****15 3.5.11** Let W_1 and W_2 be subspaces of a finite-dimensional vector space V.

a. Prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

b. Prove that $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$.

Solution a. Let $f \in (W_1 + W_2)^0$. Then by definition, for all $c, d \in F$ and $\alpha \in W_1, \beta \in W_2$, we have $f(c\alpha + d\beta) = 0$. Set $d = 0$ and $c = 1$. Then $f(\alpha) = 0$ for all $\alpha \in W_1 \implies f \in W_1^0$.
Similarly, putting $c = 0$ and $d = 1$, we have $f(\beta) = 0$ for all $\beta \in W_2 \implies f \in W_2^0$.
Thus, $(W_1 + W_2)^0 \subseteq W_1^0 \cap W_2^0$.
Let $f \in W_1^0 \cap W_2^0$. Then for all $\alpha \in W_1$ and $\beta \in W_2$, $f(\alpha) = f(\beta) = 0$. Then for all $c, d \in F$, $f(c\alpha + d\beta) = cf(\alpha) + df(\beta) = 0 \implies f \in (W_1 + W_2)^0$. Thus, $W_1^0 \cap W_2^0 \subseteq (W_1 + W_2)^0$.
Hence, $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

b. Let $f \in (W_1 \cap W_2)^0$. Then let

$$\begin{aligned} \alpha_1, \ldots, \alpha_k &\text{ be a basis of } W_1 \cap W_2 \\ \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m &\text{ be a basis of } W_1 \\ \alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_n &\text{ be a basis of } W_2 \\ \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_l &\text{ be a basis of } V \end{aligned}$$

Then consider its dual basis, α_1^*, α_2^*, etc. As $f \in (W_1 \cap W_2)^0$, $f \in V^*$ also, so we can write f as

$$\begin{aligned} f &= b_1\beta_1^* + \cdots + b_m\beta_m^* + c_1\gamma_1^* + \cdots + c_n\gamma_n^* + d_1\delta_1^* + \cdots + d_l\delta_l^* \\ &= \underbrace{(c_1\gamma_1^* + \cdots + c_n\gamma_n^* + d_1\delta_1^* + \cdots + d_l\delta_l^*)}_{:=f_1} + \underbrace{(b_1\beta_1^* + \cdots + b_m\beta_m^*)}_{:=f_2} \end{aligned}$$

f_1 is clearly an annihilator of W_1 and f_2 is clearly an annihilator of W_2, and $f = f_1 + f_2$, so $f \in W_1^0 + W_2^0 \implies (W_1 \cap W_2)^0 \subseteq W_1^0 + W_2^0$.
If $f \in W_1^0 + W_2^0$, then $f = f_1 + f_2$, where $f_1 \in W_1^0$ and $f_2 \in W_2^0$. Then if $\alpha \in W_1 \cap W_2$, we have $f(\alpha) = f_1(\alpha) + f_2(\alpha) = 0$. Thus, $W_1^0 + W_2^0 \subseteq (W_1 \cap W_2)^0$.
Hence, $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$.

****16 3.5.17** Let W be the space of $n \times n$ matrices over the field F, and let W_0 be the subspace spanned by the matrices C of the form $C = AB - BA$. Prove that W_0 is exactly the subspace of matrices which have trace zero. (*Hint:* What is the dimension of the space matrices of trace zero? Use the matrix 'units,' i.e., matrices with exactly one non-zero entry to construct enough linearly independent matrixes of the form $AB - BA$.)

Solution Define E^{ij} to be the matrix such that $E^{ij}_{ij} = \delta_{ij}$. The E^{ij}'s are clearly linearly independent and if we consider the matrices where $i \neq j$, then they span the set of matrices with 0's on the diagonal. There are $n^2 - n$ of these matrices, since an $n \times n$ matrix has n^2 entries, and if we fix the diagonal to be all 0's, then there are n fewer entries we need to account for. Thus, the dimension of the matrices with a diagonal of 0's is $n^2 - n$.

To get all the matrices with a trace zero, we add to the basis above with the set of matrices with all 0 entries except on the diagonal. The x_i be the i-th diagonal entry. We want $x_1 + \cdots + x_n = 0$. The set of solutions to this system have dimension $n - 1$. Thus, the matrices with trace zero have dimension $n^2 - n + n - 1 = n^2 - 1$.

Note that $E_{ij} = E_{ij}Ejj - E_{jj}E_{ij}$ if $i \neq j$. We can generate all of the matrices described in the first paragraph like this. Similarly, $E_{ii} - E_{jj} = E_{ij}E_{ji} - E_{ji}E_{ij}$ will produce $n - 1$ linearly independent matrices as in the second paragraph.

Thus, matrices of the form $C = AB - BA$ span the set of matrices with trace zero.

****17 6.2.11** Let N be a 2×2 complex matrix such that $N^2 = 0$. Prove that either $N = 0$ or N similar over $\mathbb{C}$ to

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Solution There are three cases:

rank $N = 0$:

Clearly $N = 0$ in this case.

rank $N = 1$:

Let $\alpha \in \mathbb{C}^2$ such that $N\alpha \neq 0$, which exists since $\operatorname{rank} N = 1$. Then $N\alpha \in \ker N$ since $N^2\alpha = 0$. They are linearly independent:

$$\begin{gathered} c_1\alpha + c_2 N\alpha = 0 \\ N(c_1\alpha + c_2 N\alpha) = c_1 N\alpha = 0 \implies c_1 = 0 \\ c_2 N\alpha = 0 \implies c_2 = 0 \end{gathered}$$

Thus, $\mathfrak{B} = \{\alpha, N\alpha\}$ is a basis of $\mathbb{C}^2$. Let $U = \begin{pmatrix} | & | \\ \alpha & N\alpha \\ | & | \end{pmatrix}$. U takes $\mathfrak{B}$ coordinates to standard coordinates, so U^{-1} does the opposite. Thus,

$$\begin{aligned} [N]_{\mathfrak{B}} &= U^{-1}NU \\ &= U^{-1}N \begin{pmatrix} | & | \\ \alpha & N\alpha \\ | & | \end{pmatrix} \\ &= U^{-1} \begin{pmatrix} | & | \\ N\alpha & N^2\alpha \\ | & | \end{pmatrix} \\ &= U^{-1} \begin{pmatrix} | & | \\ N\alpha & 0 \\ | & | \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

as desired.

rank $N = 2$:

This is not possible since $N^2 = 0 \implies (\det N)^2 = 0 \implies \det N = 0 \iff \operatorname{rank} N < 2$.