Math 515
Essential Perturbation Theory and Asymptotic Analysis
Chapter 05

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Chapter 05, Part 1 of 3
Matched Asymptotics and Boundary Layers

Matched Asymptotic Methods

Matched asymptotic methods are right at the heart of modern perturbation theory and asymptotic analysis. Essential stuff, in our opinion. These are an admixture of intuition, rigor, voodoo, elaborate computations, and spice. Our treatment of this subject follows what may be found in the classics, including the books by Holmes (2013), Hinch (1991), Kevorkian and Cole (1996), Miller (2006), and several others. Let us pick up the topic that we started in Chapter 1, but now use what we have learned from Chapter 2, especially the section on matching. We will assume that the reader has these sections fresh in the mind.

A Linear, Singularly-Perturbed BVP

A Linear Problem to Get us Restarted

Suppose that $y(\,\cdot\,;\varepsilon) : [0,1] \to \mathbb{R}$ satisfies the BVP consisting of the ordinary differential equation (ODE)

$$\varepsilon y''(x) + 2y'(x) + 2y(x) = 0, \quad \text{for} \quad 0 < x < 1, \tag{1}$$

where $\varepsilon > 0$ is a small parameter, with the boundary conditions

$$y(0) = 0 \quad \text{and} \quad y(1) = 1. \tag{2}$$

We observe immediately that, as in the example in Chapter 1, if $\varepsilon = 0$, the problem becomes a first-order ordinary differential equation with two boundary conditions. There is no solution, in general, for such a problem, which motivates the term singular perturbation problem.

Exact Solution

As a result of the singular nature, for small, positive ε, the solution possesses a boundary layer at $x = 0$, as we may verify by examining the analytic expression for the exact solution, which is

$$y(x;\varepsilon) = \frac{e^{r_1 x} - e^{r_2 x}}{e^{r_1} - e^{r_2}}, \tag{3}$$

where r_1 and r_2 are the two solutions of

$$\varepsilon r = -1 \pm \sqrt{1 - 2\varepsilon}.$$

See the figure on the next slide. Of course, we will not typically possess an exact solution, and in that case, the typical case, we will seek and find some general rules for locating the layer. But, for now, we will simply be grateful to know this inside information.

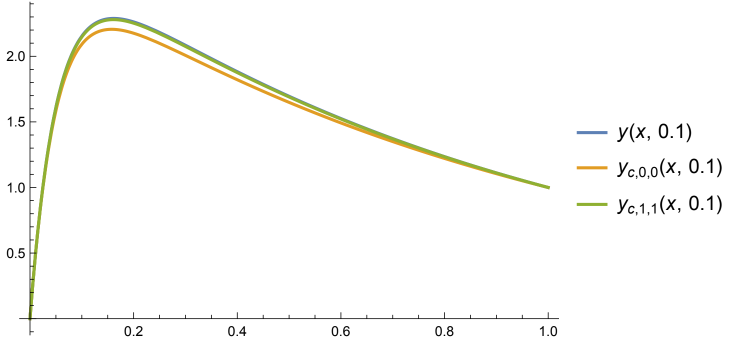

Figure: Plots of the solution $y(x;\varepsilon)$ (blue) to the problem (1) and (2) with the leading-order composite approximation $y_{\mathrm{c},0,0}(x;\varepsilon)$ (yellow) and composite approximation $y_{\mathrm{c},1,1}(x;\varepsilon)$ (green), for $\varepsilon = 0.1$. Note that $y(x;0.1)$ and $y_{\mathrm{c},1,1}(x;0.1)$ are almost indistinguishable.

Outer Expansion

Let us begin by supposing that, for some $\varepsilon_0 > 0$, $y(\cdot\,;\varepsilon) \in C^2((0,1);\mathbb{R}) \cap C([0,1];\mathbb{R})$, for each $\varepsilon \in (0,\varepsilon_0]$. Assume that $y(\cdot\,;\varepsilon)$ has an asymptotic expansion of the form

$$y(x;\varepsilon) = y_{\mathrm{out}}(x;\varepsilon) \sim y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots, \quad \text{as} \quad \varepsilon \searrow 0, \tag{4}$$

for all $x \in [x_1, 1]$, for some $x_1 \in (0,1)$. In the present case, we are explicitly preparing for the situation that the boundary layer is at the point $x = x_0 = 0$. Notice that the outer solution expansion in (4) is assumed to be valid in the region away from (outside of) the boundary layer. Suppose further that, for $x \in (x_1, 1)$,

$$\frac{\mathrm{d}y_{\mathrm{out}}}{\mathrm{d}x}(x;\varepsilon) \sim y_0'(x) + \varepsilon y_1'(x) + \varepsilon^2 y_2'(x) + \cdots, \quad \text{as} \quad \varepsilon \searrow 0, \tag{5}$$

and, for $x \in (x_1, 1)$,

$$\frac{\mathrm{d}^2 y_{\mathrm{out}}}{\mathrm{d}x^2}(x;\varepsilon) \sim y_0''(x) + \varepsilon y_1''(x) + \varepsilon^2 y_2''(x) + \cdots, \quad \text{as} \quad \varepsilon \searrow 0. \tag{6}$$

Outer Sequence of ODEs

Substituting these expansions into Equation (1), we obtain, for $x \in (x_1, 1)$,

$$\begin{aligned} &\varepsilon\left(y_0''(x) + \varepsilon y_1''(x) + \varepsilon^2 y_2''(x) + \cdots\right) \\ &+2\left(y_0'(x) + \varepsilon y_1'(x) + \varepsilon^2 y_2'(x) + \cdots\right) \\ &+2\left(y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots\right) \sim 0. \end{aligned}$$

From this, we have the following sequence of equations:

$$\begin{aligned} O(1) &: y_0'(x) + y_0(x) = 0, \\ O(\varepsilon) &: y_1'(x) + y_1(x) = -\frac{1}{2} y_0''(x), \\ O(\varepsilon^2) &: y_2'(x) + y_2(x) = -\frac{1}{2} y_1''(x), \\ &\vdots \end{aligned}$$

Outer Sequence of Boundary Conditions

The boundary conditions are

$$
\begin{aligned}
O(1) &: y_0(0) = 0 \quad \text{and} \quad y_0(1) = 1, \\
O(\varepsilon) &: y_1(0) = 0 \quad \text{and} \quad y_1(1) = 0, \\
O(\varepsilon^2) &: y_2(0) = 0 \quad \text{and} \quad y_2(1) = 0, \\
&\vdots
\end{aligned}
$$

We enforce only the boundary condition(s) at $x = 1$, because, of course, there is a boundary layer at $x = 0$. In fact, our outer expansion is not even assumed valid near $x = 0$.

Leading-Order Term

The general solution of the $O(1)$ equation is

$$y_0(x) = C_0 e^{-x}, \tag{7}$$

where $C_0 \in \mathbb{R}$ is a constant to be determined. Applying the boundary condition $y_0(1) = 1$, we have

$$y_0(x) = e^{1-x}.$$

Thus, the outer solution at leading order is uniquely and completely determined.

First-Order Correction Term

Next, we can determine the first-order correction to the outer solution, $y_1(x)$. This satisfies the equation

$$y_1'(x) + y_1(x) = -\frac{1}{2}e^{1-x}.$$

Let us try a solution of the form

$$y_1(x) = y_{1,\mathrm{h}}(x) + y_{1,\mathrm{p}}(x),$$

where $y_{1,\mathrm{h}}$ is the homogeneous solution and $y_{1,\mathrm{p}}$ is the particular solution. We guess

$$y_{1,\mathrm{h}}(x) = C_{1,\mathrm{h}}e^{-x},$$
$$y_{1,\mathrm{p}}(x) = C_{1,\mathrm{p}}xe^{-x},$$

where $C_{1,\mathrm{h}}$ and $C_{1,\mathrm{p}}$ are constants. Plugging in the guessed form and using the right-hand boundary condition $y_1(1) = 0$, we obtain

$$y_1(x) = \frac{1}{2}(1-x)e^{1-x}.$$

Boundary Layer Thickness

With the knowledge that there is a boundary layer at $x = 0$, we need only to determine its thickness. For this task, we use the principle of dominant balance. Let us define the stretched variable

$$z = \frac{x}{\delta}, \quad \delta = \delta(\varepsilon) > 0,$$

and the function

$$Y(z) := y(\delta z).$$

It follows, as before, that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}x}\frac{\mathrm{d}Y}{\mathrm{d}z} = \frac{1}{\delta}\frac{\mathrm{d}Y}{\mathrm{d}z},$$

and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{\delta^2}\frac{\mathrm{d}^2 Y}{\mathrm{d}z^2}.$$

Equation (1) transforms into

$$\frac{\varepsilon}{\delta^2}\frac{\mathrm{d}^2 Y}{\mathrm{d}z^2} + \frac{2}{\delta}\frac{\mathrm{d}Y}{\mathrm{d}z} + 2Y = 0. \tag{8}$$

Dominant Balance

We assume, in the principle of dominant balance, that $\frac{d^2Y}{dz^2}$, $\frac{dY}{dz}$, and Y are all $O(1)$, when expressed in the in the proper stretched variable. The coefficients of term 1 and one another term must balance, in the asymptotic sense, and must dominate the remaining term in the limit as $\varepsilon \searrow 0$. We have the following possibilities in dominant balance:

1. Terms 1 and 2 of Equation (8) balance. This implies that

$$\frac{\varepsilon}{\delta^2} = \frac{1}{\delta} \implies \delta = \varepsilon.$$

   In this case, terms 1 and 2, which are $O(\varepsilon^{-1})$, dominate term 3, which is $O(1)$, as $\varepsilon \searrow 0$. This is a viable situation.

2. Terms 1 and 3 of Equation (8) balance. This implies that

$$\frac{\varepsilon}{\delta^2} = 1 \implies \delta = \sqrt{\varepsilon}.$$

   In this case, terms 1 and 3, which are $O(1)$, do not dominate term 2, which is $O(\varepsilon^{-1/2})$, as $\varepsilon \searrow 0$. This is not a viable option.

So, we choose case 1, and

$$\delta = \delta(\varepsilon) = \varepsilon.$$

Inner Expansion

Assume that, for some $z_1 > 0$, and all $z \in [0, z_1]$,

$$y(x;\varepsilon) = Y_{\text{in}}(z;\varepsilon) \sim Y_0(z) + \varepsilon Y_1(z) + \varepsilon^2 Y_2(z) + \cdots, \quad \text{as} \quad \varepsilon \searrow 0. \tag{9}$$

In other words, the inner solution expansion in Equation (9) is valid inside the boundary layer. We also assume that, for all $z \in (0, z_1]$,

$$\frac{\mathrm{d}Y_{\text{in}}}{\mathrm{d}z}(z;\varepsilon) \sim Y_0'(z) + \varepsilon Y_1'(z) + \varepsilon^2 Y_2'(z) + \cdots, \quad \text{as} \quad \varepsilon \searrow 0, \tag{10}$$

and

$$\frac{\mathrm{d}^2 Y_{\text{in}}}{\mathrm{d}z^2}(z;\varepsilon) \sim Y_0''(z) + \varepsilon Y_1''(z) + \varepsilon^2 Y_2''(z) + \cdots, \quad \text{as} \quad \varepsilon \searrow 0, \tag{11}$$

With $\delta = \varepsilon$, the inner equation can be written as

$$Y''(z) + 2Y'(z) + 2\varepsilon Y(z) = 0, \tag{12}$$

with the boundary condition

$$Y(0) = 0.$$

Inner Sequence of ODEs

Substituting our inner expansions into Equation (12), we obtain

$$\begin{aligned}
&\left(Y_0''(z) + \varepsilon Y_1''(z) + \varepsilon^2 Y_2''(z) + \cdots\right) \\
&+2\left(Y_0'(z) + \varepsilon Y_1'(z) + \varepsilon^2 Y_2'(z) + \cdots\right) \\
&+2\varepsilon\left(Y_0(z) + \varepsilon Y_1(z) + \varepsilon^2 Y_2(z) + \cdots\right) \sim 0.
\end{aligned}$$

From this, we have the following sequence of equations:

$$\begin{aligned}
O(1):&\ Y_0''(z) + 2Y_0'(z) = 0, \\
O(\varepsilon):&\ Y_1''(z) + 2Y_1'(z) = -2Y_0(z), \\
O(\varepsilon^2):&\ Y_2''(z) + 2Y_2'(z) = -2Y_1(z), \\
&\ \vdots
\end{aligned}$$

Inner Sequence of Boundary Conditions

For the inner expansion, we only apply the boundary condition at $x = 0$. We have

$$\begin{aligned} O(1) &: \; Y_0(0) = 0, \\ O(\varepsilon) &: \; Y_1(0) = 0, \\ O(\varepsilon^2) &: \; Y_2(0) = 0, \\ &\vdots \end{aligned}$$

Leading-Order Inner Term

The general solution of the $O(1)$ problem is

$$Y_0(z) = C_{0,1} + C_{0,2}e^{-2z},$$

where $C_{0,1}$ and $C_{0,2}$ are constants. Applying the boundary condition $Y_0(0) = 0$, we have

$$C_{0,1} = -C_{0,2} =: C_0,$$

where C_0 is a constant to be determined. Thus,

$$Y_0(z) = C_0(1 - e^{-2z}).$$

We will leave C_0 undetermined for the moment. It will be determined later using matching with the outer solution.

First-Order Correction Term

The $O(\varepsilon)$ inner problem is

$$Y_1''(z) + 2Y_1'(z) = -2C_0(1 - e^{-2z}), \quad Y_1(0) = 0.$$

We guess a solution of the form

$$Y_1(z) = Y_{1,\mathrm{h}}(z) + Y_{1,\mathrm{p}}(z),$$

where the homogeneous solution has the form

$$Y_{1,\mathrm{h}}(z) = C_{1,1} + C_{1,2}e^{-2z},$$

and the particular solution has the form

$$Y_{1,\mathrm{p}}(z) = \alpha z e^{-2z} + \beta_1 z + \beta_2 z^2.$$

Inserting the guessed form of the solution into the equation we obtain

$$Y_1(z) = C_{1,1} + C_{1,2}e^{-2z} - C_0 z \left(e^{-2z} + 1\right).$$

First-Order Correction Term

The general solution is

$$Y_1(z) = C_{1,1} + C_{1,2}e^{-2z} - C_0 z\left(e^{-2z} + 1\right).$$

Using the boundary condition $Y_1(0) = 0$, we get

$$C_{1,1} = -C_{1,2} =: C_1,$$

where C_1 is a constant that we will determine via matching. Thus

$$Y_1(z) = C_1\left(1 - e^{-2z}\right) - C_0 z\left(e^{-2z} + 1\right).$$

Matching

Now, we use the method from Chapter 2 to carry out the matching. We assume the existence of the intermediate variable

$$w = \frac{x}{\eta(\varepsilon)} = \frac{\delta(\varepsilon)}{\eta(\varepsilon)} z. \tag{13}$$

Recall that the function δ is called the boundary layer scale, and η is the intermediate scale. The variable z is the stretched boundary-layer variable that we introduced earlier, and w is the intermediate variable. Let us recall that η must satisfy

1. $\eta(\varepsilon) > 0$, for all $\varepsilon > 0$,
2. $\eta(\varepsilon) \searrow 0$, as $\varepsilon \searrow 0$, and
3. $\frac{\delta(\varepsilon)}{\eta(\varepsilon)} \searrow 0$, as $\varepsilon \searrow 0$.

As before, the following shorthand represents our assumptions:

$$0 < \delta(\varepsilon) \ll \eta(\varepsilon) \ll 1, \quad \text{as} \quad \varepsilon \searrow 0.$$

Matching

Taking $(N, M) = (1, 1)$ and using $\phi_k(\varepsilon) = \varepsilon^k$, we look for matching functions with respect to the inner and outer expansions that satisfy

$$\begin{aligned} y_0(x) + y_1(x)\varepsilon &= \sum_{k=0}^{1} y_k(w \cdot \eta(\varepsilon))\varepsilon^k \\ &= y_{\mathrm{match}}^{1,1,\mathrm{out}}(w;\varepsilon) + o\left(\varepsilon\right), \end{aligned} \tag{14}$$

as $\varepsilon \searrow 0$, with $w > 0$ fixed and finite, and

$$\begin{aligned} Y_0(z) + Y_1(z)\varepsilon &= \sum_{k=0}^{1} Y_k\left(w\frac{\eta(\varepsilon)}{\delta(\varepsilon)}\right)\varepsilon^k \\ &= y_{\mathrm{match}}^{1,1,\mathrm{in}}(w;\varepsilon) + o\left(\varepsilon\right), \end{aligned} \tag{15}$$

as $\varepsilon \searrow 0$, with $w > 0$ fixed and finite.

The Matching Function and the Composite Approximation

We will choose our remaining free parameters so that a common matching term may be found to satisfy

$$y_{\mathrm{match}}^{1,1}(w;\varepsilon) = y_{\mathrm{match}}^{1,1,\mathrm{in}}(w;\varepsilon) = y_{\mathrm{match}}^{1,1,\mathrm{out}}(w;\varepsilon).$$

Then, we expect a composite, uniformly-valid approximation of the true solution $y(x;\varepsilon)$ may be obtained from

$$y_{\mathrm{c},1,1}(x;\varepsilon) = \sum_{k=0}^{1} y_k(x)\varepsilon^k + \sum_{k=0}^{1} Y_k\left(\frac{x}{\delta(\varepsilon)}\right)\varepsilon^k - y_{\mathrm{match}}^{1,1}\left(\frac{x}{\eta(\varepsilon)};\varepsilon\right),$$

with

$$y(x;\varepsilon) - y_{\mathrm{c},1,1}(x;\varepsilon) = O(\varepsilon^2), \quad \text{as} \quad \varepsilon \searrow 0,$$

uniformly on $[0,1]$. It only remains to find $y_{\mathrm{match}}^{1,1}$.

Outer Expansion in the Intermediate Variable

Observe that

$$y_0(x) + y_1(x)\varepsilon = e^{1-x} + \frac{1}{2}(1-x)e^{1-x}\varepsilon.$$

Writing this outer expansion in the intermediate variable, and using Taylor's Theorem, we have

$$\begin{aligned}
y_0(x) + y_1(x)\varepsilon &= y_0(w\eta) + y_1(w\eta)\varepsilon \\
&= e^{1-w\eta}\left(1 + \frac{1}{2}(1-w\eta)\varepsilon\right) \\
&= e^{1}\left(1 - w\eta + \frac{w^2}{2}\eta^2 + o(\eta^2)\right)\left(1 + \frac{1}{2}\varepsilon - \frac{1}{2}w\eta\varepsilon\right) \\
&= e^{1}\left[1 - w\eta + \frac{w^2}{2}\eta^2 + \frac{1}{2}\varepsilon - \frac{w}{2}\eta\varepsilon \right. \\
&\qquad \left. + \frac{w}{4}\eta^2\varepsilon - \frac{w}{2}\eta\varepsilon + \frac{w^2}{2}\eta^2\varepsilon - \frac{w^3}{4}\eta^3\varepsilon + o(\eta^2)\right],
\end{aligned}$$

for fixed, finite $w > 0$, as $\varepsilon \searrow 0$.

Outer Expansion in the Intermediate Variable

Continuing

$$y_0(x) + y_1(x)\varepsilon = e^1\left[1 - w\eta + \frac{w^2}{2}\eta^2 + \frac{1}{2}\varepsilon - \frac{w}{2}\eta\varepsilon \right.$$
$$\left. + \frac{w}{4}\eta^2\varepsilon - \frac{w}{2}\eta\varepsilon + \frac{w^2}{2}\eta^2\varepsilon - \frac{w^3}{4}\eta^3\varepsilon + o(\eta^2)\right]$$
$$= e^1\left[1 - w\eta + \frac{1}{2}\varepsilon + o(\varepsilon)\right],$$

for fixed, finite $w > 0$, as $\varepsilon \searrow 0$, provided

$$\lim_{\varepsilon \searrow 0} \frac{\eta^2}{\varepsilon} = 0. \tag{16}$$

This implies that

$$y^{1,1,\mathrm{out}}_{\mathrm{match}}(w;\varepsilon) = e^1 - e^1 w\eta + \frac{e^1}{2}\varepsilon.$$

Inner Expansion in the Intermediate Variable

Observe that

$$Y_0(z) + Y_1(z)\varepsilon = C_0(1 - e^{-2z}) + \left[C_1\left(1 - e^{-2z}\right) - C_0 z\left(e^{-2z} + 1\right)\right]\varepsilon.$$

Wring the inner expansion in the intermediate variable, we have

$$\begin{aligned} Y_0(z) + Y_1(z)\varepsilon &= Y_0\left(w\frac{\eta}{\varepsilon}\right) + Y_1\left(w\frac{\eta}{\varepsilon}\right)\varepsilon \\ &= C_0\left(1 - e^{-2w\eta/\varepsilon}\right) + \Bigg[C_1\left(1 - e^{-2w\eta/\varepsilon}\right) \\ &\qquad - C_0 w\frac{\eta}{\varepsilon}\left(1 + e^{-2w\eta/\varepsilon}\right)\Bigg]\varepsilon \\ &= C_0 + o\left(\frac{\varepsilon^r}{\eta^r}\right) + \Bigg[C_1 + o\left(\frac{\varepsilon^r}{\eta^r}\right) \\ &\qquad - C_0 w\frac{\eta}{\varepsilon}\left(1 + o\left(\frac{\varepsilon^r}{\eta^r}\right)\right)\Bigg]\varepsilon, \end{aligned}$$

for any $r > 0$, for fixed, finite $w > 0$, as $\varepsilon \searrow 0$.

Inner Expansion in the Intermediate Variable

We can conclude from

$$Y_0(z) + Y_1(z)\varepsilon = C_0 + o\left(\frac{\varepsilon^r}{\eta^r}\right) + \left[C_1 + o\left(\frac{\varepsilon^r}{\eta^r}\right) - C_0 w \frac{\eta}{\varepsilon}\left(1 + o\left(\frac{\varepsilon^r}{\eta^r}\right)\right)\right]\varepsilon, \quad \text{as} \quad \varepsilon \searrow 0,$$

that

$$Y_0(z) + Y_1(z)\varepsilon = C_0 + C_1\varepsilon - C_0 w\eta + o(\varepsilon),$$

for fixed, finite $w > 0$, as $\varepsilon \searrow 0$, which implies that

$$y^{1,1,\mathrm{in}}_{\mathrm{match}}(w;\varepsilon) = C_0 + C_1\varepsilon - C_0 w\eta.$$

Matching Conditions

Since

$$y_{\mathrm{match}}^{1,1,\mathrm{out}}(w;\varepsilon) = e^1 - e^1 w\eta + \frac{e^1}{2}\varepsilon \quad \text{and} \quad y_{\mathrm{match}}^{1,1,\mathrm{in}}(w;\varepsilon) = C_0 + C_1\varepsilon - C_0 w\eta,$$

we see clearly that

$$y_{\mathrm{match}}^{1,1,\mathrm{out}}(w;\varepsilon) \equiv y_{\mathrm{match}}^{1,1,\mathrm{in}}(w;\varepsilon)$$

if and only if

$$C_0 = e^1 \quad \text{and} \quad C_1 = \frac{e^1}{2}.$$

In this case, the matching term is precisely

$$y_{\mathrm{match}}^{1,1}(w;\varepsilon) = e^1 - e^1 w\eta + \frac{e^1}{2}\varepsilon.$$

The $(N, M) = (1, 1)$ Composite Approximation

The composite, uniformly-valid approximation is

$$\begin{aligned}
y_{\mathrm{c},1,1}(x;\varepsilon) &= \sum_{k=0}^{1} y_k(x)\varepsilon^k + \sum_{k=0}^{1} Y_k\left(\frac{x}{\varepsilon}\right)\varepsilon^k - y_{\mathrm{match}}^{1,1}\left(\frac{x}{\eta};\varepsilon\right) \\
&= e^{1-x} + \frac{1}{2}(1-x)e^{1-x}\varepsilon + e^1(1-e^{-2x/\varepsilon}) \\
&\quad + \left[\frac{e^1}{2}\left(1-e^{-2x/\varepsilon}\right) - e^1\frac{x}{\varepsilon}\left(e^{-2x/\varepsilon}+1\right)\right]\varepsilon \\
&\quad - e^1 + e^1 x - \frac{e^1}{2}\varepsilon \\
&= e^{1-x} - e^{1-2x/\varepsilon} - e^{1-2x/\varepsilon}\left[\frac{1}{2}+\frac{x}{\varepsilon}\right]\varepsilon + \frac{1}{2}(1-x)e^{1-x}\varepsilon \\
&= e^{1-x}\left[1+\frac{1}{2}(1-x)\varepsilon\right] - e^{1-2x/\varepsilon}\left[1+x+\frac{\varepsilon}{2}\right].
\end{aligned}$$

The $(N, M) = (0, 0)$ Composite Approximation

For the sake of comparison, the leading-order composite approximation, $y_{\mathrm{c},0,0}(x;\varepsilon)$, is, as the reader should confirm,

$$\begin{aligned} y_{\mathrm{c},0,0}(x;\varepsilon) &= y_0(x) + Y_0\left(\frac{x}{\varepsilon}\right) - y_{\mathrm{match}}^{0,0}\left(\frac{x}{\eta};\varepsilon\right) \\ &= e^{1-x} + e^{1}(1 - e^{-2x/\varepsilon}) - e^{1} \\ &= e^{1-x} - e^{1-2x/\varepsilon}. \end{aligned}$$

A comparison of the composite solutions is provided in the figure on the next slide.

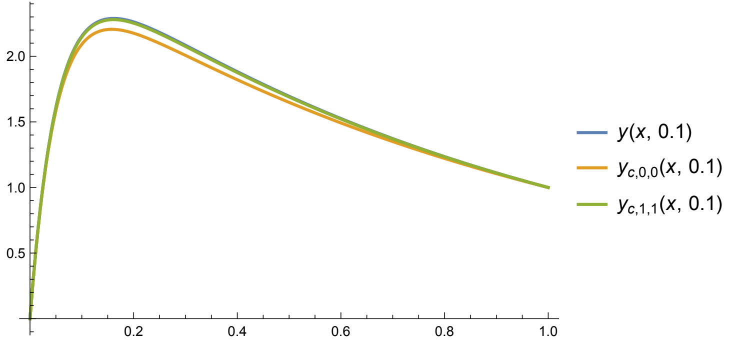

Figure: Plots of the solution $y(x;\varepsilon)$ (blue) to the problem (1) and (2) with the leading-order composite approximation $y_{\mathrm{c},0,0}(x;\varepsilon)$ (yellow) and composite approximation $y_{\mathrm{c},1,1}(x;\varepsilon)$ (green), for $\varepsilon = 0.1$. Note that $y(x;0.1)$ and $y_{\mathrm{c},1,1}(x;0.1)$ are almost indistinguishable.

Existence of Solutions

Definition

Suppose that $-\infty < x_0 < x_2 < \infty$, $\varepsilon > 0$, and $p, q \in C([x_0, x_2]; \mathbb{R})$ are given functions. Define the linear differential operator

$$L[u](x) := \varepsilon \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(x) + p(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x) + q(x)u(x), \tag{17}$$

for all $x \in (x_0, x_2)$, for any function $u \in C^2((x_0, x_2); \mathbb{R})$.

Definition

Suppose that $-\infty < x_0 < x_2 < \infty$, $\varepsilon > 0$, and $p, q \in C([x_0, x_2]; \mathbb{R})$ are given functions. Consider the homogenous boundary value problem consisting of the equation

$$L[u](x) = 0, \quad x \in (x_0, x_2), \tag{18}$$

where L is the differential operator defined in Equation (17), and the boundary conditions

$$u(x_0) = 0 \quad \text{and} \quad u(x_2) = 0. \tag{19}$$

We say that the homogeneous problem has **only the trivial solution** iff $u \equiv 0$ is the only function that satisfies Equation (18) and the boundary conditions (19). We say that the homogeneous problem has a **non-trivial solution** iff there is a function $u \in C^2((x_0, x_2); \mathbb{R}) \cap C([0, 1]; \mathbb{R})$, $u \not\equiv 0$, that satisfies Equation (18) and the boundary conditions (19).

Theorem (Fredholm Alternative)

Suppose that $-\infty < x_0 < x_2 < \infty$, $\varepsilon > 0$, $p, q \in C([x_0, x_2]; \mathbb{R})$ *are given functions, and* $A, B \in \mathbb{R}$. *The boundary value problem, consisting of the equation*

$$L[y](x) = 0, \quad x \in (x_0, x_2), \tag{20}$$

where L *is the differential operator defined in Equation* (17), *and the inhomogeneous boundary conditions*

$$y(x_0) = A \quad \text{and} \quad y(x_2) = B, \tag{21}$$

has a unique solution $y \in C^2((x_0, x_2); \mathbb{R}) \cap C([0, 1]; \mathbb{R})$ *iff the homogeneous problem, consisting of Equation* (18) *and the boundary conditions* (19) *has only the trivial solution.*

Proposition

Suppose that $-\infty < x_0 < x_2 < \infty$, $\varepsilon > 0$, *and* $p \in C^1([x_0, x_2]; \mathbb{R})$ *and* $q \in C([x_0, x_2]; \mathbb{R})$ *are given functions. Define the function*

$$f(x;\varepsilon) := q(x) - \frac{p'(x)}{2} - \frac{p^2(x)}{4\varepsilon}, \quad x \in [x_0, x_2], \quad \varepsilon > 0. \tag{22}$$

If $u \in C^2((x_0, x_2); \mathbb{R}) \cap C([0,1]; \mathbb{R})$ *is a solution to the homogenous boundary value problem, consisting of Equation* (18) *and the boundary conditions* (19), *then the function* $w \in C^2((x_0, x_2); \mathbb{R}) \cap C([0,1]; \mathbb{R})$, *defined by*

$$w(x) = u(x) \exp\left(\frac{1}{2\varepsilon}\int_{x_0}^{x} p(t)\,\mathrm{d}t\right), \tag{23}$$

satisfies the homogeneous boundary value problem consisting of the equation

$$\varepsilon \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(x) + f(x;\varepsilon) w(x) = 0 \tag{24}$$

and the homogeneous boundary conditions

$$w(x_0) = 0 \quad \textit{and} \quad w(x_2) = 0. \tag{25}$$

Proposition (Cont.)

Conversely, if the function $w \in C^2((x_0, x_2); \mathbb{R}) \cap C([0,1]; \mathbb{R})$ *is a solution to the boundary value problem consisting of Equation* (25) *and the homogeneous boundary conditions* (25), *then* $u \in C^2((x_0, x_2); \mathbb{R}) \cap C([0,1]; \mathbb{R})$ *defined via*

$$u(x) = w(x) \exp\left(-\frac{1}{2\varepsilon} \int_{x_0}^{x} p(t)\,\mathrm{d}t\right), \tag{26}$$

is a solution to the homogenous boundary value problem, consisting of Equation (18) *and the boundary conditions* (19).

Proof.

This simple computation is left as an exercise for the reader. $\square$

Theorem

Suppose that $-\infty < x_0 < x_2 < \infty$, $\varepsilon > 0$, $p \in C^1([x_0, x_2]; \mathbb{R})$ *and* $q \in C([x_0, x_2]; \mathbb{R})$. *The homogeneous boundary value problem consisting of the differential equation* (18) *and the boundary conditions* (19) *has only the trivial solution if one of the two following sufficient conditions is satisfied:*

1. $p(x) \neq 0$, *for* $x \in [x_0, x_2]$, *and* $\varepsilon > 0$ *is sufficiently small that*

$$q(x) - \frac{p'(x)}{2} \leq \frac{p^2(x)}{4\varepsilon}, \tag{27}$$

   for all $x \in [x_0, x_2]$.

2. *For all* $x \in [x_0, x_2]$,

$$q(x) - \frac{p'(x)}{2} \leq 0. \tag{28}$$

Proof.

Suppose that $w \in C^2((x_0, x_2); \mathbb{R}) \cap C([0,1]; \mathbb{R})$ is a solution to the boundary value problem consisting of Equation (25) and the homogeneous boundary conditions (25). Multiplying Equation (25) by w and integrating we get

$$\begin{aligned} 0 &= \int_{x_0}^{x_2} w(t) \left(\varepsilon \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(t) + f(t;\varepsilon) w(t) \right) \mathrm{d}t \\ &= -\varepsilon \int_{x_0}^{x_2} \left(\frac{\mathrm{d}w}{\mathrm{d}x}(t) \right)^2 \mathrm{d}t + \int_{x_0}^{x_2} f(t;\varepsilon) w^2(t) \, \mathrm{d}t, \end{aligned}$$

where we used integration by parts and the boundary conditions $w(x_0) = 0 = w(x_2)$. Thus,

$$0 = \varepsilon \int_{x_0}^{x_2} \left(\frac{\mathrm{d}w}{\mathrm{d}x}(t) \right)^2 \mathrm{d}t - \int_{x_0}^{x_2} f(t;\varepsilon) w^2(t) \, \mathrm{d}t.$$

Proof (Cont.)

Now, either of the conditions above, when assumed, imply that

$$f(x;\varepsilon) \leq 0, \quad \forall\, x \in [x_0, x_2].$$

If this is the case, it follows that

$$\int_{x_0}^{x_2} \left(\frac{\mathrm{d}w}{\mathrm{d}x}(t) \right)^2 \mathrm{d}t = 0 \tag{29}$$

and

$$\int_{x_0}^{x_2} f(t;\varepsilon) w^2(t)\, \mathrm{d}t. \tag{30}$$

Because $w \in C^2((x_0, x_2); \mathbb{R}) \cap C([0,1]; \mathbb{R})$, these conditions are only satisfied if $w \equiv 0$. Using Proposition 2.1, we conclude that the homogeneous boundary value problem consisting of the differential equation (18) and the boundary conditions (19) has only the trivial solution, $u \equiv 0$. □

Corollary

Suppose that $-\infty < x_0 < x_2 < \infty$, $\varepsilon > 0$, $p \in C^1([x_0, x_2]; \mathbb{R})$, $q \in C([x_0, x_2]; \mathbb{R})$, *and* $A, B \in \mathbb{R}$. *The boundary value problem consisting of the differential equation* (20) *and the boundary conditions* (21) *has a unique solution if one of the two sufficient conditions in the Theorem holds.*

Proof.

We can just apply the Fredholm Alternative Theorem. □

Location of the Boundary Layer

Proposition

Suppose that $0 < \varepsilon < 1$, $A, B \in \mathbb{R}$, *and* $-\infty < x_0 < x_2 < \infty$. *Assume that* $p \in C^1([x_0, x_2]; \mathbb{R})$ *and* $q \in C([x_0, x_2]; \mathbb{R})$ *satisfy*

$$p(\zeta + \alpha) = p(\zeta) + o(1) \quad \text{and} \quad q(\zeta + \alpha) = q(\zeta) + o(1), \quad \text{as} \quad \varepsilon \searrow 0,$$

for any $\zeta \in [x_0, x_2]$ *and any* $\alpha = \alpha(\varepsilon) > 0$, *with* $\alpha \searrow 0$, *as* $\varepsilon \searrow 0$. *Suppose further that* $p(x) > 0$, *for all* $x \in [x_0, x_2]$. *Let* $y = y(\,\cdot\,; \varepsilon) \in C^2((0, 1); \mathbb{R}) \cap C([0, 1]; \mathbb{R})$ *denote the unique solution of the boundary value problem consisting of Equation* (20) *and the boundary conditions* (21), *which is guaranteed to exist, provided* $\varepsilon > 0$ *is sufficiently small. Then, there is a boundary layer of thickness* $\delta = \varepsilon$ *at* $x = x_0$.

Proposition (Cont.)

Furthermore, a composite, uniformly-valid approximation of $y(\,\cdot\,;\varepsilon)$ *is given by*

$$y_{c,0,0}(x;\varepsilon) = y_0(x) + Y_0\left(\frac{x - x_0}{\varepsilon}\right) - C_1, \quad x_0 \le x \le x_2, \tag{31}$$

where

$$y_0(x) = B\exp\left(\int_x^{x_2} \frac{q(t)}{p(t)}\,\mathrm{d}t\right),$$

$$Y_0(z) = C_1 + (A - C_1)e^{-p(x_0)z},$$

$$C_1 = B\exp\left(\int_{x_0}^{x_2} \frac{q(t)}{p(t)}\,\mathrm{d}t\right),$$

and

$$\max_{x_0 \le x \le x_2} |y(x;\varepsilon) - y_{c,0,0}(x;\varepsilon)| = O(\varepsilon), \quad \textit{as} \quad \varepsilon \searrow 0.$$

Proof.

Let $\zeta \in [x_0, x_2]$ denote the location of a boundary layer. Inside the boundary layer we define the stretched variable

$$z := \frac{x - \zeta}{\delta(\varepsilon)} \quad \Longleftrightarrow \quad x = \zeta + \delta(\varepsilon) z,$$

where $\delta(\varepsilon) > 0$ and $\delta(\varepsilon) \searrow 0$, as $\varepsilon \searrow 0$. Rewriting the differential equation in terms of the stretched variable, we have

$$\frac{\varepsilon}{\delta^2} \frac{\mathrm{d}^2 Y}{\mathrm{d}z^2}(z) + \frac{p(\zeta + \delta z)}{\delta} \frac{\mathrm{d}Y}{\mathrm{d}z}(z) + q(\zeta + \delta z) Y(z) = 0,$$

where $Y(z) := y(\zeta + \delta(\varepsilon) z)$. For fixed z and ζ, as $\varepsilon \searrow 0$, the coefficients are of the order

$$\frac{\varepsilon}{\delta^2}, \quad \frac{p(\zeta)}{\delta}, \quad \text{and} \quad q(\zeta).$$

A dominant balance is achieved by balancing terms 1 and 2, which then dominate term 3 as $\varepsilon \searrow 0$, as is easily seen. From this reasoning, one is tempted to take $\delta = \frac{\varepsilon}{p(\zeta)}$, recalling that $p(\zeta) > 0$ and $p(\zeta) = O(1)$, as $\varepsilon \searrow 0$. But this level of fidelity is unnecessary. It is enough to take $\delta = \varepsilon$, so that the asymptotic orders match.

Proof (Cont.)

Now, we assume that the inner solution, Y_{in}, has the asymptotic expansion

$$y(x;\varepsilon) = Y_{\mathrm{in}}(z;\varepsilon) = Y_0(z) + o(1), \quad \text{as} \quad \varepsilon \searrow 0,$$

such that

$$\frac{\mathrm{d}Y_{\mathrm{in}}}{\mathrm{d}z}(z;\varepsilon) = Y_0'(z) + o(1), \quad \text{as} \quad \varepsilon \searrow 0,$$

and

$$\frac{\mathrm{d}^2 Y_{\mathrm{in}}}{\mathrm{d}z^2}(z;\varepsilon) = Y_0''(z) + o(1), \quad \text{as} \quad \varepsilon \searrow 0.$$

Thus, for the leading-order inner problem, we have

$$Y_0''(z) + p(\zeta) Y_0'(z) = 0.$$

The general solution is

$$Y_0(z) = C_1 + C_2 e^{-p(\zeta) z}.$$

Proof (Cont.)

Since $p(\zeta) > 0$, the solution,

$$Y_0(z) = C_1 + C_2 e^{-p(\zeta)z},$$

decays exponentially, for $z > 0$, and grows exponentially, for $z < 0$. The inner solution must decay as $|z|$ increases through the layer. The only possibility, since $p(\zeta) > 0$, is that $\zeta = x_0$. There can be no interior boundary layer or a boundary layer at $x = x_2$. Applying the boundary condition at $x = x_0$ $(z = 0)$, we observe for the inner solution that $C_2 = A - C_1$, so that

$$Y_0(z) = C_1 + (A - C_1)e^{-p(0)z}.$$

This is the expected, consistent behavior of the inner solution in the boundary layer, namely, that, at leading order, the solution has exponential decay through the boundary layer, as one moves from the location of the boundary layer toward the outer solution region.

Proof (Cont.)

Since the boundary layer is at $x = x_0$, we should use the right boundary condition at $x = x_2$ for the leading-order outer problem. Let us start by assuming the outer solution, $y_{\rm out}$, has the asymptotic expansion

$$y(x;\varepsilon) = y_{\rm out}(x;\varepsilon) = y_0(x) + o(1), \quad \text{as} \quad \varepsilon \searrow 0,$$

such that

$$\frac{\mathrm{d}y_{\rm out}}{\mathrm{d}x}(x;\varepsilon) = y_0'(x) + o(1), \quad \text{as} \quad \varepsilon \searrow 0,$$

and

$$\frac{\mathrm{d}^2 y_{\rm out}}{\mathrm{d}x^2}(x;\varepsilon) = y_0''(x) + o(1), \quad \text{as} \quad \varepsilon \searrow 0.$$

Proof (Cont.)

The $O(1)$ outer problem is

$$p(x)y_0'(x) + q(x)y_0(x) = 0, \quad x_1 \le x \le x_2,$$

where $x_1 \in (x_0, x_2)$, with the boundary condition

$$y_0(x_2) = B.$$

Since $p > 0$ on $[x_0, x_2]$, it is easy to see that

$$y_0(x) = B \exp\left(\int_x^{x_2} \frac{q(t)}{p(t)}\, \mathrm{d}t\right).$$

Proof (Cont.)

For the matching procedure, let us introduce the intermediate variable $w = \frac{x - x_0}{\eta}$, where

$$0 < \delta(\varepsilon) \ll \eta(\varepsilon) \ll 1, \quad \text{as} \quad \varepsilon \searrow 0.$$

Now, for fixed, finite w, as $\varepsilon \searrow 0$,

$$\begin{aligned} y_0(x) &= y(x_0 + \eta w) \\ &= B \exp\left(\int_{x_0 + \eta w}^{x_2} \frac{q(t)}{p(t)} \, \mathrm{d}t \right) \\ &= B \exp\left(\int_{x_0}^{x_2} \frac{q(t)}{p(t)} \, \mathrm{d}t \right) + o(1). \end{aligned}$$

Proof (Cont.)

Similarly, for fixed, finite w, as $\varepsilon \searrow 0$,

$$\begin{aligned} Y_0(z) &= Y_0\left(w\frac{\eta}{\varepsilon}\right) \\ &= C_1 + (A - C_1)e^{-p(x_0)\eta w/\varepsilon} \\ &= C_1 + o\left(\frac{\varepsilon^r}{\eta^r}\right), \end{aligned}$$

for any $r > 0$. Thus, as $\varepsilon \searrow 0$,

$$Y_0(z) = C_1 + o(1).$$

It follows that $C_1 = B\exp\left(\int_{x_0}^{x_2} \frac{q(t)}{p(t)}\,\mathrm{d}t\right)$

$$y_{\mathrm{match}}^{0,0}(w;\varepsilon) = B\exp\left(\int_{x_0}^{x_2} \frac{q(t)}{p(t)}\,\mathrm{d}t\right).$$

The proof of the error estimate will be taken up later. □

Proposition

Suppose that $0 < \varepsilon < 1$, $A, B \in \mathbb{R}$, *and* $-\infty < x_0 < x_2 < \infty$. *Assume that* $p \in C^1([x_0, x_2]; \mathbb{R})$ *and* $q \in C([x_0, x_2]; \mathbb{R})$ *satisfy*

$$p(\zeta + \alpha) = p(\zeta) + o(1) \quad \text{and} \quad q(\zeta + \alpha) = q(\zeta) + o(1), \quad \text{as} \quad \varepsilon \searrow 0,$$

for any $\zeta \in [x_0, x_2]$ *and any* $\alpha = \alpha(\varepsilon) > 0$, *with* $\alpha \searrow 0$, *as* $\varepsilon \searrow 0$. *Suppose further that* $p(x) < 0$, *for all* $x \in [x_0, x_2]$. *Let* $y = y(\,\cdot\,; \varepsilon) \in C^2((0,1); \mathbb{R}) \cap C([0,1]; \mathbb{R})$ *denote the unique solution of the boundary value problem consisting of Equation* (20) *and the boundary conditions* (21), *which is guaranteed to exist, provided* $\varepsilon > 0$ *is sufficiently small. Then, there is a boundary layer of thickness* $\delta = \varepsilon$ *at* $x = x_2$.

Proposition (Cont.)

Furthermore, a composite, uniformly-valid approximation of $y(\,\cdot\,;\varepsilon)$ *is given by*

$$y_{c,0,0}(x;\varepsilon) = y_0(x) + Y_0\left(\frac{x - x_2}{\varepsilon}\right) - C_1, \quad x_0 \le x \le x_2, \tag{32}$$

where

$$y_0(x) = A\exp\left(-\int_{x_0}^{x} \frac{q(t)}{p(t)}\,\mathrm{d}t\right),$$

$$Y_0(z) = C_1 + (B - C_1)e^{-p(x_2)z},$$

$$C_1 = A\exp\left(-\int_{x_0}^{x_2} \frac{q(t)}{p(t)}\,\mathrm{d}t\right),$$

and

$$\max_{x_0\le x\le x_2} |y(x;\varepsilon) - y_{c,0,0}(x;\varepsilon)| = O(\varepsilon), \quad \text{as} \quad \varepsilon \searrow 0.$$

Proof.

The argument is similar. □

Remark

Suppose that $a(x)$ *has a simple zero at an interior point of* $[x_0, x_2]$*, at, say,* $\zeta \in (x_0, x_2)$*, with* $a(x) < 0$ *on* $[x_0, \zeta)$ *and* $a(x) > 0$ *on* $(\zeta, x_2]$*. Then, it is not too difficult to see that* ζ *is the location of a boundary layer, generically. We will not prove this case, but will explore an example of this type in Section* **??**.