Math 515
Essential Perturbation Theory and Asymptotic Analysis
Chapter 04

Steven M. Wise

swise1@.utk.edu
University of Tennessee

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Chapter 04, Part 2 of 2

Contour Integration in the Complex Plane and the Method of Steepest Descent

Cauchy's Theorem and Integral Formulae

Theorem (Simplified Version of Cauchy's Theorem)

Suppose that $D \subset \mathbb{C}$ is an open, bounded, simply-connected set, and $f : D \to \mathbb{C}$ is analytic on D. Assume, additionally, that $f' : D \to \mathbb{C}$ is continuous. Suppose that $\gamma : [a, b] \to D$ is a smooth, simple, (that is, non-self-intersecting), counter-clockwise-oriented, closed curve (that is, satisfying $\gamma(a) = \gamma(b)$). Then,

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof.

Recall that, if $u = \Re(f)$ and $v = \Im(f)$, then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\boldsymbol{\gamma}} \left[u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y\right] + \mathrm{i}\int_{\boldsymbol{\gamma}} \left[v(x,y)\,\mathrm{d}x + u(x,y)\,\mathrm{d}y\right].$$

Proof (Cont.)

Denote by $D_\gamma \subset D$, the subset of D whose boundary is the image of γ, that is,

$$\partial D_\gamma = \gamma([a, b]).$$

By Green's Theorem and the Cauchy-Riemann Theorem,

$$\int_{\boldsymbol{\gamma}} [u(x, y)\,\mathrm{d}x - v(x, y)\,\mathrm{d}y] = \int_{D_\gamma} \left(-\frac{\partial v}{\partial x}(x, y) - \frac{\partial u}{\partial y}(x, y) \right) \mathrm{d}\mathbf{x} = 0.$$

Similarly,

$$\int_{\boldsymbol{\gamma}} [v(x, y)\,\mathrm{d}x + u(x, y)\,\mathrm{d}y] = \int_{D_\gamma} \left(\frac{\partial u}{\partial x}(x, y) - \frac{\partial v}{\partial y}(x, y) \right) \mathrm{d}\mathbf{x} = 0.$$

$\square$

Theorem (Cauchy's Theorem)

Suppose that $D \subset \mathbb{C}$ *is an open, simply connected, bounded set, and* $f : D \to \mathbb{C}$ *analytic on* D. *For any simple, piecewise smooth, closed contour* $\gamma : [a, b] \to D$

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Theorem (Path Independence)

Suppose that $D \subset \mathbb{C}$ is an open, simply connected, bounded set, and $f : D \to \mathbb{C}$ is analytic on D. Suppose that $z_0, z_1 \in D$ are arbitrary points in D, and let $\gamma : [a, b] \to D$ and $\chi : [c, d] \to D$ be any two simple, piecewise smooth contours with

$$\gamma(a) = z_0 = \chi(c) \quad \textit{and} \quad \gamma(b) = z_1 = \chi(b).$$

Then,

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\chi f(z)\,\mathrm{d}z.$$

Example

Suppose that $\gamma : [0, 2\pi] \to \mathbb{C}$ is a circular, counterclockwise oriented contour of radius r around the point $z_0 \in \mathbb{C}$. Then we can show that

$$\int_\gamma \frac{1}{z - z_0}\, \mathrm{d}z = 2\pi \mathrm{i}. \tag{1}$$

This result is particularly famous. We refer to the integral in (1) as the **residue integral**. To perform this calculation, note that the contour can be expressed as

$$\gamma(t) = z_0 + re^{\mathrm{i}t}.$$

It follows that

$$\begin{aligned}\int_\gamma \frac{1}{z - z_0}\, \mathrm{d}z &= \int_0^{2\pi} \frac{1}{z_0 + re^{\mathrm{i}t} - z_0} r\mathrm{i}e^{\mathrm{i}t}\, \mathrm{d}t \\ &= \int_0^{2\pi} \frac{r\mathrm{i}e^{\mathrm{i}t}}{re^{\mathrm{i}t}}\, \mathrm{d}t \\ &= 2\pi\mathrm{i}.\end{aligned}$$

Example (Cont.)

What if, on the other hand, the contour goes clockwise round the point z_0? In this case, the contour can be expressed as

$$\gamma(t) = z_0 + re^{-\mathrm{i}t},$$

and

$$\begin{aligned} \int_\gamma \frac{1}{z - z_0}\,\mathrm{d}z &= \int_0^{2\pi} \frac{-r\mathrm{i}e^{-\mathrm{i}t}}{z_0 + re^{-\mathrm{i}t} - z_0}\,\mathrm{d}t \\ &= \int_0^{2\pi} \frac{-r\mathrm{i}e^{-\mathrm{i}t}}{re^{-\mathrm{i}t}}\,\mathrm{d}t \\ &= -2\pi\mathrm{i}. \end{aligned}$$

Theorem (Cauchy's Integral Formula)

Suppose that $D \subset \mathbb{C}$ is an open, simply connected, bounded set, and $f : D \to \mathbb{C}$ analytic on D. Assume that $\gamma : [a, b] \to D$ is a simple, smooth, closed, counter-clockwise-oriented contour. Define $D_\gamma \subset D$ to be the set enclosed by γ and whose boundary is γ. Then, for any point $z_0 \in D_\gamma$,

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(z)}{z - z_0} \, \mathrm{d}z.$$

Theorem (Cauchy's Integral Formula for Derivatives)

Suppose that $D \subset \mathbb{C}$ is an open, simply connected, bounded set, and $f : D \to \mathbb{C}$ analytic on D. Assume that $\gamma : [a, b] \to D$ is a simple, smooth, closed, counter-clockwise-oriented contour. Define $D_\gamma \subset D$ to be the set enclosed by γ and whose boundary is γ. Then, for any point $z_0 \in D_\gamma$ and any $n \in \mathbb{N}$, f is n-times differentiable at z_0 and

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z.$$

Example

Suppose that $\gamma(t) = e^{\mathrm{i}t}$, for $0 \leq t \leq 2\pi$. This is a smooth, simple, closed, counterclockwise-oriented curve of radius 1 around the origin $z = 0$. Consider the integral

$$\int_{\gamma} \frac{z+1}{z^4 + 4z^3} \, \mathrm{d}z.$$

Define

$$f(z) = \frac{z+1}{z+4}, \quad \forall \, z \in \mathbb{C} \setminus \{-4\}.$$

Observe that $f(z)$ has no singularities inside the set enclosed by γ, that is, the unit disc. In fact f is analytic at every point inside the unit disc. By Cauchy's Integral Formula for Derivatives, f has derivatives to any order. Now, notice that the integrand can be written as

$$\frac{z+1}{z^4 + 4z^3} = \frac{f(z)}{z^3}.$$

Example (Cont.)

The theorem implies, using $n = 2$, that

$$\int_\gamma \frac{z+1}{z^4+4z^3}\,\mathrm{d}z = \int_\gamma \frac{f(z)}{z^3}\,\mathrm{d}z = \frac{2\pi\mathrm{i}}{2!}f''(0).$$

Complex differentiation obeys the same rules, for the most part, as regular differentiation. The chain rule, power rule, quotient rule all still hold. Thus,

$$f'(z) = \frac{3}{(z+4)^2} \quad \text{and} \quad f''(z) = \frac{-6}{(z+4)^3}.$$

Consequently,

$$\int_\gamma \frac{z+1}{z^4+4z^3}\,\mathrm{d}z = \frac{2\pi\mathrm{i}}{2!}f''(0) = \frac{2\pi\mathrm{i}}{2}\frac{(-6)}{4^3} = \frac{-3\pi\mathrm{i}}{32}.$$

Theorem (Taylor's Theorem for Analytic Functions)

Suppose that $D \subset \mathbb{C}$ is an open, simply connected, bounded set, and $f : D \to \mathbb{C}$ analytic on D. For any point $z_0 \in D$, f is differentiable at z_0 to any order, and

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k,$$

for all z in the closed disc $\overline{D_R}(z_0)$, where

$$\overline{D_r}(w) = \{z \in \mathbb{C} \mid |z - w| \leq r\},$$

and $R > 0$ is any positive number such that the closed disc is entirely contained in D, that is, $\overline{D_R}(z_0) \subset D$. Moreover the power series converges absolutely on $\overline{D_R}(z_0)$.

Method of Steepest Descent

Contour Integrals with a Large Parameter

We just spent several slides introducing, or, perhaps, reintroducing, the reader to the subject of contour integration in the complex plane. That work will payoff in the next two sections, for herein we will be interested in integrals of the form

$$I(\lambda) = \int_{\chi} h(z) e^{\lambda \rho(z)} \, \mathrm{d}z, \quad \lambda \in (0, \infty), \tag{2}$$

for large values of λ, where $\rho : D \to \mathbb{C}$ is an analytic function on a bounded, open, simply connected set $D \subset \mathbb{C}$, $\chi : [a, b] \to D$ is a smooth, simple contour in D, and $h(z) : D \to \mathbb{C}$ is also analytic in D. Suppose that $\phi = \Re(\rho)$ and $\psi = \Im(\rho)$, or, in other words,

$$\rho(z) = \phi(z) + \mathrm{i}\psi(z), \quad \forall\, z \in D.$$

Elements of the Method of Steepest Descent

In the simplest setting, we seek a closed contour $\gamma : [a, c] \to D$, such that the following hold:

1. The contour γ is piecewise-smooth and traverses χ as its first segment: $\gamma = \chi + \gamma_1 + \gamma_2 + \gamma_3$.
2. $\psi(\gamma_j) = C_j$, where $C_j \in \mathbb{R}$ is a constant, for $j \in \{1, 3\}$. Such segments of γ are called **constant-ψ contour segments**.
3. ψ is not constant on γ_2. Such segments of γ are called **bridge contour segments**.
4. The integral $\int_{\gamma_2} h(z)e^{\lambda\rho(z)}\,dz$ is "controllable," in some way to be made precise. For example, the value of the integral over the bridge may become vanishingly small as we "infinitely extend" the contour.
5. $\phi(\gamma_j)$, for $j \in \{1, 3\}$, has a maximum at the point where γ_j connects to the original contour χ. In other words, ϕ is decreasing along γ_j, for $j \in \{1, 3\}$, as we move away from χ.

Elements of the Method of Steepest Descent

By Cauchy's Theorem,

$$\int_{\gamma} h(z)e^{\lambda\rho(z)}\,\mathrm{d}z = 0,$$

since the integrand is analytic on D. Consequently,

$$\begin{aligned}\int_{\chi} h(z)e^{\lambda\rho(z)}\,\mathrm{d}z &= -\sum_{j=1,3}\int_{\gamma_j} h(z)e^{\lambda\rho(z)}\,\mathrm{d}z - \int_{\gamma_2} h(z)e^{\lambda\rho(z)}\,\mathrm{d}z \\ &= \sum_{j=1,3} e^{\lambda \mathrm{i} C_j}\int_{-\gamma_j} h(z)e^{\lambda\phi(z)}\,\mathrm{d}z + \int_{-\gamma_2} h(z)e^{\lambda\rho(z)}\,\mathrm{d}z,\end{aligned}$$

where $-\boldsymbol{\gamma}_j$ is the same contour as $\boldsymbol{\gamma}_j$, but run in the reverse direction.

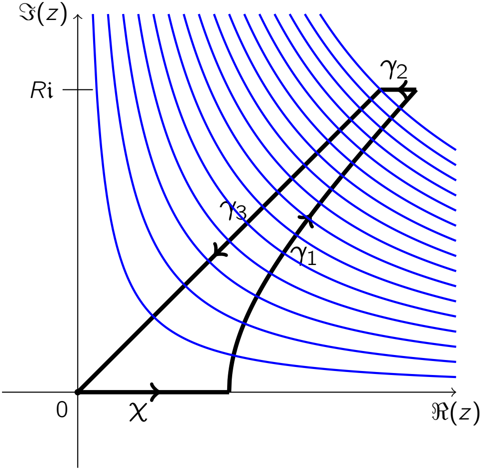

Figure: The piecewise smooth, simple, closed contour γ used in the following example.

Example

Let us consider the specific example

$$I(\lambda) = \int_0^1 e^{\mathrm{i}\lambda s^2}\,\mathrm{d}s, \quad \lambda \in (0,\infty). \tag{3}$$

We want an asymptotic approximation of this integral as $\lambda \to \infty$. This integral can be interpreted as

$$I(\lambda) = \int_\chi e^{\mathrm{i}\lambda z^2}\,\mathrm{d}z, \tag{4}$$

where $\chi : [0,1] \to \mathbb{C}$ is the straight line contour connecting $z = 0$ and $z = 1$ in the complex plane. Now, if $z = x + \mathrm{i}y$. Then,

$$\rho(z) := iz^2 = -2xy + \mathrm{i}\left(x^2 - y^2\right),$$

and we identify

$$\phi(x,y) = -2xy \quad \text{and} \quad \psi(x,y) = x^2 - y^2.$$

Example (Cont.)

Using a slight abuse of notation,

$$\phi(z) = -2xy \quad \text{and} \quad \psi(z) = x^2 - y^2,$$

when $z = x + \mathrm{i}y$. Now, consider the following contour segments:

1. γ_1 connects the point $z = 1$ to the point $z = \sqrt{R^2+1} + \mathrm{i}R$ along the curve $y = \sqrt{x^2-1}$. $\psi(x, y) = 1$ along γ_1.
2. The bridge contour segment γ_2 connects the point $z = \sqrt{R^2+1} + \mathrm{i}R$ to the point $z = R + \mathrm{i}R$ in a straight line.
3. γ_3 connects $z = R + \mathrm{i}R$ to $z = 0$ in a straight line. $\psi(x, y) = 0$ along γ_3.

Thus, $\gamma = \chi + \gamma_1 + \gamma_2 + \gamma_3$ is a piecewise smooth, simple, closed contour. See the figure on the next slide. The integrand of (4) is analytic on any bounded, simply-connected set containing γ.

Now, $R > 0$ is arbitrary in the construction of our contour γ. We intend to send $R \to \infty$. In doing so, the bridge integral will become vanishingly small. Let us consider that integral first.

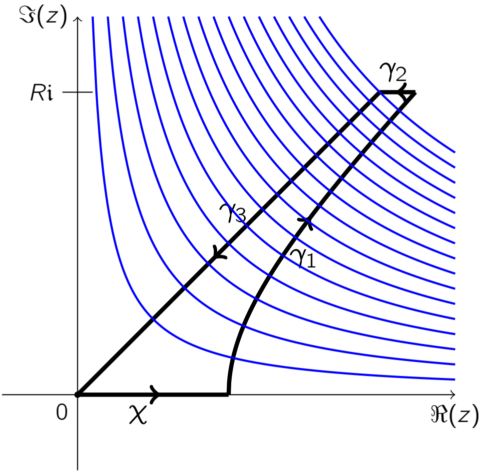

Figure: The piecewise smooth, simple, closed contour γ. Note that $\psi(\gamma_i(t)) = C_1$, $i = 1, 3$.

Example (Cont.)

Bridge integral over γ_2: Consider the following parameterization:

$$\gamma_2(t) = t + \mathrm{i}R, \quad t = \sqrt{R^2+1} \quad \text{to} \quad t = R.$$

Thus,

$$\begin{aligned} I_2(\lambda; R) &:= \int_{\gamma_2} e^{\mathrm{i}\lambda z^2} \, \mathrm{d}z \\ &= \int_{t=\sqrt{R^2+1}}^{t=R} e^{\mathrm{i}\lambda \gamma_2^2(t)} \gamma_2'(t) \, \mathrm{d}t \\ &= \int_{t=\sqrt{R^2+1}}^{t=R} e^{-2\lambda t R} e^{\mathrm{i}\lambda(t^2-R^2)} \, \mathrm{d}t. \end{aligned}$$

Example (Cont.)

Now, let us take the modulus of the integral:

$$
\begin{aligned}
|I_2(\lambda; R)| &= \left| \int_{t=\sqrt{R^2+1}}^{t=R} e^{-2\lambda t R} e^{\mathrm{i}\lambda(t^2-R^2)} \, \mathrm{d}t \right| \\
&\leq \int_{t=\sqrt{R^2+1}}^{t=R} \left| e^{-2\lambda t R} \right| \cdot \left| e^{\mathrm{i}\lambda(t^2-R^2)} \right| \, \mathrm{d}t \\
&= \int_{t=R}^{t=\sqrt{R^2+1}} e^{-2\lambda t R} \, \mathrm{d}t \\
&= \frac{1}{2\lambda R} \left[e^{-2\lambda R^2} - e^{-2\lambda R\sqrt{R^2+1}} \right] \\
&\xrightarrow{R\to\infty} 0.
\end{aligned}
$$

Thus,

$$
I_2(\gamma; R) = \int_{\gamma_2} e^{\mathrm{i}\lambda z^2} \, \mathrm{d}z \xrightarrow{R\to\infty} 0.
$$

Example (Cont.)

Integral over γ_3: Recall that γ_3 connects $z = R + \mathrm{i}R$ to $z = 0$ in a straight line, and $\psi(x, y) = 0$ along γ_3. Consider the following parameterization of γ_3:

$$\gamma_3(t) = t + \mathrm{i}t, \quad t = R \quad \text{to} \quad t = 0.$$

Then,

$$\begin{aligned} I_3(\lambda; R) &:= \int_{\gamma_3} e^{\mathrm{i}\lambda z^2}\,\mathrm{d}z \\ &= \int_{t=R}^{t=0} e^{\mathrm{i}\lambda\gamma_3^2(t)}\gamma_3'(t)\,\mathrm{d}t \\ &= \int_{t=R}^{t=0} e^{-2\lambda t^2} e^{\mathrm{i}\lambda(t^2-t^2)}(1+\mathrm{i})\,\mathrm{d}t \\ &= (1+\mathrm{i})\int_{t=R}^{t=0} e^{-2\lambda t^2}\,\mathrm{d}t. \end{aligned}$$

Example (Cont.)

Observe that $\phi(\gamma_3(t)) = -2t^2$ is a maximum at $t = 0$, where γ_3 connects to χ, as required. In any case. it follows that, as $R \to \infty$,

$$-I_3(\lambda; R) = (1 + \mathfrak{i}) \int_0^R e^{-2\lambda t^2} \, \mathrm{d}t \to (1 + \mathfrak{i}) \int_0^\infty e^{-2\lambda t^2} \, \mathrm{d}t =: -I_3(\lambda).$$

This Gaussian integral can be easily evaluated as

$$-I_3(\lambda) = \frac{1}{2} e^{\mathfrak{i}\pi/4} \sqrt{\frac{\pi}{\lambda}}.$$

Example (Cont.)

Integral over γ_1: Recall that γ_1 connects the point $z = 1$ to the point $z = \sqrt{R^2+1} + \mathrm{i}R$ along the curve $y = \sqrt{x^2-1}$. $\psi(x, y) = 1$ along γ_1. Consider the following parameterization of γ_1:

$$\gamma_1(t) = \sqrt{t^2+1} + \mathrm{i}t, \quad t = 0 \quad \text{to} \quad t = R.$$

Thus,

$$\gamma_1'(t) = \frac{t}{\sqrt{t^2+1}} + \mathrm{i},$$

and it follows that

$$\begin{aligned} I_1(\lambda; R) &:= \int_{\gamma_1} e^{\mathrm{i}\lambda z^2} \, \mathrm{d}z \\ &= \int_{t=0}^{t=R} e^{\mathrm{i}\lambda\gamma_1^2(t)} \gamma_1'(t) \, \mathrm{d}t \\ &= e^{\mathrm{i}\lambda} \int_{t=0}^{t=R} e^{-2\lambda t\sqrt{t^2+1}} \left[\frac{t}{\sqrt{t^2+1}} + \mathrm{i} \right] \mathrm{d}t. \end{aligned}$$

Example (Cont.)

Taking $R \to \infty$, we have

$$I_1(\lambda) = \lim_{R\to\infty} I_1(\lambda; R) = e^{\mathrm{i}\lambda} \int_0^\infty e^{-2\lambda t\sqrt{t^2+1}} \left[\frac{t}{\sqrt{t^2+1}} + \mathrm{i}\right] \mathrm{d}t.$$

We can use a Laplace-type method to approximate the integral $I_1(\lambda)$, as $\lambda \nearrow \infty$. The integrand is of the appropriate form:

$$e^{\lambda\tilde{\phi}(t)}, \quad \tilde{\phi}(t) = -2t\sqrt{t^2+1}.$$

Observe that $\tilde{\phi}(t)$ is a maximum at $t = 0$, which represents the point where γ_1 and χ connect.

Example (Cont.)

Let us make a sophisticated change of variables:

$$\mathrm{i}\gamma_1^2(t) = \mathrm{i} - s.$$

Recall that

$$\gamma_1(t) = \sqrt{t^2+1} + \mathrm{i}t,$$

so that

$$\mathrm{i}\gamma_1^2(t) = \mathrm{i}\left(t^2 + 1 + 2\mathrm{i}t\sqrt{t^2+1} - t^2\right) = \mathrm{i} - 2t\sqrt{t^2+1}.$$

Thus,

$$s = 2t\sqrt{t^2+1}.$$

Furthermore,

$$2\mathrm{i}\gamma_1(t)\gamma_1'(t)\,\mathrm{d}t = -\mathrm{d}s.$$

Thus,

$$\gamma_1'(t)\,\mathrm{d}t = \frac{\mathrm{i}\,\mathrm{d}s}{2\gamma_1(t)}.$$

Example (Cont.)

But,

$$\gamma_1(t) = \sqrt{1 + \mathrm{i}s}.$$

Therefore, the transformed integral is

$$I_1(\lambda) = \frac{1}{2}\mathrm{i}e^{\mathrm{i}\lambda} \int_0^\infty \frac{e^{-\lambda s}}{\sqrt{1+\mathrm{i}s}}\, \mathrm{d}s.$$

This can be expanded using Watson's Lemma. We need the power series expansion

$$\frac{1}{\sqrt{1+\mathrm{i}z}} = \sum_{k=0}^{\infty} z^k \mathrm{i}^k \binom{-\frac{1}{2}}{k} = 1 - \frac{\mathrm{i}z}{2} - \frac{3z^2}{8} + \frac{5\mathrm{i}z^3}{16} + \frac{35z^4}{128} - \cdots .$$

which is valid for $|z| < 1$. Thus, for real values of $s \in [0, 1/2]$,

$$\frac{1}{\sqrt{1+\mathrm{i}s}} = \sum_{k=0}^{\infty} s^k \mathrm{i}^k \binom{-\frac{1}{2}}{k}$$

Example (Cont.)

By Watson's Lemma,

$$I_1(\lambda) \sim \frac{1}{2}\mathrm{i}e^{\mathrm{i}\lambda}\sum_{k=0}^{\infty}\mathrm{i}^k\binom{-\frac{1}{2}}{k}\int_0^{\infty}e^{-\lambda s}s^k\,\mathrm{d}s, \quad \text{as} \quad \lambda\to\infty,$$

or, equivalently,

$$I_1(\lambda) \sim \frac{1}{2}\mathrm{i}e^{\mathrm{i}\lambda}\sum_{k=0}^{\infty}\mathrm{i}^k\binom{-\frac{1}{2}}{k}\frac{\Gamma(k+1)}{\lambda^{k+1}}, \quad \text{as} \quad \lambda\to\infty,$$

after evaluating the integrals. To conclude,

$$\begin{aligned} I(\lambda) &= \int_0^1 e^{\mathrm{i}\lambda s^2}\,\mathrm{d}s \\ &= -I_1(\lambda) - I_3(\lambda) \\ &\sim \frac{1}{2}e^{\mathrm{i}\pi/4}\sqrt{\frac{\pi}{\lambda}} - \frac{1}{2}\mathrm{i}e^{\mathrm{i}\lambda}\sum_{k=0}^{\infty}\mathrm{i}^k\binom{-\frac{1}{2}}{k}\frac{\Gamma(k+1)}{\lambda^{k+1}}, \quad \text{as} \quad \lambda\to\infty. \end{aligned}$$

Example

Here is the Mathematica code for the previous problem.

In[1]:=
```
AsymptoticIntegrate[Exp[I*λ*s^2],{s,0,1},{λ,∞,3}]
```

Out[1]=

$$e^{i\lambda}\left(\frac{3i}{8\lambda^3}-\frac{1}{4\lambda^2}-\frac{i}{2\lambda}\right)-\frac{i e^{\frac{1}{4} i\pi\left(3+4\ \mathrm{Floor}\left[\frac{3}{4}-\frac{\mathrm{Arg}[\lambda]}{2\pi}\right]\right)}\sqrt{\pi}}{2\sqrt{\lambda}}$$

The leading order term is a bit puzzling. But, it all seems to agree with our calculation.

The Saddle Point Method

Basic Assumptions

The saddle point method is essentially the same as the method of steepest descent. We are still interested integrals of the form

$$I(\lambda) = \int_{\chi} h(z) e^{\lambda \rho(z)} \, \mathrm{d}z, \quad \lambda \in (0, \infty), \tag{5}$$

for large values of λ. Again, we assume that $\rho : D \to \mathbb{C}$ is an analytic function on a bound, simply-connected, open set $D \subset \mathbb{C}$, $\chi : [a, b] \to D$ is a smooth, simple contour in D, and $h(z) : D \to \mathbb{C}$ is also analytic in D. Recall our definitions $\phi = \Re(\rho)$ and $\psi = \Im(\rho)$, so that

$$\rho(z) = \phi(z) + \mathrm{i}\psi(z), \quad \forall\, z \in D.$$

In the saddle point method, we seek a new contour that goes through a **saddle point** of the function ϕ along paths of constant ψ.

Contour Closure

In the simplest setting, we ideally seek a closed contour $\gamma : [a, c] \to D$, such that the following hold:

1. The contour γ is piecewise-smooth and traverses χ as its first segment: $\gamma = \chi + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5$.
2. $\psi(\gamma_j) = C_j$, where $C_j \in \mathbb{R}$ is a constant, for $j \in \{1, 3, 5\}$, that is, γ_j, for $j \in \{1, 3, 5\}$, are constant-ψ contour segments.
3. ψ is not constant on the bridge contour segments γ_2 and γ_4.
4. The integrals $\int_{\gamma_j} h(z) e^{\lambda\rho(z)}\, dz$, $j = 2, 4$, become vanishingly small as we deform the contours in some way.
5. $\phi(\gamma_j)$, for $j \in \{1, 5\}$, has a maximum at the point where γ_j connects to the original contour χ. In other words, ϕ is decreasing along γ_j, for $j \in \{1, 3\}$, as we move away from χ.
6. Along γ_3, ϕ has a maximum at some point in the interior of γ_3 at a saddle point of ϕ.

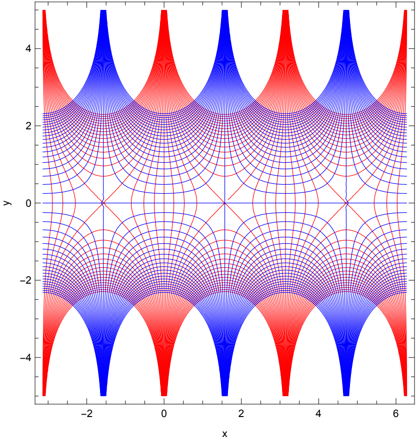

Figure: Level curves of $\phi(z) = \Re(\mathrm{i}\sin(z))$ (blue) and $\psi(z) = \Im(\mathrm{i}\sin(z))$ (red).

Example

Let us consider the integral

$$I(\lambda) = \frac{1}{\pi}\int_0^\pi e^{\mathrm{i}\lambda\sin(t)}\,\mathrm{d}t, \tag{6}$$

for $\lambda \in (0,\infty)$. Of course, as we have seen, this Fourier integral is important for the asymptotic approximation of the Bessel function, J_0, since

$$J_0(\lambda) = \frac{1}{\pi}\Re\left(\int_0^\pi e^{\mathrm{i}\lambda\sin(t)}\,\mathrm{d}t\right),$$

and, obviously, it can be regarded as the contour integral

$$I(\lambda) = \frac{1}{\pi}\int_\chi e^{\mathrm{i}\lambda\sin(z)}\,\mathrm{d}z,$$

where χ is the straight line contour connecting the points $z = 0$ and $z = \pi$ in the complex plane.

Example (Cont.)

We need to understand all we can about the analytic function

$$\rho(z) = \mathrm{i}\sin(z)$$

and the contours of its real and imaginary parts: $\phi(z) = \Re(\mathrm{i}\sin(z))$ and $\psi(z) = \Im(\mathrm{i}\sin(z))$. To identify ϕ and ψ, observe that

$$\begin{aligned}\sin(x+\mathrm{i}y) &= \sin(x)\cos(\mathrm{i}y) + \cos(x)\sin(\mathrm{i}y)\\ &= \sin(x)\cosh(y) + \mathrm{i}\cos(x)\sinh(y).\end{aligned}$$

Thus,

$$\phi(x,y) = -\cos(x)\sinh(y) \quad \text{and} \quad \psi(x,y) = \sin(x)\cosh(y).$$

We will continue our abuse of notation, writing

$$\phi(z) = \phi(x,y) \quad \text{and} \quad \psi(z) = \psi(x,y),$$

when $z = x + \mathrm{i}y$. The level curves of ϕ and ψ are plotted in the figure on the next slide.

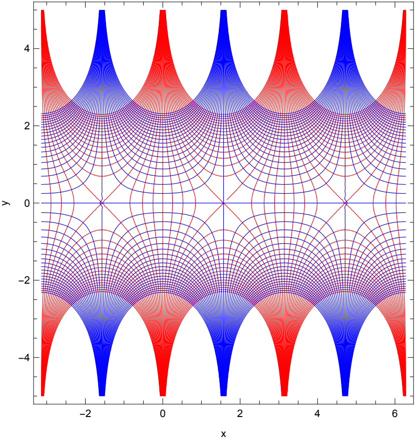

Figure: Level curves of $\phi(z) = \Re(\mathrm{i}\sin(z))$ (blue) and $\psi(z) = \Im(\mathrm{i}\sin(z))$ (red).

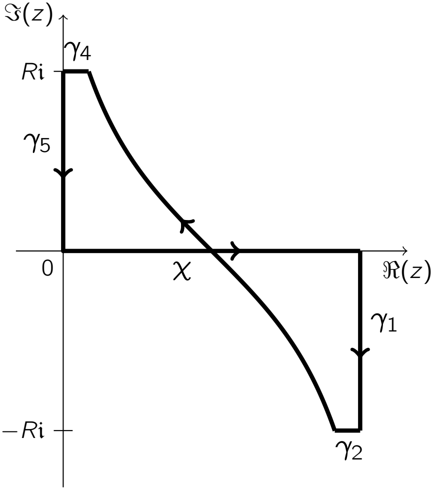

Figure: The piecewise smooth, non-simple, closed contour γ.

Example (Cont.)

Let us consider the following contour segments depicted in the figure:

1. γ_1 connects the point $z = \pi$ to the point $z = \pi - \mathrm{i}R$, where $R > 0$, in a straight line. $\psi(x, y) = 0$ along γ_1, and $\phi(x, y)$ is decreasing along γ_1.
2. The bridge contour segment γ_2 connects the point $z = \pi - \mathrm{i}R$ to the point $z = x_{R,\pi} - \mathrm{i}R$, where

$$\psi(x_{R,0}, R) = 1, \quad 0 < x_{R,0} < \frac{\pi}{2}, \quad \psi(x_{R,\pi}, -R) = 1, \quad \frac{\pi}{2} < x_{R,\pi} < \pi.$$

3. γ_3 connects $z = x_{R,\pi} - \mathrm{i}R$ to $z = x_{R,0} + \mathrm{i}R$, along the contour $\psi(x, y) = 1$ that goes through the point $z = \frac{\pi}{2}$. Along this contour, $\phi(x, y)$ is at first increasing, until it reaches the saddle point $z = \frac{\pi}{2}$, that is $x = \frac{\pi}{2}$, $y = 0$. After that, $\phi(x, y)$ is decreasing along γ_3. In other words, the saddle point $z = \frac{\pi}{2}$ is a global maximum for ϕ along γ_3.
4. The bridge contour segment γ_4 connects the point $z = x_{R,0} + \mathrm{i}R$ to the point $z = \mathrm{i}R$ in a straight line.
5. Finally, γ_5 connects the point $z = \mathrm{i}R$ to the origin, $z = 0$, in a straight line. $\psi(x, y) = 0$ along γ_5, and $\phi(x, y)$ is increasing along γ_1.

Example (Cont.)

Thus, $\gamma = \chi + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5$ is a piecewise smooth, non-simple, closed contour. The contour crosses itself at exactly one point, the saddle point $z = \frac{\pi}{2}$. The integrand of (6) is analytic on any bounded, simply-connected set containing γ. By Cauchy's Theorem,

$$\frac{1}{\pi}\int_{\gamma} e^{\lambda i \sin(z)}\, \mathrm{d}z = 0.$$

Example (Cont.)

Let us attempt to get a power series expansion for γ_3, since this may be instructive in our analysis. We start with the relation

$$\psi(x, y) = \sin(x)\cosh(y) = 1,$$

which does not uniquely define the contour we seek. In fact, there are an infinite number of curves that satisfy this expression. There are two such curves in the domain $0 \leq x \leq \pi$, both of which pass through the saddle point $z = \frac{\pi}{2}$. The relation above is equivalent to

$$\operatorname{sech}(y) = \sin\left(t + \frac{\pi}{2}\right), \quad t = x - \frac{\pi}{2}.$$

Example (Cont.)

Observe that

$$\operatorname{sech}(y) = \sum_{k=0}^{\infty} \frac{E_{2k} y^{2k}}{(2k)!}, \quad |y| \leq \frac{\pi}{2},$$

where E_k is the k^{th} Euler number, the first few being

$$E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61,\ E_8 = 1385,\ E_{10} = -50521,\ \ldots,$$

and

$$\sin\left(t + \frac{\pi}{2}\right) = \sum_{k=0}^{\infty} (-1) \frac{t^{2k}}{(2k)!}, \quad -\infty < t < \infty.$$

Thus,

$$1 - \frac{1}{2} y^2 + \frac{5}{24} y^4 - \frac{61}{720} y^6 + \cdots = 1 - \frac{1}{2} t^2 + \frac{1}{4} t^4 - \frac{1}{24} t^6 + \cdots .$$

Example (Cont.)

For small values of y and t, to leading order,

$$y^2 = t^2 + O(t^4), \quad \text{as} \quad t \to 0.$$

which has two solutions. To traverse the saddle point in the correct direction, we need to take

$$y = -t + O(t^2), \quad \text{as} \quad t \to 0.$$

Let us develop a power series expansion for the variable y in terms of t. To do this, we guess that the correct branch has the expansion

$$y_\star = -t + c_3 t^3 + c_5 t^5 + c_7 t^7 + c_9 t^9 + c_{11} t^{11} + \cdots ,$$

and subsequently determine the constants c_j, $j = 3, 5, 7, \ldots$ from the relation

$$y^2 - \frac{5}{12} y^4 + \frac{61}{360} y^6 + \cdots = t^2 - \frac{1}{2} t^4 + \frac{1}{12} t^6 + \cdots .$$

Example (Cont.)

Thus, we obtain

$$
\begin{aligned}
-\frac{1}{3} - 2c_3 = 0 &\implies c_3 = -\frac{1}{6},\\
\frac{1}{6} + \frac{5c_3}{3} + c_3^2 - 2c_5 = 0 &\implies c_5 = -\frac{1}{24},\\
-\frac{5}{12}\left(6{c_3}^2 - 4c_5\right) + 2c_3c_5 - \frac{61c_3}{60} - 2c_7 - \frac{173}{2520} = 0 &\implies c_7 = -\frac{61}{5040}\\
\cdots = 0 &\implies c_9 = -\frac{277}{72576},\\
&\vdots
\end{aligned}
$$

Hence, as $t \to 0$,

$$
y_\star(t) = -t - \frac{1}{6}t^3 - \frac{1}{24}t^5 - \frac{61}{5040}t^7 - \frac{277}{72576}t^9 + O(t^{11}),
$$

or

$$
y_\star(t) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{E_{2k}}{(2k-1)!} t^{2k-1}.
$$

Example (Cont.)

Integrals over γ_1 and γ_5: These integrals turn out to be purely imaginary, and their values are finite in the limit as $R \to \infty$. The details are left to the reader.

Integrals over the bridge contours γ_2 and γ_4: These integrals go to zero in the limit as $R \to \infty$. The details are again left to the reader.

Example (Cont.)

Integral over γ_3: On γ_3, recall that $\psi(x,y)=1$ and $\phi(x,y)$ is at first increasing, until it reaches the saddle point $z=\frac{\pi}{2}$, where it has a value of zero:

$$\phi\left(\frac{\pi}{2},0\right)=0.$$

After that, $\phi(x,y)$ is decreasing along γ_3. Let us take the limit as $R\to\infty$, a priori, as this will simplify our calculations. Suppose that a parameterization of $\gamma_3:[-a,a]\to\mathbb{C}$ is given so that $\gamma_3(0)=\frac{\pi}{2}$. Then

$$\begin{aligned}
I_3(\lambda) &= \frac{1}{\pi}\int_{\gamma_3} e^{\mathrm{i}\lambda\sin(z)}\,\mathrm{d}z,\\
&= \frac{1}{\pi}\int_{\gamma_3} e^{\lambda\phi(z)}e^{\lambda\mathrm{i}\psi(z)}\,\mathrm{d}z\\
&= \frac{e^{\mathrm{i}\lambda}}{\pi}\int_{-a}^{a} e^{\lambda\phi(\gamma_3(t))}\gamma_3'(t)\,\mathrm{d}t\\
&= 2\frac{e^{\mathrm{i}\lambda}}{\pi}\int_{0}^{a} e^{\lambda\phi(\gamma_3(t))}\gamma_3'(t)\,\mathrm{d}t,
\end{aligned}$$

where we used symmetry in the last step.

Example (Cont.)

Now, let us make a change of variable. Observe that

$$\rho(\gamma_3(t)) = \mathrm{i}\sin(\gamma_3(t)) = \phi(\gamma_3(t)) + \mathrm{i},$$

where we have used the fact that $\psi(\gamma_3(t)) = 1$. Now, set

$$q(t) := -\phi(\gamma_3(t)) = \mathrm{i}\left(1 - \sin(\gamma_3(t))\right).$$

ϕ is non-positive real on γ_3. Therefore, q is non-negative real on γ_3. Furthermore,

$$\frac{\mathrm{d}q}{\mathrm{d}t}(t) = -\mathrm{i}\cos(\gamma_3(t))\gamma_3'(t),$$

and

$$\frac{-\mathrm{d}q}{\mathrm{i}\cos(\gamma_3(t))} = \gamma_3'(t)\,\mathrm{d}t.$$

For the limits,

$$q(0) := -\phi(\gamma_3(0)) = 0 \quad \text{and} \quad q(a) := -\phi(\gamma_3(a)) = \infty.$$

Example (Cont.)

Thus, we have

$$I_3(\lambda) = -2\frac{e^{i\lambda}}{\pi}\int_{q=0}^{q=\infty} e^{-\lambda q}\frac{dq}{i\cos(\gamma_3(t))}.$$

We see that

$$\cos^2(\gamma_3(t)) = 1 - \sin^2(\gamma_3(t))$$

and

$$\sin^2(\gamma_3(t)) = (1 + iq(t))^2 = 1 + 2iq(t) - q^2(t).$$

Therefore,

$$\cos^2(\gamma_3(t)) = q^2(t) - 2iq(t).$$

We need to add a minus sign when taking the square root:

$$\begin{aligned}\cos(\gamma_3(t)) &= -\sqrt{q^2(t) - 2iq(t)} \\ &= -\sqrt{-2iq(t)}\sqrt{1 - \frac{q(t)}{2i}} \\ &= -i\sqrt{2}e^{i\frac{\pi}{4}}\sqrt{q(t)}\sqrt{1 + i\frac{q(t)}{2}}.\end{aligned}$$

Example (Cont.)

It follows that

$$I_3(\lambda) = -2\frac{e^{\mathrm{i}\lambda}}{\pi}\int_0^\infty \frac{e^{-\lambda q}}{-\mathrm{i}^2\sqrt{2}e^{\mathrm{i}\frac{\pi}{4}}\sqrt{q}\sqrt{1+\mathrm{i}\frac{q}{2}}}\,\mathrm{d}q$$
$$= -\sqrt{2}\frac{e^{\mathrm{i}\left(\lambda-\frac{\pi}{4}\right)}}{\pi}\int_0^\infty \frac{e^{-\lambda q}}{\sqrt{q}\sqrt{1+\mathrm{i}\frac{q}{2}}}\,\mathrm{d}q,$$

an integral in the real variable q. We can estimate this using Watson's Lemma. We again make use of the power series expansion

$$\frac{1}{\sqrt{1+\mathrm{i}z}} = \sum_{k=0}^\infty z^k \mathrm{i}^k \binom{-\frac{1}{2}}{k} = 1 - \frac{\mathrm{i}z}{2} - \frac{3z^2}{8} + \frac{5\mathrm{i}z^3}{16} + \frac{35z^4}{128} - \cdots .$$

which is valid for $|z| < 1$. Therefore, the following is valid for $|q| \le \frac{1}{2}$:

$$\frac{1}{\sqrt{1+\mathrm{i}\frac{q}{2}}} = \sum_{k=0}^\infty \frac{q^k \mathrm{i}^k}{2^k}\binom{-\frac{1}{2}}{k}.$$

Example (Cont.)

Thus, as $\lambda \to \infty$,

$$I_3(\lambda) = -\sqrt{2}\frac{e^{i\left(\lambda - \frac{\pi}{4}\right)}}{\pi}\int_0^\infty \frac{e^{-\lambda q}}{\sqrt{q}\sqrt{1+i\frac{q}{2}}}\,dq`$$

$$\sim -\sqrt{2}\frac{e^{i\left(\lambda - \frac{\pi}{4}\right)}}{\pi}\sum_{k=0}^{\infty}\frac{i^k}{2^k}\binom{-\frac{1}{2}}{k}\int_0^\infty \frac{e^{-\lambda q}}{\sqrt{q}}q^k\,dq`$$

$$= -\sqrt{2}\frac{e^{i\left(\lambda - \frac{\pi}{4}\right)}}{\pi}\sum_{k=0}^{\infty}\frac{i^k}{2^k}\binom{-\frac{1}{2}}{k}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\lambda^{k+1/2}}.$$

To finish up, when we take $R \to \infty$,

$$I(\lambda) = \frac{1}{\pi}\int_0^\pi e^{i\lambda\sin(t)}\,dt = -I_1(\lambda) - I_5(\lambda) - I_3(\lambda),$$

where $I_1(\lambda)$ and $I_5(\lambda)$ are purely imaginary.

Example (Cont.)

Consequently,

$$\begin{aligned} J_0(\lambda) &= \Re\left(I(\lambda)\right) \\ &\sim \Re\left(\sqrt{2}\frac{e^{\mathrm{i}\left(\lambda-\frac{\pi}{4}\right)}}{\pi}\sum_{k=0}^{\infty}\frac{\mathrm{i}^k}{2^k}\binom{-\frac{1}{2}}{k}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\lambda^{k+1/2}}\right), \quad \text{as} \quad \lambda \to \infty. \end{aligned}$$

Let us compute the first few terms of the expansion.

$$\sum_{k=0}^{\infty}\frac{\mathrm{i}^k}{2^k}\binom{-\frac{1}{2}}{k}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\lambda^{k+1/2}} = \frac{\sqrt{\pi}}{\lambda^{1/2}} - \mathrm{i}\frac{\sqrt{\pi}}{8\lambda^{3/2}} - \frac{9\sqrt{\pi}}{128\lambda^{5/2}} + \mathrm{i}\frac{75\sqrt{\pi}}{1024\lambda^{7/2}} + \cdots,$$

Thus,

$$J_0(\lambda) \sim \sqrt{\frac{2}{\pi\lambda}}\cos\left(\lambda-\frac{\pi}{4}\right) + \sqrt{\frac{2}{\pi\lambda}}\frac{1}{8\lambda}\sin\left(\lambda-\frac{\pi}{4}\right), \quad \text{as} \quad \lambda \to \infty.$$

Example

Let us reconsider the last example. But, we will compute the saddle point contribution with a clever trick. The idea is to use an approximation of the contour γ_3. We know, by our expansion work, that to leading order, the contour going through the saddle point, γ_3, may be approximated as

$$\tilde{\gamma}_3(t) = \frac{\pi}{2} + t - \mathrm{i}t,$$

from $t = \varepsilon$ to $t = -\varepsilon$, where $\varepsilon > 0$ is small. In other words, $\tilde{\gamma}_3$ is a line of slope -1, going through the saddle point $z = \frac{\pi}{2}$. Only this contribution nearest to the saddle point is expected to be important for the value of the γ_3 contour integral, using our standard arguments in Laplace's method. Using the approximate steepest descent contour

$$\tilde{\gamma}_3'(t) = 1 - \mathrm{i}.$$

Example (Cont.)

As $\lambda \to \infty$,

$$
\begin{aligned}
I_3(\lambda) &\sim \frac{1}{\pi}\int_{\varepsilon}^{-\varepsilon} e^{i\lambda \sin(\tilde{\gamma}_3(t))}\tilde{\gamma}_3'(t)\,dt \\
&= \frac{1}{\pi}\int_{\varepsilon}^{-\varepsilon} e^{i\lambda \cos(t-it)}(1-i)\,dt \\
&= -2\frac{(1-i)}{\pi}\int_0^{\varepsilon} e^{i\lambda\left(1+it^2-\frac{1}{6}t^4+\cdots\right)}\,dt \\
&\sim -2\sqrt{2}\frac{e^{i\left(\lambda-\frac{\pi}{4}\right)}}{\pi}\int_0^{\infty} e^{-\lambda t^2}e^{-i\lambda\frac{1}{6}t^4}\,dt \\
&\sim -2\sqrt{2}\frac{e^{i\left(\lambda-\frac{\pi}{4}\right)}}{\pi}\int_0^{\infty} e^{-\lambda t^2}\left(1-i\lambda\frac{1}{6}t^4+\cdots\right)dt \\
&\sim -2\sqrt{2}\frac{e^{i\left(\lambda-\frac{\pi}{4}\right)}}{\pi}\left(\frac{\sqrt{\pi}}{2\sqrt{\lambda}}-i\lambda\frac{1}{6}\cdot\frac{3\sqrt{\pi}}{8\lambda^{5/2}}\right) \\
&= -\sqrt{\frac{2}{\pi\lambda}}e^{i\left(\lambda-\frac{\pi}{4}\right)}\left(1-\frac{i}{8\lambda}\right).
\end{aligned}
$$

Example (Cont.)

Finally, as $\lambda \to \infty$,

$$J_0(\lambda) \sim \Re\left(\sqrt{\frac{2}{\pi\lambda}} e^{i\left(\lambda - \frac{\pi}{4}\right)}\left(1 - \frac{i}{8\lambda}\right)\right)$$
$$= \sqrt{\frac{2}{\pi\lambda}}\left(\cos\left(\lambda - \frac{\pi}{4}\right) + \frac{1}{8\lambda}\sin\left(\lambda - \frac{\pi}{4}\right)\right).$$