Math 515
Essential Perturbation Theory and Asymptotic Analysis
Chapter 03

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Chapter 03, Part 1 of 2
Asymptotic Expansion of Integrals

In this and the next chapter, we will dive into the world of asymptotic expansions by looking at certain integrals with a large parameter.

Definition

Suppose that $D \subset \mathbb{C}$ is an open, bounded set, and $f, g : D \to \mathbb{C}$ are continuous functions. Assume that $\gamma : [a, b] \to D$ is a smooth complex curve, and $\lambda \in (0, \infty)$. The object

$$I(\lambda) = \int_\gamma f(z) e^{\lambda g(z)}\, \mathrm{d}z \tag{1}$$

is called a **general exponential contour integral**.

Integration By Parts

Example

Consider

$$I = \int_0^{\pi} \theta \cos(\theta)\, d\theta.$$

Set

$$v = \theta \quad \text{and} \quad du = \cos(\theta)\, d\theta.$$

Therefore,

$$dv = d\theta \quad \text{and} \quad u = \sin(\theta),$$

and

$$I = uv\Big|_{\theta=0}^{\theta=\pi} - \int_{\theta=0}^{\theta=\pi} u\, dv = \theta \sin(\theta)\Big|_{\theta=0}^{\theta=\pi} - \int_{\theta=0}^{\theta=\pi} \sin(\theta)\, d\theta = \cos(\theta)\Big|_{\theta=0}^{\theta=\pi}.$$

Thus

$$I = \int_0^{\pi} \theta \cos(\theta)\, d\theta = 2.$$

Let us recall a couple of definitions before we get started.

Definition (Asymptotic Approximation)

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f, g : D \times I_\delta(\varepsilon_o) \to \mathbb{C}$ are continuous functions. We say that g **is an asymptotic approximation of** f **at** $\mathbf{x} \in D$**, as** $\varepsilon \to \varepsilon_o$, and we write $f(\mathbf{x}, \varepsilon) \sim g(\mathbf{x}, \varepsilon)$, as $\varepsilon \to \varepsilon_o$, iff $f(\mathbf{x}, \varepsilon) - g(\mathbf{x}, \varepsilon) = o(g(\mathbf{x}, \varepsilon))$, as $\varepsilon \to \varepsilon_o$.

We immediately have the following: $f(\mathbf{x}, \varepsilon) \sim g(\mathbf{x}, \varepsilon)$, as $\varepsilon \to \varepsilon_o$, iff

$$\lim_{\varepsilon \to \varepsilon_o} \frac{f(\mathbf{x}, \varepsilon) - g(\mathbf{x}, \varepsilon)}{g(\mathbf{x}, \varepsilon)} = 0 \iff \lim_{\varepsilon \to \varepsilon_o} \frac{f(\mathbf{x}, \varepsilon)}{g(\mathbf{x}, \varepsilon)} = 1.$$

Definition (Asymptotic Series)

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, $\delta > 0$, and $n \in \mathbb{N}_0$. Assume that $f : D \times I_\delta(\varepsilon_o) \to \mathbb{C}$ is a continuous function, and $a_k : D \to \mathbb{C}$ is a continuous function, for each $k \in \{0, \ldots, n\}$. Suppose that $\{\phi_k\}_{k=0}^{\infty}$ is an asymptotic sequence, as $\varepsilon \to \varepsilon_o$, on $I_\delta(\varepsilon_o)$. We say that $\sum_{k=0}^{n} a_k(\cdot)\phi_k(\cdot)$ **is a finite asymptotic series approximation of f at $\mathbf{x} \in D$, as $\varepsilon \to \varepsilon_o$**, and we write

$$f(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{n} a_k(\mathbf{x})\phi_k(\varepsilon), \quad \text{as} \quad \varepsilon \to \varepsilon_o,$$

iff $f(\mathbf{x}, \varepsilon) - \sum_{k=0}^{m} a_k(\mathbf{x})\phi_k(\varepsilon) = o(\phi_m(\mathbf{x}, \varepsilon))$, as $\varepsilon \to \varepsilon_o$, for each $m \in \{0, \ldots, n\}$. We say that $\sum_{k=0}^{\infty} a_k(\cdot)\phi_k(\cdot)$ **is an infinite asymptotic series approximation of f at $\mathbf{x} \in D$, as $\varepsilon \to \varepsilon_o$**, and we write

$$f(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{\infty} a_k(\mathbf{x})\phi_k(\varepsilon), \quad \text{as} \quad \varepsilon \to \varepsilon_o,$$

iff $f(\mathbf{x}, \varepsilon) - \sum_{k=0}^{m} a_k(\mathbf{x})\phi_k(\varepsilon) = o(\phi_m(\mathbf{x}, \varepsilon))$, as $\varepsilon \to \varepsilon_o$, for each $m \in \mathbb{N}_0$.

Taylor's Theorem and Integration By Parts

Suppose that $f \in C^{n+1}([0, T]; \mathbb{R})$, for some $n \in \mathbb{N}_0$. By the Fundamental Theorem of Calculus, we have

$$f(s) = f(0) + \int_0^s f'(t)\,\mathrm{d}t.$$

Now, set

$$v = f'(t) \quad \text{and} \quad \mathrm{d}u = \mathrm{d}t.$$

Therefore,

$$\mathrm{d}v = f''(t)\,\mathrm{d}t \quad \text{and} \quad u = t - s,$$

and

$$\begin{aligned} f(s) &= f(0) + (t-s)f'(t)\Big|_{t=0}^{t=s} - \int_0^s (t-s)f''(t)\,\mathrm{d}t, \mathrm{d}t \\ &= f(0) + sf'(0) + \int_0^s (s-t)f''(t)\,\mathrm{d}t, \mathrm{d}t. \end{aligned}$$

Taylor's Theorem and Integration By Parts

Let us do one more step of integration by parts with

$$v = f''(t) \quad \text{and} \quad \mathrm{d}u = (s-t)\,\mathrm{d}t.$$

Then,

$$\mathrm{d}v = f'''(t)\,\mathrm{d}t \quad \text{and} \quad u = -\frac{(s-t)^2}{2},$$

and

$$\begin{aligned} f(s) &= f(0) + sf'(0) + \int_0^s (s-t)f''(t)\,\mathrm{d}t, \mathrm{d}t \\ &= f(0) + sf'(0) + \left(-\frac{(s-t)^2}{2}\right) f''(t)\Big|_{t=0}^{t=s} - \int_0^s \left(-\frac{(s-t)^2}{2}\right) f'''(t)\,\mathrm{d}t \\ &= f(0) + sf'(0) + \frac{s^2}{2} f''(0) + \int_0^s \frac{(s-t)^2}{2} f'''(t)\,\mathrm{d}t. \end{aligned}$$

Taylor's Theorem and Integration By Parts

Repeated application of integration by parts yields

$$f(s) = \sum_{k=0}^{n} \frac{s^k}{k!} f^{(k)}(0) + \frac{1}{n!} \int_0^s (s-t)^n f^{(n+1)}(t)\,\mathrm{d}t. \tag{2}$$

Congratulations! You have just proved one of the most important theorems in all of mathematics, with nothing more than integration by parts. It is powerful. In fact, it is not difficult to also prove the following.

Theorem

Suppose that $T > 0$ and $f \in C^\infty([0,T];\mathbb{R})$. Then,

$$f(\varepsilon) \sim \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} f^{(k)}(0), \quad \textit{as} \quad \varepsilon \searrow 0.$$

How is the statement of the previous theorem different from the statement that f is equal to its Taylor series expansion?

An Exponential Integral

We will see that the integration-by-parts methods works well for exponential integrals. Consider the following:

$$I(\lambda) := \lambda e^{\lambda} \int_{\lambda}^{\infty} \frac{e^{-t}}{t} \, \mathrm{d}t. \tag{3}$$

The well know exponential integral, E_1, is defined simply as

$$E_1(\lambda) = \int_{\lambda}^{\infty} \frac{e^{-t}}{t} \, \mathrm{d}t = \frac{e^{-\lambda}}{\lambda} I(\lambda).$$

We again seek an asymptotic expansion for $I(\lambda)$, or, equivalently, $E_1(\lambda)$ for large real values of λ, and we will use the method of integration by parts to generate our desired expansion.

Proposition

$$I(\lambda) := \lambda e^{\lambda} \int_{\lambda}^{\infty} \frac{e^{-t}}{t} \, \mathrm{d}t \sim \sum_{k=0}^{\infty} \frac{(-1)^k k!}{\lambda^k}, \quad as \quad \lambda \to \infty.$$

Proof.

By repeated integration by parts and induction, we find the exact expression

$$I(\lambda) = S_{N-1}(\lambda) + R_N(\lambda),$$

where

$$S_{N-1}(\lambda) = \sum_{k=0}^{N-1} \frac{(-1)^k k!}{\lambda^k} \quad \text{and} \quad R_N(\lambda) = (-1)^N N! \, \lambda e^{\lambda} \int_{\lambda}^{\infty} \frac{e^{-t}}{t^{N+1}} \, \mathrm{d}t.$$

Proof (Cont.)

Let us do the base case to see how the process goes. Consider the integral

$$E_1(\lambda) = \int_\lambda^\infty \frac{e^{-t}}{t}\,\mathrm{d}t,$$

and set

$$v = \frac{-1}{t} \quad \text{and} \quad \mathrm{d}u = -e^{-t}\,\mathrm{d}t.$$

Then

$$\mathrm{d}v = \frac{1}{t^2}\,\mathrm{d}t \quad \text{and} \quad u = e^{-t},$$

and

$$\begin{aligned} E_1(\lambda) &= \left.\frac{-1}{t}e^{-t}\right|_{t=\lambda}^{t=\infty} - \int_\lambda^\infty e^{-t}\frac{1}{t^2}\,\mathrm{d}t \\ &= \frac{1}{\lambda}e^{-\lambda} - \int_\lambda^\infty e^{-t}\frac{1}{t^2}\,\mathrm{d}t. \end{aligned}$$

This leads to the first term in the asymptotic expansion. The remaining details of the induction procedure are left to the reader.

Proof (Cont.)

Clearly the sequence of functions $\{\lambda^{-k}\}_{k=0}^{\infty}$ is an asymptotic sequence as $\lambda \to \infty$. We now only have to check that

$$R_N(\lambda) = o\left(\frac{1}{\lambda^{N-1}}\right), \quad \text{as} \quad \lambda \to \infty \quad \Longleftrightarrow \quad \lim_{\lambda\to\infty} \lambda^{N-1} R_N(\lambda) = 0.$$

To prove this, observe that

$$|R_N(\lambda)| = N!\,\lambda e^{\lambda} \int_{\lambda}^{\infty} \frac{e^{-t}}{t^{N+1}} \mathrm{d}t \leq N!\,\lambda e^{\lambda} \frac{e^{-\lambda}}{\lambda^{N+1}} = \frac{N!}{\lambda^N},$$

where we use the fact that $\frac{1}{t^{N+1}} \leq \frac{1}{\lambda^{N+1}}$, for all $t \geq \lambda$. Thus

$$\lim_{\lambda\to\infty} \lambda^{N-1} R_N(\lambda) = 0,$$

and we can conclude the result. $\square$

The Error Function

The error function can be defined as

$$\mathrm{Erf}(\lambda) = \frac{2}{\sqrt{\pi}} \int_0^{\lambda} e^{-t^2}\, \mathrm{d}t, \quad \lambda \in [0, \infty). \tag{4}$$

Since e^{-z^2} is complex analytic in the whole complex plane $\mathbb{C}$ it can be represented by its Taylor series at every point, that is,

$$e^{-z^2} = \sum_{k=0}^{\infty} \frac{(-z^2)^k}{k!},$$

and the radius of convergence is infinite. We can integrate this power series term-by-term to obtain

$$\mathrm{Erf}(\lambda) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k+1}}{(2k+1)k!},$$

which is valid for all $\lambda \in [0, \infty)$. The problem with this series is that it converges relatively slowly and many terms are needed to get good accuracy.

The Error Function

Observe that

$$\mathrm{Erf}(\lambda) = 1 - \frac{2}{\sqrt{\pi}} \int_{\lambda}^{\infty} e^{-t^2} \, \mathrm{d}t, \tag{5}$$

since

$$\int_0^{\infty} e^{-t^2} \, \mathrm{d}t = \frac{\sqrt{\pi}}{2}.$$

Let us focus on the following integral:

$$I(\lambda) := \int_{\lambda}^{\infty} e^{-t^2} \, \mathrm{d}t.$$

We want an asymptotic approximation for this integral as $\lambda \to \infty$. With this, we can understand the behavior of the error function for large values of λ.

One Step of Integration By Parts

Write

$$I(\lambda) := \int_{\lambda}^{\infty} e^{-t^2}\,\mathrm{d}t = \int_{\lambda}^{\infty} \frac{1}{-2t}(-2t)e^{-t^2}\,\mathrm{d}t.$$

Let us perform one iteration of integration by parts with the variables

$$v = \frac{1}{-2t} \quad \text{and} \quad \mathrm{d}u = (-2t)e^{-t^2}\,\mathrm{d}t.$$

Thus,

$$\mathrm{d}v = \frac{1}{2t^2}\,\mathrm{d}t \quad \text{and} \quad u = e^{-t^2},$$

and

$$\begin{aligned} I(\lambda) &= \left.\frac{1}{-2t}e^{-t^2}\right|_{t=\lambda}^{t=\infty} - \int_{\lambda}^{\infty} \frac{1}{2t^2}e^{-t^2}\,\mathrm{d}t \\ &= \frac{1}{2\lambda}e^{-\lambda^2} + \int_{\lambda}^{\infty} \frac{1}{4t^3}(-2t)e^{-t^2}\,\mathrm{d}t. \end{aligned}$$

A Second Integration By Parts

Now, perform one more iteration of integration by parts with the variables

$$v = \frac{1}{4t^3} \quad \text{and} \quad \mathrm{d}u = (-2t)e^{-t^2}\,\mathrm{d}t.$$

Thus,

$$\mathrm{d}v = -\frac{3}{4t^4}\,\mathrm{d}t \quad \text{and} \quad u = e^{-t^2},$$

and

$$\begin{aligned} I(\lambda) &= \frac{1}{2\lambda}e^{-\lambda^2} + \int_\lambda^\infty \frac{1}{4t^3}(-2t)e^{-t^2}\,\mathrm{d}t \\ &= \frac{1}{2\lambda}e^{-\lambda^2} + \frac{1}{4t^3}e^{-t^2}\Big|_{t=\lambda}^{t=\infty} + \int_\lambda^\infty \frac{3}{4t^4}e^{-t^2}\,\mathrm{d}t \\ &= \frac{1}{2\lambda}e^{-\lambda^2} - \frac{1}{4\lambda^3}e^{-\lambda^2} + \int_\lambda^\infty \frac{3}{-8t^5}(-2t)e^{-t^2}\,\mathrm{d}t. \end{aligned}$$

A Third Integration By Parts

Next, let us use

$$v = -\frac{3}{8t^5} \quad \text{and} \quad \mathrm{d}u = (-2t)e^{-t^2}\,\mathrm{d}t.$$

Thus,

$$\mathrm{d}v = \frac{1\cdot 3\cdot 5}{8t^6}\,\mathrm{d}t \quad \text{and} \quad u = e^{-t^2},$$

and

$$\begin{aligned} I(\lambda) &= \frac{1}{2\lambda}e^{-\lambda^2} - \frac{1}{4\lambda^3}e^{-\lambda^2} + \int_\lambda^\infty \frac{3}{-8t^5}(-2t)e^{-t^2}\,\mathrm{d}t \\ &= \frac{1}{2\lambda}e^{-\lambda^2} - \frac{1}{4\lambda^3}e^{-\lambda^2} + \frac{1\cdot 3}{8\lambda^5}e^{-\lambda^2} - \int_\lambda^\infty \frac{1\cdot 3\cdot 5}{8t^6}e^{-t^2}\,\mathrm{d}t \\ &= \frac{1}{2\lambda}e^{-\lambda^2} - \frac{1}{4\lambda^3}e^{-\lambda^2} + \frac{1\cdot 3}{8\lambda^5}e^{-\lambda^2} + \int_\lambda^\infty \frac{1\cdot 3\cdot 5}{16t^7}(-2t)e^{-t^2}\,\mathrm{d}t. \end{aligned}$$

Estimating the Remainder Term R_4

Continuing in this fashion, we get

$$I(\lambda) = \frac{e^{-\lambda^2}}{2\lambda}\left(1 - \frac{1}{2\lambda^2} + \frac{1\cdot 3}{(2\lambda^2)^2} - \frac{1\cdot 3\cdot 5}{(2\lambda^2)^3}\right) + R_4,$$

where

$$R_4(\lambda) = -\int_\lambda^\infty \frac{1\cdot 3\cdot 5\cdot 7}{32t^9}(-2t)e^{-t^2}\,\mathrm{d}t.$$

Clearly,

$$|R_4(\lambda)| \le \frac{1\cdot 3\cdot 5\cdot 7}{32\lambda^9}\int_\lambda^\infty 2te^{-t^2}\,\mathrm{d}t = \frac{1\cdot 3\cdot 5\cdot 7}{32\lambda^9}e^{-\lambda^2},$$

since $\frac{1}{t^9} \le \frac{1}{\lambda^9}$, for all $t \ge \lambda$.

Estimating the Remainder Term R_4

Let us now show that we have generated an asymptotic expansion. To do this, we will show that

$$R_4(\lambda) = o\left(\frac{e^{-\lambda^2}}{\lambda^7}\right), \quad \text{as} \quad \lambda \to \infty.$$

In other words, we want to show that

$$\lim_{\lambda\to\infty} \frac{R_4(\lambda)}{\frac{e^{-\lambda^2}}{\lambda^7}} = 0,$$

but this is clearly true, since

$$|R_4(\lambda)| \le C\frac{e^{-\lambda^2}}{\lambda^9}.$$

Four- and Five-Term Asymptotic Expansions

Thus we have

$$I(\lambda) - \frac{e^{-\lambda^2}}{2\lambda}\left(1 - \frac{1}{2\lambda^2} + \frac{1\cdot 3}{(2\lambda^2)^2} - \frac{1\cdot 3\cdot 5}{(2\lambda^2)^3}\right) = o\left(\frac{e^{-\lambda^2}}{\lambda^7}\right), \quad \text{as} \quad \lambda \to \infty,$$

or, in other words,

$$I(\lambda) \sim \frac{e^{-\lambda^2}}{2\lambda}\left(1 - \frac{1}{2\lambda^2} + \frac{1\cdot 3}{(2\lambda^2)^2} - \frac{1\cdot 3\cdot 5}{(2\lambda^2)^3}\right), \quad \text{as} \quad \lambda \to \infty.$$

We can keep going:

$$I(\lambda) - \frac{e^{-\lambda^2}}{2\lambda}\left(1 - \frac{1}{2\lambda^2} + \frac{1\cdot 3}{(2\lambda^2)^2} - \frac{1\cdot 3\cdot 5}{(2\lambda^2)^3} + \frac{1\cdot 3\cdot 5\cdot 7}{(2\lambda^2)^4}\right)$$
$$= o\left(\frac{e^{-\lambda^2}}{\lambda^9}\right), \quad \text{as} \quad \lambda \to \infty.$$

Proposition

Consider the integral

$$I(\lambda) = \int_\lambda^\infty e^{-t^2}\,\mathrm{d}t,$$

for $\lambda > 0$. *Then*

$$I(\lambda) \sim \frac{e^{-\lambda^2}}{2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k b_k}{(2\lambda^2)^k}, \quad \textit{as} \quad \lambda \to \infty,$$

where

$$b_0 = 1 \quad \textit{and} \quad b_k = \prod_{\ell=1}^{k} (2\ell - 1), \quad k \in \mathbb{N}.$$

In other words, as $\lambda \to \infty$,

$$I(\lambda) \sim \frac{e^{-\lambda^2}}{2\lambda} \left(1 - \frac{1}{2\lambda^2} + \frac{1 \cdot 3}{(2\lambda^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2\lambda^2)^3} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{(2\lambda^2)^4} + \cdots \right).$$

Proposition (Cont.)

Since

$$I(\lambda) \sim \frac{e^{-\lambda^2}}{2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k b_k}{(2\lambda^2)^k}, \quad \text{as} \quad \lambda \to \infty,$$

we, therefore, have

$$\mathrm{Erf}(\lambda) \sim 1 - \frac{e^{-\lambda^2}}{\sqrt{\pi}\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k b_k}{(2\lambda^2)^k}, \quad \text{as} \quad \lambda \to \infty.$$

Proof.

By integration by parts and induction, we can show that, for each $m \in \mathbb{N}_0$,

$$I(\lambda) = \frac{e^{-\lambda^2}}{2\lambda} \sum_{k=0}^{m} \frac{(-1)^k b_k}{(2\lambda^2)^k} + R_{m+1}(\lambda),$$

where

$$R_{m+1}(\lambda) = (-1)^m \int_{\lambda}^{\infty} \frac{b_{m+1}}{2t(2t^2)^{m+1}} (-2t) e^{-t^2} \, dt.$$

Proof. (Cont.)

Estimating the remainder,

$$R_{m+1}(\lambda) = (-1)^m \int_\lambda^\infty \frac{b_{m+1}}{2t(2t^2)^{m+1}}(-2t)e^{-t^2}\,\mathrm{d}t,$$

we have

$$|R_{m+1}(\lambda)| \le \frac{C_{m+1}}{\lambda^{2m+3}} e^{-\lambda^2},$$

from which it follows that

$$R_{m+1}(\lambda) = o\left(\frac{e^{-\lambda^2}}{\lambda^{2m+1}}\right), \quad \text{as} \quad \lambda \to \infty.$$

The remaining details are left to the reader. □

Yet Another Exponential Integral

Consider the integral

$$I(\lambda) = \int_{\lambda}^{\infty} e^{-t^4} \, \mathrm{d}t. \tag{6}$$

We seek an approximation of this integral for large values of λ, or, in the language of asymptotic analysis, as $\lambda \to \infty$. For small positive values of λ, we could proceed as follows: rewriting the integral and using the Taylor expansion for the exponential function, we get

$$\begin{aligned} I(\lambda) &= \int_0^{\infty} e^{-t^4} \, \mathrm{d}t - \int_0^{\lambda} e^{-t^4} \, \mathrm{d}t \\ &= \Gamma\left(\frac{5}{4}\right) - \int_0^{\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k t^{4k}}{k!} \, \mathrm{d}t \\ &= \Gamma\left(\frac{5}{4}\right) - \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{4k+1}}{(4k+1)k!}, \end{aligned}$$

where Γ is the Gamma function, which is defined later in the slides. This series converges for all λ. But, it is the same story as before: slow convergence.

One Step of IBP

We will apply the method of integration by parts to get a better approximation for large λ. As usual, write

$$I(\lambda) = \int_\lambda^\infty \frac{1}{-4t^3}(-4t^3)e^{-t^4}\,\mathrm{d}t = -\frac{1}{4}\int_\lambda^\infty \frac{1}{t^3}(-4t^3)e^{-t^4}\,\mathrm{d}t.$$

Now, set

$$v = \frac{1}{t^3} \quad \text{and} \quad \mathrm{d}u = (-4t^3)e^{-t^4}\,\mathrm{d}t.$$

Then,

$$\mathrm{d}v = -\frac{3}{t^4}\,\mathrm{d}t \quad \text{and} \quad u = e^{-t^4},$$

and

$$\begin{aligned} I(\lambda) &= -\frac{1}{4}\,\frac{1}{t^3}e^{-t^4}\Big|_{t=\lambda}^{t=\infty} - \frac{1}{4}\int_\lambda^\infty \frac{3}{t^4}e^{-t^4}\,\mathrm{d}t \\ &= \frac{1}{4}\frac{e^{-\lambda^4}}{\lambda^3} - \frac{3}{4}\int_\lambda^\infty \frac{1}{t^4}e^{-t^4}\,\mathrm{d}t. \end{aligned}$$

A Leading-Order Asymptotic Approximation

From

$$I(\lambda) = \frac{1}{4}\frac{e^{-\lambda^4}}{\lambda^3} - \frac{3}{4}\int_\lambda^\infty \frac{1}{t^4} e^{-t^4}\, \mathrm{d}t$$

we can easily show that

$$I(\lambda) \sim \frac{1}{4}\frac{e^{-\lambda^4}}{\lambda^3}, \quad \text{as} \quad \lambda \to \infty,$$

because

$$\left| \int_\lambda^\infty \frac{1}{t^4} e^{-t^4}\, \mathrm{d}t \right| \le C\frac{e^{-\lambda^4}}{\lambda^7}.$$

But as usual, we can do better.

Proposition

Consider, for any $\lambda > 0$, the integral

$$I(\lambda) = \int_{\lambda}^{\infty} e^{-t^4}\, \mathrm{d}t.$$

Then,

$$I(\lambda) \sim \frac{e^{-\lambda^4}}{4\lambda^3} \sum_{k=0}^{\infty} (-1)^k \frac{b_k}{(4\lambda^4)^k},$$

where

$$b_0 = 1 \quad \text{and} \quad b_k = \prod_{\ell=1}^{k} (4\ell - 1), \quad k \in \mathbb{N}.$$

Proof.

The proof is an exercise. □

Laplace Integrals and Watson's Lemma

Simple Laplace Integrals

Next, we want to investigate exponential integrals of the following form.

Definition

Suppose that $\lambda \in (0,\infty)$, $T \in (0,\infty]$, and $f : [0,T] \to \mathbb{R}$ is continuous. The integral

$$I(\lambda) = \int_0^T e^{-\lambda t} f(t)\,\mathrm{d}t \tag{7}$$

is called a **simple Laplace integral of the first kind**. The integral

$$I(\lambda) = \int_0^T e^{-\lambda t^2} f(t)\,\mathrm{d}t \tag{8}$$

is called a **simple Laplace integral of the second kind**.

Laplace's Method and Watson's Lemmas

In particular, we are interested in an asymptotic expansions for the case that $\lambda \to \infty$. To obtain such expansions, we will use Laplace's method, which is rigorously justified by (i) Watson's Lemma (for simple Laplace integrals of the first kind), and (ii) a modified version of Watson's Lemma (for simple Laplace integrals of the second kind).

In this section, we will state and prove Watson's Lemma for Laplace integrals of the first kind and then summarize the steps for approximating integrals suggested by the result. But before get to the theorem, we need a definition and a technical lemma.

The Gamma Function

Definition

For any $z \in \mathbb{C}$, with $\Re(z) > 0$, define

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,\mathrm{d}t. \tag{9}$$

This function is called the **Gamma function**.

Proposition

The Gamma function, defined in (9)*, satisfies*

$$\Gamma(z+1) = z\Gamma(z),$$

for any $z \in \mathbb{C}$*, with* $\Re(z) > 0$*. In particular,*

$$\Gamma(n) = (n-1)!, \quad \forall\, n \in \mathbb{N}.$$

Proof.

This follows from integration by parts, and the proof is left for an exercise. □

Gaussian Integrals and the Gamma Function

We often see other exponential integrals written in terms of the Gamma function.

Example

Consider the Gaussian integral

$$I_r = \int_0^\infty e^{-t^2} t^r \, \mathrm{d}t, \quad r > -1.$$

Then, using the change of variables

$$s = t^2 \quad \Longrightarrow \quad dt = \frac{1}{2} s^{-1/2} \, \mathrm{d}s,$$

we have, for $r > -1$,

$$I_r = \frac{1}{2} \int_0^\infty e^{-s} s^{r/2-1/2} \, \mathrm{d}s = \frac{1}{2} \int_0^\infty e^{-s} s^{(r/2+1/2)-1} \, \mathrm{d}s = \frac{1}{2} \Gamma\left(\frac{r}{2} + \frac{1}{2}\right).$$

Lemma

Fix $s \in (0,\infty)$ and $q \in (-1,\infty)$. Define, for any $\lambda \in (0,\infty)$,

$$I_{q,s}(\lambda) := \int_0^s e^{-\lambda t} t^q \, \mathrm{d}t. \tag{10}$$

If $\lambda \geq 1$, then,

$$I_{q,s}(\lambda) - \frac{\Gamma(q+1)}{\lambda^{q+1}} \sim o\left(\frac{1}{\lambda^r}\right), \quad \text{as} \quad \lambda \to \infty,$$

for any $r \geq 0$. In other words,

$$I_{q,s}(\lambda) - \lambda^{-(q+1)}\Gamma(q+1) = \mathrm{TST}, \quad \text{as} \quad \lambda \to \infty. \tag{11}$$

Proof.

Observe that

$$\begin{aligned} I_{q,s}(\lambda) &= \int_0^\infty e^{-\lambda t} t^q \,\mathrm{d}t - \int_s^\infty e^{-\lambda t} t^q \,\mathrm{d}t \\ &= \lambda^{-(q+1)} \int_0^\infty e^{-\tau} \tau^q \,\mathrm{d}\tau - \int_s^\infty e^{-\lambda t} t^q \,\mathrm{d}t \\ &= \lambda^{-(q+1)} \Gamma(q+1) - \int_s^\infty e^{-\lambda t} t^q \,\mathrm{d}t, \end{aligned}$$

where we used the change of variable $\tau = \lambda t$, in the second line.

Proof. (Cont.)

Using the Cauchy-Schwartz inequality, and assuming that $\lambda \geq 1$,

$$
\begin{aligned}
\int_s^\infty e^{-\lambda t} t^q \, \mathrm{d}t &= \int_s^\infty e^{-\lambda t/2} e^{-\lambda t/2} t^q \, \mathrm{d}t \\
&\leq \sqrt{\int_s^\infty e^{-\lambda t} \, \mathrm{d}t} \sqrt{\int_s^\infty e^{-\lambda t} t^{2q} \, \mathrm{d}t} \\
&\leq e^{-\lambda s/2} \sqrt{\int_s^\infty e^{-\lambda t} t^{2q} \, \mathrm{d}t} \\
&\leq e^{-\lambda s/2} \sqrt{\int_s^\infty e^{-t} t^{2q} \, \mathrm{d}t} \\
&= C(s) e^{-\lambda s/2}.
\end{aligned}
$$

Proof. (Cont.)

Since

$$\lim_{\lambda\to\infty} \frac{e^{-\lambda s/2}}{\frac{1}{\lambda^r}} = \lim_{\lambda\to\infty} \lambda^r e^{-\lambda s/2} = 0,$$

for any $r \geq 0$,

$$\int_s^\infty e^{-\lambda t} t^q \, \mathrm{d}t = o\left(\frac{1}{\lambda^r}\right), \quad \text{as} \quad \lambda \to \infty,$$

and the result follows from the fact that

$$I_{q,s}(\lambda) - \lambda^{-(q+1)} \Gamma(q+1) = -\int_s^\infty e^{-\lambda t} t^q \, \mathrm{d}t.$$

$\square$

Theorem (Watson's Lemma)

Suppose that $T > 0$ *and* $f \in L^1(0, T; \mathbb{C})$, *that is,*

$$\int_0^T |f(t)|\,\mathrm{d}t < \infty.$$

Assume that $g \in C^\infty([0, s], \mathbb{C})$, *for some* $s \in (0, T]$, *and* $f(t) = t^\sigma g(t)$, *where* $\sigma > -1$, *for all* $t \in (0, s]$. *Then* (7) *is finite, for all* $\lambda \geq 0$, *and*

$$I(\lambda) = \int_0^T e^{-\lambda t} f(t)\,\mathrm{d}t \sim \sum_{k=0}^{\infty} \frac{g^{(k)}(0)\,\Gamma(\sigma + k + 1)}{k!\,\lambda^{\sigma+k+1}}, \quad \text{as} \quad \lambda \to \infty. \tag{12}$$

Proof of Watson's Lemma.

Our proof follows rather closely that found in Miller (2006). First write

$$I(\lambda) = \int_0^s e^{-\lambda t} f(t)\,\mathrm{d}t + \int_s^T e^{-\lambda t} f(t)\,\mathrm{d}t. \tag{13}$$

Observe that

$$\left|\int_s^T e^{-\lambda t} f(t)\,\mathrm{d}t\right| \le \int_s^T e^{-\lambda t}\,|f(t)|\,\mathrm{d}t \le e^{-\lambda s}\int_0^T |f(t)|\,\mathrm{d}t.$$

Now, it is clear that, since $\lim_{\lambda\to\infty} \lambda^r e^{-\lambda s} = 0$, for any $r \ge 0$,

$$\int_s^T e^{-\lambda t} f(t)\,\mathrm{d}t = o\left(\frac{1}{\lambda^r}\right), \quad \text{as} \quad \lambda \to \infty.$$

In other words,

$$\int_s^T e^{-\lambda t} f(t)\,\mathrm{d}t = \mathrm{TST}, \quad \text{as} \quad \lambda \to \infty. \tag{14}$$

Proof. (Cont.)

By Taylor's Theorem, since $g \in C^\infty([0,s],\mathbb{C})$, we certainly have, for any $n \in \mathbb{N}$,

$$g(t) = \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} t^k + R_{n+1}(t),$$

where, for some $\xi_n(t) \in (0,t)$,

$$R_{n+1}(t) = \frac{g^{(n+1)}(\xi_n(t))}{(n+1)!} t^{n+1}.$$

We have the following estimate:

$$|R_{n+1}(t)| \le \max_{0\le\tau\le s} \left| g^{(n+1)}(\tau) \right| \frac{t^{n+1}}{(n+1)!}, \quad \forall\, t \in [0,s].$$

Proof. (Cont.)

It follows that

$$\int_0^s e^{-\lambda t} t^\sigma g(t)\,\mathrm{d}t = \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} \int_0^s e^{-\lambda t} t^{\sigma+k}\,\mathrm{d}t + \int_0^s e^{-\lambda t} t^\sigma R_{n+1}(t)\,\mathrm{d}t$$
$$= \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} I_{\sigma+k,s}(\lambda) + \int_0^s e^{-\lambda t} t^\sigma R_{n+1}(t)\,\mathrm{d}t,$$

and

$$\left|\int_0^s e^{-\lambda t} t^\sigma R_{n+1}(t)\,\mathrm{d}t\right| \le \int_0^s e^{-\lambda t} t^\sigma |R_{n+1}(t)|\,\mathrm{d}t$$
$$\le \max_{0\le\tau\le s} \left|g^{(n+1)}(\tau)\right| \frac{1}{(n+1)!} \int_0^s e^{-\lambda t} t^{\sigma+n+1}\,\mathrm{d}t$$
$$\le \max_{0\le\tau\le s} \left|g^{(n+1)}(\tau)\right| \frac{1}{(n+1)!} I_{\sigma+n+1,s}(\lambda), \tag{15}$$

where we have used the definition in (10) (in the technical lemma).

Proof. (Cont.)

Next, define

$$E_{1,q}(\lambda) := I_{q,s}(\lambda) - \lambda^{-(q+1)}\Gamma(q+1) = \mathrm{TST}, \quad \text{as} \quad \lambda \to \infty,$$

where we have used (11). Then,

$$\begin{aligned}
\int_0^s e^{-\lambda t} t^\sigma g(t)\,\mathrm{d}t - \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!\lambda^{\sigma+k+1}}\Gamma(\sigma+k+1) \\
&= \underbrace{\sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!\lambda^{\sigma+k+1}} E_{1,\sigma+k}(\lambda)}_{=:E_{2,1}(\lambda)} + \underbrace{\int_0^s e^{-\lambda t} t^\sigma R_{n+1}(t)\,\mathrm{d}t}_{=:E_{2,2}(\lambda)} \\
&=: E_2(\lambda).
\end{aligned}$$

Proof. (Cont.)

We want to show that

$$E_2(\lambda) = o\left(\frac{1}{\lambda^{\sigma+n+1}}\right), \quad \text{as} \quad \lambda \to \infty \quad \iff \quad \lim_{\lambda\to\infty} E_2(\lambda)\lambda^{\sigma+n+1} = 0,$$

so that we can conclude

$$\int_0^s e^{-\lambda t} t^\sigma g(t)\,\mathrm{d}t \sim \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!\lambda^{\sigma+k+1}}\Gamma(\sigma+k+1), \quad \text{as} \quad \lambda \to \infty. \tag{16}$$

Since $E_{1,\sigma+k}(\lambda) = \mathrm{TST}$, as $\lambda \to \infty$, clearly,

$$\lim_{\lambda\to\infty} E_{2,1}(\lambda)\lambda^{\sigma+n+1} = 0,$$

and we only need to check $E_{2,2}(\lambda)$. Notice that, using (15) and (11), it follows that

$$\lim_{\lambda\to\infty} E_{2,2}(\lambda)\lambda^{\sigma+n+1} = 0,$$

and we can conclude (16).

Proof. (Cont.)

From (13) and (14) we have

$$I(\lambda) - \int_0^s e^{-\lambda t} f(t)\,\mathrm{d}t = \mathrm{TST}, \quad \text{as} \quad \lambda \to \infty. \tag{17}$$

Recall (16):

$$\int_0^s e^{-\lambda t} t^\sigma g(t)\,\mathrm{d}t \sim \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!\lambda^{\sigma+k+1}} \Gamma(\sigma+k+1), \quad \text{as} \quad \lambda \to \infty.$$

Thus, we can conclude that (12) holds:

$$I(\lambda) = \int_0^T e^{-\lambda t} f(t)\,\mathrm{d}t \sim \sum_{k=0}^{\infty} \frac{g^{(k)}(0)\,\Gamma(\sigma+k+1)}{k!\,\lambda^{\sigma+k+1}}, \quad \text{as} \quad \lambda \to \infty.$$

□

Corollary

Suppose that $T > 0$ *and* $f \in C([0, T]; \mathbb{C})$, *and*

$$f(t) \sim t^{\sigma} \sum_{k=0}^{\infty} a_n t^{\beta n}, \quad \text{as} \quad t \searrow 0,$$

for $\sigma > 1$, *and* $\eta > 0$. *Then the integral*

$$I(\lambda) := \int_0^T e^{-\lambda t} f(t)\, \mathrm{d}t \tag{18}$$

exists and is finite, for all $\lambda \geq 0$, *and*

$$I(\lambda) \sim \sum_{k=0}^{\infty} \frac{a_k \Gamma(\sigma + \beta k + 1)}{\lambda^{\sigma+\beta k+1}}, \quad \text{as} \quad \lambda \to \infty. \tag{19}$$

Example

Let us apply Watson's Lemma to approximate

$$I(\lambda) = \int_0^{10} \frac{e^{-\lambda t}}{1+t} \, \mathrm{d}t, \tag{20}$$

for large values of $\lambda \in (0, \infty)$. This integral is like one from Chapter 1. The only difference is that the upper limit in the later is infinite. In any case, Watson's Lemma can be used to approximate both integrals without difficulty. For any $0 < t \leq s < 1$, the series

$$\frac{1}{1+t} = \sum_{k=0}^{\infty} (-1)^k t^k$$

converges. In the context of Watson's Lemma, $\sigma = 0$. Therefore,

$$I(\lambda) \sim \sum_{k=0}^{\infty} (-1)^k \frac{k!}{\lambda^{k+1}}, \quad \text{as} \quad \lambda \to \infty,$$

since $\Gamma(k+1) = k!$. This is the result that we computed in Chapter 1.

Example

Consider the integral

$$I(\lambda) = \int_0^5 \frac{e^{-\lambda t}}{1+t^2}\,\mathrm{d}t.$$

The Taylor series for $(1+t^2)^{-1}$ at $t = 0$ is

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots,$$

which converges for $t < 1$. By Watson's Lemma, we can integrate term by term to obtain

$$I(\lambda) \sim \frac{1}{\lambda} - \frac{2!}{\lambda^3} + \frac{4!}{\lambda^5} - \frac{6!}{\lambda^7} + \cdots, \quad \text{as} \quad \lambda \to \infty.$$

Laplace Integrals of the Second Kind

Definition

Suppose that $\lambda \in (0, \infty)$, $f \in C^0([a, b]; \mathbb{R})$, and $\phi \in C^2([a, b]; \mathbb{R})$. An integral of the form

$$\int_a^b f(t) e^{\lambda \phi(t)} \, \mathrm{d}t \tag{21}$$

is called a **Laplace integral of the second kind** iff there is a single, isolated point $t_o \in [a, b]$, such that $\phi'(t_o) = 0$, but $\phi''(t_o) < 0$. The point t_o is called the **Laplace max point**.

Goals

In this section we are interested in approximating integrals of the form

$$\int_a^b f(t)e^{\lambda\phi(t)}\,\mathrm{d}t$$

for large values of the parameter λ.

In particular, we will look for points in $t_o \in [a, b]$ where $\phi'(t_o) = 0$ and $\phi''(t_0) < 0$. In the neighborhood of t_o, the exponential part of the integral will be well approximated by a Gaussian that is becoming more and more sharply peaked as $\lambda \to \infty$. Away from the peak, as $\lambda \to \infty$, we expect only transcendentally small terms to contribute. We will utilize some techniques like those used in the proof of Watson's Lemma.

Example

Let us start with a simple example. Consider the integral

$$I(\lambda) = \int_{-10}^{10} e^{-\lambda t^2}\, \mathrm{d}t,$$

for large values of $\lambda \in (0, \infty)$. This is similar to another integral, namely the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\lambda t^2}\, \mathrm{d}t = \sqrt{\frac{\pi}{\lambda}}.$$

Observe that, since $t^2 \geq 10t$, for $t \geq 10$, it follows that

$$\int_{10}^{\infty} e^{-\lambda t^2}\, \mathrm{d}t < \int_{10}^{\infty} e^{-10zt}\, \mathrm{d}t = \frac{e^{-100\lambda}}{10\lambda}.$$

Thus,

$$\int_{-\infty}^{-10} e^{-\lambda t^2}\, \mathrm{d}t = \int_{10}^{\infty} e^{-\lambda t^2}\, \mathrm{d}t = \mathrm{TST}, \quad \text{as} \quad \lambda \to \infty$$

Example (Cont.)

Consequently,

$$I(\lambda) = \int_{-10}^{10} e^{-\lambda t^2}\,\mathrm{d}t \sim \sqrt{\frac{\pi}{\lambda}}, \quad \text{as} \quad \lambda \to \infty.$$

There is another way to arrive at this conclusion. Consider, for $\tau \geq 0$, the change of variable

$$t^2 = \tau \quad \Longrightarrow \quad t = \sqrt{\tau} \quad \Longrightarrow \quad \mathrm{d}t = \frac{1}{2}\tau^{-\frac{1}{2}}\mathrm{d}\tau,$$

so that

$$I(\lambda) = 2\int_0^{10} e^{-\lambda t^2}\,\mathrm{d}t = \int_0^{100} e^{-\lambda\tau}\tau^{-\frac{1}{2}}\,\mathrm{d}\tau.$$

Using Watson's Lemma, with $g \equiv 1$, $\sigma = -\frac{1}{2}$, we have

$$I(\lambda) \sim \frac{\Gamma\left(\frac{1}{2}\right)}{0!\lambda^{\frac{1}{2}}} = \sqrt{\frac{\pi}{\lambda}}, \quad \text{as} \quad \lambda \to \infty.$$

Example

Consider

$$I(\lambda) = \int_{-\infty}^{\infty} f(t) e^{-\lambda t^2} \, \mathrm{d}t,$$

for large values of $\lambda \in (0, \infty)$. Suppose that

$$\int_{-\infty}^{\infty} |f(t)| \, \mathrm{d}t < \infty,$$

and assume the following expansion is valid in the interval $[-s, s]$, for some $s > 0$:

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k.$$

We leave it to the reader to prove that

$$\int_{-\infty}^{-s} f(t) e^{-\lambda t^2} \, \mathrm{d}t = \mathrm{TST}, \quad \int_{s}^{\infty} f(t) e^{-\lambda t^2} \, \mathrm{d}t = \mathrm{TST}, \quad \text{as} \quad \lambda \to \infty.$$

Example (Cont.)

Then, using arguments like those in the proof of Watson's Lemma,

$$I(\lambda) \sim \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k e^{-\lambda t^2}\, \mathrm{d}t, \quad \text{as} \quad \lambda \to \infty.$$

We may integrate term-by-term, and using the fact that $e^{-\lambda t^2}$ is even, we have

$$I(\lambda) \sim \sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!} \int_{-\infty}^{\infty} t^{2k} e^{-\lambda t^2}\, \mathrm{d}t, \quad \text{as} \quad \lambda \to \infty.$$

Since

$$\int_{-\infty}^{\infty} t^{2k} e^{-\lambda t^2}\, \mathrm{d}t = \frac{\Gamma\left(k+\frac{1}{2}\right)}{\lambda^{k+\frac{1}{2}}}, \quad k \in \mathbb{N}_0,$$

we have

$$I(\lambda) \sim \sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!} \cdot \frac{\Gamma\left(k+\frac{1}{2}\right)}{\lambda^{k+\frac{1}{2}}}, \quad \text{as} \quad \lambda \to \infty.$$

Theorem (Watson's Lemma for Integrals of the 2nd Kind)

Let $T > 0$ be given, and suppose that $f \in C^\infty([-T, T]; \mathbb{C})$. Then, the integral

$$I(\lambda) = \int_{-T}^{T} f(t) e^{-\lambda t^2} \, \mathrm{d}t, \tag{22}$$

is finite for each $\lambda > 0$, and

$$I(\lambda) \sim \sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!} \frac{\Gamma\left(k + \frac{1}{2}\right)}{\lambda^{k+\frac{1}{2}}}, \quad as \quad \lambda \to \infty. \tag{23}$$

Proof.

We leave the finiteness of the integral as an exercise for the reader. Let us consider the following integral tails; observe that, for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$\int_{-\infty}^{-T} t^{2k} e^{-\lambda t^2} \, \mathrm{d}t = \int_{T}^{\infty} t^{2k} e^{-\lambda t^2} \, \mathrm{d}t = \mathrm{TST}, \quad \text{as} \quad \lambda \to \infty.$$

The arguments are similar to those above.

Proof (Cont.)

For the Gaussian integrals, for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we have

$$\int_{-\infty}^{\infty} t^{2k} e^{-\lambda t^2} \, \mathrm{d}t = \frac{\Gamma\left(k + \frac{1}{2}\right)}{\lambda^{k+\frac{1}{2}}}.$$

Thus, for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$\int_{-T}^{T} t^{2k} e^{-\lambda t^2} \, \mathrm{d}t - \frac{\Gamma\left(k + \frac{1}{2}\right)}{\lambda^{k+\frac{1}{2}}} = \mathrm{TST}, \quad \text{as} \quad \lambda \to \infty.$$

For $k \in \mathbb{N}$, since the integrand is an odd function, it follows that

$$\int_{-T}^{T} t^{2k-1} e^{-\lambda t^2} \, \mathrm{d}t = 0.$$

Proof (Cont.)

By Taylor's Theorem, for each $t \in [-T, T]$,

$$f(t) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} t^k + R_{n+1}(t),$$

where

$$R_{n+1}(t) = \frac{f^{(n+1)}(\eta_n(t))}{(n+1)!} t^{n+1},$$

for some $\eta_n(t) \in (-T, T)$. We have the estimate

$$|R_{n+1}(t)| \leq \max_{-T \leq \eta \leq T} \left| f^{(n+1)}(\eta) \right| \frac{|t|^{n+1}}{(n+1)!}.$$

Proof (Cont.)

Suppose that $n = 2M$, for some $M \in \mathbb{N}_0$. Then,

$$\begin{aligned}
I(\lambda) &= \int_{-T}^{T} f(t) e^{-\lambda t^2} \, dt \\
&= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \int_{-T}^{T} t^k e^{-\lambda t^2} \, dt + \int_{-T}^{T} R_{n+1}(t) e^{-\lambda t^2} \, dt \\
&= \sum_{k=0}^{M} \frac{f^{(2k)}(0)}{(2k)!} \int_{-T}^{T} t^{2k} e^{-\lambda t^2} \, dt + \int_{-T}^{T} R_{2M+1}(t) e^{-\lambda t^2} \, dt,
\end{aligned}$$

since the integrals with odd powers of t vanish. Similarly, if $n = 2M + 1$, for some $M \in \mathbb{N}_0$,

$$I(\lambda) = \sum_{k=0}^{M} \frac{f^{(2k)}(0)}{(2k)!} \int_{-T}^{T} t^{2k} e^{-\lambda t^2} \, dt + \int_{-T}^{T} R_{2M+2}(t) e^{-\lambda t^2} \, dt.$$

The last integral in the sum vanishes, since the integrand is odd.

Proof (Cont.)

Thus, for any $M \in \mathbb{N}_0$, we have the curious fact that

$$\int_{-T}^{T} R_{2M+1}(t) e^{-\lambda t^2} \, \mathrm{d}t = \int_{-T}^{T} R_{2M+2}(t) e^{-\lambda t^2} \, \mathrm{d}t.$$

In any case, define, for any $k \in \mathbb{N}_0$,

$$\mathcal{M}_k := \max_{-T \le \eta \le T} \left| f^{(k)}(\eta) \right|.$$

We estimate the remainder integral of order $2M+1$ as

$$\begin{aligned}
\left| \int_{-T}^{T} R_{2M+1}(t) e^{-\lambda t^2} \, \mathrm{d}t \right| &\le \frac{\mathcal{M}_{2M+1}}{(2M+1)!} \int_{-T}^{T} |t|^{2M+1} e^{-\lambda t^2} \, \mathrm{d}t \\
&\le \frac{\mathcal{M}_{2M+1}}{(2M+1)!} \int_{-\infty}^{\infty} |t|^{2M+1} e^{-\lambda t^2} \, \mathrm{d}t \\
&= \frac{2\mathcal{M}_{2M+1}}{(2M+1)!} \int_{0}^{\infty} t^{2M+1} e^{-\lambda t^2} \, \mathrm{d}t \\
&= \frac{\mathcal{M}_{2M+1}}{(2M+1)!} \frac{\Gamma(M+1)}{\lambda^{M+1}}.
\end{aligned}$$

Proof (Cont.)

Thus,

$$\int_{-T}^{T} R_{2M+1}(t) e^{-\lambda t^2} \, \mathrm{d}t = o\left(\frac{1}{\lambda^{M+1/2}}\right), \quad \text{as} \quad \lambda \to \infty.$$

Consequently,

$$I(\lambda) - \sum_{k=0}^{M} \frac{f^{(2k)}(0)}{(2k)!} \frac{\Gamma\left(k+\frac{1}{2}\right)}{\lambda^{k+\frac{1}{2}}} = o\left(\frac{1}{\lambda^{M+1/2}}\right), \quad \text{as} \quad \lambda \to \infty,$$

which implies (23). □

Corollary

Let $T > 0$ be given, and suppose that $f \in L^1([-T, T]; \mathbb{C})$. Suppose that, for some $s \in (0, T)$, $\sigma > -1$, and $g \in C^\infty([-s, s]; \mathbb{C})$, $f(t) = |t|^\sigma g(t)$, for all $t \in [-s, s]$. Then the integral

$$I(\lambda) = \int_{-T}^{T} f(t) e^{-\lambda t^2} \, \mathrm{d}t, \tag{24}$$

is finite, for all $\lambda > 0$, and

$$I(\lambda) \sim \sum_{k=0}^{\infty} \frac{g^{(2k)}(0)}{(2k)!} \frac{\Gamma\left(k + \frac{\sigma}{2} + \frac{1}{2}\right)}{\lambda^{k + \frac{\sigma}{2} + \frac{1}{2}}}, \quad \text{as} \quad \lambda \to \infty. \tag{25}$$

Proof Sketch.

The core idea is to pretend that f can be written as a convergent power series that can be integrated term by term. Specifically, take

$$f(t) = |t|^\sigma \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} t^k.$$

Proof Sketch (Cont.)

Then, adjusting our limits of integration to introduce, at worst, transcendentally small terms, as $\lambda \to \infty$,

$$\begin{aligned}
I(\lambda) &= \int_{-T}^{T} f(t) e^{-\lambda t^2}\, \mathrm{d}t \\
&\sim \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} \int_{-s}^{s} |t|^{\sigma} t^k e^{-\lambda t^2}\, \mathrm{d}t \\
&\sim \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} \int_{-\infty}^{\infty} |t|^{\sigma} t^k e^{-\lambda t^2}\, \mathrm{d}t \\
&= \sum_{k=0}^{\infty} \frac{g^{(2k)}(0)}{(2k)!} \int_{-\infty}^{\infty} |t|^{\sigma} t^{2k} e^{-\lambda t^2}\, \mathrm{d}t \\
&= \sum_{k=0}^{\infty} \frac{g^{(2k)}(0)}{(2k)!} \frac{\Gamma\left(k + \frac{\sigma}{2} + \frac{1}{2}\right)}{\lambda^{k+\frac{\sigma}{2}+\frac{1}{2}}}.
\end{aligned}$$

$\square$

Example

Consider the Laplace integral

$$I(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda \cosh(t)}\,\mathrm{d}t.$$

Since ϕ is an even function, we have

$$I(\lambda) = 2\int_{0}^{\infty} e^{-\lambda \cosh(t)}\,\mathrm{d}t.$$

Here $\phi(t) = -\cosh(t)$, and, again, $\phi'(0) = 0$ with $\phi''(0) < 0$. As $\lambda \to \infty$, the following approximations become better and better:

$$-\lambda \cosh(t) \approx -\lambda\left(1 + \frac{1}{2}t^2\right) \quad \text{and} \quad e^{-\lambda\cosh(t)} \approx e^{-\lambda\left(1+\frac{1}{2}t^2\right)}.$$

Example (Cont.)

Inspired by this observation, for $t \geq 0$, let us utilize the change of variable

$$\tau = \sqrt{\cosh(t) - 1} \quad \Longrightarrow \quad 1 + \tau^2 = \cosh(t),$$

which implies that

$$\begin{aligned} dt &= \frac{2\tau \, d\tau}{\sinh(t)} \\ &= \frac{2\tau \, d\tau}{\sqrt{\cosh^2(t) - 1}} \\ &= \frac{2\tau \, d\tau}{\sqrt{(1+\tau^2)^2 - 1}} \\ &= \frac{2\tau \, d\tau}{\sqrt{2\tau^2 + \tau^4}} \\ &= \frac{\sqrt{2} \, d\tau}{\sqrt{1 + \frac{1}{2}\tau^2}}. \end{aligned}$$

Example (Cont.)

Thus,

$$I(\lambda) = 2\int_0^\infty e^{-\lambda\cosh(t)}\,\mathrm{d}t = 2\sqrt{2}e^{-\lambda}\int_0^\infty \frac{e^{-\lambda\tau^2}\,\mathrm{d}\tau}{\sqrt{1+\frac{1}{2}\tau^2}}.$$

This problem is now exactly like the one in a previous example. We need only expand the function

$$f(\tau) = \frac{1}{\sqrt{1+\frac{1}{2}\tau^2}} = 1 - \frac{1}{4}\tau^2 + \frac{3}{32}\tau^4 - \frac{5}{128}\tau^6 + \cdots,$$

using the binomial expansion, which is valid for

$$0 \le \frac{\tau^2}{2} < 1.$$

Now, even though this expansion is valid for only a small part of the integration domain, we may utilize it, arguing that the errors we incur are only transcendentally small in nature.

Example (Cont.)

Finally, we have

$$I(\lambda) \sim 2\sqrt{2}e^{-\lambda}\Bigg(\int_0^\infty e^{-\lambda\tau^2}\,d\tau - \frac{1}{4}\int_0^\infty \tau^2 e^{-\lambda\tau^2}\,d\tau + \frac{3}{32}\int_0^\infty \tau^4 e^{-\lambda\tau^2}\,d\tau - \frac{5}{128}\int_0^\infty \tau^6 e^{-\lambda\tau^2}\,d\tau + \cdots\Bigg), \quad \text{as} \quad \lambda \to \infty.$$

Evaluating these Gaussian integrals, we have

$$I(\lambda) \sim 2\sqrt{2}e^{-\lambda}\Bigg(\frac{1}{2}\frac{\sqrt{\pi}}{\lambda^{\frac{1}{2}}} - \frac{1}{4}\cdot\frac{1}{4}\frac{\sqrt{\pi}}{\lambda^{\frac{3}{2}}} + \frac{3}{32}\cdot\frac{3}{8}\frac{\sqrt{\pi}}{\lambda^{\frac{5}{2}}} - \frac{5}{128}\cdot\frac{15}{16}\frac{\sqrt{\pi}}{\lambda^{\frac{7}{2}}} + \cdots\Bigg), \quad \text{as} \quad \lambda \to \infty,$$

and simplifying, we get

$$I(\lambda) \sim \frac{\sqrt{2\pi}e^{-\lambda}}{\sqrt{\lambda}}\left(1 - \frac{1}{8\lambda} + \frac{9}{128\lambda^2} - \frac{75}{1024\lambda^3} + \cdots\right), \quad \text{as} \quad \lambda \to \infty.$$

Example

Let us use Mathematica to confirm the last result. When we ask for a single term in the asymptotic expansion, Mathematica delivers the following:

```
In[1]:= AsymptoticIntegrate[Exp[-λ*Cosh[t]],{t,-∞,∞},{λ,∞,1}]
```

Out[1]= $e^{-\lambda}\sqrt{\frac{2\pi}{\lambda}}$

This agrees with the first term of our approximation above. However, if we ask Mathematica for more terms, it fails to deliver. For example, asking for 3 terms in the expansion, the software give the same one-term answer, with no explanation.

```
In[2]:= AsymptoticIntegrate[Exp[-λ*Cosh[t]],{t,-∞,∞},{λ,∞,3}]
```

Out[2]= $e^{-\lambda}\sqrt{\frac{2\pi}{\lambda}}$

Example

In this example, we will derive Stirling's approximation of the Gamma function,

$$\Gamma(\lambda+1) = \int_0^\infty t^\lambda e^{-t}\,\mathrm{d}t,$$

for large values of $\lambda \in (0,\infty)$. At first glance, it is not clear that this can be viewed as a Laplace integral. But, if we employ the following identity we get some clarity: for $t, \lambda \in (0,\infty)$,

$$t^\lambda = e^{\lambda \log(t)}.$$

Then,

$$\Gamma(\lambda+1) = \int_0^\infty e^{-t} e^{\lambda \log(t)}\,\mathrm{d}t,$$

which appears to be a Laplace integral with the identifications $g(t) = e^{-t}$ and $\phi(t) = \log(t)$. But, unfortunately, there is no point $t_o \in [0,\infty)$ such that $\phi'(t_o) = 0$.

Example (Cont.)

So, let us make a change of variable:

$$s = \frac{t}{\lambda} \quad \Longrightarrow \quad \mathrm{d}t = \lambda\, \mathrm{d}s, .$$

Then

$$\Gamma(\lambda+1) = \lambda^{\lambda+1} \int_0^\infty e^{-\lambda(s-\log(s))}\, \mathrm{d}s,$$

which is a Laplace integral with the identifications $f(s) \equiv 1$ and $\phi(s) = -s + \log(s)$. Indeed, we observe that $\phi(1) = 1$, $\phi'(1) = 0$, and $\phi''(1) < 0$. We expect that, as $\lambda \to \infty$, the following approximations become better and better:

$$-\lambda(s - \log(s)) \approx -\lambda\left(1 + \frac{1}{2}(s-1)^2\right) \quad \text{and} \quad e^{-\lambda(s-\log(s))} \approx e^{-\left(1+\frac{1}{2}(s-1)^2\right)}.$$

Example (Cont.)

We have identified $s_o = 1$ as the Laplace max point. Let us shift the problem so that this point is at the origin. We use the change of variable

$$t = s - 1 \quad \Longrightarrow \quad \mathrm{d}t = \mathrm{d}s,$$

so that

$$\Gamma(\lambda+1) = \lambda^{\lambda+1} \int_{-1}^{\infty} e^{-\lambda(t+1-\log(t+1))} \, \mathrm{d}t.$$

Now, for $t \geq 0$, we make the change of variable

$$\tau = \sqrt{t - \log(t+1)} \quad \Longrightarrow \quad 1 + \tau^2 = t + 1 - \log(t+1).$$

Using the power series expansion for $\log(t+1)$, we know that

$$\tau^2 = \sum_{k=2}^{\infty} \frac{(-1)^k}{k} t^k, \quad -1 < t < 1.$$

One way to proceed, it turns out, is to invert the power series.

Example (Cont.)

In other words, we seek an expansion of the form

$$t = \sum_{\ell=1}^{\infty} a_\ell \tau^\ell$$

that is valid in some neighborhood of $\tau = 0$. Notice that we have set $a_0 = 0$, since $\tau = 0$ iff $t = 0$. Formally, we may compute this term by term, using

$$\tau^2 = \frac{1}{2}\left(\sum_{\ell=1}^{\infty} a_\ell \tau^\ell\right)^2 - \frac{1}{3}\left(\sum_{\ell=1}^{\infty} a_\ell \tau^\ell\right)^3 + \frac{1}{4}\left(\sum_{\ell=1}^{\infty} a_\ell \tau^\ell\right)^4 - \cdots$$

and equate coefficients.

Example (Cont.)

We get

$$
\begin{aligned}
1 &= \frac{1}{2}a_1^2, \\
0 &= -\frac{1}{3}a_1^3 + a_1 a_2, \\
0 &= \frac{1}{4}a_1^4 - a_1^2 a_2 + \frac{1}{2}a_2^2 + a_1 a_3, \\
0 &= -\frac{1}{5}a_1^5 + a_1^3 a_2 - a_1 a_2^2 - a_1^2 a_3 + a_2 a_3 + a_1 a_4, \\
&\vdots
\end{aligned}
$$

which is solved by

$$
a_1 = \sqrt{2}, \quad a_2 = \frac{2}{3}, \quad a_3 = \frac{\sqrt{2}}{18}, \quad a_4 = -\frac{2}{135}, \quad a_5 = \frac{\sqrt{2}}{1080}, \quad \cdots .
$$

Example (Cont.)

Mathematica can do this automatically:

$$t = \sqrt{2}\tau + \frac{2\tau^2}{3} + \frac{\tau^3}{9\sqrt{2}} - \frac{2\tau^4}{135} + \frac{\tau^5}{540\sqrt{2}} + \frac{4\tau^6}{8505} - \frac{139\tau^7}{340200\sqrt{2}} + \frac{2\tau^8}{25515} - \frac{571\tau^9}{73483200\sqrt{2}} + \cdots .$$

In any case, we have

$$\mathrm{d}t = \mathrm{d}\tau\left(\sqrt{2} + \frac{4\tau}{3} + \frac{3\tau^2}{9\sqrt{2}} - \frac{8\tau^3}{135} + \frac{5\tau^4}{540\sqrt{2}} + \frac{24\tau^5}{8505} - \frac{973\tau^6}{340200\sqrt{2}} + \frac{16\tau^7}{25515} - \frac{5139\tau^8}{73483200\sqrt{2}} + \cdots\right).$$

Example (Cont.)

Using our usual arguments,

$$\Gamma(\lambda+1) \sim \lambda^{\lambda+1}e^{-\lambda}\left(\sqrt{2}\int_{-\infty}^{\infty} e^{-\lambda\tau^2}\,\mathrm{d}\tau + \frac{1}{3\sqrt{2}}\int_{-\infty}^{\infty} e^{-\lambda\tau^2}\tau^2\,\mathrm{d}\tau \right.$$
$$\left. + \frac{1}{108\sqrt{2}}\int_{-\infty}^{\infty} e^{-\lambda\tau^2}\tau^4\,\mathrm{d}\tau - \frac{973}{340200\sqrt{2}}\int_{-\infty}^{\infty} e^{-\lambda\tau^2}\tau^6\,\mathrm{d}\tau + \cdots\right),$$

as $\lambda \to \infty$. Simplifying, we have

$$\Gamma(\lambda+1) \sim \frac{\lambda^{\lambda+1}e^{-\lambda}}{\sqrt{2}}\left(2\frac{\sqrt{\pi}}{\lambda^{\frac{1}{2}}} + \frac{1}{3}\frac{\sqrt{\pi}}{2\lambda^{\frac{3}{2}}} \right.$$
$$\left. + \frac{1}{108}\frac{3\sqrt{\pi}}{4\lambda^{\frac{5}{2}}} - \frac{973}{340200}\frac{15\sqrt{\pi}}{8\lambda^{\frac{7}{2}}} + \cdots\right), \quad \text{as} \quad \lambda \to \infty,$$

or, equivalently, as $\lambda \to \infty$,

$$\Gamma(\lambda+1) \sim e^{-\lambda}\lambda^{\lambda+1}\sqrt{\frac{2\pi}{\lambda}}\left(1 + \frac{1}{12\lambda} + \frac{1}{288\lambda^2} - \frac{139}{51840\lambda^3} + \cdots\right).$$

Example

Let us check our last calculation using Mathematica.

```
In[3]:= AsymptoticIntegrate[t^λ*Exp[-t],{t,0,∞},{λ,∞,3}]
```

Out[3]= $$e^{\lambda(-1+\mathrm{Log}[\lambda])}\left(-\frac{139\sqrt{\frac{\pi}{2}}}{25920\lambda^{5/2}}+\frac{\sqrt{\frac{\pi}{2}}}{144\lambda^{3/2}}+\frac{\sqrt{\frac{\pi}{2}}}{6\sqrt{\lambda}}+\sqrt{\lambda}\sqrt{2\pi}\right)$$

This agrees perfectly, after some simplification.