Math 515
Essential Perturbation Theory and Asymptotic Analysis
Chapter 06

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Chapter 06, Part 1 of 2
The WKB Method

WKB or WKBJ

The WKB method, named in honor of Wentzel (1926), Kramer (1926), and Brillouin (1926), and sometimes the WKBJ method to honor the earlier contribution of Jeffreys (1924), is an approximation scheme for computing solutions to a general class of linear ODE problems with a small parameter. The method can generalize some of the work that we did for singularly perturbed boundary value problems in the last chapter, as well as some of the work that we will do in the next chapter on multiple scales problems.

In a nutshell, the method was originally developed for approximating solutions to Schrödinger-type equations in quantum mechanics, and is designed to handle both dispersive and diffusive phenomena in the modeling of problems in and around potential barriers. Our treatment will follow that in Bender and Orszag (1978) and in Miller (2006).

WKB Expansion for the Schrödinger Equation

The ODE

Consider the following linear ODE

$$\varepsilon^2 y''(x) - q(x)y(x) = 0, \quad x \in \Omega, \tag{1}$$

where $\varepsilon \in (0,1)$ and $\Omega \subseteq \mathbb{R}$ is some open interval, which could be infinite or semi-infinite. In this section, we will assume that $q \in C(\Omega;\mathbb{R})$ and $q(x) \neq 0$, for all $x \in \Omega$. Following Bender and Orszag (1978), we will refer to (1) as a Schrödinger Equation.

Depending upon the sign of q, this equation can model dispersive (wave-like) phenomena, on the one hand, and dissipative (decay-type) phenomena, on the other hand. Later, we will investigate cases where q changes sign, in which case the solutions of Equation (1) can display wave-like features on one part of the domain and decay-like features on another, with a transition region between.

The WKB Expansion

In the WKB method we suppose that the solution has the asymptotic form

$$y(x;\varepsilon) \sim \exp\left(\frac{s(x;\delta)}{\delta}\right), \quad \text{as} \quad \varepsilon \searrow 0, \tag{2}$$

for all $x \in \Omega$, where

$$s(x;\delta) \sim \sum_{k=0}^{\infty} s_k(x)\delta^k, \quad \text{as} \quad \varepsilon \searrow 0,$$

and $\delta = \delta(\varepsilon) > 0$, with $\delta(\varepsilon) \searrow 0$, as $\varepsilon \searrow 0$. Notice that the explicit exponential dependence built into the solution is intended to capture either dispersive or dissipative behaviors described above. In the case of wave-like behavior, we expect that s will be complex.

Derivative Expansions

Let us further assume, as usual, that, our expansion can be differentiated term-by-term. In other words, the following are valid: for all $x \in \Omega$, and as $\varepsilon \searrow 0$,

$$\frac{\mathrm{d}}{\mathrm{d}x} y(x;\varepsilon) \sim \exp\left(\frac{s(x;\delta)}{\delta}\right) \frac{1}{\delta} \frac{\mathrm{d}}{\mathrm{d}x} s(x;\delta), \tag{3}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} y(x;\varepsilon) \sim \exp\left(\frac{s(x;\delta)}{\delta}\right) \left[\frac{1}{\delta^2}\left(\frac{\mathrm{d}}{\mathrm{d}x} s(x;\delta)\right)^2 + \frac{1}{\delta}\frac{\mathrm{d}^2}{\mathrm{d}x^2} s(x;\delta)\right], \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} s(x;\delta) \sim \sum_{k=0}^{\infty} s_k'(x)\delta^k, \tag{5}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} s(x;\delta) \sim \sum_{k=0}^{\infty} s_k''(x)\delta^k. \tag{6}$$

The Transformed Equation

Substituting this ansatz into (1) we obtain, after cancelling the exponential terms,

$$q(x) \sim \frac{\varepsilon^2}{\delta^2}\left(\frac{\mathrm{d}}{\mathrm{d}x}s(x;\delta)\right)^2 + \frac{\varepsilon^2}{\delta}\frac{\mathrm{d}^2}{\mathrm{d}x^2}s(x;\delta), \quad \text{as} \quad \varepsilon \searrow 0. \tag{7}$$

Next, let use attempt to use the standard form of the principle of dominant balance to determine the size of δ, viewing $s(x;\varepsilon)$ as the new dependent variable.

Dominant Balance: Failed First Attempt

Typically, we would wish to balance the coefficient of the $\frac{d^2}{dx^2}s(x;\delta)$ term with the the coefficient of one of the other two remaining lower-order terms asymptotically, as $\varepsilon \searrow 0$. Here is what happens, if we try this procedure.

1. The coefficient of the $\frac{d^2}{dx^2}s(x;\delta)$ term balances with that of the $\frac{d}{dx}s(x;\delta)$ term. In this case, $\delta = 1$. The coefficients of the $\frac{d^2}{dx^2}s(x;\delta)$ and $\frac{d}{dx}s(x;\delta)$ terms are of order ε, which cannot dominate the coefficient of the $q(x)$ term, which is of order 1. This is not an acceptable case.
2. The coefficient of the $\frac{d^2}{dx^2}s(x;\delta)$ term balances with that of the $q(x)$ term. In this case, $\delta = \varepsilon^2$. The coefficients of the $\frac{d^2}{dx^2}s(x;\delta)$ and $q(x)$ terms are of order 1, which cannot dominate the coefficient of the $\frac{d}{dx}s(x;\delta)$ term, which is of order ε^{-2}. This is not an acceptable case either.

Dominant Balance:

What do we do in this situation? In the present setting, the only way to get an acceptable dominant balance is to balance the coefficients of the $q(x)$ term and the $\frac{\mathrm{d}}{\mathrm{d}x} s(x;\delta)$ term. This yields

$$\frac{\varepsilon^2}{\delta^2} = 1 \quad \Longleftrightarrow \quad \delta = \varepsilon.$$

In this case, the coefficients of the $q(x)$ term and the $\frac{\mathrm{d}}{\mathrm{d}x} s(x;\delta)$ are of order 1, and they dominate the coefficient of the $\frac{\mathrm{d}^2}{\mathrm{d}x^2} s(x;\delta)$ term, which is of order ε, asymptotically, as $\varepsilon \searrow 0$.

In claiming that $q(x)$ is of order 1, we have explicitly used the fact that $q \in C(\Omega;\mathbb{R})$ and $q(x) \neq 0$, for all $x \in \Omega$. But, in fact, we need a little more. In particular, we should assume that either

$$q(x) \geq q_0 > 0, \quad \forall\, x \in \Omega,$$

or

$$q(x) \leq q_0 < 0, \quad \forall\, x \in \Omega.$$

q is Positive on Ω

Here, let us explicitly assume that

$$q(x) \geq q_0 > 0, \quad \forall x \in \Omega.$$

This will represent the dissipative case. With $\delta = \varepsilon$, expanding, we have

$$q(x) \sim \left(s_0'(x)\right)^2 + 2s_0'(x)s_1'(x)\varepsilon + \left(2s_0'(x)s_2'(x) + \left(s_1'(x)\right)^2\right)\varepsilon^2 + \cdots \tag{8}$$

$$+ s_0''(x)\varepsilon + s_1''(x)\varepsilon^2 + s_2''(x)\varepsilon^3 + \cdots . \tag{9}$$

Equating coefficients of like powers, we obtain the following recursive system of equations:

$$O(1): \quad \left(s_0'(x)\right)^2 = q(x), \tag{10}$$

$$O(\varepsilon): \quad 2s_0'(x)s_1'(x) + s_0''(x) = 0, \tag{11}$$

$$O(\varepsilon^2): \quad 2s_0'(x)s_2'(x) + s_1''(x) + \left(s_1'(x)\right)^2 = 0, \tag{12}$$

$$\vdots$$

$$O(\varepsilon^k): \quad 2s_0'(x)s_k'(x) + s_{k-1}''(x) + \sum_{j=1}^{k-1} s_j'(x)s_{k-j}'(x) = 0, \quad k \geq 3. \tag{13}$$

Leading-Order and First-Order Correction Terms

The solution to the leading-order equation (10), called the Eikonal equation, is

$$s_0(x) = \pm \int_a^x \sqrt{q(t)}\,\mathrm{d}t,$$

where $a \in \mathbb{R}$ is an arbitrary point of integration. If $q(x) < 0$, for all $x \in \Omega$, s_0 is complex, as we shall see, but here we deal first with the case that $q \geq q_0 > 0$ on Ω. The second equation, Equation (11), has the solution

$$s_1(x) = -\frac{1}{4}\ln\left(q(x)\right) = \ln\left(\frac{1}{\sqrt[4]{q(x)}}\right).$$

Leading-Order Asymptotic Approximation

The leading-order asymptotic approximation is

$$y(x) \sim \frac{C_{0,0}}{\sqrt[4]{q(x)}} \exp\left(\frac{1}{\varepsilon}\int_a^x \sqrt{q(t)}\,\mathrm{d}t\right) + \frac{C_{0,1}}{\sqrt[4]{q(x)}} \exp\left(-\frac{1}{\varepsilon}\int_a^x \sqrt{q(t)}\,\mathrm{d}t\right), \tag{14}$$

where $C_{0,0}$ and $C_{0,1}$ are determined by boundary and/or initial conditions.

Higher-Order Terms

Higher-order corrections can be added when needed. Terms s_2, s_3, s_4 and s_5 can be found in Bender and Orszag (1978). Here we give

$$s_2(x) = \pm \int_a^x \left(\frac{q''(x)}{8(q(x))^{3/2}} - \frac{5(q'(x))^2}{32(q(x))^{5/2}} \right) \mathrm{d}t,$$

$$s_3(x) = -\frac{q''(x)}{16(q(x))^2} + \frac{5(q'(x))^2}{64(q(x))^3}.$$

q is Negative on Ω

Now, let us assume that

$$q(x) \le q_0 < 0, \quad \forall x \in \Omega.$$

In this case, the leading-order equation is

$$O(1): \quad \left(s_0'(x)\right)^2 = -|q(x)| \tag{15}$$

and

$$s_0(x) = \pm \mathrm{i} \int_a^x \sqrt{|q(t)|}\, \mathrm{d}t.$$

The first-order equation is '

$$O(\varepsilon): \quad \pm 2\mathrm{i}\sqrt{|q(x)|} s_1'(x) \pm \frac{\mathrm{i}}{2\sqrt{|q(x)|}} \frac{d}{\mathrm{d}x}\left[|q(x)|\right] = 0, \tag{16}$$

which implies that

$$s_1(x) = -\frac{1}{4} \ln\left(|q(x)|\right) = \ln\left(\frac{1}{\sqrt[4]{|q(x)|}}\right).$$

Leading-Order Asymptotic Approximation

The leading-order asymptotic approximation is

$$\begin{aligned} y(x) &\sim \frac{C_{0,0}}{\sqrt[4]{|q(x)|}} \exp\left(\frac{\mathrm{i}}{\varepsilon}\int_a^x \sqrt{|q(t)|}\,\mathrm{d}t\right) \\ &\quad + \frac{C_{0,1}}{\sqrt[4]{|q(x)|}} \exp\left(-\frac{\mathrm{i}}{\varepsilon}\int_a^x \sqrt{|q(t)|}\,\mathrm{d}t\right) \qquad (17) \\ &= \frac{\tilde{C}_{0,0}}{\sqrt[4]{|q(x)|}} \cos\left(\frac{1}{\varepsilon}\int_a^x \sqrt{|q(t)|}\,\mathrm{d}t\right) \\ &\quad + \frac{\tilde{C}_{0,1}}{\sqrt[4]{|q(x)|}} \sin\left(\frac{1}{\varepsilon}\int_a^x \sqrt{|q(t)|}\,\mathrm{d}t\right), \qquad (18) \end{aligned}$$

where $\tilde{C}_{0,0}$ and $\tilde{C}_{0,1}$ are determined by boundary and/or initial conditions.

Example

In lieu of a rigorous analysis of the approximation just derived, let us compare against some numerically computed "true solutions" for a few specific examples. In particular, let us assume that

$$q(x) = \left(1 + x^2\right)^2, \quad x \in [0, \infty),$$

in Equation (1) with the initial conditions $y(0) = 0$ and $y'(0) = 1$. The WKB method produces the following approximation, via (14), to this linear initial value problem:

$$y(x;\varepsilon) \sim \frac{\varepsilon}{\sqrt{1+x^2}} \sinh\left(\frac{3x + x^3}{3\varepsilon}\right), \quad \text{as} \quad \varepsilon \searrow 0. \tag{19}$$

We compare numerically calculated solutions with those from the WKB approximation in the figures on the next few slides. As expected, the relative errors decrease as ε is made smaller.

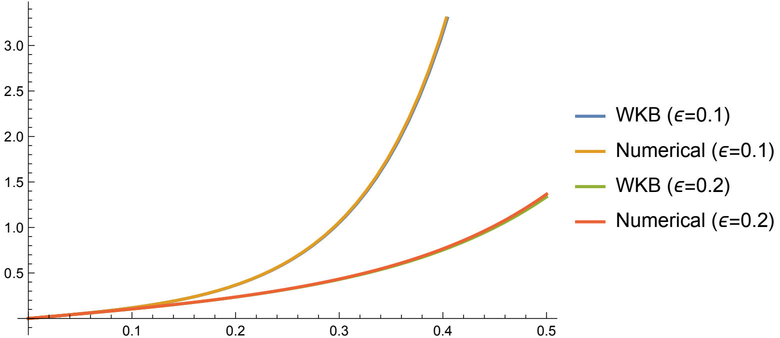

Figure: Comparison of the WKB approximation in (19) and the numerically computed solution to Equation 1, with $q(x) = \left(1 + x^2\right)^2$ and the initial conditions $y(0) = 0$ and $y'(0) = 1$, using $\varepsilon = 0.1, 0.2$.

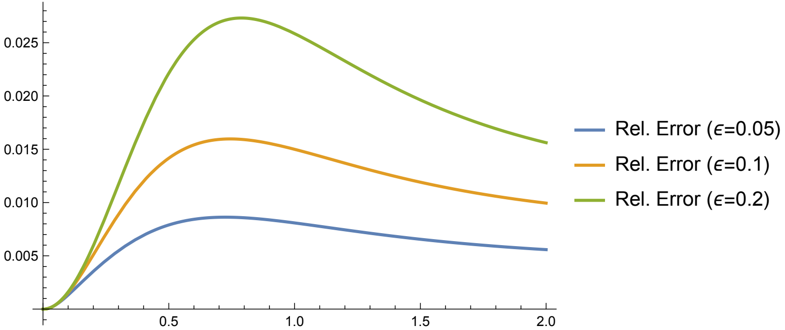

Figure: Relative errors between the WKB approximation in (19) and the numerically computed solution to Equation 1, with $q(x) = \left(1 + x^2\right)^2$ and the initial conditions $y(0) = 0$ and $y'(0) = 1$, using $\varepsilon = 0.05, 0.1, 0.2$.

Example

Let us return to a problem that we analyzed in detail in the last Chapter. Consider the singularly perturbed boundary value problem consisting of the linear ordinary differential equation

$$\varepsilon y''(x) + p(x)y'(x) + q(x)y(x) = 0, \tag{20}$$

where $0 < \varepsilon < 1$, with the boundary conditions

$$y(0) = A \quad \text{and} \quad y(1) = B. \tag{21}$$

We will assume that p is smooth and $p(x) > 0$ for all $x \in [0, 1]$. According to theory, for every $\varepsilon > 0$ that is sufficiently small, there is a unique solution, $y(\,\cdot\,; \varepsilon)$, to this problem.

Example (Cont.)

There is a boundary layer of thickness $\delta = \varepsilon$ at $x = 0$, and a composite, uniformly-valid approximation of $y(\cdot\,; \varepsilon)$ is given by

$$y_{c,0,0}(x; \varepsilon) = y_0(x) + Y_0\left(\frac{x}{\varepsilon}\right) - C_1, \quad 0 \le x \le 1,$$

where

$$\begin{aligned} y_0(x) &= B \exp\left(\int_x^1 \frac{q(t)}{p(t)}\, \mathrm{d}t\right), \\ Y_0(z) &= C_1 + (A - C_1)e^{-p(0)z}, \\ C_1 &= B \exp\left(\int_0^1 \frac{q(t)}{p(t)}\, \mathrm{d}t\right). \end{aligned}$$

Example (Cont.)

In other words,

$$y_{\mathrm{c},0,0}(x;\varepsilon) = B\exp\left(\int_x^1 \frac{q(t)}{p(t)}\,\mathrm{d}t\right) + \left(A - B\exp\left[\int_0^1 \frac{q(t)}{p(t)}\,\mathrm{d}t\right]\right)e^{-p(0)x/\varepsilon}. \tag{22}$$

In this example, we will show that the WKB method will reproduce this approximation, but without the need for a formal matching procedure between an inner solution and an outer solution. The WKB method is a global approach, as we shall see.

Example (Cont.)

To begin, let us insert (7) into equation (20) to obtain

$$\varepsilon\left[\frac{1}{\delta^2}\left(\frac{\mathrm{d}}{\mathrm{d}x}s(x;\delta)\right)^2+\frac{1}{\delta}\frac{\mathrm{d}^2}{\mathrm{d}x^2}s(x;\delta)\right]+p(x)\frac{1}{\delta}\frac{\mathrm{d}}{\mathrm{d}x}s(x;\delta)+q(x)=0,$$

after canceling the exponential term. There are four terms in this equation. To determine δ, we look for a dominant balance among these terms, keeping in mind that, in the WKB approach, the dominant balance need not involve the highest-order derivative term.

Two different dominant balance scenarios are possible. First, the third and fourth terms above may balance, that is, $\frac{1}{\delta}=1$. (Here we assume that both p and q are of order 1.) This implies $\delta=1$, and terms three and four are of order 1, provided q has no zeros. Terms one and two are thus of order ε, which are dominated by order 1 terms, clearly. This balance, it turns out, can only reproduce the outer solution, as the interested reader can show. Thus we reject it.

Example (Cont.)

The other possible dominant balance follows from equating the coefficients of terms one and three: $\frac{\varepsilon}{\delta^2} = \frac{1}{\delta}$, which implies that $\delta = \varepsilon$. This suggests that terms one and three are of order ε^{-1}, and terms two and four are of order 1. This is a viable dominant balance and the one we will use. Thus the operative equation is

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}s(x;\varepsilon)\right)^2 + \varepsilon\frac{\mathrm{d}^2}{\mathrm{d}x^2}s(x;\varepsilon) + p(x)\frac{\mathrm{d}}{\mathrm{d}x}s(x;\varepsilon) + \varepsilon q(x) \sim 0.$$

Example (Cont.)

Inserting the expansions for $s(x;\varepsilon)$ and its first and second derivatives into

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}s(x;\varepsilon)\right)^2+\varepsilon\frac{\mathrm{d}^2}{\mathrm{d}x^2}s(x;\varepsilon)+p(x)\frac{\mathrm{d}}{\mathrm{d}x}s(x;\varepsilon)+\varepsilon q(x)=0,$$

we get the following leading-order and first-order equations:

$$\begin{aligned} O(1): &\quad (s_0'(x))^2+p(x)s_0'(x)=0;\\ O(\varepsilon): &\quad 2s_0'(x)s_1'(x)+s_0''(x)+p(x)s_1'(x)+q(x)=0. \end{aligned}$$

The $O(1)$ equation has two possible solutions, which emerge from

$$s_0'(x)=0 \quad \text{and} \quad s_0'(x)=-p(x).$$

Example (Cont.)

Suppose that $s_0'(x) = 0$, for $x \in [0, 1]$. Then, s_0 is a constant, and the $O(\varepsilon)$ equation simplifies to become

$$p(x)s_1'(x) = -q(x),$$

which has the solution

$$s_1(x) = -\int_0^x \frac{q(t)}{p(t)}\,\mathrm{d}t.$$

This produces the approximation

$$y_1(x;\varepsilon) \sim C_1 \exp\left(-\int_0^x \frac{q(t)}{p(t)}\,\mathrm{d}t\right),$$

where C_1 is a constant to be determined.

Example (Cont.)

The other solution branch is obtained by assuming that $s_0'(x) = -p(x)$. The $O(\varepsilon)$ equation then becomes

$$p(x)s_1'(x) + p'(x) = q(x).$$

Thus,

$$s_1(x) = -\ln(p(x)) + \int_0^x \frac{q(t)}{p(t)}\,\mathrm{d}t.$$

This branch produces the approximation

$$y_2(x;\varepsilon) \sim \frac{C_2}{p(x)} \exp\left[\int_0^x \frac{q(t)}{p(t)}\,\mathrm{d}t - \frac{1}{\varepsilon}\int_0^x p(t)\,\mathrm{d}t\right].$$

where C_2 is an arbitrary constant.

Example (Cont.)

The approximate solution should be a linear combination of $y_1(x;\varepsilon)$ and $y_2(x;\varepsilon)$:

$$y(x;\varepsilon) \sim C_1 \exp\left(-\int_0^x \frac{q(t)}{p(t)}\,\mathrm{d}t\right) + \frac{C_2}{p(x)} \exp\left[\int_0^x \frac{q(t)}{p(t)}\,\mathrm{d}t - \frac{1}{\varepsilon}\int_0^x p(t)\,\mathrm{d}t\right].$$

Imposing the boundary conditions, we find

$$A = C_1 + \frac{C_2}{p(0)} \quad \text{and} \quad B = C_1 \exp\left[-\int_0^1 \frac{q(t)}{p(t)}\,\mathrm{d}t\right] + \mathrm{TST},$$

as $\varepsilon \searrow 0$. Solving for C_1 and C_2 and dropping TST terms, we get

$$\begin{aligned} y(x;\varepsilon) \sim{}& B\exp\left[\int_x^1 \frac{q(t)}{p(t)}\,\mathrm{d}t\right] \\ &+ \frac{p(0)}{p(x)}\left(A - B\exp\left[\int_0^1 \frac{q(t)}{p(t)}\,\mathrm{d}t\right]\right)\exp\left[\int_0^x \frac{q(t)}{p(t)}\,\mathrm{d}t - \frac{1}{\varepsilon}\int_0^x p(t)\,\mathrm{d}t\right], \end{aligned}$$

as $\varepsilon \searrow 0$.

Example (Cont.)

Whenever $x = O(1)$, the second term becomes transcendentally small, as $\varepsilon \searrow 0$. Therefore, in evaluating the second term we may assume that $x = O(\varepsilon)$, as $\varepsilon \searrow 0$, in which case we find

$$p(x) = p(0) + O(\varepsilon),$$

and

$$y(x;\varepsilon) \sim B \exp\left[\int_x^1 \frac{q(t)}{p(t)}\,\mathrm{d}t\right] + \left(A - B\exp\left[\int_0^1 \frac{q(t)}{p(t)}\,\mathrm{d}t\right]\right)\exp\left[-\frac{p(0)x}{\varepsilon}\right], \tag{23}$$

as $\varepsilon \searrow 0$. The expression on the right hand side of (23) is the same as the expression in (22).

Example

Consider the following model for a slowly aging spring:

$$y''(t) + e^{-\varepsilon t} y(t) = 0, \tag{24}$$

where $\varepsilon > 0$ is a small parameter, with the initial conditions

$$y(0) = 1 \quad \text{and} \quad y'(0) = 0. \tag{25}$$

Let us introduce the slow time scale

$$\tau = \varepsilon t,$$

and the dependent variable $Y(\tau) := y\left(\frac{\tau}{\varepsilon}\right)$. Then the equation transforms as

$$\varepsilon^2 Y''(\tau) = -e^{-\tau} Y(\tau),$$

with the boundary conditions

$$Y(0) = 1 \quad \text{and} \quad Y'(0) = 0.$$

Example (Cont.)

The problem

$$\varepsilon^2 Y''(\tau) = -e^{-\tau} Y(\tau), \quad Y(0) = 1 \quad \text{and} \quad Y'(0) = 0,$$

fits the form whose solutions we can approximate via the WKB method. In this case,

$$q(t) = -e^{-\tau} < 0.$$

Example (Cont.)

The WKB approximation at leading order is

$$
\begin{aligned}
y(t) &\sim C_{0,0}\,|q(t)|^{-1/4}\cos\left(\frac{1}{\varepsilon}\int_a^t \sqrt{|q(s)|}\,\mathrm{d}s\right) \\
&\quad + C_{0,1}\,|q(t)|^{-1/4}\sin\left(\frac{1}{\varepsilon}\int_a^t \sqrt{|q(s)|}\,\mathrm{d}s\right) \\
&= C_{0,0}e^{t/4}\cos\left(\frac{1}{\varepsilon}\int_a^t e^{-s/2}\,\mathrm{d}s\right) \\
&\quad + C_{0,1}e^{t/4}\sin\left(\frac{1}{\varepsilon}\int_a^t e^{-s/2}\,\mathrm{d}s\right) \\
&= \tilde{C}_{0,0}e^{t/4}\sin\left(\frac{2}{\varepsilon}e^{-t/2}\right) + \tilde{C}_{0,1}e^{t/4}\cos\left(\frac{2}{\varepsilon}e^{-t/2}\right). \qquad (26)
\end{aligned}
$$

Example (Cont.)

The constants are determined using the initial conditions:

$$\tilde{C}_{0,0} \sin\left(\frac{2}{\varepsilon}\right) + \tilde{C}_{0,1} \cos\left(\frac{2}{\varepsilon}\right) = 1,$$

$$-\tilde{C}_{0,0} \cos\left(\frac{2}{\varepsilon}\right) + \tilde{C}_{0,1} \sin\left(\frac{2}{\varepsilon}\right) = -\frac{\varepsilon}{4}.$$

A comparison of the WKB approximation (26) and the numerically computed "true" solution to the slowly aging spring problem comprised of the ODE (24) and the initial conditions (25) is shown in the figure on the next slide.

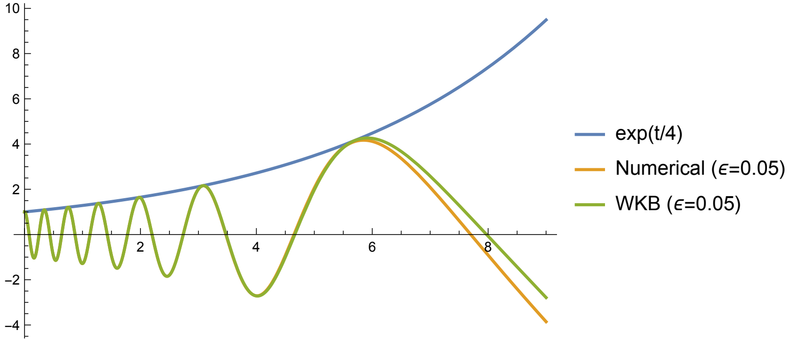

Figure: Comparison of the WKB approximation (26) and the numerically computed "true" solution to the slowly aging spring problem comprised of the ODE (24) and the initial condtions (25) for $\varepsilon = 0.05$.

Approximation of Sturm-Liouville Problems

Definition

Suppose that $-\infty < a < b < \infty$. Assume that $q, w \in C([a, b]; \mathbb{R})$ and $p \in C^1([a, b]; \mathbb{R})$. Suppose that $w(x), p(x) > 0$, for all $x \in [a, b]$. The linear differential operator defined via

$$L[y](x) := -\frac{1}{w(x)}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}(x)\right] + q(x)y(x)\right], \quad x \in (a, b), \tag{27}$$

is called a **regular Sturm-Liouville operator**. We say that $\lambda \in \mathbb{C}$ is an **eigenvalue** of the regular Sturm-Liouville operator with the associated boundary conditions

$$L_a[y] := \alpha_1 y(a) + \beta_1 y'(a) = 0, \quad [\alpha_1, \beta_1]^T \in \mathbb{R}^2 \setminus \{\mathbf{0}\}, \tag{28}$$

$$L_b[y] := \alpha_2 y(b) + \beta_2 y'(b) = 0, \quad [\alpha_2, \beta_2]^T \in \mathbb{R}^2 \setminus \{\mathbf{0}\}, \tag{29}$$

iff there is a nontrivial function $y \in C^2((a, b); \mathbb{C}) \cap C^1([a, b]; \mathbb{C})$, such that, for all $x \in (a, b)$,

$$L[y](x) = \lambda y(x), \tag{30}$$

and the boundary conditions (28) and (29) hold for y. We call y an **eigenfunction associated to** λ. We refer to the pair (λ, y) as an **eigen-pair**.

Proposition

Suppose that $y_i \in C^2((a,b);\mathbb{C}) \cap C^1([a,b];\mathbb{C})$, $i = 1,2$ *are given functions and the regular Sturm-Liouville operator is defined as in (27), with the same assumptions on* p, q, w *as above. Assume that the following boundary conditions hold:*

$$L_a[y_i] = 0 = L_b[y_i], \quad i = 1,2.$$

Then,

$$\int_a^b L[y_1](x)\overline{y_2(x)}w(x)\,\mathrm{d}x = \int_a^b y_1(x)\overline{L[y_2](x)}w(x)\,\mathrm{d}x.$$

In other words, defining the weighted inner product

$$(f,g)_{L^2_w(a,b)} = \int_a^b f(x)\overline{g(x)}w(x)\,\mathrm{d}x, \quad \forall\, f,g \in C([a,b];\mathbb{C}),$$

we have

$$(L[y_1], y_2)_{L^2_w(a,b)} = (y_1, L[y_2])_{L^2_w(a,b)}.$$

In this case, we say that L *is a hermitian (or self-adjoint) operator with respect to the weighted inner product* $(\,\cdot\,,\,\cdot\,)_{L^2_w(a,b)}$.

Proof.

$$
\begin{aligned}
(L[y_1], y_2)_{L^2_w(a,b)} &= \int_a^b L[y_1](x)\overline{y_2(x)}w(x)\,\mathrm{d}x \\
&= -\int_a^b \left[\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y_1}{\mathrm{d}x}(x)\right] + q(x)y_1(x)\right]\overline{y_2(x)}\,\mathrm{d}x \\
&= \left[p(x)\left(y_1(x)\overline{y_2'(x)} - y_1'(x)\overline{y_2(x)}\right)\right]\Big|_{x=a}^{x=b} \\
&\quad - \int_a^b \left[\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\overline{\frac{\mathrm{d}y_2}{\mathrm{d}x}(x)}\right] + q(x)\overline{y_2(x)}\right] y_1(x)\,\mathrm{d}x
\end{aligned}
$$

where we have used integration by parts twice. The reader should verify that the boundary term is zero, because of the assumed boundary conditions:

$$
\left[p(x)\left(y_1(x)\overline{y_2'(x)} - y_1'(x)\overline{y_2(x)}\right)\right]\Big|_{x=a}^{x=b} = 0.
$$

Proof (Cont.)

Finally, since p and q are real-valued,

$$-\left[\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\overline{\frac{\mathrm{d}y_2}{\mathrm{d}x}(x)}\right] + q(x)\overline{y_2(x)}\right] = \overline{L[y_2](x)}w(x).$$

Therefore,

$$\left(L[y_1], y_2\right)_{L^2_w(a,b)} = \left(y_1, L[y_2]\right)_{L^2_w(a,b)}.$$

□

Proposition

Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of the Sturm-Liouville operator (27) *subject to the boundary conditions* (28) *and* (29). *Then, in fact, λ must be real. If, we additionally assume that $q(x) \geq 0$, for all $x \in [a, b]$, and*

$$\left[-p(x)y'(x)\overline{y(x)}\right]\Big|_{x=a}^{x=b} \geq 0$$

then it follows that $\lambda \geq 0$.

Proof.

Since λ is an eigenvalue, there must be a corresponding, non-trivial function y, such that $L[y] = \lambda y$ and y satisfies the boundary conditions (28) and (29). Since y is non-trivial

$$(y, y)_{L^2_w(a,b)} = \int_a^b |y(x)|^2 w(x)\, dx > 0.$$

Proof (Cont.)

Thus,

$$\begin{aligned}\lambda(y, y)_{L^2_w(a,b)} &= (\lambda y, y)_{L^2_w(a,b)} \\ &= (L[y], y)_{L^2_w(a,b)} \\ &= (y, L[y])_{L^2_w(a,b)} \\ &= (y, \lambda y)_{L^2_w(a,b)} \\ &= \overline{\lambda}(y, y)_{L^2_w(a,b)}.\end{aligned}$$

By cancellation,

$$\lambda = \overline{\lambda}.$$

Now, since λ is an eigenvalue, it is equal to its corresponding Rayleigh quotient:

$$\lambda = R[y] := \frac{(L[y], y)_{L^2_w(a,b)}}{(y, y)_{L^2_w(a,b)}}.$$

Proof (Cont.)

The result will follow if we can show that $(L[y], y)_{L^2_w(a,b)} \geq 0$. Indeed, using integration by parts,

$$\begin{aligned}
(L[y], y)_{L^2_w(a,b)} &= \int_a^b L[y](x)\overline{y(x)}w(x)\,\mathrm{d}x \\
&= -\int_a^b \left[\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}(x)\right] + q(x)y(x)\right]\overline{y(x)}\,\mathrm{d}x \\
&= \left[-p(x)y'(x)\overline{y(x)}\right]\Big|_{x=a}^{x=b} \\
&\quad + \int_a^b \left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}(x)\overline{\frac{\mathrm{d}y}{\mathrm{d}x}(x)} + q(x)y(x)\overline{y(x)}\right]\,\mathrm{d}x \\
&= \left[-p(x)y'(x)\overline{y(x)}\right]\Big|_{x=a}^{x=b} \\
&\quad + \int_a^b \left[p(x)\left|\frac{\mathrm{d}y}{\mathrm{d}x}(x)\right|^2 + q(x)\left|y(x)\right|^2\right]\,\mathrm{d}x \\
&\geq 0.
\end{aligned}$$

$\square$

Proposition

Suppose that $\lambda_i \in \mathbb{R}$, $i = 1, 2$ are distinct eigenvalues of the Sturm-Liouville operator (27) *subject to the boundary conditions* (28) *and* (29)*. Assume that $y_i \in C^2((a, b); \mathbb{C}) \cap C^1([a, b]; \mathbb{C})$, $i = 1, 2$, are the associated eigenfunctions, respectively. Then,*

$$(y_1, y_2)_{L^2_w(a,b)} = 0.$$

In other words, y_1 and y_2 are orthogonal with respect to the weighted inner product.

Proof.

Since L is hermitian and λ_i are real,

$$\begin{aligned}
\lambda_1 (y_1, y_2)_{L^2_w(a,b)} &= (L[y_1], y_2)_{L^2_w(a,b)} \\
&= (y_1, L[y_2])_{L^2_w(a,b)} \\
&= (y_1, \lambda_2 y_2)_{L^2_w(a,b)} \\
&= \overline{\lambda_2} (y_1, y_2)_{L^2_w(a,b)} \\
&= \lambda_2 (y_1, y_2)_{L^2_w(a,b)}.
\end{aligned}$$

Proof (Cont.)

Thus,

$$(\lambda_1 - \lambda_2)(y_1, y_2)_{L^2_w(a,b)} = 0.$$

Since λ_1 and λ_2 are distinct, the result follows. $\square$

Theorem

Consider the Sturm-Liouville operator (27) subject to the boundary conditions (28) and (29). The following hold:

1. *The eigenvalues are real and countable. There is a smallest eigenvalue, but not a largest one. If the eigenvalues are ordered*

$$-\infty < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_k < \cdots ,$$

then

$$\lim_{k\to\infty} \lambda_k = \infty.$$

2. *For each eigenvalue, λ_k, there exists a corresponding eigenfunction, y_k, which is unique up to a multiplicative constant, and y_k has $k-1$ zeros in (a, b).*
3. *Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weighted inner product $(\cdot\,,\cdot)_{L^2_w(a,b)}$.*

Theorem (Cont.)

4. *The set of eigenfunctions,* $\{y_1, y_2, \ldots\}$*, which may be normalized so that*

$$(y_i, y_j)_{L^2_w(a,b)} = \delta_{i,j}, \quad i, j \in \mathbb{N},$$

is complete, that is, for any $f \in L^2_w(a,b)$*,*

$$f(x) = \sum_{k=1}^{\infty} c_k y_k(x),$$

where

$$c_k = \frac{(f, y_k)_{L^2_w(a,b)}}{(y_k, y_k)_{L^2_w(a,b)}}, \quad k \in \mathbb{N}.$$

Proof.

See for example the book by Teschl (2012). □

An Eigenvalue Problem

Consider the following linear ODE

$$y''(x) + Ew(x)y(x) = 0, \quad x \in \Omega = (0, \pi), \tag{31}$$

where $E \in \mathbb{R}$ and $w(x) > 0$, for all $x \in [0, \pi]$, with the boundary conditions

$$y(0) = 0 = y(\pi). \tag{32}$$

Here we use E to denote the unknown eigenvalue, and, therefore, $y \not\equiv 0$ is the associated eigenfunction. This is a Sturm-Liouville eigenvalue problem with the choices

$$p \equiv 1, \quad q \equiv 0.$$

We now know that there are an infinite number of eigenvalues

$$0 < E_1 < E_2 < \cdots,$$

and there is a complete orthonormal set of associated eigenfunctions:

$$(y_i, y_j)_{L^2_w(0,\pi)} = \delta_{i,j}, \quad i, j \in \mathbb{N}.$$

Notice that, in the present case, it is easy to show that $E_1 > 0$. In other words, all of the eigenvalues are positive.

WKB Eigenvalue Approximation

The WKB method may be used to approximate E_k and y_k, when k is large, as we now show. Indeed, the leading-order WKB approximation, valid for large values of the eigenvalues, is

$$y(x) \sim \frac{A}{\sqrt[4]{w(x)}} \cos\left(\sqrt{E}\int_0^x \sqrt{w(t)}\,\mathrm{d}t\right) + \frac{B}{\sqrt[4]{w(x)}} \sin\left(\sqrt{E}\int_0^x \sqrt{w(t)}\,\mathrm{d}t\right), \quad \text{as} \quad E \to \infty.$$

The boundary condition $y(0) = 0$ implies that $A = 0$. We therefore have

$$y(x) \sim \frac{B}{\sqrt[4]{w(x)}} \sin\left(\sqrt{E}\int_0^x \sqrt{w(t)}\,\mathrm{d}t\right), \quad \text{as} \quad E \to \infty.$$

The second boundary condition determines the eigenvalues

$$\sqrt{E_k}\int_0^\pi \sqrt{w(t)}\,\mathrm{d}t = \pi k, \quad k \in \mathbb{N}.$$

Equivalently,

$$E_k \sim \left[\frac{\pi k}{\int_0^\pi \sqrt{w(t)}\,\mathrm{d}t}\right]^2, \quad \text{as} \quad k \to \infty.$$

WKB Eigenfunction Approximation

We now write

$$y_k(x) \sim \frac{B_k}{\sqrt[4]{w(x)}} \sin\left(\sqrt{E_k}\int_0^x \sqrt{w(t)}\,\mathrm{d}t\right), \quad \text{as} \quad k \to \infty.$$

The constant B_k can be determined via normalization:

$$\int_0^\pi |y_k(x)|^2 w(x)\,\mathrm{d}x = (y_k, y_k)_{L^2_Q(0,\pi)} \sim 1, \quad \text{as} \quad k \to \infty.$$

This results in

$$\begin{aligned} 1 &\sim \int_0^\pi \frac{B_k^2}{\sqrt{w(x)}} \sin^2\left(\sqrt{E_k}\int_0^x \sqrt{w(t)}\,\mathrm{d}t\right) w(x)\,\mathrm{d}x \\ &= \int_0^\pi B_k^2 \sin^2\left(\sqrt{E_k}\int_0^x \sqrt{w(t)}\,\mathrm{d}t\right)\sqrt{w(x)}\,\mathrm{d}x \\ &= \frac{B_k^2}{\sqrt{E_k}}\int_0^{\pi k} \sin^2(s)\,\mathrm{d}s, \quad \text{as} \quad k \to \infty, \end{aligned}$$

where we employed the change of variable

$$s = \sqrt{E_k}\int_0^x \sqrt{w(t)}\,\mathrm{d}t.$$

WKB Eigenfunction Approximation

Thus,

$$1 \sim \frac{B_k^2}{\sqrt{E_k}} \frac{k\pi}{2} \sim \frac{B_k^2 \int_0^\pi \sqrt{w(t)}\,\mathrm{d}t}{2}, \quad \text{as} \quad k \to \infty,$$

and

$$B_k \sim \sqrt{\frac{2}{\int_0^\pi \sqrt{w(t)}\,\mathrm{d}t}}, \quad \text{as} \quad k \to \infty.$$

The eigenfunction approximation is

$$y_k(x) \sim \sqrt{\frac{2}{\sqrt{w(x)}\int_0^\pi \sqrt{w(t)}\,\mathrm{d}t}}\,\sin\left(\pi k \frac{\int_0^x \sqrt{w(t)}\,\mathrm{d}t}{\int_0^\pi \sqrt{w(t)}\,\mathrm{d}t}\right), \quad \text{as} \quad k \to \infty.$$

Example

If $w(x) \equiv 1$ in equation (31), with the boundary conditions (32), then the approximations are exact:

$$E_k = k^2, \quad k = 1, 2, \ldots,$$

and

$$y_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx), \quad k = 1, 2, \ldots.$$

Example

Suppose that $w(x) = (x+\pi)^4$ in equation (31), with the boundary conditions (32). Then,

$$E_k \sim \frac{9k^2}{49\pi^4}, \quad \text{as} \quad k \nearrow \infty, \tag{33}$$

and

$$y_k(x) \sim \sqrt{\frac{6}{7\pi^3}} \frac{\sin\left[\frac{k}{7\pi^2}\left(x^3 + 3x^2\pi + 3\pi^2 x\right)\right]}{\pi + x}, \quad \text{as} \quad k \nearrow \infty. \tag{34}$$

We have computed the eigenfunctions numerically in Mathematica. See the figures on upcoming slides for a comparison of the "exact" solutions with the WKB approximations from (34), for $k = 2$ and $k = 3$, respectively. The accuracy is remarkable even for small values of k. Recall that the asymptotic approximations are developed for $k \nearrow \infty$. A comparison of these "true" eigenvalues with the WKB approximations from (33) is given in the table on the next slide.

k	E_k	E_k^{WKB}	$\frac{\left\lvert E_k - E_k^{\mathrm{WKB}} \right\rvert}{E_k}$
1	0.00174402	0.00188559	0.0811767
2	0.00734887	0.00754235	0.0263278
3	0.0167551	0.0169703	0.0128413
4	0.0299541	0.0301694	0.00718985
5	0.0469603	0.0471397	0.00382079

Table: A comparison of the numerically computed "true" eigenvalues for $w(x) = (x + \pi)^4$ in equation (31), with the boundary conditions (32), and the WKB approximations from (33) for $k = 1, \ldots, 5$.

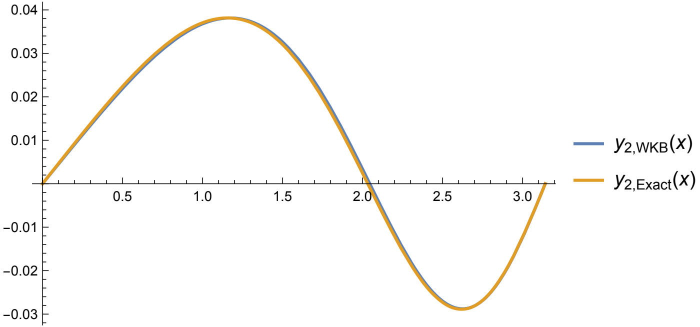

Figure: Comparison of the WKB approximation (34) and the numerically computed "true" eigenfunction of order $k = 2$ when $w(x) = (x + \pi)^4$.

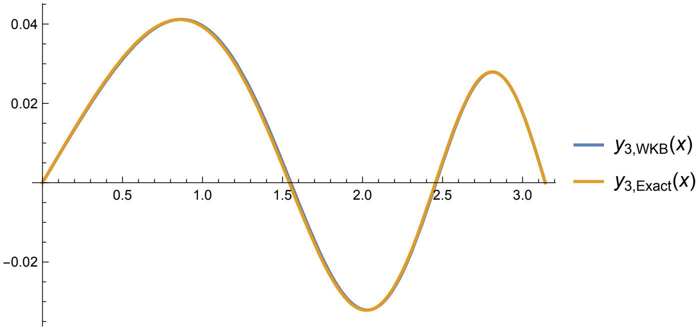

Figure: Comparison of the WKB approximation (34) and the numerically computed "true" eigenfunction of order $k = 3$ when $w(x) = (x + \pi)^4$.

One-Turning-Point Problems

Turning Point Problem

Consider the following Schrödinger equation

$$\varepsilon^2 y''(x) - q(x)y(x) = 0, \quad -\infty < x < \infty, \tag{35}$$

where $\varepsilon \in (0,1)$, with the "boundary" conditions

$$y(0) = 1 \quad \text{and} \quad \lim_{x \nearrow \infty} y(x) = 0. \tag{36}$$

In this section, we address the case that q changes sign once on the domain $\Omega = (-\infty, \infty)$. Specifically, we assume that $q \in C^\infty((-\infty,\infty);\mathbb{R})$,

$$q(0) = 0, \quad q(x) < 0,\ \forall\, x \in (-\infty, 0), \quad \text{and} \quad q(x) > 0,\ \forall\, x \in (0,\infty),$$

Furthermore, we assume that, uniformly in $[-1,1]$,

$$q(x) = ax + O(x^2), \quad \text{as} \quad x \searrow 0.$$

where $a = q'(0) > 0$. In other words, we consider the case that q has a *simple turning point*.

Three Solution Regions

We will build an approximate solution in three regions: (i) the right-hand outer-solution region, $(x_{1,R}, \infty)$, where $x_{1,R} > 0$, (ii) the left-hand outer-solution region, $(-\infty, x_{1,L})$, where $x_{1,L} < 0$, and (*iii*) an inner-solution region $(-z_1, z_1)$, where $z_1 > 0$. As with the boundary layer problems of the last chapter, we will assume the existence of overlapping regions and we will use a matching procedure to stitch the approximate solutions together.

Right-Hand Outer-Solution Region

About q, let us further assume that

$$q(x) \geq q_{0,R} > 0, \quad \text{on} \quad \Omega_R := (x_{1,+}, \infty),$$

for some $q_{0,R} > 0$. Then, we immediately see that an approximation of form

$$y_R(x) = \frac{A}{\sqrt[4]{q(x)}} \exp\left(-\frac{1}{\varepsilon}\int_0^x \sqrt{q(t)}\,\mathrm{d}t\right), \quad x \in \Omega_R, \tag{37}$$

is the asymptotically correct one. In other words, we keep only the term that is exponentially decaying as $x \nearrow \infty$, since the exponentially growing term will not satisfy our condition at infinity.

Inner Solution Region

For constructing the inner solution, let us propose a change of independent variable of the form

$$z := \frac{a^{\alpha}}{\delta(\varepsilon)} x, \tag{38}$$

where $\alpha > 0$ must be determined. As usual, the idea is that the variable z is $O(1)$, as $\varepsilon \searrow 0$, and $\delta(\varepsilon) > 0$ and $\delta(\varepsilon) \searrow 0$, as $\varepsilon \searrow 0$. We introduce the dependent variable

$$Y(z) := y\left(\frac{\delta(\varepsilon)}{a^{\alpha}} z\right).$$

It follows that

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x) = \frac{a^{\alpha}}{\delta(\varepsilon)} \frac{\mathrm{d}Y}{\mathrm{d}z}(z) \quad \text{and} \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(x) = \frac{a^{2\alpha}}{\delta^2(\varepsilon)} \frac{\mathrm{d}^2 Y}{\mathrm{d}z^2}(z).$$

Inner Solution Region

Next, it follows for $|z| > 0$, fixed and finite,

$$q(x) = q\left(\frac{\delta(\varepsilon)}{a^\alpha} z\right) = a^{1-\alpha}\delta(\varepsilon)z + O(\delta^2(\varepsilon)), \quad \text{as} \quad \varepsilon \searrow 0.$$

The differential equation becomes

$$a^{2\alpha}\frac{\varepsilon^2}{\delta^2(\varepsilon)}Y''(z) = \left(a^{1-\alpha}\delta(\varepsilon)z + O(\delta^2(\varepsilon))\right)Y(z), \quad \text{as} \quad \varepsilon \searrow 0.$$

Dominant Balance

Dominant balance suggests that

$$\delta(\varepsilon) = \varepsilon^{2/3}.$$

Though it does not affect the asymptotic scaling, we can take

$$\alpha = \frac{1}{3},$$

which eliminates a in the equation. The differential equation in the inner region is

$$\varepsilon^{2/3} Y''(z) = \left(\varepsilon^{2/3} z + O(\varepsilon^{4/3})\right) Y(z), \quad \text{as} \quad \varepsilon \searrow 0.$$

The Inner Expansion

This next assumption is quite important for matching. In particular, we will assume that

$$Y(z) = \varepsilon^{\gamma} \sum_{k=0}^{\infty} Y_k(z) \varepsilon^{rk}, \quad \text{as} \quad \varepsilon \searrow 0, \tag{39}$$

uniformly in the inner region, where $\gamma \in \mathbb{R}$ and $r > 0$ must be determined via matching. We have not used such an asymptotic power series in any of our examples, but it is perfectly legitimate. And, we will see, exactly what is needed for this situation. Since we will only be interested in the leading-order matching, the value of r will not be important for this example.

Airy's Equation

Regardless what is chosen for the values of γ and r, the Y_0 equation is

$$Y_0''(z) = zY_0(z),$$

which is called Airy's equation. The generic solution to this is

$$Y_0(z) = C\mathrm{Ai}(z) + D\mathrm{Bi}(z),$$

where Ai is the Airy function of first kind and Bi is Airy function of the second kind, and C and D are constants. See the figure on the next slide.

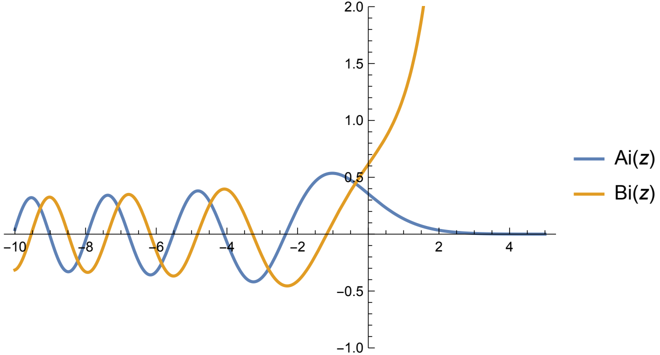

Figure: Plots of the Airy Functions, Ai and Bi.

Asymptotic Behavior of the Airy Functions

The Airy functions are oscillating for $z < 0$ and asymptotically decaying (Ai) and growing (Bi), respectively, for $z > 0$. For matching, we need to know what are asymptotic behaviors of the Airy functions for $|z| \nearrow \infty$. According to Bender and Orszag (1978), these are

$$\mathrm{Ai}(z) \sim \frac{\exp\left(-\frac{2}{3}z^{3/2}\right)}{2\sqrt{\pi}z^{1/4}}, \quad \text{as} \quad z \nearrow \infty, \tag{40}$$

$$\mathrm{Bi}(z) \sim \frac{\exp\left(\frac{2}{3}z^{3/2}\right)}{\sqrt{\pi}z^{1/4}}, \quad \text{as} \quad z \nearrow \infty, \tag{41}$$

$$\mathrm{Ai}(-z) \sim \frac{1}{\sqrt{\pi}z^{1/4}} \sin\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right), \quad \text{as} \quad z \nearrow \infty, \tag{42}$$

$$\mathrm{Bi}(-z) \sim \frac{1}{\sqrt{\pi}z^{1/4}} \cos\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right), \quad \text{as} \quad z \nearrow \infty, \tag{43}$$

which are well-known. These asymptotic behaviors are apparent in plots.

Left-Hand Outer-Solution Region

In the left-hand region, let us make the further assumption that

$$q(x) \leq q_{0,L} < 0, \quad \text{on} \quad \Omega_L := (-\infty, x_{1,L}).$$

It is clear that the solution on this region should be of the form

$$\begin{aligned} y_L(x) = & \frac{E}{\sqrt[4]{-q(x)}} \cos\left(\frac{1}{\varepsilon}\int_0^x \sqrt{-q(t)}\,\mathrm{d}t\right) \\ & + \frac{F}{\sqrt[4]{-q(x)}} \sin\left(\frac{1}{\varepsilon}\int_0^x \sqrt{-q(t)}\,\mathrm{d}t\right), \quad x \in \Omega_L, \end{aligned} \tag{44}$$

where E and F are real constants that will be determined via matching. Observe that, for $x < 0$,

$$\int_0^x \sqrt{-q(t)}\,\mathrm{d}t < 0.$$

Matching the Right-Hand and Inner Solutions

As usual, we introduce the intermediate variable

$$w = \frac{x}{\eta(\varepsilon)},$$

where $w > 0$ is fixed and finite, and we assume that

$$0 < \varepsilon^{2/3} = \delta(\varepsilon) \ll \eta(\varepsilon) \ll 1,$$

as $\varepsilon \searrow 0$. Thus, from (38), we

$$z = \frac{a^{1/3}}{\varepsilon^{2/3}} x = \frac{a^{1/3}\eta(\varepsilon)}{\varepsilon^{2/3}} w.$$

Let us perform an $N = 0 = M$ matching, according to the usual fashion. Starting with the right outer solution at leading order,

$$y_R(x) = y_R(w\eta) = \frac{A}{\sqrt[4]{q(w\eta)}} \exp\left(-\frac{1}{\varepsilon}\int_0^{w\eta} \sqrt{q(t)}\,\mathrm{d}t\right),$$

we look for a matching function that satisfies

$$y_R(w\eta) \sim y_{\mathrm{match}}^{0,0,R}(w;\varepsilon),$$

for fixed, finite $w > 0$, as $\varepsilon \searrow 0$.

Right-Hand Match Function

Now, observe that

$$q(u) = au + O(u^2), \quad \text{as} \quad u \searrow 0.$$

We assume that this holds uniformly on the matching region. Next, using the binomial series, it follows that

$$\sqrt{q(u)} = \sqrt{au} + O(u^{3/2}), \quad \text{as} \quad u \searrow 0.$$

It follows that

$$Q(u) := \int_0^u \sqrt{q(t)}\,\mathrm{d}t = \frac{2}{3}\sqrt{a}u^{3/2} + O(u^{5/2}), \quad \text{as} \quad u \searrow 0.$$

Again, we assume that this holds uniformly on the matching region. Likewise

$$\frac{1}{\sqrt[4]{q(u)}} = \frac{1}{\sqrt[4]{au}} + O(u^{3/4}), \quad \text{as} \quad u \searrow 0,$$

uniformly on the matching region. It now follows that

$$y^{0,0,R}_{\mathrm{match}}(w;\varepsilon) = \frac{A}{\sqrt[4]{aw\eta}} \exp\left(-\frac{2}{3\varepsilon}\sqrt{a}w^{3/2}\eta^{3/2}\right).$$

Inner Match Function

Next, using the inner solution, we seek a matching function that satisfies

$$Y_0(z)\varepsilon^{\gamma} = Y_0\left(\frac{a^{1/3}\eta(\varepsilon)}{\varepsilon^{2/3}}w\right)\varepsilon^{\gamma} \sim y_{\mathrm{match}}^{0,0,\mathrm{in},+}(w;\varepsilon),$$

for fixed, finite $w > 0$, as $\varepsilon \searrow 0$. Using (40) and (43), we find

$$y_{\mathrm{match}}^{0,0,\mathrm{in},+}(w;\varepsilon) = \frac{C\varepsilon^{1/6+\gamma}}{2\sqrt{\pi}a^{1/12}\eta^{1/4}w^{1/4}}\exp\left(-\frac{2}{3\varepsilon}\sqrt{a}\eta^{3/2}w^{3/2}\right) + \frac{D\varepsilon^{1/6+\gamma}}{\sqrt{\pi}a^{1/12}\eta^{1/4}w^{1/4}}\exp\left(\frac{2}{3\varepsilon}\sqrt{a}\eta^{3/2}w^{3/2}\right).$$

Matching Conditions

Matching thus requires that

$$\gamma = -\frac{1}{6},$$

$$D = 0,$$

$$\frac{A}{\sqrt[4]{aw\eta}} = \frac{C}{2\sqrt{\pi}a^{1/12}\eta^{1/4}w^{1/4}}.$$

The last condition implies that

$$C = \frac{2\sqrt{\pi}A}{a^{1/6}}.$$

The matching function is, therefore,

$$y_{\mathrm{match}}^{0,0,\mathrm{in},R}(w;\varepsilon) = \frac{C}{2\sqrt{\pi}a^{1/12}\eta^{1/4}w^{1/4}} \exp\left(-\frac{2}{3\varepsilon}\sqrt{a}\eta^{3/2}w^{3/2}\right). \tag{45}$$

Matching the Left-Hand and Inner Solutions

Next, we introduce the intermediate variable

$$w = \frac{x}{\eta(\varepsilon)},$$

where $w < 0$ is fixed and finite, and we assume that

$$0 < \varepsilon^{2/3} = \delta(\varepsilon) \ll \eta(\varepsilon) \ll 1,$$

as $\varepsilon \searrow 0$. Observe that w and x are now negative. The matching scale, η, is not necessarily the same as above. As before, from (38), we have

$$z = \frac{a^{1/3}}{\varepsilon^{2/3}} x = \frac{a^{1/3}\eta(\varepsilon)}{\varepsilon^{2/3}} w.$$

The variable z is also negative herein.

Left-Hand Match Function

Let us perform an $N = 0 = M$ matching between the inner solution and the left-hand outer solution. Starting with the left outer solution at leading order,

$$\begin{aligned} y_L(x) &= y_L(w\eta) \\ &= \frac{E}{\sqrt[4]{-q(w\eta)}} \cos\left(\frac{1}{\varepsilon}\int_0^{w\eta} \sqrt{-q(t)}\,\mathrm{d}t\right) \\ &\quad + \frac{F}{\sqrt[4]{-q(w\eta)}} \sin\left(\frac{1}{\varepsilon}\int_0^{w\eta} \sqrt{-q(t)}\,\mathrm{d}t\right). \end{aligned}$$

We look for a matching function that satisfies

$$y_L(w\eta) \sim y_{\mathrm{match}}^{0,0,L}(w;\varepsilon),$$

for fixed, finite $w < 0$, as $\varepsilon \searrow 0$. It follow from an analysis similar to what was used for the right-hand matching that

$$\begin{aligned} y_{\mathrm{match}}^{0,0,L}(w;\varepsilon) &= \frac{E}{\sqrt[4]{-aw\eta}} \cos\left(-\frac{2}{3\varepsilon}\sqrt{a}(-w)^{3/2}\eta^{3/2}\right) \\ &\quad + \frac{F}{\sqrt[4]{-aw\eta}} \sin\left(-\frac{2}{3\varepsilon}\sqrt{a}(-w)^{3/2}\eta^{3/2}\right). \end{aligned}$$

Inner Match Function

For the inner solution with $z < 0$, we seek a matching function that satisfies

$$Y_0(z)\varepsilon^\gamma = Y_0\left(\frac{a^{1/3}\eta(\varepsilon)}{\varepsilon^{2/3}}w\right)\varepsilon^\gamma \sim y_{\mathrm{match}}^{0,0,\mathrm{in},-}(w;\varepsilon),$$

for fixed, finite $w < 0$, as $\varepsilon \searrow 0$. Using (42) and the sine angle summation formula, we find

$$\begin{aligned}
y_{\mathrm{match}}^{0,0,\mathrm{in},-}(w;\varepsilon) &= \frac{C\varepsilon^{1/6+\gamma}}{\sqrt{\pi}a^{1/12}\eta^{1/4}(-w)^{1/4}}\sin\left(\frac{2}{3\varepsilon}\sqrt{a}\eta^{3/2}(-w)^{3/2} + \frac{\pi}{4}\right) \\
&= \frac{C\varepsilon^{1/6+\gamma}}{\sqrt{2\pi}a^{1/12}\eta^{1/4}(-w)^{1/4}}\sin\left(\frac{2}{3\varepsilon}\sqrt{a}\eta^{3/2}(-w)^{3/2}\right) \\
&\quad + \frac{C\varepsilon^{1/6+\gamma}}{\sqrt{2\pi}a^{1/12}\eta^{1/4}(-w)^{1/4}}\cos\left(\frac{2}{3\varepsilon}\sqrt{a}\eta^{3/2}(-w)^{3/2}\right) \\
&= \frac{-C\varepsilon^{1/6+\gamma}}{\sqrt{2\pi}a^{1/12}\eta^{1/4}(-w)^{1/4}}\sin\left(-\frac{2}{3\varepsilon}\sqrt{a}\eta^{3/2}(-w)^{3/2}\right) \\
&\quad + \frac{C\varepsilon^{1/6+\gamma}}{\sqrt{2\pi}a^{1/12}\eta^{1/4}(-w)^{1/4}}\cos\left(-\frac{2}{3\varepsilon}\sqrt{a}\eta^{3/2}(-w)^{3/2}\right).
\end{aligned}$$

Matching Conditions

Matching requires,

$$\gamma = -\frac{1}{6}, \quad \text{(consistent)}$$

$$\frac{E}{\sqrt[4]{-aw\eta}} = \frac{C}{\sqrt{2\pi}a^{1/12}\eta^{1/4}(-w)^{1/4}},$$

$$\frac{F}{\sqrt[4]{-aw\eta}} = \frac{-C}{\sqrt{2\pi}a^{1/12}\eta^{1/4}(-w)^{1/4}}.$$

The constants E and F are,

$$E = \frac{Ca^{1/6}}{\sqrt{2\pi}} = \frac{a^{1/6}}{\sqrt{2\pi}}\frac{2\sqrt{\pi}A}{a^{1/6}} = \sqrt{2}A,$$

$$F = -\frac{Ca^{1/6}}{\sqrt{2\pi}} = -\frac{a^{1/6}}{\sqrt{2\pi}}\frac{2\sqrt{\pi}A}{a^{1/6}} = -\sqrt{2}A.$$

The Connection Formulae

Suppose that Ω_L, Ω_I and Ω_R are overlapping domains, as described above. To sum up, the piecewise WKB approximation for the single-turning-point problem is as follows: as $\varepsilon \searrow 0$,

$$
\begin{aligned}
y(x;\varepsilon) &\sim \begin{cases}
\frac{\sqrt{2}A}{\sqrt[4]{-q(x)}} \cos\left(\frac{1}{\varepsilon}\int_0^x \sqrt{-q(t)}\,\mathrm{d}t\right) \\
\quad - \frac{\sqrt{2}A}{\sqrt[4]{-q(x)}} \sin\left(\frac{1}{\varepsilon}\int_0^x \sqrt{-q(t)}\,\mathrm{d}t\right), & x \in \Omega_L, \\
\frac{2\sqrt{\pi}A}{(\varepsilon a)^{1/6}} \mathrm{Ai}\left(\frac{a^{1/3}}{\varepsilon^{2/3}}x\right), & x \in \Omega_I, \\
\frac{A}{\sqrt[4]{q(x)}} \exp\left(-\frac{1}{\varepsilon}\int_0^x \sqrt{q(t)}\,\mathrm{d}t\right), & x \in \Omega_R,
\end{cases} \\
&= \begin{cases}
\frac{2A}{\sqrt[4]{-q(x)}} \sin\left(\frac{1}{\varepsilon}\int_x^0 \sqrt{-q(t)}\,\mathrm{d}t + \frac{\pi}{4}\right), & x \in \Omega_L, \\
\frac{2\sqrt{\pi}A}{(\varepsilon a)^{1/6}} \mathrm{Ai}\left(\frac{a^{1/3}}{\varepsilon^{2/3}}x\right), & x \in \Omega_I, \\
\frac{A}{\sqrt[4]{q(x)}} \exp\left(-\frac{1}{\varepsilon}\int_0^x \sqrt{q(t)}\,\mathrm{d}t\right), & x \in \Omega_R,
\end{cases} \qquad (46)
\end{aligned}
$$

which, though our derivation is somewhat different, agrees with the result in Bender and Orszag (1978). These formulae are called the *Connection Formulae* in the quantum mechanics literature.

Langer's Uniform Approximation

In a 1931 paper, Langer gave a single composite, uniformly-valid version of result above. See, for example, Bender and Orszag (1978), for details on the derivation. Langer's expression is

$$y(x;\varepsilon) \sim 2\sqrt{\pi}A\left(\frac{3}{2\varepsilon}s_0(x)\right)^{1/6}(q(x))^{-1/4}\mathrm{Ai}\left[\left(\frac{3}{2\varepsilon}s_0(x)\right)^{2/3}\right], \quad \text{as} \quad \varepsilon \searrow 0,$$

where

$$s_0(x) = \int_0^x \sqrt{q(t)}\,\mathrm{d}t.$$

Finally, we mention that, to obtain a value of the unknown constant A, one merely has to invoke the remaining condition, namely that

$$y(0) = 1.$$