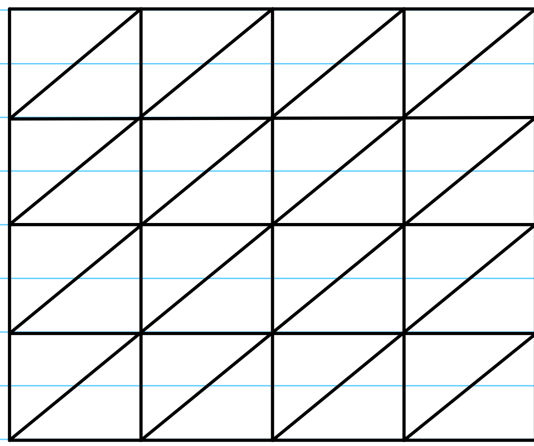
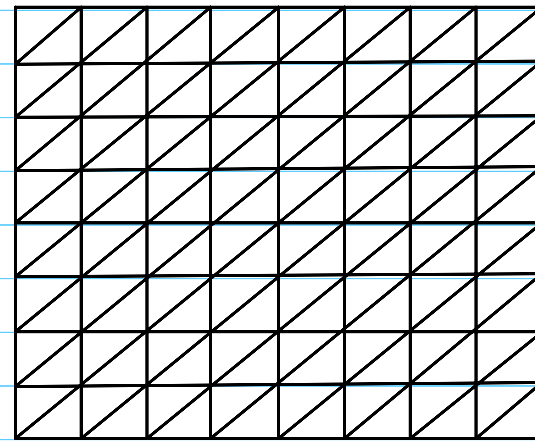
Math 574
class # 14
10/09/2025

This family of triangulations is shape regular.





Recall that, if the mesh is shape regular, we have

Corollary (13.6): If $\{\mathcal{T}_h\}_h$ is a shape regular family, $\exists\ C>0$, independent of the shape and size of K, such that

$$\|v-\Pi_K v\|_{H^m(K)} \le C\, h_K^{r+1-m}\, |v|_{H^{r+1}(K)}$$

for all $v \in H^{r+1}(K)$, $\forall\ K \in \mathcal{T}_h$.

Proof:

$$|v-\Pi_K v|_{H^m(K)} \overset{\text{Thm 13.4}}{\le} C\, \frac{h_K^{r+1}}{\rho_K^m}\, |v|_{H^{r+1}(K)}$$

$$= C \left(\frac{h_K}{\rho_K}\right)^m h_K^{r+1-m}\, |v|_{H^{r+1}(K)}$$

$$\leq c\,\sigma^m h_K^{r+1-m} |v|_{H^{r+1}(K)}.$$

To finish the proof, write

$$\|v-\Pi_K v\|^2_{H^m(K)} = \sum_{t=0}^{m} |v-\Pi_K v|^2_{H^t(K)}. \quad ///$$

Global Interpolation Error

Theorem (14.1): Assume the conditions of Cor (12.6) hold. Then $\exists C>0$, independent of

$$h = \max_{K\in\mathcal{T}_h} h_K,$$

such that

(14.1) $$\|v-\Pi_h v\|_{H^m(\Omega)} \leq C\, h^{r+1-m} |v|_{H^{r+1}(\Omega)},$$

for all $v\in H^{r+1}(\Omega)$.

Proof: Recall that

$$\Pi_h v\big|_K = \Pi_K v\big|_K$$

$$\|u-\Pi_h u\|^2_{H^m(\Omega)} = \sum_{K\in\mathcal{T}_h} \|u-\Pi_K u\|^2_{H^m(K)}$$

$$\leq \sum_{K\in\mathcal{T}_h} C h_K^{2(r+1-m)} |u|^2_{H^{r+1}(K)}$$

$$\leq C h^{2(r+1-m)} \sum_{K\in\mathcal{T}_h} |u|^2_{H^{r+1}(K)}$$

$$= C h^{2(r+1-m)} |u|^2_{H^{r+1}(\Omega)}$$

Now, take square roots. ///

Remark: Here we have used the useful property that

$$\int_\Omega f(\vec{x})\, d\vec{x} = \sum_{K \in \mathcal{T}_h} \int_K f(\vec{x})\, d\vec{x}$$

Error Estimates for the Model Problem

Find $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that

(14.2)
$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

The weak formulation is given by the following: given $f \in L^2(\Omega)$, find $u \in H^1_0(\Omega)$ such that

(14.3)
$$a(u,v) = (f,v)_{L^2}, \quad \forall\, v \in H^1_0(\Omega),$$

where

$$a(u,v) = (\nabla u, \nabla v)_{L^2}$$

The Galerkin approx is given as follows: find $u_h \in \in V_h = \mathcal{M}_{0,r}$ such that

(14.4) $$a(u_h, v) = (f, v)_{L^2} \quad \forall\, v \in V_h.$$

Recall Cea's Lemma for the model problem:

(14.5) $$|u - u_h|_{H^1} = \min_{v \in V_h} |u - v|_{H^1}$$

<u>Corollary</u> (14.2) Suppose that $V_h = \mathcal{M}_{0,r}$, for $r \in \mathbb{N}$. Assume that $u \in H_0^1(\Omega) \cap H^{r+1}(\Omega)$ is the solution of the model problem in the weak formulation (14.3). Suppose that $u_h \in V_h$ is the Galerkin (Finite Element) approximation. Assume that $\mathcal{T}_h$ is a shape regular family of triangulations. Then

$$|u - u_h|_{H^1} \le C h^r |u|_{H^{r+1}},$$

for some constant that is independent of h.

<u>Proof</u>:

$$|u - u_h|_{H^1} \overset{(14.5)}{=} \min_{v \in V_h} |u - v|_{H^1}$$

$$\le |u - \Pi_h u|_{H^1} \quad (\Pi_h u \in V_h)$$

$$\overset{(14.1)}{\le} C h^{r+1-1} |u|_{H^{r+1}} \quad (m = 1)$$

$$= C h^r |u|_{H^{r+1}} \quad ///$$

<u>Example</u>:

$r = 1$ (piecewise linear FEM):

$$|u - u_h|_{H^1} \le C h |u|_{H^2}$$

$r = 2$ (piecewise quadratic FEM):

$$|u - u_h|_{H^1} \leq C h^2 |u|_{H^3}$$

$r = 3$ (piecewise cubic FEM):

$$|u - u_h|_{H^1} \leq C h^3 |u|_{H^4}$$

"The currency of approximation is derivatives."

— Ohannes Karakashian

Question: Can we improve our approximations by measuring error in the L^2 norm?

Theorem (14.3): Suppose that $\Omega \subset \mathbb{R}^2$ is a convex polygonal domain and $f \in L^2(\Omega)$. Let $u \in H_0^1(\Omega)$ be the solution to (14.3) and suppose that $u_h \in V_h = \mathcal{M}_{0,1}$ is the solution to (14.4). Assume that $\mathcal{T}_h$ is a shape regular family of triangulations. Then

$$\|u - u_h\|_{L^2} \leq C_1 h \, |u - u_h|_{H^1},$$

for some constant $C_1 > 0$ that is independent of h. Consequently,

$$\|u - u_h\|_{L^2} \leq C_1 C h^2 |u|_{H^2},$$

for some $C > 0$ that is independent of h.

Proof: By elliptic regularity theory, since Ω is convex polygonal and $f \in L^2(\Omega)$, it follows that

$$u \in H_0^1(\Omega) \cap H^2(\Omega)$$

and there is a constant $C_R > 0$ such that

(14.6) $$\|u\|_{H^2} \leq C_R \|f\|_{L^2} = C_R \|\Delta u\|_{L^2}.$$

We are going to use a "duality argument," in this context known as Nitsche's Trick.

First, recall the fundamental Galerkin orthogonality (FGO):

(14.7) $$0 = a(u - u_h, v) = a(e, v), \quad \forall v \in V_h,$$

where

$$e := u - u_h \in H_0^1(\Omega)$$

Consider the following problem: Find $\phi \in H_0^1(\Omega)$ such that

(14.8) $$\begin{aligned} -\Delta \phi &= e \quad &\text{in } \Omega \\ \phi &= 0 \quad &\text{on } \partial\Omega \end{aligned}$$

which, in weak form, is expressed as

(14.9) $$a(\phi, v) = (e, v) \qquad \forall v \in H_0^1(\Omega)$$

By elliptic regularity theory, we know that

$$\phi \in H_0^1(\Omega) \cap H^2(\Omega)$$

and

(14.10) $$\|\phi\|_{H^2} \leq C_R \|e\|_{L^2}$$

Now,

$$\|e\|_{L^2}^2 = (e,e)_{L^2}$$
$$\overset{(14.9)}{=} a(\phi, e)$$
$$\overset{\text{sym}}{=} a(e, \phi)$$
$$\overset{\text{FGO}}{=} a(e, \phi - v)$$

for any $v \in V_h$. Thus, for arbitrary $v \in V_h$,

$$\|e\|_{L^2}^2 = a(e, \phi - v)$$
$$\overset{\text{C.S.}}{\leq} |e|_{H^1} |\phi - v|_{H^1}.$$

Now, since $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$, $\Pi_h \phi \in V_h$ is well-defined and

$$\|e\|_{L^2}^2 \leq |e|_{H^1} \, |\phi - \Pi_h \phi|_{H^1}$$
$$\overset{(14.1)}{\leq} |e|_{H^1} C h |\phi|_{H^2} \qquad (m=1, r=1)$$
$$\overset{(14.10)}{\leq} \underbrace{C_R C}_{C_1} h \|e\|_{L^2} |e|_{H^1}$$

Hence

$$\|e\|_{L^2} \leq C_1 h |e|_{H^1}. \quad //$$

Now, we know that $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and

$$|e|_{H^1} \le C h |u|_{H^2}$$

by Corollary 14.2. Combining the last two estimates gives

$$\|u-u_h\|_{L^2} \le C C_1 h^2 |u|_{H^2}. \qquad ///$$

We'll prove a more general result shortly...

A Non-Symmetric Problem

Given $f \in C^0(\Omega)$, $\vec{b} \in \mathbb{R}^d$ and $c > 0$. Find $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that

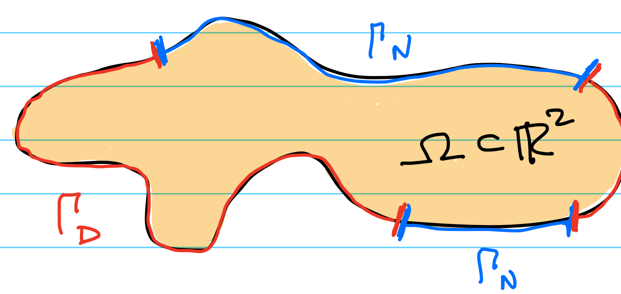
$$(14.11) \quad \begin{cases} -\Delta u + \vec{b}\cdot\nabla u + cu = f & \text{in } \Omega \subset \mathbb{R}^d \\ u = 0 & \text{on } \Gamma_D \\ \dfrac{\partial u}{\partial n} = g & \text{on } \Gamma_N \end{cases}$$

Here

$$\Gamma_D \cup \Gamma_N = \partial\Omega,$$

$$\Gamma_D \cap \Gamma_N = \emptyset,$$

$$\mu_1(\Gamma_D), \mu_1(\Gamma_N) > 0$$



Let us formulate a weak version of the problem. Let

$$v \in V := \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D \}$$

be arbitrary. Then, using integration-by-parts,

$$(f,v)_{L^2} - c(u,v)_{L^2} - (\vec{b}\nabla u, v)_{L^2}$$

$$= -(\Delta u, v)_{L^2}$$

$$= (\nabla u, \nabla v)_{L^2} - \int_{\partial\Omega} \overbrace{\hat{n}\cdot\nabla u}^{\frac{\partial u}{\partial n}}\, v\, dS$$

$$= (\nabla u, \nabla v)_{L^2} - \int_{\Gamma_N} \frac{\partial u}{\partial n} v\, ds$$

$$- \int_{\Gamma_D} \frac{\partial u}{\partial n} v\, ds$$

$$= (\nabla u, \nabla v)_{L^2} - \int_{\Gamma_N} g v\, ds$$

Thus,

(14.12) $$a(u,v) = L(v), \quad \forall\, v \in V$$

where

(14.13) $$\begin{cases} a(u,v) := (\nabla u, \nabla v)_{L^2} + (\vec{b}\cdot\nabla u, v)_{L^2} + c(u,v)_{L^2}, \\ L(v) := (f,v)_{L^2} + (g,v)_{L^2(\Gamma_N)}. \end{cases}$$

<u>Definition</u> (14.4): The following is a weak formulation of the non-symmetric problem (14.11). Given $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$, $c > 0$, and $\vec{b} \in \mathbb{R}^d$, find $u \in V$, where

(14.14) $$V := \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_N \}$$

such that (14.12) holds. We call such a $u \in V$, if it exists, a <u>weak solution to (14.11)</u>.

<u>Proposition</u> (14.5): The space defined in (14.4) is a Hilbert space. The functional $L : V \to \mathbb{R}$ is linear and bounded. The bilinear form

$$a : V \times V \longrightarrow \mathbb{R}$$

is, in general, non-symmetric, but is unconditionally continuous. In particular,

$$|a(u,v)| \leq \gamma \|u\|_{H^1} \|v\|_{H^1}$$

where

$$\gamma = 1 + \|\vec{b}\|_2 + c.$$

a is coercive, provided that

(14.15) $$\beta := \min\{1, c\} > \frac{\|\vec{b}\|_2}{2}.$$

In particular, in this case,

$$\alpha \|u\|_{H^1} \leq a(u,u),$$

where

$$\alpha = \beta - \frac{\|\vec{b}\|_2}{2} > 0.$$

Under condition (14.15) there exists a unique weak solution $u \in V$ to (14.11).

<u>Proof</u>: The conclusion follows from the Lax-Milgram lemma, provided we can prove the other points.

1) <u>L is a bnded linear operator on V.</u>

L is clearly linear. To show that it is bounded, let $v \in V$ be arbitrary. Then

$$|L(v)| \leq \left|\int_\Omega f\, v\, d\vec{x}\right| + \left|\int_{\Gamma_N} g\, v\, d\vec{x}\right|$$

$$\overset{C.S}{\leq} \|f\|_{L^2}\|v\|_{L^2} + \|g\|_{L^2(\Gamma_N)}\|v\|_{L^2(\Gamma_N)}$$

Using the Trace Theorem (Lecture 06)

$$\|v\|_{L^2(\Gamma_N)} \leq C_T\|v\|_{H^1(\Omega)}$$

we have

$$|L(v)| \leq \left(\|f\|_{L^2} + C_T\|g\|_{L^2(\Gamma_N)}\right)\|v\|_{H^1}$$

$$= C\|v\|_{H^1}. \quad //$$

(To use the Trace Theorem we need to assume that the boundary Γ_N is sufficiently regular. Let us assume that $\partial\Omega$ is Lipschitz cont. This should be enough.)

2) <u>a is continuous</u>: Let $u, v \in V$ be arbitrary. Then

$$|a(u,v)| \leq |(\nabla u, \nabla v)_{L^2}| + |(\vec{b}\cdot\nabla u, c)_{L^2}|$$

$$+ c|(u,v)_{L^2}|$$

$$\leq |u|_{H^1}|v|_{H^1} + \|\vec{b}\|_2 |u|_{H^1}\|v\|_{L^2}$$

$$+ c \|u\|_{L^2} \|v\|_{L^2}$$

$$\leq (1 + \|\vec{b}\|_2 + c) \|u\|_{H^1} \|v\|_{H^1}$$

$$= \gamma \|u\|_{H^1} \|v\|_{H^1} \quad //$$

3| <u>a is coercive</u>:

$$a(u,u) = |u|_{H^1}^2 + (\vec{b}\cdot\nabla u, u) + c\|u\|_{L^2}^2$$

For non-symmetric terms,

$$-(\vec{b}\cdot\nabla\vec{u}, u) \leq |(\vec{b}\cdot\nabla u, u)|$$

$$\leq \|\vec{b}\|_2 |u|_{H^1} \|u\|_{L^2}$$

$$\leq \|\vec{b}\|_2 \|u\|_{H^1}$$

This estimate is not optimal. We can do just a bit better by using the AGMI:

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

$$-(\vec{b}\cdot\nabla\vec{u}, u) \leq |(\vec{b}\cdot\nabla u, u)|$$

$$\leq \|\vec{b}\|_2 |u|_{H^1} \|u\|_{L^2}$$

$$= \left(\|\vec{b}\|_2^{1/2} |u|_{H^1}\right)\cdot\left(\|\vec{b}\|_2^{1/2} \|u\|_{L^2}\right)$$

$$\overset{AGMI}{\leq} \frac{\|\vec{b}\|_2}{2} |u|_{H^1}^2 + \frac{\|\vec{b}\|_2}{2} \|u\|_{L^2}^2$$

$$= \frac{\|\vec{b}\|_2}{2} \left(\|u\|_{L^2}^2 + |u|_{H^1}^2\right)$$

$$= \frac{\|\vec{b}\|_2}{2} \|u\|_{H^1}^2 .$$

Thus,

$$(\vec{b} \cdot \nabla u, u) \geq -\frac{\|\vec{b}\|_2}{2} \|u\|_{H^1}^2$$

so that

$$\begin{aligned} a(u,u) &\geq |u|_{H^1}^2 - \frac{\|\vec{b}\|_2}{2} \|u\|_{H^1}^2 + c\|u\|_{L^2}^2 \\ &\geq \beta\left(\|u\|_{L^2}^2 + |u|_{H^1}^2\right) - \frac{\|\vec{b}\|_2}{2} \|u\|_{H^1}^2 \\ &= \alpha \|u\|_H^2 \end{aligned}$$

where

$$\alpha = \beta - \frac{\|\vec{b}\|_2}{2} > 0 . \quad //$$

Using the Lax-Milgram lemma, a unique solution $u \in V$ to (14.12) exists. ///