get bored.

**Solution** Addition axioms:

Closure $(f+g)(s) = f(s) + g(s) \in V$, since $f(s)$ and $g(s)$ are in $V$.

(a) $(f+g)(s) = f(s) + g(s) \overset{(3a)}{=} g(s) + f(s) = (g+f)(s)$

(b) $(f+(g+h))(s) = f(s) + (g+h)(s) = f(s) + g(s) + h(s) \overset{(3b)}{=} (f(s)+g(s)) + h(s) = ((f+g)+h)(s)$

(c) $(f+0)(s) = f(s) + 0(s) = f(s) + 0 \overset{(3c)}{=} f(s)$

(d) $(f+(-f))(s) = f(s) + (-f(s)) \overset{(3d)}{=} 0$

The verification for the scalar multiplication axioms are similar to the above.

---

****2 2.2.7** Let $W_1$ and $W_2$ be subspaces of a vector space $V$ such that the set-theoretic union of $W_1$ and $W_2$ is is also a subspace. Prove that one of the subspaces $W_i$ is contained in the other.

**Solution** Let $\alpha \in W_1$ and $\beta \in W_2$. Then both $\alpha$ and $\beta$ are in $W_1 \cup W_2$, which means $\alpha + \beta \in W_1 \cup W_2$. Then there are two cases:

$\alpha + \beta \in W_1$:

Then since $W_1$ is a subspace, $\alpha + \beta - \alpha = \beta \in W_1$. Thus, if $\beta \in W_2$, then $\beta \in W_1$ also, which implies that $W_2 \subseteq W_1$.

$\alpha + \beta \in W_2$:

Then since $W_2$ is a subspace, $\alpha + \beta - \beta = \alpha \in W_2$. Thus, if $\alpha \in W_1$, then $\alpha \in W_2$ also, which implies that $W_1 \subseteq W_2$.

Thus, if $W_1 \cup W_2$ is a subspace, then $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

---

****3 2.2.8** Let $V$ be the vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$; let $V_e$ be the subset of even functions, $f(-x) = f(x)$; let $V_o$ be the subset of odd functions, $f(-x) = -f(x)$.

a. Prove that $V_e$ and $V_o$ are subspaces of $V$.

b. Prove that $V_e + V_o = V$.

c. Prove that $V_e \cap V_o = \{0\}$.

**Solution** a. Suppose $f, g \in V_e$. Then it suffices to show that for all $a \in \mathbb{R}$, $af + g \in V_e$.

$$(af+g)(-x) = (af)(-x) + g(-x) = af(-x) + g(-x) = af(x) + g(x) = (af+g)(x) \implies af + g \in V_e.$$

Similarly, suppose $f, g \in V_o$ and $a \in \mathbb{R}$. Then

$$(af+g)(-x) = af(-x) + g(-x) = -af(x) - g(x) = -(af+g)(x) \implies af + g \in V_o.$$

b. Let $f \in V_e$, $g \in V_o$, and $h \in V$. Since $V$ is a vector space, $f + g \in V \implies V_e + V_o \subseteq V$. Now we need to show that $V \subseteq V_e + V_o$; that is, we can write $h = f + g$. Notice that

$$\frac{h(x) + h(-x)}{2} \in V_e \text{ and } \frac{h(x) - h(-x)}{2} \in V_o.$$

Then

$$\frac{h(x) + h(-x)}{2} + \frac{h(x) - h(-x)}{2} = \frac{2h(x)}{2} = h(x).$$

Thus, we can indeed write any function in $V$ as the sum of a function in $V_e$ and a function in $V_o$, so $V \subseteq V_e + V_o$. Hence, $V = V_e + V_o$.

c. Let $f \in V_e \cap V_o$. Then we have

$$f(-x) = f(x) = -f(x) \implies 2f(x) = 0 \implies f(x) = 0$$

Thus, $V_e \cap V_o = \{0\}$.

---

****4 2.2.9** Let $W_1$ and $W_2$ be subspaces of a vector space $V$ such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector $\alpha$ in V there are *unique* vectors $\alpha_1$ in $W_1$ and $\alpha_2$ in $W_2$ such that $\alpha = \alpha_1 + \alpha_2$.

**Solution** Let $\alpha \in V$. Then since $V = W_1 + W_2$, we can find $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ such that $\alpha = \alpha_1 + \alpha_2$. Suppose $\alpha_1$ and $\alpha_2$ are not unique; that is, there exists $\tilde{\alpha_1} \in W_1 \setminus \{\alpha_1\}$ and $\tilde{\alpha_2} \in W_2 \setminus \{\alpha_2\}$ such that $\alpha = \tilde{\alpha_1} + \tilde{\alpha_2}$. Then

$$\begin{aligned} \alpha_1 + \alpha_2 &= \tilde{\alpha_1} + \tilde{\alpha_2} \\ \beta \coloneqq \alpha_1 - \tilde{\alpha_1} &= \tilde{\alpha_2} - \alpha_2 \end{aligned}$$

Since $W_1$ is a subspace, $\beta = \alpha_1 - \tilde{\alpha_1} \in W_1$. Similarly, $\beta = \alpha_2 - \tilde{\alpha_2} \in W_2$ also. Thus,

$$\begin{aligned} \beta = \alpha_1 - \tilde{\alpha_1} = \alpha_2 - \tilde{\alpha_2} = 0 &\implies \tilde{\alpha_1} = \alpha_1 \\ &\implies \tilde{\alpha_2} = \alpha_2 \end{aligned}$$

Thus, $\alpha_1$ and $\alpha_2$ are unique.