Math 574
class # 12
9/30/2025

Quadrature Rules

How do we compute integrals of the form

$$\int_{\hat{K}} f(\hat{x})\,d\hat{x} \;?$$

We wish to approximate

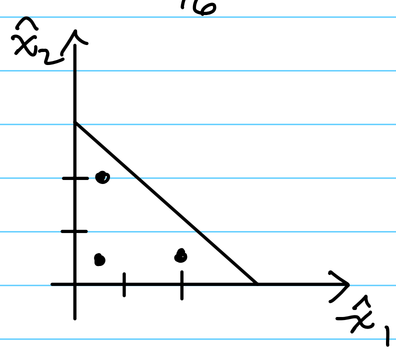
$$\int_{\hat{K}} f(\hat{x}_1,\hat{x}_2)\,d\hat{x}_1\,d\hat{x}_2 \doteq \sum_{n=1}^{\alpha} w_n f(q_1^{(n)}, q_2^{(n)}) \qquad (12.1)$$

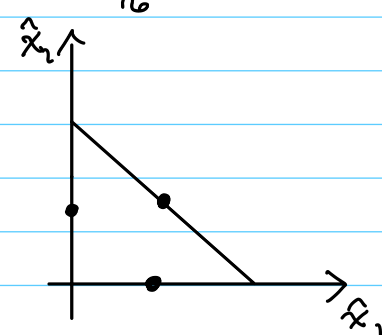
such that the approximation is exact for all

$$f \in \mathbb{P}_r(\hat{K})$$

	Rule 1	Rule 2
$(q_1^{(1)}, q_2^{(1)})$	$(1/6, 1/6)$	$(1/2, 0)$
$(q_1^{(2)}, q_2^{(2)})$	$(2/3, 1/6)$	$(1/2, 1/2)$
$(q_1^{(3)}, q_2^{(3)})$	$(1/6, 2/3)$	$(0, 1/2)$
w_1	$1/6$	$1/6$
w_2	$1/6$	$1/6$
w_3	$1/6$	$1/6$

$(\alpha = 3)$





Rules 1 and 2 are exact for polynomials of degree 2 or less.

Suppose that $\hat{\varphi}_i, \hat{\varphi}_j \in \mathbb{P}_2(\hat{K})$. Then,

$$B_K^{-1} \hat{\nabla} \hat{\varphi}_i \cdot B_K^{-T} \hat{\nabla} \hat{\varphi}_j \in \mathbb{P}_2(\hat{K})$$

The approximation of

$$\int_{\hat{K}} (B_K^{-T} \hat{\nabla} \hat{\varphi}_i) \cdot (B_K^{-T} \hat{\nabla} \hat{\varphi}_j) \, d\hat{x}$$

via the quadrature (12.1) is exact.

Suppose that $\hat{\varphi}_i, \hat{\varphi}_j \in \mathbb{P}_2(\hat{K})$. Then the product $\hat{\varphi}_i \hat{\varphi}_j \in \mathbb{P}_4(\hat{K})$ and we need a higher-order quadrature rule to compute

$$\int_{\hat{K}} \hat{\varphi}_i \hat{\varphi}_j \, d\hat{x}$$

exactly using quadrature rule is (12.1).

For example, rules for $r=3$ and 4 are below.

α	r	n	$\xi_1^{(n)}$	$\xi_2^{(n)}$	w_n
4	3	1	$1/3$	$1/3$	$-9/32$
		2	$3/5$	$1/5$	$25/96$
		3	$1/5$	$3/5$	$25/96$
		4	$1/5$	$1/5$	$25/96$

α	r	n	$\xi_1^{(n)}$	$\xi_2^{(n)}$	w_n
7	4	1	0	0	1/40
		2	1/2	0	1/15
		3	1	0	1/40
		4	1/2	1/2	1/15
		5	0	1	1/40
		6	0	1/2	1/15
		7	1/3	1/3	1/40

Observe that, we always require that

$$\sum_{n=1}^{\alpha} w_n = \frac{1}{2}.$$

Why is this so?

Now, if $f \in L^2(\Omega)$ is not a polynomial then we have a small dilemma. What will be the effect of the approximation

$$\left(\hat{f}(\hat{\vec{x}}), \hat{\varphi}_i(\hat{\vec{x}})\right)_{L^2(\hat{K})} = \int_{\hat{K}} \hat{f}(\hat{\vec{x}})\, \hat{\varphi}_i(\hat{\vec{x}})\, d\hat{\vec{x}}$$

$$\doteq \sum_{n=1}^{\alpha} \hat{f}(\vec{\xi}^{(n)})\, \hat{\varphi}_i(\vec{\xi}^{(n)})\, w_n \; ?$$

There will be an error generated that can be accounted for in the global error analysis.

See the book by Braess for a discussion.

We will <u>not</u> discuss this error herein.

The book by Gockenbach has quadrature rules for $\mathbb{P}_r$, $r = 1, \dots, 6$. See Chapter 8.

Mark Gockenbach, Understanding and Implementing the Finite Element Method, SIAM, 2006

Basic Error Analysis

Let us recall Cea's lemma. There is some $C > 0$, st

$$\|u - u_h\|_V \le C \inf_{v \in V_h} \|u - v\|_V .$$

For the model problem, $V = H_0^1(\Omega)$ and

$$V_h = M_{0,r} \overset{\text{v.ss.}}{\subset} H_0^1(\Omega)$$

We can use

$$\|\cdot\|_V = \|\cdot\|_{H^1}$$

or

$$\|\cdot\|_V = \|\cdot\|_{H_0^1} = |\cdot|_{H^1}$$

Recall that these are equivalent on $H_0^1(\Omega)$. Also recall

$$\|v\|_{H_0^1} = |v|_{H^1} = \sqrt{(\nabla v, \nabla v)_{L^2}} = \sqrt{\sum_{|\alpha|=1} (\partial^\alpha v, \partial^\alpha v)_{L^2}},$$

and

$$\|v\|_{H^1} = \sqrt{\|v\|_{L^2}^2 + \|v\|_{H_0^1}^2} = \sqrt{\sum_{|\alpha| \le 1} (\partial^\alpha v, \partial^\alpha v)_{L^2}} .$$

<u>Key Idea</u>: We will examine the difference between $u \in V$ and its piecewise polynomial interpolant

$$\Pi_h u \in V_h = \mathcal{M}_{0,r}$$

$$\|u - u_h\|_V \le \inf_{v \in V_h} \|u - v\|_V \le \|u - \Pi_h u\|_V .$$

Suppose that $u \in H^m(\Omega)$, $m \geq 2$. Then

$$H^m(\Omega) \hookrightarrow C^0(\bar{\Omega}) \qquad \Omega \subset \mathbb{R}^d \leftarrow \text{space dim}$$

for $d = 1, 2, 3$, via Sobolev embedding. In other words, u has well defined point values everywhere in $\bar{\Omega}$.

<u>Defn</u>: Let $\{\varphi_1^K, \dots \varphi_d^K\}$ be a local Lagrange nodal basis for $\mathbb{P}_r(K)$,

$$d = \dim(\mathbb{P}_r(K)) = \frac{(r+1)(r+2)}{2}$$

Define

$$\Pi_K : C(\overline{K}) \longrightarrow \mathbb{P}_r(K)$$

via

$$\Pi_K u(\vec{x}) := \sum_{i=1}^{d} u(\vec{a}_{K,i}) \varphi_i^K(\vec{x})$$

where $\{\vec{a}_{K,i}\}_{i=1}^{d} \subset \overline{K}$ are the local nodes satisfying

$$\varphi_i^K(\vec{a}_{K,j}) = \delta_{i,j} \qquad i,j = 1, \dots, d$$

Π_K is called the <u>local Lagrange nodal interpolation operator</u>.

<u>Def</u>: Let $\{\phi_1, \dots, \phi_{N_r}\}$ be a global Lagrange nodal basis for

$$V_h = \mathcal{M}_r = \{ u \in C(\overline{\Omega}) \mid u|_K \in \mathbb{P}_r(K),\ K \in \mathcal{T}_h \}$$

Suppose that $\{\vec{z}_j\}_{j=1}^{N_r} \subset \overline{\Omega}$ is the set of nodes satisfying

$$\phi_i(\vec{z}_j) = \delta_{i,j}$$

Define

$$\Pi_h : C(\overline{\Omega}) \to V_h$$

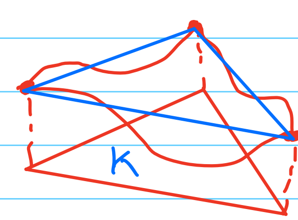
via

$$\Pi_h u(\vec{x}) := \sum_{j=1}^{N_r} u(\vec{z}_j)\, \phi_j(\vec{x})$$

Π_h is called the (global) <u>Lagrange nodal interpolation operator</u>.

<u>Proposition</u>: For every $u \in C(\overline{K})$, $\Pi_K u \in \mathbb{P}_r(\overline{K})$ and

i) $\Pi_K u = u \quad \forall\, u \in \mathbb{P}_r(\overline{K})$



ii) $\Pi_K(\Pi_K w) = \Pi_K u$

iii) $u(\vec{a}_{K,i}) = \Pi_K u(\vec{a}_{K,i}), \quad i = 1, \dots, d$

For every $u \in C(\bar{\Omega})$, $\Pi_h u \in V_h = \mathcal{M}_r$ and

i) $\Pi_h u = u \quad \forall\, u \in \mathcal{M}_r$,

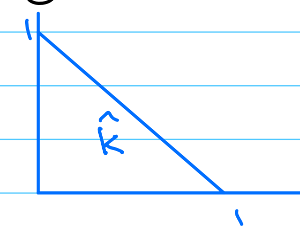
ii) $\Pi_h(\Pi_h u) = \Pi_h u$,

iii) $u(\vec{z}_j) = \Pi_h u(\vec{z}_j)$, $j = 1, \dots, N_r$

iv) $(\Pi_h u)|_K = \Pi_K u$

<u>Lemma</u>: Suppose that K is non-degenerate and

$$\vec{F}_K : \hat{K} \xrightarrow[\text{onto}]{1-1} K$$



via

$$\vec{F}_K(\hat{\vec{x}}) = B_K \hat{\vec{x}} + \vec{a}_K,$$

Set

$$h_K = \operatorname{diam}(K), \quad \hat{h} = \operatorname{diam}(\hat{K}) = \sqrt{2}$$

Let ρ_K and $\hat{\rho}$ be the diameters of the largest inscribed balls in K and $\hat{K}$, resp.
Then

$$\|B_K\|_2 \leq \frac{h_K}{\hat{\rho}}$$

$$\|B_K^{-1}\|_2 \leq \frac{\hat{h}}{\rho_K}$$

<u>Proof</u>: Recall that

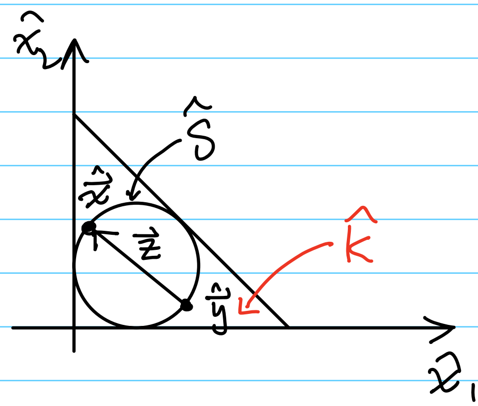
$$\|B_K\|_2 = \sup_{\hat{\vec{x}} \neq 0} \frac{\|B_K \hat{\vec{x}}\|_2}{\|\hat{\vec{x}}\|_2}$$

$$= \sup_{\|\vec{z}\|_2=1} \|B_K \vec{z}\|_2$$

$$= \sup_{\|\vec{z}\|=\hat{\rho}} \hat{\rho}^{-1} \|B_K \vec{z}\|_2$$

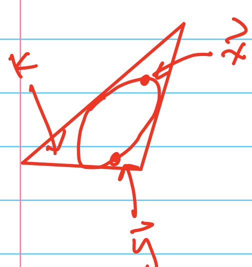
Now, let $\hat{\vec{x}}$ and $\hat{\vec{y}}$ be any two vectors on $\hat{S}$ such that

$$\vec{z} = \hat{\vec{x}} - \hat{\vec{y}}, \quad \|\vec{z}\|_2 = \hat{\rho}$$



Then

$$\|B_K\|_2 = \sup_{\substack{\hat{\vec{x}},\hat{\vec{y}} \in \hat{S} \\ \|\hat{\vec{x}}-\hat{\vec{y}}\|_2=\hat{\rho}}} \|B_K(\hat{\vec{x}}-\hat{\vec{y}})\|_2 \, \hat{\rho}^{-1}$$



$$= \sup_{\substack{\hat{\vec{x}},\hat{\vec{y}} \in \hat{S} \\ \|\hat{\vec{x}}-\hat{\vec{y}}\|_2=\hat{\rho}}} \|\vec{F}_K(\hat{\vec{x}})-\vec{F}_K(\hat{\vec{y}})\|_2 \, \hat{\rho}^{-1}$$

$$\leq \sup_{\vec{x},\vec{y} \in K} \|\vec{x}-\vec{y}\|_2 \, \hat{\rho}^{-1}$$

$$= \frac{h_K}{\hat{\rho}}.$$

The other inequality is gotten analogously. ///

<u>Defn</u>: let $\hat{v} \in C(\overline{\hat{K}})$ and suppose that

$$\{\hat{\psi}_1, \dots, \hat{\psi}_d\} \subset \mathbb{P}_r(\hat{K})$$

is the local Lagrange nodal basis for $\mathbb{P}_r(\hat{K})$ satisfying

$$\hat{\psi}_i(\hat{\vec{a}}_j) = \delta_{i,j}, \quad i,j = 1, \dots, d.$$

Then

$$\hat{\Pi}\hat{v}(\hat{\vec{x}}) := \sum_{i=1}^{d} \hat{v}(\hat{\vec{a}}_i)\hat{\psi}_i(\hat{\vec{x}})$$

<u>Theorem</u>: Suppose that $v \in C(\overline{K})$ and

$$\hat{v}(\hat{\vec{x}}) := v(\vec{F}_K(\hat{\vec{x}}))$$

Then $\hat{v} \in C(\overline{\hat{K}})$ and

$$\hat{\Pi}\hat{v}(\hat{\vec{x}}) = \Pi_K v(\vec{F}_K(\hat{\vec{x}})) \qquad \forall \hat{\vec{x}} \in \hat{K}.$$

<u>Proof</u>: By defn

$$\Pi_K v(\vec{x}) = \sum_{i=1}^{d} v(\vec{a}_{K,i})\psi_i^K(\vec{x})$$

Recall that

$$\vec{F}_K(\hat{\vec{a}}_i) = \vec{a}_{K,i} \qquad i = 1, \dots, d$$

and

$$\hat{\psi}_i(\hat{\vec{x}}) = \psi_i^K(\vec{F}_K(\hat{\vec{x}})) \qquad i = 1, \dots, d \quad \forall \hat{\vec{x}} \in \hat{K}$$

So

$$\Pi_K v(\vec{F}_K(\hat{\vec{x}})) = \sum_{i=1}^{d} v(\vec{F}_K(\hat{\vec{a}}_i))\, \psi_i^K(F_K(\hat{\vec{x}}))$$

$$= \sum_{i=1}^{d} \hat{v}(\hat{\vec{a}}_i)\, \hat{\psi}_i(\hat{\vec{x}})$$

$$= \hat{\Pi}\hat{v}(\hat{\vec{x}}). \quad ///$$

A Quotient Space Estimate

We will need the following estimate for our error analysis of Lagrange nodal interpolation.

Theorem: Let Ω be an open, bounded, Lipschitz domain in $\mathbb{R}^d$. There is a constant $C = C(\Omega, k) > 0$ such that

$$\inf_{p \in \mathbb{P}_k(\Omega)} \|v + p\|_{H^{k+1}(\Omega)} \leq C\, |v|_{H^{k+1}(\Omega)}$$

for all $v \in H^{k+1}(\Omega)$.

Recall that

$$\|\vec{v}\|_{H^{k+1}(\Omega)} = \sqrt{\sum_{|\alpha|\le k+1} (\partial^\alpha v, \partial^\alpha v)_{L^2}}$$

$$|\vec{v}|_{H^{k+1}(\Omega)} = \sqrt{\sum_{|\alpha|= k+1} (\partial^\alpha v, \partial^\alpha v)_{L^2}}$$

Proposition: Let $\Omega \subset \mathbb{R}^d$ be an open, bounded, Lipschitz domain the object

$$\inf_{p \in \mathbb{P}_k(\Omega)} \|v+p\|_{H^{k+1}(\Omega)},$$

the so called quotient norm is a bona fide norm on the space

$$V = H^{k+1}(\Omega) / \mathbb{P}_k(\Omega)$$

$$= \{ [v] \mid [v] = \{ v+q \mid q \in \mathbb{P}_k(\Omega) \}, v \in H^{k+1}(\Omega) \}$$

which is the so called quotient space.

See the book by Atkinson and Han for details.