Math 574
class #02
8/21/2025

Cea's Lemma, the Stiffness Matrix, and Hilbert Spaces

Proposition (FGO): Suppose that $u \in V$ solves (1.6) and $u_h \in V_h \subseteq V$ solves (1.9). Then

(2.1) $$a(u-u_h, v) = 0 \quad \forall v \in V_h.$$

The identity in (2.1) is called the fundamental Galerkin orthogonality (FGO).

Proof: Because u solves (1.6),

$$a(u,v) \overset{(1.6)}{=} F(v), \quad \forall v \in V.$$

As $V_h \subseteq V$,

$$a(u,v) = F(v), \quad \forall v \in V_h.$$

Because u_h solves (1.9)

$$a(u_h,v) = F(v), \quad \forall v \in V_h.$$

Let $v \in V_h$ be arbitrary, using the linearity of a in its first argument,

$$a(u-u_h, v) = F(v) - F(v) = 0. \qquad ///$$

Remark: We have not yet proven that solutions to (1.6) and (1.9) can be found!

<u>Theorem</u> (Cea's Lemma): Suppose $u \in V$ solves (1.6) and $u_n \in V$ solves (1.9). Then

(2.2) $$\|u-u_n\|_E = \inf_{v \in V_n} \|u-v\|_E$$

<u>Proof</u>: Let $v \in V_n$ be arbitrary. Then

$$\begin{aligned}
\|u-u_n\|_E^2 &= a(u-u_n, u-u_n) \\
&= a(u-u_n, u-u_n-v+v) \\
&\overset{\text{lin.}}{=} a(u-u_n, u-v) + a(u-u_n, \underbrace{v-u_n}_{\in V_n}) \\
&\overset{\text{FGO}}{=} a(u-u_n, u-v) \\
&\overset{\text{C.S.}}{\leq} \|u-u_n\|_E \, \|u-v\|_E .
\end{aligned}$$

Assume that $u \neq u_n$. Then,

$$\|u-u_n\|_E \leq \|u-v\|_E, \quad \forall v \in V_n.$$

Then,

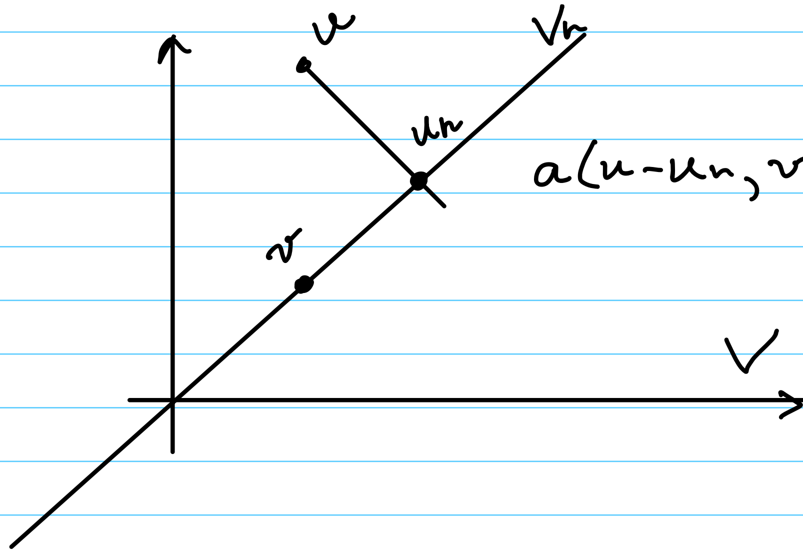
$$\|u-u_n\|_E \leq \inf_{v \in V_n} \|u-v\|_E$$

$$\leq \|u-u_n\|_E$$

The only way this can happen is that

$$\|u-u_n\|_E = \inf_{v \in V_n} \|u-v\|_E \quad ///$$

Fundamental Galerkin Orthogonality



$a(u-u_h, v) = 0 \quad \forall\, v \in V_h$

Is the Galerkin approx computable?

Proposition: $u_h \in V_h$ is the Galerkin approx

(2.3) $$a(u_h, v) = (f, v)_{L^2} =: \int_\Omega f\, v\, d\vec{x} \qquad \forall\, v \in V_h$$

iff u_h solves

(2.4) $$a(u_h, \phi_i) = (f, \phi_i)_{L^2} \qquad i = 1, \cdots, M,$$

where
$$B = \{\phi_i\}_{i=1}^{M}$$
is a basis for V_h.

Proof: ($\Rightarrow$): Suppose that u_h solves (2.3). Then it must clearly solve (2.4). (Why?) //

($\Leftarrow$): Suppose that u_h solves (2.4). Let $v \in V_h$ be arbitrary. Then, $\exists!\ c_i \in \mathbb{R}$, $i = 1, \dots, M$, such that

$$v = \sum_{i=1}^{M} c_i \phi_i \qquad \text{(unique basis expansion)}$$

But

$$\begin{aligned} a(u_h, v) &= a\left(u_h, \sum_{i=1}^{n} c_i \phi_i\right) \\ &= \sum_{i=1}^{n} c_i\, a(u_h, \phi_i) \\ &\overset{(2.4)}{=} \sum_{i=1}^{n} c_i (f, \phi_i)_{L^2} \\ &= \left(f, \sum_{i=1}^{n} c_i \phi_i\right)_{L^2} \\ &= (f, v)_{L^2} \qquad /// \end{aligned}$$

Defn: Suppose that $B = \{\phi_i\}_{i=1}^{M}$ is a basis for V_h. The matrix $A = [a_{i,j}]_{i,j=1}^{M} \in \mathbb{R}^{M \times M}$ whose entries are

(2.5) $$a_{i,j} = a(\phi_j, \phi_i), \qquad i,j = 1, \dots, M$$

is called the <u>stiffness matrix</u>. The vector $f \in \mathbb{R}^M$ whose entries are

(2.6) $$f_i = (f, \phi_i)_{L^2}, \qquad i = 1, \dots, M$$

is called the <u>load vector</u>.

<u>Remark</u>: Clearly A is symmetric.

Proposition: Let $B = \{\phi_i\}_{i=1}^{M}$ be a basis for V_h and suppose that A and $\vec{f}$ are assembled as above. Then $u_h \in V_h$ satisfies the Galerkin approx (2.3) iff $\vec{u} \in \mathbb{R}^M$ solves

$$A\vec{u} = \vec{f}$$

where $\vec{u} = [u_i]_{i=1}^{M}$ is the vector whose components are the coefficients in the basis expansion for $u_h \in V_h$, ie,

$$u_h = \sum_{i=1}^{M} u_i \phi_i = \sum_{i=1}^{M} [\vec{u}]_i \phi_i \qquad \left([\vec{u}]_i = u_i \right)$$

Proof: ($\Longrightarrow$): Suppose u_h solves (2.3). Then

$$a(u_h, \phi_i) = (f, \phi_i)_{L^2} \quad i = 1, \cdots, M.$$

Since $u_h \in V_h$ and B is a basis, there are unique coefficients $u_j \in \mathbb{R}$, $j = 1, \cdots, M$, such that

$$u_h = \sum_{j=1}^{M} u_j \phi_j$$

Set

$$\vec{u} = [u_j]_{j=1}^{M} \in \mathbb{R}^M$$

Then

$$[A\vec{u}]_i = \sum_{j=1}^{M} a_{ij} u_j$$

$$= \sum_{j=1}^{M} a(\phi_j, \phi_i) u_j$$

$$= a\left(\sum_{j=1}^{M} u_j \phi_j , \phi_i\right)$$

$$= a(u_h, \phi_i)$$

$$= (f, \phi_i)_{L^2}$$

$$= f_i \quad //$$

($\Leftarrow$): Now suppose that $\vec{u} \in \mathbb{R}^M$ solves

$$A\vec{u} = \vec{f}.$$

Set

$$u_h = \sum_{j=1}^{M} u_j \phi_j$$

$$= \sum_{j=1}^{M} [\vec{u}]_j \phi_j .$$

Run the last argument in reverse. ///

<u>Proposition</u>: The stiffness matrix $A \in \mathbb{R}^{M \times M}$, computed as above, is SPD.

Proof: A is clearly symmetric. Let $\vec{v} \in \mathbb{R}^M$ be arbitrary but $\vec{v} \neq \vec{0}$. Then

$$\vec{v}^T A \vec{v} = \sum_{i=1}^{M} \sum_{j=1}^{M} v_i a_{i,j} v_j$$

$$= a\left(\sum_{j=1}^{M} v_j \phi_j , \sum_{i=1}^{M} v_i \phi_i\right)$$

$$= a(v, v)$$

where

$$v = \sum_{i=1}^{M} v_i \phi_i \in V_h$$

Thus

$$\vec{v}^T A \vec{v} = \|v\|_E^2$$

Since $v \not\equiv 0 \in V_h$, $\|v\|_E^2 > 0$. Hence A is SPD. ///

<u>Corollary</u>: Problem (2.3) has a unique solution.

<u>Proof</u>: This follows since A is SPD. ///

<u>Hilbert Spaces</u>

Let us talk about some Functional Analysis.

<u>Defn</u>: Let V be a linear (vector) space over $\mathbb{R}$. An <u>inner products</u>

$$(\cdot, \cdot) : V \times V \longrightarrow \mathbb{R}$$

is a function with the properties

1) <u>Positivity</u>: For all $v \in V$,

$$(v, v) \geq 0$$

and

$$(v, v) = 0 \quad \text{iff} \quad v = 0 \in V$$

2) Symmetry: For all $u, v \in V$

$$(u, v) = (v, u)$$

3) Linearity: For all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$,

$$(\alpha u + \beta v, w) = \alpha (u, w) + \beta (v, w)$$

Defn: Let V be a linear space over $\mathbb{R}$. A norm, $\| \cdot \| : V \to \mathbb{R}$, is a function with the properties

1) Positivity: For all $v \in V$

$$\|v\| \geq 0$$

and

$$\|v\| = 0 \quad \text{iff} \quad v = 0.$$

2) Non-Negative Homogeneity:

$$\|\alpha v\| = |\alpha| \cdot \|v\|$$

for all $\alpha \in \mathbb{R}$ and all $v \in V$.

3) Triangularity: For all $u, v \in V$,

$$\|u + v\| \leq \|u\| + \|v\|.$$

A linear space V with an inner product is called an inner product space. One with a norm is called a normed linear space.

Theorem: (Cauchy-Schwartz) Let $(V, (\cdot,\cdot))$ be an inner product space. Then, for all $u, v \in V$

$$|(u,v)| \leq \sqrt{(u,u)}\,\sqrt{(v,v)}$$

Equality holds iff u and v are linearly dependents.

Proof: Similar to that of the last lecture. ///

Theorem: Let V be an inner product space. Then

$$\|v\| = \sqrt{(v,v)}, \quad v \in V$$

defines a norm. Thus $(V, \|\cdot\|)$ is a normed linear space.

Proof: Exercise. Use C-S ineq. ///

Defn: Let $(V, \|\cdot\|)$ be a normed linear space. A sequence $\{v_n\}_{n=1}^{\infty} \subset V$ converges in V iff there is a point $v \in V$ such that, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$, such that if $n \geq N$

$$\|v_n - v\| \leq \varepsilon$$

We write $v_n \xrightarrow[n\to\infty]{V} v$, $\|v_n - v\| \xrightarrow{n\to\infty} 0$.

$\{v_n\}_{n=1}^{\infty} \subset V$ is called <u>Cauchy</u> iff for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that when $n, m \geq N$

$$\|v_n - v_m\| \leq \varepsilon$$

A sequence $\{v_n\}_{n=1}^{\infty} \subset V$ is <u>bounded</u> iff there is an $M \in [0, \infty)$ such that for all $n \in \mathbb{N}$

$$\|v_n\| \leq M.$$

<u>Theorem</u>: Let $(V, \|\cdot\|)$ be a normed linear space. Let $\{v_n\}_{n=1}^{\infty} \subset V$ be a convergent sequence. Then, $\{v_n\}$ is bounded. If

$$v_n \longrightarrow v \quad \text{and} \quad v_n \longrightarrow w$$

it follows that $v = w$, i.e., limits are unique. Finally, $\{v_n\}$ is Cauchy.

<u>Proof</u>: Exercise ///

<u>Defn</u>: A linear space for which every Cauchy sequence is convergent is called <u>complete</u>. A complete normed linear space is called a <u>Banach space</u>. A complete inner product space is called a <u>Hilbert space</u>.