Math 515
Essential Perturbation Theory and Asymptotic Analysis
Chapter 05

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Spring 2024

Chapter 05, Part 2 of 3
Matched Asymptotics and Boundary Layers

A Linear, Non-Constant Coefficient BVP

Example

Suppose that $y(\,\cdot\,;\varepsilon) : [0,1] \to \mathbb{R}$ satisfies the BVP

$$\varepsilon y''(x) + (1+x)y'(x) + y(x) = 0, \quad \text{for} \quad 0 < x < 1, \tag{1}$$

where $\varepsilon > 0$ is a small parameter, with the boundary conditions

$$y(0) = 1 \quad \text{and} \quad y(1) = 1. \tag{2}$$

Since $p(x) = 1 + x > 0$, for all $x \in [0,1]$, there is a boundary layer at the left end point $x = 0$, and its thickness is $\delta(\varepsilon) = \varepsilon$. In this example, our mission is to find the composite approximation $y_{\mathrm{c},1,1}$. The higher-order composite approximation, $y_{\mathrm{c},2,2}$, is computed in the follow on example.

Example (Cont.)

Using the usual techniques, the outer expansion of the solution is found to satisfy

$$y_{\text{out}}(x;\varepsilon) \sim \frac{2}{1+x} + \left[\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right]\varepsilon + \left[\frac{6}{(1+x)^5} - \frac{1}{2(1+x)^3} - \frac{1}{4(1+x)}\right]\varepsilon^2 + \cdots, \tag{3}$$

as $\varepsilon \searrow 0$, for all $x \in [x_1, 1]$, for some $x_1 \in (0,1)$.

Example (Cont.)

Meanwhile, the inner expansion, written in terms of the stretched variable $z = \frac{x}{\varepsilon}$, satisfies

$$
\begin{aligned}
Y_{\mathrm{in}}(z;\varepsilon) \sim{} & 1 + C_0(e^{-z} - 1) \\
& + \left[-z + C_0 \left(z - \frac{1}{2} z^2 e^{-z} \right) + C_1 \left(e^{-z} - 1 \right) \right] \varepsilon \\
& + \left[z^2 - 2z + C_0 \left(\frac{1}{8} z^4 e^{-z} - z^2 + 2z \right) \right. \\
& \left. + C_1 \left(z - \frac{1}{2} z^2 e^{-z} \right) + C_2 \left(e^{-z} - 1 \right) \right] \varepsilon^2 \\
& + \cdots ,
\end{aligned} \tag{4}
$$

as $\varepsilon \searrow 0$, for all $z \in [0, z_1]$, for some $z_1 > 0$.

Example (Cont.)

To remind ourselves what is the task at hand, we first introduce the intermediate variable

$$w = \frac{x}{\eta(\varepsilon)} = \frac{\varepsilon}{\eta(\varepsilon)} z. \tag{5}$$

where we have already set $\delta(\varepsilon) = \varepsilon$ for the present case. Recall that the function η must satisfy

1. $\eta(\varepsilon) > 0$, for all $\varepsilon > 0$,
2. $\eta(\varepsilon) \searrow 0$, as $\varepsilon \searrow 0$, and
3. $\frac{\varepsilon}{\eta(\varepsilon)} \searrow 0$, as $\varepsilon \searrow 0$,

or, in short-hand form,

$$0 < \varepsilon \ll \eta(\varepsilon) \ll 1, \quad \text{as} \quad \varepsilon \searrow 0.$$

Example (Cont.)

To this end, with $N = M$, and using $\phi_k(\varepsilon) = \varepsilon^k$, we look for matching functions with respect to the inner and outer expansions that satisfy

$$\sum_{k=0}^{N} y_k(x)\varepsilon^k = \sum_{k=0}^{N} y_k(w \cdot \eta(\varepsilon) + x_0)\varepsilon^k$$
$$= y_{\mathrm{match}}^{N,N,\mathrm{out}}(w;\varepsilon) + o\left(\varepsilon^N\right), \tag{6}$$

as $\varepsilon \searrow 0$, with $w > 0$ fixed and finite, and

$$\sum_{k=0}^{N} Y_k(z)\varepsilon^k = \sum_{k=0}^{N} Y_k\left(w\frac{\eta(\varepsilon)}{\varepsilon}\right)\varepsilon^k$$
$$= y_{\mathrm{match}}^{N,N,\mathrm{in}}(w;\varepsilon) + o\left(\varepsilon^N\right), \tag{7}$$

as $\varepsilon \searrow 0$, with $w > 0$ fixed and finite.

Example (Cont.)

If possible, we will choose the remaining free parameters in the inner expansion so that the matching functions are equal:

$$y_{\mathrm{match}}^{N,N}(w;\varepsilon) = y_{\mathrm{match}}^{N,N,\mathrm{in}}(w;\varepsilon) = y_{\mathrm{match}}^{N,N,\mathrm{out}}(w;\varepsilon).$$

To do this, recall, we must adjust width of the matching layer appropriately. Then, we expect a composite, uniformly-valid approximation of the true solution $y(x;\varepsilon)$ may be obtained from

$$y_{\mathrm{c},N,N}(x;\varepsilon) = \sum_{k=0}^{N} y_k(x)\varepsilon^k + \sum_{k=0}^{N} Y_k\left(\frac{x}{\varepsilon}\right)\varepsilon^k - y_{\mathrm{match}}^{N,N}\left(\frac{x}{\eta(\varepsilon)};\varepsilon\right),$$

with

$$y(x;\varepsilon) - y_{\mathrm{c},N,N}(x;\varepsilon) = O(\varepsilon^{N+1}), \quad \text{as} \quad \varepsilon \searrow 0,$$

uniformly on $[0,1]$.

Example (Cont.)

Inner Matching Function, $y_{\text{match}}^{1,1,\text{in}}$: Recall that

$$Y_0(z) + Y_1(z)\varepsilon = 1 + C_0(e^{-z} - 1) + \left[-z + C_0\left(z - \frac{1}{2}z^2 e^{-z}\right) + C_1\left(e^{-z} - 1\right)\right]\varepsilon.$$

Wring the inner expansion in the intermediate variable, we have

$$\begin{aligned}
Y_0(z) + Y_1(z)\varepsilon &= 1 + C_0\left(e^{-w\eta/\varepsilon} - 1\right) + \Bigg[-w\frac{\eta}{\varepsilon} \\
&\quad + C_0\left(w\frac{\eta}{\varepsilon} - \frac{1}{2}w^2\frac{\eta^2}{\varepsilon^2}e^{-w\eta/\varepsilon}\right) + C_1\left(e^{-w\eta/\varepsilon} - 1\right)\Bigg]\varepsilon \\
&= 1 + C_0\left(o\left(\frac{\varepsilon^r}{\eta^r}\right) - 1\right) + \Bigg[-w\frac{\eta}{\varepsilon} \\
&\quad + C_0\left(w\frac{\eta}{\varepsilon} + o\left(\frac{\varepsilon^r}{\eta^r}\right)\right) + C_1\left(o\left(\frac{\varepsilon^r}{\eta^r}\right) - 1\right)\Bigg]\varepsilon,
\end{aligned}$$

for any $r > 0$, for fixed, finite $w > 0$, as $\varepsilon \searrow 0$.

Example (Cont.)

We can conclude from this that

$$Y_0(z) + Y_1(z)\varepsilon = 1 - C_0 + (C_0 - 1)w\eta - C_1\varepsilon + o(\varepsilon),$$

for fixed, finite $w > 0$, as $\varepsilon \searrow 0$, which implies that

$$y^{1,1,\mathrm{in}}_{\mathrm{match}}(w;\varepsilon) = 1 - C_0 + (C_0 - 1)w\eta - C_1\varepsilon.$$

Example (Cont.)

Outer Matching Function, $y_{\text{match}}^{1,1,\text{out}}$: Recall that

$$y_0(x) + y_1(x)\varepsilon = \frac{2}{1+x} + \left[\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right]\varepsilon.$$

For the outer expansion, we will make use of the following binomial expansions:

$$\begin{aligned}
\frac{1}{1+w\eta} &= 1 - w\eta + \frac{(-1)(-2)}{2!}w^2\eta^2 + O(\eta^3) \\
&= 1 - w\eta + w^2\eta^2 + O(\eta^3), \\
\frac{1}{(1+w\eta)^3} &= 1 - 3w\eta + \frac{(-3)(-4)}{2!}w^2\eta^2 + O(\eta^3) \\
&= 1 - 3w\eta + 6w^2\eta^2 + O(\eta^3).
\end{aligned}$$

Example (Cont.)

Writing the two-term outer expansion in the intermediate variable, and using the binomial expansions above, we have

$$
\begin{aligned}
y_0(x) + y_1(x)\varepsilon &= y_0(w\eta) + y_1(w\eta)\varepsilon \\
&= \frac{2}{1+w\eta} + \left[\frac{2}{(1+w\eta)^3} - \frac{1}{2(1+w\eta)}\right]\varepsilon \\
&= \left(2 - \frac{\varepsilon}{2}\right)\left[1 - w\eta + w^2\eta^2 + O(\eta^3)\right] \\
&\quad + 2\varepsilon\left[1 - 3w\eta + 6w^2\eta^2 + O(\eta^3)\right] \\
&= 2 - 2w\eta + \left(2 - \frac{1}{2}\right)\varepsilon + o(\varepsilon),
\end{aligned}
$$

for fixed, finite $w > 0$, as $\varepsilon \searrow 0$, provided

$$
\lim_{\varepsilon \searrow 0} \frac{\eta^2}{\varepsilon} = 0. \tag{8}
$$

Example (Cont.)

This last choice sets the width of the matching scale for the $(N, M) = (1, 1)$ case. Thus, we find that

$$y_{\mathrm{match}}^{1,1,\mathrm{out}}(w; \varepsilon) = 2 - 2w\eta + \frac{3}{2}\varepsilon.$$

Example (Cont.)

$y_{c,1,1}$ Composite Approximation: Matching $y^{1,1,\mathrm{out}}_{\mathrm{match}}$ and $y^{1,1,\mathrm{in}}_{\mathrm{match}}$ at the appropriate orders requires that

$$
\begin{aligned}
O(1):&\quad 2 = 1 - C_0 \implies C_0 = -1,\\
O(\eta):&\quad -2w = w(C_0 - 1) \text{ (consistent)},\\
O(\varepsilon):&\quad \frac{3}{2} = -C_1 \implies C_1 = -\frac{3}{2},
\end{aligned}
$$

where, by consistent, we mean that no new information is added, but the matching condition is consistent with information at lower orders. Therefore,

$$
y^{1,1}_{\mathrm{match}}(w;\varepsilon) = 2 - 2w\eta + \frac{3}{2}\varepsilon.
$$

Example (Cont.)

The composite approximation is

$$
\begin{aligned}
y_{\mathrm{c},1,1}(x;\varepsilon) &= \sum_{k=0}^{1} y_k(x)\varepsilon^k + \sum_{k=0}^{1} Y_k\left(\frac{x}{\varepsilon}\right)\varepsilon^k - y_{\mathrm{match}}^{1,1}\left(\frac{x}{\eta};\varepsilon\right) \\
&= \frac{2}{1+x} + \left[\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right]\varepsilon \\
&\quad + 1 - (e^{-x/\varepsilon} - 1) \\
&\quad + \left[-\frac{x}{\varepsilon} - \left(\frac{x}{\varepsilon} - \frac{1}{2}\frac{x^2}{\varepsilon^2}e^{-x/\varepsilon}\right) - \frac{3}{2}\left(e^{-x/\varepsilon} - 1\right)\right]\varepsilon \\
&\quad - 2 + 2x - \frac{3}{2}\varepsilon \\
&= \frac{2}{1+x} + \left[\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right]\varepsilon - e^{-x/\varepsilon} \\
&\quad + \left[\frac{1}{2}\frac{x^2}{\varepsilon^2} - \frac{3}{2}\right] e^{-x/\varepsilon}\varepsilon. \qquad (9)
\end{aligned}
$$

Example

In this example, we continue with the problem posed in the last. Specifically, our task now is to find the composite approximation $y_{c,2,2}$ for the singularly-perturbed BVP comprised of the second-order ordinary differential equation (1) with the boundary conditions (2). The three terms that we will need in the outer and inner solution expansions can be found in (3) and (4), respectively. As before, let us start with the inner matching function.

Example (Cont.)

Inner Matching Function, $y^{2,2,\mathrm{in}}_{\mathrm{match}}$: Recall from (4) that

$$
\begin{aligned}
\sum_{k=0}^{2} Y_k(z)\varepsilon^k = 1 + C_0(e^{-z} - 1)& \\
&+ \left[-z + C_0\left(z - \frac{1}{2}z^2e^{-z}\right) + C_1\left(e^{-z} - 1\right)\right]\varepsilon \\
&+ \Bigg[z^2 - 2z + C_0\left(\frac{1}{8}z^4e^{-z} - z^2 + 2z\right) \\
&+ C_1\left(z - \frac{1}{2}z^2e^{-z}\right) + C_2\left(e^{-z} - 1\right)\Bigg]\varepsilon^2.
\end{aligned}
$$

Example (Cont.)

Wring the inner expansion in the intermediate variable, we have

$$\begin{aligned}\sum_{k=0}^{2} Y_k(z)\varepsilon^k = 1 &+ C_0\left(o\left(\frac{\varepsilon^r}{\eta^r}\right) - 1\right) + \Bigg[- w\frac{\eta}{\varepsilon} \\ &+ C_0\left(w\frac{\eta}{\varepsilon} + o\left(\frac{\varepsilon^r}{\eta^r}\right)\right) + C_1\left(o\left(\frac{\varepsilon^r}{\eta^r}\right) - 1\right)\Bigg]\varepsilon \\ &+ \Bigg[w^2\frac{\eta^2}{\varepsilon^2} - 2w\frac{\eta}{\varepsilon} + C_0\left(o\left(\frac{\varepsilon^r}{\eta^r}\right) - w^2\frac{\eta^2}{\varepsilon^2} + 2w\frac{\eta}{\varepsilon}\right) \\ &+ C_1\left(w\frac{\eta}{\varepsilon} + o\left(\frac{\varepsilon^r}{\eta^r}\right)\right) + C_2\left(o\left(\frac{\varepsilon^r}{\eta^r}\right) - 1\right)\Bigg]\varepsilon^2,\end{aligned}$$

for any $r > 0$, for fixed, finite $w > 0$, as $\varepsilon \searrow 0$.

Example (Cont.)

We can conclude from this expansion that

$$\sum_{k=0}^{2} Y_k(z)\varepsilon^k = 1 - C_0 + (C_0 - 1)w\eta - C_1\varepsilon + w^2\eta^2 - 2w\eta\varepsilon - C_0 w^2\eta^2$$

$$+ 2C_0 w\eta\varepsilon + C_1 w\eta\varepsilon - C_2\varepsilon^2 + o(\varepsilon^2)$$

$$= 1 - C_0 + (C_0 - 1)w\eta - C_1\varepsilon + (1 - C_0)w^2\eta^2$$

$$+ (2C_0 - 2 + C_1)w\eta\varepsilon - C_2\varepsilon^2 + o(\varepsilon^2),$$

for fixed, finite $w > 0$, as $\varepsilon \searrow 0$, which implies that

$$y_{\mathrm{match}}^{2,2,\mathrm{in}}(w;\varepsilon) = 1 - C_0 + (C_0 - 1)w\eta - C_1\varepsilon + (1 - C_0)w^2\eta^2$$

$$+ (2C_0 - 2 + C_1)w\eta\varepsilon - C_2\varepsilon^2.$$

Example (Cont.)

Outer Matching Function, $y_{\text{match}}^{2,2,\text{out}}$: Next, recall from (3) that

$$\sum_{k=0}^{2} y_k(x)\varepsilon^k = \frac{2}{1+x} + \left[\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right]\varepsilon + \left[\frac{6}{(1+x)^5} - \frac{1}{2(1+x)^3} - \frac{1}{4(1+x)}\right]\varepsilon^2.$$

For the outer expansion, we will make use of the following binomial expansion, in addition to those introduced in the previous example:

$$\frac{1}{(1+w\eta)^5} = 1 - 5w\eta + \frac{(-5)(-6)}{2!}w^2\eta^2 + O(\eta^3)$$
$$= 1 - 5w\eta + 15w^2\eta^2 + O(\eta^3).$$

Example (Cont.)

Writing the three-term outer expansion in the intermediate variable, and using the binomial expansions, we have

$$
\begin{aligned}
\sum_{k=0}^{2} y_k(x)\varepsilon^k &= \left(2 - \frac{\varepsilon}{2} - \frac{\varepsilon^2}{4}\right)\left[1 - w\eta + w^2\eta^2 + O(\eta^3)\right] \\
&\quad + \left(2\varepsilon - \frac{\varepsilon^2}{2}\right)\left[1 - 3w\eta + 6w^2\eta^2 + O(\eta^3)\right] \\
&\quad + 6\varepsilon^2\left[1 - 5w\eta + 15w^2\eta^2 + O(\eta^3)\right] \\
&= 2 - 2w\eta + \left(-\frac{1}{2} + 2\right)\varepsilon + 2w^2\eta^2 \\
&\quad + \left(\frac{1}{2} - 6\right) w\eta\varepsilon + \left(-\frac{1}{2} + 12\right) w^2\eta^2\varepsilon \\
&\quad + \left(-\frac{1}{4} - \frac{1}{2} + 6\right)\varepsilon^2 + o(\varepsilon^2),
\end{aligned}
$$

for fixed, finite $w > 0$, as $\varepsilon \searrow 0$.

Example (Cont.)

Continuing,

$$\sum_{k=0}^{2} y_k(x)\varepsilon^k = 2 - 2w\eta + \frac{3}{2}\varepsilon + 2w^2\eta^2 - \frac{11}{2}w\eta\varepsilon + \frac{21}{4}\varepsilon^2 + o(\varepsilon^2),$$

for fixed, finite $w > 0$, as $\varepsilon \searrow 0$, provided

$$\lim_{\varepsilon \searrow 0} \frac{\eta^3}{\varepsilon^2} = 0 \quad \text{and} \quad \lim_{\varepsilon \searrow 0} \frac{\eta^2 \varepsilon}{\varepsilon^2} = \lim_{\varepsilon \searrow 0} \frac{\eta^2}{\varepsilon} = 0. \tag{10}$$

The latter requirement is the same as that in (8), which could be satisfied by taking

$$\eta(\varepsilon) = \varepsilon^{2/3},$$

for example. The first requirement is a bit stronger. It can be satisfied by taking, for example,

$$\eta(\varepsilon) = \varepsilon^{3/4}.$$

Example (Cont.)

As usual, we see that the width of the matching scale decreases as $N = M$ increases. In any case, we find that

$$y^{2,2,\mathrm{out}}_{\mathrm{match}}(w;\varepsilon) = 2 - 2w\eta + \frac{3}{2}\varepsilon + 2w^2\eta^2 - \frac{11}{2}w\eta\varepsilon + \frac{21}{4}\varepsilon^2.$$

Example (Cont.)

$y_{c,2,2}$ Composite Approximation: Matching $y^{2,2,\text{out}}_{\text{match}}$ and $y^{2,2,\text{in}}_{\text{match}}$ at the appropriate orders requires that

$$\begin{aligned}
O(1):&\quad 2 = 1 - C_0 \implies C_0 = -1,\\
O(\eta):&\quad -2w = w(C_0 - 1) \text{ (consistent)},\\
O(\varepsilon):&\quad \frac{3}{2} = -C_1 \implies C_1 = -\frac{3}{2}\\
O(\eta^2):&\quad 2w^2 = (1 - C_0)w^2 \text{ (consistent)},\\
O(\eta\varepsilon):&\quad -\frac{11}{2}w = (2C_0 - 2 + C_1)w \text{ (consistent)},\\
O(\varepsilon^2):&\quad \frac{21}{4} = -C_2 \implies C_2 = -\frac{21}{4}.
\end{aligned}$$

Therefore,

$$y^{2,2}_{\text{match}}(w;\varepsilon) = 2 - 2w\eta + \frac{3}{2}\varepsilon + 2w^2\eta^2 - \frac{11}{2}w\eta\varepsilon + \frac{21}{4}\varepsilon^2.$$

Example (Cont.)

The composite approximation we seek is defined as

$$y_{\mathrm{c},2,2}(x;\varepsilon) = \sum_{k=0}^{2} y_k(x)\varepsilon^k + \sum_{k=0}^{2} Y_k\left(\frac{x}{\varepsilon}\right)\varepsilon^k - y_{\mathrm{match}}^{2,2}\left(\frac{x}{\eta};\varepsilon\right).$$

Example (Cont.)

Thus,

$$
\begin{aligned}
y_{\mathrm{c},2,2}(x;\varepsilon) = {}& \frac{2}{1+x} + \left[\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right]\varepsilon \\
&+ \left[\frac{6}{(1+x)^5} - \frac{1}{2(1+x)^3} - \frac{1}{4(1+x)}\right]\varepsilon^2 \\
&+ 1 - (e^{-x/\varepsilon} - 1) \\
&+ \left[-\frac{x}{\varepsilon} - \left(\frac{x}{\varepsilon} - \frac{1}{2}\frac{x^2}{\varepsilon^2}e^{-x/\varepsilon}\right) - \frac{3}{2}\left(e^{-x/\varepsilon} - 1\right)\right]\varepsilon \\
&+ \left[\frac{x^2}{\varepsilon^2} - 2\frac{x}{\varepsilon} - \left(\frac{1}{8}\frac{x^4}{\varepsilon^4}e^{-x/\varepsilon} - \frac{x^2}{\varepsilon^2} + 2\frac{x}{\varepsilon}\right)\right. \\
&\left. - \frac{3}{2}\left(\frac{x}{\varepsilon} - \frac{1}{2}\frac{x^2}{\varepsilon^2}e^{-x/\varepsilon}\right) - \frac{21}{4}\left(e^{-x/\varepsilon} - 1\right)\right]\varepsilon^2 \\
&- \left[2 - 2x + \frac{3}{2}\varepsilon + 2x^2 - \frac{11}{2}x\varepsilon + \frac{21}{4}\varepsilon^2\right].
\end{aligned}
$$

Example (Cont.)

And, after some simplification, we arrive at

$$\begin{aligned} y_{\mathrm{c},2,2}(x;\varepsilon) = {} & \frac{2}{1+x} + \left[\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right]\varepsilon \\ & + \left[\frac{6}{(1+x)^5} - \frac{1}{2(1+x)^3} - \frac{1}{4(1+x)}\right]\varepsilon^2 \\ & - e^{-x/\varepsilon} + \left[\frac{1}{2}\frac{x^2}{\varepsilon^2} - \frac{3}{2}\right] e^{-x/\varepsilon}\varepsilon \\ & + \left[-\frac{1}{8}\frac{x^4}{\varepsilon^4} + \frac{3}{4}\frac{x^2}{\varepsilon^2} - \frac{21}{4}\right] e^{-x/\varepsilon}\varepsilon^2. \end{aligned} \tag{11}$$

Now that we have two approximate solutions, $y_{\mathrm{c},1,1}(x;\varepsilon)$ and $y_{\mathrm{c},2,2}(x;\varepsilon)$, let us compare them against the true solution. See the figure on the next slide.

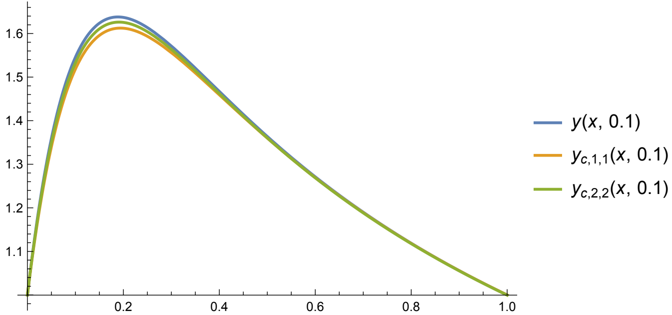

Figure: Plots of the numerically computed "true" solution $y(x;\varepsilon)$ (blue) to the problem (3) and (4) with the $(N, M) = (1, 1)$ composite approximation $y_{\mathrm{c},1,1}(x;\varepsilon)$ (yellow) from (9) and $(N, M) = (2, 2)$ composite approximation $y_{\mathrm{c},2,2}(x;\varepsilon)$ (green) from (11), for $\varepsilon = 0.1$.

A Corner Layer Problem

Example

Consider the singularly-perturbed BVP consisting of the ODE

$$\varepsilon y''(x) + xy'(x) - y(x) = 0, \quad \text{for} \quad -1 < x < 1, \tag{12}$$

where $\varepsilon > 0$ is a small parameter, with the boundary conditions

$$y(-1) = 1 \quad \text{and} \quad y(1) = 2. \tag{13}$$

This problem is discussed in detail in the book by Kevorkian and Cole (1996). Similar problems are presented in the books by Holmes (2013) and Miller (2006). In this case, $p(x) = x$, which has a simple zero at $x = 0$ in the domain of interest $[-1, 1]$. Based on an earlier remark, we expect that there will be a boundary layer at $x = 0$.

Example (Cont.)

Luckily, we can obtain an exact solution of this problem by noting that $y = C_0 x$ is a generic solution of Equation (12), where C_0 is a constant. To find the other linearly independent solution we use the method of reduction of order, which yields the general solution

$$y(x;\varepsilon) = C_0 x + C_1 \left(e^{-x^2/(2\varepsilon)} + \frac{x}{\varepsilon} \int_{-1}^{x} e^{-s^2/(2\varepsilon)} \, \mathrm{d}s \right). \tag{14}$$

Enforcing the boundary conditions, we obtain values for our two constants:

$$C_0 = -1 + \frac{3e^{-1/(2\varepsilon)}}{2e^{-1/(2\varepsilon)} + \frac{1}{\varepsilon} I(\varepsilon)} \quad \text{and} \quad C_1 = \frac{3}{2e^{-1/(2\varepsilon)} + \frac{1}{\varepsilon} I(\varepsilon)},$$

where

$$I(\varepsilon) = \int_{-1}^{1} e^{-2s^2/(2\varepsilon)} \, \mathrm{d}s = 2\int_{0}^{\infty} e^{-s^2/(2\varepsilon)} \, \mathrm{d}s - 2\int_{1}^{\infty} e^{-s^2/(2\varepsilon)} \, \mathrm{d}s.$$

In other words, $I(\varepsilon)$ is a Gaussian integral, with a small correction.

Example (Cont.)

Recall that, from our work in Chapter 3,

$$I(\varepsilon) = \sqrt{2\pi\varepsilon} + \mathrm{TST}, \quad \text{as} \quad \varepsilon \searrow 0.$$

This implies that

$$C_0 = -1 + \mathrm{TST}, \quad \text{as} \quad \varepsilon \searrow 0,$$

and

$$C_1 = 3\sqrt{\frac{\varepsilon}{2\pi}} + \mathrm{TST}, \quad \text{as} \quad \varepsilon \searrow 0.$$

Example (Cont.)

From this information, the reader can prove that,

$$y(0;\varepsilon) = 3\sqrt{\frac{\varepsilon}{2\pi}} + \mathrm{TST}, \quad \text{as} \quad \varepsilon \searrow 0;$$

for $x < 0$,

$$y(x;\varepsilon) = -x + \mathrm{TST}, \quad \text{as} \quad \varepsilon \searrow 0;$$

and, for $x > 0$,

$$y(x;\varepsilon) = 2x + \mathrm{TST}, \quad \text{as} \quad \varepsilon \searrow 0.$$

The exact solution confirms that there is a boundary layer at $x = 0$. In fact, this particular type of boundary layer is what is known as a *corner layer*. The choice of this name will be obvious from the shape of the solution. See the figure on the next page for approximate solutions.

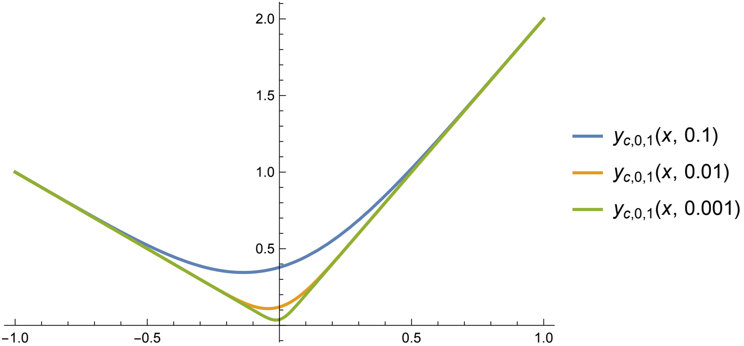

Figure: Plots of the $N = 0$, $M = 1$ composite approximation $y_{c,1,1}(x; \varepsilon)$, defined in (16), for three values of ε, $\varepsilon = 0.1$ (blue), $\varepsilon = 0.01$ (yellow), and $\varepsilon = 0.001$ (green).

Example

Let us again consider the singularly-perturbed corner layer problem consisting of the ODE (12) with the boundary conditions (13). Now, we perform our usual matched asymptotic analysis, but, because it is now second nature, we will abbreviate some of the descriptions.

Since we expect a boundary/transition layer at $x = 0$, we will have two outer regions, effectively. We assume that

$$y(x;\varepsilon) = y_{\mathrm{out}}(x;\varepsilon) \sim \sum_{k=0}^{\infty} y_k(x)\varepsilon^k, \quad \text{as} \quad \varepsilon \searrow 0,$$

for $x \in \overline{\Omega_{\mathrm{out}}}$, where $\Omega_{\mathrm{out}} := (-1, x_{1,-}) \cup (x_{1,+}, 1)$, where $x_{1,-} \in (-1, 0)$ and $x_{1,+} \in (0, 1)$. We will only be interested in the leading-order term in this example, namely, $y_0(x)$, which satisfies

$$x y_0'(x) - y_0(x) = 0, \quad x \in \Omega_{\mathrm{out}}.$$

Example (Cont.)

The general solution is, of course, $y_0 = C_{0,0}x$. Using the boundary conditions at the two end points, we find that

$$y_0(x) = \begin{cases} -x, & x \in [-1, x_{1,-}]. \\ 2x, & x \in [x_{1,+}, 1]. \end{cases}$$

It is easy to see, and the reader should show, that $y_k \equiv 0$ on $\overline{\Omega_{\mathrm{out}}}$, for $k \geq 1$. In other words, the y_0 term is the outer solution to all orders.

Notice for the current problem that, if we extent the leading-order outer solution to all of $[-1, 1]$, y_0 would be continuous. There is no point-wise jump in the solution. However, the extended outer solution is not differentiable at $x = 0$. For this reason, a corner layer forms instead of a boundary layer.

Example (Cont.)

Next, let us introduce the stretched variable and the inner solution:

$$z = \frac{x}{\delta(\varepsilon)} \quad \text{and} \quad Y(z;\varepsilon) := y(z\delta;\varepsilon).$$

The ODE transforms as

$$\frac{\varepsilon}{\delta^2} Y'' + zY' - Y = 0.$$

Using the principle of dominant balance, we discover that $\delta(\varepsilon) = \varepsilon^{1/2}$, in which case all three terms balance, and the inner problem is

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} Y(z;\varepsilon) + z\frac{\mathrm{d}}{\mathrm{d}z} Y(z;\varepsilon) - Y(z;\varepsilon) = 0.$$

Example (Cont.)

What expansion should we use for the inner solution? It is not clear. In this case, let us guess that

$$Y(z;\varepsilon) = Y_{\mathrm{in}}(z;\varepsilon) \sim \sum_{k=0}^{\infty} Y_k(z)\varepsilon^{k/2}.$$

This choice is based on the thickness of the corner layer, but no other information. Unusually, there is no explicit ε dependence in the two lower-order terms of the inner ODE. We have the following sequence of equations:

$$O(\varepsilon^{k/2}): \quad Y_k''(z) + zY_k'(z) - Y_k(z) = 0, \quad k \in \mathbb{N}_0.$$

This is also a bit strange. All of the equations are the same, and the solution of the Y_k equation does not depend upon any of the lower order terms, $Y_{k-1}, Y_{k-2}, \ldots, Y_0$.

Example (Cont.)

The solution of the leading-order equation is

$$Y_0(z) = C_{0,0}z + C_{0,1}\left(e^{-z^2/2} + z\int_{-\infty}^{z} e^{-s^2/2}\,\mathrm{d}s\right), \tag{15}$$

where $C_{0,0}$ and $C_{0,1}$ are undetermined coefficients. In fact, all terms of all orders have exactly this form.

Example (Cont.)

Let us perform matching for the leading-order case, that is, $N = 0 = M$, if possible. We need to an intermediate variable,

$$w = \frac{x}{\eta(\varepsilon)},$$

where

$$0 < \delta(\varepsilon) = \varepsilon^{1/2} \ll \eta(\varepsilon) \ll 1, \quad \text{as} \quad \varepsilon \searrow 0.$$

In the present case, w can be positive or negative, and $\phi_k(\varepsilon) = \varepsilon^{k/2}$ for the inner expansion sequence. To see if a $N = 0 = M$ matching is possible, using the standard procedure, we look for the following:

$$y_0(x) = y_0(w \cdot \eta(\varepsilon)) = y_{\mathrm{match}}^{0,0,\mathrm{out}}(w; \varepsilon) + o\left(1\right),$$

as $\varepsilon \searrow 0$, with $|w| > 0$ fixed and finite, and

$$Y_0(z) = Y_0\left(w\frac{\eta(\varepsilon)}{\varepsilon^{1/2}}\right) = y_{\mathrm{match}}^{0,0,\mathrm{in}}(w; \varepsilon) + o\left(1\right),$$

as $\varepsilon \searrow 0$, with $|w| > 0$ fixed and finite. Again, w can be either positive or negative.

Example (Cont.)

We have

$$y_0(w\eta) = \begin{cases} -w\eta, & w < 0, \\ 2w\eta, & w > 0, \end{cases}$$
$$= y_{\mathrm{match}}^{0,0,\mathrm{out}}(w;\varepsilon).$$

In fact, since $y_k \equiv 0$, for all $k \geq 1$, we have

$$y_{\mathrm{match}}^{N,M,\mathrm{out}}(w;\varepsilon) = y_{\mathrm{match}}^{0,0,\mathrm{out}}(w;\varepsilon),$$

for any N and M.

Example (Cont.)

For the inner solution, we split the computation into two cases, explicitly. First, suppose that $w < 0$. Then,

$$\begin{aligned} Y_0\left(w\frac{\eta}{\varepsilon^{1/2}}\right) &= C_{0,0}w\frac{\eta}{\varepsilon^{1/2}} + C_{0,1}\left(e^{-w^2\eta^2/(2\varepsilon)} + w\frac{\eta}{\varepsilon^{1/2}}\int_{-\infty}^{w\eta/\varepsilon^{1/2}} e^{-s^2/2}\,\mathrm{d}s\right) \\ &= C_{0,0}w\frac{\eta}{\varepsilon^{1/2}} + o\left(\frac{\varepsilon^{r/2}}{\eta^r}\right), \end{aligned}$$

as $\varepsilon \searrow 0$, for any $r > 0$, for fixed finite $w < 0$. Thus, for $w < 0$,

$$y_{\mathrm{match}}^{0,0,\mathrm{in}}(w;\varepsilon) = C_{0,0}w\frac{\eta}{\varepsilon^{1/2}}.$$

Clearly, matching is impossible at this order, $N = 0$, $M = 0$!

Example (Cont.)

No worries. Let us proceed to the next possibility, $N = 0$, $M = 1$. We already have our matching function for the outer solution:

$$y_{\mathrm{match}}^{0,1,\mathrm{out}}(w;\varepsilon) = \begin{cases} -w\eta, & w < 0, \\ 2w\eta, & w > 0. \end{cases}$$

The reader can easily show that the inner matching function is

$$y_{\mathrm{match}}^{0,1,\mathrm{in}}(w;\varepsilon) = C_{0,0}w\frac{\eta}{\varepsilon^{1/2}} + C_{1,0}w\eta,$$

for $w < 0$.

Example (Cont.)

Let us work on the inner matching function for $w > 0$. To this end, we need to expand the following terms for $k = 0, 1$ and for $w > 0$:

$$Y_k\left(w\frac{\eta}{\varepsilon^{1/2}}\right) = C_{k,0}w\frac{\eta}{\varepsilon^{1/2}} + C_{k,1}\left(e^{-w^2\eta^2/(2\varepsilon)} + w\frac{\eta}{\varepsilon^{1/2}}\int_{-\infty}^{w\eta/\varepsilon^{1/2}} e^{-s^2/2}\,\mathrm{d}s\right).$$

First,

$$\begin{aligned}\int_{-\infty}^{w\eta/\varepsilon^{1/2}} e^{-s^2/2}\,\mathrm{d}s &= \int_{-\infty}^{\infty} e^{-s^2/2}\,\mathrm{d}s - \int_{w\eta/\varepsilon^{1/2}}^{\infty} e^{-s^2/2}\,\mathrm{d}s.\\ &= \sqrt{2\pi} + o\left(\frac{\varepsilon^{r/2}}{\eta^r}\right),\end{aligned}$$

as $\varepsilon \searrow 0$, for any $r > 0$.

Example (Cont.)

Therefore, we can show that, for $w > 0$,

$$\begin{aligned} Y_0\left(w\frac{\eta}{\varepsilon^{1/2}}\right) + Y_1\left(w\frac{\eta}{\varepsilon^{1/2}}\right)\varepsilon^{1/2} = {} & C_{0,0}w\frac{\eta}{\varepsilon^{1/2}} + C_{1,0}w\eta \\ & + C_{0,1}w\frac{\eta}{\varepsilon^{1/2}}\sqrt{2\pi} + C_{1,1}w\eta\sqrt{2\pi} \\ & + o\left(\frac{\varepsilon^{r/2}}{\eta^r}\right), \end{aligned}$$

as $\varepsilon \searrow 0$, for any $r > 0$. Thus, all together,

$$y_{\text{match}}^{0,1,\text{in}}(w;\varepsilon) = \begin{cases} \left(\frac{C_{0,0}}{\varepsilon^{1/2}} + C_{1,0}\right) w\eta, & w < 0, \\ \left(\frac{C_{0,0}}{\varepsilon^{1/2}} + C_{1,0} + \sqrt{2\pi}\left[\frac{C_{0,1}}{\varepsilon^{1/2}} + C_{1,1}\right]\right) w\eta, & w > 0. \end{cases}$$

Example (Cont.)

We now see that a match at this order, $N = 0$, $M = 1$, is possible:

$$
\begin{aligned}
O\left(\frac{\eta}{\varepsilon^{1/2}}\right),\ w < 0: &\quad C_{0,0} = 0;\\
O\left(\frac{\eta}{\varepsilon^{1/2}}\right),\ w > 0: &\quad C_{0,0} + \sqrt{2\pi}\, C_{0,1} = 0 \implies C_{0,1} = 0;\\
O(\eta),\ w < 0: &\quad C_{1,0} = -1;\\
O(\eta),\ w > 0: &\quad C_{1,0} + \sqrt{2\pi}\, C_{1,1} = 2 \implies C_{1,1} = \frac{3}{\sqrt{2\pi}}.
\end{aligned}
$$

It follows that the matching function is

$$
y_{\mathrm{match}}^{0,1}(w;\varepsilon) = \begin{cases} -w\eta, & w < 0,\\ 2w\eta, & w > 0. \end{cases}
$$

Example (Cont.)

The uniformly valid composite solution is, as the reader should confirm,

$$\begin{aligned} y_{\mathrm{c},0,1}(x;\varepsilon) &= -x + 3\sqrt{\frac{\varepsilon}{2\pi}}\left(e^{-x^2/(2\varepsilon)} + \frac{x}{\varepsilon^{1/2}}\int_{-\infty}^{x/\varepsilon^{1/2}} e^{-s^2/2}\,\mathrm{d}s\right) \\ &= -x + 3\sqrt{\frac{\varepsilon}{2\pi}}\left(e^{-x^2/(2\varepsilon)} + \frac{x}{\varepsilon}\int_{-\infty}^{x} e^{-t^2/(2\varepsilon)}\,\mathrm{d}t\right). \qquad (16) \end{aligned}$$

We plot the composite solution in the figure on the next slide for three value of ε.

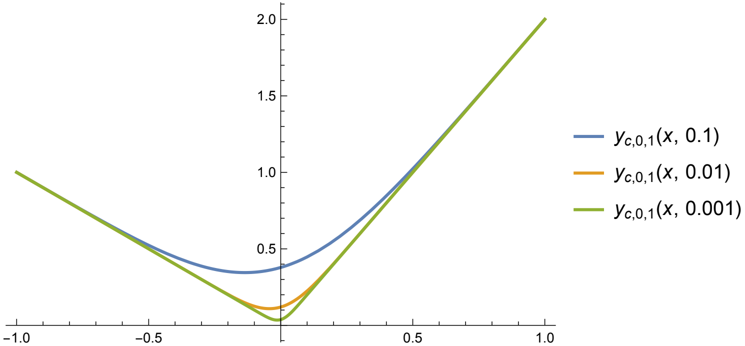

Figure: Plots of the $N = 0$, $M = 1$ composite approximation $y_{c,1,1}(x; \varepsilon)$, defined in (16), for three values of ε, $\varepsilon = 0.1$ (blue), $\varepsilon = 0.01$ (yellow), and $\varepsilon = 0.001$ (green).