9.3.10 $\mathbf{y}(t) = C_1 e^{-t}\begin{pmatrix}2\\1\end{pmatrix} + C_2 e^{-2t}\begin{pmatrix}-1\\1\end{pmatrix}$

Solution Nodal sink

9.3.12 $\mathbf{y}(t) = C_1 e^{t}\begin{pmatrix}1\\1\end{pmatrix} + C_2 e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix}$

Solution Saddle

9.3.20 $\mathbf{y}' = \begin{pmatrix}-2 & 2\\-1 & 0\end{pmatrix}\mathbf{y}$

Solution

9.4.14 $\mathbf{y}(t) = e^{\lambda t}[(C_1 + C_2 t)\mathbf{v}_1 + C_2\mathbf{v}_2]$

Assume that the eigenvalue λ is negative.

(a) Describe the exponential solution.

(b) Describe the behavior of the general solution as $t \to \infty$.

(c) Describe the behavior of the general solution as $t \to -\infty$.

(d) Is the equilibrium point at the origin a degenerate nodal sink or source?

Solution (a) The graph of the solution is somewhere between a nodal sink and a spiral sink.

(b) As $t \to \infty$, $\mathbf{y}(t)$ approaches the equilibrium solution $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(c) As $t \to -\infty$, $\mathbf{y}(t)$ is tangent to $\mathbf{v}_1$.

(d) The equilibrium is a degenerate sink.

9.4.26 Pure water enters Tank A in Figure 7 at a rate r_1 gallons per minute. Two pipes connect the tanks. Through the top pipe, salt solution enters Tank A from Tank B at a rate r_B gal/min. Through the bottom pipe, salt solution enters Tank B from Tank A at a rate r_A gal/min. Finally, there is a drain on Tank B that drains salt solution from Tank B at a rate of r_D gal/min, so as to keep the level of solution in each tank at a constant volume V.

(a) Set up a system of equations that models the salt content $x_A(t)$ and $x_B(t)$ in each tank over time.

(b) Use qualitative analysis to show that the equilibrium point of the system in part (a) is either a nodal sink or a degenerate sink, regardless of the values r_A, r_B, r_1, and r_D. That is, show that the salt content in each tank has no chance of oscillation as the salt content goes to zero.

Solution (a)

$$x_A'(t) = -r_A\frac{x_A}{V} + r_B\frac{x_B}{V} = -\frac{r_A}{V}x_A + \frac{r_B}{V}x_B$$

$$x_B'(t) = r_A\frac{x_A}{V} - r_B\frac{x_B}{V} - r_D\frac{x_B}{V} = \frac{r_A}{V}x_A - \frac{r_B + r_D}{V}x_B$$

(b)

$$\mathbf{y}' = \frac{1}{V}\begin{pmatrix} -r_A & r_B \\ r_A & -r_B - r_D \end{pmatrix}\mathbf{y}$$

$$p(\lambda) = \lambda^2 + (r_A + r_B + r_D)\lambda + r_A(r_B + r_D) - r_A r_B$$

$$T^2 - 4D = (r_A + r_B + r_D)^2 - 4[r_A(r_B + r_D) - r_A r_B]$$

Since the volumes remain constant, we have the equalities $r_1 + r_B - r_A = 0$ and $r_A - r_B - r_D = 0$. Rewriting the discriminant, we get

$$\begin{aligned} &(2r_B + 2r_D)^2 - 4(r_B + r_D)(r_D) \\ &= 4(r_B + r_D)^2 - 4(r_B + r_D)(r_D) > 0 \end{aligned}$$

since $r_B + r_D > r_D$, meaning our eigenvalues will be real valued. Also, $T = -r_a - r_B - r_D < 0$, and $D = (r_B + r_D)(r_D) > 0$, meaning both eigenvalues are negative. Thus, since the eigenvalues are real there is no possibility of oscillation, and since the eigenvalues are negative, we have either a nodal or degenerate sink.