Homework 01, Math 515
15 February, 2024

Name: ______________________________

1. Suppose that $f(\varepsilon) = 1 - \cos^2(\varepsilon)$, for $\varepsilon > 0$.
   (a) Show that
   $$f(\varepsilon) = O(\varepsilon^2), \quad \text{as} \quad \varepsilon \searrow 0.$$
   (b) Show that
   $$f(\varepsilon) = o(\varepsilon), \quad \text{as} \quad \varepsilon \searrow 0.$$
2. Show that
   $$\log(1+\varepsilon) = O(\varepsilon), \quad \text{as} \quad \varepsilon \searrow 0.$$
3. Consider the problem
   $$y'(t) + y(t) = \varepsilon y^2(t), \quad y(0) = 1,$$
   for $t \geq 0$, and $\varepsilon > 0$.
   (a) Determine a three-term expansion for small $\varepsilon > 0$ using the perturbation approach. Assume the standard asymptotic power series approximation is valid, as $\varepsilon \searrow 0$.
   (b) Show that the exact solution is
   $$y(t) = e^{-t}\left[1 + \varepsilon(e^{-t} - 1)\right]^{-1}.$$
   (c) Expand the exact solution for small ε and compare with the result from part (a).
   (d) Are the expansions you obtained valid for all $t \geq 0$?
4. Consider the solution of the equation
   $$y'(x) + y(x) = \frac{1}{x},$$
   for $x > 0$. In seeking an approximation for large x, assume that
   $$y(x) \sim \sum_{k=1}^{\infty} \frac{a_k}{x^k}, \quad \text{as} \quad x \nearrow \infty.$$
   (a) Substitute this expansion into the equation to determine the coefficients.

(b) Does the series you have produced converge?

5. Consider the van der Pol oscillator problem:

$$u''(t) + u(t) = \varepsilon(1 - u^2(t))u'(t),$$

for $t \geq 0$ and $\varepsilon > 0$. Suppose that

$$u(t, \varepsilon) \sim \sum_{k=0}^{\infty} u_k(t)\varepsilon^k, \quad \text{as} \quad \varepsilon \searrow 0.$$

(a) Substitute this expansion in the differential equation and find u_0 and u_1 for general initial data.

(b) Plot your two-term composite approximate solution for a couple of interesting initial data choices.

6. Determine the second-order (three term) asymptotic expansion for the solution of

$$u''(t) + u(t) = \varepsilon u^2(t),$$

for $t \geq 0$, with $u(0) = a$ and $u'(0) = 0$, as $\varepsilon \searrow 0$. Is this expansion uniformly valid for $t \geq 0$? If not, can you restrict the domain of the solution/approximation so that it is?

7. Find an asymptotic series expansion for the integral

$$I(\lambda) = \lambda \int_0^{\infty} \frac{e^{-t}}{\lambda + t}\, \mathrm{d}t,$$

as $\lambda \nearrow \infty$.

8. Suppose that

$$f(\lambda) \sim \sum_{k=0}^{\infty} \frac{a_k}{\lambda^k}, \quad \text{as} \quad \lambda \nearrow \infty.$$

(a) Find the first few terms in the asymptotic expansion, as $\lambda \to \infty$ of $\frac{1}{f(\lambda)}$.

(b) Find the first few terms in the asymptotic expansion, as $\lambda \to \infty$ of $p(f(\lambda))$, where p is a polynomial.