Math 515
Essential Perturbation Theory and Asymptotic Analysis
Chapter 02

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Chapter 02, Part 1 of 2

Notation and Fundamental Definitions

Order Symbols

Big-Oh

In the following definitions, we will use the following common notation: for any numbers $a \in \mathbb{R}$ and $r > 0$,

$$I_r(a) := (a - r, a + r).$$

Definition (Big-Oh)

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f, g : D \times I_\delta(\varepsilon_o) \to \mathbb{C}$ are continuous functions. We say that f **is big-oh of** g **at** $\mathbf{x} \in D$**, as** $\varepsilon \to \varepsilon_o$, and we write $f(\mathbf{x}, \varepsilon) = O(g(\mathbf{x}, \varepsilon))$, as $\varepsilon \to \varepsilon_o$, iff there exists numbers $\delta_o = \delta_o(\mathbf{x}) \in (0, \delta)$ and $C = C(\mathbf{x}) \in (0, \infty)$, such that,

$$|f(\mathbf{x}, \varepsilon)| \le C \, |g(\mathbf{x}, \varepsilon)| \, , \quad \forall \varepsilon \in I_{\delta_o}(\varepsilon_o). \tag{1}$$

We say that f **is big-oh of** g **uniformly in** D**, as** $\varepsilon \to \varepsilon_o$ iff there exist numbers $\delta_o \in (0, \delta)$ and $C \in (0, \infty)$, both independent of $\mathbf{x} \in D$, such that,

$$|f(\mathbf{x}, \varepsilon)| \le C \, |g(\mathbf{x}, \varepsilon)| \, , \quad \forall \varepsilon \in I_{\delta_o}(\varepsilon_o). \tag{2}$$

Example

Consider the function

$$f(\varepsilon) = 2\varepsilon^3 - \varepsilon,$$

for $\varepsilon > 0$. Then

$$f(\varepsilon) = O(\varepsilon), \quad \text{as} \quad \varepsilon \searrow 0.$$

This is because the ε dominates the ε^3 term, for small values of $\varepsilon > 0$. In particular, if $0 < \varepsilon \leq 1$, then

$$\varepsilon^3 \leq \varepsilon.$$

Thus, if $0 < \varepsilon \leq \delta_o = 1$,

$$|f(\varepsilon)| \leq 2\varepsilon^3 + \varepsilon \leq 3\varepsilon.$$

The result holds with $C = 3$.

Big-Oh at Infinity

Definition (Big-Oh at Inftinity)

Suppose that $M > 0$, and assume that $f, g : D \times (M, \infty) \to \mathbb{C}$ are continuous functions. We say that f **is big-oh of** g **at** $\mathbf{x} \in D$**, as** $\lambda \to \infty$, and we write $f(\mathbf{x}, \lambda) = O(g(\mathbf{x}, \lambda))$, as $\lambda \to \infty$, iff there exist numbers $M_o = M_o(\mathbf{x}) \in [M, \infty)$ and $C = C(\mathbf{x}) \in (0, \infty)$, such that,

$$|f(\mathbf{x}, \lambda)| \leq C\,|g(\mathbf{x}, \lambda)|\,, \quad \forall\,\lambda \in (M_o, \infty). \tag{3}$$

Similarly, we say that f **is big-oh of** g **uniformly in** D**, as** $\lambda \to \infty$ iff there exist numbers $M_o \in [M, \infty)$ and $C \in (0, \infty)$, both independent of $\mathbf{x} \in D$, such that

$$|f(\mathbf{x}, \lambda)| \leq C\,|g(\mathbf{x}, \lambda)| \quad \forall\,\lambda \in (M_o, \infty). \tag{4}$$

Lemma

For any $m \in \mathbb{N}$, there is a constant $M_o = M_o(m) > 0$, such that, if $\lambda > M_o$, it follows that

$$e^{\lambda} \geq \lambda^{m}. \tag{5}$$

Proof.

Suppose that $\lambda > 0$. Using Taylor's Theorem,

$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6} + \cdots + \frac{\lambda^m}{m!} + \frac{\lambda^{m+1}}{(m+1)!} + \frac{\lambda^{m+2}}{(m+2)!}e^{\eta},$$

for some $\eta \in (0, \lambda)$. Since $e^{\eta} \geq 1$, it follows that

$$e^{\lambda} \geq 1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6} + \cdots + \frac{\lambda^m}{m!} + \frac{\lambda^{m+1}}{(m+1)!} =: p(\lambda).$$

Since $p(\lambda)$ is a polynomial of degree $m+1$, there is an $M_o = M_o(m) > 0$, such that, if $\lambda > M_o$,

$$p(\lambda) \geq \lambda^{m}.$$

$\square$

Corollary

For any $r > 0$*, there is an* $M_o = M_o(r) > 0$ *such that, if* $\lambda > M_o$*, then*

$$e^{\lambda} \geq \lambda^{r}, \tag{6}$$

and

$$e^{-\lambda} \leq \frac{1}{\lambda^{r}}. \tag{7}$$

Example

Suppose that

$$g(\lambda) = -e^{-\lambda} + \frac{4}{\lambda}.$$

Then,

$$g(\lambda) = O\left(\frac{1}{\lambda}\right), \quad \text{as} \quad \lambda \to \infty.$$

To see this, observe that, for all $\lambda > 0$

$$\frac{1}{\lambda} \geq e^{-\lambda}.$$

Hence, for, say, $\lambda > 1 =: M_o$,

$$\begin{aligned} |g(\lambda)| &\leq e^{-\lambda} + \frac{4}{\lambda} \\ &\leq \frac{1}{\lambda} + \frac{4}{\lambda} \\ &= \frac{5}{\lambda}. \end{aligned}$$

The result holds with $C = 5$ and $M_o = 1$.

Example

In fact, we have the following more general result. Suppose that $p > 0$ and

$$g(\lambda) = C_1 e^{-x} + \frac{C_2}{\lambda^p},$$

where $C_1, C_2 \in \mathbb{R}$ are constants. Then

$$g(\lambda) = O\left(\frac{1}{\lambda^p}\right), \quad \text{as} \quad \lambda \to \infty.$$

The reader should provide the details.

Example

Suppose that

$$a_n := a(n) = \frac{2}{3}n^2 - \frac{1}{2}n^3.$$

Then,

$$a_n = O\left(n^3\right), \quad \text{as} \quad n \to \infty.$$

We want to show that there is an integer $N_o \geq 1$, such that, if $n \geq N_o$, then

$$|a_n| \leq Cn^3,$$

for some constant $C > 0$. Let us prove this by finding $N_o \geq 1$ and $C > 0$. Clearly, if $n \geq N_o = 1$, it follows that

$$n^3 \geq n^2.$$

So, if $n \geq N_o$,

$$|a_n| \leq \frac{2}{3}n^2 + \frac{1}{2}n^3 \leq \frac{2}{3}n^3 + \frac{1}{2}n^3 \leq \frac{7}{6}n^3.$$

The result follows with $C = \frac{7}{6}$ and $N_o = 1$.

Little-Oh

Definition (Little-Oh)

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f, g : D \times I_\delta(\varepsilon_o) \to \mathbb{C}$ are continuous functions. We say that f **is little-oh of** g **at** $\mathbf{x} \in D$**, as** $\varepsilon \to \varepsilon_o$, and we write $f(\mathbf{x}, \varepsilon) = o(g(\mathbf{x}, \varepsilon))$, as $\varepsilon \to \varepsilon_o$, iff, for every $\alpha > 0$, there exists a number $\delta_o = \delta_o(\mathbf{x}, \alpha) \in (0, \delta)$, such that

$$|f(\mathbf{x}, \varepsilon)| \le \alpha \, |g(\mathbf{x}, \varepsilon)| \, , \quad \forall \varepsilon \in I_{\delta_o}(\varepsilon_o). \tag{8}$$

We say that f **is little-oh of** g **uniformly in** D**, as** $\varepsilon \to \varepsilon_o$ iff, for every $\alpha > 0$, there exists a number $\delta_o = \delta_o(\alpha) \in (0, \delta)$, independent of $\mathbf{x} \in D$, such that

$$|f(\mathbf{x}, \varepsilon)| \le \alpha \, |g(\mathbf{x}, \varepsilon)| \, , \quad \forall \varepsilon \in I_{\delta_o}(\varepsilon_o). \tag{9}$$

Little-Oh at Infinity

Definition (Little-Oh at Infinity)

Suppose that $M > 0$, and assume that $f, g : D \times (M, \infty) \to \mathbb{C}$ are continuous functions. We say that f **is little-oh of** g **at** $\mathbf{x} \in D$**, as** $\lambda \to \infty$, and we write $f(\mathbf{x}, \lambda) = o(g(\mathbf{x}, \lambda))$, as $\lambda \to \infty$, iff, for every $\alpha > 0$, there exists a number $M_o = M_o(\mathbf{x}, \alpha) \in [M, \infty)$, such that

$$|f(\mathbf{x}, \lambda)| \leq \alpha \, |g(\mathbf{x}, \lambda)| \, , \quad \forall \, \lambda \in (M_0, \infty). \tag{10}$$

Similarly, we say that f **is little-oh of** g **uniformly in** D**, as** $\lambda \to \infty$ iff, for every $\alpha > 0$, there exists a number $M_o = M_o(\alpha) \in [M, \infty)$, independent of $\mathbf{x} \in D$, such that

$$|f(\mathbf{x}, \lambda)| \leq \alpha \, |g(\mathbf{x}, \lambda)| \, , \quad \forall \, \lambda \in (M_0, \infty). \tag{11}$$

Remark

To save needless writing, and reading as well, we will state and prove a few properties only involving our definitions for the case that $\varepsilon \to \varepsilon_o$*, the finite limit case. However, the reader should be able to prove analogous results for the case that* $\lambda \to \infty$*, the infinite limit case, with only minor changes to the assumptions and arguments.*

Proposition

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f, g : D \times I_\delta(\varepsilon_o) \to \mathbb{C}$ are continuous functions and $\mathbf{x} \in D$. Suppose that

$$g(\mathbf{x}, \varepsilon) \neq 0, \quad \forall\, \varepsilon \in I_\delta^\star(\varepsilon_o) := I_\delta(\varepsilon_o) \setminus \{\varepsilon_o\},$$

and there exists some number $L = L(\mathbf{x}) \in \mathbb{C}$ such that

$$\lim_{\varepsilon \to \varepsilon_o} \frac{f(\mathbf{x}, \varepsilon)}{g(\mathbf{x}, \varepsilon)} = L.$$

Then, $f(\mathbf{x}, \varepsilon) = O(g(\mathbf{x}, \varepsilon))$, as $\varepsilon \to \varepsilon_o$.

Proof.

Using the definition of limit, for any $\varrho > 0$, there exists a number $\beta = \beta(\mathbf{x}, \varrho) > 0$, such that, if $\varepsilon \in I_\delta^\star(\varepsilon_o) \cap I_\beta(\varepsilon_o)$, it follows that

$$\left| \frac{f(\mathbf{x}, \varepsilon)}{g(\mathbf{x}, \varepsilon)} - L \right| < \varrho.$$

Proof Cont.

Using the reverse triangle inequality, we find

$$|f(\mathbf{x},\varepsilon)| - |L| \cdot |g(\mathbf{x},\varepsilon)| \leq |f(\mathbf{x},\varepsilon) - L \cdot g(\mathbf{x},\varepsilon)| < \varrho \cdot |g(\mathbf{x},\varepsilon)|,$$

which, in turn, implies that

$$|f(\mathbf{x},\varepsilon)| < (|L| + \varrho)\,|g(\mathbf{x},\varepsilon)|,$$

provided $\varepsilon \in I^{\star}_{\delta}(\varepsilon_o) \cap I_{\beta}(\varepsilon_o)$. Take $\delta_o := \min(\delta, \beta)$ and $C = |L| + \varrho$. $\square$

Proposition (Little-Oh Implies Big-Oh)

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f, g : D \times I_\delta(\varepsilon_o) \to \mathbb{C}$ are continuous functions and, for some $\mathbf{x} \in D$, $f(\mathbf{x}, \varepsilon) = o(g(\mathbf{x}, \varepsilon))$, as $\varepsilon \to \varepsilon_o$. Then, it follows that $f(\mathbf{x}, \varepsilon) = O(g(\mathbf{x}, \varepsilon))$, as $\varepsilon \to \varepsilon_o$.

Proof.

Exercise. $\square$

Proposition

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f, g : D \times I_\delta(\varepsilon_o) \to \mathbb{C}$ are continuous, $\mathbf{x} \in D$, and

$$g(\mathbf{x}, \varepsilon) \neq 0, \quad \forall\, \varepsilon \in I_\delta^\star(\varepsilon_o) := I_\delta(\varepsilon_o) \setminus \{\varepsilon_o\}.$$

Then, $f(\mathbf{x}, \varepsilon) = o(g(\mathbf{x}, \varepsilon))$ at $\mathbf{x} \in D$, as $\varepsilon \to \varepsilon_o$ iff

$$\lim_{\varepsilon \to \varepsilon_o} \frac{f(\mathbf{x}, \varepsilon)}{g(\mathbf{x}, \varepsilon)} = 0.$$

Proof.

($\Longrightarrow$): Suppose that $f(\mathbf{x}, \varepsilon) = o(g(\mathbf{x}, \varepsilon))$ at $\mathbf{x} \in D$, as $\varepsilon \to \varepsilon_o$. For every $\alpha > 0$, there exists a number $\delta_o = \delta_o(\mathbf{x}, \alpha) \in (0, \delta)$, such that

$$|f(\mathbf{x}, \varepsilon)| \leq \alpha\, |g(\mathbf{x}, \varepsilon)|\,, \quad \forall\, \varepsilon \in I_{\delta_o}(\varepsilon_o).$$

Proof Cont.

Thus, it follows that

$$\left| \frac{f(\mathbf{x}, \varepsilon)}{g(\mathbf{x}, \varepsilon)} - 0 \right| \leq \alpha, \quad \forall \varepsilon \in I^{\star}_{\delta_o}(\varepsilon_o).$$

Since $\alpha > 0$ is arbitrary,

$$\lim_{\varepsilon \to \varepsilon_o} \frac{f(\mathbf{x}, \varepsilon)}{g(\mathbf{x}, \varepsilon)} = 0.$$

($\Longleftarrow$): This direction is similarly straightforward. $\square$

Example

Suppose that $p > 0$. Then

$$e^{-1/\varepsilon} = o\left(\varepsilon^p\right), \quad \text{as} \quad \varepsilon \searrow 0,$$

for any . To establish this fact, it suffices to show that

$$\lim_{\varepsilon \searrow 0} \frac{e^{-1/\varepsilon}}{\varepsilon^p} = 0,$$

for any $p > 0$. This follows from a previous corollary, since

$$e^{-1/\varepsilon} \le \varepsilon^{p+1},$$

for all $0 < \varepsilon < \varepsilon_o$, for some $\varepsilon_o = \varepsilon(p+1) > 0$. A similar argument shows that

$$e^{-\lambda} = o\left(\frac{1}{\lambda^p}\right), \quad \text{as} \quad \lambda \to \infty.$$

Transcendentally Small Terms (TST)

Definition

Suppose that $\varepsilon_o \in \mathbb{R}$, $\delta > 0$, and $f : I_\delta(\varepsilon_o) \to \mathbb{C}$ is continuous. We say that f **is a transcendentally small term, as** $\varepsilon \to \varepsilon_o$, abbreviated TST, iff $f = o((\varepsilon - \varepsilon_o)^r)$, as $\varepsilon \to \varepsilon_o$, for all $r \geq 0$.

Definition

Suppose that $M > 0$ and $f : (M, \infty) \to \mathbb{C}$ is continuous. We say that f **is a transcendentally small term, as** $\lambda \to \infty$, abbreviated TST, iff $f = o(\lambda^{-r})$, as $\lambda \to \infty$, for all $r \geq 0$.

Example

Consider the function

$$f(\lambda) = e^{-\lambda},$$

for $\lambda > 0$. Then

$$f(\lambda) = \mathrm{TST}, \quad \text{as} \quad \lambda \to \infty.$$

Definition (Asymptotic Approximation)

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f, g : D \times I_\delta(\varepsilon_o) \to \mathbb{C}$ are continuous functions. We say that g **is an asymptotic approximation of** f **at** $\mathbf{x} \in D$**, as** $\varepsilon \to \varepsilon_o$, and we write $f(\mathbf{x},\varepsilon) \sim g(\mathbf{x},\varepsilon)$, as $\varepsilon \to \varepsilon_o$, iff $f(\mathbf{x},\varepsilon) - g(\mathbf{x},\varepsilon) = o(g(\mathbf{x},\varepsilon))$, as $\varepsilon \to \varepsilon_o$.

We immediately have the following: $f(\mathbf{x},\varepsilon) \sim g(\mathbf{x},\varepsilon)$, as $\varepsilon \to \varepsilon_o$, iff

$$\lim_{\varepsilon \to \varepsilon_o} \frac{f(\mathbf{x},\varepsilon) - g(\mathbf{x},\varepsilon)}{g(\mathbf{x},\varepsilon)} = 0 \iff \lim_{\varepsilon \to \varepsilon_o} \frac{f(\mathbf{x},\varepsilon)}{g(\mathbf{x},\varepsilon)} = 1.$$

Proposition

Suppose that $D \subset \mathbb{C}^d$ *is an open set,* $\varepsilon_o \in \mathbb{R}$*, and* $\delta > 0$*. Assume that* $f, g : D \times I_\delta(\varepsilon_o) \to \mathbb{C}$ *are continuous functions and* $f(\mathbf{x}, \varepsilon) \sim g(\mathbf{x}, \varepsilon)$*, as* $\varepsilon \to \varepsilon_o$*. If*

$$f(\mathbf{x}, \varepsilon) \neq 0, \quad g(\mathbf{x}, \varepsilon) \neq 0, \quad \forall \varepsilon \in I_\delta^\star(\varepsilon_o) := I_\delta(\varepsilon_o) \setminus \{\varepsilon_o\},$$

then $g(\mathbf{x}, \varepsilon) \sim f(\mathbf{x}, \varepsilon)$*, as* $\varepsilon \to \varepsilon_o$*.*

Proof.

Exercise. $\square$

Asymptotic Sequences and Series

Asymptotic Sequences

Definition (Asymptotic Sequence)

Suppose that $\varepsilon_o \in \mathbb{R}$ and $\delta > 0$. Assume that $\phi_k : I_\delta(\varepsilon_o) \to \mathbb{C}$ is a continuous function, for each $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The sequence $\{\phi_k\}_{k=0}^{\infty}$ is called an **asymptotic sequence, as** $\varepsilon \to \varepsilon_o$ iff $\phi_{k+1}(\varepsilon) = o(\phi_k(\varepsilon))$, as $\varepsilon \to \varepsilon_o$, for all $k \in \mathbb{N}_0$.

Example

Let us give two common examples. First, for $\varepsilon \to 0$, we commonly use

$$\phi_k = \varepsilon^k, \quad k \in \mathbb{N}_0.$$

For $\lambda \to \infty$, we will commonly use

$$\phi_k = \frac{1}{\lambda^k}, \quad k \in \mathbb{N}_0.$$

These are both asymptotic sequences, as is easy to check.

Definition (Asymptotic Series)

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, $\delta > 0$, and $n \in \mathbb{N}_0$. Assume that $f : D \times I_\delta(\varepsilon_o) \to \mathbb{C}$ is a continuous function, and $a_k : D \to \mathbb{C}$ is a continuous function, for each $k \in \{0, \ldots, n\}$. Suppose that $\{\phi_k\}_{k=0}^{\infty}$ is an asymptotic sequence, as $\varepsilon \to \varepsilon_o$, on $I_\delta(\varepsilon_o)$. We say that $\sum_{k=0}^{n} a_k(\,\cdot\,)\phi_k(\,\cdot\,)$ **is a finite asymptotic series approximation of f at $\mathbf{x} \in D$, as $\varepsilon \to \varepsilon_o$**, and we write

$$f(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{n} a_k(\mathbf{x})\phi(\varepsilon), \quad \text{as} \quad \varepsilon \to \varepsilon_o,$$

iff $f(\mathbf{x}, \varepsilon) - \sum_{k=0}^{m} a_k(\mathbf{x})\phi_k(\varepsilon) = o(\phi_m(\mathbf{x}, \varepsilon))$, as $\varepsilon \to \varepsilon_o$, for each $m \in \{0, \ldots, n\}$. We say that $\sum_{k=0}^{\infty} a_k(\,\cdot\,)\phi_k(\,\cdot\,)$ **is an infinite asymptotic series approximation of f at $\mathbf{x} \in D$, as $\varepsilon \to \varepsilon_o$**, and we write

$$f(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{\infty} a_k(\mathbf{x})\phi_k(\varepsilon), \quad \text{as} \quad \varepsilon \to \varepsilon_o,$$

iff $f(\mathbf{x}, \varepsilon) - \sum_{k=0}^{m} a_k(\mathbf{x})\phi_k(\varepsilon) = o(\phi_m(\mathbf{x}, \varepsilon))$, as $\varepsilon \to \varepsilon_o$, for each $m \in \mathbb{N}_0$.

Example

The most common example that we will encounter involves approximations of the form

$$f(\mathbf{x},\varepsilon) \sim \sum_{k=0}^{\infty} a_k(\mathbf{x})(\varepsilon-\varepsilon_o)^k, \quad \text{as} \quad \varepsilon \to \varepsilon_o.$$

The expansion $\sum_{k=0}^{\infty} a_k(\mathbf{x})(\varepsilon-\varepsilon_o)^k$ is called an asymptotic power series. We leave it to the reader to check that $\{(\varepsilon-\varepsilon_o)^k\}_{k=0}^{\infty}$ is an asymptotic sequence as $\varepsilon \to \varepsilon_o$. But, we will also encounter approximations of the form

$$f(\mathbf{x},\varepsilon) \sim \sum_{k=0}^{\infty} a_k(\mathbf{x})(\varepsilon-\varepsilon_o)^{\alpha\cdot k}, \quad \text{as} \quad \varepsilon \to \varepsilon_o,$$

where $\alpha \in (0,1)$ is a given fraction. Common examples are $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{3}$. The expansion $\sum_{k=0}^{\infty} a_k(\mathbf{x})(\varepsilon-\varepsilon_o)^{\alpha\cdot k}$ is called a generalized asymptotic power series. The reader can check that $\{(\varepsilon-\varepsilon_o)^{\alpha\cdot k}\}_{k=0}^{\infty}$ is also an asymptotic sequence as $\varepsilon \to \varepsilon_o$.

Uniqueness of the Coefficients

Proposition

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f : D \times I_\delta(\varepsilon_o) \to \mathbb{C}$ is a continuous function, and $a_k : D \to \mathbb{C}$ is a continuous function, for each $k \in \{0, \ldots, n\}$. Suppose that $\{\phi_k\}_{k=0}^{\infty}$ is an asymptotic sequence, as $\varepsilon \to \varepsilon_o$, on $I_\delta(\varepsilon_o)$, with $\phi_n(\varepsilon_o) \neq 0$. Assume that $f(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{\infty} a_k(\mathbf{x})\phi(\varepsilon)$, as $\varepsilon \to \varepsilon_o$. Then,

$$a_0(\mathbf{x}) = \lim_{\varepsilon \to \varepsilon_o} \frac{f(\mathbf{x}, \varepsilon)}{\phi_0(\varepsilon)} \tag{12}$$

and

$$a_m(\mathbf{x}) = \lim_{\varepsilon \to \varepsilon_o} \frac{f(\mathbf{x}, \varepsilon) - \sum_{k=0}^{m-1} a_k(\mathbf{x})\phi_k(\varepsilon)}{\phi_m(\varepsilon)}, \quad \forall\, m \in \mathbb{N}. \tag{13}$$

Thus, for a given asymptotic sequence, the coefficient terms in the series expansion are uniquely determined.

Proof of the Last Proposition.

If $f(\mathbf{x},\varepsilon) \sim \sum_{k=0}^{\infty} a_k(\mathbf{x})\phi(\varepsilon)$, as $\varepsilon \to \varepsilon_o$, then, by definition, this implies that

$$\lim_{\varepsilon\to\varepsilon_o} \frac{f(\mathbf{x},\varepsilon) - \sum_{k=0}^{m} a_k(\mathbf{x})\phi_k(\varepsilon)}{\phi_m(\varepsilon)} = 0,$$

which is equivalent to

$$\lim_{\varepsilon\to\varepsilon_o} \frac{f(\mathbf{x},\varepsilon) - \sum_{k=0}^{m-1} a_k(\mathbf{x})\phi_k(\varepsilon) - a_m(\mathbf{x})\phi_m(\varepsilon)}{\phi_m(\varepsilon)} = 0,$$

which, in turn, is equivalent to

$$\lim_{\varepsilon\to\varepsilon_o} \frac{f(\mathbf{x},\varepsilon) - \sum_{k=0}^{m-1} a_k(\mathbf{x})\phi_k(\varepsilon)}{\phi_m(\varepsilon)} = a_m(\mathbf{x}),$$

□

Proposition

Suppose that $D \subset \mathbb{C}^d$ *is an open set,* $\varepsilon_o \in \mathbb{R}$, *and* $\delta > 0$. *Assume that* $f, g : D \times I_\delta(\varepsilon_o) \to \mathbb{C}$ *are continuous functions and* $a_k, b_k : D \to \mathbb{C}$ *are continuous function, for each* $k \in \{0, \ldots, n\}$. *Suppose that* $\{\phi_k\}_{k=0}^{\infty}$ *is an asymptotic sequence, as* $\varepsilon \to \varepsilon_o$, *on* $I_\delta(\varepsilon_o)$, *with* $\phi_n(\varepsilon_o) \neq 0$. *Assume that*

$$f(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{\infty} a_k(\mathbf{x})\phi(\varepsilon) \quad \text{and} \quad g(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{\infty} b_k(\mathbf{x})\phi(\varepsilon), \quad \text{as} \quad \varepsilon \to \varepsilon_o.$$

Then,

$$C \cdot f(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{\infty} C a_k(\mathbf{x})\phi(\varepsilon), \quad \text{as} \quad \varepsilon \to \varepsilon_o,$$

$$f(\mathbf{x}, \varepsilon) + g(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{\infty} (a_k(\mathbf{x}) + b_k(\mathbf{x}))\phi(\varepsilon), \quad \text{as} \quad \varepsilon \to \varepsilon_o,$$

Proposition (Cont.)

and

$$f(\mathbf{x},\varepsilon)g(\mathbf{x},\varepsilon) \sim \sum_{k=0}^{\infty} c_k(\mathbf{x})\phi(\varepsilon), \quad \textit{as} \quad \varepsilon \to \varepsilon_o,$$

where

$$c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0.$$

Proof.

The proof is an exercise. □

Proposition

Suppose that $\varepsilon_0 \in \mathbb{R}$, $\delta > 0$, and $f : I_\delta(\varepsilon_o) \to \mathbb{C}$ is a continuous function satisfying

$$f(\varepsilon) \sim \sum_{k=0}^{\infty} a_k(\varepsilon - \varepsilon_o)^k, \quad \textit{as} \quad \varepsilon \to \varepsilon_o.$$

If $\phi : I_\delta(\varepsilon_o) \to \mathbb{C}$ is a TST, as $\varepsilon \to \varepsilon_o$, then

$$f(\varepsilon) + \phi(\varepsilon) \sim \sum_{k=0}^{\infty} a_k(\varepsilon - \varepsilon_o)^k, \quad \textit{as} \quad \varepsilon \to \varepsilon_o.$$

Proof.

This follows immediately from the fact that

$$\phi(\varepsilon) \sim \sum_{k=0}^{\infty} b_k(\varepsilon - \varepsilon_o)^k, \quad \text{as} \quad \varepsilon \to \varepsilon_o, \quad \text{where} \quad b_k = 0, \quad \forall\, k \in \mathbb{N}_0.$$

$\square$

Two Motivating Problems Revisited

Proposition

$$\log(n!) \sim n\log(n) - n + 1, \quad \textit{as} \quad n \to \infty. \tag{14}$$

Proof.

Set, as before, for any $n \in \mathbb{N}$,

$$L(n) := \log(n!)$$

and

$$I(n) := \int_1^n \log(x)\,\mathrm{d}x = [x\log(x) - x]\big|_{x=1}^{x=n} = n\log(n) - n + 1.$$

By definition (or, better, a natural extension of the definition), $L(n) \sim I(n)$, as $n \to \infty$, iff

$$\lim_{n\to\infty} \frac{L(n) - I(n)}{I(n)} \to 0.$$

Proof Cont.

In Chapter 1, we showed, using an argument involving Riemann sums, that

$$0 \leq \frac{L(n) - l(n)}{l(n)} \leq \frac{\log(n)}{l(n)}, \quad \forall\, n \in \mathbb{N}.$$

Since l'Hôpital's rule guarantees that

$$\frac{\log(n)}{l(n)} \to 0, \quad \text{as} \quad n \to \infty,$$

we can conclude the asymptotic result using the Squeeze Theorem. $\square$

Proposition

Define, for each $\lambda \in [0, \infty)$*,*

$$I(\lambda) = \int_0^\infty \frac{e^{-\lambda t}}{1+t}\,\mathrm{d}t. \tag{15}$$

Then

$$I(\lambda) \sim \sum_{k=0}^{\infty} (-1)^k \frac{k!}{\lambda^{k+1}}, \quad \textit{as} \quad \lambda \to \infty.$$

Proof.

Here $a_k = (-1)^k k!$ and $\phi_k(\lambda) = \frac{1}{\lambda^{k+1}}$. First of all, let us confirm that $\{\phi_k(\lambda)\}_{k=0}^{\infty}$ is an asymptotic sequence, as $\lambda \to \infty$. In particular, we must show that

$$\phi_{k+1}(\lambda) = o(\phi_k(\lambda)), \quad \text{as} \quad \lambda \to \infty, \quad \forall\, k \in \mathbb{N}_0,$$

or, equivalently,

$$\lim_{\lambda \to \infty} \frac{\phi_{k+1}(\lambda)}{\phi_k(\lambda)} = 0, \quad \forall\, k \in \mathbb{N}_0.$$

Proof Cont.

This clearly follows, since

$$\frac{\phi_{k+1}(\lambda)}{\phi_k(\lambda)} = \frac{\lambda^{k+1}}{\lambda^{k+2}} = \frac{1}{\lambda}.$$

Next, recall that, in the construction of our approximation in Chapter 1, we showed that, precisely,

$$I(\lambda) = \sum_{k=0}^{n-1}(-1)^k \frac{k!}{\lambda^{k+1}} + R_n(\lambda),$$

where

$$R_n(\lambda) = \int_0^\infty (-1)^n \frac{t^n}{1+t} e^{-\lambda t}\, \mathrm{d}t.$$

Proof Cont.

Now,

$$I(\lambda) \sim \sum_{k=0}^{\infty} a_k \phi_k(\lambda), \quad \text{as} \quad \lambda \to \infty,$$

iff

$$I(\lambda) - \sum_{k=0}^{m-1} a_k \phi_k(\lambda) = o(\phi_{m-1}(x)), \quad \text{as} \quad \lambda \to \infty,$$

for each $m \in \mathbb{N}$. Notice that

$$R_m(\lambda) = I(\lambda) - \sum_{k=0}^{m-1} a_k \phi_k(\lambda).$$

Therefore, we only need to show that

$$R_m(\lambda) = \int_0^{\infty} (-1)^m \frac{t^m}{1+t} e^{-\lambda t} \, \mathrm{d}t = o(\phi_{m-1}(\lambda)) = o\left(\frac{1}{\lambda^m}\right), \quad \text{as } \lambda \to \infty.$$

Proof Cont.

But, we know that

$$R_m(\lambda) = \int_0^\infty (-1)^m \frac{t^m}{1+t} e^{-\lambda t} \, \mathrm{d}t = o(\phi_{m-1}(\lambda)) = o\left(\frac{1}{\lambda^m}\right), \quad \text{as } \lambda \to \infty,$$

is true, by definition, iff

$$\lim_{\lambda \to \infty} \lambda^m \int_0^\infty (-1)^m \frac{t^m}{1+t} e^{-\lambda t} \, \mathrm{d}t = 0. \tag{16}$$

Since, as we have already shown in Chapter 1,

$$|R_m(\lambda)| \le \frac{m!}{\lambda^{m+1}},$$

(16) must be satisfied, and the result follows. □

Mathematica

The Mathematica software package has functions that make various asymptotic approximation in the analysis of integral, or differential equations. For instance, we can obtain a five-term approximation of the integral (15) for large values of $\lambda = L$ using the following code:

```
In[1]:= AsymptoticIntegrate[Exp[-x*t]/(1 + t),
```

$$\text{Out[1]=} \quad \frac{24}{L^5} - \frac{6}{L^4} + \frac{2}{L^3} - \frac{1}{L^2} + \frac{1}{L}$$

Without breaking a sweat, Mathematica can obtain a nine-term approximation via the code

```
In[2]:= AsymptoticIntegrate[Exp[-x*t]/(1 + t),
```

$$\text{Out[2]=} \quad \frac{40320}{L^9} - \frac{5040}{L^8} + \frac{720}{L^7} - \frac{120}{L^6} + \frac{24}{L^5} - \frac{6}{L^4} + \frac{2}{L^3} - \frac{1}{L^2} + \frac{1}{L}$$