Math 515
Essential Perturbation Theory and Asymptotic Analysis
Chapter 01

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Chapter 01, Part 1 of 2
Motivating Themes

Stirling's Approximation

The Factorial Function and its Natural Logarithm, $L(n)$

Arguably, no approximation is as important to physics and, specifically, statistical mechanics, as Stirling's approximation of the factorial function. Often attributed to James Stirling, the approximation was actually first given by Abraham De Moivre in 1730 and later (in the same year) refined by Stirling.

First, recall that, for any $n \in \mathbb{N}$,

$$n! := 1 \times 2 \times 3 \times \cdots \times (n-1) \times n.$$

Therefore,

$$L(n) := \log(n!) = \sum_{k=1}^{n} \log(k) = \sum_{k=2}^{n} \log(k).$$

In this book we will exhibit at least three versions of Stirling's famous approximation, which seeks to estimate $n!$ or, equivalently, $\log(n!)$, for large n.

A Related Integral, $I(n)$

This first one will be the most elementary. It relies upon the observation that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x\log(x) - x\right] = \log(x).$$

In fact, the function $f(x) = x\log(x) - x$ appears again and again in statistical physics and is called the *ideal gas model*. In any case, for any $n \in \mathbb{N}$, or, more generally, for any $n \in \mathbb{R}$,

$$I(n) := \int_1^n \log(x)\,\mathrm{d}x = \left[x\log(x) - x\right]\Big|_1^n = n\log(n) - n + 1.$$

Our goal is to show that, in some sense to be made precise later in the book,

$$L(n) \approx I(n), \quad \text{for large } n \in \mathbb{N}.$$

Upper and Lower Riemann Sum Approximations to $I(n)$

To establish this approximation, let us consider two Riemann sums with uniform partition spacing $h = 1$,

$$R_U(n) := \sum_{k=2}^{n} \log(k) = L(n),$$

and

$$R_L(n) := \sum_{k=1}^{n-1} \log(k) = \sum_{k=2}^{n-1} \log(k) = L(n) - \log(n).$$

The function $\log(x)$ is an increasing function, and $R_U(n)$ can be interpreted as the Riemann sum over $[1, n]$ using the right hand endpoint of each sub-interval. Similarly, $R_L(n)$ can be interpreted as the Riemann sum over $[1, n]$ using the left hand endpoint of each sub-interval. It follows that

$$R_L(n) \le I(n) \le R_U(n).$$

Upper and Lower Riemann Sum Approximations to $I(n)$

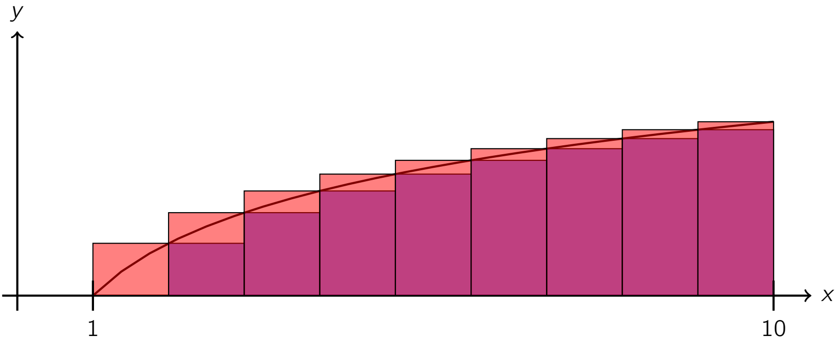

Figure: Upper ($R_U(n)$, red) and lower ($R_L(n)$, blue) Riemann sum approximations, with uniform partition spacing $h = 1$, to the definite integral $I(n) = \int_1^n \log(x)\,dx$, where $n = 10$. Clearly, $R_L(n) \leq I(n) \leq R_U(n)$, for any $n \in \mathbb{N}$.

But The Error is Not Getting Smaller

Recall,

$$R_L(n) \leq I(n) \leq R_U(n).$$

This implies

$$L(n) - \log(n) \leq I(n) \leq L(n).$$

Equivalently, since $n \in \mathbb{N}$ is arbitrary,

$$I(n) \leq L(n) \leq I(n) + \log(n),$$

which implies that

$$0 \leq L(n) - I(n) \leq \log(n), \quad \forall\, n \in \mathbb{N}.$$

Unfortunately, $\log(n) \nearrow +\infty$, as $n \to \infty$. We can not apply the squeeze theorem to conclude that

$$L(n) - I(n) \to 0, \quad \text{as} \quad n \to \infty.$$

Relative Error to the Rescue

The error in the approximation $L(n) \approx I(n)$ does not diminish as $n \to \infty$. But that is not the only measure of error that may be useful. Instead of the considering the **exact error**, let us evaluate this approximation in the sense of the **relative error**, or, more precisely, a kind of relative error. In particular, observe that

$$\lim_{n\to\infty} \frac{L(n) - I(n)}{I(n)} = 0. \tag{1}$$

To see this, note that

$$0 \leq \frac{L(n) - I(n)}{I(n)} \leq \frac{\log(n)}{I(n)}, \quad \forall\, n \in \mathbb{N},$$

and

$$\frac{\log(n)}{I(n)} \to 0, \quad \text{as} \quad n \to \infty.$$

Stirling's Approximation as an Asymptotic Approximation

Using notation that will be made precise later, we will write

$$L(n) \sim I(n), \quad \text{as} \quad n \to \infty$$

as a way of expressing the approximation (1) in symbols. In other words, we write Stirling's approximation as

$$\log(n!) \sim n \log(n) - n + 1, \quad \text{as} \quad n \to \infty. \tag{2}$$

Computed Error in Stirling's Approximation

n	$L(n) - I(n)$	$\frac{L(n)-I(n)}{I(n)}$
10	1.078561643135055	0.076898125363124
30	1.622314898965499	0.022212561528085
50	1.876616680365714	0.012800831895872
70	2.044376622321806	0.008951069872520
90	2.169769290485476	0.006866730744680
110	2.269936289771522	0.005562848890973
130	2.353346782809183	0.004671382783474
150	2.424811734985383	0.004023947346865
170	2.487327947242761	0.003532706069838

Table: The error in Stirling's approximation (2) for some values of n.

Expansion of an Integral with a Large Parameter

An Integral with a Large Parameter

Suppose that $x \in \mathbb{R}$ is a large, positive parameter, that is, it is much larger than 1. Define

$$I(x) = \int_0^\infty \frac{e^{-xt}}{1+t}\,\mathrm{d}t. \tag{3}$$

There is no known analytic solution for this integral in terms of simple algebraic or transcendental functions. Instead of seeking a numerical approximation, let us construct a series expansion for the solution. First, try writing

$$\frac{1}{1+t} = \sum_{k=0}^{\infty}(-1)^k t^k = 1 - t + t^2 - t^3 + t^4 - \cdots \ .$$

This geometric/binomial series expansion converges for $-1 < t < 1$. The limited radius of convergence is problematical for us, since we plan to integrate over the non-negative real numbers, i.e., t satisfying $0 \le t < \infty$.

An Exact Binomial Series with Remainder

Instead of the infinite series, let us use the following exact expression.

Proposition

For all $t \geq 0$, the following expression holds:

$$\frac{1}{1+t} = \sum_{k=0}^{n-1} (-1)^k t^k + (-1)^n \frac{t^n}{1+t}. \tag{4}$$

Proof.

The proof is an exercise. $\square$

A Finite Series Approximation with a Remainder Term

Making use of (4), we have

$$I(x) = \int_0^\infty e^{-xt} \left(\sum_{k=0}^{n-1} (-t)^k + (-1)^n \frac{t^n}{1+t} \right) \mathrm{d}t = S_{n-1}(x) + R_n(x), \tag{5}$$

where

$$S_{n-1}(x) := \sum_{k=0}^{n-1} a_k(x), \quad a_k(x) := (-1)^k \frac{k!}{x^{k+1}}, \tag{6}$$

and

$$R_n(x) := \int_0^\infty (-1)^n \frac{t^n}{1+t} e^{-xt} \, \mathrm{d}t. \tag{7}$$

In the computation of the terms of the partial sum, $S_{n-1}(x)$, we have used the identity

$$\int_0^\infty t^k e^{-xt} \, \mathrm{d}t = \frac{k!}{x^{k+1}}, \quad \forall\, k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \tag{8}$$

which can be obtained by repeated integration by parts.

Trading a Hard Problem for a Really Hard Problem

Our manipulations so far are exact; we have made no approximations yet. But, the calculation of the reminder term,

$$R_n(x) := \int_0^\infty (-1)^n \frac{t^n}{1+t} e^{-xt}\, dt.$$

is still a challenge, and it seems that all we have done is kick the can down the street a bit.

In fact, the computation of the remainder is a harder problem than the one with which we started. If calculating the remainder exactly were easy, this would not be such an interesting problem.

Estimating the Remainder

The remainder may be more complicated than our starting problem, but it has a structure that will be useful for us. To see this, let us now estimate the size of this term:

$$\begin{aligned}|R_n(x)| &= \int_0^\infty \frac{t^n}{1+t} e^{-xt}\,\mathrm{d}t \\ &\le \int_0^\infty t^n e^{-xt}\,\mathrm{d}t \\ &= \frac{n!}{x^{n+1}}, \\ &= |a_n(x)|,\end{aligned}$$

where we have used the fact that

$$\frac{1}{1+t} \le 1, \quad \forall\, t > 0,$$

and the identity (8) again. Thus, for each fixed $x > 0$,

$$|R_n(x)| \le |a_n(x)|. \tag{9}$$

Understanding the Remainder

This is interesting. It indicates that, for fixed $x > 0$, the remainder is not larger in magnitude than the modulus of the next term in the sequence of partial sums. In other words,

$$\begin{aligned} |I(x) - S_{n-1}(x)| &= \left| \int_0^\infty \frac{e^{-xt}}{1+t}\,\mathrm{d}t - \sum_{k=0}^{n-1} (-1)^k \frac{k!}{x^{k+1}} \right| \\ &= |R_n(x)| \\ &\le |a_n(x)| \\ &= \frac{n!}{x^{n+1}}. \end{aligned}$$

In principle, we know how good we can make our approximation. For fixed values of $x > 0$, we can use $S_{n-1}(x)$ as an approximation of $I(x)$, stopping for that value of n such that $a_n(x)$ becomes sufficiently small in magnitude. For large values of x, sometimes written as $x \gg 1$, it is conceivable that $a_n(x)$ might be very small for $n = 1$, 2, or 3.

We Constructed a Divergent Series

But, we do have to be careful, because the sequence of partial sums $\{S_n(x)\}$ does not converge as $n \to \infty$, for any value of $x > 0$. In other words, series

$$S(x) = \lim_{n\to\infty} S_n(x) = \sum_{k=0}^{\infty} (-1)\frac{k!}{x^{k+1}} = \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots$$

diverges! We can use the ratio test to show that the series diverges, for any $x > 0$. Since

$$|a_n(x)| = \frac{n!}{x^{n+1}},$$

we find

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{x}.$$

For any fixed $x > 0$, letting $n \to \infty$, we see that

$$\left|\frac{a_{n+1}}{a_n}\right| \to \infty.$$

Is that bad?

Maybe It's Not All Bad

We are conditioned to believe that all divergent series are useless and to be avoided. But, our partial sum approximations are still useful, owing to the estimate (9). We just need a bit of careful consideration. Indeed, let us suppose that $x = 10$. The "true" value of $I(10)$ can be found using Mathematica:

$$I(10) = 0.0915633339397881\ldots .$$

Using four terms of the expansion we derived above, we get

$$S_3(10) = 0.1000 - 0.0100 + 0.0020 - 0.0006 = 0.0914,$$

which is a good approximation.

Sizing up the Error

So

$$I(10) = 0.0915633339397881\ldots \quad \text{and} \quad S_3(10) = 0.0914.$$

Considering the size of the next term, the fifth term, in our sequence of partial sums, which is precisely,

$$a_4(10) = \frac{4!}{10^5} = 0.00024,$$

based on our error estimate (9), we know that

$$|I(10) - S_3(10)| \leq 0.00024.$$

Likewise,

$$a_5(10) = -0.000120,$$

and

$$S_4(10) = 0.0914,$$

so that

$$|I(10) - S_4(10)| \leq 0.000120.$$

Making the Error as Small as Possible

But, the party does not last, alas, because $|a_n(10)| \to \infty$, as $n \to \infty$. For each fixed $x > 0$, the best that we can hope for is that there is a minimum $a_n(10)$ in magnitude, which indicates where we should terminate our sequence of partial sums. See Table 2. We observe that $a_n(10)$ is smallest in magnitude at term 9.

Approximations for $x = 10$

n	$a_n(10)$	$S_n(10)$
0	0.1	0.1
1	-0.01	0.09
2	0.002	0.092
3	-0.0006	0.0914
4	0.00024	0.09164
5	-0.000120	0.091520
6	0.0000720	0.0915920
7	-0.00005040	0.09154160
8	0.000040320	0.091581920
9	-0.0000362880	0.0915456320
10	0.00003628800	0.09158192000
11	-0.000039916800	0.091542003200
12	0.0000479001600	0.0915899033600

Table: Partial sums $S_n(10)$ for the approximation of the integral in (3). The terms $a_n(10)$ start to grow in magnitude after the 10^{th} term, and the partial sums get worse after that point.

Approximations for $x = 20$

n	$a_n(20)$	$S_n(20)$
11	-9.745312500000000E-09	4.771854191093750E-02
12	5.847187500000000E-09	4.771854775812500E-02
13	-3.800671875000000E-09	4.771854395745310E-02
14	2.660470312500000E-09	4.771854661792340E-02
15	-1.995352734375000E-09	4.771854462257070E-02
16	1.596282187500000E-09	4.771854621885290E-02
17	-1.356839859375000E-09	4.771854486201300E-02
18	1.221155873437500E-09	4.771854608316890E-02
19	-1.160098079765630E-09	4.771854492307080E-02
20	1.160098079765630E-09	4.771854608316890E-02
21	-1.218102983753910E-09	4.771854486506590E-02
22	1.339913282129300E-09	4.771854620497920E-02
23	-1.540900274448690E-09	4.771854466407890E-02
24	1.849080329338430E-09	4.771854651315930E-02

Table: Partial sums $S_n(20)$ for the approximation of the integral in (3). The terms $a_n(20)$ start to grow in magnitude after the 20^{th} term, and the partial sums get worse after that point.

Alternating Approximations

Another interesting fact. Observe that, since $R_n(x)$ alternates in sign with n, the exact value of $I(x)$ must lie between any two consecutive partial sums. Indeed, suppose that $R_n(x) > 0$, so that $R_{n+1}(x) < 0$. Then,

$$\begin{aligned} I(x) &= S_{n-1}(x) + R_n(x) &\implies&\quad I(x) > S_{n-1}(x), \\ I(x) &= S_n(x) + R_{n+1}(x) &\implies&\quad I(x) < S_n(x). \end{aligned}$$

In other words,

$$S_{n-1}(x) < I(x) < S_n(x).$$

Thus, for our example with $x = 10$,

$$S_9(10) = 0.0915456320 < I(10) < 0.09158192000 = S_{10}(10).$$

Since $I(10) \approx 0.09156$, we see that the approximation after 9 or 10 terms is pretty good.

Asymptotic Approximations: Fix n, Let $x \to \infty$

If we fix n, observe that the approximations get better and better as x gets larger and larger:

$$
\begin{aligned}
|I(x) - S_{n-1}(x)| &= \left| \int_0^\infty \frac{e^{-xt}}{1+t}\,\mathrm{d}t - \sum_{k=0}^{n-1} (-1)^k \frac{k!}{x^{k+1}} \right| \\
&= |R_n(x)| \\
&\le |a_n(x)| \\
&= \frac{n!}{x^{n+1}} \xrightarrow{x\to\infty} 0.
\end{aligned}
$$

Algebraic Equations with a Small Parameter

A Quadratic Equation with a Small Parameter

Now, let us talk a bit about algebraic equations. Consider, in particular, the equation

$$x^2 - 2x + \varepsilon = 0, \tag{10}$$

where $0 < \varepsilon < 1$ is a small parameter, signified by writing $0 < \varepsilon \ll 1$. We will make notions precise later. Of course, this equation has exactly two solutions, counting multiplicities, which can be found using the quadratic formula:

$$x_1(\varepsilon) = 1 - \sqrt{1-\varepsilon} \quad \text{and} \quad x_2(\varepsilon) = 1 + \sqrt{1-\varepsilon}.$$

Of course,

$$x_1(\varepsilon = 0) = 0 \quad \text{and} \quad x_2(\varepsilon = 0) = 2,$$

which we can see by inspection.

Just Use a Calculator

Otherwise, we can plug in values of $\varepsilon > 0$, fire up our handheld calculator, and get very good approximate solutions:

$$x_1(\varepsilon) = 1 - \sqrt{1-\varepsilon} \quad \text{and} \quad x_2(\varepsilon) = 1 + \sqrt{1-\varepsilon}.$$

The square root process on your calculator will probably be computed using Newton's method, or something akin to that. But, suppose, $\varepsilon > 0$ is small, really small. Is there another way to get a good approximate solution?

The Binomial Series, Again

Recall, if you have seen it before, the binomial series:

$$(1+t)^{\alpha} = 1 + \alpha t + \frac{\alpha(\alpha-1)}{2!}t^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}t^3 + \cdots , \tag{11}$$

which converges for all $|t| < 1$, even if $\alpha \in \mathbb{R} \setminus \mathbb{N}$. The reader can use the ratio test to prove this fact quite easily. Of course, if $\alpha \in \mathbb{N}$, the expansion terminates in a finite number of steps, and there is nothing to worry about. For $0 < \varepsilon < 1$ and $\alpha = \frac{1}{2}$, we find

$$\sqrt{1-\varepsilon} = 1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 - \frac{1}{16}\varepsilon^3 + \cdots .$$

What we mean, rigorously, of course, is that the sequence of partial sums converges. In any case,

$$x_1(\varepsilon) = \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \frac{1}{16}\varepsilon^3 + \cdots$$
$$x_2(\varepsilon) = 2 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 - \frac{1}{16}\varepsilon^3 + \cdots .$$

Using our Expansions

So, for example, by the calculator, the "exact solution" is, to 16 digits of accuracy,

$$x_1(0.1) = 0.05131670194948623\ldots.$$

Using three terms of the corresponding expansion,

$$x_1(0.1) \approx 0.0513125,$$

which has four digits of accuracy.

A Perturbation Method

There is another way to proceed as well, one that you might not have seen before, especially if you are new to perturbation theory. Let us suppose that the following expansion is valid:

$$x_i(\varepsilon) = x_{i,0} + \varepsilon x_{i,1} + \varepsilon^2 x_{i,2} + \cdots , \tag{12}$$

where $i = 1, 2$, anticipating two solution expansions. Let us insert the expansion (12) into (10) and equate like powers of ε to obtain

$$\begin{aligned}
\varepsilon^0 &: \quad x_{i,0}^2 - 2x_{i,0} = 0,\\
\varepsilon^1 &: \quad 2x_{i,0}x_{i,1} - 2x_{i,1} + 1 = 0,\\
\varepsilon^2 &: \quad 2x_{i,0}x_{i,2} + x_{i,1}^2 - 2x_{i,2} = 0,\\
\varepsilon^3 &: \quad 2x_{i,0}x_{i,3} + 2x_{i,1}x_{i,2} - 2x_{i,3} = 0.
\end{aligned}$$

The leading-order (ε^0) equation implies that $x_{i,0} = 0, 2$. Let us agree that $x_{1,0} = 0$ and $x_{2,0} = 2$, to remain consistent with the computations above.

Solving for the Unknown Coefficients

Then, solving, in order, we find that

$$x_{1,1} = \frac{1}{2}, \quad x_{1,2} = \frac{1}{8}, \quad x_{1,3} = \frac{1}{16}, \ldots,$$

in agreement with our prior results. Likewise,

$$x_{2,1} = -\frac{1}{2}, \quad x_{2,2} = -\frac{1}{8}, \quad x_{2,3} = -\frac{1}{16}, \ \ldots,$$

also in agreement. This powerful method, **which does not require any knowledge of the quadratic formula or the binomial series**, is called a perturbation method, or perturbation argument. It will be the basis of much of what we will do in the course.

Open Questions

Of course, there are a few open questions associated to the method. Is the expansion that we introduced above justified? Are our the manipulations we performed justified? Suppose we chose, instead, to use

$$x_i(\varepsilon) = x_{i,0} + \varepsilon^{\frac{1}{2}} x_{i,1} + \varepsilon x_{i,2} + \varepsilon^{\frac{3}{2}} x_{i,3} + \varepsilon^2 x_{i,4} + \cdots , \tag{13}$$

or

$$x_i(\varepsilon) = x_{i,0} + \varepsilon^2 x_{i,1} + \varepsilon^4 x_{i,2} + \varepsilon^6 x_{i,3} + \varepsilon^8 x_{i,4} + \cdots , \tag{14}$$

Would either of these expansions work? Is there anything special about the form in (12)?

A Cubic Equation with a Small Parameter

Similarly, let us consider the cubic equation

$$x^3 - x + \varepsilon = 0, \tag{15}$$

where $\varepsilon > 0$ is a small parameter. The solution of this equation is, of course, a bit more challenging than the preceding quadratic equation. We do not wish to produce a numerical solution, and we will not purse the computation of an exact solution. Instead, we go straightaway to the perturbation method.
Let us assume again that solutions can be expanded as follows:

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots . \tag{16}$$

We expect that there will be three solution expansions, but we neglect to introduce the cumbersome second index i.

Equating Like Terms: Recursive Formulea

Substituting this expansion into the cubic equation and collecting terms of powers of ε, we have the following:

$$\varepsilon^0: \quad x_0^3 - x_0 = 0, \tag{17}$$

$$\varepsilon^1: \quad 3x_0^2 x_1 - x_1 + 1 = 0, \tag{18}$$

$$\varepsilon^2: \quad 3x_0 x_2 + 3x_0 x_1^2 - x_2 = 0, \tag{19}$$

$$\varepsilon^3: \quad 3x_0^2 x_3 + 6x_0 x_1 x_2 + x_1^3 - x_3 = 0. \tag{20}$$

The leading-order equation reveals that

$$x_0 = 0, \pm 1.$$

We will have three separate expansions, which is desirable, since we expect to have three roots, depending upon which of the three starting values we use for x_0.

The $x_0 = 1$ Branch

Let us start with the $x_0 = 1$ expansion branch. Then, the first-order equation gives

$$3x_1 - x_1 + 1 = 0 \quad \Longrightarrow \quad x_1 = -\frac{1}{2};$$

the second order equation gives

$$3x_2 + \frac{3}{4} - x_2 = 0 \quad \Longrightarrow \quad x_2 = -\frac{3}{8};$$

and, finally, the third order equation gives

$$3x_3 + \frac{9}{8} - \frac{1}{8} - x_3 = 0 \quad \Longrightarrow \quad x_3 = -\frac{1}{2}.$$

Thus, for the $x_0 = 1$ expansion branch, we have

$$x_1(\varepsilon) = 1 - \varepsilon\frac{1}{2} - \varepsilon^2\frac{3}{8} - \varepsilon^3\frac{1}{2} + \cdots .$$

The $x_0 = 0$ and $x_0 = -1$ Branches

For the $x_0 = 0$ branch, we find

$$x_2(\varepsilon) = \varepsilon + \varepsilon^3 + \cdots ,$$

and, for the $x_0 = -1$ branch, we have

$$x_3(\varepsilon) = -1 - \varepsilon\frac{1}{2} - \varepsilon^2\frac{3}{16} + \varepsilon^3\frac{11}{32} + \cdots .$$

We have stopped at the cubic term. But it is clear that we can go on until our hearts are content.

The beauty of the perturbation method is that each update equation is linear beyond the leading-order equation, and this process could be iterated indefinitely, obtaining expansions to any order in ε.

Regular Versus Singular Perturbation Problems

The last two problems encountered above are called *regular perturbation problems*, since the limiting case, $\varepsilon = 0$, results in an equation of the same polynomial order as the original problem. When the limiting case results in a problem of reduced polynomial order, we call the problem a *singular perturbation problem*, and a different strategy is needed.

A Singularly Perturbed Cubic Problem

Now, consider the equation

$$\varepsilon x^3 - x + 1 = 0, \tag{21}$$

where $\varepsilon > 0$ is a small parameter. If, as before, we assume that

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots \tag{22}$$

and we substitute the expansion into the equation, we get the following system of equations:

$$\varepsilon^0 : \quad -x_0 + 1 = 0, \tag{23}$$

$$\varepsilon^1 : \quad x_0^3 - x_1 = 0, \tag{24}$$

$$\varepsilon^2 : \quad 3x_0^2 x_1 - x_2 = 0, \tag{25}$$

and so on. In this case, we see that we will only obtain one solution rather than the desired three. What is happening?

Escape to Infinity

Well, the total polynomial order degenerates to one, when $\varepsilon = 0$, and the degenerate problem has only one solution. What is happening is this: as $\varepsilon \searrow 0$, two of the solutions are escaping to infinity, and this is a difficulty for our regular perturbation method. We simply cannot capture this process of escape.

Dominant Balance

To remedy this situation, define a new variable

$$y := \varepsilon^{\alpha} x,$$

where $\alpha > 0$ will be determined. Substituting this in the equation, we have

$$\frac{\varepsilon}{\varepsilon^{3\alpha}} y^3 - \frac{1}{\varepsilon^{\alpha}} y + 1 = 0. \tag{26}$$

To produce a nontrivial solution for the transformed problem, we need to invoke the **Principle of Dominant Balance**:

1. The coefficients of the first term and one other term should balance in terms of their powers ε, as $\varepsilon \to 0$.
2. The coefficients of these two terms should dominate the coefficients of all other terms in size, as $\varepsilon \to 0$.

Dominant Balance

$$\varepsilon^{1-3\alpha}y^3 - \varepsilon^{-\alpha}y + \varepsilon^0 = 0.$$

Here are our balance options:

1. The first and second terms balance in their powers of ε. This implies that $\alpha = \frac{1}{2}$. Assuming that $y = O(1)$, with the notation to be made precise below, then the first and second terms are $O\left(\varepsilon^{-\frac{1}{2}}\right)$, and the last term is $O(1)$. This scenario satisfies dominant balance, because $\varepsilon^{-\frac{1}{2}}$ becomes much larger than 1, as $\varepsilon \searrow 0$.
2. The first and third terms balance in their powers of ε. This implies that $\varepsilon = \frac{1}{3}$, and the first and third terms are $O(1)$. The second term is $O\left(\varepsilon^{-\frac{1}{3}}\right)$. This does not satisfy dominant balance.
3. Finally, the second and third terms balance in their powers of ε. This implies that $\alpha = 0$, which yields the original problem. Dominant balance is not satisfied.

The Singular Problem Becomes Regular

So, dominant balance suggests that $\alpha = \frac{1}{2}$, and the problem transforms to become

$$\varepsilon^{-\frac{1}{2}} y^3 - \varepsilon^{-\frac{1}{2}} y + 1 = 0, \tag{27}$$

or, equivalently,

$$y^3 - y + \varepsilon^{\frac{1}{2}} = 0. \tag{28}$$

This is, more or less, the regular perturbation problem studied above, though ε is raised to a different power. Now, a new question clearly arises. What expansion should we use for y in terms of powers of ε? Since the last term appears with ε to the power $\frac{1}{2}$, let us try the following:

$$y(\varepsilon) = y_0 + \varepsilon^{\frac{1}{2}} y_1 + \varepsilon y_2 + \varepsilon^{\frac{3}{2}} y_3 + \cdots .$$

Equating Coefficients

Plugging

$$y(\varepsilon) = y_0 + \varepsilon^{\frac{1}{2}} y_1 + \varepsilon y_2 + \varepsilon^{\frac{3}{2}} y_3 + \cdots .$$

into the equation and equating coefficients leads to the following system of equations:

$$\varepsilon^0 : \quad y_0^3 - y_0 = 0, \tag{29}$$

$$\varepsilon^{\frac{1}{2}} : \quad 3y_0^2 y_1 - y_1 + 1 = 0, \tag{30}$$

$$\varepsilon^1 : \quad 3y_0 y_2 + 3y_0 y_1^2 - y_2 = 0, \tag{31}$$

$$\varepsilon^{\frac{3}{2}} : \quad 3y_0^2 y_3 + 6y_0 y_1 y_2 + y_1^3 - y_3 = 0. \tag{32}$$

This is the same system that we derived for the cubic regular perturbation problem. We have three expansion branches:

$$y_1(\varepsilon) = 1 - \varepsilon^{\frac{1}{2}} \frac{1}{2} - \varepsilon \frac{3}{8} - \varepsilon^{\frac{3}{2}} \frac{1}{2} + \cdots .$$

$$y_2(\varepsilon) = \varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{3}{2}} + \cdots ,$$

$$y_3(\varepsilon) = -1 - \varepsilon^{\frac{1}{2}} \frac{1}{2} - \varepsilon \frac{3}{16} + \varepsilon^{\frac{3}{2}} \frac{11}{32} + \cdots .$$

Transforming Back to x

Transforming to x via the relation $y = \varepsilon^{\frac{1}{2}} x$, we have

$$x_1(\varepsilon) = \varepsilon^{-\frac{1}{2}} - \frac{1}{2} - \varepsilon^{\frac{1}{2}}\frac{3}{8} - \varepsilon\frac{1}{2} + \cdots .$$
$$x_2(\varepsilon) = 1 + \varepsilon + \cdots ,$$
$$x_3(\varepsilon) = -\varepsilon^{-\frac{1}{2}} - \frac{1}{2} - \varepsilon^{\frac{1}{2}}\frac{3}{16} + \varepsilon\frac{11}{32} + \cdots .$$

Now, we clearly see how two of the solutions are escaping to infinity as $\varepsilon \searrow 0$.