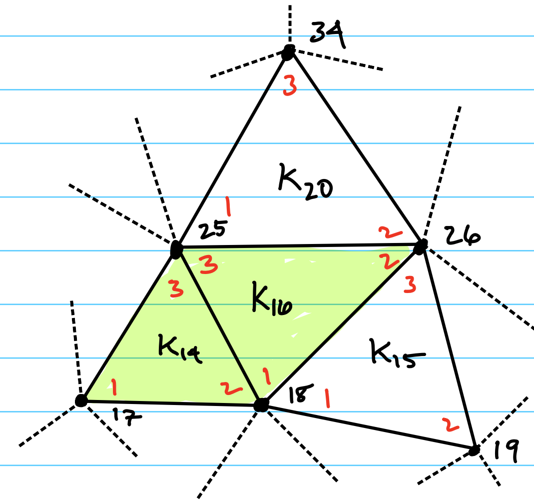
Math 574
class # 11
09/23/2025



Example: ($r = 1$): let us consider the following mesh in 2D. The green triangles are deep in the interior of the mesh. Suppose we wish to compute $a_{25,18}$:

global stiffness matrix

$$
\begin{aligned}
a_{25,18} &= \int_{\Omega} \nabla \phi_{18} \cdot \nabla \phi_{25} \, d\vec{x} \\
&= \int_{S_{18} \cap S_{25}} \nabla \phi_{18} \cdot \nabla \phi_{25} \, d\vec{x} \\
&= \int_{K_{14}} \nabla \phi_{18} \cdot \nabla \phi_{25} \, d\vec{x} + \int_{K_{16}} \nabla \phi_{18} \cdot \nabla \phi_{25} \, d\vec{x} \\
&= \int_{K_{14}} \nabla \psi_2^{14} \cdot \nabla \psi_3^{14} \, d\vec{x} + \int_{K_{16}} \nabla \psi_1^{16} \cdot \nabla \psi_3^{16} \, d\vec{x} \\
&= a_{3,2}^{(14)} + a_{3,1}^{(16)}
\end{aligned}
$$

local stiffness matrices for triangles 14 and 16.

Theory of Local Lagrange Nodal Bases

Theorem: Let $I = [a,b] \subset \mathbb{R}$, $p \in \mathbb{P}_r(I)$. Suppose that

$$a \le x_1 < x_2 < \cdots < x_r < x_{r+1} \le b.$$

Then, $p \equiv 0$ on I iff $p(x_i) = 0$, for all $i = 1, \cdots, r+1$.

Proof: ($\Longrightarrow$): This direction is obvious.

($\Longleftarrow$): Since $p(x_1) = 0$.

$$p(x) = (x - x_1)\, p_1(x)$$

for some $p_1 \in \mathbb{P}_{r-1}(I)$. Continuing in this way,

$$p(x) = (x - x_1) \cdots (x - x_r)\, p_r(x)$$

for some $p_r \in \mathbb{P}_0(I)$. Thus,

$$p(x) = c(x - x_1) \cdots (x - x_r)$$

But

$$0 = p(x_{r+1}) = c\, \underbrace{(x_{r+1} - x_1)}_{>0} \cdots \underbrace{(x_{r+1} - x_r)}_{>0}$$

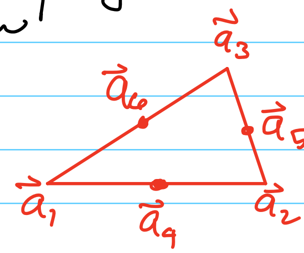
This implies that $c = 0$. Thus, $p \equiv 0$ on I. ///

Theorem: Let $v \in \mathbb{P}_1(K)$, where K is a non-degenerate triangle. Then v is uniquely determined by the 3 degrees of freedom $v(\vec{a}_j)$, $j = 1, 2, 3$.

Furthermore,
$$v(\vec{x}) = \sum_{i=1}^{3} v(\vec{a}_i)\,\psi_i(\vec{x}).$$
and $B_1 = \{\psi_i\}_{i=1}^{3}$ is a basis for $\mathbb{P}_1(K)$.

Proof: Exercise. ///

Theorem: A function $v \in \mathbb{P}_2(K)$ is uniquely determined by the 6 degrees of freedom
$$v(\vec{a}_i), \qquad i = 1, 2, \cdots, 6.$$

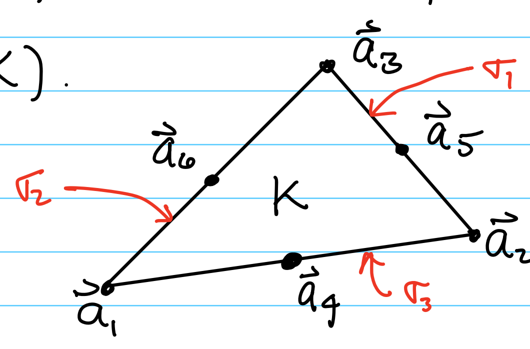


Furthermore,
$$v(\vec{x}) = \sum_{i=1}^{6} v(\vec{a}_i)\,\psi_i(\vec{x})$$
and $B_2 = \{\psi_i\}_{i=1}^{6}$ is a basis for $\mathbb{P}_2(K)$.

Proof: We first observe (without proof) that
$$\dim(\mathbb{P}_2(K)) = 6$$
Since
$$\{1, x, y, xy, x^2, y^2\}$$
is a basis for $\mathbb{P}_2(K)$.

why?

Now, it is sufficient to show that if $v \in \mathbb{P}_2(K)$ and
$$v(\vec{a}_i) = 0, \quad i = 1, 2, \cdots, 6,$$
then $v \equiv 0$ on $\mathbb{P}_2(K)$.



Suppose that

$$v(\vec{a}_2) = v(\vec{a}_5) = v(\vec{a}_3) = 0$$

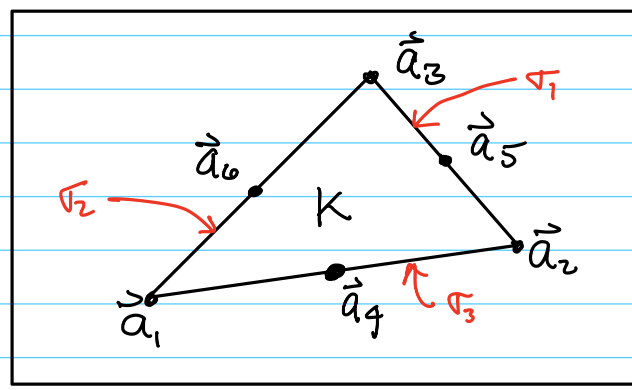
Since $v|_{\sigma_1} \in \mathbb{P}_2(\mathbb{R})$, $v|_{\sigma_1} \equiv 0$ on σ_1

Consequently

$$\vec{v}(\vec{x}) = \lambda_1(\vec{x})\, w_1(\vec{x})$$

for some $w_1 \in \mathbb{P}_1(K)$.

By a similar argument $v|_{\sigma_3} \equiv 0$. But, since $\lambda_1(\vec{x}) = 0$ for $\vec{x} \in \sigma_3$ iff $\vec{x} = \vec{a}_2$, it must be that $\lambda_3(\vec{x})$ is a factor of $w_1(\vec{x})$. Therefore



$$v(\vec{x}) = \lambda_1(\vec{x})\lambda_3(\vec{x})\, w_0(x),$$

for some $w_0 \in \mathbb{P}_0(K)$.

In other words, $w_0 \equiv c \in \mathbb{R}$. Hence

$$v(\vec{x}) = c\,\lambda_1(\vec{x})\lambda_3(\vec{x}) \qquad \exists\, c \in \mathbb{R}$$

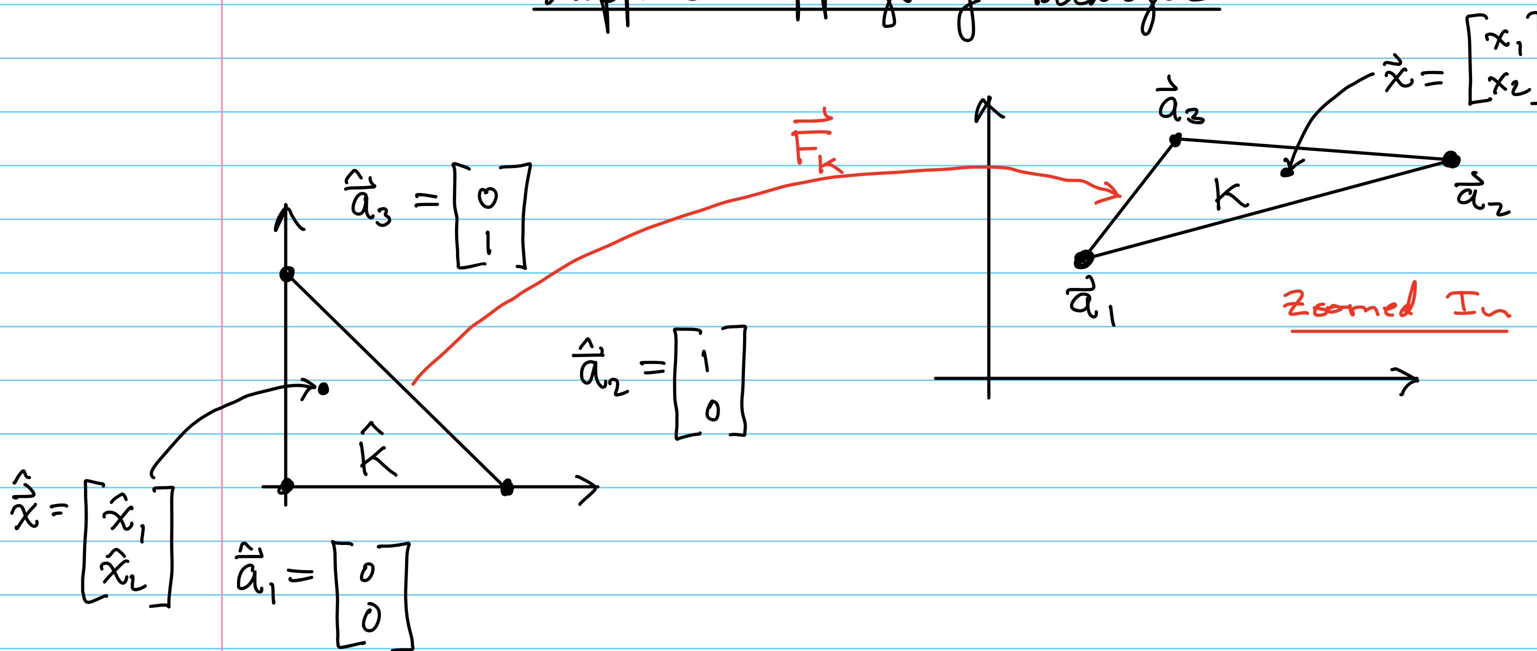
Finally, we note that

$$0 = v(\vec{a}_6) = c\,\lambda_1(\vec{a}_6)\lambda_3(\vec{a}_6)$$

$$= c\,\frac{1}{2}\cdot\frac{1}{2}$$

Hence $c = 0$. Therefore, $v \equiv 0$ on K. ///

Affine Mappings of Triangles



The mapping

$$F_K : \hat{K} \longrightarrow K$$

is affine, meaning

$$F_K(\hat{\vec{x}}) = B_K \hat{\vec{x}} + \vec{c}_K$$

for some $B_K \in \mathbb{R}^{2\times 2}$ and $\vec{c}_K \in \mathbb{R}^2$.

We require, if possible that

(11.1) $$\vec{F}(\hat{\vec{a}}_j) = \vec{a}_j\,, \quad j = 1, 2, 3.$$

Define

$$B_K := \begin{bmatrix} (a_{2,1} - a_{1,1}) & (a_{3,1} - a_{1,1}) \\ (a_{2,2} - a_{1,2}) & (a_{3,2} - a_{1,2}) \end{bmatrix}$$

and

$$\vec{c}_K := \vec{a}_1$$

Here, we use the convention that

$$[\vec{a}_j]_i = a_{j,i}, \quad j = 1,2,3, \quad i = 1,2.$$

It is easy to see that (11.1) holds.

<u>Proposition</u>: The matrix B_K is invertible and

$$B_K^{-1} = \frac{1}{\det(B_K)} \begin{bmatrix} a_{3,2} - a_{1,2} & a_{1,1} - a_{3,1} \\ a_{1,2} - a_{2,2} & a_{2,1} - a_{1,1} \end{bmatrix}$$

provided the triangle K is non-degenerate, that is, it has positive area, and

$$\begin{aligned} \det(B_K) &= (a_{2,1} - a_{1,1})(a_{3,2} - a_{1,2}) \\ &\quad - (a_{2,2} - a_{1,2})(a_{3,1} - a_{1,1}) \\ &= 2|K| > 0 \end{aligned}$$

Proof: The only difficulty in this proof is to show

$$\det(B_K) = 2|K|.$$

This is left for an exercise. ///

Our goal is to use this affine mapping to compute integrals of the form

$$\int_K f(\vec{x})\, dx$$

and

$$\int_K \nabla f(\vec{x}) \cdot \nabla g(\vec{x})\, d\vec{x}$$

Recall the following useful result from your calculus classes:

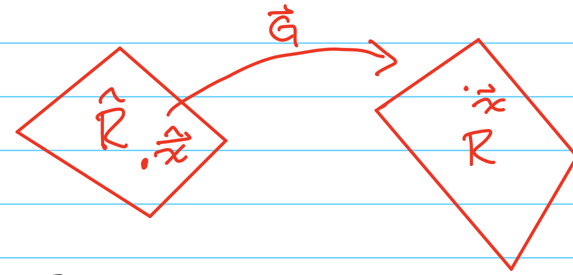
<u>Theorem</u>: Suppose that

$$\vec{G} : \hat{R} \to R$$

is a C^1, 1-1, onto mapping from the bounded, closed set $\hat{R} \subset \mathbb{R}^2$ to the bounded, closed set $R \subset \mathbb{R}^2$. Assume that R is Riemann integrable. Let us write

$$\vec{x} = \vec{G}(\hat{\vec{x}}), \quad \forall \hat{\vec{x}} \in \hat{R}.$$

Suppose that



$$f : R \to \mathbb{R}$$

is a continuous function. Then

$$\int_R f(\vec{x})\,d\vec{x} = \int_{\hat{R}} f(\vec{G}(\hat{\vec{x}}))\,|\det(J(\hat{\vec{x}}))|\,d\hat{\vec{x}}$$

where J is the Jacobian matrix

$$J(\hat{\vec{x}}) = \begin{bmatrix} \frac{\partial G_1}{\partial \hat{x}_1} & \frac{\partial G_1}{\partial \hat{x}_2} \\ \frac{\partial G_2}{\partial \hat{x}_1} & \frac{\partial G_2}{\partial \hat{x}_2} \end{bmatrix}$$

and $\det(J(\hat{\vec{x}}))$ is its determinant.

<u>Corollary</u>: let $\vec{F}_K : \hat{K} \to K$ be the affine mapping from above and assume that $|K|>0$. Then, if $f : K \to \mathbb{R}$ is continuous,

$$\int_K f(\vec{x})\,d\vec{x} = 2|K| \int_{\hat{K}} f(\vec{F}_K(\hat{\vec{x}}))\,d\hat{\vec{x}}.$$

<u>Proposition</u>: let $\psi_i \in \mathbb{P}_r(K)$ and $\hat{\psi}_i \in \mathbb{P}_r(\hat{K})$, $i = 1, \cdots, d = \dim(\mathbb{P}_r(K))$, which, one may show satisfies

$$\dim(\mathbb{P}_r(K)) = \frac{(r+1)(r+2)}{2}.$$

These are the local Lagrange nodal basis functions on K and $\hat{K}$ respectively. Let

$$\vec{F}_K : \hat{K} \xrightarrow[\text{onto}]{1-1} K$$

be the canonical affine mapping from $\hat{K}$, the reference triangle to $K \in \mathcal{T}_h$. Then

$$\hat{\psi}_i(\hat{\vec{x}}) = \psi_i(\vec{F}_K(\hat{\vec{x}})), \quad \forall \hat{\vec{x}} \in \hat{K}, \quad i = 1, \cdots, d.$$

For brevity, I will write

$$\hat{\psi}_i(\hat{\vec{x}}) = \psi_i(\vec{x}).$$

Proof: It suffices to prove that

$$\hat{\lambda}_i(\hat{\vec{x}}) = \lambda_i(\vec{x}), \quad \forall \hat{\vec{x}} \in \hat{K}, \quad i = 1, \cdots, 3,$$

that is the barycentric coordinates transform as expected. This is left as an exercise. ///

Corollary: Affine mappings preserve polynomials.

Proposition: Suppose $\vec{F}_K : \hat{K} \to K$ is the canonical affine mapping from the reference triangle $\hat{K}$ to $K \in \mathcal{T}_h$. Then if

$$f : K \to \mathbb{R}$$

is a C^1 function, it follows that

$$\nabla f(\vec{x}) = B_K^{-T} \hat{\nabla} \hat{f}(\hat{\vec{x}})$$

where

$$\hat{\nabla} := \nabla_{\hat{\vec{x}}} := \left(\frac{\partial}{\partial \hat{x}_1}, \frac{\partial}{\partial \hat{x}_2} \right)$$

and

$$\hat{f}(\hat{\vec{x}}) := f(\vec{F}_K(\hat{\vec{x}})) = f \circ \vec{F}_K(\hat{\vec{x}})$$

Proof: let us compute $\hat{\nabla}\hat{f}$. By definition

$$\frac{\partial}{\partial \hat{x}_i}\hat{f}(\hat{\vec{x}}) = \frac{\partial}{\partial \hat{x}_i} f(\vec{F}_K(\hat{\vec{x}}))$$
$$= \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial \hat{x}_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial \hat{x}_i}$$

Now, since

$$\vec{x} = \vec{F}_K(\hat{\vec{x}}) = B_{\hat{K}}\hat{\vec{x}} + \vec{a}_1$$
$$= \begin{bmatrix} b_{11}\hat{x}_1 + b_{12}\hat{x}_2 \\ b_{21}\hat{x}_1 + b_{22}\hat{x}_2 \end{bmatrix} + \vec{a}_1$$

$$\frac{\partial x_1}{\partial \hat{x}_i} = b_{1,i} \qquad \frac{\partial x_2}{\partial \hat{x}_i} = b_{2,i}$$

Thus
$$\frac{\partial}{\partial \hat{x}_i}\hat{f}(\hat{\vec{x}}) = \frac{\partial f}{\partial x_1} b_{1,i} + \frac{\partial f}{\partial x_2} b_{2,i}$$

or, equivalently,

$$\hat{\nabla}\hat{f}(\hat{\vec{x}}) = B_K^T \nabla f(\vec{x}) \quad ///$$

Proposition: let $\vec{F}_K : \hat{K} \to K$ be defined as above. Then

$$\int_K \nabla\varphi_i(\vec{x})\cdot\nabla\varphi_j(\vec{x})\, d\vec{x} = 2|K| \int_{\hat{K}} (B_K^{-T}\hat{\nabla}\hat{\varphi}_i(\hat{\vec{x}}))\cdot(B_K^{-T}\hat{\nabla}\hat{\varphi}_j(\hat{\vec{x}}))\, d\hat{\vec{x}}$$

Proof: Exercise ///