Math 515
Essential Perturbation Theory and Asymptotic Analysis
Chapter 04

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Spring 2024

Chapter 04, Part 1 of 2

Contour Integration in the Complex Plane and the Method of Steepest Descent

Contour Integration in the Plane

References

In this chapter we will review some basic theory from Complex Analysis, which will help us in the asymptotic approximation of certain integrals with a large parameter. We will loosely follow the treatment in the book by Marsden and Hoffman (1987). The student can also get an excellent introduction to the subject – without frills, but with loads of examples – using the undergraduate text by Zill and Wright (2011).

Definition

Suppose that $D \subset \mathbb{R}^2$ is an open, bounded set, and $u, v : D \to \mathbb{R}$ are continuous functions. Assume that $\boldsymbol{\gamma} = (\alpha, \beta) : [a, b] \to D$ is a smooth planar curve in D. Then, the **contour integral of** (u, v) **along** $\boldsymbol{\gamma}$ is defined as

$$\int_{\boldsymbol{\gamma}} [u(x, y)\,\mathrm{d}x + v(x, y)\,\mathrm{d}y] := \int_a^b \left[u(\alpha(t), \beta(t)) \frac{\mathrm{d}\alpha}{\mathrm{d}t} + v(\alpha(t), \beta(t)) \frac{\mathrm{d}\beta}{\mathrm{d}t} \right] \mathrm{d}t. \tag{1}$$

Often times $\boldsymbol{\gamma}$ is not smooth. It may have corners, for example. We can extend the definition, in a very natural way, for $\boldsymbol{\gamma}$ that is continuous and piecewise smooth. All that we have to do is stop and restart the contour integral at a corner. Consequently, we will keep the simpler, more-restrictive assumption that $\boldsymbol{\gamma}$ is smooth.

Vector Version of Contour Integration

If we write $\mathbf{U}(\mathbf{x}) = (u(\mathbf{x}), v(\mathbf{x}))$, where $\mathbf{x} = (x, y)$, then we can express the contour integral of (u, v) along $\boldsymbol{\gamma}$ in compact form as

$$\int_{\boldsymbol{\gamma}} [u(x, y)\,\mathrm{d}x + v(x, y)\,\mathrm{d}y] = \int_{\boldsymbol{\gamma}} \mathbf{U}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = \int_a^b \mathbf{U}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,\mathrm{d}t. \tag{2}$$

Simply Connected Sets

Definition

Let $D \subset \mathbb{R}^2$ be an open, bounded domain. D is called a **path-connected set** iff between any two points $\mathbf{x}, \mathbf{y} \in D$, there is a continuous curve (a path) $\boldsymbol{\gamma} : [a, b] \to D$, such that $\boldsymbol{\gamma}(a) = \mathbf{x}$ and $\boldsymbol{\gamma}(b) = \mathbf{y}$. D is called **simply connected** iff it is (i) a path-connected set and (ii) every closed curve in D can be continuously deformed to a point in D, that is, there are no holes.

Convex Sets

Definition

A set $K \subseteq \mathbb{R}^2$ is called **convex** iff whenever $\mathbf{x}$ and $\mathbf{y}$ are element of K, the point $\mathbf{w}(t)$, defined as

$$\mathbf{w}(t) = t\mathbf{x} + (1-t)\mathbf{y},$$

is also an element of K, for all $t \in (0, 1)$.

Example

Every convex set $K \subseteq \mathbb{R}^2$ is simply connected.

Theorem (Green's Theorem)

Suppose that $D \subset \mathbb{R}^2$ is an open, bounded, simply connected set, and assume that $u, v \in C^1(D; \mathbb{R})$. Suppose that $\boldsymbol{\gamma} : [a, b] \to D$ is any smooth, simple, (that is, non-self-intersecting), counter-clockwise-oriented, closed curve (that is, satisfying $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$). Denote by $D_{\boldsymbol{\gamma}} \subset D$, the subset of D whose boundary is the image of γ, that is,

$$\partial D_{\boldsymbol{\gamma}} = \boldsymbol{\gamma}([a, b]).$$

Then,

$$\int_{\boldsymbol{\gamma}} [u(x, y)\,\mathrm{d}x + v(x, y)\,\mathrm{d}y] = \int_{D_{\boldsymbol{\gamma}}} \left(\frac{\partial v}{\partial x}(x, y) - \frac{\partial u}{\partial y}(x, y)\right) \mathrm{d}\mathbf{x}. \qquad (3)$$

Complex Contour Integration

Definition

Suppose that $D \subset \mathbb{C}$ is an open, bounded set, and $f : D \to \mathbb{C}$ is a continuous function. Assume that $\gamma : [a, b] \to D$ is a smooth complex curve. Then, the **complex contour integral** of f along γ, denoted $\int_\gamma f(z)\,\mathrm{d}z$, is defined as

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t. \tag{4}$$

We need to know how to interpret the obtect on the right hand side of (4), because that object should tell us what complex contour integration is really all about. To do this translation, we need some definitions that we will use repeatedly.

Definition

Suppose that $D \subset \mathbb{C}$ is a set and $f : D \to \mathbb{C}$ is a given function. Assume that $\gamma : [a, b] \to D$ is a continuous complex contour. The **realification of the complex set** D, denoted $D(\mathbb{R}^2)$, is a set defined as

$$D(\mathbb{R}^2) := \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; x + \mathrm{i}y \in D \right\}.$$

The **real part of** f, denoted $u : D(\mathbb{R}^2) \to \mathbb{R}$, and the **imaginary part of** f, denoted $v : D(\mathbb{R}^2) \to \mathbb{R}$, are functions defined via the relation

$$f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y), \quad \forall\, (x, y) \in D(\mathbb{R}^2).$$

Equivalently,

$$u(x, y) = \Re(f(x + \mathrm{i}y)) \qquad v(x, y) = \Im(f(x + \mathrm{i}y)), \quad \forall\, (x, y) \in D(\mathbb{R}^2).$$

Similarly, if $\gamma(t) = \alpha(t) + \mathrm{i}\beta(t)$, for all $t \in [a, b]$, the **realification of the complex contour** γ is the real euclidean contour, $\boldsymbol{\gamma} : [a, b] \to D(\mathbb{R}^2)$, defined as

$$\boldsymbol{\gamma}(t) = (\alpha(t), \beta(t)), \quad \forall\, t \in [a, b].$$

Proposition

Suppose that $D \subset \mathbb{C}$ is an open, bounded set, and $f : D \to \mathbb{C}$ is a continuous function, with $u = \Re(f)$ and $v = \Im(f)$. Assume that $\gamma : [a, b] \to D$ is a smooth complex curve, with $\alpha = \Re(\gamma)$ and $\beta = \Im(\gamma)$, and $\boldsymbol{\gamma} = (\alpha, \beta) : [a, b] \to D(\mathbb{R}^2)$ is its realification. Define

$$\mathbf{S}(x, y) := (u(x, y), -v(x, y)), \quad \mathbf{T}(x, y) := (v(x, y), u(x, y)),$$

for all $(x, y) \in D(\mathbb{R}^2)$. Then, the complex contour integral $\int_\gamma f(z)\,\mathrm{d}z$ can be expressed in terms of real contour integrals as follows:

$$\boxed{\begin{aligned} \int_\gamma f(z)\,\mathrm{d}z &= \int_{\boldsymbol{\gamma}} [u(x, y)\,\mathrm{d}x - v(x, y)\,\mathrm{d}y] \\ &\quad + \mathrm{i} \int_{\boldsymbol{\gamma}} [v(x, y)\,\mathrm{d}x + u(x, y)\,\mathrm{d}y] \\ &= \int_a^b \mathbf{S}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,\mathrm{d}t + \mathrm{i} \int_a^b \mathbf{T}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,\mathrm{d}t. \end{aligned}} \tag{5}$$

Proof.

Using the definitions above, we have

$$
\begin{aligned}
f(\gamma(t))\gamma'(t) &= \left[u(\alpha(t),\beta(t)) + \mathrm{i}v(\alpha(t),\beta(t))\right]\left[\alpha'(t) + \mathrm{i}\beta'(t)\right] \\
&= u(\alpha(t),\beta(t))\alpha'(t) - v(\alpha(t),\beta(t))\beta'(t) \\
&\quad + \mathrm{i}\left(u(\alpha(t),\beta(t))\beta'(t) + v(\alpha(t),\beta(t))\alpha'(t)\right) \\
&= \mathbf{S}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t) + \mathrm{i}\mathbf{T}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t).
\end{aligned} \tag{6}
$$

Therefore,

$$
\int_\gamma f(z)\,\mathrm{d}z = \int_a^b \mathbf{S}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,\mathrm{d}t + \mathrm{i}\int_a^b \mathbf{T}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,\mathrm{d}t. \tag{7}
$$

In other words, complex contour integrals can be computed using two real contour integrals.

Proof (Cont.)

Going further, using a previous definition, we can write

$$\int_a^b \mathbf{S}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,\mathrm{d}t = \int_{\boldsymbol{\gamma}} [u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y]$$

and

$$\int_a^b \mathbf{T}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,\mathrm{d}t = \int_{\boldsymbol{\gamma}} [v(x,y)\,\mathrm{d}x + u(x,y)\,\mathrm{d}y] \,.$$

□

Proposition

Suppose that $D \subset \mathbb{C}$ is an open, bounded set, and $f, g : D \to \mathbb{C}$ are continuous function. Assume that $\gamma, \gamma_1, \gamma_2 : [a, b] \to D$ are continuous, piecewise smooth complex curves, and $c_1, c_2 \in \mathbb{C}$ are arbitrary constants. Then,

$$\int_\gamma (c_1 f(z) + c_2 g(z))\,\mathrm{d}z = c_1 \int_\gamma f(z)\,\mathrm{d}z + c_2 \int_\gamma g(z)\,\mathrm{d}z; \tag{8}$$

$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_\gamma f(z)\,\mathrm{d}z; \tag{9}$$

$$\int_{\gamma_1+\gamma_2} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z. \tag{10}$$

Proposition

Suppose that $D \subset \mathbb{C}$ is an open, bounded set, and $f : D \to \mathbb{C}$ is a continuous function. Assume that $\gamma : [a, b] \to D$ is a smooth complex curve. Suppose that $\eta : [c, d] \to [a, b]$ is smooth, monotonically increasing function, with $\eta(c) = a$ and $\eta(d) = b$, and define $\chi(t) = \gamma(\eta(t))$. Then,

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\chi f(z)\,\mathrm{d}z.$$

Proof.

Let $\boldsymbol{\gamma}$ and $\boldsymbol{\chi}$ be the realifications of γ and χ, respectively. By definition, on one hand, we have

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b \mathbf{S}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,\mathrm{d}t + \mathrm{i}\int_a^b \mathbf{T}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,\mathrm{d}t.$$

Proof (Cont.)

On the other, we have

$$\begin{aligned}
\int_{\chi} f(z)\,\mathrm{d}z &= \int_c^d \mathbf{S}(\boldsymbol{\chi}(s)) \cdot \boldsymbol{\chi}'(s)\,\mathrm{d}s + \mathrm{i}\int_c^d \mathbf{T}(\boldsymbol{\chi}(s)) \cdot \boldsymbol{\chi}'(s)\,\mathrm{d}s \\
&= \int_c^d \mathbf{S}(\boldsymbol{\gamma}(\eta(s))) \cdot \boldsymbol{\gamma}'(\eta(s))\eta'(s)\,\mathrm{d}s \\
&\quad + \mathrm{i}\int_c^d \mathbf{T}(\boldsymbol{\gamma}(\eta(s))) \cdot \boldsymbol{\gamma}'(\eta(s))\eta'(s)\,\mathrm{d}s \\
&= \int_a^b \mathbf{S}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,\mathrm{d}t + \mathrm{i}\int_a^b \mathbf{T}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,\mathrm{d}t,
\end{aligned}$$

using the usual change of variables relations for standard (real) integrals. □

Proposition

Suppose that $D \subset \mathbb{C}$ is an open, bounded set, and $f : D \to \mathbb{C}$ is a continuous function. Assume that $\gamma : [a, b] \to D$ is a continuous, piecewise smooth complex curve. Suppose that there is some $M > 0$, such that $|f(z)| \leq M$, for all $z \in D$. Then,

$$\left| \int_\gamma f(z) \, \mathrm{d}z \right| \leq ML(\gamma),$$

where $L(\gamma)$ is the length of γ, that is,

$$L(\gamma) := \int_a^b \left| \gamma'(t) \right| \, \mathrm{d}t.$$

More generally,

$$\left| \int_\gamma f(z) \, \mathrm{d}z \right| \leq \int_a^b |f(\gamma(t))| \cdot \left| \gamma'(t) \right| \, \mathrm{d}t \leq ML(\gamma).$$

Proof.

See Marsden and Hoffman (1987). □

Analytic Functions and Harmonic Conjugates

Definition

Suppose that $D \subset \mathbb{C}$ is an open, bounded set and $f : D \to \mathbb{C}$. We say that f is **(complex) differentiable at** $z_0 \in D$ iff the following limit exists and is finite:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

If the limit exists and is finite, we call $f'(z_0)$ the **derivative of** f **at** z_0. We say that f is **complex analytic in** D iff f is complex differentiable at each point $z \in D$.

Limits are a bit different in the complex case, because, for example, in the above definition, h can approach 0 from any direction.

Theorem (Cauchy-Riemann Theorem)

Suppose that $D \subset \mathbb{C}$ *is an open, bounded set and* $f : D \to \mathbb{C}$*, with* $u = \Re(f)$ *and* $v = \Im(f)$*. Then,* $f'(z_0)$ *exists at* $z_0 = x_0 + \mathrm{i}y_0 \in D$ *iff* $u, v : D(\mathbb{R}^2) \to \mathbb{R}$ *are differentiable (in the real variables sense) at* $(x_0, y_0) \in D(\mathbb{R}^2)$ *and satisfy* **Cauchy-Riemann equations** *at* (x_0, y_0)*:*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \tag{11}$$

Furthermore, if $u, v \in C^1(D(\mathbb{R}^2); \mathbb{R})$ *and satisfy the Cauchy-Riemann equations on* $D(\mathbb{R}^2)$*, then* f *is complex analytic on* D *and*

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + \mathrm{i}\frac{\partial v}{\partial x}(x_0, y_0) \tag{12}$$

and

$$f'(z_0) = \frac{\partial v}{\partial y}(x_0, y_0) - \mathrm{i}\frac{\partial u}{\partial y}(x_0, y_0). \tag{13}$$

Proposition

Suppose that $D \subset \mathbb{C}$ is an open, bounded set and $f : D \to \mathbb{C}$, with $u = \Re(f)$ and $v = \Im(f)$. If $u, v \in C^2(D(\mathbb{R}^2); \mathbb{R})$, then u and v are harmonic on $D(\mathbb{R}^2)$, that is,

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0, \quad \forall (x, y) \in \tilde{D},$$

and

$$\frac{\partial^2 v}{\partial x^2}(x, y) + \frac{\partial^2 v}{\partial y^2}(x, y) = 0, \quad \forall (x, y) \in \tilde{D}.$$

Proof.

The Cauchy-Riemann Theorem guarantees that $u, v : D(\mathbb{R}^2) \to \mathbb{R}$ are differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}$ and satisfy the Cauchy-Riemann equations. If the regularity of u and v are improved to C^2, then we can go a little further.

Proof (Cont.)

In particular, recall from advanced calculus that, if $u, v \in C^2(D(\mathbb{R}^2); \mathbb{R})$, then

$$\frac{\partial^2 u}{\partial x \partial y}(x, y) = \frac{\partial^2 u}{\partial y \partial x}(x, y), \quad \forall\, (x, y) \in D(\mathbb{R}^2),$$

and

$$\frac{\partial^2 v}{\partial x \partial y}(x, y) = \frac{\partial^2 v}{\partial y \partial x}(x, y), \quad \forall\, (x, y) \in D(\mathbb{R}^2).$$

In other words, the order of partial differentiation is irrelevant. Combining these facts with the Cauchy-Riemann equations yields the result. The simple details are left to reader. $\square$

Definition

Suppose that $D \subset \mathbb{C}$ is an open, bounded subset of the complex plane and $D(\mathbb{R}^2)$ is its realification. Assume that $u, v \in C^1(D(\mathbb{R}^2); \mathbb{R})$ are given functions. If there is a function $f : D \to \mathbb{C}$ that is (i) analytic on D and (ii) has the property that $u = \Re(f)$ and $v = \Im(f)$, then the pair u and v are called **harmonic conjugates**.

Example

Let $D \subset \mathbb{C}$ be any bounded, open subset of the complex plane, and let, as usual, $D(\mathbb{R}^2)$ represent its realification. Define, for any $(x, y) \in D(\mathbb{R}^2)$,

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

Then, u and v are harmonic conjugates. To see this, consider $f(z) = z^2$. Then, writing $z = x + \mathrm{i}y$, as usual,

$$f(z) = z^2 = (x + \mathrm{i}y)(x + \mathrm{i}y) = x^2 - y^2 + 2\mathrm{i}xy.$$

See the figure two slides after this one. It only remains to show that this function f is analytic on any D. We can check this via the Cauchy-Riemann Theorem:

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y},$$

and

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

Example (Cont.)

What is the complex derivative of $f(z) = z^2$? We use the Cauchy-Riemann Theorem again to answer this question.

$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = 2x + \mathrm{i}2y = 2z.$$

Likewise,

$$f'(z) = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y} = 2x - \mathrm{i}(-2y) = 2z.$$

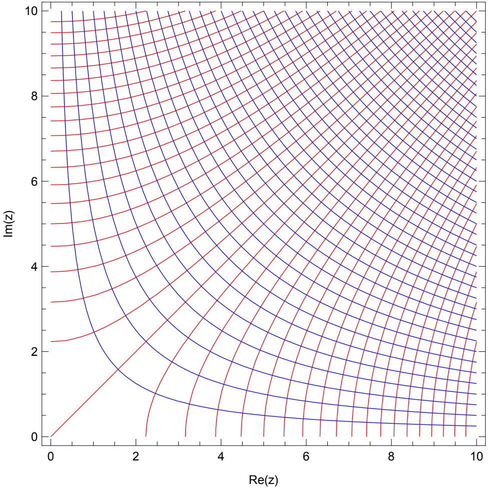

Figure: Contour plots of $\Re(z^2)$ (red) and $\Im(z^2)$ (blue), where $z = x + \mathrm{i}y$. Observe that the blue and red contours are orthogonal.

Definition

Suppose that $D \subset \mathbb{R}^2$ is an open, bounded set and $u : D \to \mathbb{R}$. For each $r \in \mathrm{range}(u)$, define the set

$$\mathcal{C}_{u,r} = \{(x, y) \in D \mid u(x, y) = r\} .$$

The set $\mathcal{C}_{u,r} \subset D$ is called a **level curve of** u or an **iso-contour of** u. We say that $\mathcal{C}_{u,r} \subset D$ **is of class** C^k iff there is a curve $\boldsymbol{\gamma} \in C^k([a, b]; \mathbb{R}^2)$ such that, for some interval $I \subseteq [a, b]$, $\boldsymbol{\gamma} : I \to \mathcal{C}_{u,r}$ and this mapping is one-to-one from I onto $\mathcal{C}_{u,r}$. We say that $\mathcal{C}_{u,r} \subset D$ **is smooth**, if the curve $\boldsymbol{\gamma}$ above is smooth.

Proposition

Suppose that $D \subset \mathbb{C}$ is an open, bounded set and $f : D \to \mathbb{C}$. Assume that f is analytic on D, with $u = \Re(f)$ and $v = \Im(f)$. Suppose that, for the point $(x_0, y_0) \in D(\mathbb{R}^2)$, $u(x_0, y_0) = r_0$ and $v(x_0, y_0) = s_0$. If the level curves $\mathcal{C}_{u,r_0}$ and $\mathcal{C}_{v,s_0}$ are smooth, then, they intersect orthogonally at the point (x_0, y_0).

Proof.

That the curves intersect is clear; indeed, both level curves contain the point (x_0, y_0). Since both curves are smooth, there are functions $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}$ and $\boldsymbol{\chi} : [c, d] \to \mathbb{R}$ and intervals $I \subseteq [a, b]$ and $J \subseteq [c, d]$, such that $\boldsymbol{\gamma}$ maps I one-to-one and onto $\mathcal{C}_{u,r_0}$, and $\boldsymbol{\chi}$ maps J one-to-one and onto $\mathcal{C}_{v,s_0}$. Furthermore, there are unique points $t_0 \in I$ and $t_1 \in J$, such that

$$\boldsymbol{\gamma}(t_0) = (x_0, y_0) = \boldsymbol{\chi}(t_1).$$

Now, since $u(\boldsymbol{\gamma}(t)) = r_0$, for all $t \in I$, by the chain rule,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[u(\boldsymbol{\gamma}(t))\right] = \nabla u(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t).$$

Proof (Cont.)

Likewise, since $v(\boldsymbol{\chi}(t)) = s_0$, for all $t \in J$, by the chain rule,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[v(\boldsymbol{\chi}(t))\right] = \nabla v(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\chi}'(t).$$

By the Cauchy-Riemann Theorem, it is easy to see that

$$\nabla u(x_0, y_0) \cdot \nabla v(x_0, y_0) = 0.$$

The only possibility, then, is that

$$\boldsymbol{\gamma}'(t_0) \cdot \boldsymbol{\chi}'(t_1) = 0.$$

To finish up, recall a fact from vector calculus that $\boldsymbol{\gamma}'(t_0)$ is tangent to the level curve $\mathcal{C}_{u,r_0}$ at (x_0, y_0) and, likewise, $\boldsymbol{\chi}'(t_1)$ is tangent to the level curve $\mathcal{C}_{v,s_0}$ at (x_0, y_0). The proof is complete. $\square$

The Fundamental Theorem of Calculus and Path Independence

Theorem (Fundamental Theorem for Contour Integrals)

Suppose that $D \subset \mathbb{R}^2$ is an open, bounded set, and $F \in C^1(D;\mathbb{R})$. Assume that $\boldsymbol{\gamma} : [a,b] \to D$ is a smooth planar curve in D. Then

$$\int_{\boldsymbol{\gamma}} \nabla F(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = F(\boldsymbol{\gamma}(b)) - F(\boldsymbol{\gamma}(b)).$$

Proof.

This is an easy application of the standard Fundamental Theorem of Calculus and the chain rule in two dimensions:

$$\begin{aligned}\int_{\boldsymbol{\gamma}} \nabla F(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} &= \int_a^b \nabla F(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, \mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left[F(\boldsymbol{\gamma}(t))\right] \mathrm{d}t \\ &= F(\boldsymbol{\gamma}(b)) - F(\boldsymbol{\gamma}(a)).\end{aligned}$$

$\square$

Theorem (Fundamental Theorem for Contour Integrals)

Suppose that $D \subset \mathbb{C}$ is an open, bounded set, and $F : D \to \mathbb{C}$ is analytic on D. Assume that $\gamma : [a, b] \to D$ is a smooth complex curve. Then,

$$\int_\gamma F'(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$

Proof.

We have

$$\int_\gamma F'(z)\,\mathrm{d}z = \int_a^b \mathbf{S}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,\mathrm{d}t + \mathrm{i}\int_a^b \mathbf{T}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,\mathrm{d}t,$$

where $\boldsymbol{\gamma} : [a, b] \to D(\mathbb{R}^2)$ is the realification of γ;

$$\mathbf{S}(x, y) := (U(x, y), -V(x, y)), \quad \mathbf{T}(x, y) := (V(x, y), U(x, y)),$$

for all $(x, y) \in D(\mathbb{R}^2)$; and $U = \Re(F')$ and $V = \Im(F')$.

Proof (Cont.)

Using the Cauchy-Riemann Theorem, specifically, Equations (12) and (13), we know that

$$U = \frac{\partial u}{\partial x} \quad \text{and} \quad V = -\frac{\partial u}{\partial y},$$

where $u = \Re(F)$. On the other hand, we also have

$$V = \frac{\partial v}{\partial x} \quad \text{and} \quad U = \frac{\partial v}{\partial y},$$

where $v = \Im(F)$. Therefore, we can write

$$\mathbf{S}(x, y) = \nabla u(x, y) \quad \text{and} \quad \mathbf{T}(x, y) = \nabla v(x, y),$$

and, as a consequence, using the chain rule,

$$\int_\gamma F'(z)\,\mathrm{d}z = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left[u(\boldsymbol{\gamma}(t))\right]\mathrm{d}t + \mathrm{i}\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left[v(\boldsymbol{\gamma}(t))\right]\mathrm{d}t.$$

By the standard Fundamental Theorem of Calculus, the result follows. $\square$

Corollary

Suppose that $D \subset \mathbb{C}$ is an open, bounded, path-connected set, and $F : D \to \mathbb{C}$ is a analytic on D. If $F'(z) = 0$ at every $z \in D$, then F is constant in D.

Proof.

Let $z_1, z_2 \in D$ be arbitrary. Suppose that $\gamma : [a, b] \to D$ satisfies $\gamma(a) = z_1$ and $\gamma(b) = z_2$. By the formula

$$0 = \int_\gamma F'(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)) = F(z_1) - F(z_2).$$

Hence, $F(z_1) = F(z_2)$, and the result follows. □

Theorem (Path Independence)

Suppose that $D \subset \mathbb{C}$ is an open, bounded, path-connected set, and $f : D \to \mathbb{C}$ is continuous on D. The following are equivalent:

1. *Integrals are path independent. Specifically, for any two points $z_1, z_2 \in D$, and any two contours $\gamma : [a, b] \to D$ and $\chi : [c, d] \to D$, with the property that $\gamma(a) = z_1 = \chi(c)$ and $\gamma(b) = z_2 = \chi(d)$, it holds that*

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\chi f(z)\,\mathrm{d}z.$$

2. *Integrals around closed contours are zero. In other words, if $\gamma : [a, b] \to D$ is any smooth, closed curve in D, then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

3. *There is an antiderivative for f in D. In particular, there is a function $F : D \to \mathbb{C}$ that is analytic on D and*

$$F'(z) = f(z), \quad \forall\, z \in \mathbb{C}.$$