Math 515
Essential Perturbation Theory and Asymptotic Analysis
Chapter 04

Steven M. Wise

swise1@.utk.edu
University of Tennessee

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Chapter 04, Part 2 of 2

Contour Integration in the Complex Plane and the Method of Steepest Descent

Cauchy's Theorem and Integral Formulae

Theorem (Simplified Version of Cauchy's Theorem)

Suppose that $D \subset \mathbb{C}$ is an open, bounded, simply-connected set, and $f : D \to \mathbb{C}$ is analytic on D. Assume, additionally, that $f' : D \to \mathbb{C}$ is continuous. Suppose that $\gamma : [a, b] \to D$ is a smooth, simple, (that is, non-self-intersecting), counter-clockwise-oriented, closed curve (that is, satisfying $\gamma(a) = \gamma(b)$). Then,

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof.

Recall that, if $u = \Re(f)$ and $v = \Im(f)$, then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\boldsymbol{\gamma}} [u(x, y)\,\mathrm{d}x - v(x, y)\,\mathrm{d}y] + \mathrm{i}\int_{\boldsymbol{\gamma}} [v(x, y)\,\mathrm{d}x + u(x, y)\,\mathrm{d}y] \,.$$

Proof (Cont.)

Denote by $D_\gamma \subset D$, the subset of D whose boundary is the image of γ, that is,

$$\partial D_\gamma = \gamma([a,b]).$$

By Green's Theorem and the Cauchy-Riemann Theorem,

$$\int_{\boldsymbol{\gamma}} [u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y] = \int_{D_\gamma} \left(-\frac{\partial v}{\partial x}(x,y) - \frac{\partial u}{\partial y}(x,y) \right) \mathrm{d}\mathbf{x} = 0.$$

Similarly,

$$\int_{\boldsymbol{\gamma}} [v(x,y)\,\mathrm{d}x + u(x,y)\,\mathrm{d}y] = \int_{D_\gamma} \left(\frac{\partial u}{\partial x}(x,y) - \frac{\partial v}{\partial y}(x,y) \right) \mathrm{d}\mathbf{x} = 0.$$

$\square$

Theorem (Cauchy's Theorem)

Suppose that $D \subset \mathbb{C}$ is an open, simply connected, bounded set, and $f : D \to \mathbb{C}$ analytic on D. For any simple, piecewise smooth, closed contour $\gamma : [a, b] \to D$

$$\int_{\gamma} f(z)\,\mathrm{d}z = 0.$$

Theorem (Path Independence)

Suppose that $D \subset \mathbb{C}$ is an open, simply connected, bounded set, and $f : D \to \mathbb{C}$ is analytic on D. Suppose that $z_0, z_1 \in D$ are arbitrary points in D, and let $\gamma : [a, b] \to D$ and $\chi : [c, d] \to D$ be any two simple, piecewise smooth contours with

$$\gamma(a) = z_0 = \chi(c) \quad \textit{and} \quad \gamma(b) = z_1 = \chi(b).$$

Then,

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\chi f(z)\,\mathrm{d}z$$

Example

Suppose that $\gamma : [0, 2\pi] \to \mathbb{C}$ is a circular, counterclockwise oriented contour of radius r around the point $z_0 \in \mathbb{C}$. Then we can show that

$$\int_\gamma \frac{1}{z - z_0} \, \mathrm{d}z = 2\pi \mathrm{i}. \tag{1}$$

This result is particularly famous. We refer to the integral in (1) as the **residue integral**. To perform this calculation, note that the contour can be expressed as

$$\gamma(t) = z_0 + re^{\mathrm{i}t}.$$

It follows that

$$\begin{aligned} \int_\gamma \frac{1}{z - z_0} \, \mathrm{d}z &= \int_0^{2\pi} \frac{1}{z_0 + re^{\mathrm{i}t} - z_0} r\mathrm{i}e^{\mathrm{i}t} \, \mathrm{d}t \\ &= \int_0^{2\pi} \frac{r\mathrm{i}e^{\mathrm{i}t}}{re^{\mathrm{i}t}} \, \mathrm{d}t \\ &= 2\pi \mathrm{i}. \end{aligned}$$

Example (Cont.)

What if, on the other hand, the contour goes clockwise round the point z_0? In this case, the contour can be expressed as

$$\gamma(t) = z_0 + re^{-\mathrm{i}t},$$

and

$$\begin{aligned}\int_\gamma \frac{1}{z - z_0}\,\mathrm{d}z &= \int_0^{2\pi} \frac{-r\mathrm{i}e^{-\mathrm{i}t}}{z_0 + re^{-\mathrm{i}t} - z_0}\,\mathrm{d}t \\ &= \int_0^{2\pi} \frac{-r\mathrm{i}e^{-\mathrm{i}t}}{re^{-\mathrm{i}t}}\,\mathrm{d}t \\ &= -2\pi\mathrm{i}.\end{aligned}$$

Theorem (Cauchy's Integral Formula)

Suppose that $D \subset \mathbb{C}$ is an open, simply connected, bounded set, and $f : D \to \mathbb{C}$ analytic on D. Assume that $\gamma : [a, b] \to D$ is a simple, smooth, closed contour. Define $D_\gamma \subset D$ to be the set enclosed by γ and whose boundary is γ. Then, for any point $z_0 \in D_\gamma$,

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(z)}{z - z_0} \, \mathrm{d}z = 0.$$

Theorem (Cauchy's Integral Formula for Derivatives)

Suppose that $D \subset \mathbb{C}$ is an open, simply connected, bounded set, and $f : D \to \mathbb{C}$ analytic on D. Assume that $\gamma : [a, b] \to D$ is a simple, smooth, closed contour. Define $D_\gamma \subset D$ to be the set enclosed by γ and whose boundary is γ. Then, for any point $z_0 \in D_\gamma$ and any $n \in \mathbb{N}$, f is n-times differentiable at z_0 and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} \, dz = 0.$$

Example

Suppose that $\gamma(t) = e^{\mathrm{i}t}$, for $0 \le t \le 2\pi$. This is a smooth, simple, closed, counterclockwise-oriented curve of radius 1 around the origin $z = 0$. Consider the integral

$$\int_\gamma \frac{z+1}{z^4 + 4z^3}\,\mathrm{d}z.$$

Define

$$f(z) = \frac{z+1}{z+4}, \quad \forall\, z \in \mathbb{C} \setminus \{-4\}.$$

Observe that $f(z)$ has no singularities inside the set enclosed by γ, that is, the unit disc. In fact f is analytic at every point inside the unit disc. By Cauchy's Integral Formula for Derivatives, f has derivatives to any order. Now, notice that the integrand can be written as

$$\frac{z+1}{z^4 + 4z^3} = \frac{f(z)}{z^3}.$$

Example (Cont.)

The theorem implies, using $n = 2$, that

$$\int_\gamma \frac{z+1}{z^4+4z^3}\,\mathrm{d}z = \int_\gamma \frac{f(z)}{z^3}\,\mathrm{d}z = \frac{2\pi\mathrm{i}}{2!}f''(0).$$

Complex differentiation obeys the same rules, for the most part, as regular differentiation. The chain rule, power rule, quotient rule all still hold. Thus,

$$f'(z) = \frac{3}{(z+4)^2} \quad \text{and} \quad f''(z) = \frac{-6}{(z+4)^3}.$$

Consequently,

$$\int_\gamma \frac{z+1}{z^4+4z^3}\,\mathrm{d}z = \frac{2\pi\mathrm{i}}{2!}f''(0) = \frac{2\pi\mathrm{i}}{2}\frac{(-6)}{4^3} = \frac{-3\pi\mathrm{i}}{32}.$$

Theorem (Taylor's Theorem for Analytic Functions)

Suppose that $D \subset \mathbb{C}$ is an open, simply connected, bounded set, and $f : D \to \mathbb{C}$ analytic on D. For any point $z_0 \in D$, f is differentiable at z_0 to any order, and

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k,$$

for all z in the closed disc $\overline{D_R}(z_0)$, where

$$\overline{D_r}(w) = \{z \in \mathbb{C} \mid |z - w| \leq r\},$$

and $R > 0$ is any positive number such that the closed disc is entirely contained in D, that is, $\overline{D_R}(z_0) \subset D$. Moreover the power series converges absolutely on $\overline{D_R}(z_0)$.

Method of Steepest Descent

Contour Integrals with a Large Parameter

We just spent several slides introducing, or, perhaps, reintroducing, the reader to the subject of contour integration in the complex plane. That work will payoff in the next two sections, for herein we will be interested in integrals of the form

$$I(\lambda) = \int_{\chi} h(z) e^{\lambda \rho(z)} \, \mathrm{d}z, \quad \lambda \in (0, \infty), \tag{2}$$

for large values of λ, where $\rho : D \to \mathbb{C}$ is an analytic function on a bounded, open, simply connected set $D \subset \mathbb{C}$, $\chi : [a, b] \to D$ is a smooth, simple contour in D, and $h(z) : D \to \mathbb{C}$ is also analytic in D. Suppose that $\phi = \Re(\rho)$ and $\psi = \Im(\rho)$, or, in other words,

$$\rho(z) = \phi(z) + \mathrm{i}\psi(z), \quad \forall\, z \in D.$$

Elements of the Method of Steepest Descent

In the simplest setting, we seek a closed contour $\gamma : [a, c] \to D$, such that the following hold:

1. The contour γ is piecewise-smooth and traverses χ as its first segment: $\gamma = \chi + \gamma_1 + \gamma_2 + \gamma_3$.
2. $\psi(\gamma_j) = C_j$, where $C_j \in \mathbb{R}$ is a constant, for $j \in \{1, 3\}$. Such segments of γ are called **constant-ψ contour segments**.
3. ψ is not constant on γ_2. Such segments of γ are called **bridge contour segments**.
4. The integral $\int_{\gamma_2} h(z)e^{\lambda\rho(z)}\,\mathrm{d}z$ is "controllable," in some way to be made precise. For example, the value of the integral over the bridge may become vanishingly small as we "infinitely extend" the contour.
5. $\phi(\gamma_j)$, for $j \in \{1, 3\}$, has a maximum at the point where γ_j connects to the original contour χ. In other words, ϕ is decreasing along γ_j, for $j \in \{1, 3\}$, as we move away from χ.

Elements of the Method of Steepest Descent

By Cauchy's Theorem,

$$\int_{\gamma} h(z)e^{\lambda\rho(z)}\,\mathrm{d}z = 0,$$

since the integrand is analytic on D. Consequently,

$$\begin{aligned}\int_{\chi} h(z)e^{\lambda\rho(z)}\,\mathrm{d}z &= -\sum_{j=1,3}\int_{\gamma_j} h(z)e^{\lambda\rho(z)}\,\mathrm{d}z - \int_{\gamma_2} h(z)e^{\lambda\rho(z)}\,\mathrm{d}z \\ &= \sum_{j=1,3} e^{\lambda \mathrm{i} C_j}\int_{-\gamma_j} h(z)e^{\lambda\phi(z)}\,\mathrm{d}z + \int_{-\gamma_2} h(z)e^{\lambda\rho(z)}\,\mathrm{d}z,\end{aligned}$$

where $-\boldsymbol{\gamma}_j$ is the same contour as $\boldsymbol{\gamma}_j$, but run in the reverse direction.

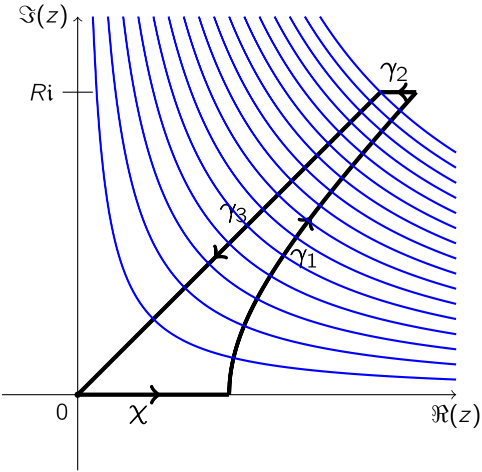

Figure: The piecewise smooth, simple, closed contour γ used in the following example.

Example

Let us consider the specific example

$$I(\lambda) = \int_0^1 e^{\mathrm{i}\lambda s^2}\,\mathrm{d}s, \quad \lambda \in (0,\infty). \tag{3}$$

We want an asymptotic approximation of this integral as $\lambda \to \infty$. This integral can be interpreted as

$$I(\lambda) = \int_\chi e^{\mathrm{i}\lambda z^2}\,\mathrm{d}z, \tag{4}$$

where $\chi : [0,1] \to \mathbb{C}$ is the straight line contour connecting $z = 0$ and $z = 1$ in the complex plane. Now, if $z = x + \mathrm{i}y$. Then,

$$\rho(z) := iz^2 = -2xy + \mathrm{i}\left(x^2 - y^2\right),$$

and we identify

$$\phi(x,y) = -2xy \quad \text{and} \quad \psi(x,y) = x^2 - y^2.$$

Example (Cont.)

Using a slight abuse of notation,

$$\phi(z) = -2xy \quad \text{and} \quad \psi(z) = x^2 - y^2,$$

when $z = x + \mathrm{i}y$. Now, consider the following contour segments:

1. γ_1 connects the point $z = 1$ to the point $z = \sqrt{R^2+1} + \mathrm{i}R$ along the curve $y = \sqrt{x^2-1}$. $\psi(x, y) = 1$ along γ_1.
2. The bridge contour segment γ_2 connects the point $z = \sqrt{R^2+1} + \mathrm{i}R$ to the point $z = R + \mathrm{i}R$ in a straight line.
3. γ_3 connects $z = R + \mathrm{i}R$ to $z = 0$ in a straight line. $\psi(x, y) = 0$ along γ_3.

Thus, $\gamma = \chi + \gamma_1 + \gamma_2 + \gamma_3$ is a piecewise smooth, simple, closed contour. See the figure on the next slide. The integrand of (4) is analytic on any bounded, simply-connected set containing γ.

Now, $R > 0$ is arbitrary in the construction of our contour γ. We intend to send $R \to \infty$. In doing so, the bridge integral will become vanishingly small. Let us consider that integral first.

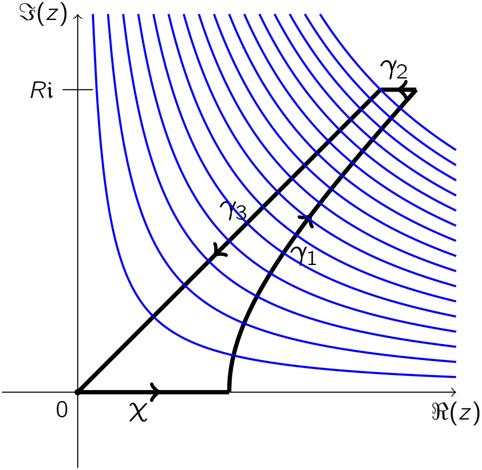

Figure: The piecewise smooth, simple, closed contour γ. Note that $\psi(\gamma_i(t)) = C_1$, $i = 1, 3$.

Example (Cont.)

Bridge integral over γ_2: Consider the following parameterization:

$$\gamma_2(t) = t + \mathrm{i}R, \quad t = \sqrt{R^2+1} \quad \text{to} \quad t = R.$$

Thus,

$$\begin{aligned} I_2(\lambda; R) &:= \int_{\gamma_2} e^{\mathrm{i}\lambda z^2}\, \mathrm{d}z \\ &= \int_{t=\sqrt{R^2+1}}^{t=R} e^{\mathrm{i}\lambda\gamma_2^2(t)}\gamma_2'(t)\, \mathrm{d}t \\ &= \int_{t=\sqrt{R^2+1}}^{t=R} e^{-2\lambda tR} e^{\mathrm{i}\lambda(t^2-R^2)}\, \mathrm{d}t. \end{aligned}$$

Example (Cont.)

Now, let us take the modulus of the integral:

$$
\begin{aligned}
|I_2(\lambda; R)| &= \left| \int_{t=\sqrt{R^2+1}}^{t=R} e^{-2\lambda t R} e^{i\lambda(t^2-R^2)} \, dt \right| \\
&\leq \int_{t=\sqrt{R^2+1}}^{t=R} \left| e^{-2\lambda t R} \right| \cdot \left| e^{i\lambda(t^2-R^2)} \right| \, dt \\
&= \int_{t=R}^{t=\sqrt{R^2+1}} e^{-2\lambda t R} \, dt \\
&= \frac{1}{2\lambda R} \left[e^{-2\lambda R^2} - e^{-2\lambda R \sqrt{R^2+1}} \right] \\
&\xrightarrow{R\to\infty} 0.
\end{aligned}
$$

Thus,

$$
I_2(\gamma; R) = \int_{\gamma_2} e^{i\lambda z^2} \, dz \xrightarrow{R\to\infty} 0.
$$

Example (Cont.)

Integral over γ_3: Recall that γ_3 connects $z = R + \mathrm{i}R$ to $z = 0$ in a straight line, and $\psi(x, y) = 0$ along γ_3. Consider the following parameterization of γ_3:

$$\gamma_3(t) = t + \mathrm{i}t, \quad t = R \quad \text{to} \quad t = 0.$$

Then,

$$\begin{aligned} I_3(\lambda; R) &:= \int_{\gamma_3} e^{\mathrm{i}\lambda z^2}\,\mathrm{d}z \\ &= \int_{t=R}^{t=0} e^{\mathrm{i}\lambda\gamma_3^2(t)}\gamma_3'(t)\,\mathrm{d}t \\ &= \int_{t=R}^{t=0} e^{-2\lambda t^2} e^{\mathrm{i}\lambda(t^2-t^2)}(1+\mathrm{i})\,\mathrm{d}t \\ &= (1+\mathrm{i})\int_{t=R}^{t=0} e^{-2\lambda t^2}\,\mathrm{d}t. \end{aligned}$$

Example (Cont.)

Observe that $\phi(\gamma_3(t)) = -t^2$ is a maximum at $t = 0$, where γ_3 connects to χ, as required. In any case. it follows that,

$$-I_3(\lambda; R) = (1 + \mathrm{i}) \int_0^R e^{-2\lambda t^2}\, \mathrm{d}t \to (1 + \mathrm{i}) \int_0^\infty e^{-2\lambda t^2}\, \mathrm{d}t =: -I_3(\lambda).$$

This Gaussian integral can be easily evaluated as

$$-I_3(\lambda) = \frac{1}{2} e^{\mathrm{i}\pi/4} \sqrt{\frac{\pi}{\lambda}}.$$

Example (Cont.)

Integral over γ_1: Recall that γ_1 connects the point $z = 1$ to the point $z = \sqrt{R^2+1} + \mathrm{i}R$ along the curve $y = \sqrt{x^2-1}$. $\psi(x, y) = 1$ along γ_1. Consider the following parameterization of γ_1:

$$\gamma_1(t) = \sqrt{t^2+1} + \mathrm{i}t, \quad t = 0 \quad \text{to} \quad t = R.$$

Thus,

$$\gamma_1'(t) = \frac{t}{\sqrt{t^2+1}} + \mathrm{i},$$

and it follows that

$$\begin{aligned} I_1(\lambda; R) &:= \int_{\gamma_1} e^{\mathrm{i}\lambda z^2}\,\mathrm{d}z \\ &= \int_{t=0}^{t=R} e^{\mathrm{i}\lambda\gamma_1^2(t)}\gamma_1'(t)\,\mathrm{d}t \\ &= e^{\mathrm{i}\lambda}\int_{t=0}^{t=R} e^{-2\lambda t\sqrt{t^2+1}}\left[\frac{t}{\sqrt{t^2+1}} + \mathrm{i}\right]\mathrm{d}t. \end{aligned}$$

Example (Cont.)

Taking $R \to \infty$, we have

$$I_1(\lambda) = \lim_{R\to\infty} I_1(\lambda; R) = e^{i\lambda} \int_0^\infty e^{-2\lambda t\sqrt{t^2+1}} \left[\frac{t}{\sqrt{t^2+1}} + i \right] dt.$$

We can use a Laplace-type method to approximate the integral $I_1(\lambda)$, as $\lambda \nearrow \infty$. The integrand is of the appropriate form:

$$e^{\lambda\tilde{\phi}(t)}, \quad \tilde{\phi}(t) = -2t\sqrt{t^2+1}.$$

Observe that $\tilde{\phi}(t)$ is a maximum at $t = 0$, which represents the point where γ_1 and χ connect.

Example (Cont.)

Let us make a sophisticated change of variables:

$$\mathrm{i}\gamma_1^2(t) = \mathrm{i} - s.$$

Recall that

$$\gamma_1(t) = \sqrt{t^2+1} + \mathrm{i}t,$$

so that

$$\mathrm{i}\gamma_1^2(t) = \mathrm{i}\left(t^2 + 1 + 2\mathrm{i}t\sqrt{t^2+1} - t^2\right) = \mathrm{i} - 2t\sqrt{t^2+1}.$$

Thus,

$$s = 2t\sqrt{t^2+1}.$$

Furthermore,

$$2\mathrm{i}\gamma_1(t)\gamma_1'(t)\,\mathrm{d}t = -\mathrm{d}s.$$

Thus,

$$\gamma_1'(t)\,\mathrm{d}t = \frac{\mathrm{i}\,\mathrm{d}s}{2\gamma_1(t)}.$$

Example (Cont.)

But,

$$\gamma_1(t) = \sqrt{1 + \mathrm{i}s}.$$

Therefore, the transformed integral is

$$I_1(\lambda) = \frac{1}{2}\mathrm{i}e^{\mathrm{i}\lambda} \int_0^\infty \frac{e^{-\lambda s}}{\sqrt{1+\mathrm{i}s}}\, \mathrm{d}s.$$

This can be expanded using Watson's Lemma. We need the power series expansion

$$\frac{1}{\sqrt{1+\mathrm{i}z}} = \sum_{k=0}^\infty z^k \mathrm{i}^k \binom{-\frac{1}{2}}{k} = 1 - \frac{\mathrm{i}z}{2} + \frac{3\mathrm{i}z}{8} - \frac{5\mathrm{i}z}{16} + \frac{35\mathrm{i}z}{128} + \cdots .$$

which is valid for $|z| < 1$. Thus, for real values of $s \in [0, 1/2]$,

$$\frac{1}{\sqrt{1+\mathrm{i}s}} = \sum_{k=0}^\infty s^k \mathrm{i}^k \binom{-\frac{1}{2}}{k}$$

Example (Cont.)

By Watson's Lemma,

$$I_1(\lambda) \sim \frac{1}{2}\mathrm{i}e^{\mathrm{i}\lambda}\sum_{k=0}^{\infty}\mathrm{i}^k\binom{-\frac{1}{2}}{k}\int_0^{\infty}e^{-\lambda s}s^k\,\mathrm{d}s, \quad \text{as} \quad \lambda\to\infty,$$

or, equivalently,

$$I_1(\lambda) \sim \frac{1}{2}\mathrm{i}e^{\mathrm{i}\lambda}\sum_{k=0}^{\infty}\mathrm{i}^k\binom{-\frac{1}{2}}{k}\frac{\Gamma(k+1)}{\lambda^{k+1}}, \quad \text{as} \quad \lambda\to\infty,$$

after evaluating the integrals. To conclude,

$$\begin{aligned} I(\lambda) &= \int_0^1 e^{\mathrm{i}\lambda s^2}\,\mathrm{d}s \\ &= -I_1(\lambda) - I_3(\lambda) \\ &\sim \frac{1}{2}e^{\mathrm{i}\pi/4}\sqrt{\frac{\pi}{\lambda}} - \frac{1}{2}\mathrm{i}e^{\mathrm{i}\lambda}\sum_{k=0}^{\infty}\mathrm{i}^k\binom{-\frac{1}{2}}{k}\frac{\Gamma(k+1)}{\lambda^{k+1}}, \quad \text{as} \quad \lambda\to\infty. \end{aligned}$$

Example

Here is the Mathematica code for the previous problem.

```
In[1]:= AsymptoticIntegrate[Exp[I*λ*s^2],{s,0,1},{λ,∞,3}]
```

$$\text{Out[1]=}\quad e^{i\lambda}\left(\frac{3i}{8\lambda^3}-\frac{1}{4\lambda^2}-\frac{i}{2\lambda}\right)-\frac{i e^{\frac{1}{4}i\pi\left(3+4\,\mathrm{Floor}\left[\frac{3}{4}-\frac{\mathrm{Arg}[\lambda]}{2\pi}\right]\right)}\sqrt{\pi}}{2\sqrt{\lambda}}$$

The leading order term is a bit puzzling. But, it all seems to agree with our calculation.