Q1

3.16 Given $\mathbf{B} = \hat{\mathbf{x}}(z - 3y) + \hat{\mathbf{y}}(2x - 3z) - \hat{\mathbf{z}}(x + y)$, find a unit vector parallel to $\mathbf{B}$ at point $P(1, 0, -1)$.

$$B = \hat{x}(-1) + \hat{y}(2+3) - \hat{z}(1)$$

$$= -\hat{x} + 5\hat{y} - \hat{z}$$

$$|B| = \sqrt{(-1)^2 + 5^2 + (-1)^2}$$

$$\text{Unit vector} = \frac{B}{|B|} = \boxed{\frac{1}{\sqrt{27}}(-\hat{x} + 5\hat{y} - \hat{z})}$$

Q2

2) Solve only items (c) and (d)

3.48* Determine if each of the following vector fields is solenoidal, conservative, or both:

(a) $\mathbf{A} = \hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}2xy$

(b) $\mathbf{B} = \hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z$

(c) $\mathbf{C} = \hat{\mathbf{r}}(\sin\phi)/r^2 + \hat{\boldsymbol{\phi}}(\cos\phi)/r^2$

(d) $\mathbf{D} = \hat{\mathbf{R}}/R$

r^{-1}

$-r^{-2}$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}}r & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

c) $\nabla \cdot C = 0$ is solenoidal $\qquad \nabla \times C = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\sin\varphi \cdot r^{-2}\right) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(\cos\varphi\, r^{-2}\right)$$

$$= \frac{-\sin\varphi}{r^3} + \frac{-\sin}{r^3} \neq 0$$

$\nabla \times C = 0$ is conservative

$$= \hat{r}\left(\frac{1}{r}\frac{\partial C_z}{\partial \phi} - \frac{\partial C_\varphi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial C_r}{\partial z} - \frac{\partial C_z}{\partial r}\right) + \hat{z}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\varphi) - \frac{\partial C_r}{\partial \varphi}\right]$$

$$= \hat{r}\left(0 - \underbrace{\frac{\partial}{\partial z}\frac{\cos\varphi}{r^2}}_{0}\right) + \hat{\varphi}\left(\underbrace{\frac{\partial}{\partial z}\frac{\sin\varphi}{r^2}}_{0} - 0\right) + \hat{z}\left[\frac{\partial}{\partial r}\left(\frac{\cos\varphi}{r}\right) - \frac{\partial}{\partial \varphi}\left(\frac{\sin\varphi}{r^2}\right)\right]$$

$$= \hat{z}\left[\frac{-\cos\varphi}{r^2} - \frac{\cos\varphi}{r^2}\right] \cdot \frac{1}{r} \neq 0$$

neither solenoidal nor conservative

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}}R & \hat{\boldsymbol{\phi}}R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial\theta}\right]$$

d) $D = \hat{R}\frac{1}{R}$

$$\nabla\cdot D = \frac{1}{R^2}\cdot\frac{\partial}{\partial R}\left(R^2\cdot\frac{1}{R}\right) = \frac{1}{R^2}$$

$$\nabla\times D = \hat{R}\frac{1}{R\sin\theta}\left[\underbrace{\frac{\partial}{\partial\theta}}_{0} - \underbrace{\frac{\partial A_\theta}{\partial\phi}}_{0}\right] + \hat{\theta}\frac{1}{R}\left[\underbrace{\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{R}\right)}_{0} - \underbrace{\frac{\partial}{\partial R}(0)}_{0}\right]$$
$$+\hat{\phi}\frac{1}{R}\left[\underbrace{\frac{\partial}{\partial R}\,0}_{0} - \underbrace{\frac{\partial}{\partial\theta}\frac{1}{R}}_{0}\right]$$

$= 0$

So D is not solenoidal
but conservative

Q3

3) Solve only items (b) and (e)

3.49 Find the Laplacian of the following scalar functions:

(a) $V = 4xy^2z^3$

(b) $V = xy + yz + zx$

(c) $V = 3/(x^2 + y^2)$

(d) $V = 5e^{-r}\cos\phi$

(e) $V = 10e^{-R}\sin\theta$

Scalar Laplacian operator

– Applies to a scalar and produces a scalar

$$\nabla^2 V \triangleq \nabla\cdot(\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

b) $V = xy + yz + zx$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= 0 + 0 + 0 = 0$$

e)

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(-R^2\cdot 10e^{-R}\sin\theta\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot 10e^{-R}\cos\theta\right) + 0$$

$$= \frac{1}{R^2}\left(-2R\,10e^{-R}\sin\theta + 10R^2e^{-R}\sin\theta\right)$$

$$+ \frac{1}{R^2\sin\theta}\left(10e^{-R}\cdot\cos 2\theta\right)$$

$$= \boxed{10e^{-R}\left(-\frac{2}{R}\sin\theta + \sin\theta\right) + \frac{10e^{-R}\cos 2\theta}{R^2\sin\theta}}$$

4) Solve only items (a) and (b)

4.5* Find the total charge on a circular disk defined by $r \leq a$ and $z = 0$ if:

(a) $\rho_s = \rho_{s0} \cos\phi$ (C/m^2)

(b) $\rho_s = \rho_{s0} \sin^2\phi$ (C/m^2)

(c) $\rho_s = \rho_{s0} e^{-r}$ (C/m^2)

(d) $\rho_s = \rho_{s0} e^{-r} \sin^2\phi$ (C/m^2)

where ρ_{s0} is a constant.

$$Q = \int_s \rho_s \, ds$$

a) $$Q = \int_s \rho_s ds = \rho_{s0} \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \cos\phi \, r \cdot dr \, d\varphi$$

$$= \rho_{s0} \int_{r=0}^{a} r \cdot \sin \Big|_0^{2\pi} dr \quad = 0$$

b) $$Q = \int_s \rho_s ds = \rho_{s0} \int_{r=0}^{a} \int_0^{2\pi} \sin^2\varphi \; r \; dr \, d\varphi$$

$$= \rho_{s0} \int_{r=0}^{a} \frac{\varphi - \sin(2\varphi)}{2} \Big|_0^{2\pi} \cdot r \, dr$$

$$= \rho_{s0} \; \frac{1}{2} r^2 \Big|_0^{a} \cdot \left(\frac{2\pi}{2} - 0 \right)$$

$$= \rho_{s0} \frac{a^2}{2} \cdot \pi$$

5)

4.7* If $\mathbf{J} = \hat{\mathbf{R}}5/R$ (A/m^2), find I through the surface $R = 5$ m.

$$I = \int_s J \cdot ds$$

$$= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{5}{R} \cdot \underbrace{R^2 \sin\theta \; d\theta \; d\varphi}^{\text{surface area}}$$

$$= 5R \cdot 2\pi \cdot (-\cos\theta)\Big|_0^{\pi}$$

$$= 10R\pi \cdot (1-(-1)) = 20R\pi = 100\pi$$

$$= 314 \text{ A}$$

4.23* The electric flux density inside a dielectric sphere of radius a centered at the origin is given by

$$\mathbf{D} = \hat{\mathbf{R}}\rho_0 R \quad (\text{C/m}^2)$$

where ρ_0 is a constant. Find the total charge inside the sphere.

$$\oiint \mathbf{D} \cdot d\mathbf{s} = Q$$

$$\oiint_S D \cdot ds = Q$$

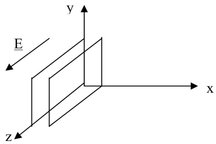
$$= \rho_0 \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} R \cdot R^2 \sin\theta \, d\theta \, d\varphi$$

$$= \rho_0 \cdot R^3 \cdot 2\pi \underbrace{\left[-\cos\theta\right]\Big|_0^{\pi}}_{1-(-1)}$$

$$= 4\pi \rho_0 \cdot R^3$$

$$= 4\pi \rho_0 \cdot a^3$$

7) Consider two infinite parallel plates with separation d, which are parallel to the $y-z$ plane. The electric field between the plates is $\underline{E} = \hat{z}A\sin(\pi x/d)\cos(\pi ct/d)$. Find the magnetic field, as well as the electric current and charge density in the region between the plates. Show that $\underline{\nabla}\cdot\underline{E} = 0$



$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\underline{\nabla}\cdot\underline{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$
$$= 0 + 0 + 0$$
$$= 0$$

Magnetic field

$$\nabla \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & A\sin(\frac{\pi x}{d})\cos(\frac{\pi ct}{d}) \end{vmatrix}$$

$$\nabla \times E = \begin{matrix} \frac{\partial}{\partial y}\left(A\sin(\frac{\pi x}{d})\cos(\frac{\pi ct}{d})\right) - \frac{\partial}{\partial z} 0 & \hat{x} \\ \frac{\partial}{\partial z}(0) - \left(\frac{\partial}{\partial x}\left(A\sin(\frac{\pi x}{d})\cos(\frac{\pi ct}{d})\right)\right) & \hat{y} \\ \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(0) & \hat{z} \end{matrix}$$

$$= 0\ \hat{x} + \left(0 - \frac{\pi}{d}A\cos(\frac{\pi x}{d})\cos(\frac{\pi ct}{d})\right)\hat{y} + 0\ \hat{z}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\int \frac{\partial B}{\partial t}dt = \int \frac{\pi}{d}A\cos(\frac{\pi x}{d})\cos(\frac{\pi ct}{d})dt$$

$$\underline{B} = \frac{\pi}{d}A\cos(\frac{\pi x}{d})\sin(\frac{\pi ct}{d})\cdot\frac{d}{\pi c}$$
$$= \frac{A}{c}\cos(\frac{\pi x}{d})\sin(\frac{\pi ct}{d})\ \hat{y}$$

$$\nabla \cdot \mathbf{D} = \rho \qquad \text{Gauss's law}$$

Charge Density

in vaccum $D = \varepsilon_0 E \Rightarrow D = \hat{z}\, \varepsilon_0 \left(A \sin\left(\frac{\pi x}{d}\right) \cos\left(\frac{\pi c t}{d}\right)\right)$

$$\nabla \cdot D = \nabla \cdot \varepsilon_0 E = \varepsilon_0 \underbrace{\nabla \cdot E}_{0} = 0$$

Electric current

$$B = \mu_0 H$$

$$H = \frac{B}{\mu_0} = \frac{A}{\mu_0 c} \cos\left(\frac{\pi x}{d}\right) \sin\left(\frac{\pi c t}{d}\right) \hat{y}$$

Sol ①

$$\oint_C H \cdot dl =$$

$$\int_0^z \frac{A}{\mu_0 c} \cos\left(\frac{\pi x}{d}\right) \sin\left(\frac{\pi c t}{d}\right) dz$$

$$= \frac{A \cdot z}{\mu_0 c} \cos\left(\frac{\pi x}{d}\right) \sin\left(\frac{\pi c t}{d}\right)$$

$$\oint H \cdot dl = \frac{d}{dt} \iint_S D \cdot ds + I$$

$$= \varepsilon_0 \frac{d}{dt} \int_0^d \int_0^z A \sin\left(\frac{\pi x}{d}\right) \cos\left(\frac{\pi c t}{d}\right) dx\, dz$$

$$= \varepsilon_0 \frac{d}{dt} \int_0^d A z \sin\left(\frac{\pi x}{d}\right) \cos\left(\frac{\pi c t}{d}\right) dx$$

$$= \varepsilon_0 \frac{d}{dt} \left(-\frac{Az\, d}{\pi} \cos\left(\frac{\pi x}{d}\right) \cos\left(\frac{\pi c t}{d}\right)\right)\Bigg|_0^d$$

$$= \varepsilon_0 \frac{d}{dt} \left(\frac{-Az\, d}{\pi} \cdot (-1) \cos\left(\frac{\pi c t}{d}\right) - \left(-\frac{Az \sin d}{\pi} \cos\left(\frac{\pi c t}{d}\right)\right)\right)$$

$$= \varepsilon_0 \frac{d}{dt} \left(\frac{2Azd}{\pi} \cdot \cos\left(\frac{\pi c t}{d}\right)\right)$$

$$= -\varepsilon_0 \frac{2Azd}{\pi} \sin\left(\frac{\pi ct}{d}\right) \cdot \frac{\pi c}{d}$$

$$= -\varepsilon_0 \; 2Az \cdot c \; \sin\left(\frac{\pi ct}{d}\right)$$

$$I = \oint H \, dl - \left(-\varepsilon_0 \, 2Azc \, \sin\left(\frac{\pi ct}{d}\right)\right)$$

Sol 2

$$\nabla \times H = \frac{\partial D}{\partial t} + J \qquad \iint_S J \, ds = I$$

$$H = \frac{A}{u_0 c} \cos\left(\frac{\pi x}{d}\right) \sin\left(\frac{\pi ct}{d}\right) \hat{y}$$

$$\nabla \times H = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix}$$

$$\nabla \times E = \begin{array}{ll} \frac{\partial}{\partial y}(0) - \underbrace{\frac{\partial}{\partial z} H_y}_{0} & \hat{x} \\ \frac{\partial}{\partial z}(0) - \left(\frac{d}{\partial x} \; 0\right) & \hat{y} \\ \frac{\partial}{\partial x}(H_y) - \frac{\partial}{\partial y}(0) & \hat{z} \end{array}$$

$$= 0 \; \hat{x} + 0 \; \hat{y} + \hat{z} \frac{A}{u_0 c} \cdot \frac{\pi}{d} \left(-\sin\left(\frac{\pi x}{d}\right) \sin\left(\frac{\pi ct}{d}\right)\right)$$

$$= \hat{z} - \frac{A\pi}{u_0 c d} \sin\left(\frac{\pi x}{d}\right) \sin\left(\frac{\pi ct}{d}\right)$$

$$D = \hat{z}\,\varepsilon_0\left(A\sin\left(\frac{\pi x}{d}\right)\cos\left(\frac{\pi c t}{d}\right)\right)$$

$$\frac{\partial D}{\partial t} = \hat{z}\,\varepsilon_0 - A\sin\left(\frac{\pi x}{d}\right)\sin\left(\frac{\pi c t}{d}\right)\cdot\frac{d}{\pi c}$$

$$J = \nabla \times H - \frac{\partial D}{\partial t}$$

$$\hat{z} - \frac{A\pi}{u_0 c d}\sin\left(\frac{\pi x}{d}\right)\sin\left(\frac{\pi c t}{d}\right) - \left(\hat{z} - \varepsilon_0 A\sin\left(\frac{\pi x}{d}\right)\sin\left(\frac{\pi c t}{d}\right)\cdot\frac{d}{\pi c}\right)$$

$$= \hat{z}\left[A\sin\left(\frac{\pi x}{d}\right)\sin\left(\frac{\pi c t}{d}\right)\left(\frac{\varepsilon_0 d}{\pi c} - \frac{\pi}{u_0 c d}\right)\right]$$

$$= \hat{z}\left[A\sin\left(\frac{\pi x}{d}\right)\sin\left(\frac{\pi c t}{d}\right)\left(\frac{\varepsilon_0 d^2\cdot u_0 - \pi^2}{\pi c u_0 d}\right)\right]$$

$$\iint_S J\,ds = I$$