Math 515
Essential Perturbation Theory and Asymptotic Analysis
Chapter 03

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Chapter 03, Part 2 of 2
Asymptotic Expansion of Integrals

Method of Stationary Phase

Theorem (Riemann-Lebesgue Lemma)

Suppose that $-\infty < a < b < \infty$, $f \in C^0([a,b];\mathbb{R})$, *and* $k \in (0,\infty)$. *Then,*

$$\lim_{k\to\infty} \int_a^b f(x)\cos(kx)\,\mathrm{d}x = 0$$

and

$$\lim_{k\to\infty} \int_a^b f(x)\sin(kx)\,\mathrm{d}x = 0.$$

Proof.

We only prove the first result. The second follows by an analogous argument. Since f is continuous on $[a, b]$, given any $\varepsilon > 0$, there is a partition of $[a, b]$, that is,

$$a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b,$$

and step function

$$g(x) = \sum_{i=1}^{m} c_i \chi_{(x_{i-1}, x_i]}(x),$$

such that $g(x) \leq f(x)$, for all $x \in (a, b]$ and

$$0 \leq \int_a^b (f(x) - g(x))\, \mathrm{d}x < \frac{\varepsilon}{2}.$$

In fact, we can use

$$c_i = \min_{x_{i-1} \leq x \leq x_i} f(x).$$

Proof (Cont.)

Now,

$$
\begin{aligned}
\left|\int_a^b f(x)\cos(kx)\,\mathrm{d}x\right| &= \left|\int_a^b (f(x)-g(x)+g(x))\cos(kx)\,\mathrm{d}x\right| \\
&\le \left|\int_a^b (f(x)-g(x))\cos(kx)\,\mathrm{d}x\right| + \left|\int_a^b g(x)\cos(kx)\,\mathrm{d}x\right| \\
&\le \int_a^b |(f(x)-g(x))|\cdot|\cos(kx)|\,\mathrm{d}x + \left|\int_a^b g(x)\cos(kx)\,\mathrm{d}x\right| \\
&\le \int_a^b |(f(x)-g(x))|\,\mathrm{d}x + \left|\int_a^b g(x)\cos(kx)\,\mathrm{d}x\right| \\
&\le \frac{\varepsilon}{2} + \left|\frac{1}{k}\sum_{k=1}^{m} c_i\left(\sin(kx_i)-\sin(kx_{i-1})\right)\right| \\
&\le \frac{\varepsilon}{2} + \frac{1}{k}\sum_{k=1}^{m} 2|c_i|.
\end{aligned}
$$

Proof (Cont.)

We just showed that

$$\left| \int_a^b f(x) \cos(kx) \, \mathrm{d}x \right| \le \frac{\varepsilon}{2} + \frac{1}{k} \sum_{k=1}^{m} 2|c_i|.$$

With the function g chosen and fixed, it is clear that we can take k large enough so that

$$\frac{1}{k} \sum_{k=1}^{m} 2|c_i| \le \frac{\varepsilon}{2},$$

and the result follows. □

Theorem (Generalized Riemann-Lebesgue Lemma I)

Suppose that $-\infty < a < b < \infty$ *and* $f \in C^1([a, b]; \mathbb{R})$*. There exists a constant* $C > 0$*, independent of* k*, and a positive number* k_0*, such that, if* $k \geq k_0$*, it follows that*

$$\left| \int_a^b f(x) \cos(kx) \, \mathrm{d}x \right| \leq \frac{C}{k}.$$

In other words,

$$\int_a^b f(x) \cos(kx) \, \mathrm{d}x = O(k^{-1}), \quad \text{as} \quad k \to \infty.$$

Proof.

Let us apply integration by parts:

$$\int_a^b f(x)\cos(kx)\,\mathrm{d}x = \frac{1}{k}\left(f(b)\sin(kb) - f(a)\sin(ka)\right) - \frac{1}{k}\int_a^b f'(x)\sin(kx)\,\mathrm{d}x.$$

Now, by the standard Riemann-Lebesgue Lemma there is a k_0 such that, if $k \geq k_0$,

$$\left|\int_a^b f'(x)\sin(kx)\,\mathrm{d}x\right| \leq 1.$$

Therefore, it follows that, if $k \geq k_0$,

$$\left|\int_a^b f(x)\cos(kx)\,\mathrm{d}x\right| \leq \frac{2}{k}\max_{a\leq x\leq b}|f(x)| + \frac{1}{k}.$$

The result follows by taking

$$C = 2\max_{a\leq x\leq b}|f(x)| + 1.$$

$\square$

In this section we will consider approximations of

$$I(\lambda) = \int_a^b f(t) e^{\mathrm{i}\lambda\psi(t)}\,\mathrm{d}t,$$

which is a type of generalized Fourier integral, as $\lambda \to \infty$. Let us try integration by parts for a special case, using the ideas conceived in the proof of the Riemann-Lebesque Lemmas.

Proposition (Generalized Riemann-Lebesgue Lemma II)

Suppose that $-\infty < a < b < \infty$, $f \in C^2([a,b];\mathbb{R})$, *and* $\psi \in C^3([a,b];\mathbb{R})$. *Assume that* $\psi'(t) \neq 0$, *for all* $t \in [a,b]$. *Then,*

$$\int_a^b f(t) e^{\mathrm{i}\lambda\psi(t)}\,\mathrm{d}t = \left.\frac{f(t)}{\mathrm{i}\lambda\psi'(t)} e^{\mathrm{i}\lambda\psi(t)}\right|_{t=a}^{t=b} + O(\lambda^{-2}), \quad \text{as} \quad \lambda \to \infty.$$

and

$$\int_a^b f(t) e^{\mathrm{i}\lambda\psi(t)}\,\mathrm{d}t \sim \left.\frac{f(t)}{\mathrm{i}\lambda\psi'(t)} e^{\mathrm{i}\lambda\psi(t)}\right|_{t=a}^{t=b}, \quad \text{as} \quad \lambda \to \infty.$$

Proof.

Use integration by parts twice. The details are left for an exercise. □

Definition

Suppose that $-\infty < a < b < \infty$, $f \in C([a,b];\mathbb{C})$, $\psi \in C^1([a,b];\mathbb{R})$, and $\lambda \in (0,\infty)$. The integral

$$I(\lambda) = \int_a^b f(t) e^{i\lambda\psi(t)}\,\mathrm{d}t \tag{1}$$

is called a **Fourier integral**. The integral in (1) is called a **Fourier integral of stationary phase** iff there is a single isolated point $t_o \in [a,b]$ such that $\psi'(t_o) = 0$. Assuming $\psi \in C^{k+1}([a,b];\mathbb{R})$, the point t_o is called a **point of stationary phase of order** k iff $\psi'(t_o) = \cdots = \psi^{(k)}(t_o) = 0$ and $\psi^{(k+1)}(t_o) \neq 0$.

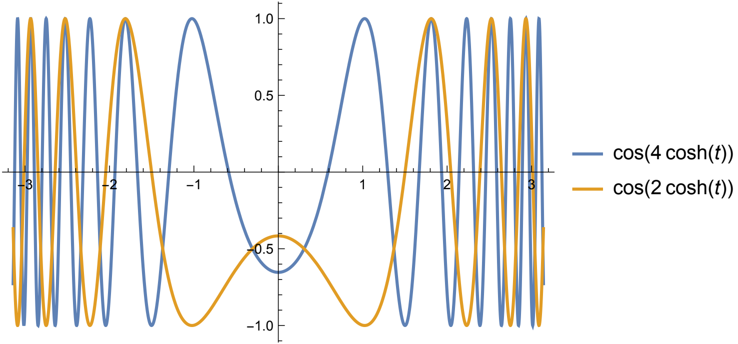

Figure: Oscillations "slow down" at a point of stationary phase. The function $\cos(\lambda \cosh(t))$ has a point of stationary phase of order $k = 1$ at $t = 0$. As $\lambda \to \infty$, the oscillations away from the point of stationary phase become more and more rapid.

Example

Consider the Bessel function

$$J_0(\lambda) = \frac{1}{\pi}\int_0^{\pi} \cos(\lambda \sin(t))\, \mathrm{d}t, \tag{2}$$

where $\lambda \in (0, \infty)$. Note that

$$J_0(\lambda) = \frac{1}{\pi}\Re\left(\int_0^{\pi} e^{\mathrm{i}\lambda \sin(t)}\, \mathrm{d}t\right).$$

In this example, $\psi(t) = \sin(t)$. Clearly, $\psi'(t_o) = 0$, where $t_o = \frac{\pi}{2}$, and $\psi''(t_o) = 1$. Thus t_o is a point of stationary phase of order one. Let us write

$$\begin{aligned} I(\lambda) &= \frac{1}{\pi}\int_0^{\pi} e^{\mathrm{i}\lambda \sin(t)}\, \mathrm{d}t \\ &= \frac{1}{\pi}\int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} e^{\mathrm{i}\lambda \sin(t)}\, \mathrm{d}t + \frac{1}{\pi}\int_0^{\pi/2-\varepsilon} e^{\mathrm{i}\lambda \sin(t)}\, \mathrm{d}t + \frac{1}{\pi}\int_{\pi/2+\varepsilon}^{\pi} e^{\mathrm{i}\lambda \sin(t)}\, \mathrm{d}t, \end{aligned}$$

where $\varepsilon > 0$ is small.

Example (Cont.)

Using one of the previous generalized Riemann-Lesbegue results, we are guaranteed that

$$E_1(\lambda) := \frac{1}{\pi}\int_0^{\pi/2-\varepsilon} e^{\mathrm{i}\lambda\sin(t)}\,\mathrm{d}t + \frac{1}{\pi}\int_{\pi/2+\varepsilon}^{\pi} e^{\mathrm{i}\lambda\sin(t)}\,\mathrm{d}t = O(\lambda^{-1}), \tag{3}$$

as $\lambda \to \infty$. Expanding about t_o, we have

$$\sin(t) = 1 - \frac{1}{2}\left(t-\frac{\pi}{2}\right)^2 + \frac{1}{24}\left(t-\frac{\pi}{2}\right)^4 - \cdots .$$

For the integral centered on the point of stationary phase, replacing $\sin(t)$ by the first two terms of the expansion and enlarging the domain of integration to $(-\infty,\infty)$, we expect

$$\frac{1}{\pi}\int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} e^{\mathrm{i}\lambda\sin(t)}\,\mathrm{d}t \sim \frac{1}{\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\lambda\left(1-\frac{1}{2}\left(t-\frac{\pi}{2}\right)^2\right)}\,\mathrm{d}t,$$

as $\lambda \to \infty$.

Example (Cont.)

Continuing our calculation,

$$\begin{aligned}
\frac{1}{\pi}\int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} e^{i\lambda\sin(t)}\,\mathrm{d}t &\sim \frac{1}{\pi}\int_{-\infty}^{\infty} e^{i\lambda\left(1-\frac{1}{2}\left(t-\frac{\pi}{2}\right)^2\right)}\,\mathrm{d}t\\
&= \sqrt{\frac{2}{\lambda}}\frac{1}{\pi}e^{i\lambda}\int_{-\infty}^{\infty} e^{-is^2}\,\mathrm{d}s\\
&= \sqrt{\frac{2}{\lambda}}\frac{1}{\pi}e^{i\lambda}\sqrt{\pi}e^{-i\frac{\pi}{4}}\\
&= \sqrt{\frac{2}{\pi\lambda}}e^{i\left(\lambda-\frac{\pi}{4}\right)}\\
&= O\left(\lambda^{-\frac{1}{2}}\right), \quad \text{as} \quad \lambda\to\infty,
\end{aligned}$$

where we used a convenient change of variable going from the second to the third integral.

Example (Cont.)

To finish up, we need to estimate the error,

$$E_2(\lambda) := \frac{1}{\pi}\int_0^{\pi} e^{\mathrm{i}\lambda \sin(t)}\,\mathrm{d}t - \frac{1}{\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\lambda\left(1-\frac{1}{2}\left(t-\frac{\pi}{2}\right)^2\right)}\,\mathrm{d}t,$$

as $\lambda \to \infty$, that is, the error incurred by replacing the **sin** term and increasing integration limits. The theory behind the method of stationary phase, which is tedious and difficult, as we will see momentarily, indicates that

$$E_2(\lambda) = O(\lambda^{-1}), \quad \text{as} \quad \lambda \to \infty.$$

In other words, the error term $E_2(\lambda)$ is no larger that than the size of the discarded tails in (3), $E_1(\lambda)$. Since the contribution from the integral centered on the point of stationary phase is $O\left(\lambda^{-\frac{1}{2}}\right)$, we can neglect both $E_1(\lambda)$ and $E_2(\lambda)$ in the leading order asymptotic approximation to obtain

$$I(\lambda) = \sqrt{\frac{2}{\pi\lambda}}e^{\mathrm{i}\left(\lambda-\frac{\pi}{4}\right)} + O(\lambda^{-1}), \quad \text{as} \quad \lambda \to \infty.$$

Example (Cont.)

We just argued that

$$I(\lambda) = \sqrt{\frac{2}{\pi\lambda}} e^{\mathrm{i}\left(\lambda - \frac{\pi}{4}\right)} + O(\lambda^{-1}), \quad \text{as} \quad \lambda \to \infty.$$

If this holds, we have

$$I(\lambda) \sim \sqrt{\frac{2}{\pi\lambda}} e^{\mathrm{i}\left(\lambda - \frac{\pi}{4}\right)}, \quad \text{as} \quad \lambda \to \infty.$$

Taking the real part of our approximation, we get the classical and well-known result

$$J_0(\lambda) \sim \sqrt{\frac{2}{\pi\lambda}} \cos\left(\lambda - \frac{\pi}{4}\right), \quad \text{as} \quad \lambda \to \infty.$$

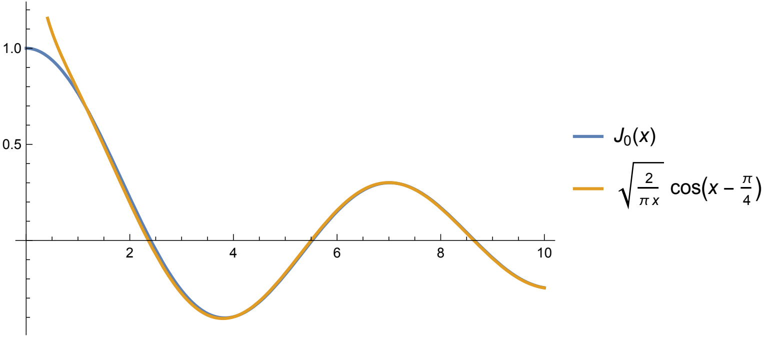

Figure: Comparison of the Bessel Function, J_0, with its asymptotic approximation

The Method of Stationary Phase

Let us generalize the process that we just used to approximate the Bessel function of order zero. Again, we want to approximate the Fourier integral

$$I(\lambda) = \int_a^b f(t) e^{\mathrm{i}\lambda\psi(t)} \, \mathrm{d}t,$$

under the assumption that there is a point $t_o \in (a, b)$, such that $\psi'(t_o) = 0$, and $\psi'(t) \neq 0$, for all $t \in [a, b] \setminus \{t_o\}$. Let us assume that $\psi''(t_o) = \alpha > 0$, to be specific. Then, as above, we write

$$I(\lambda) = \int_{t_o-\varepsilon}^{t_o+\varepsilon} f(t) e^{\mathrm{i}\lambda\psi(t)} \, \mathrm{d}t + \int_a^{t_o-\varepsilon} f(t) e^{\mathrm{i}\lambda\psi(t)} \, \mathrm{d}t + \int_{t_o+\varepsilon}^b f(t) e^{\mathrm{i}\lambda\psi(t)} \, \mathrm{d}t.$$

The Method of Stationary Phase

For the integrals whose domains of integration do not contain the point of stationary phase, the tails, we define

$$E_1(\lambda) := \int_a^{t_o-\varepsilon} f(t)e^{\mathrm{i}\lambda\psi(t)}\,\mathrm{d}t + \int_{t_o+\varepsilon}^{b} f(t)e^{\mathrm{i}\lambda\psi(t)}\,\mathrm{d}t,$$

and we expect, because of the generalized Riemann-Lebesgue Lemma, that

$$E_1(\lambda) = O(\lambda^{-1}), \quad \text{as} \quad \lambda \to \infty.$$

The Method of Stationary Phase

For the integral at the stationary phase point, we expect a contribution that is $O\left(\lambda^{-\frac{1}{2}}\right)$, as $\lambda \to \infty$. In particular, we will show that

$$\begin{aligned}
\int_{t_o-\varepsilon}^{t_o+\varepsilon} f(t) e^{\mathrm{i}\lambda\psi(t)}\,\mathrm{d}t &\sim \int_{t_o-\varepsilon}^{t_o+\varepsilon} f(t_o) e^{\mathrm{i}\lambda\left(\psi(t_o)+\frac{\alpha}{2}(t-t_o)^2\right)}\,\mathrm{d}t \\
&= f(t_o) e^{\mathrm{i}\lambda\psi(t_o)} \int_{t_o-\varepsilon}^{t_o+\varepsilon} e^{\mathrm{i}\lambda\frac{\alpha}{2}(t-t_o)^2}\,\mathrm{d}t \\
&\sim f(t_o) e^{\mathrm{i}\lambda\psi(t_o)} \sqrt{\frac{2}{\alpha\lambda}} \int_{-\infty}^{\infty} e^{\mathrm{i}s^2}\,\mathrm{d}s \\
&= f(t_o) e^{\mathrm{i}\lambda\psi(t_o)} \sqrt{\frac{2}{\alpha\lambda}} \sqrt{\pi} e^{\mathrm{i}\frac{\pi}{4}},
\end{aligned}$$

where, specifically, the error

$$E_2(\lambda) := \int_{t_o-\varepsilon}^{t_o+\varepsilon} f(t) e^{\mathrm{i}\lambda\psi(t)}\,\mathrm{d}t - f(t_o) e^{\mathrm{i}\lambda\psi(t_o)} \sqrt{\frac{2}{\alpha\lambda}} \sqrt{\pi} e^{\mathrm{i}\frac{\pi}{4}}$$

satisfies

$$E_2(\lambda) = O(\lambda^{-1}), \quad \text{as} \quad \lambda \to \infty.$$

The Method of Stationary Phase

As a consequence, we have

$$I(\lambda) - f(t_o)\sqrt{\frac{2\pi}{\alpha\lambda}}e^{\mathrm{i}\left(\lambda\psi(t_o)+\frac{\pi}{4}\right)} = O(\lambda^{-1}), \quad \text{as} \quad \lambda \to \infty,$$

which implies that

$$I(\lambda) \sim f(t_o)\sqrt{\frac{2\pi}{\alpha\lambda}}e^{\mathrm{i}\left(\lambda\psi(t_o)+\frac{\pi}{4}\right)}, \quad \text{as} \quad \lambda \to \infty.$$

Here we have tacitly assumed that $f(t_o) \neq 0$. If, to the contrary, $f(t_o) = 0$, we must determine whether or not it is still true that primary contribution is $O\left(\lambda^{-\frac{1}{2}}\right)$, as $\lambda \to \infty$. Note also that this result applies for the case that the stationary point is in the center of the domain of integration. The result must be modified if $t_o = a$ or $t_o = b$.

A Technical Lemma

Lemma

Suppose that $h : D \to \mathbb{C}$ *is analytic on an open set* $D \subset \mathbb{C}$ *that contains the finite real interval* $[a, b]$, h *is real valued for every* $t \in [a, b]$, $h'(t) = 0$ *for exactly one point* $t_o \in (a, b)$, *and* $h''(t_o) = \alpha > 0$. *Then, for some* $\delta > 0$, *there is an analytic function* $w : \overline{B_\delta(t_o)} \to \mathbb{C}$, *where*

$$B_\delta(t_o) = \{z \in \mathbb{C} \mid |z - t_o| < \delta\},$$

such that w *is real valued, strictly increasing, and continuous on* $[t_o - \delta, t_o + \delta]$, *one-to-one on* $\overline{B_\delta(t_o)}$, *and*

$$h(z) - h(t_o) = w^2(z), \quad \forall\, z \in \overline{B_\delta(t_o)}.$$

Proof.

A proof can be found in the book by Marsden and Hoffman and requires some machinery from complex analysis. □

Theorem

Suppose that $-\infty < a < b < \infty$ *and* $\lambda \in (0, \infty)$*. Assume that* ψ *is complex analytic in an open disk that contains the interval* $[a, b]$*, such that the restriction of* ψ *to* $[a, b]$ *is real-valued. Suppose that* $f \in C^1([a, b]; \mathbb{R})$*. Assume further that* $t_o \in (a, b)$ *is a point of stationary phase of order* 1*, that is,* $\psi'(t_o) = 0$ *and* $\psi''(t_o) = \alpha > 0$*. Then,*

$$I(\lambda) = \int_a^b f(t) e^{\mathrm{i}\lambda\psi(t)}\,\mathrm{d}t \sim f(t_o)\sqrt{\frac{2\pi}{\alpha\lambda}}\, e^{\mathrm{i}\left(\lambda\psi(t_o) + \frac{\pi}{4}\right)}, \quad \text{as} \quad \lambda \to \infty. \tag{4}$$

If $\psi'(t_o) = 0$ *and* $\psi''(t_o) = \alpha < 0$*. Then,*

$$I(\lambda) \sim f(t_o)\sqrt{\frac{2\pi}{-\alpha\lambda}}\, e^{\mathrm{i}\left(\lambda\psi(t_o) - \frac{\pi}{4}\right)}, \quad \text{as} \quad \lambda \to \infty. \tag{5}$$

Proof.

Our proof follows that in Marsden and Hoffman. We will assume, for simplicity, that $\alpha > 0$. The case $\alpha < 0$ is similar. Using the technical lemma, for some $\delta > 0$, there is an analytic function $w : \overline{B_\delta(t_o)} \to \mathbb{C}$, such that w is real valued, strictly increasing, and continuous on $[t_o - \delta, t_o + \delta]$, one-to-one on $\overline{B_\delta(t_o)}$, and

$$\psi(z) - \psi(t_o) = w^2(z), \quad \forall\, z \in \overline{B_\delta(t_o)}.$$

Let us write

$$I_\delta(\lambda) := \int_{t_o-\delta}^{t_o+\delta} f(t) e^{\mathrm{i}\lambda\psi(t)}\, \mathrm{d}t.$$

Then, using the generalized Riemann-Lebesgue Lemma,

$$I(\lambda) - I_\delta(\lambda) = O\left(\lambda^{-1}\right), \quad \text{as} \quad \lambda \to \infty.$$

Proof (Cont.)

Let us make a change of variables in I_δ: $x = w(t)$. Set $w(t_o - \delta) =: c$ and $w(t_o + \delta) =: d$. Then, $c < 0 < d$ and

$$I_\delta = e^{\mathrm{i}\lambda\psi(t_o)} \int_c^d g(x) e^{\mathrm{i}\lambda x^2} \, \mathrm{d}x,$$

where

$$g(x) := f\left(w^{-1}(x)\right) \frac{\mathrm{d}w^{-1}}{\mathrm{d}x}(x)$$

and we use the fact that w is invertible on $[t_o - \delta, t_o + \delta]$. The point $t = t_o$ corresponds to $x = 0$ and

$$\alpha = \psi''(t_o) = 2w(t_o)w''(t_o) + 2(w'(t_o))^2 = 2(w'(t_o))^2.$$

Proof (Cont.)

Therefore,

$$\frac{\mathrm{d}w^{-1}}{\mathrm{d}x}(0) = \frac{1}{w^{-1}(t_o)} = \sqrt{\frac{2}{\alpha}},$$

and

$$g(0) = f(t_o)\sqrt{\frac{2}{\alpha}}.$$

Since g' is continuous, g has bounded variation and can be written as the difference of two increasing functions:

$$g(x) = g_1(x) - g_2(x), \quad \forall\, x \in [c, d].$$

Let $\varepsilon > 0$ be given. Since $c, d \to 0$ as $\delta \to 0$, by a technical lemma in the appendix, there is a $\delta > 0$ small enough so that

$$|g_1(c) - g_1(0)|,\ |g_1(d) - g_1(0)|,\ |g_2(c) - g_2(0)|,\ |g_2(d) - g_2(0)| < \varepsilon.$$

Proof (Cont.)

Recall

$$I_\delta = e^{\mathrm{i}\lambda\psi(t_o)} \int_c^d g(x) e^{\mathrm{i}\lambda x^2} \, \mathrm{d}x.$$

Now consider

$$\begin{aligned} J_\delta(\lambda) &:= \sqrt{\lambda} e^{-\mathrm{i}\lambda\psi(t_o)} I_\delta(\lambda) \\ &= \sqrt{\lambda} \int_c^d e^{\mathrm{i}\lambda x^2} g(x) \, \mathrm{d}x \\ &= \sqrt{\lambda} \int_a^b g_1(x) e^{\mathrm{i}\lambda x^2} \, \mathrm{d}x - \sqrt{\lambda} \int_a^b g_2(x) e^{\mathrm{i}\lambda x^2} \, \mathrm{d}x. \end{aligned}$$

Proof (Cont.)

By the Second Integral Mean Value Theorem (see the appendix, again), there is a point $m_i \in [c, d]$ such that

$$\begin{aligned} J_{\delta,i}(\lambda) &:= \sqrt{\lambda} \int_c^d e^{\mathrm{i}\lambda x^2} g_i(x)\, \mathrm{d}x \\ &= g_i(c)\sqrt{\lambda} \int_c^{m_i} e^{\mathrm{i}\lambda x^2}\, \mathrm{d}x + g_i(d)\sqrt{\lambda} \int_{m_i}^d e^{\mathrm{i}\lambda x^2}\, \mathrm{d}x \\ &= g_i(c) \int_{c\sqrt{\lambda}}^{m_i\sqrt{\lambda}} e^{\mathrm{i}u^2}\, \mathrm{d}u + g_i(d) \int_{m_i\sqrt{\lambda}}^{d\sqrt{\lambda}} e^{\mathrm{i}u^2}\, \mathrm{d}u, \end{aligned}$$

using the change of variable $u = \sqrt{\lambda}x$.

Proof (Cont.)

Using the known integral,

$$\int_{-\infty}^{\infty} e^{\mathrm{i}u^2}\,\mathrm{d}u = (1+\mathrm{i})\sqrt{\frac{\pi}{2}} = \sqrt{\pi}e^{\mathrm{i}\frac{\pi}{4}},$$

and taking $\lambda \to \infty$, we get

$$\lim_{\lambda\to\infty} J_{\delta,i}(\lambda) = g_{i,\star}(c,d)\sqrt{\pi}e^{\mathrm{i}\frac{\pi}{4}},$$

where

$$g_{i,\star}(c,d) = \begin{cases} g_i(c), & \text{if} \quad m_i > 0, \\ g_i(d), & \text{if} \quad m_i < 0, \\ \frac{1}{2}\left(g_i(c) + g_i(d)\right), & \text{if} \quad m_i = 0. \end{cases}$$

Consequently,

$$\lim_{\lambda\to\infty} J_{\delta}(\lambda) = \left(g_{1,\star}(c,d) + g_{2,\star}(c,d)\right)\sqrt{\pi}e^{\mathrm{i}\frac{\pi}{4}}.$$

Proof (Cont.)

At the same time,

$$\lim_{\delta \searrow 0} \left(g_{1,\star}(c,d) + g_{2,\star}(c,d)\right) = g(0) = f(t_o)\sqrt{\frac{2}{\alpha}}.$$

Putting everything, together, since $\delta > 0$ was arbitrary and independent of λ, we have

$$\begin{aligned}
\lim_{\lambda\to\infty} \sqrt{\lambda} e^{-\mathrm{i}\psi(t_o)} I(\lambda) &= \lim_{\lambda\to\infty} J_\delta(\lambda) + \lim_{\lambda\to\infty} \sqrt{\lambda} e^{-\mathrm{i}\psi(t_o)} \left(I(\lambda) - I_\delta(\lambda)\right) \\
&= f(t_o)\sqrt{\frac{2}{\alpha}}\sqrt{\pi} e^{\mathrm{i}\frac{\pi}{4}} + \lim_{\lambda\to\infty} O(\lambda^{-1/2}) \\
&= f(t_o)\sqrt{\frac{2}{\alpha}}\sqrt{\pi} e^{\mathrm{i}\frac{\pi}{4}}.
\end{aligned}$$

We conclude that

$$I(\lambda) \sim f(t_o)\sqrt{\frac{2\pi}{\lambda\alpha}} e^{\mathrm{i}\psi(t_o)} e^{\mathrm{i}\frac{\pi}{4}}, \quad \text{as} \quad \lambda \to \infty,$$

which is what we wanted to prove. $\square$

Theorem

Suppose that $-\infty < a < b < \infty$, $p \in \mathbb{N}$, $p > 1$, *and* $\lambda \in (0, \infty)$. *Assume that* f *and* ψ *are complex analytic in a disk that contains the interval* $[a, b]$, *such that the restriction of* ψ *to* $[a, b]$ *is real-valued. Assume further that* $t_o = a$ *is a point of stationary phase of order* $p - 1$, *that is,* $\psi'(a) = \cdots \psi^{(p-1)}(a) = 0$ *and* $\psi^{(p)}(a) = \alpha > 0$. *Suppose that* $f(a) \neq 0$. *Then, if* $p = 2$,

$$I(\lambda) = \int_a^b f(t) e^{\mathrm{i}\lambda\psi(t)} \, \mathrm{d}t \sim f(a) \sqrt{\frac{\pi}{2\alpha\lambda}} e^{\mathrm{i}\left(\lambda\psi(a) + \frac{\pi}{4}\right)}, \quad \textit{as} \quad \lambda \to \infty. \tag{6}$$

and, more generally, for $p > 2$,

$$I(\lambda) \sim f(a) \left[\frac{p!}{\alpha\lambda}\right]^{\frac{1}{p}} e^{\mathrm{i}\left(\lambda\psi(a) + \frac{\pi}{2p}\right)} \frac{\Gamma\left(\frac{1}{p}\right)}{p}, \quad \textit{as} \quad \lambda \to \infty. \tag{7}$$

Proof.

The proof can be found in the book Copson (1965). □