



Math 515

Essential Perturbation Theory and Asymptotic Analysis

Chapter 02

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Chapter 02, Part 1 of 2

Notation and Fundamental Definitions



Order Symbols



Big-Oh

In the following definitions, we will use the following common notation: for any numbers $a \in \mathbb{R}$ and $r > 0$,

$$I_r(a) := (a - r, a + r).$$

Definition (Big-Oh)

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f, g : D \times I_\delta(\varepsilon_o) \rightarrow \mathbb{C}$ are continuous functions. We say that f **is big-oh of g at $\mathbf{x} \in D$, as $\varepsilon \rightarrow \varepsilon_o$** , and we write $f(\mathbf{x}, \varepsilon) = O(g(\mathbf{x}, \varepsilon))$, as $\varepsilon \rightarrow \varepsilon_o$, iff there exists numbers $\delta_o = \delta_o(\mathbf{x}) \in (0, \delta)$ and $C = C(\mathbf{x}) \in (0, \infty)$, such that,

$$|f(\mathbf{x}, \varepsilon)| \leq C |g(\mathbf{x}, \varepsilon)|, \quad \forall \varepsilon \in I_{\delta_o}(\varepsilon_o). \quad (1)$$

We say that f **is big-oh of g uniformly in D , as $\varepsilon \rightarrow \varepsilon_o$** iff there exist numbers $\delta_o \in (0, \delta)$ and $C \in (0, \infty)$, both independent of $\mathbf{x} \in D$, such that,

$$|f(\mathbf{x}, \varepsilon)| \leq C |g(\mathbf{x}, \varepsilon)|, \quad \forall \varepsilon \in I_{\delta_o}(\varepsilon_o). \quad (2)$$



Example

Consider the function

$$f(\varepsilon) = 2\varepsilon^3 - \varepsilon,$$

for $\varepsilon > 0$. Then

$$f(\varepsilon) = O(\varepsilon), \quad \text{as } \varepsilon \searrow 0.$$

This is because the ε dominates the ε^3 term, for small values of $\varepsilon > 0$. In particular, if $0 < \varepsilon \leq 1$, then

$$\varepsilon^3 \leq \varepsilon.$$

Thus, if $0 < \varepsilon \leq \delta_o = 1$,

$$|f(\varepsilon)| \leq 2\varepsilon^3 + \varepsilon \leq 3\varepsilon.$$

The result holds with $C = 3$.

Big-Oh at Infinity



Definition (Big-Oh at Infinity)

Suppose that $M > 0$, and assume that $f, g : D \times (M, \infty) \rightarrow \mathbb{C}$ are continuous functions. We say that f is **big-oh of g at $\mathbf{x} \in D$, as $\lambda \rightarrow \infty$** , and we write $f(\mathbf{x}, \lambda) = O(g(\mathbf{x}, \lambda))$, as $\lambda \rightarrow \infty$, iff there exist numbers $M_o = M_o(\mathbf{x}) \in [M, \infty)$ and $C = C(\mathbf{x}) \in (0, \infty)$, such that,

$$|f(\mathbf{x}, \lambda)| \leq C |g(\mathbf{x}, \lambda)|, \quad \forall \lambda \in (M_o, \infty). \quad (3)$$

Similarly, we say that f is **big-oh of g uniformly in D , as $\lambda \rightarrow \infty$** iff there exist numbers $M_o \in [M, \infty)$ and $C \in (0, \infty)$, both independent of $\mathbf{x} \in D$, such that

$$|f(\mathbf{x}, \lambda)| \leq C |g(\mathbf{x}, \lambda)| \quad \forall \lambda \in (M_o, \infty). \quad (4)$$



Lemma

For any $m \in \mathbb{N}$, there is a constant $M_o = M_o(m) > 0$, such that, if $\lambda > M_o$, it follows that

$$e^\lambda \geq \lambda^m. \quad (5)$$

Proof.

Suppose that $\lambda > 0$. Using Taylor's Theorem,

$$e^\lambda = 1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6} + \cdots + \frac{\lambda^m}{m!} + \frac{\lambda^{m+1}}{(m+1)!} + \frac{\lambda^{m+2}}{(m+2)!} e^\eta,$$

for some $\eta \in (0, \lambda)$. Since $e^\eta \geq 1$, it follows that

$$e^\lambda \geq 1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6} + \cdots + \frac{\lambda^m}{m!} + \frac{\lambda^{m+1}}{(m+1)!} =: p(\lambda).$$

Since $p(\lambda)$ is a polynomial of degree $m+1$, there is an $M_o = M_o(m) > 0$, such that, if $\lambda > M_o$,

$$p(\lambda) \geq \lambda^m.$$





Corollary

For any $r > 0$, there is an $M_o = M_o(r) > 0$ such that, if $\lambda > M_o$, then

$$e^\lambda \geq \lambda^r, \quad (6)$$

and

$$e^{-\lambda} \leq \frac{1}{\lambda^r}. \quad (7)$$



Example

Suppose that

$$g(\lambda) = -e^{-\lambda} + \frac{4}{\lambda}.$$

Then,

$$g(\lambda) = O\left(\frac{1}{\lambda}\right), \quad \text{as } \lambda \rightarrow \infty.$$

To see this, observe that, for all $\lambda > 0$

$$\frac{1}{\lambda} \geq e^{-\lambda}.$$

Hence, for, say, $\lambda > 1 =: M_o$,

$$\begin{aligned} |g(\lambda)| &\leq e^{-\lambda} + \frac{4}{\lambda} \\ &\leq \frac{1}{\lambda} + \frac{4}{\lambda} \\ &= \frac{5}{\lambda}. \end{aligned}$$

The result holds with $C = 5$ and $M_o = 1$.



Example

In fact, we have the following more general result. Suppose that $p > 0$ and

$$g(\lambda) = C_1 e^{-\lambda} + \frac{C_2}{\lambda^p},$$

where $C_1, C_2 \in \mathbb{R}$ are constants. Then

$$g(\lambda) = O\left(\frac{1}{\lambda^p}\right), \quad \text{as } \lambda \rightarrow \infty.$$

The reader should provide the details.



Example

Suppose that

$$a_n := a(n) = \frac{2}{3}n^2 - \frac{1}{2}n^3.$$

Then,

$$a_n = O(n^3), \quad \text{as } n \rightarrow \infty.$$

We want to show that there is an integer $N_o \geq 1$, such that, if $n \geq N_o$, then

$$|a_n| \leq Cn^3,$$

for some constant $C > 0$. Let us prove this by finding $N_o \geq 1$ and $C > 0$. Clearly, if $n \geq N_o = 1$, it follows that

$$n^3 \geq n^2.$$

So, if $n \geq N_o$,

$$|a_n| \leq \frac{2}{3}n^2 + \frac{1}{2}n^3 \leq \frac{2}{3}n^3 + \frac{1}{2}n^3 \leq \frac{7}{6}n^3.$$

The result follows with $C = \frac{7}{6}$ and $N_o = 1$.

Little-Oh



Definition (Little-Oh)

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f, g : D \times I_\delta(\varepsilon_o) \rightarrow \mathbb{C}$ are continuous functions. We say that f **is little-oh of g at $\mathbf{x} \in D$, as $\varepsilon \rightarrow \varepsilon_o$** , and we write $f(\mathbf{x}, \varepsilon) = o(g(\mathbf{x}, \varepsilon))$, as $\varepsilon \rightarrow \varepsilon_o$, iff, for every $\alpha > 0$, there exists a number $\delta_o = \delta_o(\mathbf{x}, \alpha) \in (0, \delta)$, such that

$$|f(\mathbf{x}, \varepsilon)| \leq \alpha |g(\mathbf{x}, \varepsilon)|, \quad \forall \varepsilon \in I_{\delta_o}(\varepsilon_o). \quad (8)$$

We say that f **is little-oh of g uniformly in D , as $\varepsilon \rightarrow \varepsilon_o$** iff, for every $\alpha > 0$, there exists a number $\delta_o = \delta_o(\alpha) \in (0, \delta)$, independent of $\mathbf{x} \in D$, such that

$$|f(\mathbf{x}, \varepsilon)| \leq \alpha |g(\mathbf{x}, \varepsilon)|, \quad \forall \varepsilon \in I_{\delta_o}(\varepsilon_o). \quad (9)$$

Little-Oh at Infinity



Definition (Little-Oh at Infinity)

Suppose that $M > 0$, and assume that $f, g : D \times (M, \infty) \rightarrow \mathbb{C}$ are continuous functions. We say that f **is little-oh of g at $\mathbf{x} \in D$, as $\lambda \rightarrow \infty$** , and we write $f(\mathbf{x}, \lambda) = o(g(\mathbf{x}, \lambda))$, as $\lambda \rightarrow \infty$, iff, for every $\alpha > 0$, there exists a number $M_o = M_o(\mathbf{x}, \alpha) \in [M, \infty)$, such that

$$|f(\mathbf{x}, \lambda)| \leq \alpha |g(\mathbf{x}, \lambda)|, \quad \forall \lambda \in (M_o, \infty). \quad (10)$$

Similarly, we say that f **is little-oh of g uniformly in D , as $\lambda \rightarrow \infty$** iff, for every $\alpha > 0$, there exists a number $M_o = M_o(\alpha) \in [M, \infty)$, independent of $\mathbf{x} \in D$, such that

$$|f(\mathbf{x}, \lambda)| \leq \alpha |g(\mathbf{x}, \lambda)|, \quad \forall \lambda \in (M_o, \infty). \quad (11)$$



Remark

To save needless writing, and reading as well, we will state and prove a few properties only involving our definitions for the case that $\varepsilon \rightarrow \varepsilon_0$, the finite limit case. However, the reader should be able to prove analogous results for the case that $\lambda \rightarrow \infty$, the infinite limit case, with only minor changes to the assumptions and arguments.



Proposition

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f, g : D \times I_\delta(\varepsilon_o) \rightarrow \mathbb{C}$ are continuous functions and $\mathbf{x} \in D$. Suppose that

$$g(\mathbf{x}, \varepsilon) \neq 0, \quad \forall \varepsilon \in I_\delta^*(\varepsilon_o) := I_\delta(\varepsilon_o) \setminus \{\varepsilon_o\},$$

and there exists some number $L = L(\mathbf{x}) \in \mathbb{C}$ such that

$$\lim_{\varepsilon \rightarrow \varepsilon_o} \frac{f(\mathbf{x}, \varepsilon)}{g(\mathbf{x}, \varepsilon)} = L.$$

Then, $f(\mathbf{x}, \varepsilon) = O(g(\mathbf{x}, \varepsilon))$, as $\varepsilon \rightarrow \varepsilon_o$.

Proof.

Using the definition of limit, for any $\varrho > 0$, there exists a number $\beta = \beta(\mathbf{x}, \varrho) > 0$, such that, if $\varepsilon \in I_\delta^*(\varepsilon_o) \cap I_\beta(\varepsilon_o)$, it follows that

$$\left| \frac{f(\mathbf{x}, \varepsilon)}{g(\mathbf{x}, \varepsilon)} - L \right| < \varrho.$$



Proof Cont.

Using the reverse triangle inequality, we find

$$|f(\mathbf{x}, \varepsilon)| - |L| \cdot |g(\mathbf{x}, \varepsilon)| \leq |f(\mathbf{x}, \varepsilon) - L \cdot g(\mathbf{x}, \varepsilon)| < \varrho \cdot |g(\mathbf{x}, \varepsilon)|,$$

which, in turn, implies that

$$|f(\mathbf{x}, \varepsilon)| < (|L| + \varrho) |g(\mathbf{x}, \varepsilon)|,$$

provided $\varepsilon \in I_\delta^*(\varepsilon_o) \cap I_\beta(\varepsilon_o)$. Take $\delta_o := \min(\delta, \beta)$ and $C = |L| + \varrho$. □



Proposition (Little-Oh Implies Big-Oh)

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f, g : D \times I_\delta(\varepsilon_o) \rightarrow \mathbb{C}$ are continuous functions and, for some $\mathbf{x} \in D$, $f(\mathbf{x}, \varepsilon) = o(g(\mathbf{x}, \varepsilon))$, as $\varepsilon \rightarrow \varepsilon_o$. Then, it follows that $f(\mathbf{x}, \varepsilon) = O(g(\mathbf{x}, \varepsilon))$, as $\varepsilon \rightarrow \varepsilon_o$.

Proof.

Exercise.





Proposition

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f, g : D \times I_\delta(\varepsilon_o) \rightarrow \mathbb{C}$ are continuous, $\mathbf{x} \in D$, and

$$g(\mathbf{x}, \varepsilon) \neq 0, \quad \forall \varepsilon \in I_\delta^*(\varepsilon_o) := I_\delta(\varepsilon_o) \setminus \{\varepsilon_o\}.$$

Then, $f(\mathbf{x}, \varepsilon) = o(g(\mathbf{x}, \varepsilon))$ at $\mathbf{x} \in D$, as $\varepsilon \rightarrow \varepsilon_o$ iff

$$\lim_{\varepsilon \rightarrow \varepsilon_o} \frac{f(\mathbf{x}, \varepsilon)}{g(\mathbf{x}, \varepsilon)} = 0.$$

Proof.

(\implies): Suppose that $f(\mathbf{x}, \varepsilon) = o(g(\mathbf{x}, \varepsilon))$ at $\mathbf{x} \in D$, as $\varepsilon \rightarrow \varepsilon_o$. For every $\alpha > 0$, there exists a number $\delta_o = \delta_o(\mathbf{x}, \alpha) \in (0, \delta)$, such that

$$|f(\mathbf{x}, \varepsilon)| \leq \alpha |g(\mathbf{x}, \varepsilon)|, \quad \forall \varepsilon \in I_{\delta_o}(\varepsilon_o).$$



Proof Cont.

Thus, it follows that

$$\left| \frac{f(\mathbf{x}, \varepsilon)}{g(\mathbf{x}, \varepsilon)} - 0 \right| \leq \alpha, \quad \forall \varepsilon \in I_{\delta_o}^*(\varepsilon_o).$$

Since $\alpha > 0$ is arbitrary,

$$\lim_{\varepsilon \rightarrow \varepsilon_o} \frac{f(\mathbf{x}, \varepsilon)}{g(\mathbf{x}, \varepsilon)} = 0.$$

(\Leftarrow): This direction is similarly straightforward. □



Example

Suppose that $p > 0$. Then

$$e^{-1/\varepsilon} = o(\varepsilon^p), \quad \text{as } \varepsilon \searrow 0,$$

for any p . To establish this fact, it suffices to show that

$$\lim_{\varepsilon \searrow 0} \frac{e^{-1/\varepsilon}}{\varepsilon^p},$$

for any $p > 0$. This follows from a previous corollary, since

$$e^{-1/\varepsilon} \leq \varepsilon^p,$$

for all $0 < \varepsilon < \varepsilon_o$, for some $\varepsilon_o = \varepsilon(p) > 0$. A similar argument shows that

$$e^{-\lambda} = o\left(\frac{1}{\lambda^p}\right), \quad \text{as } \lambda \rightarrow \infty.$$



Transcendentally Small Terms (TST)

Definition

Suppose that $\varepsilon_o \in \mathbb{R}$, $\delta > 0$, and $f : I_\delta(\varepsilon_o) \rightarrow \mathbb{C}$ is continuous. We say that f **is a transcendentally small term, as $\varepsilon \rightarrow \varepsilon_o$** , abbreviated TST, iff $f = o((\varepsilon - \varepsilon_o)^r)$, as $\varepsilon \rightarrow \varepsilon_o$, for all $r \geq 0$.

Definition

Suppose that $M > 0$ and $f : (M, \infty) \rightarrow \mathbb{C}$ is continuous. We say that f **is a transcendentally small term, as $\lambda \rightarrow \infty$** , abbreviated TST, iff $f = o(\lambda^{-r})$, as $\lambda \rightarrow \infty$, for all $r \geq 0$.

Example

Consider the function

$$f(\lambda) = e^{-\lambda},$$

for $\lambda > 0$. Then

$$f(\lambda) = \text{TST}, \quad \text{as } \lambda \rightarrow \infty.$$



Definition (Asymptotic Approximation)

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f, g : D \times I_\delta(\varepsilon_o) \rightarrow \mathbb{C}$ are continuous functions. We say that g **is an asymptotic approximation of f at $\mathbf{x} \in D$, as $\varepsilon \rightarrow \varepsilon_o$** , and we write $f(\mathbf{x}, \varepsilon) \sim g(\mathbf{x}, \varepsilon)$, as $\varepsilon \rightarrow \varepsilon_o$, iff $f(\mathbf{x}, \varepsilon) - g(\mathbf{x}, \varepsilon) = o(g(\mathbf{x}, \varepsilon))$, as $\varepsilon \rightarrow \varepsilon_o$.

We immediately have the following: $f(\mathbf{x}, \varepsilon) \sim g(\mathbf{x}, \varepsilon)$, as $\varepsilon \rightarrow \varepsilon_o$, iff

$$\lim_{\varepsilon \rightarrow \varepsilon_o} \frac{f(\mathbf{x}, \varepsilon) - g(\mathbf{x}, \varepsilon)}{g(\mathbf{x}, \varepsilon)} = 0 \iff \lim_{\varepsilon \rightarrow \varepsilon_o} \frac{f(\mathbf{x}, \varepsilon)}{g(\mathbf{x}, \varepsilon)} = 1.$$



Proposition

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f, g : D \times I_\delta(\varepsilon_o) \rightarrow \mathbb{C}$ are continuous functions and $f(\mathbf{x}, \varepsilon) \sim g(\mathbf{x}, \varepsilon)$, as $\varepsilon \rightarrow \varepsilon_o$. If

$$f(\mathbf{x}, \varepsilon) \neq 0, \quad g(\mathbf{x}, \varepsilon) \neq 0, \quad \forall \varepsilon \in I_\delta^*(\varepsilon_o) := I_\delta(\varepsilon_o) \setminus \{\varepsilon_o\},$$

then $g(\mathbf{x}, \varepsilon) \sim f(\mathbf{x}, \varepsilon)$, as $\varepsilon \rightarrow \varepsilon_o$.

Proof.

Exercise. □



Asymptotic Sequences and Series



Asymptotic Sequences

Definition (Asymptotic Sequence)

Suppose that $\varepsilon_o \in \mathbb{R}$ and $\delta > 0$. Assume that $\phi_k : I_\delta(\varepsilon_o) \rightarrow \mathbb{C}$ is a continuous function, for each $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The sequence $\{\phi_k\}_{k=0}^\infty$ is called an **asymptotic sequence**, as $\varepsilon \rightarrow \varepsilon_o$ iff $\phi_{k+1}(\varepsilon) = o(\phi_k(\varepsilon))$, as $\varepsilon \rightarrow \varepsilon_o$, for all $k \in \mathbb{N}_0$.

Example

Let us give two common examples. First, for $\varepsilon \rightarrow 0$, we commonly use

$$\phi_k = \varepsilon^k, \quad k \in \mathbb{N}_0.$$

For $\lambda \rightarrow \infty$, we will commonly use

$$\phi_k = \frac{1}{\lambda^k}, \quad k \in \mathbb{N}_0.$$

These are both asymptotic sequences, as is easy to check.



Definition (Asymptotic Series)

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, $\delta > 0$, and $n \in \mathbb{N}_0$. Assume that $f : D \times I_\delta(\varepsilon_o) \rightarrow \mathbb{C}$ is a continuous function, and $a_k : D \rightarrow \mathbb{C}$ is a continuous function, for each $k \in \{0, \dots, n\}$. Suppose that $\{\phi_k\}_{k=0}^\infty$ is an asymptotic sequence, as $\varepsilon \rightarrow \varepsilon_o$, on $I_\delta(\varepsilon_o)$. We say that $\sum_{k=0}^n a_k(\cdot)\phi_k(\cdot)$ **is a finite asymptotic series approximation of f at $\mathbf{x} \in D$, as $\varepsilon \rightarrow \varepsilon_o$** , and we write

$$f(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^n a_k(\mathbf{x})\phi_k(\varepsilon), \quad \text{as } \varepsilon \rightarrow \varepsilon_o,$$

iff $f(\mathbf{x}, \varepsilon) - \sum_{k=0}^m a_k(\mathbf{x})\phi_k(\varepsilon) = o(\phi_m(\mathbf{x}, \varepsilon))$, as $\varepsilon \rightarrow \varepsilon_o$, for each $m \in \{0, \dots, n\}$. We say that $\sum_{k=0}^\infty a_k(\cdot)\phi_k(\cdot)$ **is an infinite asymptotic series approximation of f at $\mathbf{x} \in D$, as $\varepsilon \rightarrow \varepsilon_o$** , and we write

$$f(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^\infty a_k(\mathbf{x})\phi_k(\varepsilon), \quad \text{as } \varepsilon \rightarrow \varepsilon_o,$$

iff $f(\mathbf{x}, \varepsilon) - \sum_{k=0}^m a_k(\mathbf{x})\phi_k(\varepsilon) = o(\phi_m(\mathbf{x}, \varepsilon))$, as $\varepsilon \rightarrow \varepsilon_o$, for each $m \in \mathbb{N}_0$.



Example

The most common example that we will encounter involves approximations of the form

$$f(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{\infty} a_k(\mathbf{x})(\varepsilon - \varepsilon_o)^k, \quad \text{as } \varepsilon \rightarrow \varepsilon_o.$$

The expansion $\sum_{k=0}^{\infty} a_k(\mathbf{x})(\varepsilon - \varepsilon_o)^k$ is called an asymptotic power series. We leave it to the reader to check that $\{(\varepsilon - \varepsilon_o)^k\}_{k=0}^{\infty}$ is an asymptotic sequence as $\varepsilon \rightarrow \varepsilon_o$. But, we will also encounter approximations of the form

$$f(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{\infty} a_k(\mathbf{x})(\varepsilon - \varepsilon_o)^{\alpha \cdot k}, \quad \text{as } \varepsilon \rightarrow \varepsilon_o,$$

where $\alpha \in (0, 1)$ is a given fraction. Common examples are $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{3}$. The expansion $\sum_{k=0}^{\infty} a_k(\mathbf{x})(\varepsilon - \varepsilon_o)^{\alpha \cdot k}$ is called a generalized asymptotic power series. The reader can check that $\{(\varepsilon - \varepsilon_o)^{\alpha \cdot k}\}_{k=0}^{\infty}$ is also an asymptotic sequence as $\varepsilon \rightarrow \varepsilon_o$.



Uniqueness of the Coefficients

Proposition

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f : D \times I_\delta(\varepsilon_o) \rightarrow \mathbb{C}$ is a continuous function, and $a_k : D \rightarrow \mathbb{C}$ is a continuous function, for each $k \in \{0, \dots, n\}$. Suppose that $\{\phi_k\}_{k=0}^\infty$ is an asymptotic sequence, as $\varepsilon \rightarrow \varepsilon_o$, on $I_\delta(\varepsilon_o)$, with $\phi_n(\varepsilon_o) \neq 0$. Assume that $f(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^\infty a_k(\mathbf{x})\phi_k(\varepsilon)$, as $\varepsilon \rightarrow \varepsilon_o$. Then,

$$a_0(\mathbf{x}) = \lim_{\varepsilon \rightarrow \varepsilon_o} \frac{f(\mathbf{x}, \varepsilon)}{\phi_0(\varepsilon)} \quad (12)$$

and

$$a_m(\mathbf{x}) = \lim_{\varepsilon \rightarrow \varepsilon_o} \frac{f(\mathbf{x}, \varepsilon) - \sum_{k=0}^{m-1} a_k(\mathbf{x})\phi_k(\varepsilon)}{\phi_m(\varepsilon)}, \quad \forall m \in \mathbb{N}. \quad (13)$$

Thus, for a given asymptotic sequence, the coefficient terms in the series expansion are uniquely determined.



Proof of the Last Proposition.

If $f(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{\infty} a_k(\mathbf{x})\phi_k(\varepsilon)$, as $\varepsilon \rightarrow \varepsilon_0$, then, by definition, this implies that

$$\lim_{\varepsilon \rightarrow \varepsilon_0} \frac{f(\mathbf{x}, \varepsilon) - \sum_{k=0}^m a_k(\mathbf{x})\phi_k(\varepsilon)}{\phi_m(\varepsilon)} = 0,$$

which is equivalent to

$$\lim_{\varepsilon \rightarrow \varepsilon_0} \frac{f(\mathbf{x}, \varepsilon) - \sum_{k=0}^{m-1} a_k(\mathbf{x})\phi_k(\varepsilon) - a_m(\mathbf{x})\phi_m(\varepsilon)}{\phi_m(\varepsilon)} = 0,$$

which, in turn, is equivalent to

$$\lim_{\varepsilon \rightarrow \varepsilon_0} \frac{f(\mathbf{x}, \varepsilon) - \sum_{k=0}^{m-1} a_k(\mathbf{x})\phi_k(\varepsilon)}{\phi_m(\varepsilon)} = a_m(\mathbf{x}),$$





Proposition

Suppose that $D \subset \mathbb{C}^d$ is an open set, $\varepsilon_o \in \mathbb{R}$, and $\delta > 0$. Assume that $f, g : D \times I_\delta(\varepsilon_o) \rightarrow \mathbb{C}$ are continuous functions and $a_k, b_k : D \rightarrow \mathbb{C}$ are continuous function, for each $k \in \{0, \dots, n\}$. Suppose that $\{\phi_k\}_{k=0}^\infty$ is an asymptotic sequence, as $\varepsilon \rightarrow \varepsilon_o$, on $I_\delta(\varepsilon_o)$, with $\phi_n(\varepsilon_o) \neq 0$. Assume that

$$f(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{\infty} a_k(\mathbf{x})\phi(\varepsilon) \quad \text{and} \quad g(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{\infty} b_k(\mathbf{x})\phi(\varepsilon), \quad \text{as } \varepsilon \rightarrow \varepsilon_o.$$

Then,

$$C \cdot f(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{\infty} C a_k(\mathbf{x})\phi(\varepsilon), \quad \text{as } \varepsilon \rightarrow \varepsilon_o,$$

$$f(\mathbf{x}, \varepsilon) + g(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{\infty} (a_k(\mathbf{x}) + b_k(\mathbf{x}))\phi(\varepsilon), \quad \text{as } \varepsilon \rightarrow \varepsilon_o,$$



Proposition (Cont.)

and

$$f(\mathbf{x}, \varepsilon)g(\mathbf{x}, \varepsilon) \sim \sum_{k=0}^{\infty} c_k(\mathbf{x})\phi(\varepsilon), \quad \text{as } \varepsilon \rightarrow \varepsilon_o,$$

where

$$c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0.$$

Proof.

The proof is an exercise.





Proposition

Suppose that $\varepsilon_0 \in \mathbb{R}$, $\delta > 0$, and $f : I_\delta(\varepsilon_0) \rightarrow \mathbb{C}$ is a continuous function satisfying

$$f(\varepsilon) \sim \sum_{k=0}^{\infty} a_k (\varepsilon - \varepsilon_0)^k, \quad \text{as } \varepsilon \rightarrow \varepsilon_0.$$

If $\phi : I_\delta(\varepsilon_0) \rightarrow \mathbb{C}$ is a TST, as $\varepsilon \rightarrow \varepsilon_0$, then

$$f(\varepsilon) + \phi(\varepsilon) \sim \sum_{k=0}^{\infty} a_k (\varepsilon - \varepsilon_0)^k, \quad \text{as } \varepsilon \rightarrow \varepsilon_0.$$

Proof.

This follows immediately from the fact that

$$\phi(\varepsilon) \sim \sum_{k=0}^{\infty} b_k (\varepsilon - \varepsilon_0)^k, \quad \text{as } \varepsilon \rightarrow \varepsilon_0, \quad \text{where } b_k = 0, \quad \forall k \in \mathbb{N}_0.$$





Two Motivating Themes Revisited



Proposition

$$\log(n!) \sim n \log(n) - n + 1, \quad \text{as } n \rightarrow \infty. \quad (14)$$

Proof.

Set, as before, for any $n \in \mathbb{N}$,

$$L(n) := \log(n!)$$

and

$$I(n) := \int_1^n \log(x) \, dx = [x \log(x) - x]_{x=1}^{x=n} = n \log(n) - n + 1.$$

By definition (or, better, a natural extension of the definition), $L(n) \sim I(n)$, as $n \rightarrow \infty$, iff

$$\lim_{n \rightarrow \infty} \frac{L(n) - I(n)}{I(n)} \rightarrow 0.$$



Proof Cont.

In Chapter 1, we showed, using an argument involving Riemann sums, that

$$0 \leq \frac{L(n) - I(n)}{I(n)} \leq \frac{\log(n)}{I(n)}, \quad \forall n \in \mathbb{N}.$$

Since l'Hôpital's rule guarantees that

$$\frac{\log(n)}{I(n)} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

we can conclude the asymptotic result using the Squeeze Theorem. □



Proposition

Define, for each $\lambda \in [0, \infty)$,

$$I(\lambda) = \int_0^\infty \frac{e^{-\lambda t}}{1+t} dt. \quad (15)$$

Then

$$I(\lambda) \sim \sum_{k=0}^{\infty} (-1)^k \frac{k!}{\lambda^{k+1}}, \quad \text{as } \lambda \rightarrow \infty.$$

Proof.

Here $a_k = (-1)^k k!$ and $\phi_k(\lambda) = \frac{1}{\lambda^{k+1}}$. First of all, let us confirm that $\{\phi_k(\lambda)\}_{k=0}^\infty$ is an asymptotic sequence, as $\lambda \rightarrow \infty$. In particular, we must show that

$$\phi_{k+1}(\lambda) = o(\phi_k(\lambda)), \quad \text{as } \lambda \rightarrow \infty, \quad \forall k \in \mathbb{N}_0,$$

or, equivalently,

$$\lim_{\lambda \rightarrow \infty} \frac{\phi_{k+1}(\lambda)}{\phi_k(\lambda)} = 0, \quad \forall k \in \mathbb{N}_0.$$



Proof Cont.

This clearly follows, since

$$\frac{\phi_{k+1}(\lambda)}{\phi_k(\lambda)} = \frac{\lambda^{k+1}}{\lambda^{k+2}} = \frac{1}{\lambda}.$$

Next, recall that, in the construction of our approximation in Chapter 1, we showed that, precisely,

$$I(\lambda) = \sum_{k=0}^{n-1} (-1)^k \frac{k!}{\lambda^{k+1}} + R_n(\lambda),$$

where

$$R_n(\lambda) = \int_0^\infty (-1)^n \frac{t^n}{1+t} e^{-\lambda t} dt.$$



Proof Cont.

Now,

$$I(\lambda) \sim \sum_{k=0}^{\infty} a_k \phi_k(\lambda), \quad \text{as } \lambda \rightarrow \infty,$$

iff

$$I(\lambda) - \sum_{k=0}^{m-1} a_k \phi_k(\lambda) = o(\phi_{m-1}(\lambda)), \quad \text{as } \lambda \rightarrow \infty,$$

for each $m \in \mathbb{N}$. Notice that

$$R_m(\lambda) = I(\lambda) - \sum_{k=0}^{m-1} a_k \phi_k(\lambda).$$

Therefore, we only need to show that

$$R_m(\lambda) = \int_0^{\infty} (-1)^m \frac{t^m}{1+t} e^{-\lambda t} dt = o(\phi_{m-1}(\lambda)) = o\left(\frac{1}{\lambda^m}\right), \quad \text{as } \lambda \rightarrow \infty.$$



Proof Cont.

But, we know that

$$R_m(\lambda) = \int_0^\infty (-1)^m \frac{t^m}{1+t} e^{-\lambda t} dt = o(\phi_{m-1}(\lambda)) = o\left(\frac{1}{\lambda^m}\right), \quad \text{as } \lambda \rightarrow \infty,$$

is true, by definition, iff

$$\lim_{\lambda \rightarrow \infty} \lambda^m \int_0^\infty (-1)^m \frac{t^m}{1+t} e^{-\lambda t} dt = 0. \quad (16)$$

Since, as we have already shown in Chapter 1,

$$|R_m(\lambda)| \leq \frac{m!}{\lambda^{m+1}},$$

(16) must be satisfied, and the result follows. □