Q1

1) Consider a line charge distributed on straight line $-l < z < l$. Find the electric field $\mathbf{E}$ using two methods:

a) Find the electric potential V based on superposition (integration) and then the electric field as $\mathbf{E} = -\nabla V$

b) Find $\mathbf{E}$ directly based on superposition

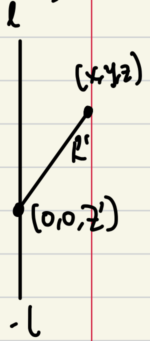
Assume observation point is at (x, y, z)

$$\therefore R' = \sqrt{(z-z')^2 + \underbrace{x^2+y^2}_{r^2}}$$

$$= \sqrt{(z-z') + r^2}$$

For a single charge, q

a) Potential:



$$V = \frac{q}{4\pi\varepsilon_0 r} = \frac{1}{4\pi\varepsilon_0}\int_l \frac{\rho_l dl}{R'}$$

$$V_p = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{R'}\, dq = \frac{1}{4\pi\varepsilon_0}\int_{-l}^{l} \frac{1}{R'} \cdot \rho_l \cdot dz'$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{-l}^{l} \frac{1}{\sqrt{(z-z')^2 + r^2}} \cdot \rho_l \cdot dz'$$

$$= \frac{\rho_l}{4\pi\varepsilon_0} \ln\left[\frac{(l-z) + \sqrt{r^2 + (l-z)^2}}{(-l-z) + \sqrt{r^2 + (l+z)^2}}\right]$$

From Discussion note

Due to symmetry, $\frac{\partial}{\partial \varphi} V = 0$, $\frac{\partial}{\partial z} V|_{z=0} = 0$

$$\therefore \quad E = -\hat{r}\frac{\partial V}{\partial r}$$

$$= -\hat{r}\frac{\partial}{\partial r}\frac{\rho_l}{4\pi\varepsilon_0}\left[\ln(l + \sqrt{r^2+l^2}) - \ln(-l + \sqrt{r^2+l^2})\right]$$

$$= -\hat{r}\frac{\rho_l}{4\pi\varepsilon_0}\frac{r}{\sqrt{r^2+l^2}}\left(\frac{1}{l+\sqrt{r^2+l^2}} - \frac{1}{-l+\sqrt{r^2+l^2}}\right)$$

$$E = \frac{\rho_l \cdot l}{2\pi\varepsilon_0} \cdot \frac{1}{r\sqrt{r^2+l^2}}\,\hat{r}$$

From Discussion Note:

b) Electric field for a single charge is

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi \varepsilon R^2}$$

$$E = \frac{q}{4\pi\varepsilon_0 R^2} \qquad R^2 = x^2 + y^2 + (z - z')^2$$

$$E = \hat{r} \frac{\rho_\ell}{4\pi\varepsilon_0} \int_{-\ell}^{\ell} \frac{1}{r^2 + z'^2} \cdot \frac{r}{\sqrt{r^2 + z'^2}} \, dz'$$

$$= \hat{r} \frac{\rho_\ell \cdot r}{4\pi\varepsilon_0} \int_{-\ell}^{\ell} \frac{dz'}{(r^2 + z'^2)^{\frac{3}{2}}}$$

$$= \hat{r} \frac{\rho_\ell \cdot r}{4\pi\varepsilon_0} \left(\frac{1}{r^2} \cdot \frac{z'}{\sqrt{r^2 + z'^2}} \right) \Big|_{-\ell}^{+\ell}$$

$$= \hat{r} \frac{\rho_\ell \cdot r}{4\pi\varepsilon_0} \cdot \frac{1}{r^2} \cdot \frac{2\ell}{\sqrt{r^2 + \ell^2}}$$

$$= \hat{r} \frac{\rho_\ell \cdot \ell}{2\pi\varepsilon_0} \cdot \frac{1}{r\sqrt{r^2 + \ell^2}} \hat{r}$$

Q2

2)

4.28 The circular disk of radius a shown in Fig.4-7 has uniform charge density ρ_s across its surface.

(a) Obtain an expression for the electric potential V at a point $P(0, 0, z)$ on the z-axis.

(b) Use your result to find $\mathbf{E}$ and then evaluate it for $z = h$. Compare your final expression with (4.24), which was obtained on the basis of Coulomb's law.

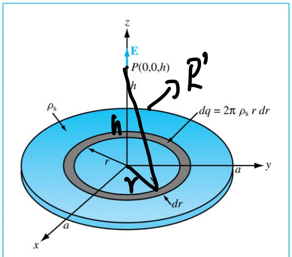


Figure 4-7: Circular disk of charge with surface charge density ρ_s. The electric field at $P(0, 0, h)$ points along the z-direction (Example 4-5).

a)

$$V = \frac{1}{4\pi\varepsilon} \int_0^a \frac{\rho_s \cdot 2\pi r\ dr}{R'}$$

$$R' = \sqrt{r^2 + h^2}$$

$$V = \frac{\rho_s \cdot 2\pi}{4\pi\varepsilon} \int_0^a \frac{r}{\sqrt{r^2+h^2}}\, dr = \frac{\rho_s \cdot 2\pi}{4\pi\varepsilon} \left(\sqrt{h^2+r^2}\right)\Big|_0^a$$

$$= \frac{\rho_s 2\pi}{4\pi\varepsilon} \left(\sqrt{h^2+a^2} - \sqrt{h^2}\right)$$

b)

$$(h^2+a^2)^{\frac{1}{2}} \qquad h$$

$$\mathbf{E} = -\nabla V$$

$$= -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$$

$$= 0 \qquad 0 \qquad \boxed{-\frac{\rho_s 2\pi}{4\pi\varepsilon}\left(\frac{h}{\sqrt{h^2+a^2}} - 1\right)\hat{z}}$$

3)

4.42 A 2×10^{-3}-mm-thick square sheet of aluminum has 5 cm × 5 cm faces. Find the following:

(a) The resistance between opposite edges on a square face.

(b) The resistance between the two square faces. (See Appendix B for the electrical constants of materials.)

$$\sigma_{AL} = 3.538 \cdot 10^{7} \frac{S}{m}$$

a)

$$R = \frac{l}{\sigma A} \quad (\Omega)$$

$$A = 2 \cdot 10^{-3} \cdot 10^{-3} \cdot 5 \cdot 10^{-2}$$

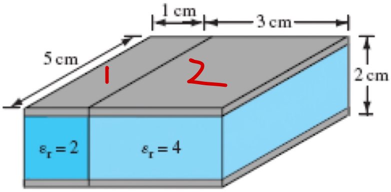
$$R = \frac{l}{\sigma_{AL} \cdot A} = 0.014 \ \Omega$$

b)

$$R = \frac{l = 2 \cdot 10^{-3} \cdot 10^{-3}}{\sigma_{AL} \cdot (5 \cdot 10^{-2})^2} = 2.26 \cdot 10^{-11} \ \Omega$$

Q4

4) Find the capacitance of the following capacitor



Parallel plate capacitor

$$\mathbf{E} = -\hat{z}E$$

$$V = -\int_0^d \mathbf{E}\cdot d\mathbf{l} = -\int_0^d (-\hat{z}E)\cdot\hat{z}\,dz = Ed,$$

$$C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{\varepsilon A}{d}$$

$$\varepsilon_0 = 8.854\cdot 10^{-12}$$

$$C_{tot} = C_1 + C_2$$

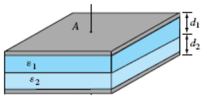
$$= \frac{\varepsilon_0 \varepsilon_{r_1}\cdot 5\cdot 10^{-2}}{2} + \frac{\varepsilon_0 \varepsilon_{r4}\cdot 15\cdot 10^{-2}}{2}$$

$$= 8.854\cdot 10^{-12}\left(5\cdot 10^{-2} + 2\cdot 15\cdot 10^{-2}\right)$$

$$= \boxed{3.0989\cdot 10^{-12}\ F}$$

Q5

5) Find the capacitance of the following capacitor



where $d_1 = 5mm, d_2 = 3mm, A = 400mm^2$

$$C_{tot} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} = \frac{1}{\frac{5}{400\cdot\varepsilon_0\varepsilon_1\cdot 10^{-3}} + \frac{3}{400\cdot 10^{-3}\varepsilon_0\varepsilon_2}}$$

$$= \frac{1}{\frac{5\varepsilon_2 + 3\varepsilon_1}{400\cdot 10^{-3}\varepsilon_0\varepsilon_1\varepsilon_2}}$$

$$= \boxed{\frac{400\cdot 10^{-3}\varepsilon_0\varepsilon_1\varepsilon_2}{5\varepsilon_2 + 3\varepsilon_1}}\ F$$

Q6

6) (Quiz 3, 2008) In a certain region of space, the volumetric charge density is given in the spherical coordinates by

$$\rho_v = \begin{cases} \rho_0; & a < R < b \\ 0; & \text{otherwise} \end{cases},$$

where R is the radial coordinate of the spherical coordinate system, whereas a and b are given constants.

a) Find the electric field distribution everywhere in space (i.e. for $R < a$, $a \le R < b$ and $R \ge b$).

$R < a$ $\quad \rho_v = 0 \qquad E = 0$

$\rho_v \cdot V$

$a \le R < b$ $\quad \rho_v = \rho_0$

$$\oiint_S D\,ds = Q = \rho_0 \cdot \frac{4}{3}\pi R^3$$

$$4\pi R^2 \varepsilon E_R = \left(\frac{4}{3}\pi R^3 - \frac{4}{3}\pi a^3\right)\rho_0$$

$$E_R = \frac{\rho_0}{\varepsilon}\left(\frac{R}{3} - \frac{a^3}{3R^2}\right)\hat{R}$$

$R > b$

$$4\pi R^2 \varepsilon E_R = \frac{4}{3}\pi b^3 \cdot \rho_0 - \frac{4}{3}\pi a^3 \rho_0$$

$$E_R = \frac{\rho_0}{3\varepsilon R^2}\left(b^3 - a^3\right)\hat{R}$$

b) Now, you are required to have a vanishing electric field outside the charge distribution, i.e. $\mathbf{E} = 0$ for $R > b$. Find the surface charge density ρ_s on the outer interface (i.e. at $R = b$).

$$\varepsilon_0 E_R \cdot 4\pi R^2 = \rho_0 \tfrac{4}{3}\pi (b^3 - a^3) + \rho_s \cdot 4\pi b^2$$

$$E_R = \frac{\rho_0 \frac{1}{3}(b^3 - a^3) + \rho_s b^2}{\varepsilon_0 R^2} \hat{R} = 0$$

$$\rho_s b^2 = \left(\rho_0 \tfrac{1}{3} b^3 - \rho_0 \tfrac{1}{3} a^3\right) \hat{R}$$

$$\rho_s = \boxed{\rho_0 \tfrac{1}{3} b - \rho_0 \tfrac{1}{3} \frac{a^3}{b^2}} \hat{R}$$

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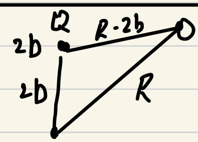
c) Now you are given a system that in addition to the volumetric charge density given in Eq. (1) and the surface charge density found in item 2, comprises a point charge q placed at the location $(x = 0, y = 0, z = 2b)$. Find the field distribution everywhere is space for this system.



Z

Electric field due to a single charge $\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2}$

$$E = \hat{R} \frac{q}{4\pi\varepsilon R^2}$$

$$4\pi R^2 \varepsilon_0 = q$$

$R < a$ $\quad E = 0$

$a < R < b$ $\quad E = E_R = \frac{\rho_0}{\varepsilon}\left(\frac{R}{3} - \frac{a^3}{3R^2}\right)\hat{R}$

$b < R < 2b$ $\quad E_R = \frac{\rho_0 \frac{1}{3}(b^3 - a^3) + \rho_s b^2}{\varepsilon_0 R^2}\hat{R} = 0$

$R > 2b$ $\quad E_R = \frac{\rho_0 \frac{1}{3}(b^3 - a^3) + \rho_s b^2}{\varepsilon_0 R^2}\hat{R} + \frac{q}{4\pi\varepsilon(R-2b)^2}\hat{R}$