THE LAPLACE TRANSFORM

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ABSTRACT. We re-visit the Laplace transform and give a brief exposition on its origins. Afterwards, we will state and prove some basic properties and applications of the transform to ODEs, after which we will discuss the inverse Laplace transform in detail. Finally, we end the paper with a mathematical puzzle which utilizes the Laplace transform.

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1. INTRODUCTION

Definition 1. Let $f\colon [0,\infty) \to \mathbb{R}$ be a function. Then the **Laplace transform** of f is $\mathcal{L}\{f\}\colon U \to \mathbb{C}$, defined by

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

where $U \subseteq \mathbb{C}$ is the set of parameters where the integral exists.

Example 1. The Laplace transform of $f(t) = e^{at}$ is given by

$$\begin{aligned}\mathcal{L}\{e^{at}\}(s) = \int_0^\infty e^{-st}e^{at}\,\mathrm{d}t &= \int_0^\infty e^{(a-s)t}\,\mathrm{d}t \\ &= \left.\frac{e^{(a-s)t}}{a-s}\right|_0^\infty \\ &= \frac{1}{s-a},\end{aligned}$$

where $s \in \mathbb{C}$ is such that

$$\lim_{t\to\infty}\left|e^{(a-s)t}\right| = 0 \iff \lim_{t\to\infty} e^{t\,\mathrm{Re}(a-s)} = 0 \iff \mathrm{Re}(s) > \mathrm{Re}(a).$$

Contrary to what one might suspect, the earliest use of the Laplace transform is attributed to Euler in [3], where the transform appears in the following form to study differential equations:

$$z(x,a) = \int e^{ax} X(x)\,\mathrm{d}x$$

Here, z is an *indefinite* integral, which is why z is actually a function of both x and the parameter a, unlike the modern transform. The transform was expanded upon by Lagrange in [1], where he studies integrals of the forms

$$\int y a^x\,\mathrm{d}x \quad \text{and} \quad \int \frac{X(x)a^x}{e^{xa}}\,\mathrm{d}x,$$

where y is some function. A version of the transform appears in Laplace's work for the first time in 1779 in his paper [9]. However, it is not until his 1882 paper [8] does he lay the foundation for the theory familiar to mathematicians today, and his contributions are the reason why the transform is named after him.

2. Basic Properties and Applications to ODEs

The Laplace transform arose in the study of differential equations, so it is no surprise that its most popular applications remain in this field. To help motivate why this is so, we begin with the following proposition:

Proposition 1. *Let $f\colon [0,\infty) \to \mathbb{R}$.*

(i) Let $a, b \in \mathbb{C}$, and let $g\colon [0,\infty) \to \mathbb{R}$ be another function. Then
$$\mathcal{L}\{af + bg\}(s) = a\mathcal{L}\{f\}(s) + b\mathcal{L}\{f\}(s)$$
for $s \in \mathbb{C}$ where both Laplace transforms converge.

(ii) If $f(t) = \mathcal{O}(e^{at})$ as $t \to \infty$ for some $a > 0$, then
$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0) \tag{1}$$
for $\mathrm{Re}(s) > \mathrm{Re}(a)$.

Proof. (i) follows immediately linearity of integration.

For (ii), we simply perform integration by parts:
$$\begin{aligned}\mathcal{L}\{f'\}(s) = \int_0^\infty e^{-st} f'(t)\,\mathrm{d}t &= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,\mathrm{d}t \\ &= (0 - f(0)) + s\mathcal{L}\{f\}(s) \\ &= s\mathcal{L}\{f\}(s) - f(0).\end{aligned}$$
The upper limit vanishes since $f(t) = \mathcal{O}(e^{at})$, and so
$$|f(t)| \leq Ce^{t\,\mathrm{Re}(a-s)} \xrightarrow{t\to\infty} 0.$$
□

Example 2. As a quick application of (i), write $\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$ and apply Example 1 to get
$$\begin{aligned}\mathcal{L}\{\sin(\omega t)\}(s) &= \frac{1}{2i}\big(\mathcal{L}\{e^{i\omega t}\}(s) - \mathcal{L}\{e^{-i\omega t}\}(s)\big) \\ &= \frac{1}{2i}\left(\frac{1}{s - i\omega} - \frac{1}{s + i\omega}\right) \\ &= \frac{1}{2i}\frac{2i\omega}{s^2 + \omega^2} \\ &= \frac{\omega}{s^2 + \omega^2}\end{aligned}$$
for $s > 0$.

The more remarkable result is the equation (1), which tells us that (ignoring the conditions on the domain) the Laplace transforms of the derivatives of f are determined by the Laplace transform of f itself up to initial conditions. For example, applying (1) with f replaced with f', we obtain
$$\begin{aligned}\mathcal{L}\{f''\}(s) = s\mathcal{L}\{f'\}(s) - f'(0) &= s(s\mathcal{L}\{f\}(s) - f(0)) - f'(0) \\ &= s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0).\end{aligned}$$

As one can imagine, these two results suggest that the Laplace transform can be applied to linear ODEs.

Example 3. Consider the ODE $y'' - y = \sin 3t$ with initial conditions $y(0) = 0$ and $y'(0) = 0$. Suppose we had a solution y and that its Laplace transform $Y \coloneqq \mathcal{L}\{y\}$ exists. Applying the Laplace transform to the equation yields

$$\begin{aligned}\big(s^2Y(s) - sy(0) - y'(0)\big) - Y(s) &= \frac{3}{s^2+9} \\ \implies Y(s) &= \frac{3}{(s^2-1)(s^2+9)}.\end{aligned}$$

After applying partial fractions, we eventually end up at

$$\begin{aligned}Y(s) &= -\frac{1}{10}\frac{3}{s^2+9} - \frac{3}{20}\frac{1}{s+1} + \frac{3}{20}\frac{1}{s-1} \\ \mathcal{L}\{y\}(s) &= \mathcal{L}\left\{-\frac{1}{10}\big(\sin 3t + 3e^{-t} - 3e^{t}\big)\right\}(s).\end{aligned}$$

From here, it's easy to check that $y(t) = -\frac{1}{10}\big(\sin 3t + 3e^{-t} - 3e^{t}\big)$ is a solution to the initial value problem.

Remark. Notice that after we applied the Laplace transform, we only had to perform partial fractions to solve the problem. In other words, the Laplace transform turns initial value problems into algebraic ones.

3. The Inverse Laplace Transform

In the last step of Example 3, we did not invoke any type of inverse Laplace transform; instead, we used the Laplace transform to come up with an ansatz. However, it is a natural question to ask: could we have made the conclusion

$$\begin{aligned}\mathcal{L}\{y\}(s) &= \mathcal{L}\left\{-\frac{1}{10}\big(\sin 3t + 3e^{-t} - 3e^{t}\big)\right\}(s) \\ \implies y(t) &= -\frac{1}{10}\big(\sin 3t + 3e^{-t} - 3e^{t}\big)?\end{aligned}$$

In other words, is the Laplace transform one-to-one? Moreover, the expression we had for $\mathcal{L}\{y\}(s)$ was very convenient—it was easy to formulate it as the Laplace transform of some known function. Consequently, another natural question arises: if we are given the Laplace transform $F(s)$ of a function $f(t)$, is it possible to recover $f(t)$? We will investigate these questions in this section.

In light of Lebesgue integration theory, we can see that the Laplace transform is not one-to-one in general. Indeed, if $f(t) = g(t)$ for Lebesgue-almost every $t \geq 0$, then $\mathcal{L}\{f\}(s) = \mathcal{L}\{g\}(s)$ whenever the transforms exist. Consequently, we will want to restrict ourselves to certain classes of functions in order to define an inverse transform. For example:

Theorem 1. *Suppose $f\colon [0,\infty) \to \mathbb{R}$ is differentiable with $f(t), f'(t) \in \mathcal{O}(e^{at})$ as $t \to \infty$. Then for $t > 0$ and $\operatorname{Re}(\gamma) > \operatorname{Re}(a)$, we have*

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\mathcal{L}\{f\}(s)\,\mathrm{d}s \coloneqq \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\gamma - iT}^{\gamma + iT} e^{st}\mathcal{L}\{f\}(s)\,\mathrm{d}s. \tag{2}$$

(2) is called the ***Bromwich integral****.*

Before we prove this theorem, the following calculation will be useful:

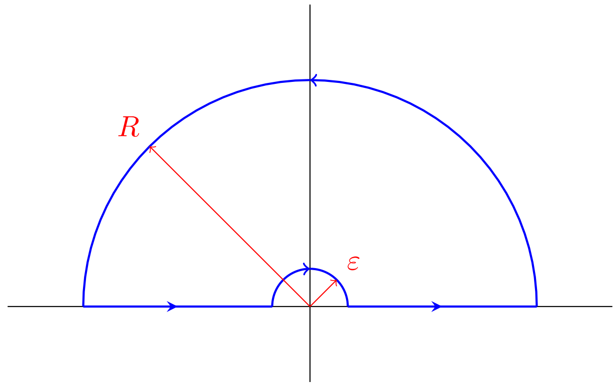
Lemma 1. *Let*

$$I(x) = \frac{1}{2} + \frac{1}{\pi}\int_0^x \frac{\sin t}{t}\,\mathrm{d}t.$$

Then

$$\lim_{T\to\infty} I(Tx) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{1}{2}, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Proof. Consider the contour integral $\int_{\gamma_{R,\varepsilon}} \frac{e^{iz}}{z}\,\mathrm{d}z$, where $\gamma_{R,\varepsilon}$ comprises of the upper semi-circles of radii R, ε centered at the origin, with segments connecting these endpoints:



On the larger arc, which we call γ_R, we have the parameterization $z = Re^{i\theta}$, which gives

$$\begin{aligned}\left|\int_{\gamma_R} \frac{e^{iz}}{z}\,\mathrm{d}z\right| \le \int_0^\pi e^{-R\sin\theta}\,\mathrm{d}\theta \le 2\int_0^{\pi/2} e^{-R\sin\theta}\,\mathrm{d}\theta \\ \le 2\int_0^{\pi/2} e^{-(2R/\pi)\theta}\,\mathrm{d}\theta \\ = \frac{\pi}{2R}\left(1-e^{-R}\right) \xrightarrow{R\to\infty} 0.\end{aligned}$$

The inequality in the second line comes from concavity of $\sin\theta$ on $[0, \frac{\pi}{2}]$:

$$\sin\frac{t\pi}{2} \ge t\sin\frac{\pi}{2} = t \implies \sin\theta \ge \frac{2\theta}{\pi}.$$

For γ_ε, parameterizing via $z = \varepsilon e^{i(\pi-\theta)}$ gives

$$\int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z = -i\int_0^\pi e^{i\varepsilon e^{i(\pi-\theta)}}\,\mathrm{d}\theta.$$

The integrand is bounded, so by dominated convergence, we get

$$\lim_{\varepsilon\to 0^+}\int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z = \int_{\gamma_\varepsilon} \lim_{\varepsilon\to 0^+} e^{i\varepsilon e^{i(\pi-\theta)}}\,\mathrm{d}z = -i\int_0^\pi \mathrm{d}\theta = -\pi i.$$

It is also clear that the segments will converge to $\int_{-\infty}^{\infty} \frac{e^{it}}{t}\,\mathrm{d}t$ as $\varepsilon \to 0^+$ and $R \to \infty$. Observe that $\frac{e^{iz}}{z}$ only has a singularity at $z = 0$, which the interior of $\gamma_{R,\varepsilon}$ avoids, so by Cauchy's theorem,

$$0 = \lim_{\varepsilon\to 0^+; R\to\infty}\int_{\gamma_{R,\varepsilon}} \frac{e^{iz}}{z}\,\mathrm{d}z = \int_{-\infty}^{\infty} \frac{e^{it}}{t}\,\mathrm{d}t - \pi i.$$

Taking imaginary parts and noting that $\frac{\sin t}{t}$ is even, we get

$$2\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \pi \implies \int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \frac{\pi}{2}.$$

Thus, returning to our original problem, if $x < 0$, we have

$$\lim_{T\to\infty} I(Tx) = \frac{1}{2} + \frac{1}{\pi}\int_0^{-\infty} \frac{\sin t}{t}\,\mathrm{d}t = \frac{1}{2} - \frac{1}{\pi}\cdot\frac{\pi}{2} = 0.$$

If $x = 0$, the integral vanishes so $\frac{1}{2}$ remains, and if $x > 0$,

$$\lim_{T\to\infty} I(Tx) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = 1.$$

$\square$

Proof of Theorem 1. Fix $t > 0$. We begin by expanding out definitions:

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\int_0^\infty e^{-sx} f(x)\,\mathrm{d}x\,\mathrm{d}s &= \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT}\int_0^\infty e^{s(t-x)} f(x)\,\mathrm{d}x\,\mathrm{d}s \\ &= \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\int_0^\infty e^{(\gamma+iu)(t-x)} f(x)\,\mathrm{d}x\,\mathrm{d}u \qquad (3)\end{aligned}$$

We would like to apply Fubini's theorem: for $x \gg 0$, our assumption on the growth rate of f gives for $(x, s) \in [0,\infty)\times(\gamma + i[-T, T])$

$$\left|e^{(\gamma+iu)(t-x)} f(x)\right| \le C\left|e^{(\gamma+iu)(t-x)} e^{ax}\right| = C\left|e^{(\gamma+iu)t} e^{(a-s)x}\right| = C\left|e^{\gamma t} e^{-x\,\mathrm{Re}(\gamma-a)}\right|.$$

γ and t are fixed, so we see that the integrand is $\mathcal{O}\big(e^{-x\,\mathrm{Re}(\gamma-a)}\big)$ as $x\to\infty$. Since $\mathrm{Re}(\gamma)-\mathrm{Re}(a) > 0$, this is integrable: by Fubini-Tonelli, we have

$$\int_{-T}^{T}\int_0^\infty e^{-x\,\mathrm{Re}(\gamma-a)}\,\mathrm{d}x\,\mathrm{d}u = \int_{-T}^{T}\frac{1}{\mathrm{Re}(\gamma)-\mathrm{Re}(a)}\,\mathrm{d}u = \frac{2T}{\mathrm{Re}(\gamma)-\mathrm{Re}(a)} < \infty.$$

Thus, applying Fubini's theorem in (3) and the fundamental theorem of calculus, we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{-T}^{T}\int_0^\infty e^{(\gamma+iu)(t-x)} f(x)\,\mathrm{d}x\,\mathrm{d}u &= \frac{1}{2\pi}\int_0^\infty f(x)\left(\int_{-T}^{T} e^{(\gamma+iu)(t-x)}\,\mathrm{d}u\right)\mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^\infty f(x)\left(\frac{e^{(\gamma+iT)(t-x)}}{i(t-x)} - \frac{e^{(\gamma-iT)(t-x)}}{i(t-x)}\right)\mathrm{d}x \\ &= \int_0^\infty f(x)e^{\gamma(t-x)}\left(\frac{\sin T(t-x)}{\pi(t-x)}\right)\mathrm{d}x. \qquad (4)\end{aligned}$$

Set $S_T(x) := \frac{1}{\pi}\int_0^x \frac{\sin T(t-u)}{t-u}\,\mathrm{d}u = \frac{1}{\pi}\int_0^x \frac{\sin T(u-t)}{u-t}\,\mathrm{d}u$, so that if we integrate by parts in (4), we get

$$f(x)e^{\gamma(t-x)}S_T(x)\Big|_{x=0}^{x=\infty} - \int_0^\infty \left[f(x)e^{\gamma(t-x)}\right]' S_T(x)\,\mathrm{d}x =: (A_\infty(T) - A_0(T)) - B(T).$$

For $A_\infty(T)$, notice that because $t > 0$, we get

$$|S_T(x)| \le \frac{1}{\pi}\int_0^x \frac{1}{u-t}\,\mathrm{d}u = \frac{x}{\pi}\log\left|\frac{x-t}{t}\right|.$$

Hence, for $x \gg 0$,

$$\left|f(x)e^{\gamma(t-x)}S_T(x)\right| \le \frac{C}{\pi}\left|xe^{\gamma t}e^{-x\,\mathrm{Re}(\gamma-a)}\log\left|\frac{x-t}{t}\right|\right| \xrightarrow{x\to\infty} 0,$$

so $A_\infty(T) = 0$. On the other hand, $A_0(T) = f(0)e^{\gamma t}S_T(0) = 0$ since the integrand in $S_T(x)$ is continuous for $0 \le u < t$, so we do not have an improper integral. In summary, we have shown thus far that

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st}\mathcal{L}\{f\}(s)\,\mathrm{d}s &= -\int_0^\infty \left[f(x)e^{\gamma(t-x)}\right]' S_T(x)\,\mathrm{d}x \\ &= -\int_0^\infty \left[f'(x)e^{\gamma(t-x)} - \gamma f(x)e^{\gamma(t-x)}\right] S_T(x)\,\mathrm{d}x. \qquad (5)\end{aligned}$$

Observe that with the change of variables $u \mapsto T(u-t)$, we get

$$S_T(x) = \frac{1}{\pi}\int_{-Tt}^{T(x-t)} \frac{\sin u}{u}\,\mathrm{d}u = I(T(x-t)) - I(-Tt),$$

where I is as in our lemma. Hence, because $t > 0$,

$$\lim_{T\to\infty} S_T(x) = \begin{cases} 0 & \text{if } x < t, \\ \frac{1}{2} & \text{if } x = t, \\ 1 & \text{if } x > t. \end{cases}$$

Thus, the integrand in (5) is bounded in T, and by the growth assumptions on f and f', we may apply the dominated convergence theorem to (5):

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st}\mathcal{L}\{f\}(s)\,\mathrm{d}s &= -\int_0^\infty \left[f(x)e^{\gamma(t-x)}\right]'\lim_{T\to\infty} S_T(x)\,\mathrm{d}x \\ &= -\int_t^\infty \left[f(x)e^{\gamma(t-x)}\right]'\mathrm{d}x \\ &= -f(x)e^{\gamma(t-x)}\Big|_{x=t}^{x=\infty} \\ &= f(t).\end{aligned}$$

□

Remark. This theorem is not optimal in that we can weaken the assumptions quite a bit. For example, we don't *need* assumptions on the growth rates for f and f'; instead, we only need that $f(t)e^{-\gamma t}$ is integrable to apply Fubini's theorem in (3), and we need further that $f'(t)e^{-\gamma t}$ is integrable to integrate by parts in (4) and apply dominated convergence in (5). We can go even further and replace f' with a function g such that $f(t) = f(0) + \int_0^t g(x)\,\mathrm{d}x$ and $g(t)e^{-\gamma t}$ is integrable and make the same conclusions. This leads to the (more cumbersome) version of the theorem thanks to [5]:

Theorem 2. *Let $f\colon [0,\infty) \to \mathbb{R}$ and suppose there exists $g\colon [0,\infty)\to\mathbb{R}$ which satisfies the following conditions:*

(i) $f(t) = f(0) + \int_0^t g(x)\,\mathrm{d}x$.
(ii) $f(t)e^{-\gamma t}$ and $g(t)e^{-\gamma t}$ are integrable on $[0,\infty)$.

Then for $t > 0$,

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\mathcal{L}\{f\}(s)\,\mathrm{d}s := \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st}\mathcal{L}\{f\}(s)\,\mathrm{d}s.$$

In the context of ODEs, where many of the applications of the Laplace transform lie, Theorem 1 is easier to use.

Example 4. We return to Example 3, where we solved the IVP $y'' - y = \sin 3t$ with $y(0) = y'(0) = 0$. If y is a solution to this problem, then

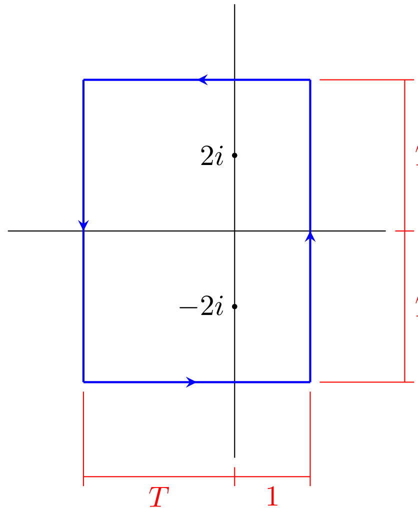
$$\mathcal{L}\{y\}(s) = \mathcal{L}\left\{-\frac{1}{10}\big(\sin 3t + 3e^{-t} - 3e^{t}\big)\right\}(s).$$

The function in the Laplace transform in the right-hand side and its derivative certainly grows like e^{2t}. If we impose the same growth condition on solutions to the IVP, then Theorem 1 tells us that we may apply the inverse Laplace transform to get $y(t) = -\frac{1}{10}\big(\sin 3t + 3e^{-t} - 3e^{t}\big)$, i.e., equality of Laplace transforms implies equality of the original functions. This also tells us that all solutions to the IVP with growth $\mathcal{O}(e^{2t})$ must be equal to y, i.e., the solution is unique.

Example 5. We will calculate $\mathcal{L}^{-1}\left\{\frac{s}{s^2+7}\right\}(t)$ using (2), and after examining the result, we will apply Theorem 1. For $t > 0$, we calculate

$$f(t) := \lim_{T\to\infty}\frac{1}{2\pi i}\int_{1-iT}^{1+iT} e^{st}\frac{s}{s^2+4}\,\mathrm{d}s.$$

The denominator factors to $(s-2i)(s+2i)$. Now suppose γ_T is the boundary of the rectangle $[-T,1]\times[-T,T]$ oriented counter-clockwise:



Let $\gamma_T^r, \gamma_T^t, \gamma_T^\ell, \gamma_T^b$ be the right, top, left, and bottom arcs, respectively. We will show that the integral on all arcs except γ_T^r will vanish as $T \to \infty$:

Parametrize γ_T^t via $u \mapsto u + iT$ to get

$$\begin{aligned}\left|\int_{\gamma_T^t} \frac{se^{st}}{s^2+4}\,\mathrm{d}s\right| &\le \frac{1}{T^2-4}\int_{-T}^{1}(u^2+T^2)e^{ut}\,\mathrm{d}u \\ &\le \frac{2T^2}{T^2-4}\int_{-T}^{1} e^{ut}\,\mathrm{d}u \\ &= \frac{2T}{T^2-4}\frac{e^t - e^{-Tt}}{t} \xrightarrow{T\to\infty} 0,\end{aligned}$$

since t is constant. Similarly, for the bottom, we use $u \mapsto u - iT$ to get the exact same estimate:

$$\left|\int_{\gamma_T^b} \frac{se^{st}}{s^2+4}\,\mathrm{d}s\right| \le \frac{2T}{T^2-4}\frac{e^t - e^{-Tt}}{t} \xrightarrow{T\to\infty} 0.$$

For the left, we have $u \mapsto -T + iu$, so

$$\begin{aligned}\left|\int_{\gamma_T^\ell} \frac{se^{st}}{s^2+4}\,\mathrm{d}s\right| &\le \frac{1}{T^2-4}\int_{-T}^{T}(T^2+u^2)e^{-Tt}\,\mathrm{d}u \\ &\le \frac{4T^3}{T^2-4}e^{-Tt} \xrightarrow{T\to\infty} 0,\end{aligned}$$

since $t > 0$. Finally, the only poles of the integrand are at $2i$ and $-2i$ which are in the interior of γ_R, and these have residue

$$\begin{aligned}\operatorname{Res}\left(\frac{se^{st}}{s^2+4}; 2i\right) &= \lim_{s\to 2i}(s-2i)\frac{se^{st}}{s^2+4} = \frac{2ie^{2it}}{4i} = \frac{e^{2it}}{2} \\ \operatorname{Res}\left(\frac{se^{st}}{s^2+4}; -2i\right) &= \lim_{s\to -2i}(s+2i)\frac{se^{st}}{s^2+4} = \frac{-2ie^{-2it}}{-4i} = \frac{e^{-2it}}{2}.\end{aligned}$$

Thus, by the residue theorem,

$$f(t) = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{1-iT}^{1+iT}\frac{se^{st}}{s^2+4}\,\mathrm{d}s = \frac{e^{2it}+e^{-2it}}{2} = \cos 2t.$$

f and f' are both bounded, so Theorem 1 applies, which means that $\cos 2t$ is the unique differentiable function whose Laplace transform is $\frac{s}{s^2+4}$.

4. Other Applications

The probabilist Tom Liggett proposed the following problem in [6], which we attempt to solve here by adapting the solution from [7]:

Problem 1. *Let $a_0, a_2, \dots$ be a bounded sequence of real numbers, and suppose that the power series*

$$f(x) \coloneqq \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

(which has an infinite radius of convergence) decays like $\mathcal{O}(e^{-x})$ as $x \to \infty$, in the sense that the function $e^x f(x)$ remains bounded as $x \to \infty$. Must the sequence a_n be of the form $a_n = C(-1)^n$ for some constant C?

Solution. First, we will prove the remark about the radius of convergence. Let $A > 0$ be an upper bound for the sequence, so by the root test,

$$\liminf_{n\to\infty} \left|\frac{a_n}{n!}\right|^{-1/n} \geq A \liminf_{n\to\infty} |n!|^{1/n}.$$

Now observe that

$$\begin{aligned}
\lim_{n\to\infty} (n!)^{1/n} = \lim_{n\to\infty} \exp\left(\frac{\ln n!}{n}\right) &= \lim_{n\to\infty} \exp\left(\frac{1}{n}\sum_{k=1}^{n} \ln k\right) \\
&\geq \lim_{n\to\infty} \exp\left(\frac{1}{n}\int_1^n \ln x \,\mathrm{d}x\right) \\
&= \lim_{n\to\infty} \exp\left(\frac{n\ln n - n}{n}\right) \\
&= \infty.
\end{aligned}$$

The inequality comes from the fact that e^x and $\ln x$ are increasing functions and observing that the sum is actually a lower Riemann sum for $\ln x$. Thus,

$$\liminf_{n\to\infty} \left|\frac{a_n}{n!}\right|^{-1/n} = \infty,$$

so the series converges on all of $\mathbb{R}$.

Now if $\mathrm{Re}(s) > -1$, then $f(x)e^{-sx}$ is integrable, i.e., its Laplace transform $\mathcal{L}\{f\}(s)$ is exists on this region. Moreover,

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-sx} f(x)\,\mathrm{d}x = \int_0^\infty \sum_{n=0}^{\infty} \frac{(-sx)^n}{n!} f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \left(\int_0^\infty \frac{(-x)^n}{n!} f(x)\,\mathrm{d}x\right) s^n.$$

Indeed, the sum converges uniformly on compact subsets: f is bounded, which leaves the power series for e^{-x}, so we may commute the integral and the sum. We conclude that $\mathcal{L}\{f\}(s)$ is holomorphic on $\mathrm{Re}(s) > -1$.

On the other hand,

$$e^{-sx} a_n \frac{x^n}{n!} \leq A e^{-sx} \frac{x^n}{n!}$$

is uniformly summable on compact sets. Thus, on compact sets, we may apply the Weierstrass M-test to interchange the Laplace transform and the summation:

$$\mathcal{L}\{f\}(s) = \sum_{n=0}^{\infty} \mathcal{L}\left\{a_n \frac{x^n}{n!}\right\}(s) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \mathcal{L}\{x^n\}(s).$$

We then calculate

$$\begin{aligned}\mathcal{L}\{x^n\}(s) = \int_0^\infty e^{-sx}x^n \,\mathrm{d}x &= -\frac{e^{-sx}}{s}x^n\bigg|_{x=0}^{x=\infty} + \frac{n}{s}\int_0^\infty e^{-sx}x^{n-1}\,\mathrm{d}x \\ &= \frac{n}{s}\int_0^\infty e^{-sx}x^{n-1}\,\mathrm{d}x \\ &= \frac{n}{s}\mathcal{L}\{x^{n-1}\}(s).\end{aligned}$$

Notice that $\mathcal{L}\{1\}(s) = \frac{1}{s}$, so we conclude by induction that $\mathcal{L}\{x^n\}(s) = \frac{n!}{s^{n+1}}$, which means

$$\mathcal{L}\{f\}(s) = \sum_{n=0}^\infty \frac{a_n}{s^{n+1}}$$

on their common domain. The left-hand side is holomorphic on $\mathrm{Re}(s) > -1$ and the right-hand side is holomorphic on $|s| > 1$, so because these two functions agree on an open, non-empty set, they are the restriction of a single holomorphic function on their connected component $\{|s| > 1\} \cup \{\mathrm{Re}(s) > -1\} = \{s \neq -1\}$. It remains to study what occurs when $s = -1$.

We will show that $g(s) := (s+1)\mathcal{L}\{f\}(s)$ has a pole at $z = -1$ by giving a bound on the annulus $0 < |s+1| < \frac{1}{2}$. Let B be an upper bound for $e^x f(x)$ on the $(-1,\infty)$. Because we have different expressions for $g(s)$ on different regions, we need to consider several cases:

Case 1: $\mathrm{Re}(s) \le -1$. In this case, we have

$$|g(s)| \le |s+1|\sum_{n=0}^\infty \frac{A}{|s|^{n+1}} \le \frac{A}{2}\frac{|s|^{-1}}{1-|s|^{-1}} \le \frac{A}{|s|-1}.$$

Moreover, if we write $s = x+iy$, then $|x| - 1 = ||x|-1| \le |x+1| < 1$ and so

$$\begin{aligned}|s+1|^2 = (|x|-1)^2 + y^2 \le |x| - 1 + y^2 &\le (|x|-1)(|x|+1) + y^2 \\ &\le |x|^2 - 1 + y^2 \\ &= |s|^2 - 1 \\ &\le (|s|-1)(|s|+1) \\ &\le 3(|s|-1) \\ \implies |g(s)| &\le \frac{3A}{|s+1|^2}.\end{aligned}$$

Case 2: $|s| \le 1$. Here,

$$\begin{aligned}|g(s)| = |s+1||\mathcal{L}\{f\}(s)| &\le \frac{1}{2}\int_0^\infty e^{-x\,\mathrm{Re}(s)}|f(x)|\,\mathrm{d}x \\ &\le \frac{B}{2}\int_0^\infty e^{-x\,\mathrm{Re}(s)}e^{-x}\,\mathrm{d}x \\ &= \frac{B}{2}\frac{1}{\mathrm{Re}(s)+1} \\ &\le \frac{B}{\mathrm{Re}(s)+1}.\end{aligned}$$

As before, write $s = x+iy$ so $x^2+y^2 = |s|^2 \le 1 \implies y^2 \le 1-x^2 = (1-x)(1+x) \le 2(x+1)$, which gives

$$|s+1|^2 = (x+1)^2 + y^2 \le (x+1)^2 + 2(x+1) \le (x+1)(x+3) \le 4(x+1).$$

Thus,

$$|g(s)| \le \frac{4B}{|s+1|^2}.$$

Case 3: $\mathrm{Re}(s) > -1$ and $|s| > 1$. Then if $s = x + iy$,

$$\begin{aligned}
|s+1|^2 = (x+1)^2 + y^2 &= x^2 + 2x + 1 + y^2 \\
&= |z|^2 + 2(x+1) - 1 \\
&= 2(x+1) + (|z|-1)(|z|+1) \\
&\le 2(x+1) + 3(|z|-1) \\
&\le \max\{2(x+1), 3(|z|-1)\} + \max\{2(x+1), 3(|z|-1)\}.
\end{aligned}$$

If the maximum is $2(x+1)$, then we obtain twice the bound as in Case 2, and if the maximum is $6(|z|-1)$, we get similarly get twice the bound in Case 1. Altogether, if we set $K = \max\{A, B\}$, then we obtain the bound

$$|g(s)| \le \frac{6K}{|s+1|^2}$$

on the annulus, so g has a pole of order at most 2 at $s = -1$. But for some $\delta > 0$, f is bounded by some $M > 0$ for $t \in (-1, -1+\delta)$ by continuity, which means

$$|g(t)| = |(t+1)\mathcal{L}\{f\}(t)| \le \left| M(t+1) \int_0^\infty e^{-tx}\, \mathrm{d}x \right| = \frac{M(t+1)}{|t|} \xrightarrow{t \to -1^+} 0.$$

Thus, the pole at $s = -1$ is actually not a pole, but a removable singularity. Away from $s = -1$, we have

$$|g(s)| \le \frac{A}{|s| - 1} \xrightarrow{s \to \infty} 0.$$

In conclusion, $g(s)$ is a bounded entire function, so by Liouville's theorem, $g(s) = C$ for some $C \in \mathbb{C}$ for all $s \in \mathbb{C}$. Hence,

$$\sum_{n=0}^\infty \frac{a_n}{s^{n+1}} = \frac{C}{s+1} = \frac{C}{s} \frac{1}{1 + \frac{1}{s}} = \sum_{n=0}^\infty \frac{C(-1)^n}{s^{n+1}}.$$

By uniqueness of Laurent series, we have $a_n = C(-1)^n$, which completes the solution. $\square$

Remark. The decay rate of $\mathcal{O}(e^{-x})$ is optimal, i.e., replacing the decay rate with $\mathcal{O}(e^{-ax})$ yields a negative answer. For any $a \in (0,1)$, set

$$f(x) = \exp\Big(-\Big(a + i\sqrt{1-a^2}\Big)x\Big) = \sum_{n=0}^\infty (-1)^n \Big(a + i\sqrt{1-a^2}\Big)^n \frac{x^n}{n!}.$$

By construction, $|f(x)| = e^{-ax}$ and the coefficients satisfy

$$\left| a + i\sqrt{1-a^2} \right|^n = \left| a^2 + 1 - a^2 \right|^n = 1,$$

i.e., they are bounded, but the coefficients of f do not have the form in the problem statement.

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