Math 574
class # 07
09/09/2024

Model Problem Revisited

Let us begin with the model problem in $\mathbb{R}^d$. Given $f \in C^0(\Omega)$. Find $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that

(7.1)
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ is an open, bounded Lipschitz domain.

Let $v \in C_0^\infty(\Omega)$ be an arbitrary test function. Then

(7.2)
$$\sum_{|\alpha|=1} \int_\Omega \partial^\alpha u \, \partial^\alpha v \, dx = \int_\Omega \nabla u \cdot \nabla v \, d\vec{x} = \int_\Omega f v \, d\vec{x},$$

using integration by parts.

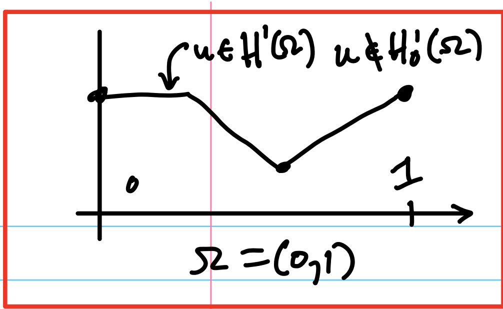
We reframe the problem in weak form. Given $f \in L^2(\Omega)$. Find $u \in H_0^1(\Omega)$ such that

(7.3)
$$a(u,v) = f(v), \quad \forall \; v \in H_0^1(\Omega),$$

where

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, d\vec{x}$$

$$= (u,v)_{H_0^1(\Omega)}$$



$$= \sum_{|\alpha|=1} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)}$$

and

$$f(v) := \int_\Omega f \, v \, d\vec{x}$$

$$= (f, v)_{L^2(\Omega)}$$

<u>Proposition</u>: Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain. The pairing $(H_0^1(\Omega), (\cdot,\cdot)_{H_0^1})$ is a Hilbert space, $H_0^1(\Omega)$ being a proper subspace of $H^1(\Omega)$. Here,

$$(u,v)_{H_0^1(\Omega)} := \sum_{|\alpha|=1} (\partial^\alpha u, \partial^\alpha v)_{L^2}$$

and

$$\|u\|_{H_0^1(\Omega)} := \sqrt{(u,u)_{H_0^1(\Omega)}}$$

$$:= |u|_{H^1(\Omega)}$$

$$= \sqrt{\sum_{|\alpha|=1} \|\partial^\alpha u\|_{L^2}^2}.$$

Furthermore, there is a constant $C > 1$, such that

(7.4) $$\|u\|_{H_0^1(\Omega)} \le \|u\|_{H^1(\Omega)} \le C\|u\|_{H_0^1(\Omega)}$$

for all $u \in H_0^1(\Omega)$.

Proof: I leave it as an exercise to show that the set $H_0^1(\Omega)$ is a topologically closed subspace

Every topologically closed subspace of a Banach is itself a Banach Space!

of $H^1(\Omega)$. This will imply that $H_0^1(\Omega)$ is complete and, therefore, a Hilbert space.

- $\left(H_0^1(\Omega), \langle \cdot, \cdot \rangle_{H^1(\Omega)}\right)$ is Hilbert

Next, the Poincare inequality gaurantees that, there is a constant $C_p > 0$, such that

$$\|u\|_{L^2(\Omega)} \le C_p |u|_{H^1(\Omega)}$$

$$= C_p \|u\|_{H_0^1(\Omega)},$$

for all $u \in H_0^1(\Omega)$. Observe that, for any $u \in H_0^1(\Omega)$

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2}^2 + |u|_{H^1(\Omega)}^2$$

$$= \|u\|_{L^2}^2 + \sum_{|\alpha|=1} \|\partial^\alpha u\|_{L^2}^2$$

$$\overset{\text{Poincare}}{\le} C_p^2 |u|_{H^1(\Omega)}^2 + |u|_{H^1(\Omega)}^2$$

$$= (1 + C_p^2) |u|_{H^1(\Omega)}^2.$$

Thus,

$$\|u\|_{H^1(\Omega)} \le \sqrt{1 + C_p^2}\, |u|_{H^1(\Omega)}.$$

Naturally, for any $u \in H^1(\Omega)$,

$$|u|_{H^1(\Omega)} \le \|u\|_{H^1(\Omega)}.$$

Therefore (7.4) is established. ///

Corollary: $(H_0^1(\Omega), (\cdot,\cdot)_{H^1(\Omega)})$ and $(H_0^1(\Omega), (\cdot,\cdot)_{H_0^1(\Omega)})$ are topologically equivalent Hilbert spaces.

Proposition: $a(\cdot,\cdot): H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is a symmetric, coercive, continuous bilinear form with respect to either $(H_0^1(\Omega), (\cdot,\cdot)_{H^1(\Omega)})$ or $(H_0^1(\Omega), (\cdot,\cdot)_{H_0^1(\Omega)})$.

$f: H_0^1(\Omega) \to \mathbb{R}$ is a bounded linear functional with respect to either $(H_0^1(\Omega), (\cdot,\cdot)_{H^1(\Omega)})$ or $(H_0^1(\Omega), (\cdot,\cdot)_{H_0^1(\Omega)})$.

Proof: Symmetry is clear

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, d\vec{x} = a(v,u).$$

coercivity: let $u \in H_0^1(\Omega)$ be arbitrary.

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + |u|_{H^1}^2$$

$$\overset{\text{Poincare}}{\leq} C_P^2 |u|_{H^1}^2 + |u|_{H^1}^2$$

$$\left(\alpha = \frac{1}{1+C_P^2}\right) \qquad = (1+C_P^2)\, a(u,u).$$

Thus,

$$\frac{1}{1+C_P^2} \|u\|_{H^1}^2 \leq a(u,u), \quad \forall\, u \in H_0^1(\Omega).$$

Continuity: let $u, v \in H_0^1(\Omega)$ be arbitrary.

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, d\vec{x}$$

$$= \sum_{|\alpha|=1} (\partial^\alpha u, \partial^\alpha v)_{L^2}$$

$$\overset{\text{C.S.}}{\leq} \sum_{|\alpha|=1} \|\partial^\alpha u\|_{L^2} \|\partial^\alpha v\|_{L^2}$$

$$\overset{\text{C.S.}}{\leq} \sqrt{\sum_{|\alpha|=1} \|\partial^\alpha u\|_{L^2}^2} \sqrt{\sum_{|\alpha|=1} \|\partial^\alpha v\|_{L^2}^2}$$

$$= |u|_{H^1(\Omega)} |v|_{H^1(\Omega)}$$

$$\overset{(\gamma=1)}{\leq} \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.$$

f is bounded: Recall, we assumed that $f \in L^2(\Omega)$.

$$|f(v)| = \left| \int_\Omega f v \, d\vec{x} \right|$$

$$\overset{\text{C.S.}}{\leq} \|f\|_{L^2} \|v\|_{L^2}$$

$$\leq \|f\|_{L^2} \|v\|_{H^1(\Omega)}$$

$$= M \|v\|_{H^1(\Omega)},$$

for all $v \in H_0^1(\Omega)$, where $M := \|f\|_{L^2}$. ///

Corollary: By the Lax-Milgram lemma or the RRT, problem (7.3) has a unique solution $u \in H_0^1(\Omega)$.

Remark: Can use RRT because $a(\cdot,\cdot)$ is an inner product on $H_0^1(\Omega)$ that is equivalent to the original inner product.

1D FEM

<u>Defn</u>: Let $\Omega = (0,1)$. A <u>partition</u> of $(0,1)$ is a sets

$$P = \{x_0, \cdots, x_{M+1}\}$$

with the property that

$$0 = x_0 < x_1 < x_2 < \cdots < x_M < x_{M+1} = 1$$

For the partition, define the <u>grid spacing</u>

$$h_i := x_i - x_{i-1}, \quad i = 1, \cdots, M+1.$$

and

$$h := \max_{1 \le i \le M+1} h_i$$

Define

$$K_i = (x_{i-1}, x_i) \qquad i = 1, \cdots, M+1$$

and

$$\mathcal{T}_h = \{K_i\}_{i=1}^{M+1}$$

$\mathcal{T}_h$ is called the <u>triangulation</u> of Ω

<u>Remark</u>: Observe that

$$\Omega = \left(\bigcup_{i=1}^{M+1} \overline{K_i} \right)^{\circ} \qquad \overset{i \neq j}{K_i \cap K_j = \emptyset}$$

$$= \left(\bigcup_{K \in \mathcal{T}_h} \overline{K} \right)^{\circ}$$

Defn: For each $r \in \mathbb{N}$, set

(7.5)
$$\mathcal{M}_r := \{ v \in C^0(\overline{\Omega}) \mid v|_{K_i} \in \mathbb{P}_r(K_i),\ i = 1, \cdots, M+1 \}$$
$$= \{ v \in C^0(\overline{\Omega}) \mid v|_K \in \mathbb{P}_r(K),\ \forall K \in \mathcal{T}_h \},$$

where $\mathbb{P}_r$ is the space of polynomials of degree at most r.

Define the subspace

(7.6)
$$\mathcal{M}_{0,r} := \{ v \in \mathcal{M}_r \mid v(0) = v(1) = 0 \}$$
$$= \{ v \in \mathcal{M}_r \mid v|_{\partial\Omega} \equiv 0 \}$$

These are the so-called piece-wise polynomial finite element spaces.

Defn: Let P be a partition of $\Omega = (0,1)$. Define $\phi_i \in \mathcal{M}_1$, $i = 0, \cdots, M+1$, via

(7.7)
$$\phi_i(x_j) = \delta_{i,j}, \qquad i, j \in \{0, 1, \cdots, M+1\}.$$

These functions $\phi_0, \cdots, \phi_{M+1} \in \mathcal{M}_1$ are called hat functions.

Proposition: The functions $\phi_i \in \mathcal{M}_1$, $i = 0, \cdots, M+1$ are uniquely defined by (7.7). Further, the sets

$$B_1 = B_{h,1} := \{ \phi_0, \cdots, \phi_{M+1} \}$$

and

$$B_{0,1} = B_{h,0,1} := \{ \phi_1, \cdots, \phi_M \}$$

are bases for $\mathcal{M}_1$ and $\mathcal{M}_{0,1}$, respectively, so that

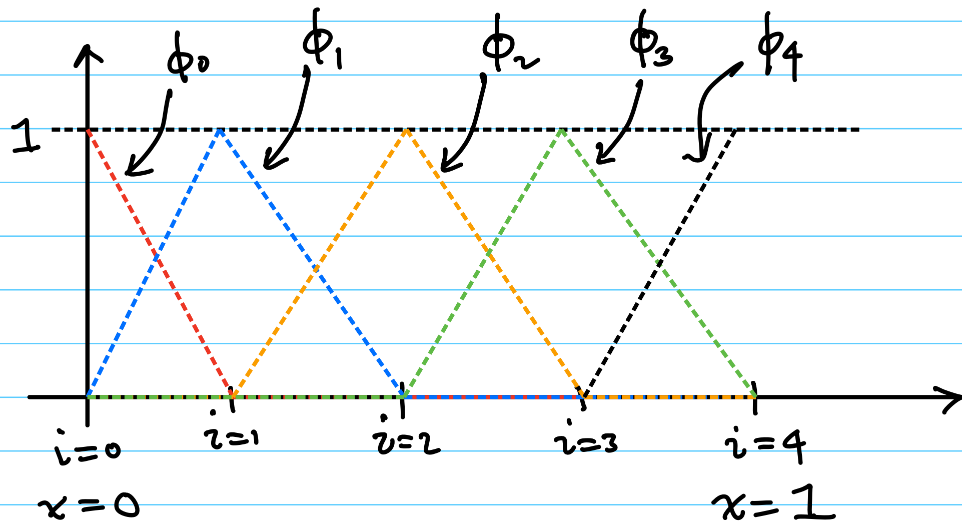
$$\dim(\mathcal{M}_1) = M+2$$

and

$$\dim(\mathcal{M}_{0,1}) = M.$$

<u>Proof</u>: Exercise.

<u>Example</u>: Suppose $M=3$.



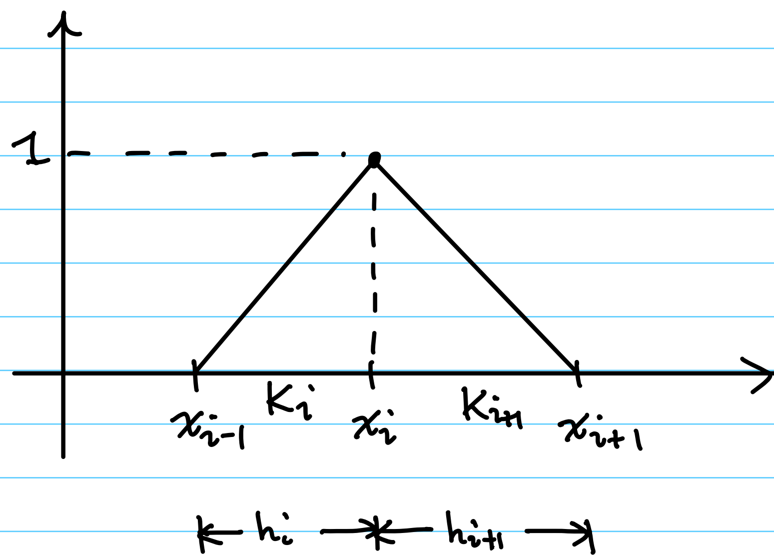
<u>Proposition</u>: $\mathcal{M}_r$ $(\mathcal{M}_{0,r})$ is a finite-dimensional linear subspace of $H^1(\Omega)$ $(H^1_0(\Omega))$.

<u>Proof</u>: Exercise. ///

<u>The Stiffness Matrix</u>

Let us compute the stiffness matrix for the model problem on $\Omega = (0,1)$ (1D) with respect to the basis $B_{0,1}$. The M basis functions ϕ_i, $i = 1, \cdots, M$ are

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_i} & x \in \overline{K_i} = [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{h_{i+1}} & x \in \overline{K_{i+1}} = [x_i, x_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$



$$\frac{d_w \phi_i}{dx}(x) = \begin{cases} h_i^{-1}, & x \in K_i \\ -h_{i+1}^{-1}, & x \in K_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

Then

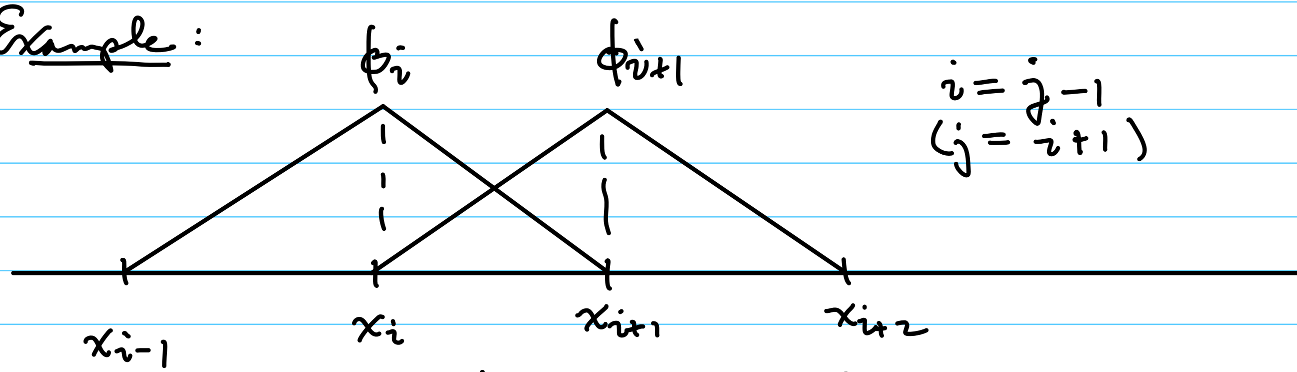
$$[A]_{i,j} = a(\phi_j, \phi_i)$$

$$= \int_0^1 \begin{Bmatrix} h_j^{-1} \text{ in } K_j \\ -h_{j+1}^{-1} \text{ in } K_{j+1} \\ 0 \text{ other} \end{Bmatrix} \begin{Bmatrix} h_i^{-1} \text{ in } K_i \\ -h_{i+1}^{-1} \text{ in } K_{i+1} \\ 0 \text{ other} \end{Bmatrix} dx$$

$$
= \begin{cases} \frac{1}{h_i^2} \cdot h_i + \frac{1}{h_{i+1}^2} h_{i+1} & i = j \\ -\frac{1}{h_{i+1}^2} \cdot h_{i+1} & i = j-1 \neq M \\ -\frac{1}{h_i^2} \cdot h_i & i = j+1 \neq 1 \\ 0 & \text{otherwise} \end{cases}
$$

$$
= \begin{cases} \frac{1}{h_i} + \frac{1}{h_{i+1}} & i = j \\ -\frac{1}{h_{i+1}} & i = j-1 \neq M \\ -\frac{1}{h_i} & i = j+1 \neq 1 \\ 0 & \text{otherwise} \end{cases}
$$

<u>Example</u>:



$i = j-1$
$(j = i+1)$

$$
\int_0^1 \phi_i \phi_j \, dx = \int_0^1 \phi_i \phi_{i+1} \, dx
$$

$$
= \int_{K_{i+1}} \phi_i \phi_{i+1} \, dx
$$

$$
= \left(-\frac{1}{h_{i+1}}\right)\left(\frac{1}{h_{i+1}}\right) h_{i+1}
$$

$$
= -\frac{1}{h_{i+1}} \quad //
$$