Math 515
Essential Perturbation Theory and Asymptotic Analysis
Chapter 02

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Chapter 02, Part 2 of 2
Notation and Fundamental Definitions

Nonuniform Asymptotic Expansions

The Prototype Boundary Layer Function

Let us revisit the solution of one of the boundary layer problems from Chapter 1. For us, this will be the prototype boundary layer function. Consider, or any $\varepsilon > 0$,

$$y(x, \varepsilon) := \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}, \quad 0 \leq x \leq 1. \tag{1}$$

In this section, we will work directly with the solution, rather than the boundary value problem that the function satisfies. In other words, we have before us a complete description of the boundary layer function, explicitly and analytically defined.

The Prototype Boundary Layer Function

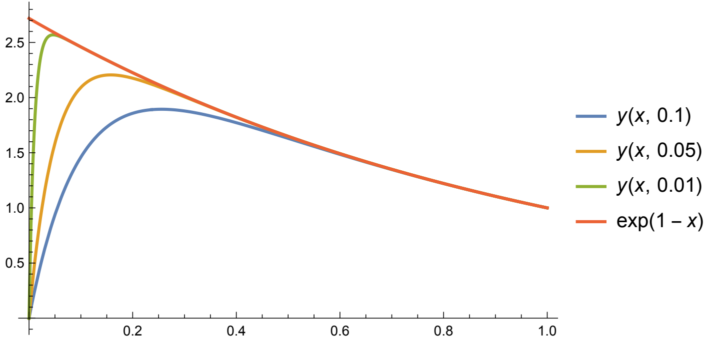

Figure: Plots of the boundary layer function defined in (1), $y(x) = y(x, \varepsilon)$, for the values $\varepsilon = 0.1$ (blue), $\varepsilon = 0.05$ (orange), and $\varepsilon = 0.01$ (green). The red line is the leading-order outer solution, $y_0(x) = e^{1-x}$.

Asymptotic Series Expansion for $0 < x \leq 1$: $y_0(x)$

Suppose that, away from the boundary layer at $x = 0$, specifically, for $0 < x \leq 1$, we have the asymptotic series expansion

$$y(x, \varepsilon) = y_{\mathrm{out}}(x, \varepsilon) \sim y_0(x) + y_1(x)\varepsilon + y_2(x)\varepsilon^2 + \cdots, \quad \text{as} \quad \varepsilon \searrow 0. \tag{2}$$

We can compute the coefficients $y_k(x)$ of this outer expansion using a previous proposition, keeping in mind that $\phi_k = \varepsilon^k$. First,

$$\begin{aligned} y_0(x) &= \lim_{\varepsilon \searrow 0} \frac{y(x, \varepsilon)}{\phi_0(\varepsilon)} \\ &= \lim_{\varepsilon \searrow 0} \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}} \\ &= e^{1-x}. \end{aligned}$$

Asymptotic Series Expansion for $0 < x \leq 1$: $y_1(x)$

Similarly,

$$\begin{aligned} y_1(x) &= \lim_{\varepsilon \searrow 0} \frac{y(x,\varepsilon) - y_0(x)\phi_0(\varepsilon)}{\phi_1(\varepsilon)} \\ &= \lim_{\varepsilon \searrow 0} \frac{\frac{e^{-x}-e^{-x/\varepsilon}}{e^{-1}-e^{-1/\varepsilon}} - e^{1-x}}{\varepsilon} \\ &= \frac{0}{0}, \end{aligned}$$

an indeterminant form. Employing, l'Hôpital's rule, for any $0 < x \leq 1$,

$$\begin{aligned} y_1(x) &= \lim_{\varepsilon \searrow 0} \frac{\frac{\partial}{\partial \varepsilon} \frac{e^{-x}-e^{-x/\varepsilon}}{e^{-1}-e^{-1/\varepsilon}} - \frac{\partial}{\partial \varepsilon} e^{1-x}}{\frac{\partial}{\partial \varepsilon}\varepsilon} \\ &= \lim_{\varepsilon \searrow 0} \frac{\partial}{\partial \varepsilon} \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}} \\ &= 0. \end{aligned}$$

The reader should confirm the last calculation.

Asymptotic Series Expansion for $0 < x \leq 1$: $y_k(x)$, $k \geq 2$

Continuing, we have

$$\begin{aligned} y_2(x) &= \lim_{\varepsilon \searrow 0} \frac{y(x,\varepsilon) - y_0(x)\phi_0(\varepsilon) - y_1(x)\phi_1(\varepsilon)}{\phi_2(\varepsilon)} \\ &= \lim_{\varepsilon \searrow 0} \frac{\frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}} - e^{1-x}}{\varepsilon^2} \\ &= 0. \end{aligned}$$

In fact, using induction, we can prove that, for any $0 < x \leq 1$,

$$y_k(x) = 0, \quad \forall\, k \in \mathbb{N}.$$

Thus,

$$y_{\mathrm{out}}(x,\varepsilon) \sim e^{1-x} + 0\,\varepsilon + 0\,\varepsilon^2 + 0\,\varepsilon^3 + \cdots, \quad \text{as} \quad \varepsilon \searrow 0.$$

Asymptotic Series Expansion for $0 < x \leq 1$

Note that

$$y_{\mathrm{out}}(x,\varepsilon) \sim e^{1-x} + 0\,\varepsilon + 0\,\varepsilon^2 + 0\,\varepsilon^3 + \cdots, \quad \text{as} \quad \varepsilon \searrow 0.$$

can be expressed more succinctly as

$$y_{\mathrm{out}}(x,\varepsilon) - e^{1-x} = o\left(\varepsilon^k\right), \quad \text{as} \quad \varepsilon \searrow 0, \quad \forall\, k \in \mathbb{N},$$

or even more succinctly as

$$y_{\mathrm{out}}(x,\varepsilon) - e^{1-x} = \mathrm{TST}, \quad \text{as} \quad \varepsilon \searrow 0.$$

Proposition

Suppose that, for each fixed $\varepsilon > 0$ $y(\cdot, \varepsilon) : [0, 1] \to \mathbb{R}$ *is defined as in* (1). *Then, for* $0 < x \leq 1$,

$$y(x, \varepsilon) \sim e^{1-x}, \quad \text{as} \quad \varepsilon \searrow 0. \tag{3}$$

More specifically, for any $0 < x_1 < 1$, *and for any* $k \in \mathbb{N}$,

$$y(x, \varepsilon) - e^{1-x} = o(\varepsilon^k), \quad \text{as} \quad \varepsilon \searrow 0,$$

uniformly for $x \in [x_1, 1]$.

Proof.

Here, we know the analytic expression; it is given in (1). Thus, we need only show that

$$\lim_{\varepsilon \searrow 0} \frac{y(x, \varepsilon) - e^{1-x}}{\varepsilon^k} = 0,$$

for each $k \in \mathbb{N}$. The fact that this limit property is uniform with respect to x in the interval $[x_1, 1]$ is left as an exercise. □

The Inner Asymptotic Series Expansion

Now, let us transform our function to one in the stretched variable, setting $z = \frac{x}{\varepsilon}$. In other words, for some $z_1 > 0$, let us define

$$Y_{\mathrm{in}}(z,\varepsilon) = y(z\varepsilon,\varepsilon) := \frac{e^{-z\varepsilon} - e^{-z}}{e^{-1} - e^{-1/\varepsilon}}, \quad 0 \le z \le z_1. \tag{4}$$

Just as before, we assume that

$$Y_{\mathrm{in}}(z,\varepsilon) \sim Y_0(z) + Y_1(z)\varepsilon + Y_2(z)\varepsilon^2 + Y_3(z)\varepsilon^3 + \cdots, \quad \text{as} \quad \varepsilon \searrow 0.$$

Terms of the Series Expansion: $Y_0(z)$

If we have the series expansion

$$Y_{\rm in}(z,\varepsilon) \sim Y_0(z) + Y_1(z)\varepsilon + Y_2(z)\varepsilon^2 + Y_3(z)\varepsilon^3 + \cdots , \quad \text{as} \quad \varepsilon \searrow 0,$$

then, as above, we have, for any $z \in [0, z_1]$,

$$\begin{aligned} Y_0(z) &= \lim_{\varepsilon \searrow 0} \frac{Y_{\rm in}(z,\varepsilon)}{\phi_0(\varepsilon)} \\ &= \lim_{\varepsilon \searrow 0} \frac{e^{-\varepsilon z} - e^{-z}}{e^{-1} - e^{-1/\varepsilon}} \\ &= e^1 \left(1 - e^{-z}\right). \end{aligned}$$

Terms of the Series Expansion: $Y_k(z)$, $k \geq 1$

Next we find that

$$
\begin{aligned}
Y_1(z) &= \lim_{\varepsilon \searrow 0} \frac{Y_{\mathrm{in}}(z,\varepsilon) - Y_0(z)\phi_0(\varepsilon)}{\phi_1(\varepsilon)} \\
&= \lim_{\varepsilon \searrow 0} \frac{\frac{e^{-\varepsilon z} - e^{-z}}{e^{-1} - e^{-1/\varepsilon}} - e^{1}\left(1 - e^{-z}\right)}{\varepsilon} \\
&= -ez.
\end{aligned}
$$

Continuing, we get, for $k \in \mathbb{N}$ and $z \in [0, z_1]$,

$$
Y_k(z) = e\frac{(-1)^k z^k}{k!}.
$$

Proposition

Suppose $0 < z_1$. *Then, for* $z \in [0, z_1]$,

$$Y_{\mathrm{in}}(z,\varepsilon) \sim e\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!}\varepsilon^k - e^{1-z} = \frac{e^{-\varepsilon z} - e^{-z}}{e^{-1}}, \quad \text{as} \quad \varepsilon \searrow 0. \tag{5}$$

How Does One Join the Inner and Outer Expansions?

Now, y_{out} and Y_{in} describe the same function, but in different parts of the x-domain $[0, 1]$. The rough idea of matching, and matched asymptotic expansions, is that these two function should agree, in some overlapping region, one that is between the boundary layer and the outer region. In the next section, we describe a general principle of matching.

Matching via an Intermediate Variable

Matching: General Assumptions

Here we describe the concept of matching in very general terms. The boundary layer could be located anywhere in the domain, and the boundary layer thickness could have various forms. Suppose that, for every $\varepsilon > 0$,

$$y(\,\cdot\,,\varepsilon) : [x_0, x_2] \to \mathbb{C} \tag{6}$$

is a given, sufficiently smooth function. Suppose that there is a boundary layer for the function $y(x,\varepsilon)$ located at $x = x_0$. Assume that $\{\phi_k(\varepsilon)\}_{k=0}^{\infty}$ is an asymptotic sequence as $\varepsilon \to 0$. For example, the most common sequence will be

$$\phi_k(\varepsilon) = \varepsilon^{\alpha\cdot k}, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

where $\alpha > 0$. Assume that the stretched variable in the boundary layer is

$$z = \frac{x - x_0}{\delta(\varepsilon)},$$

where $\delta(\varepsilon) > 0$, for each $\varepsilon > 0$, and

$$\delta(\varepsilon) \searrow 0, \quad \text{as} \quad \varepsilon \searrow 0.$$

Matching: General Assumptions

The prototypical scenario is

$$x_0 = 0 \quad \text{and} \quad \delta(\varepsilon) = \varepsilon.$$

The inner function/solution is defined as

$$Y_{\mathrm{in}}(z, \varepsilon) = y(z \cdot \delta(\varepsilon) + x_0, \varepsilon). \tag{7}$$

Assume that, as $\varepsilon \searrow 0$, the inner function has the following asymptotic series expansion

$$Y_{\mathrm{in}}(z, \varepsilon) \sim \sum_{k=0}^{\infty} Y_k(z)\phi_k(\varepsilon), \tag{8}$$

for all $z \in [0, z_1]$, for some $z_1 > 0$. Next, suppose that the outer function/solution has the asymptotic series expansion

$$y_{\mathrm{out}}(x, \varepsilon) = y(x, \varepsilon) \sim \sum_{k=0}^{\infty} y_k(x)\varepsilon^k, \tag{9}$$

as $\varepsilon \searrow 0$, for all $x \in [x_1, x_2]$, with $x_0 < x_1 < x_2$.

The Intermediate Variable

Assume the existence of the intermediate variable

$$w = \frac{x - x_0}{\eta(\varepsilon)} = \frac{\delta(\varepsilon)}{\eta(\varepsilon)} z. \tag{10}$$

The function δ is called the **boundary layer scale**, and η is called the **intermediate scale**. The variable z is called the **stretched boundary-layer variable**, or just **stretched variable**, and w is called the **intermediate variable**. We assume that η satisfies

1. $\eta(\varepsilon) > 0$, for all $\varepsilon > 0$,
2. $\eta(\varepsilon) \searrow 0$, as $\varepsilon \searrow 0$, and
3. $\frac{\delta(\varepsilon)}{\eta(\varepsilon)} \searrow 0$, as $\varepsilon \searrow 0$.

We use the following shorthand for these conditions:

$$0 < \delta(\varepsilon) \ll \eta(\varepsilon) \ll 1, \quad \text{as} \quad \varepsilon \searrow 0.$$

Constructing $y_{\text{match}}^{N,M,\text{out}}(w,\varepsilon)$ and $y_{\text{match}}^{N,M,\text{in}}(w,\varepsilon)$

Let $N, M \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be given. We assume that the following asymptotic relations hold:

$$
\begin{aligned}
\sum_{k=0}^{N} y_k(x)\varepsilon^k &= \sum_{k=0}^{N} y_k(w \cdot \eta(\varepsilon) + x_0)\varepsilon^k \\
&= y_{\text{match}}^{N,M,\text{out}}(w,\varepsilon) + o\left(\phi_M(\varepsilon)\right), \qquad (11)
\end{aligned}
$$

as $\varepsilon \searrow 0$, with $w > 0$ fixed and finite, and

$$
\begin{aligned}
\sum_{k=0}^{M} Y_k(z)\phi_k(\varepsilon) &= \sum_{k=0}^{M} Y_k\left(w\frac{\eta(\varepsilon)}{\delta(\varepsilon)}\right)\phi_k(\varepsilon) \\
&= y_{\text{match}}^{N,M,\text{in}}(w,\varepsilon) + o\left(\varepsilon^N\right), \qquad (12)
\end{aligned}
$$

as $\varepsilon \searrow 0$, with $w > 0$ fixed and finite.

General Matching Principle

For each compatible choice of $M, N \in \mathbb{N}_0$, there is an intermediate scale, $\eta(\varepsilon)$, which depends upon M and N, in general, and satisfies

$$0 < \delta(\varepsilon) \ll \eta(\varepsilon) \ll 1, \quad \text{as} \quad \varepsilon \searrow 0,$$

such that, for each fixed and finite $w > 0$,

$$y_{\mathrm{match}}^{N,M,\mathrm{out}}(w, \varepsilon) = y_{\mathrm{match}}^{N,M,\mathrm{in}}(w, \varepsilon), \tag{13}$$

where $y_{\mathrm{match}}^{N,M,\mathrm{out}}(w, \varepsilon)$ and $y_{\mathrm{match}}^{N,M,\mathrm{in}}(w, \varepsilon)$ are determined from the asymptotic relations (11) and (12), respectively, as $\varepsilon \searrow 0$. We define the common matching function

$$y_{\mathrm{match}}^{N,M}(w, \varepsilon) := y_{\mathrm{match}}^{N,M,\mathrm{out}}(w, \varepsilon) = y_{\mathrm{match}}^{N,M,\mathrm{in}}(w, \varepsilon). \tag{14}$$

A uniformly-valid, composite solution on the interval $[x_0, x_2]$ may be constructed as

$$y_{\mathrm{c},N,M}(x, \varepsilon) := \sum_{k=0}^{N} y_k(x)\varepsilon^k + \sum_{k=0}^{M} Y_k(z)\phi_k(\varepsilon) - y_{\mathrm{match}}^{N,M}(w, \varepsilon).$$

General Matching Principle (Cont.)

The uniformly-valid, composite solution

$$y_{\mathrm{c},N,M}(x,\varepsilon) := \sum_{k=0}^{N} y_k(x)\varepsilon^k + \sum_{k=0}^{M} Y_k(z)\phi_k(\varepsilon) - y_{\mathrm{match}}^{N,M}(w,\varepsilon), \quad \forall\, x \in [x_0, x_2],$$

can be written in the equivalent form

$$\boxed{\begin{aligned} y_{\mathrm{c},N,M}(x,\varepsilon) = \sum_{k=0}^{N} y_k(x)\varepsilon^k + \sum_{k=0}^{M} Y_k\left(\frac{x-x_0}{\delta(\varepsilon)}\right)\phi_k(\varepsilon) \\ - y_{\mathrm{match}}^{N,M}\left(\frac{x-x_0}{\eta(\varepsilon)},\varepsilon\right), \quad \forall\, x \in [x_0, x_2]. \end{aligned}} \tag{15}$$

Width of the Matching Scale

When we rigorously justify matching and explore examples, we will observe an interesting phenomenon in the process. To explain this, let us define the variable

$$\omega_{N,M}(\varepsilon) := \eta_{N,M}(\varepsilon) - \delta(\varepsilon), \tag{16}$$

which we call the **width of the matching scale**. Typically, the size of the intermediate scale, $\eta(\varepsilon) = \eta_{N,M}(\varepsilon)$ depends upon N and M; we have emphasized this with the added subscripts. We will observe that, for fixed $\varepsilon > 0$,

$$\lim_{N,M\to\infty} \omega_{N,M}(\varepsilon) \searrow 0. \tag{17}$$

In other words, the width of the matching scale diminishes with increasing N, M. We will illustrate this below with our prototype matching problem.

Matching in Action

Matching in the Prototype BL Function

Consider again

$$y(x,\varepsilon) := \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}, \quad 0 \le x \le 1. \tag{18}$$

We can identify some of the elements involved in the matching procedure immediately, namely,

$$\delta(\varepsilon) = \varepsilon, \quad x_0 = 0, \quad x_2 = 1, \quad \phi_k(\varepsilon) = \varepsilon^k.$$

Recall that, for $x \in [x_1, 1]$, with $0 < x_1 < 1$,

$$y_{\mathrm{out}}(x,\varepsilon) - e^{1-x} = \mathrm{TST}, \quad \text{as} \quad \varepsilon \searrow 0.$$

We can identify

$$y_0(x) = e^{1-x},\ y_1(x) = 0,\ \cdots,\ y_k(x) = 0,\ \cdots,$$

in the asymptotic series expansion

$$y_{\mathrm{out}}(x,\varepsilon) \sim \sum_{k=0}^{\infty} y_k(x)\varepsilon^k, \quad \text{as} \quad \varepsilon \searrow 0.$$

The Inner Expansion in the Boundary Layer

With $\delta = \varepsilon$ and $x_0 = 0$, we define the inner expansion via

$$Y_{\rm in}(z,\varepsilon) = y(z\varepsilon,\varepsilon) := \frac{e^{-z\varepsilon} - e^{-z}}{e^{-1} - e^{-1/\varepsilon}}, \quad 0 \le z \le z_1.$$

for some $z_1 > 0$. We previously found that

$$Y_{\rm in}(z,\varepsilon) \sim \sum_{k=0}^{\infty} Y_k(z)\varepsilon^k, \quad \text{as} \quad \varepsilon \searrow 0,$$

with

$$Y_0(z) = e^1\left(1 - e^{-z}\right), \;\; Y_1(z) = ez, \;\; \cdots, \;\; Y_k(z) = e\frac{(-1)^k z^k}{k!}, \;\; \cdots .$$

Now, let us use the general matching principle to establish a leading-order composite asymptotic approximation valid on the whole interval $[0,1]$.

$N = M = 0$ Case: Computing $y_{\text{match}}^{0,0,\text{out}}(w, \varepsilon)$

Starting with the outer solution, we are looking for an asymptotic approximation of the form

$$y_0(x) = y_0(w\eta) = y_{\text{match}}^{0,0,\text{out}}(w, \varepsilon) + o(1), \quad \text{as} \quad \varepsilon \searrow 0, \tag{19}$$

for fixed, finite $w > 0$. To determine $y_{\text{match}}^{0,0,\text{out}}(w, \varepsilon)$, for $w > 0$ fixed, observe that, using Taylor's Theorem,

$$\begin{aligned} y_0(w\eta) &= e^1 e^{-w\eta} \\ &= e^1 \left(1 - w\eta + \frac{w^2}{2}\eta^2 + o(\eta^2)\right) \\ &= e^1 + o(1), \end{aligned}$$

as $\varepsilon \searrow 0$. Thus,

$$y_{\text{match}}^{0,0,\text{out}}(w, \varepsilon) = e^1. \tag{20}$$

$N = M = 0$ Case: Computing $y^{0,0,\mathrm{in}}_{\mathrm{match}}(w,\varepsilon)$

Next, we are looking for an asymptotic approximation of the form

$$Y_0(z) = Y_0\left(w\frac{\eta}{\delta}\right) = y^{0,0,\mathrm{in}}_{\mathrm{match}}(w,\varepsilon) + o(1), \quad \text{as} \quad \varepsilon \searrow 0, \tag{21}$$

for fixed, finite $w > 0$. Observe that, since $\delta = \varepsilon$

$$\begin{aligned} Y_0(z) &= e^1\left(1 - e^{-w\cdot\eta/\varepsilon}\right) \\ &= e^1 + o\left(\frac{\varepsilon^r}{\eta^r}\right) \end{aligned}$$

for any $r > 0$, for fixed, finite $w > 0$, as $\varepsilon \searrow 0$. Thus,

$$Y_0(z) = e^1 + o(1)$$

for fixed, finite $w > 0$, as $\varepsilon \searrow 0$. Thus,

$$y^{0,0,\mathrm{in}}_{\mathrm{match}}(w,\varepsilon) = e^1. \tag{22}$$

$N = M = 0$ Case: Computing $y_{\mathrm{c},0,0}(x, \varepsilon)$

Observe that

$$y_{\mathrm{match}}^{0,0,\mathrm{in}}(w, \varepsilon) = e^1 = y_{\mathrm{match}}^{0,0,\mathrm{out}}(w, \varepsilon),$$

as desired and expected. Thus, our leading-order composite approximation, valid on the interval [0, 1], is

$$y_{\mathrm{c},0,0}(x, \varepsilon) = e^{1-x} + e^1 \left(1 - e^{-x/\varepsilon}\right) - e^1,$$

or, equivalently,

$$y_{\mathrm{c},0,0}(x, \varepsilon) = \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1}}.$$

$N = M = 1$ Case: Computing $y_{\text{match}}^{1,1,\text{out}}(w, \varepsilon)$

Using Taylor's Theorem, we have

$$
\begin{aligned}
y_0(x) + y_1(x)\varepsilon &= e^{1-x} \\
&= e^{1-w\cdot\eta} \\
&= e^{1}\left(1 - w\eta + \frac{w^2}{2}\eta^2 + o(\eta^2)\right) \\
&= e^{1} - e^{1} w\eta + o(\varepsilon),
\end{aligned}
$$

for fixed, finite $w > 0$, as $\varepsilon \searrow 0$, provided

$$
\lim_{\varepsilon \searrow 0} \frac{\eta^2}{\varepsilon} = 0. \tag{23}
$$

$N = M = 1$ Case: Computing $y_{\mathrm{match}}^{1,1,\mathrm{out}}(w,\varepsilon)$

We had no restrictions on the intermediate scale for the $N = M = 0$ case. But, in general, the intermediate scale, η, will depend upon the values on M and N, which are here equal to 1. To see what is the meaning of (23), suppose that $\eta = \varepsilon^{\alpha}$. Then clearly, $\alpha = \frac{1}{2}$ is not possible. We must have

$$2\alpha - 1 > 0 \quad \iff \quad \alpha > \frac{1}{2}.$$

Thus, $1 > \alpha > \frac{1}{2}$. So $\alpha = \frac{2}{3}$ or $\alpha = \frac{3}{4}$ would be possible. In any case, if condition (23) does hold, it means that, for fixed, finite $w > 0$,

$$\lim_{\varepsilon \searrow 0} \frac{y_0(w \cdot \eta) - (e^1 - e^1 w\eta)}{\varepsilon} = \lim_{\varepsilon \searrow 0} \frac{\frac{w^2}{2}\eta^2 + o(\eta^2)}{\varepsilon} = 0,$$

and we can conclude that, as $\varepsilon \searrow 0$,

$$y_0(x) + y_1(x)\varepsilon = e^1 - e^1 w\eta + o(\varepsilon),$$

for fixed, finite $w > 0$. This implies that

$$y_{\mathrm{match}}^{1,1,\mathrm{out}}(w,\varepsilon) = e^1 - e^1 w\eta.$$

$N = M = 1$ Case: Computing $y^{1,1,\mathrm{in}}_{\mathrm{match}}(w,\varepsilon)$

Moving on to the inner solution, we have

$$\begin{aligned} Y_0(z) + Y_1(z)\varepsilon &= Y_0\left(\frac{\eta}{\varepsilon}w\right) + Y_1\left(\frac{\eta}{\varepsilon}w\right)\varepsilon \\ &= e^1\left(1 - e^{-w\eta/\varepsilon}\right) - e^1\frac{\eta}{\varepsilon}w\varepsilon \\ &= e^1 - e^1\eta w + o\left(\frac{\varepsilon^r}{\eta^r}\right), \end{aligned}$$

for any $r > 0$, for fixed, finite $w > 0$, as $\varepsilon \searrow 0$. We can conclude that

$$Y_0(z) + Y_1(z)\varepsilon = e^1 - e^1\eta w + o(\varepsilon),$$

for fixed, finite $w > 0$, as $\varepsilon \searrow 0$, which implies that

$$y^{1,1,\mathrm{in}}_{\mathrm{match}}(w,\varepsilon) = e^1 - e^1 w\eta.$$

$N = M = 1$ Case: Computing $y_{\mathrm{c},1,1}(x, \varepsilon)$

As before, we see that

$$y_{\mathrm{match}}^{1,1,\mathrm{out}}(w, \varepsilon) = e^1 - e^1 w\eta = y_{\mathrm{match}}^{1,1,\mathrm{in}}(w, \varepsilon),$$

as desired and expected.

Thus, our first-order composite approximation, valid on the interval $[0, 1]$, is

$$y_{\mathrm{c},1,1}(x, \varepsilon) = e^{1-x} + e^1 \left(1 - e^{-x/\varepsilon}\right) + e^1 \frac{x}{\varepsilon}\varepsilon - e^1 - e^1 x,$$

or, equivalently,

$$y_{\mathrm{c},1,1}(x, \varepsilon) = \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1}},$$

which is the same approximation as that obtained at leading order.

$N = M = 2$ Case: Computing $y^{2,2,\text{in}}_{\text{match}}(w,\varepsilon)$

This time, let us start with the inner solution. Continuing in the now familiar fashion, we have

$$\begin{aligned} Y_0(z) + Y_1(z)\varepsilon + Y_2(z)\varepsilon^2 &= Y_0\left(\frac{\eta}{\varepsilon}w\right) + Y_1\left(\frac{\eta}{\varepsilon}w\right)\varepsilon + Y_2\left(\frac{\eta}{\varepsilon}w\right)\varepsilon^2 \\ &= e^1\left(1 - e^{-w\eta/\varepsilon}\right) - e^1\frac{\eta}{\varepsilon}w\varepsilon + \frac{e^1}{2}\frac{\eta^2 w^2}{\varepsilon^2}\varepsilon^2 \\ &= e^1 - e^1\eta w + \frac{e^1}{2}\eta^2 w^2 + o\left(\frac{\varepsilon^r}{\eta^r}\right), \end{aligned}$$

for any $r > 0$, for fixed, finite $w > 0$, as $\varepsilon \searrow 0$. Therefore,

$$Y_0(z) + Y_1(z)\varepsilon + Y_2(z)\varepsilon^2 = e^1 - e^1\eta w + \frac{e^1}{2}\eta^2 w^2 + o\left(\varepsilon^2\right),$$

for fixed, finite $w > 0$, as $\varepsilon \searrow 0$, which implies that

$$y^{2,2,\text{in}}_{\text{match}}(w,\varepsilon) = e^1 - e^1 w\eta + \frac{e^1}{2}\eta^2 w^2.$$

$N = M = 2$ Case: Computing $y^{2,2,\mathrm{out}}_{\mathrm{match}}(w, \varepsilon)$

For the outer solution, using Taylor's Theorem, we have

$$\begin{aligned} y_0(x) + y_1(x)\varepsilon + y_2(x)\varepsilon^2 &= e^{1-x} \\ &= e^{1-w\cdot\eta} \\ &= e^1\left(1 - w\eta + \frac{w^2}{2}\eta^2 - \frac{w^2}{3}\eta^3 + o(\eta^3)\right) \\ &= e^1 - e^1 w\eta + \frac{e^1}{2}w^2\eta^2 + o(\varepsilon^2), \end{aligned}$$

for fixed, finite $w > 0$, as $\varepsilon \searrow 0$, provided

$$\lim_{\varepsilon \searrow 0} \frac{\eta^3}{\varepsilon^2} = 0. \tag{24}$$

$N = M = 2$ Case: Computing $y_{\text{match}}^{2,2,\text{out}}(w,\varepsilon)$

Again we observe that η, will depend upon the values on M and N. To explore the meaning of (24) for the $N = M = 2$ case, suppose, as before, that $\eta = \varepsilon^{\alpha}$. We must have

$$3\alpha - 2 > 0 \quad \iff \quad \alpha > \frac{2}{3}.$$

Thus, $1 > \alpha > \frac{2}{3}$. So, while $\alpha = \frac{3}{4}$ would be possible, $\alpha = \frac{1}{2}$ and $\alpha = \frac{2}{3}$ are not. If condition (24) holds, it means that, for fixed, finite $w > 0$, as $\varepsilon \searrow 0$,

$$y_0(x) + y_1(x)\varepsilon + y_2(x)\varepsilon^2 = e^1 - e^1 w\eta + \frac{e^1}{2} w^2\eta^2 + o(\varepsilon^2).$$

This implies that

$$y_{\text{match}}^{2,2,\text{out}}(w,\varepsilon) = e^1 - e^1 w\eta + \frac{e^1}{2} w^2\eta^2.$$

$N = M = 2$ Case: Computing $y_{\mathrm{c},2,2}(x,\varepsilon)$

It is no surprise that

$$y_{\mathrm{match}}^{2,2,\mathrm{out}}(w,\varepsilon) = e^1 - e^1 w\eta + \frac{e^1}{2} w^2\eta^2 = y_{\mathrm{match}}^{2,2,\mathrm{in}}(w,\varepsilon).$$

Interestingly, we get the same composite approximation as in the $N = M = 0$ and $N = M = 1$ cases:

$$y_{\mathrm{c},2,2}(x,\varepsilon) = \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1}}.$$

Width of the Matching Scale

The Diminishing Width of the Matching Scale

Continuing the process of matching for the prototype boundary layer example in the last section, we can easily establish that, for $N = M$, a viable choice for matching is

$$\delta(\varepsilon) = \varepsilon \quad \text{and} \quad \eta(\varepsilon) = \eta_N(\varepsilon) = \varepsilon^{\alpha_N},$$

where

$$\alpha_N = \frac{N+1}{N+2} > \frac{N}{N+1}.$$

The reader can easily verify this for the cases $N = M = 1$ and $N = M = 2$. The general case is a straightforward induction argument.

Since $\alpha_N \nearrow 1$, as $N \nearrow \infty$,

$$\lim_{N,M\to\infty} \omega_{N,M}(\varepsilon) \searrow 0, \quad \omega_{N,M}(\varepsilon) := \eta_{N,M}(\varepsilon) - \delta(\varepsilon),$$

for fixed $\varepsilon > 0$.

Why does the width of matching diminish?