8 A signal of amplitude $s = 2$ is transmitted from a satellite but is corrupted by noise, and the received signal is $X = s + W$, where W is noise. When the weather is good, W is normal with zero mean and variance 1. When the weather is bad, W is normal with zero mean and variance 4. Good and bad weather are equally likely. In the absence of any weather information:

a. Calculate the PDF of X.

b. Calculate the probability that X is between 1 and 3.

Solution

1. Let $F_X(t)$ be the CDF of X, $W_g \sim \mathcal{N}(0,1)$, and $W_b \sim \mathcal{N}(0,4)$. Then

$$\begin{aligned}
F_X(t) = \mathbb{P}(X \leq t) &= \mathbb{P}(X \leq t \mid \text{weather is good})\mathbb{P}(\text{weather is good}) + \mathbb{P}(X \leq t \mid \text{weather is bad})\mathbb{P}(\text{weather is bad}) \\
&= \frac{1}{2}\mathbb{P}(W_g \leq t - s) + \frac{1}{2}\mathbb{P}(W_b \leq t - s) \\
&= \frac{1}{2}\left[\int_{-\infty}^{t-2} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u + \int_{-\infty}^{t-2} \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{u^2}{8}\right) \mathrm{d}u\right] \\
f_X(t) = \frac{\mathrm{d}}{\mathrm{d}t}F_X &= \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(t-2)^2}{2}\right) + \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{(t-2)^2}{8}\right)\right)
\end{aligned}$$

2. $$F_X(3) - F_X(1) = \frac{1}{2}\left[\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u + \int_{-1}^{1} \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{u^2}{8}\right) \mathrm{d}u\right] \approx 0.532807$$

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5 Let A be the set of all pairs (x, y) which satisfy each of the conditions

$$x \geq 0, y \geq 0, 1 \leq x + y \leq 2.$$

Let the random variables X and Y be jointly continuous with the joint PDF

$$f_{X,Y}(x, y) = \begin{cases} C(x+y), & \text{if } (x, y) \in A \\ 0, & \text{otherwise.} \end{cases}$$

Find the value of the constant C. Find the marginal PDFs and CDFs.

Solution

$$\iint_A f_{X,Y}(x, y)\,\mathrm{d}A = \int_0^1 \int_{1-x}^{2-x} C(x+y)\,\mathrm{d}y\,\mathrm{d}x + \int_1^2 \int_0^{2-x} C(x+y)\,\mathrm{d}y\,\mathrm{d}x = \frac{7}{3}C = 1 \implies C = \frac{3}{7}$$

$$f_X = \begin{cases} \displaystyle\int_{1-x}^{2-x} \frac{3}{7}(x+y)\,\mathrm{d}y = \frac{9}{14}, & 0 \leq x \leq 1 \\ \displaystyle\int_0^{2-x} \frac{3}{7}(x+y)\,\mathrm{d}y = \frac{3}{14}\left(4 - x^2\right), & 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X = \begin{cases} 0, & x < 0 \\ \dfrac{9}{14}x & 0 \leq x \leq 1 \\ \dfrac{9}{14} + \dfrac{3}{14}\left(\dfrac{-x^3}{3} + 4x - \dfrac{11}{3}\right) & 1 \leq x \leq 2 \\ 1 & x > 2 \end{cases}$$

By symmetry, $f_Y = f_X$, but with x replaced with y. Similarly, $F_Y = F_X$.