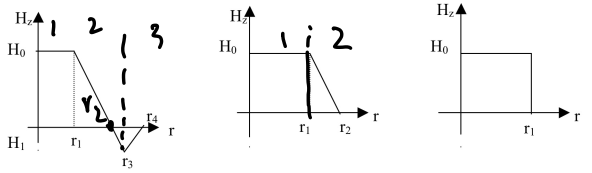
1) Three static distributions of currents (including volumetric and surface) generate corresponding magnetic fields that depend only on the cylindrical coordinate r as shown in Figures (a), (b), and (c) below and point in the z-direction. Find the (volume AND possible surface) current distributions in these three cases.



(a) (b) (c)

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\hat{\mathbf{n}}_{12} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s$$

$H = \hat{z}\, H(r)$

a) We have 3 region J_1, J_2 and J_3 also surface J_{12} and J_{23}

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} r & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

$$J_1: \nabla \times H_1 = \nabla \times (\hat{z} H_0) = \hat{\phi}\left(\frac{\partial H_0}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{z}\, 0$$
$$= \hat{\phi}(0 - 0) = 0$$

$$\frac{H_0 - H_1}{r_1 - r_3} \cdot r_2 + b = 0$$
$$b = -\frac{H_0 - H_1}{r_1 - r_3} \cdot r_2$$

$$\frac{H_1}{r_3 - r_4} \cdot r_4 + b = 0$$

$$\hat{z} H(r) \begin{cases} H_1: & H_0 \quad r < r_1 \\ H_2: & \frac{H_0 - H_1}{r_1 - r_3} \cdot r - \frac{H_0 - H_1}{r_1 - r_3} \cdot r_2 \quad r_1 < r < r_3 \\ H_3: & \frac{H_1 - 0}{r_3 - r_4} \cdot r - \frac{H_1}{r_3 - r_4} \cdot r_4 \quad r_3 < r < r_4 \end{cases}$$

$$J_2\ \nabla \times H_2 = \hat{\phi} \frac{-\partial}{\partial r}(H_2) = \boxed{\hat{\phi}\, \frac{H_0 - H_1}{r_1 - r_3}}$$

$$J_3\ \nabla \times H_3 = \boxed{\hat{\phi}\ -\frac{H_1}{r_3 - r_4}}$$

$$J_{12} = \hat{r}_{12}(H_2 - H_1)$$

$$J_{23} = \hat{r}_{23}(H_3 - H_2)$$

$$\frac{H_0}{r_1-r_2}\cdot r_2 + b = 0$$

b)

$$\hat{z}\ H(r)\begin{cases} H_0 & r<r_1 \\ \frac{H_0-0}{r_1-r_2}r - \frac{H_0}{r_1-r_2}\cdot r_2 & r_1<r<r_2 \end{cases}$$

$$J_1 = -\hat{\Phi}\frac{\partial}{\partial r} H_0 = \boxed{0}$$

$$J_2 = -\hat{\Phi}\frac{\partial}{\partial r}\left(H_2(r)\right) = \boxed{\hat{\Phi}\ -\frac{H_0}{r_1-r_2}}$$

$$J_{12} = \hat{r}_{12}\times\left(H_2 - H_1\right) = \hat{r}\times\left(\frac{H_0}{r_1-r_2}(r\cdot r_2) - H_0\right)$$

$$= \boxed{\hat{r}\times(0) = 0}$$

c)

$$\hat{z}\ H(r)\begin{cases} H_0 & r<r_1 \\ 0 & r>r_1 \end{cases}$$

$$\boxed{J_1 = \hat{\Phi} - \frac{\partial}{\partial r} H_0 = 0} \quad ; \quad \boxed{J_2 = 0}$$

$$\boxed{J_{12} = \hat{r}_{12}\times(0-H_0)}$$

Q2

2) Consider a spherical volume of radius R_0. The static electric field distribution is given by

$$\mathbf{E} = \begin{cases} (A/R_0)(-\hat{\mathbf{R}}\cos(\theta) + \hat{\boldsymbol{\theta}}\sin(\theta)); R < R_0 \\ (AR_0^2/R^3)(\hat{\mathbf{R}}2\cos(\theta) + \hat{\boldsymbol{\theta}}\sin(\theta)); R > R_0 \end{cases}$$

where A is a constant. Find the volumetric and surface charge distributions everywhere in space. Find the total charge on the surface of the sphere.

3)

ρ_v ρ_s

Q

$$\nabla \cdot \mathbf{D} = \rho$$

$D = \varepsilon E$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$R < R_0$

$\nabla \cdot D = \varepsilon \nabla \cdot E$

$$\rho_v = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \cdot \frac{A}{R_0} \cdot (-\cos(\theta))\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \sin\theta \cdot \frac{A}{R_0}\right) \cdot \varepsilon_1$$

$$= -\frac{2}{R} \cdot \frac{A}{R_0}\cos(\theta) + \frac{1}{R\sin\theta} \cdot \sin(2\theta) \cdot \frac{A}{r_0}$$

$$= \frac{A}{r_0}\left(-\frac{2}{R}\cos(\theta) + \frac{1}{R\sin\theta}2\sin(\theta)\cos(\theta)\right) = 0$$

$R > R_0$

$\nabla \cdot D = \rho_v$

$$= \frac{AR_0^2}{R^3}\left(\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \cdot 2\cos(\theta)\right)\right) + \left(\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \cdot \sin\theta\right)\right) \cdot \varepsilon_2$$

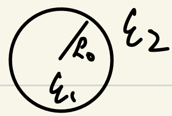
$$= \frac{AR_0^2}{R^3} \cdot \left(\frac{1}{R^2}2R \cdot 2\cos(\theta) + \frac{2}{R}\cos(\theta)\right)$$

$$= \frac{AR_0^2}{R^3}\left(\frac{2}{R} \cdot 2\cos(\theta) + \frac{2}{R}\cos(\theta)\right)$$

$$= \frac{AR_0^2}{R^3}\left(\frac{4}{R}\cos(\theta) + \frac{2}{R}\cos(\theta)\right)$$

$$= \frac{AR_0^2}{R^3} \cdot \frac{6}{R}\cos(\theta) \cdot \varepsilon_2$$

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$$



surface charge

$$\rho_s = \hat{R}_{12} \cdot$$

$$\left[\varepsilon_2 \frac{A R_0^2}{R^3} \left(\hat{R}\, 2\cos(\theta) + \hat{\theta} \sin(\theta) \right) - \varepsilon_1 \frac{A}{R_0} \left(-\hat{R} \cos(\theta) + \hat{\theta} \sin(\theta) \right) \right]$$

@ $R = R_0$

$$\rho_s = \hat{R}_{12} \cdot \left(\varepsilon_2 \frac{A}{R_0} (\hat{R}\, 2\cos\theta + \hat{\theta} \sin\theta) - \varepsilon_1 \frac{A}{R_0} (-\hat{R} \cos\theta + \hat{\theta} \sin\theta) \right)$$

$$= \varepsilon_2 \frac{A}{R_0} 2\cos\theta \, \hat{R} - \left(-\varepsilon_1 \frac{A}{R_0} \cos\theta \, \hat{R} \right)$$

$$= \frac{A}{R_0} \cos\theta \, (2\varepsilon_2 + \varepsilon_1)$$

Total charge

$$Q = \int_S \rho_s \, ds$$

$R = R_0$

$$Q = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \rho_s \cdot R^2 \sin\theta \, d\theta \, d\varphi$$

$$= \frac{A}{R_0} (2\varepsilon_2 + \varepsilon_1) \cdot R^2 \cdot 2\pi \underbrace{\int_{\theta=0}^{\pi} \sin\theta \cdot \cos\theta \, d\theta}_{0}$$

$$= 0$$

Q3

3)

4.24 In a certain region of space, the charge density is given in cylindrical coordinates by the function:

$$\rho_v = 5re^{-r} \quad (\text{C/m}^3)$$

Apply Gauss's law to find **D**.

$\nabla \cdot \mathbf{D} = \rho$ Gauss' s law

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \cdot D = 5re^{-r} = \frac{1}{r}\frac{\partial}{\partial r}(rD_r)$$

$$5r^2e^{-r} = \frac{\partial}{\partial r}(rD_r)$$

$$\int_0^r 5r^2e^{-r}dr = \int_0^r \frac{\partial}{\partial r}(rD_r)\,dr$$

By symbolab

$$5(-e^{-r}r^2+2(-e^{-r}\cdot r-e^{-r}))\Big|_0^r = D_r \cdot r$$

$$5(-e^{-r}r^2+2(-e^{-r}\cdot r-e^{-r})) - 2(-e^0) = D_r \cdot r$$

$$\boxed{D_r = \hat{r}\,\frac{5}{r}(-e^{-r}r^2+2(-e^{-r}\cdot r-e^{-r})+2)}$$

Q4

4)

4.26 If the charge density increases linearly with distance from the origin such that $\rho_v = 0$ at the origin and $\rho_v = 4\ \mathrm{C/m^3}$ at $R = 2$ m, find the corresponding variation of **D**.

$\rho_v = \frac{4}{2} \cdot R + b$

@ $\rho_v = 0$

$0 = 0 + b$

$\rho_v = 2R$

$\nabla \cdot D = \rho_v$

$$Q = \iiint_V \rho_v \, dv = \iint_S D \, ds$$

$$\iiint 2R \cdot dv = D_r \cdot 4\pi R^2$$

$$\int_0^R \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} 2R \, R^2 \sin\theta \, d\theta \, d\varphi \, dr = D_r \cdot 4\pi R^2$$

$$\int_0^R 2R^3 \cdot 4\pi \, dr = D_r \cdot 4\pi R^2$$

$$\frac{1}{4} \cdot 2R^4 \Big|_0^R \cdot 4\pi$$

$$\frac{1}{2} R^4 \cdot 4\pi = D_r \cdot 4\pi R^2$$

$$2\pi R^4 = D_r \cdot 4\pi R^2$$

$$D_r = \frac{1 \quad R^2}{2}$$

Q5

5.31* Given that a current sheet with surface current density $\mathbf{J}_s = \hat{\mathbf{x}}\,8$ (A/m) exists at $y = 0$, the interface between two magnetic media, and $\mathbf{H}_1 = \hat{\mathbf{z}}\,11$ (A/m) in medium 1 ($y > 0$), determine $\mathbf{H}_2$ in medium 2 ($y < 0$).

$$\hat{\mathbf{n}}_{12} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s$$

$$\begin{vmatrix} x & y & z \\ 0 & 1 & 0 \\ 0 & 0 & z \end{vmatrix}$$

$\hat{x} = z\hat{x} - 0 + 0$

$\hat{x}8 = \hat{x}z$

$8 = z = 11 - ?$

$? = 3$

$$\hat{y}_{21} \times (H_1 - H_2) = \hat{x}\,8$$

$$\hat{y}_{21} \times (\hat{z}\,11 - H_{2z}) = \hat{x}\,8$$

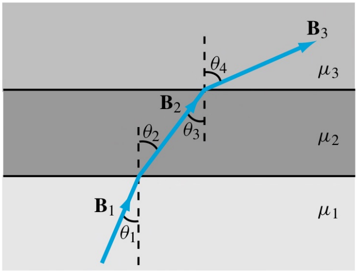
$$\boxed{H_{2z} = \hat{z}\,3\ \frac{A}{m}}$$

Q6

6)

5.34 Show that if no surface current densities exist at the parallel interfaces shown in Fig. 5-47, the relationship between θ_4 and θ_1 is independent of μ_2.

$\boxed{H_{2t} - H_{1t} = J_s}$ $\boxed{\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s}$



$B = \mu H$

If $J_s = 0$ $\therefore$ $H_{2t} = H_{1t}$ $\quad$ $H_{3t} = H_{2t}$

$$\frac{B_{2t}}{\mu_2} = \frac{B_{1t}}{\mu_1} \qquad \frac{B_{3t}}{\mu_3} = \frac{B_{2t}}{\mu_2}$$

$$\Rightarrow \frac{B_{1t}}{\mu_1} = \frac{B_{3t}}{\mu_3}$$

$$B_{1t} = \underline{B_1} \cdot \sin\theta_1 \qquad B_{3t} = \underline{B_3} \cdot \sin\theta_4$$

$$\frac{B_1 \sin\theta_1}{\mu_1} = \frac{B_3 \sin\theta_4}{\mu_3}$$ $\Leftarrow$ independent of μ_2

7) You are given an electric field (given in a phasor form)

$$\mathbf{E} = \begin{cases} \hat{\mathbf{x}} E_0 \sin(\beta z); z > 0 \\ 0; z < 0 \end{cases}$$

$B = uH$

i. Find the magnetic (phasor) field

ii. Find the volumetric charge density and current density in the entire space

iii. Find the surface charge density and current density in the entire space

$$\nabla \times \tilde{\mathbf{E}} = -j\omega \tilde{\mathbf{B}}$$ Faraday's law

i) $-\nabla \times E \cdot \frac{1}{jw} = B = uH$

$$\begin{vmatrix} x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \hat{x}\, 0 + \frac{\partial}{\partial z}(E_x)\, \hat{y} + \hat{z} \frac{\partial}{\partial y} E_x$$

$$= \hat{y}\, \beta E_0 \cos(\beta z)$$

$$-\frac{1}{jw} \nabla \times \hat{E} = \hat{y}\, -\frac{\beta E_0}{jw} \cos(\beta z) = \hat{B}$$

$$\boxed{H \begin{cases} \hat{y}\, \frac{-\beta E_0}{ujw} \cos(\beta z) & \text{for } z > 0 \\ 0 \;; z < 0 \end{cases}}$$

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ii) $\nabla \cdot D = \rho_v = \varepsilon \nabla \cdot E = \rho_v$

$$= \varepsilon \left[\frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z \right]$$

$$= \boxed{0}$$

Curren Density Js $\nabla \times \tilde{\mathbf{H}} = j\omega\tilde{\mathbf{D}} + \tilde{\mathbf{J}}$

$\nabla \times H$

$$\begin{vmatrix} x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix} = \hat{x} -\frac{\partial}{\partial z} H_y - \hat{y} 0 + \hat{z} \frac{\partial}{\partial x} H_y$$

$$= -\hat{x} \frac{\beta^2 E_0}{u j w} \sin(\beta z) + 0$$

$$j w \hat{D} = j w \varepsilon \tilde{E}$$

$$= j w \varepsilon \; \hat{x} E_0 \sin(\beta z)$$

$$\tilde{J} = \nabla \times \tilde{H} - j w \hat{D}$$

$$= \boxed{\hat{x} \left[j w \varepsilon E_0 \sin(\beta z) - \frac{\beta^2 E_0}{u j w} \sin(\beta z) \right]}$$

iii) ρ_s $\hat{\mathbf{n}}_{12} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$

D_1 for $z > 0$ $\qquad D = \varepsilon E$

D_2 for $z < 0$

$D_2 - D_1 = \hat{x}\left(0 - E_0 \sin(\beta z)\right)$

$$\boxed{\rho_s = \hat{z}_{12} \cdot \hat{x}\left(-E_0 \sin(\beta z)\right) = 0}$$

$\hat{z}_{12} \leftarrow \hat{\mathbf{n}}_{12} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s$

H_2 for $z > 0$

H_1 for $z < 0$

$H_2 - H_1 = \hat{y}\, 0 - \hat{y} \frac{-\beta E_0}{u j \omega} \cos(\beta z) = \hat{y} \frac{\beta E_0}{u j \omega} \cos(\beta z)$

$$\begin{vmatrix} x & y & z \\ 0 & 0 & 1 \\ 0 & H_{2-1,y} & 0 \end{vmatrix}$$

$J_s = -H_{2-1,y}\hat{x} - \hat{y}\, 0 + \hat{z}\, 0$

$= \frac{-\beta E_0}{u j \omega} \cos(\beta z)$

@ Boundary $z = 0$

$$\boxed{J_s = \frac{-\beta E_0}{u j \omega}}$$