Multivariate CAViaR

An Insightful Approach to Risk Modeling

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Background, Motivation, and Initial Results

- My M.S. thesis at the University of Chicago developed a novel method to understand and measure value-at-risk, a commonly accepted method to measure downside risk.
- The metric is understood as follows: a one-day 1% VaR of -10 million dollars for a portfolio means that the portfolio will lose at least 10 million dollars of its value on the 1% worst trading days. A major advantage of VaR is that it distills a distribution of returns into one number.
- As such, VaR is often used in stress testing by regulatory agencies (Holton 2014).

Background, Motivation, and Initial Results

- There have been many popular approaches in the literature such as modeling the total distribution of returns (Longerstaey and Spencer 1996), and approaches using a semiparametric or a nonparametric historical simulation (Richardson, Boudoukh, and Whitelaw 2005).
- While modeling the entire distribution is likely too simplistic, Engle and Manganelli in a 2004 paper (Engle and Manganelli 2004) argue that nonparametric methods in the other camp are usually chosen for "empirical justifications rather than on sound statistical theory".
- To balance these approaches, they propose a framework called CAViaR that directly forecasts the VaR quantile using a conditional autoregressive quantile specification.
- This approach builds upon the statistical literature that extends linear quantile models to settings amenable to financial modeling, such as with heteroskedastic and nonstationary error distributions (Portnoy 1991).

Background, Motivation, and Initial Results

- My thesis extends the model beyond a univariate setting into a multivariate setting using the diffusion index model, originally developed by Stock and Watson for predicting conditional means (Stock and Watson 2002b, 2002a).
- My model uses exchange-traded funds (ETFs) as explanatory variables that are combined into principal component vectors at the forecast origin.
- Combining these principal component vectors with transformations of lagged autoregressive response variables produces similar predictive accuracy during periods of relatively low volatility (when compared to the CAViaR model) along with more insight into the drivers of the changes in the response variable.

Proposed Future Work - First Idea

- As encouraging as the results were from my thesis, two important questions remained unanswered.
- The first is whether some sort of mixture model would be appropriate, that is, aiming to use the basket of ETFs during good times, and use the CAViaR ARMA specification during bad times.
- The approach of using ETFs allows a prediction based on forward-looking expectations of fundamental factors.

Proposed Future Work - First Idea

- Indeed, ETFs are just baskets of individual stocks or bonds, and those securities are (in theory) based on rational expectations about future resources, market conditions, etc - the microfoundations of what drives our economy.
- The ARMA specification, while practically and statistically sound, is contradicted by economic theory and practice - the weak form of the efficient market hypothesis states that it is impossible to forecast future values of asset prices using past values. But perhaps this view is incomplete.

Proposed Future Work - First Idea

- To combine these ideas, I would fit a Hidden Markov Model to infer the state of the world - the "rational" one, or the "irrational" one.
- Given the highly non-normal nature of financial data, I suspect there would be many interesting statistical and computational challenges that would arise with this approach.
- In addition, it is likely worth exploring alternative ensemble methods to further probe into the seemingly enigmatic nature that pervades financial time series.

Proposed Future Work - Second Idea

The second question is to understand shifts in the economy using a changepoint detection algorithm:

1. Using a set of ETFs, perform Principal Component Analysis at T many points for M many factors - $f_{m,t}$
2. At each time point, add the vectors together to get a resultant: $\sum_{m=1}^{M} f_{m,t} = r_t$, giving $r_1, r_2, ..., r_T$.
3. Starting with an arbitrary reference point t_0 with associated r_0 resultant, measure the angle between resultants calculated at different time steps r_t

$$\theta_t = \arccos\left(\frac{r_0 \cdot r_t}{||r_0||||r_t||}\right)$$

Proposed Future Work - Second Idea

- The angle θ could be plotted over time, and changepoints could be detected using Monte Carlo simulation, because PCA transformations are non-linear, so calculating an analytical density from the transformed data is intractable.
- Moreover, the data fed into the PCA transformation is non-normal, which further supports the notion of using Monte Carlo simulation to establish reasonable estimates of uncertainty for detected changepoints. – As with the first line of reasoning, there would certainly be interesting challenges, particularly in creating crisp null and alternative hypotheses.

Proposed Future Work - Second Idea

- Furthermore, there is value in exploring alternatives to standard PCA, namely sparse and robust PCA. Sparse PCA would generalize the algorithm to work with many covariates ($p >> t$) and robust PCA could deal with the significant issue of outliers that are common in financial time series.
- I think that with these two main questions answered, per the advice of several mentors, this paper would be ready for a journal submission.

Appendix

Background and Introduction

- There have been many popular approaches in the literature such as modeling the total distribution of returns (Longerstaey and Spencer 1996), and approaches using a semiparametric or a nonparametric historical simulation (Richardson, Boudoukh, and Whitelaw 2005). While modeling the entire distribution is likely too simplistic, Engle and Manganelli in a 2004 paper (Engle and Manganelli 2004) argue that nonparametric methods in the other camp are usually chosen for "empirical justifications rather than on sound statistical theory". To balance these approaches, they propose a framework called CAViaR that directly forecasts the VaR quantile using a conditional autoregressive quantile specification. This approach builds upon the statistical literature that extends linear quantile models to settings amenable to financial modeling, such as with heteroskedastic and nonstationary error distributions (Portnoy 1991).

Background and Introduction

- Kerry Pechter at Forbes describes it as a premium for the fact that “stocks are riskier” and “more prone to price fluctuations in the short run” compared to lower risk investments (Pechter 2020).
- A portfolio manager must indeed consider the long-run picture; a small difference in the annual rate of return can make an enormous difference in the ending value of investments.
- However, focusing entirely on long-run value generation is not the only consideration a prudent manager ought to make.

Background and Introduction

- While forecasting stock returns in the long-run is challenging, the performance of indices such as the S&P 500, despite seemingly existential threats such as the World Wars and the Great Depression, does give some confidence to investors who try to focus on long-run value generation.
- Ignoring the short-run reminds one of John Maynard Keynes' famous maxim that the "long run is a misleading guide to current affairs" because "in the long run we are all dead" (Keynes 1923), and moreover, the short-run impact of a strategy is often more difficult to understand than the long-run results, and potentially more precarious.
- An investment manager using financial leverage to magnify returns (positive or negative) could be left in dire straits if their investments fell rapidly, despite a sound long-run strategy.

Background and Introduction

- While there are other ways to understand and measure downside risk, a commonly accepted method is using value-at-risk (VaR).
- The metric is understood as follows: a one-day 1% VaR of -10 million dollars for a portfolio means that the portfolio will lose at least 10 million dollars of its value on the 1% worst trading days.
- A major advantage of VaR is that it distills a distribution of returns into one number.
- As such, VaR is often used in stress testing by regulatory agencies in the United States, the United Kingdom, and Europe (Holton 2014).

Background and Introduction

- A popular approach to modeling VaR called RiskMetrics (Longerstaey and Spencer 1996) was introduced by J.P. Morgan in 1994 and re-relased in 1996.
- The model assumed that a "portfolio or any asset's returns follow a normal distribution over time" and used this along with the "variance-covariance method" to calculate VaR (Investopedia 2019).
- While this was certainly a step forward at the time, perhaps the model's greatest downfall is the pretense of knowledge that modeling the distribution of returns in entirety is possible.

Background and Introduction

- The elegant simplicity of using a normal distribution is appealing - only having to estimate the mean and the variance to get a universal picture of returns is certainly appealing, and perhaps necessary in a time of comparatively limited computing power.
- Having said that, modeling the big picture while making clear assumptions about the nature of returns has its' perks, and is perhaps advantageous over alternatives for modeling VaR.
- Indeed, many of the approaches for modeling VaR rely on a semiparametric or a nonparametric historical simulation (Richardson, Boudoukh, and Whitelaw 2005).

Background and Introduction

- According to Robert Engle and Simone Manganelli in a 2004 paper, these methods are usually chosen for “empirical justifications rather than on sound statistical theory” (Engle and Manganelli 2004).
- They propose a framework called CAViaR that directly forecasts the VaR quantile using a conditional autoregressive quantile specification.
- This approach builds upon the statistical literature that extends linear quantile models to settings amenable to financial modeling, such as with heteroskedastic and nonstationary error distributions (Portnoy 1991).

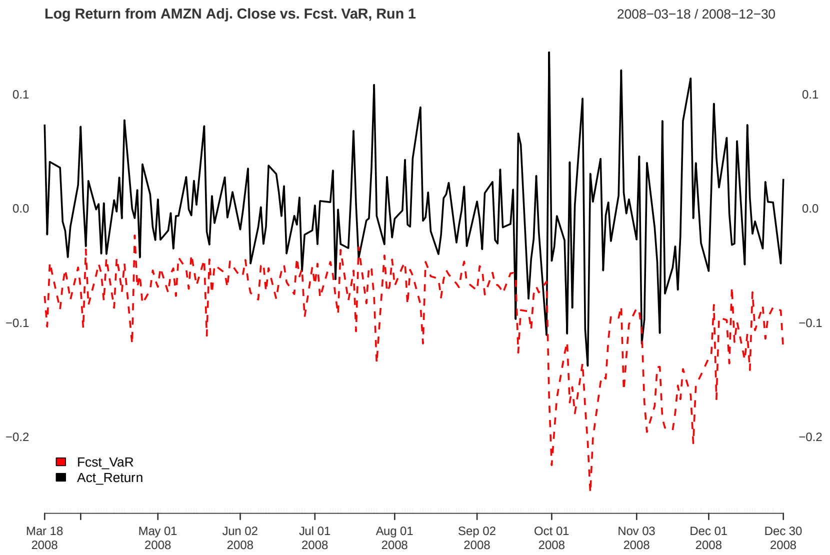
Background and Introduction

- The appeal of this model is that it combines the crisp statistical assumptions with the flexibility required to model financial returns.
- However, the model still runs into issues when a training sample is totally unrepresentative of the testing period - a common problem in statistical analysis.
- Initial motivations for this paper involved analyzing two stocks - Amazon (ticker: AMZN) and Procter & Gamble (ticker: PG) and their performance during the Great Recession (specifically, the last 200 trading days of 2008).

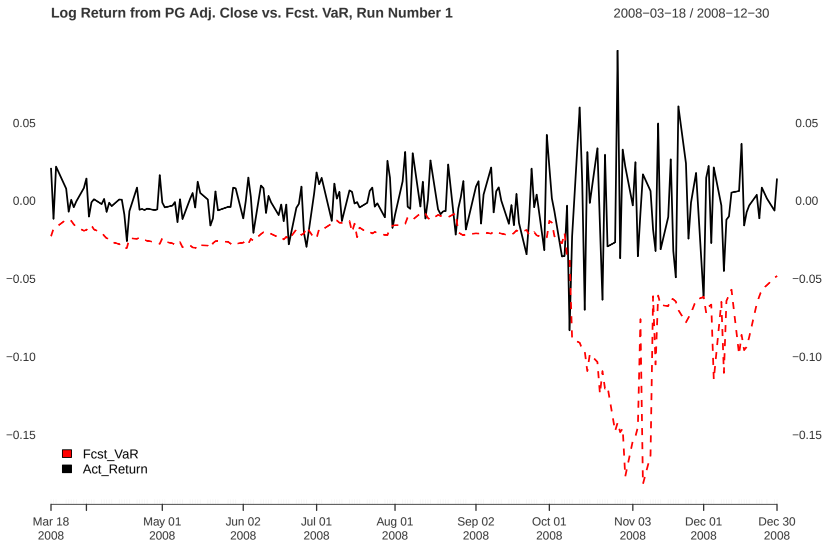
Background and Introduction

- A relevant question of a financial institution would understandably be how their risk model performed during 2008, a highly volatile period which was driven by the "most severe financial crisis since the Great Depression", according to Gary Becker (Becker 2008), a Nobel-prize winning economist.
- Interestingly, the univariate CAViaR forecast for Amazon was fairly accurate whereas the forecast for PG was not.
- One reason for this could be the fact that a stock like Amazon was highly volatile during the training sample, which included return data starting from the second quarter of 2004, but PG was fairly stable.
- How would it be possible for a univariate model such as CAViaR, that does not explicitly account for other factors, to forecast well? What if a volatile stock such as AMZN was included into the forecast for PG - would it improve the prediction?

Amazon Univariate CAViaR Results



Procter & Gamble Univariate CAViaR Results



AMZN and PG Summary Table

Table 1: Accuracy of VaR Forecast for PG Over Last 200 Trading Days in 2008

	AMZN	PG
VaR Break Rate	0.025	0.055
Theoretical VaR	0.010	0.010

Note:

Tested Using the Symmetric Absolute Value Univariate Model

Diffusion Index Model

- From these results, the idea of combining stocks into a multivariate setting to capture correlations and better forecast risk was formed.
- A natural choice appeared to be the diffusion index model, originally developed by Stock and Watson for predicting conditional means (Stock and Watson 2002b, 2002a).
- The model is a useful method of predicting stock movements in the future is the Stock and Watson diffusion index.
- The model is outlined below, which is adapted from Multivariate Time Series Analysis With R and Financial Applications by Ruey S. Tsay (Tsay 2014).

Diffusion Index Model

There are two key equations:

1. $\boldsymbol{z_t} = \boldsymbol{L}\boldsymbol{f_t} + \boldsymbol{\epsilon_t}$

- $\boldsymbol{z_t} = (z_{1t}, ..., z_{kt})'$ is an observed time series with mean 0
- $\boldsymbol{f_t}$ is an m-dimensional vector of common factors with mean 0 and identity covariance matrix
- $\boldsymbol{L}$ is a $k \times m$ loading matrix
- $\boldsymbol{\epsilon_t}$ is an independent and identically distributed (i.i.d.) sequence of random vectors with mean 0 and covariance matrix $\boldsymbol{\Sigma_e}$.

Diffusion Index Model

2. $y_{t+h} = \beta' \boldsymbol{f_t} + \boldsymbol{e_{t+h}}$.

- The above equation represents the h-step ahead prediction based on $\boldsymbol{f_t}$
- y_t is the scalar time series of interest
- h is the forecast horizon
- β represents the vector of coefficients - e_t is a sequence of uncorrelated random variables with mean 0 and constant variance

Diffusion Index Model

- To model the data, principal component analysis is performed on the covariates (described later) to obtain an estimate of $\boldsymbol{f_t}$.
- When modeling the conditional mean, the β coefficients are estimated using ordinary least squares.
- A specific formulation mentioned in the textbook is as follows, where the individuals diffusion indices are given by $f_i t$, and the goal is a one-step ahead prediction of y_t:

$$y_{t+1} = \beta_0 + \sum_{i=1}^{m} \beta_i f_{it} + e_t.$$

Univariate CAViaR Model Specifications

Work needed to be done to align the diffusion index model with the CAViaR model, which is defined below. The following variables are required for use in the CAViaR model. For ease of notation, these are sourced directly from the Engle and Manganelli 2004 CAViaR paper (Engle and Manganelli 2004), with some added description:

- $(y_t)_{t=1}^{T}$ is a "vector of portfolio returns"
- θ is the "probability associated with VaR" (a 5% VaR would mean $\theta = 0.05$)
- $\boldsymbol{x_t}$ is a "vector of time t observable variables"
- $f_t(\boldsymbol{\beta}) \equiv f_t(\boldsymbol{x_{t-1}}, \boldsymbol{\beta_\theta})$ is the "time $t\theta$ quantile of the distribution of portfolio returns formed at time $t-1$"

Univariate CAViaR Model Specifications

The authors then describe a “generic CAViaR specification” as follows:

$$f_t(\boldsymbol{\beta}) = \beta_0 + \sum_{i=1}^{q} \beta_i f_{t-1}(\boldsymbol{\beta}) + \sum_{j=1}^{r} \beta_{q+j} l(\boldsymbol{x_{t-j}})$$

- What is interesting about the general setup is that there are two main components to the model - lagged observed variables (represented by l) and lagged values of unknown parameters, which in the specification below is used as moving average terms.
- As such, it is reasonable to generalize the specifications below as nonlinear ARMA models where y_{t-1} terms refer to previous returns, whereas $f_{t-1}(\beta_1)$ terms refer to previous predictions.

Univariate CAViaR Specifications

Below is the symmetric absolute value CAViaR model:

$$f_t(\boldsymbol{\beta}) = \beta_1 + \beta_2 f_{t-1}(\boldsymbol{\beta}) + \beta_3 |y_{t-1}|.$$

Below is the asymmetric slope CAViaR model:

$$f_t(\boldsymbol{\beta}) = \beta_1 + \beta_2 f_{t-1}(\boldsymbol{\beta}) + \beta_3 (y_{t-1})^+ + \beta_4 (y_{t-1})^-.$$

Below is the Indirect GARCH (1,1) model:

$$f_t(\boldsymbol{\beta}) = (\beta_1 + \beta_2 f_{t-1}^2(\boldsymbol{\beta}) + \beta_3 y_{t-1}^2)^{1/2}.$$

Adaptive CAViaR Model

$$f_t(\beta_1) = f_{t-1}(\beta_1) + \beta_1 \left[(1 + \exp(G[y_{t-1} - f_{t-1}(\beta_1)]))^{-1} - \theta \right]$$

- Following Engle and Manganelli's 2004 paper, they choose $G = 10$, so that is what is used in the results section of this paper. - The authors state the reason for the seemingly arbitrary choice is that while "the parameter G itself could be estimated; however, this would go against the spirit of this model, which is simplicity".
- Sensitivity analysis shows that running the adaptive model with $G = 5$ did not materially affect the VaR predictions - the accuracy was not changed.
- While this model is nonlinear in G and total scale invariance in G would be surprising given the nonlinear relationship, the fact that the other fitted parameters likely adjusted is not surprising.

Multivariate CAViaR Model Specifications

The multivariate CAViaR model takes inspiration from the models described above in several specifications, as mentioned in the original specifications. The general model form looks like the specification below:

$$f_t(\beta) = \beta_0 + \sum_{i=1}^{p} \beta_i y_{t-i} + \sum_{j=1}^{m} \beta_{j+p} f_{j,t-1} + e_t.$$

Multivariate CAViaR Model Specifications

- As with the univariate CAViaR model, the object of interest is a θ percentile return and the model is fit iteratively to minimize the loss function on the training data.
- However, there are some notable differences between the univariate model and the multivariate model.
- First, there are no moving average terms (lagged error terms) - the reasoning for this is because this model aims for a clear economic interpretation, and crisp interpretations of MA models are harder to create.
- Also, moving average models require recursive estimation since error terms are not observed, and so developing a method to work with these errors in a robust regression framework is challenging.

Multivariate CAViaR Model Specifications

- Second, in some of the specifications below, there are lagged return variables. This is similar to the univariate CAViaR specification, though there is often more than 1 lag as in the univariate model - there are p lags in the dataset.
- Third, in all of the specifications below, there are m diffusion indices used in each model lagged by one time step to avoid look-ahead bias.

Multivariate CAViaR Specifications

No lags model:

$$f_t(\beta) = \beta_0 + \sum_{j=1}^{m} \beta_j f_{j,t-1} + e_t$$

Model with Autoregressive lags:

$$f_t(\beta) = \beta_0 + \sum_{i=1}^{p} \beta_i y_{t-i} + \sum_{j=1}^{m} \beta_{j+p} f_{j,t-1} + e_t$$

Multivariate CAViaR Specifications

Model with Symmetric Absolute Value AR lags:

$$f_t(\boldsymbol{\beta}) = \beta_0 + \sum_{i=1}^{p} \beta_i |y_{t-i}| + \sum_{j=1}^{m} \beta_{j+p} f_{j,t-1} + e_t$$

Model with Asymmetric Slope AR lags:

$$f_t(\boldsymbol{\beta}) = \beta_0 + \sum_{i=1}^{p} \beta_i (y_{t-i})_+ + \sum_{j=p+1}^{2p} \beta_i (y_{t-i})_- + \sum_{k=1}^{m} \beta_{k+2p} f_{k,t-1} + e_t$$

Fitting the Models

- To fit the models, an optimal value of m diffusion indices and p autoregressive terms are added (or $2p$ in the case of the asymmetric slope model).
- The optimal values of these parameters are determined using a validation dataset.
- In all of the runs below, there are a total of 5 years of trading days, or about 1,260 days assuming 252 trading days a year.
- The adjusted closing prices are logged and differenced, shortening the dataset by one.

Fitting the Models

- After doing this, the last 250 data points are reserved as test data, and the 250 data points before that are used as a validation set.
- Measured by the loss function written out below, the values of p and m that minimize losses are chosen and the optimal model is refit over both the training and the validation data combined and then evaluated on the test data.
- Note that there is an optimal model chosen for each of the four multivariate CAViaR specifications described above, so there are 4 optimal sets of p and m chosen for each set of model.
- Thus, there are 8 models compared on the test data - 4 univariate CAViaR models and 4 multivariate CAViaR models.

Fitting the Models

From the CAViaR paper, the θth regression quantile is defined as any $\hat{\beta}$ that solves the following loss function:

$$\underset{\beta}{argmin} \frac{1}{T} \sum_{t=1}^{T} [\theta - I(y_t < f_t(\beta))][y_t - f_t(\beta)]$$

Theoretical Guarantees of Consistency and Asymptotic Normality

- Part of the reason for working with the CAViaR and diffusion index is their strong theoretical guarantees about consistency and asymptotically.
- Indeed, following the results in Engle and Manganelli (Engle and Manganelli 2004), there are 8 conditions required for consistency of the β estimate and 4 required for asymptotic normality.

Theoretical Guarantees of Consistency and Asymptotic Normality

The paper states that the model specified by:

$$y_t = f(y_{t-1}, \boldsymbol{x}_{t-1}, ..., y_1, \boldsymbol{x}_1; \beta^0) + \epsilon_{t\theta}[Quant_\theta(\epsilon_{t\theta}|\Omega_t) = 0]$$

$$\equiv f_t(\beta^{\mathbf{0}}) + \epsilon_{t\theta}, t = 1, ..., T$$

"where $f_1(\beta^{\mathbf{0}})$ is some given initial condition, $\boldsymbol{x_t}$ is a vector of exogenous of predetermined variables, $\beta^{\mathbf{0}} \in \mathbb{R}^p$ is the vector of true unknown parameters that need to be estimated, and $\Omega_t = [y_{t-1}, \boldsymbol{x}_{t-1}, ..., y_1, \boldsymbol{x}_1; f_1(\beta^0)]$ is the information set available at time t", and $\hat{\beta}$ is the parameter vector that minimizes the loss function specified above. According to theorems in the paper, they state that under favorable conditions, $\hat{\beta}$ is consistent and asymptotically normal.

Consistency

Per Engle and Manganelli, under the model specified above and using 8 assumptions given below, $\hat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta}^{\mathbf{0}}$ where $\hat{\boldsymbol{\beta}}$ is the parameter vector that minimizes the loss function specified above. There are 8 assumptions listed in the paper; most seem fairly standard.

1. "(Ω, F, P) is a complete probability space, and $\{\epsilon_{t\theta}, \boldsymbol{x}_t\}$, t $=$ 1,2,. . . . are random vectors on this space"
2. "The function $f_t(\boldsymbol{\beta}) : \mathbb{R}^{k_t} \times B \rightarrow \mathbb{R}$ is such that for each $\boldsymbol{\beta} \in B$, a compact subset of $\mathbb{R}^p$, $f_t(\boldsymbol{\beta})$ is measurable with respect to the information set Ω_t and $f_t(\cdot)$ is continuous in B, $t = 1, 2, ...$, for a given choice of explanatory variables $\{y_{t-1}, \boldsymbol{x}_{t-1}, ..., y_1, \boldsymbol{x}_1\}$."
3. "Conditional on all of the past information Ω_t, the error terms $\epsilon_{t\theta}$ form a stationary process, with continuous conditional density $h_t(\epsilon|\Omega_t)$."

Consistency

4. "There exists $h > 0$ such that for all $t, h_t(0|\Omega_t) \geq h$."
5. "$|f_t(\beta)| < K(\Omega_t)$ for each $\beta \in B$ and for all t, where $K(\Omega_t)$ is some (possibly) stochastic function of variables that belong to the information set, such that $\mathbb{E}(|K(\Omega_t)|) \leq K_0 < \infty$, for some constant K_0"
6. "$\mathbb{E}[|\epsilon_{t\theta}|] < \infty$ for all t"
7. "$\{[\theta - I(y_t < f_t(\beta))][(y_t - f_t(\beta))]\}$ obeys the uniform law of large numbers"
8. "For every $\xi > 0$, there exits a $\tau > 0$ such that if $||\beta = \beta^0||$ $\geq \xi$, then $\liminf_{T\to\infty} T^{-1}\Sigma P[|f_t(\beta) - f_t(\beta^{\mathbf{0}})| > \tau] > 0$"

Consistency

When analyzing real data, it's hard to verify any assumptions exactly, but one that is most controversial might be the third assumption - indeed, it seems highly unlikely that given all the past information, there would be a stationary process.

Asymptotic Normality

Also per Engle and Manganelli, under the same assumptions required for consistency as well as the assumptions below, there is a guarantee of asymptotic normality:

$$\sqrt{T}\boldsymbol{A}_T^{-1/2}\boldsymbol{D}_T(\hat{\beta} - \beta^0) \xrightarrow{d} \mathcal{N}(0, \boldsymbol{I})$$

where

$$\boldsymbol{A}_T \equiv \mathbb{E}\left[T^{-1}\theta(1-\theta)\sum_{t=1}^{T}\nabla' f_t(\beta^0)\nabla f_t(\beta^0)\right]$$

and

$$\boldsymbol{D}_T \equiv \mathbb{E}\left[T^{-1}\sum_{t=1}^{T}h_t(0|\Omega_t)\nabla' f_t(\beta^0)\nabla f_t(\beta^0)\right]$$

Asymptotic Normality

There are 4 assumptions listed in the paper required for asymptotic normality to hold. As with the assumptions required for consistency, these seem fairly standard as well:

1. "$f_t(\beta)$ is differentiable in B and for all β and γ in a neighborhood ν_0 of $\beta^{\mathbf{0}}$, such that $||\beta - \gamma|| \leq d$ for d sufficiently small and for all t:"

a. "$||\nabla f_t(\beta)|| \leq F(\Omega_t)$, where $F(\Omega_t)$ is some (possibly) stochastic function of variables that belong to the information set and $\mathbb{E}(F(\Omega_t)^3) \leq F_0 < \infty$, for some constant F_0."

b. "$||\nabla f_t(\beta) - \nabla f_t(\gamma)|| \leq M(\Omega_t, \beta, \gamma) = \mathcal{O}(||\beta - \gamma||)$, where $M(\Omega_t, \beta, \gamma)$ is some function such that $\mathbb{E}[M(\Omega_t, \beta, \gamma)]^2 \leq M_0 ||\beta - \gamma|| < \infty$ and $\mathbb{E}[M(\Omega_t, \beta, \gamma)]F(\Omega_t)] \leq M_1 |\beta - \gamma|| < \infty$ for some constants M_0 and M_1."

Asymptotic Normality

2a. "$h(\epsilon|\Omega_t) \leq N < \infty \ \forall t$, for some constant N." b. "$h(\epsilon|\Omega_t)$ satisfies the Lipschitz condition $|h_t(\lambda_1|\Omega_t) - h_t(\lambda_2|\Omega_t)| \leq L|\lambda_1 - \lambda_2|$ for some constant $L < \infty \ \forall t$." 3. "The matrices $\boldsymbol{A}_T \equiv \mathbb{E}\left[T^{-1}\theta(1-\theta)\sum_{t=1}^{T} \nabla' f_t(\beta^0)\nabla \times f_t(\beta^0)\right]$ and $\boldsymbol{D}_T \equiv \mathbb{E}\left[T^{-1}\sum_{t=1}^{T} h_t(0|\Omega_t)\nabla' f_t(\beta^0) \times \nabla f_t(\beta^0)\right]$ have the smallest eigenvalues bounded below by a positive constant T for sufficiently large." 4. "The sequence $\{T^{-1/2}\sum_{t=1}^{T}[\theta - I(y_t < f_t(\beta^0))]\nabla' f_t(\beta^0)\}$ obeys the central limit theorem.

Asymptotic Normality

As with the consistency conditions, these seem reasonable enough - the data considered in this analysis seems well-behaved enough such that these conditions are satisfied.

Data Used

- The response variable used in this analysis is SPY, which is an exchange-traded fund that aims to track the performance of the S&P 500, which is discussed above.
- It is broadly used as a bellwether of the U.S. economy, and has the advantage of avoiding survivorship bias - while an individual stock might go bankrupt or merge with another, it is reasonable to assume that these issues do not apply with an ETF.

Data Used - U.S. ETFs

- Following this logic, there are several classes of response variables used in this analysis.
- The first group is a set of U.S. sector ETFs obtained from Seeking Alpha (NA 2020).
- As with the response variable, these ETFs were publicly traded throughout the Great Recession of 2008.

Data Used - U.S. ETFs

a. Utilities (XLU)
b. Consumer Staples (XLP)
c. Healthcare (XLV)
d. Technology (XLK)
e. Consumer Discretionary (XLY)
f. Industrial (XLI)
g. Financial Services (XLF)
h. Basic Materials (XLB)
i. Energy (XLE)

Data Used - Global Sector ETFs

- The second group is Global Sector ETFs, also from Seeking Alpha (NA 2020). - The rationale for including these is that perhaps some global exposure is useful in understanding the broader market.

Data Used - Global Sector ETFs

a. Utilities (JXI)
b. Consumer Staples (KXI)
c. Healthcare (IXJ)
d. Telecommunications (IXP)
e. Technology (IXN)
f. Consumer Discretionary (RXI)
g. Industrial (EXI)
h. Financial Services (IXG)
i. Basic Materials (MXI)
j. Energy (IXC)

Data Used - Bond ETFs

- The third group is bond ETFs.
- Like the previous two groups, these ETFs potentially contain forward-looking information about the stock market.
- These ETFs were chosen because they were the first fixed-income ETFs available in the United States, and had enough history for this paper (NA 2017).

Data Used - Bond ETFs

a. iShares 1-3 Year Treasury Bond Fund (SHY)
b. iShares 7-10 Year Treasury Bond Fund (IEF)
c. iShares 20+ Year Treasury Bond Fund (TLT)
d. iShares iBoxx $ Investment Grade Corporate Bond ETF (LQD)

Data Used

- Lastly, all of the above three groups are run together.
- One reason for having bond and stocks grouped together is the fact that bonds are somewhat of a substitute for equities, which tend to drop more in a period of crisis (Amadeo 2020).
- As such, some unexplained movements in the stock price could be picked up by bond movements.

Data Used

- In each run, the explanatory variables are lagged to avoid look-ahead bias.
- U.S. of the runs analyze the difference of the log of the adjusted closing price.
- The reason for using the differenced log is that it closely approximates the percentage change of the price for small changes.
- The reason for using the adjusted closing prices is that an adjusted closing price excludes the effects of “corporate actions such as stock splits, dividends / distributions and rights offerings” (Gant 2019).

Data Used

- While dividends are essential to study the long-term performance of a strategy, studying short-term price movements do not require understanding the effects of dividend reinvestment.
- While there are many candidate ETFs chosen, these were chosen because they all had price history going back through the beginning of 2004.

Results

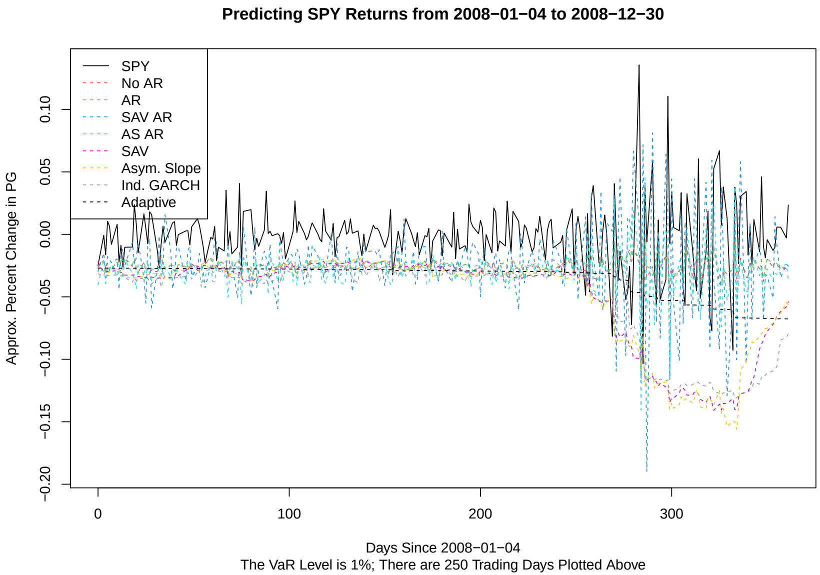
- For the sake of brevity, the results with only U.S. ETFs, global ETFs, or bonds is included in a later results section. The results from those runs are similar to the results below. - To test how well the models do at different VaR levels, 1%, 5%, and 10% are tested.

Results: Notation

Below is the notation used on future slides. Items 2 - 5 listed below are new models developed in this thesis; models 6 - 9 are from the established CAViaR model developed by Engle and Manganelli.

1. SPY: SPY ETF
2. No AR: Multivariate CAViaR Model with no lags
3. AR: Multivariate CAViaR Model with p lags
4. SAV AR: Multivariate CAViaR Model with p absolute value lags
5. AS AR: Multivariate CAViaR Model with $2p$ lags with asymmetric slopes
6. SAV: Univariate CAViaR Model with symmetric absolute framework
7. Asym. Slope: Univariate CAViaR Model with asymmetric slope framework
8. Ind. GARCH: Univariate CAViaR Model with indirect GARCH framework
9. Adaptive: Univariate CAViaR Model with adaptive slope framework

2008 Test Period - All ETFs - 1% VaR Plot



2008 Test Period - All ETFs - 1% VaR Tables

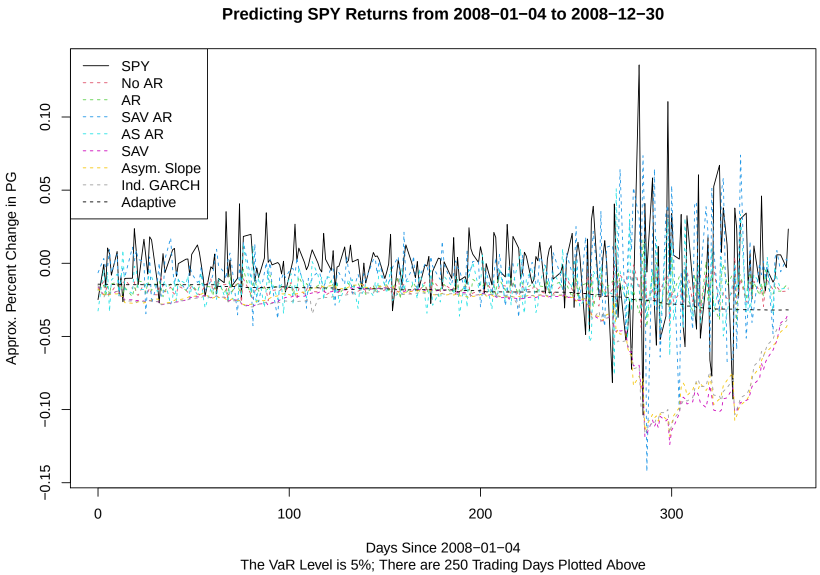
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.736	0.737	1.733	0.863
VaR Break Rate	0.104	0.108	0.216	0.104

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.208	0.213	0.219	0.355
VaR Break Rate	0.028	0.028	0.028	0.060

2008 Test Period - All ETFs - 5% VaR Plot



2008 Test Period - All ETFs - 5% VaR Tables

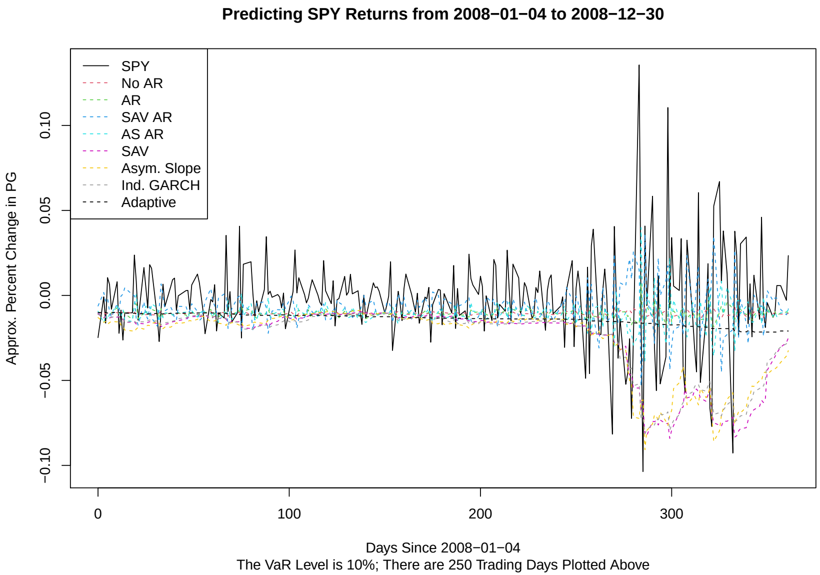
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	1.148	1.236	2.391	1.371
VaR Break Rate	0.168	0.208	0.400	0.220

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.651	0.654	0.640	0.956
VaR Break Rate	0.076	0.076	0.064	0.160

2008 Test Period - All ETFs - 10% VaR Plot



2008 Test Period - All ETFs - 10% VaR Tables

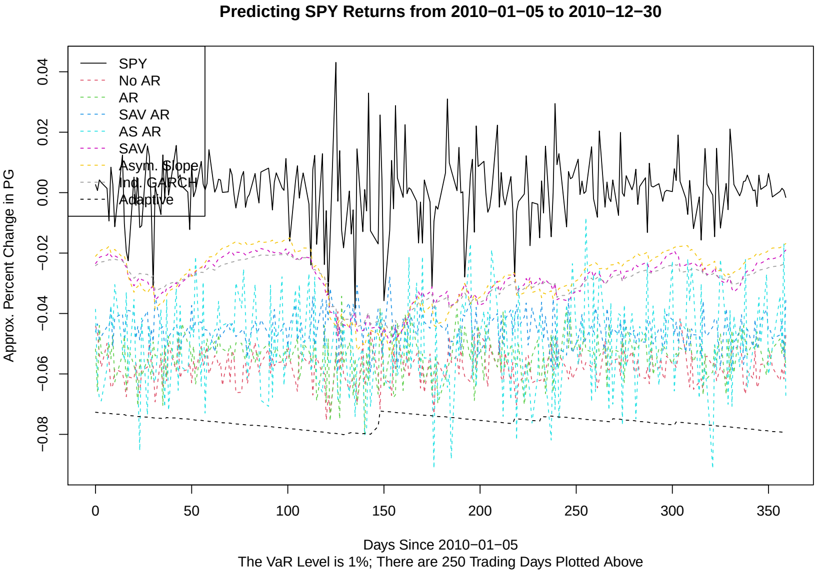
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	1.521	1.549	1.797	1.644
VaR Break Rate	0.284	0.288	0.344	0.292

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	1.077	1.066	1.068	1.366
VaR Break Rate	0.144	0.156	0.140	0.224

2010 Test Period - All ETFs - 1% VaR Plot



2010 Test Period - All ETFs - 1% VaR Tables

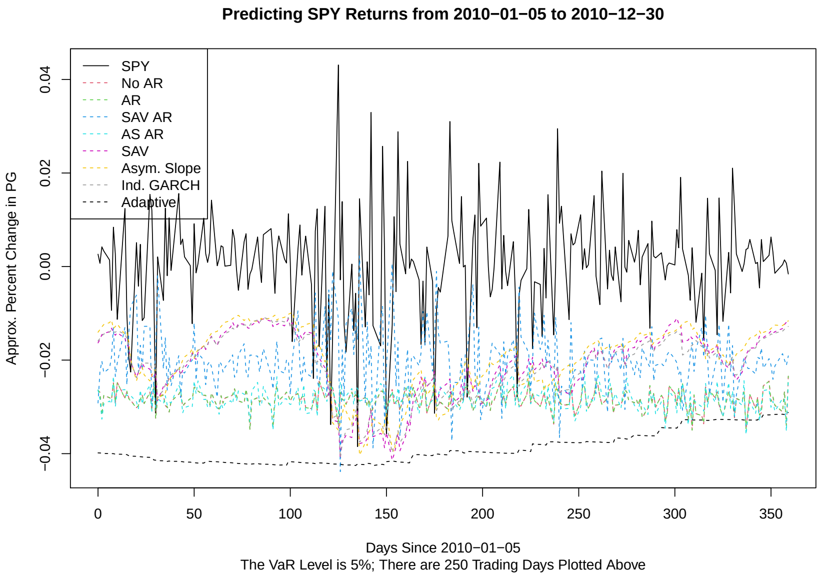
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.146	0.136	0.114	0.128
VaR Break Rate	0.000	0.000	0.000	0.000

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.079	0.08	0.086	0.191
VaR Break Rate	0.020	0.02	0.016	0.000

2010 Test Period - All ETFs - 5% VaR Plot



2010 Test Period - All ETFs - 5% VaR Tables

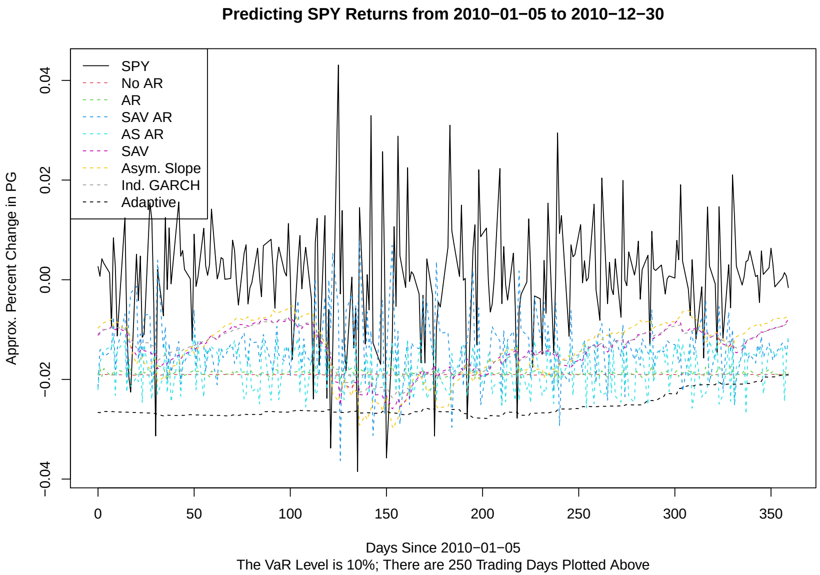
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.394	0.394	0.436	0.397
VaR Break Rate	0.024	0.024	0.068	0.028

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.336	0.336	0.343	0.492
VaR Break Rate	0.052	0.052	0.048	0.000

2010 Test Period - All ETFs - 10% VaR Plot



2010 Test Period - All ETFs - 10% VaR Tables

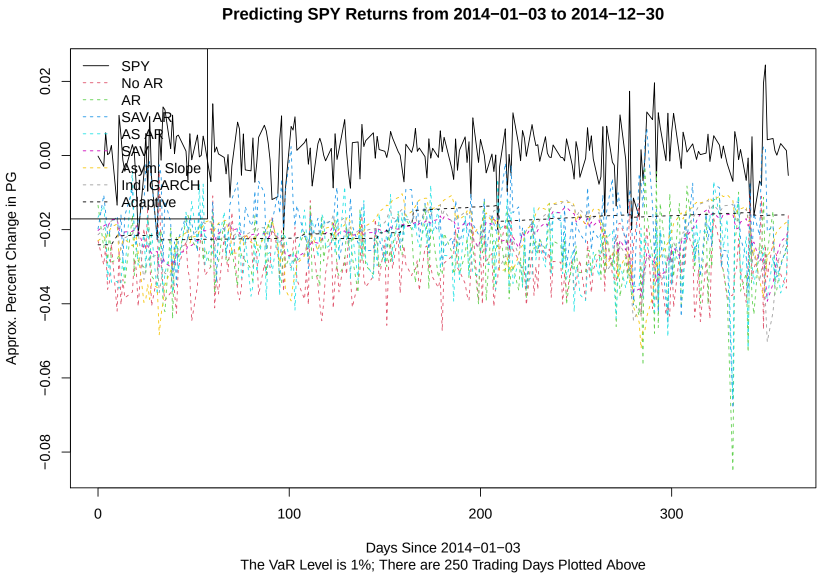
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.595	0.594	0.664	0.596
VaR Break Rate	0.044	0.044	0.120	0.076

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.547	0.549	0.546	0.690
VaR Break Rate	0.080	0.088	0.084	0.028

2014 Test Period - All ETFs - 1% VaR Plot



2014 Test Period - All ETFs - 1% VaR Tables

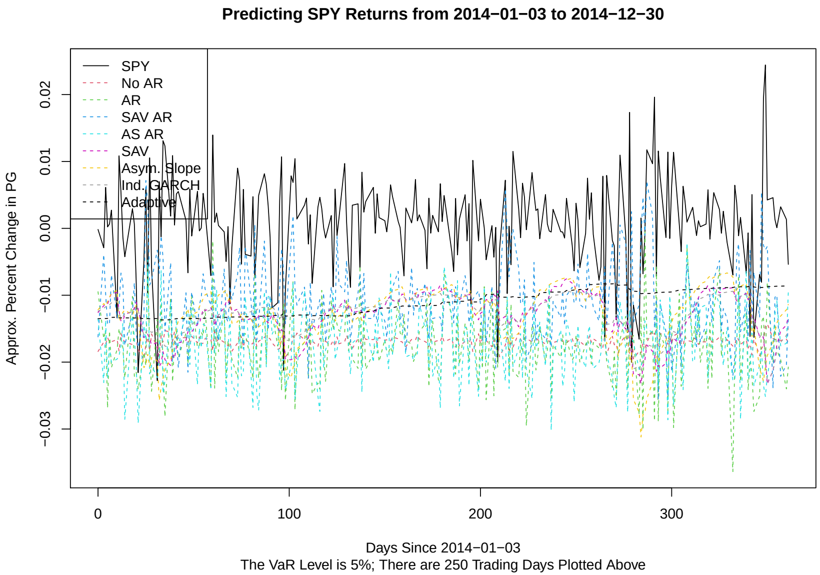
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.087	0.073	0.107	0.079
VaR Break Rate	0.008	0.008	0.052	0.008

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.061	0.057	0.063	0.061
VaR Break Rate	0.008	0.004	0.012	0.028

2014 Test Period - All ETFs - 5% VaR Plot



2014 Test Period - All ETFs - 5% VaR Tables

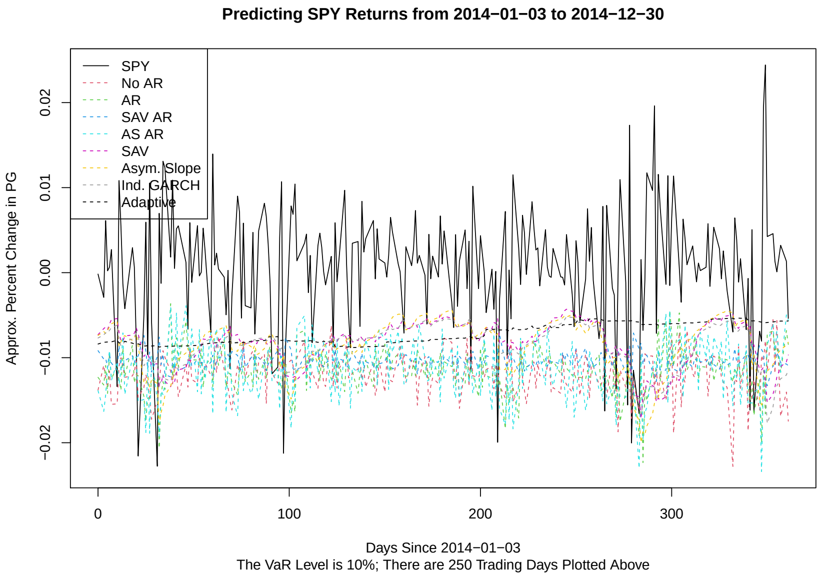
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.241	0.256	0.320	0.246
VaR Break Rate	0.024	0.032	0.084	0.028

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.226	0.218	0.225	0.240
VaR Break Rate	0.052	0.048	0.052	0.056

2014 Test Period - All ETFs - 10% VaR Plot



2014 Test Period - All ETFs - 10% VaR Tables

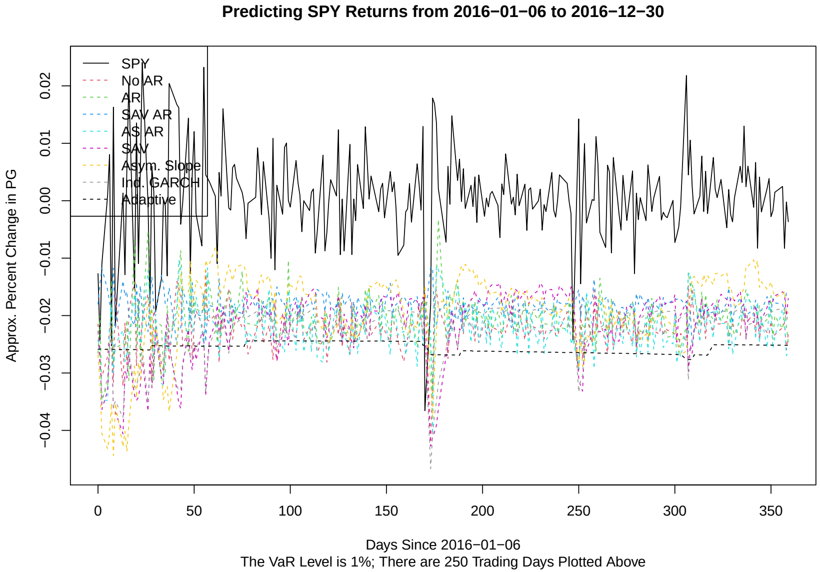
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.371	0.359	0.370	0.361
VaR Break Rate	0.056	0.044	0.072	0.056

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.367	0.359	0.364	0.368
VaR Break Rate	0.116	0.104	0.112	0.132

2016 Test Period - All ETFs - 1% VaR Plot



2016 Test Period - All ETFs - 1% VaR Tables

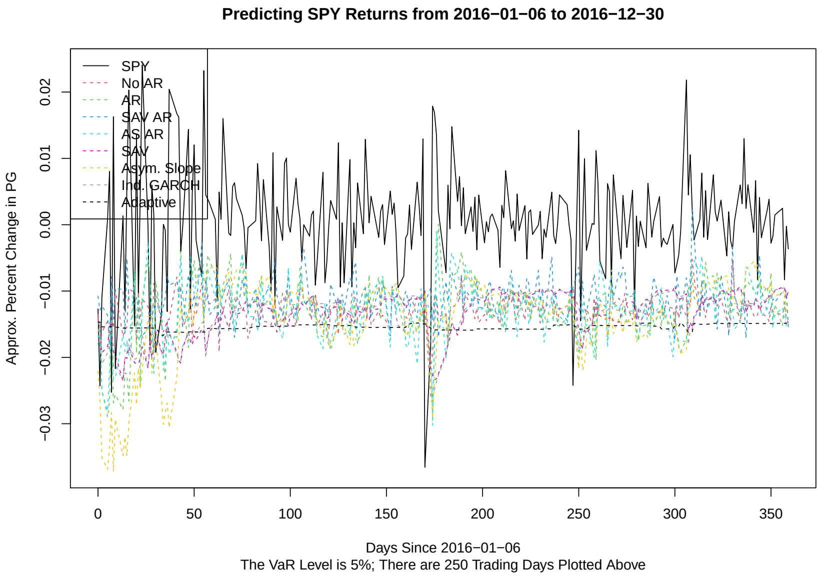
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.091	0.097	0.092	0.083
VaR Break Rate	0.020	0.020	0.028	0.016

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.078	0.082	0.078	0.077
VaR Break Rate	0.012	0.020	0.012	0.004

2016 Test Period - All ETFs - 5% VaR Plot



2016 Test Period - All ETFs - 5% VaR Tables

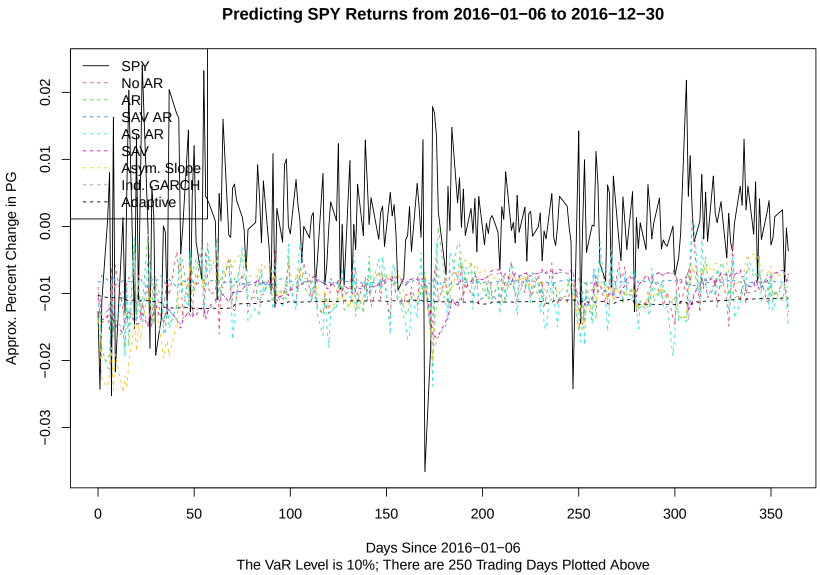
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.26	0.265	0.283	0.272
VaR Break Rate	0.06	0.048	0.092	0.056

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.238	0.238	0.234	0.264
VaR Break Rate	0.032	0.040	0.028	0.032

2016 Test Period - All ETFs - 10% VaR Plot



2016 Test Period - All ETFs - 10% VaR Tables

- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.425	0.397	0.401	0.441
VaR Break Rate	0.096	0.096	0.112	0.100

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.370	0.373	0.368	0.414
VaR Break Rate	0.088	0.092	0.096	0.072

Conclusions

- For the 2008 results, the univariate CAViaR models significantly outperform the multivariate model, particularly at the 1% level. The extreme behavior towards the end of 2008 proved difficult for the multivariate model to pick up on.
- However, for the results from 2010, 2014, and 2016, the multivariate forecast is largely in line with the univariate CAViaR model, though it seems like the univariate model does a better job tracking the response variable in the case of a large swing because of the moving average component.
- Also, while the multivariate models had a rate of VaR breaks that was too high for 2008, the rate of VaR breaks was generally too low for the multivariate models in 2010.
- There appears to be less differentiation between the models in 2014 and 2016.

Conclusions and Future Work

- The problem of how to predict a low quantile of a stock's log return when the training sample is substantially different from the test scenario is an enormously difficult problem.
- Almost axiomatically, the distribution is nonstationary over time.
- How is it possible to predict the return of an index like the S&P500 during a period of market turmoil such as the Great Recession?
- While the univariate CAViaR model performs comparatively well during times of stress, it performs about the same as the multivariate CAViaR model during more benign economic periods.

Conclusions and Future Work

- This conclusion drawn from the above results might support the notion of combining the two models in some sort of a mixture model - aiming to use the basket of ETFs during good times, and use the CAViaR ARMA specification during bad times.
- The approach of using ETFs allows a prediction based on forward-looking expectations of fundamental factors. -Indeed, ETFs are just baskets of individual stocks or bonds, and those securities are (in theory) based on rational expectations about future resources, market conditions, etc - the microfoundations of what drives our economy.
- The ARMA specification, while practically and statistically sound, is contradicted by economic theory and practice - the weak form of the efficient market hypothesis states that it is impossible to forecast future values of asset prices using past values.
- But perhaps this view is incomplete.

Conclusions and Future Work

- Any model that attempts to capture relationships in the real world will only work until an omitted variable is found.
- The elegance of the multivariate CAViaR model is that it provides insight into why a prediction is wrong; the change in the angle between resultant vectors is a sensible measurement of economic changepoints.
- However, errors in the world are costly, and it is wishful thinking to say that explaining why the error occurred is sufficient.

Conclusions and Future Work

- As such, for future work it is worth exploring the notion of weighting an ARMA-approach more heavily when predictions using fundamentals were too high, then not only would this after-the-fact recognition be achieved, but also a hierarchical model that captures fundamental relationships in the economy and potentially changes our understanding of asset prices in general - a synthesis between Keynes' animal spirits during a time of severe crisis; where a model cannot explain shifts, and a more rational world that explains other periods.
- In addition to significant predictive power because of the switching between the two worlds, there is also an elegant explanation; a way to explain changes in the usefulness of the underpinnings in the economy.

Conclusions and Future Work

- Because of the flexibility of the model, it is entirely possible that a whole gamut of variables could be tossed in and backtested to when “changepoints” occurred.
- Additional future work involves confirming the theoretical guarantees on the parameters in the multivariate CAViaR model. One advantage of both the diffusion index model and the CAViaR model is that both have theorems about asymptotic normality and consistency.

Conclusions and Future Work - Changepoint Detection Algorithm

- One advantage of the diffusion index model is the ability to detect changepoints underlying factors, which can serve as a proxy for economic changes.
- Below is a method that can be explored in future work:

1. Using a set of ETFs, perform PCA at t many points for m many factors - $f_{j,t}$
2. At each time point, add the vectors together to get a resultant: $\sum_{j=1}^{m} f_{j,t} = r_t$
3. Calculate the resultants at each time point t, giving $r_1, r_2, ..., r_T$.
4. Starting with a reference point t_0 with associated r_{t0} resultant, measure the angle between resultants calculated at different time steps r_{tk}

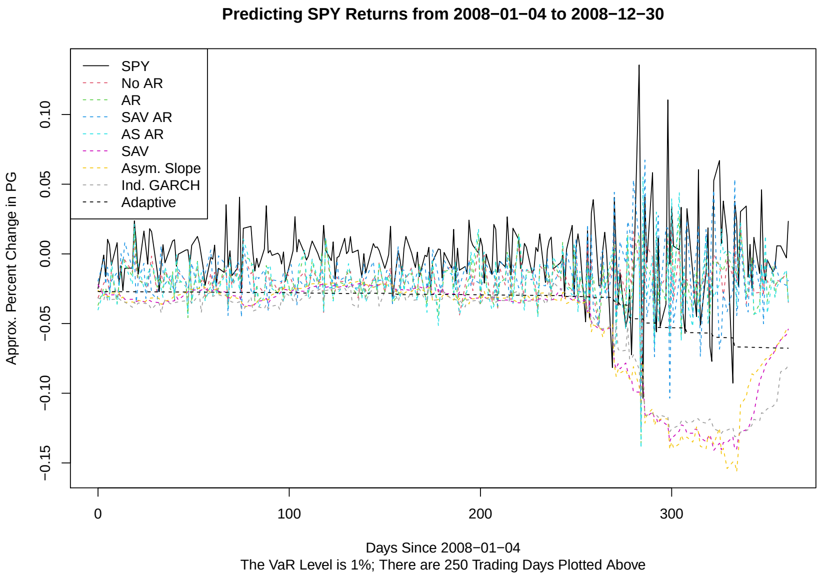
$$\theta = \arccos\left(\frac{r_{t0} \cdot r_{tk}}{||r_{t0}||||r_{tk}||}\right)$$

Conclusions and Future Work - Changepoint Detection Algorithm

- The angle θ could be plotted over time.
- Using Monte Carlo simulation (because PCA transformations are highly non-linear and thus are difficult to calculate an analytical density from), p-values could be calculated to detect a changepoint in the underlying factors.

Additional Results

2008 Test Period - U.S. ETFs - 1% VaR Plot



2008 Test Period - U.S. ETFs - 1% VaR Tables

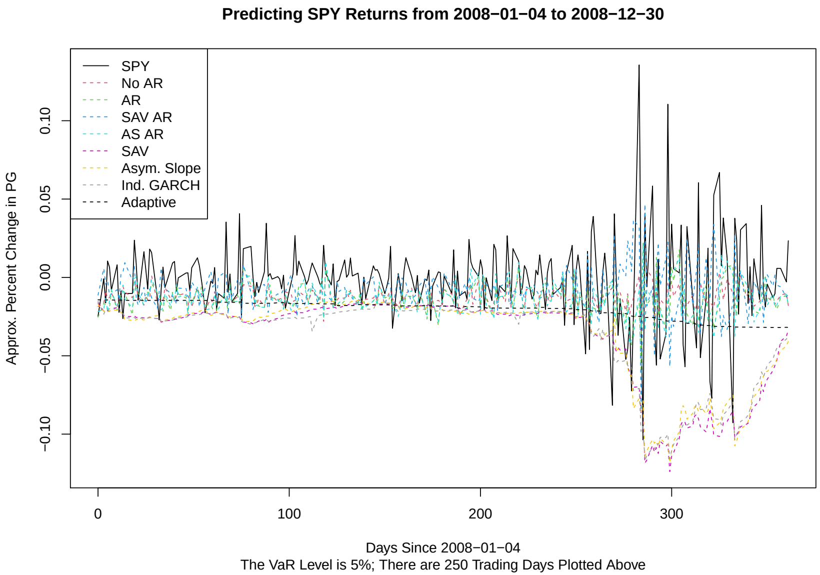
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	1.046	1.1	1.371	1.265
VaR Break Rate	0.200	0.2	0.212	0.208

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.208	0.213	0.219	0.355
VaR Break Rate	0.028	0.028	0.028	0.060

2008 Test Period - U.S. ETFs - 5% VaR Plot



2008 Test Period - U.S. ETFs - 5% VaR Tables

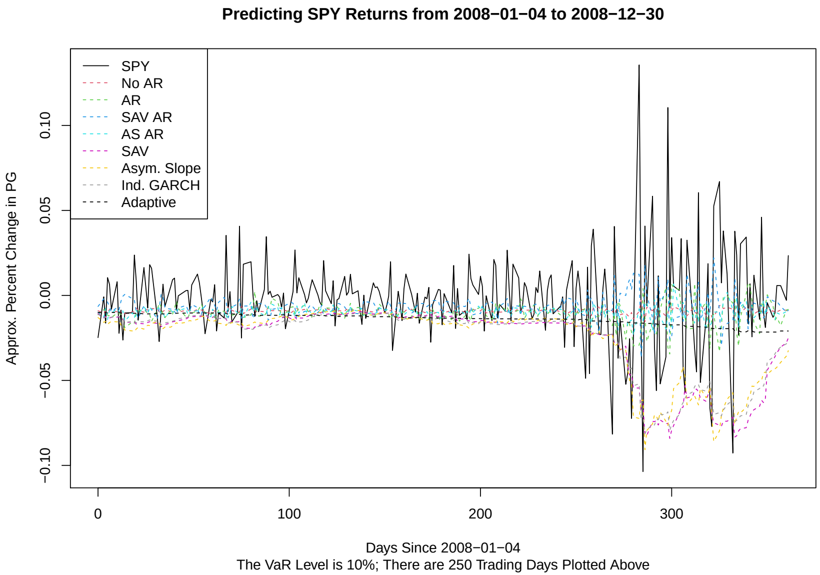
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	1.319	1.344	1.768	1.385
VaR Break Rate	0.260	0.236	0.340	0.260

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.651	0.654	0.640	0.956
VaR Break Rate	0.076	0.076	0.064	0.160

2008 Test Period - U.S. ETFs - 10% VaR Plot



2008 Test Period - U.S. ETFs - 10% VaR Tables

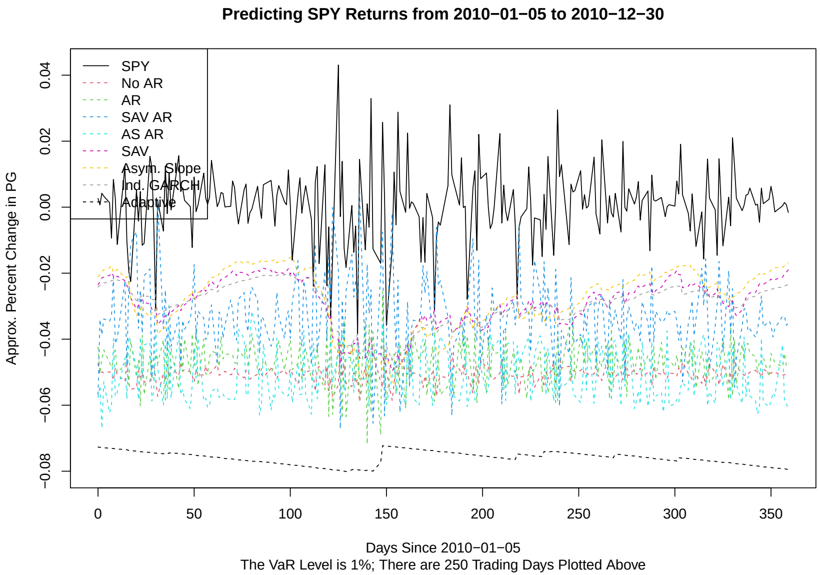
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	1.623	1.564	1.738	1.534
VaR Break Rate	0.328	0.312	0.348	0.332

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	1.077	1.066	1.068	1.366
VaR Break Rate	0.144	0.156	0.140	0.224

2010 Test Period - U.S. ETFs - 1% VaR Plot



2010 Test Period - U.S. ETFs - 1% VaR Tables

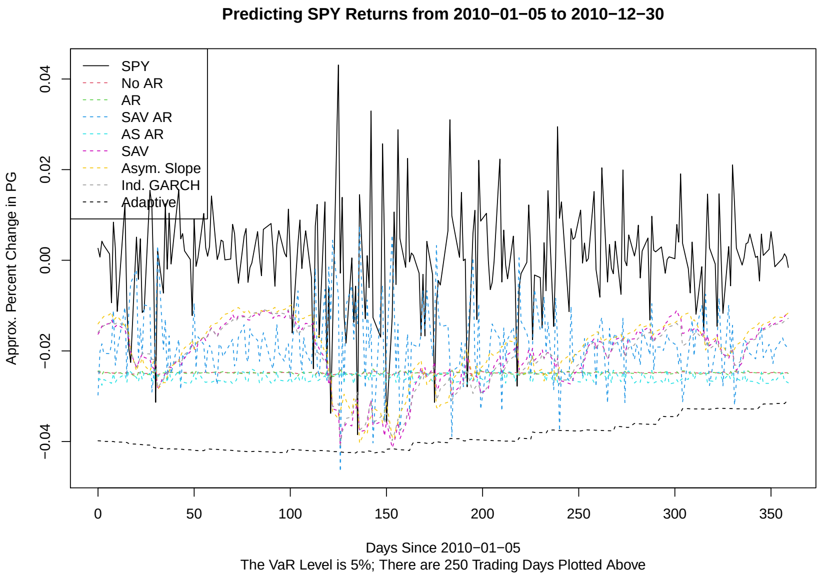
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.128	0.119	0.155	0.128
VaR Break Rate	0.000	0.000	0.044	0.000

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.079	0.08	0.086	0.191
VaR Break Rate	0.020	0.02	0.016	0.000

2010 Test Period - U.S. ETFs - 5% VaR Plot



2010 Test Period - U.S. ETFs - 5% VaR Tables

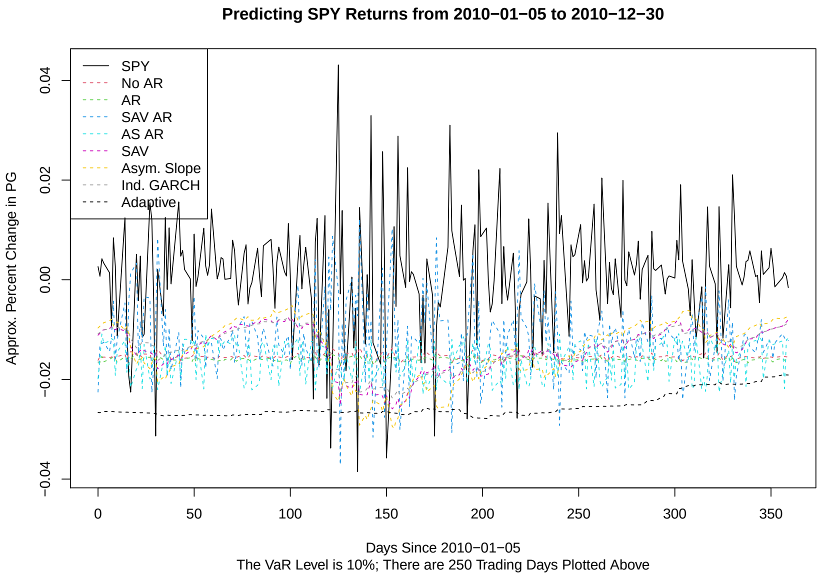
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.369	0.369	0.468	0.379
VaR Break Rate	0.028	0.028	0.084	0.028

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.336	0.336	0.343	0.492
VaR Break Rate	0.052	0.052	0.048	0.000

2010 Test Period - U.S. ETFs - 10% VaR Plot



2010 Test Period - U.S. ETFs - 10% VaR Tables

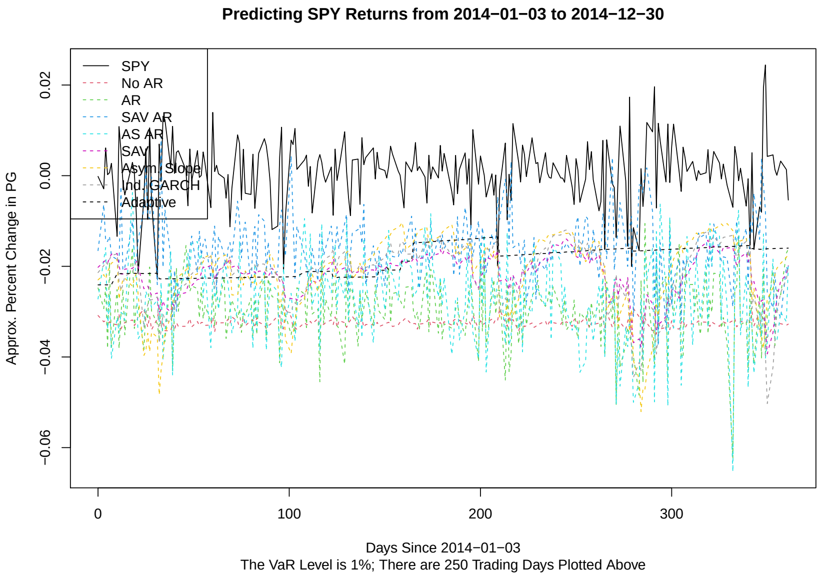
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.56	0.561	0.710	0.572
VaR Break Rate	0.08	0.072	0.132	0.080

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.547	0.549	0.546	0.690
VaR Break Rate	0.080	0.088	0.084	0.028

2014 Test Period - U.S. ETFs - 1% VaR Plot



2014 Test Period - U.S. ETFs - 1% VaR Tables

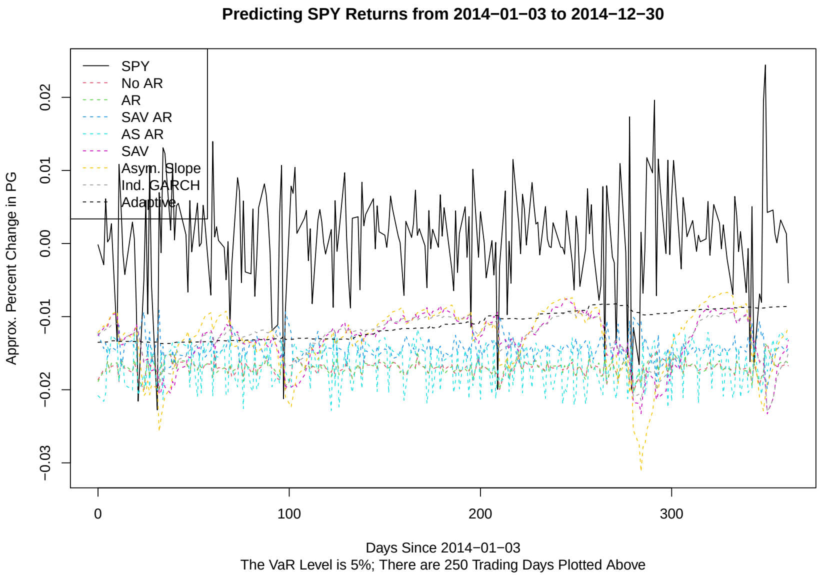
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.083	0.075	0.173	0.071
VaR Break Rate	0.000	0.000	0.072	0.004

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.061	0.057	0.063	0.061
VaR Break Rate	0.008	0.004	0.012	0.028

2014 Test Period - U.S. ETFs - 5% VaR Plot



2014 Test Period - U.S. ETFs - 5% VaR Tables

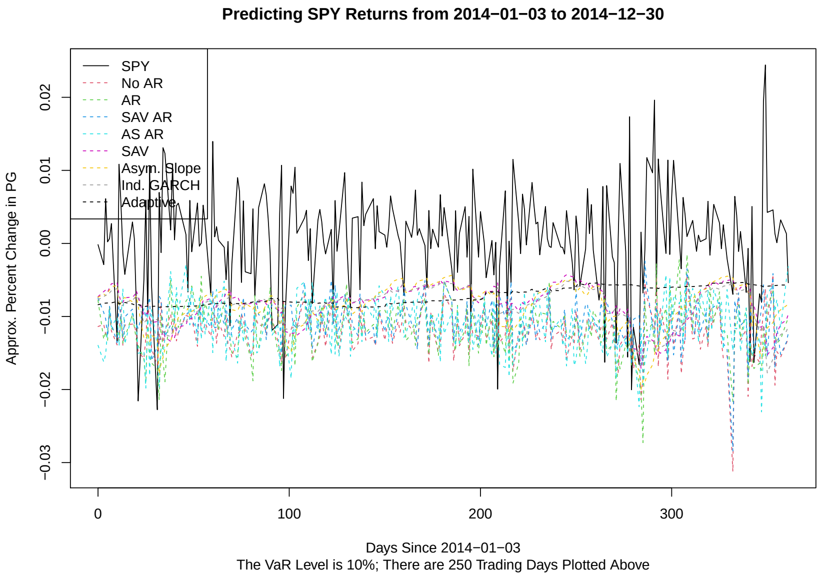
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.242	0.241	0.229	0.238
VaR Break Rate	0.024	0.028	0.044	0.032

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.226	0.218	0.225	0.240
VaR Break Rate	0.052	0.048	0.052	0.056

2014 Test Period - U.S. ETFs - 10% VaR Plot



2014 Test Period - U.S. ETFs - 10% VaR Tables

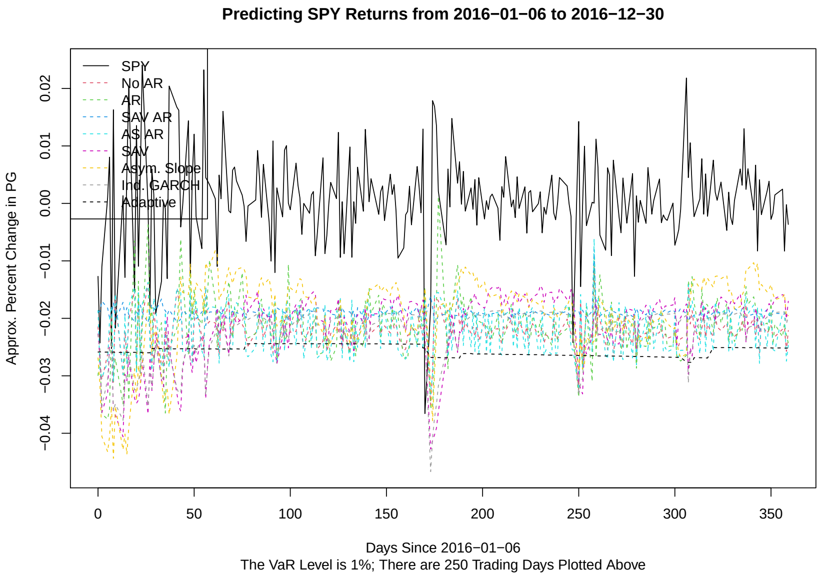
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.388	0.362	0.368	0.367
VaR Break Rate	0.060	0.056	0.080	0.076

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.367	0.359	0.364	0.368
VaR Break Rate	0.116	0.104	0.112	0.132

2016 Test Period - U.S. ETFs - 1% VaR Plot



2016 Test Period - U.S. ETFs - 1% VaR Tables

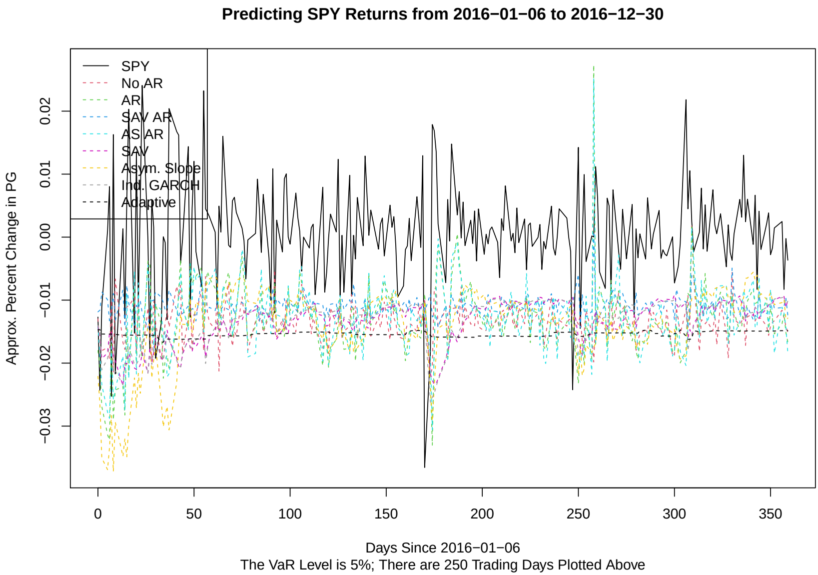
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.087	0.093	0.085	0.09
VaR Break Rate	0.020	0.020	0.028	0.02

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.078	0.082	0.078	0.077
VaR Break Rate	0.012	0.020	0.012	0.004

2016 Test Period - U.S. ETFs - 5% VaR Plot



2016 Test Period - U.S. ETFs - 5% VaR Tables

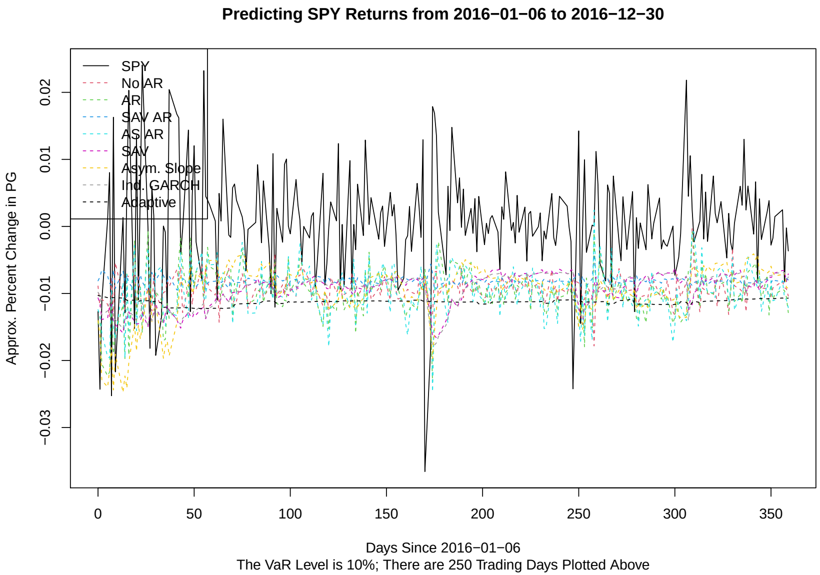
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.278	0.297	0.257	0.290
VaR Break Rate	0.064	0.056	0.080	0.068

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.238	0.238	0.234	0.264
VaR Break Rate	0.032	0.040	0.028	0.032

2016 Test Period - U.S. ETFs - 10% VaR Plot



2016 Test Period - U.S. ETFs - 10% VaR Tables

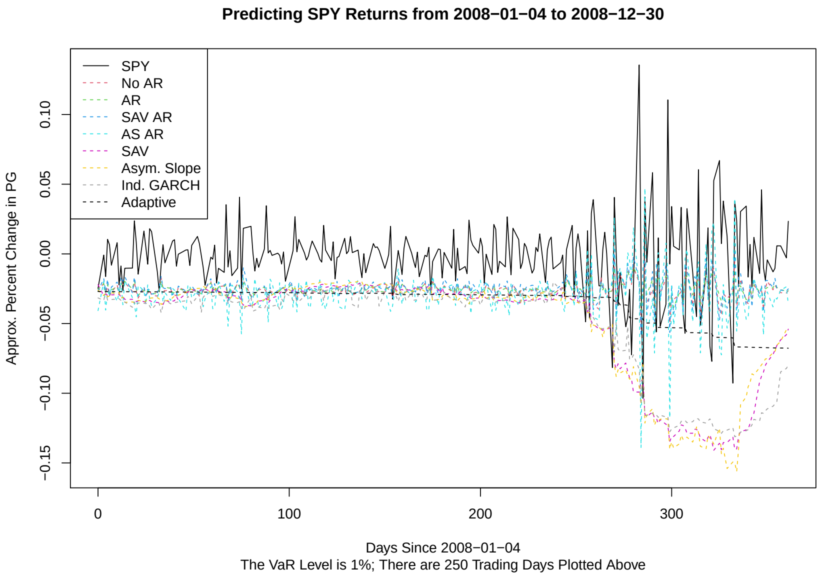
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.419	0.415	0.401	0.429
VaR Break Rate	0.104	0.100	0.120	0.108

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.370	0.373	0.368	0.414
VaR Break Rate	0.088	0.092	0.096	0.072

2008 Test Period - Global ETFs - 1% VaR Plot



2008 Test Period - Global ETFs - 1% VaR Tables

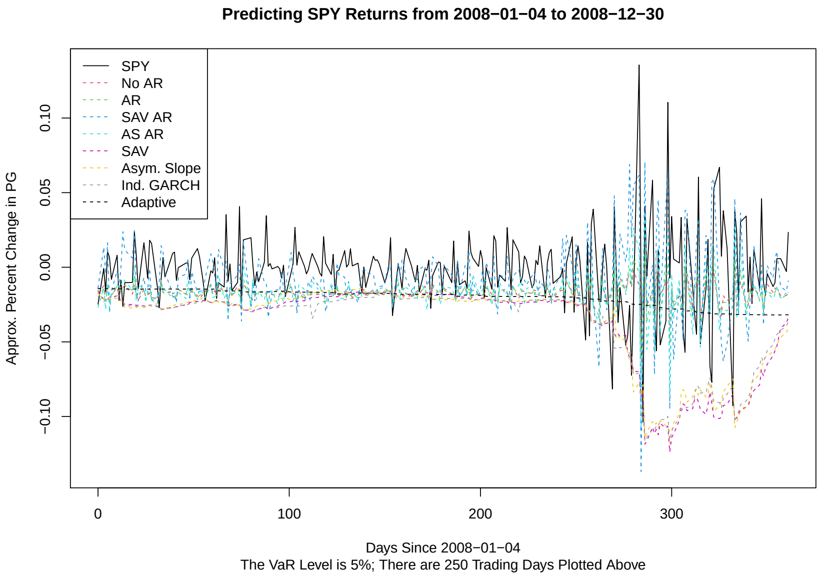
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.740	0.761	0.841	0.867
VaR Break Rate	0.108	0.112	0.120	0.108

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.208	0.213	0.219	0.355
VaR Break Rate	0.028	0.028	0.028	0.060

2008 Test Period - Global ETFs - 5% VaR Plot



2008 Test Period - Global ETFs - 5% VaR Tables

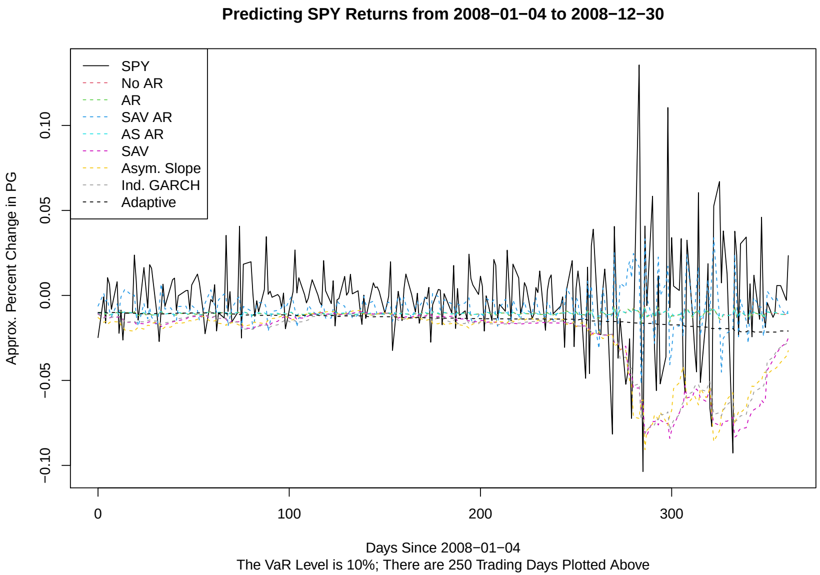
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	1.160	1.173	2.157	1.283
VaR Break Rate	0.176	0.172	0.412	0.184

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.651	0.654	0.640	0.956
VaR Break Rate	0.076	0.076	0.064	0.160

2008 Test Period - Global ETFs - 10% VaR Plot



2008 Test Period - Global ETFs - 10% VaR Tables

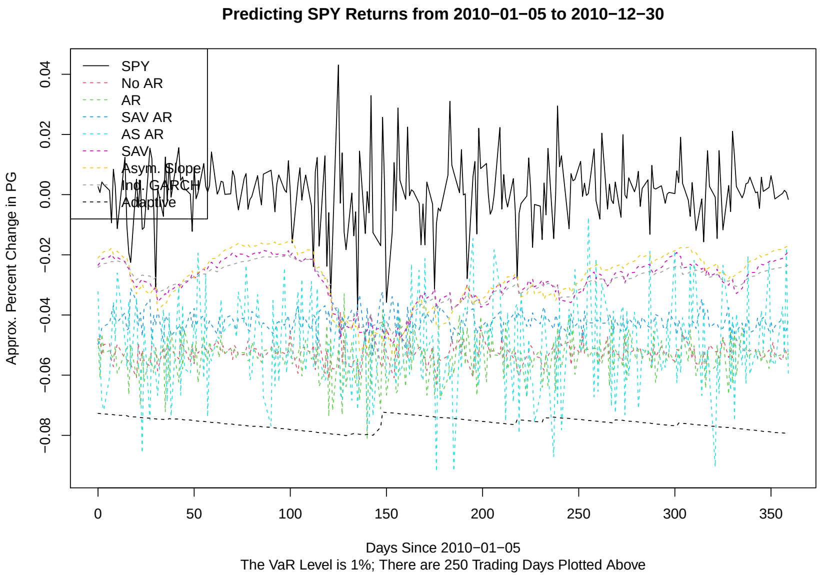
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	1.517	1.517	1.791	1.523
VaR Break Rate	0.284	0.284	0.348	0.288

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	1.077	1.066	1.068	1.366
VaR Break Rate	0.144	0.156	0.140	0.224

2010 Test Period - Global ETFs - 1% VaR Plot



2010 Test Period - Global ETFs - 1% VaR Tables

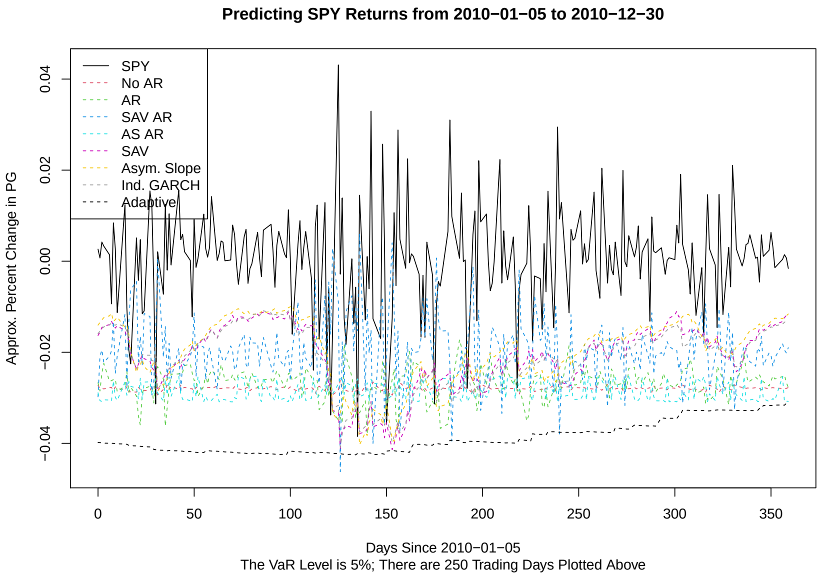
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.133	0.136	0.110	0.127
VaR Break Rate	0.000	0.000	0.004	0.000

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.079	0.08	0.086	0.191
VaR Break Rate	0.020	0.02	0.016	0.000

2010 Test Period - Global ETFs - 5% VaR Plot



2010 Test Period - Global ETFs - 5% VaR Tables

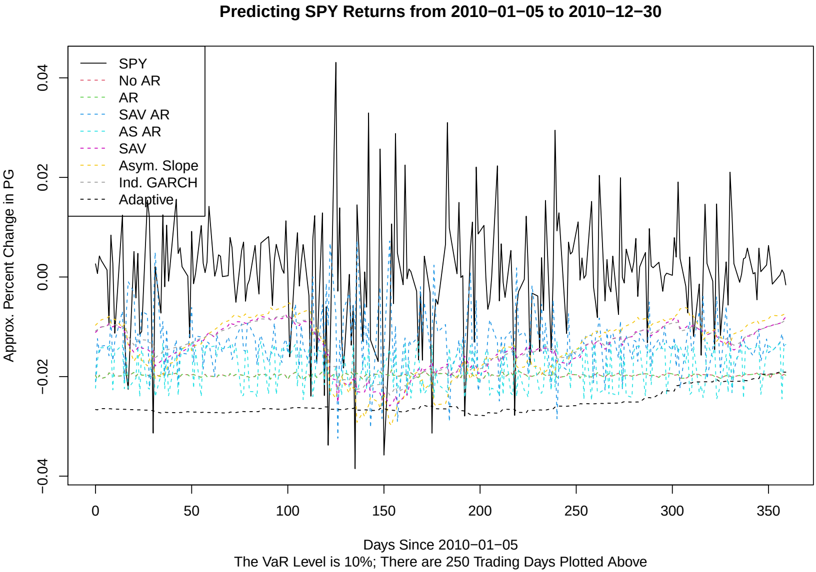
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.386	0.371	0.449	0.401
VaR Break Rate	0.024	0.024	0.068	0.024

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.336	0.336	0.343	0.492
VaR Break Rate	0.052	0.052	0.048	0.000

2010 Test Period - Global ETFs - 10% VaR Plot



2010 Test Period - Global ETFs - 10% VaR Tables

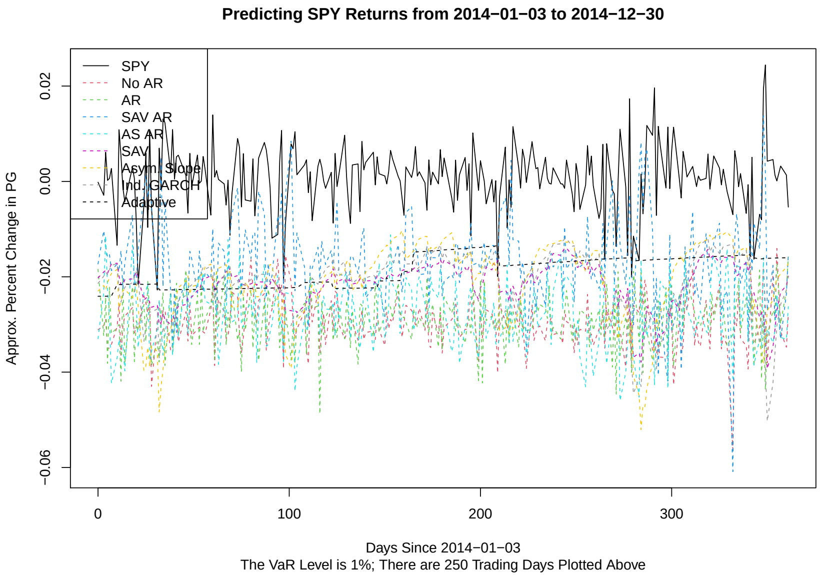
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.606	0.606	0.664	0.590
VaR Break Rate	0.040	0.040	0.128	0.068

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.547	0.549	0.546	0.690
VaR Break Rate	0.080	0.088	0.084	0.028

2014 Test Period - Global ETFs - 1% VaR Plot



2014 Test Period - Global ETFs - 1% VaR Tables

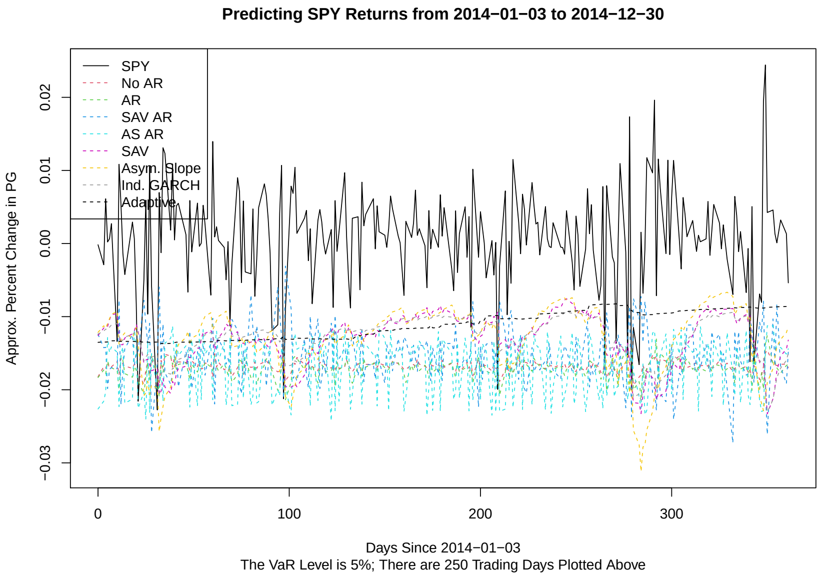
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.076	0.074	0.158	0.071
VaR Break Rate	0.000	0.000	0.096	0.000

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.061	0.057	0.063	0.061
VaR Break Rate	0.008	0.004	0.012	0.028

2014 Test Period - Global ETFs - 5% VaR Plot



2014 Test Period - Global ETFs - 5% VaR Tables

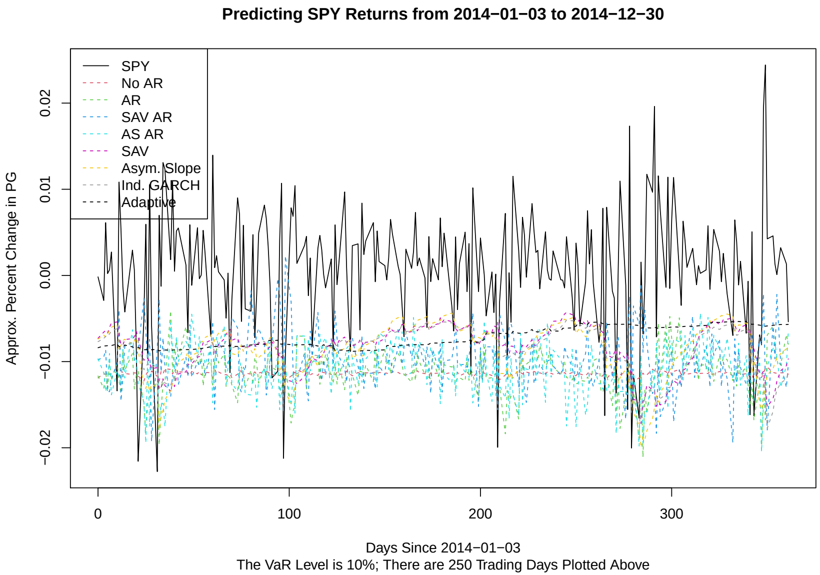
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.240	0.245	0.241	0.245
VaR Break Rate	0.024	0.028	0.040	0.024

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.226	0.218	0.225	0.240
VaR Break Rate	0.052	0.048	0.052	0.056

2014 Test Period - Global ETFs - 10% VaR Plot



2014 Test Period - Global ETFs - 10% VaR Tables

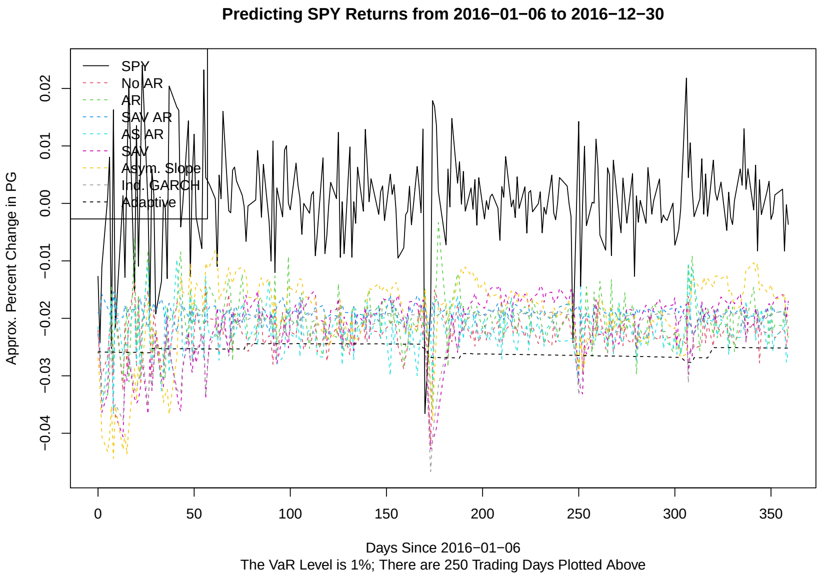
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.377	0.358	0.388	0.358
VaR Break Rate	0.056	0.048	0.084	0.068

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.367	0.359	0.364	0.368
VaR Break Rate	0.116	0.104	0.112	0.132

2016 Test Period - Global ETFs - 1% VaR Plot



2016 Test Period - Global ETFs - 1% VaR Tables

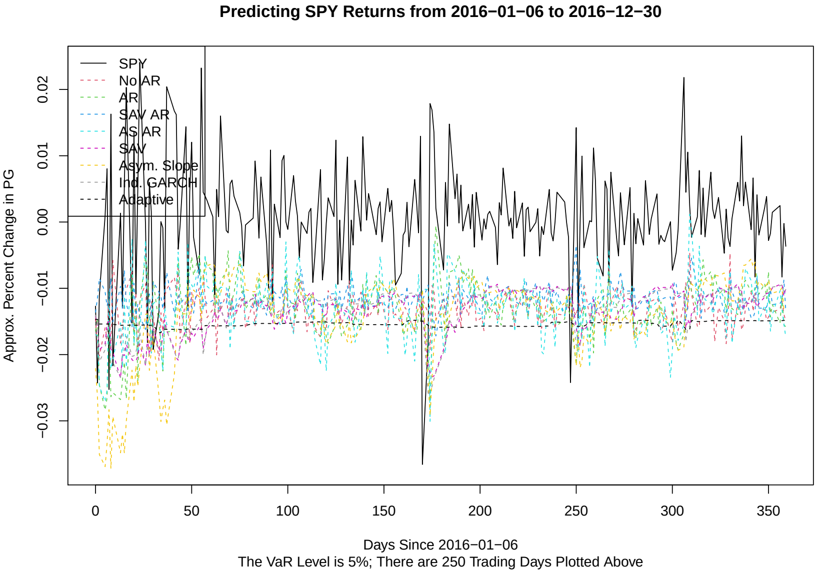
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.09	0.098	0.085	0.087
VaR Break Rate	0.02	0.020	0.028	0.016

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.078	0.082	0.078	0.077
VaR Break Rate	0.012	0.020	0.012	0.004

2016 Test Period - Global ETFs - 5% VaR Plot



2016 Test Period - Global ETFs - 5% VaR Tables

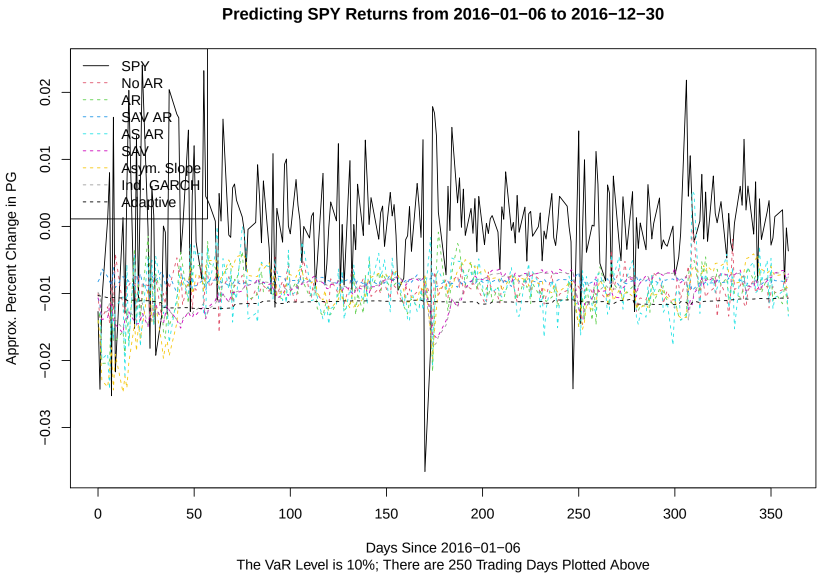
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.279	0.261	0.253	0.291
VaR Break Rate	0.068	0.040	0.048	0.080

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.238	0.238	0.234	0.264
VaR Break Rate	0.032	0.040	0.028	0.032

2016 Test Period - Global ETFs - 10% VaR Plot



2016 Test Period - Global ETFs - 10% VaR Tables

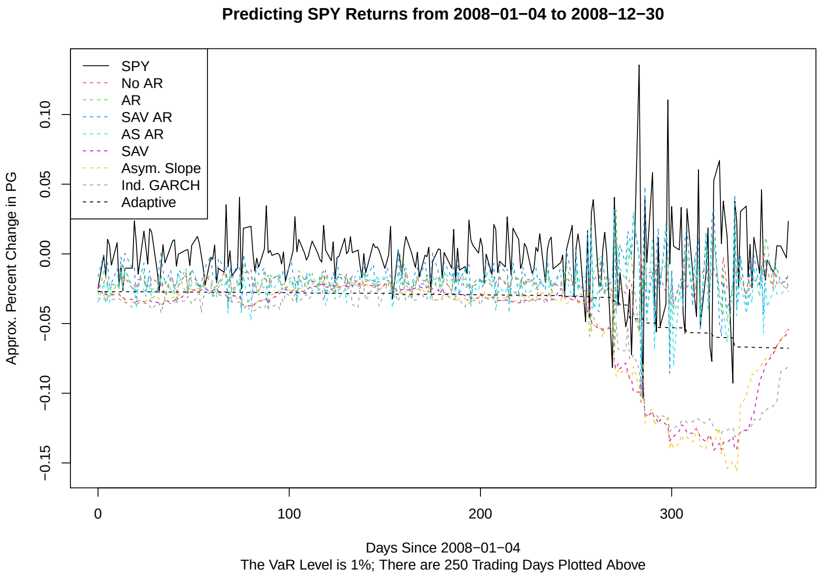
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.419	0.400	0.400	0.442
VaR Break Rate	0.104	0.096	0.116	0.112

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.370	0.373	0.368	0.414
VaR Break Rate	0.088	0.092	0.096	0.072

2008 Test Period - Bond ETFs - 1% VaR Plot



2008 Test Period - Bond ETFs - 1% VaR Tables

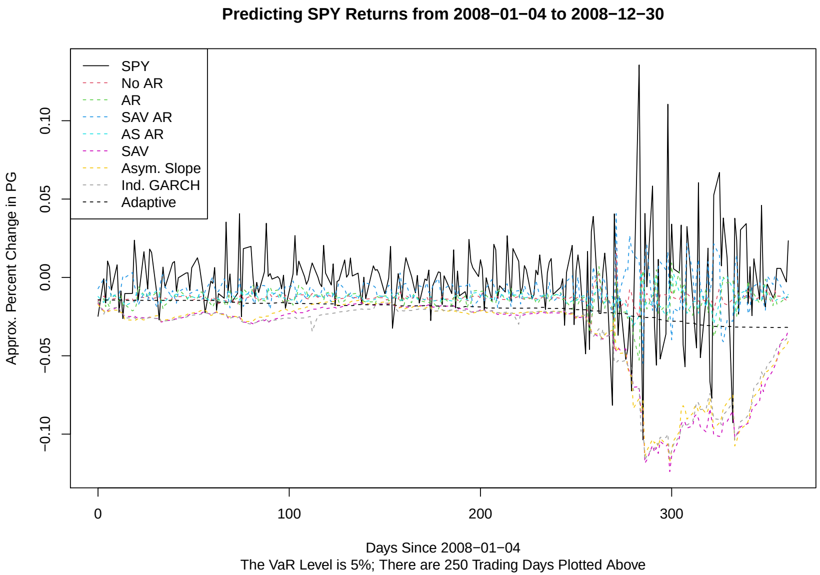
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.891	1.017	1.271	1.090
VaR Break Rate	0.156	0.168	0.224	0.128

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.208	0.213	0.219	0.355
VaR Break Rate	0.028	0.028	0.028	0.060

2008 Test Period - Bond ETFs - 5% VaR Plot



2008 Test Period - Bond ETFs - 5% VaR Tables

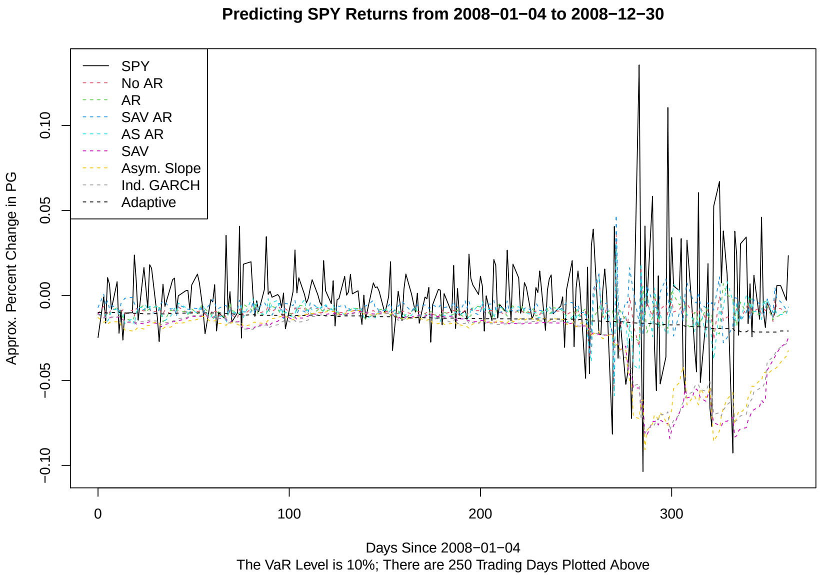
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	1.338	1.316	1.659	1.325
VaR Break Rate	0.236	0.236	0.316	0.256

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.651	0.654	0.640	0.956
VaR Break Rate	0.076	0.076	0.064	0.160

2008 Test Period - Bond ETFs - 10% VaR Plot



2008 Test Period - Bond ETFs - 10% VaR Tables

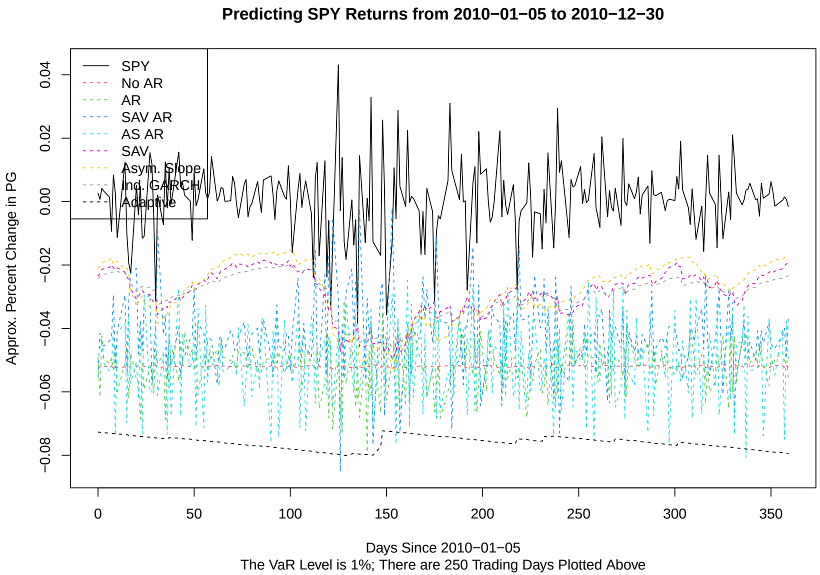
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	1.641	1.575	1.840	1.586
VaR Break Rate	0.308	0.304	0.364	0.308

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	1.077	1.066	1.068	1.366
VaR Break Rate	0.144	0.156	0.140	0.224

2010 Test Period - Bond ETFs - 1% VaR Plot



2010 Test Period - Bond ETFs - 1% VaR Tables

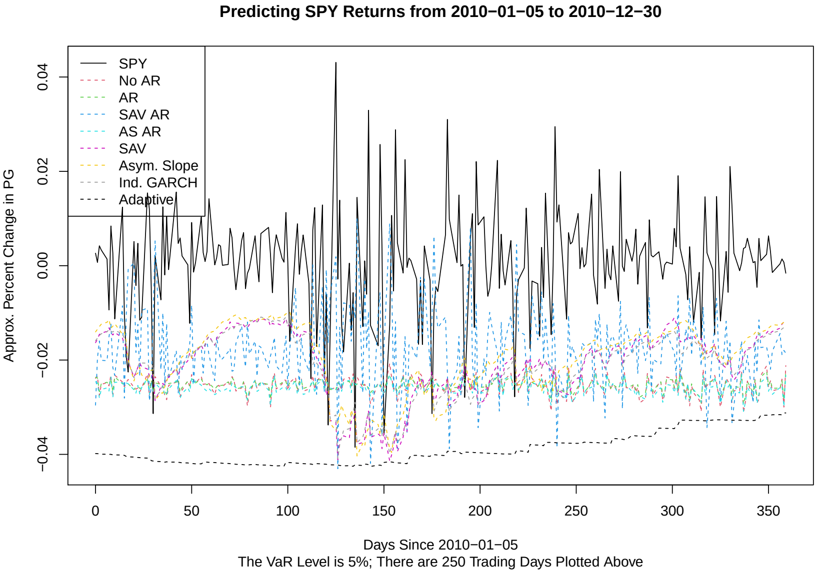
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.131	0.13	0.129	0.128
VaR Break Rate	0.000	0.00	0.016	0.004

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.079	0.08	0.086	0.191
VaR Break Rate	0.020	0.02	0.016	0.000

2010 Test Period - Bond ETFs - 5% VaR Plot



2010 Test Period - Bond ETFs - 5% VaR Tables

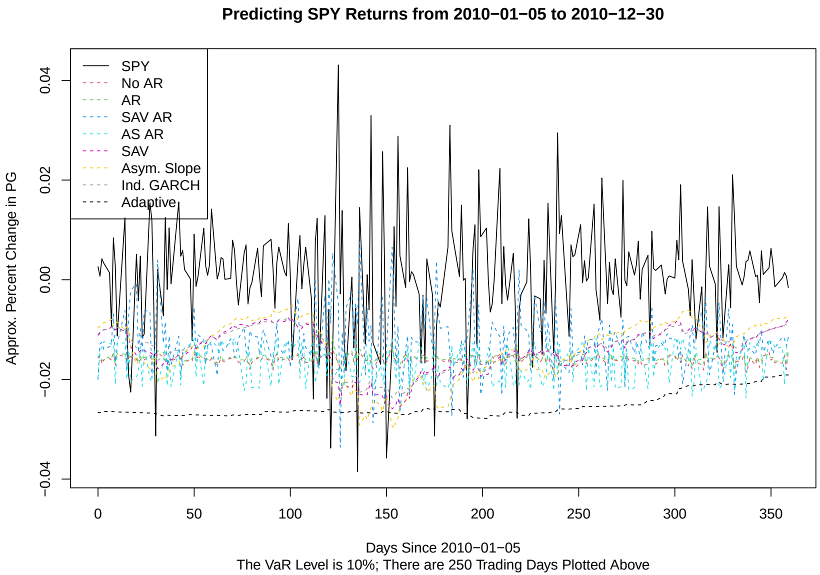
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.373	0.372	0.494	0.378
VaR Break Rate	0.028	0.028	0.104	0.028

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.336	0.336	0.343	0.492
VaR Break Rate	0.052	0.052	0.048	0.000

2010 Test Period - Bond ETFs - 10% VaR Plot



2010 Test Period - Bond ETFs - 10% VaR Tables

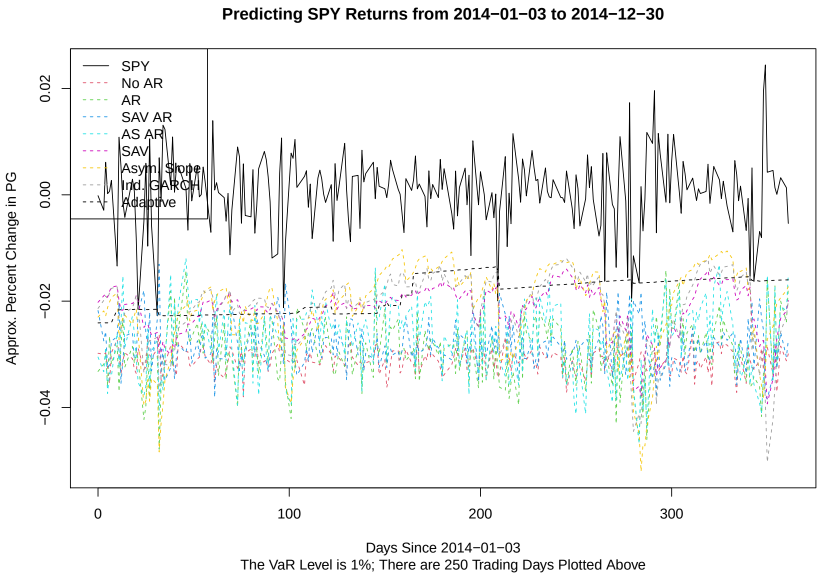
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.566	0.565	0.659	0.578
VaR Break Rate	0.076	0.076	0.128	0.080

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.547	0.549	0.546	0.690
VaR Break Rate	0.080	0.088	0.084	0.028

2014 Test Period - Bond ETFs - 1% VaR Plot



2014 Test Period - Bond ETFs - 1% VaR Tables

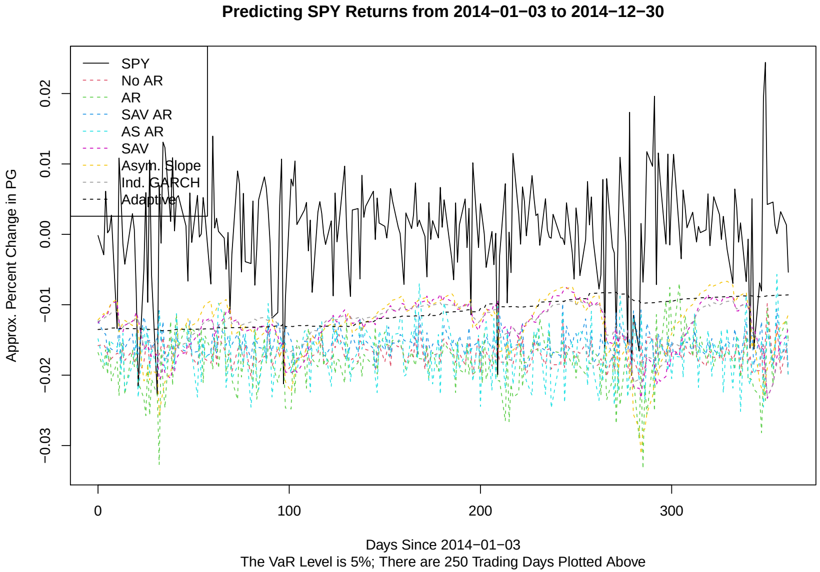
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.079	0.075	0.072	0.070
VaR Break Rate	0.000	0.000	0.000	0.004

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.061	0.057	0.063	0.061
VaR Break Rate	0.008	0.004	0.012	0.028

2014 Test Period - Bond ETFs - 5% VaR Plot



2014 Test Period - Bond ETFs - 5% VaR Tables

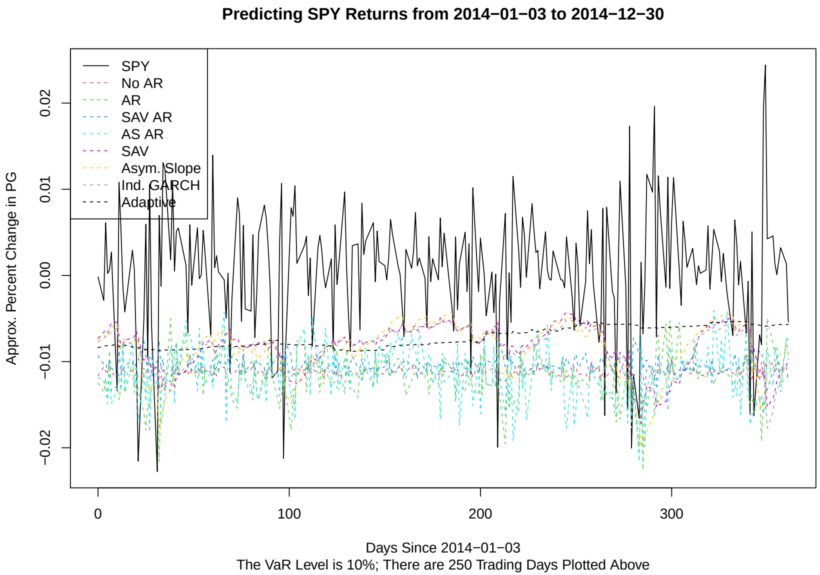
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.241	0.237	0.231	0.237
VaR Break Rate	0.028	0.012	0.040	0.024

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.226	0.218	0.225	0.240
VaR Break Rate	0.052	0.048	0.052	0.056

2014 Test Period - Bond ETFs - 10% VaR Plot



2014 Test Period - Bond ETFs - 10% VaR Tables

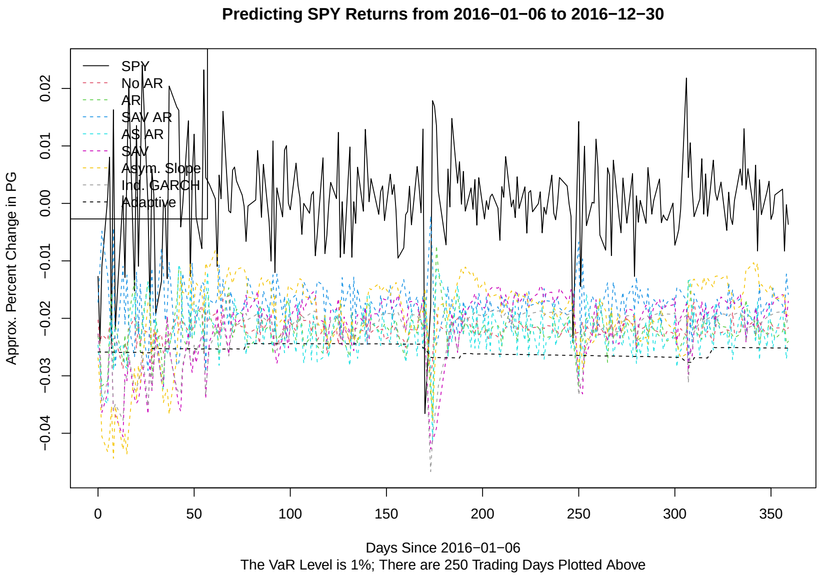
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.370	0.364	0.371	0.352
VaR Break Rate	0.056	0.044	0.072	0.064

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.367	0.359	0.364	0.368
VaR Break Rate	0.116	0.104	0.112	0.132

2016 Test Period - Bond ETFs - 1% VaR Plot



2016 Test Period - Bond ETFs - 1% VaR Tables

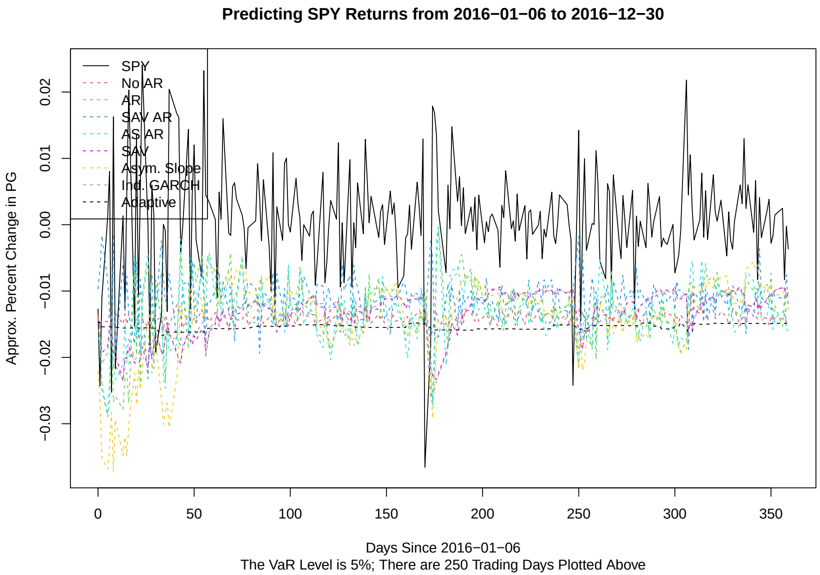
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.076	0.089	0.108	0.085
VaR Break Rate	0.012	0.024	0.028	0.016

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.078	0.082	0.078	0.077
VaR Break Rate	0.012	0.020	0.012	0.004

2016 Test Period - Bond ETFs - 5% VaR Plot



2016 Test Period - Bond ETFs - 5% VaR Tables

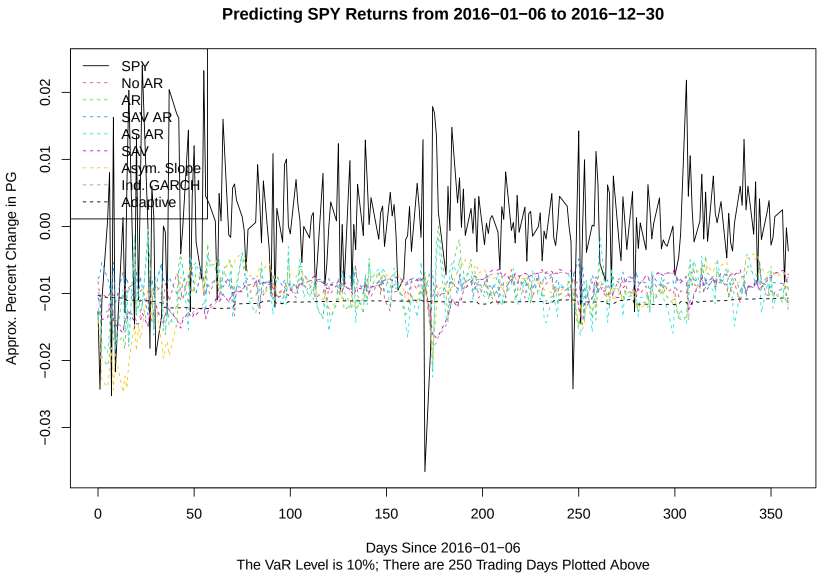
- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.257	0.265	0.273	0.273
VaR Break Rate	0.040	0.044	0.060	0.060

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.238	0.238	0.234	0.264
VaR Break Rate	0.032	0.040	0.028	0.032

2016 Test Period - Bond ETFs - 10% VaR Plot



2016 Test Period - Bond ETFs - 10% VaR Tables

- Multivariate model results

	No AR	AR	SAV AR	AS AR
Losses	0.402	0.395	0.397	0.415
VaR Break Rate	0.100	0.096	0.108	0.084

- Univariate model results

	SAV	Asym. Slope	Ind. GARCH	Adaptive
Losses	0.370	0.373	0.368	0.414
VaR Break Rate	0.088	0.092	0.096	0.072

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