Math 574
class #20
11/06/2025

The Elliptic (Ritz) Projection

<u>Defn</u> (20.1): Let Ω be an open polygonal domain in $\mathbb{R}^2$. Suppose that $u \in H_0^1(\Omega)$. The <u>elliptic (Ritz) projection</u> of u,

$$R_h u \in V_h = \mathcal{M}_{0,r}$$

is defined via

(20.1) $$a(R_h u, \psi) = a(u, \psi)$$

for all $\psi \in V_h$.

<u>Theorem</u> (20.2): $R_h : H_0^1(\Omega) \longrightarrow V_h$ is a linear projection operator and satisfies the stability

$$|R_h u|_{H^1} \leq |u|_{H^1}$$

for all $u \in H_0^1(\Omega)$. Furthermore, if elliptic regularity holds for the problem

$$-\Delta u = f \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

and $u \in H_0^1(\Omega) \cap H^{r+1}(\Omega)$, then

$$\|u - R_h u\|_{L^2} + h\|\nabla(u - R_h u)\|_{L^2} \leq C h^s \|u\|_{H^s}$$

for all $1 \leq s \leq r+1$.

Proof: That it is a linear projection is straightforward. For stability, set $\psi = R_h u$. Then

$$|R_h u|_{H^1}^2 = a(R_h u, R_h u)$$
$$= a(u, R_h u)$$
$$\overset{C.S.}{\leq} |u|_{H^1} |R_h u|_{H^1}$$

By cancelation,

$$|R_h u|_{H^1} \leq |u|_{H^1}.$$

The rest of the details are an exercise. ///

Theorem (20.3): Assume the hypotheses of Thm (20.2) are satisfied. Suppose that u is a solution to

(20.2) $$(\partial_t u, \psi) + (\nabla u, \nabla \psi) = (f, \psi), \quad \forall \psi \in H_0^1(\Omega),$$

with $u(\cdot, 0) = v$, for all $t \in (0, T]$, with the regularities

$$u \in L^2(0,T; H_0^1 \cap H^{r+1}) \cap H^1(0,T; H^{r+1}),$$

$$\Rightarrow \boxed{\begin{array}{l} \int_0^T \|u\|_{H^{r+1}}^2 \, dt < \infty \\ \int_0^T \|\partial_t u\|_{H^{r+1}}^2 \, dt < \infty \end{array}}$$

$v \in H_0^1 \cap H^{r+1}$.

Suppose that $u_h : [0,T] \longrightarrow V_h = \mathcal{M}_{0,r}$ satisfies

(20.3) $$(\partial_t u_h, \psi) + (\nabla u_h, \nabla \psi) = (f, \psi), \quad \forall \psi \in V_h$$

for all $t \in (0, T]$, and

$$u_h(\cdot, 0) = v_h = P_h v \in V_h,$$

where $P_h : L^2(\Omega) \to V_h$ is some linear projection. Then

(20.4)
$$\|u(t) - u_h(t)\|_{L^2} \leq \|v - v_h\|_{L^2} + Ch^{r+1}\left(\|v\|_{H^{r+1}} + \int_0^t \|\partial_t u\|_{H^{r+1}} ds\right)$$

for all $t \in [0,T]$.

Proof: Set

$$e(t) := u(t) - u_h(t), \quad t \in [0,T].$$

Note that $e(t) \notin V_h$. But, observe,

(20.5)
$$e(t) = \underbrace{u(t) - R_h u_h(t)}_{:= \rho(t)} + \underbrace{R_h u_h(t) - u_h(t)}_{:= \theta(t) \in V_h}$$

$$= \rho(t) + \theta(t)$$

Note that $\theta(t) \in V_h$, for all $t \in [0,T]$. This is a very important point.

By Theorem (20.2),

$$\|\rho(t)\|_{L^2} + h\|\nabla\rho(t)\|_{L^2} \leq Ch^{r+1}\|u(t)\|_{H^{r+1}}$$

Since

$$u(t) - \underbrace{u(0)}_{=v} = \int_0^t \partial_t u(s)\, ds$$

we have, by the triangle inequality,

$$\|\rho(t)\|_{L^2} \le C h^{r+1}\left(\|v\|_{H^{r+1}} + \int_0^t \|\partial_t u\|_{H^{r+1}}\, ds\right).$$

Again by the triangle inequality,

(20.6)
$$\begin{aligned}\|u-u_h(t)\|_{L^2} &\le \|\rho(t)\|_{L^2} + \|\theta(t)\|_{L^2} \\ &\le C h^{r+1}\left(\|v\|_{H^{r+1}} + \int_0^t \|\partial_t u\|_{H^{r+1}}\, ds\right) \\ &\quad + \|\theta(t)\|_{L^2}\end{aligned}$$

To complete our proof we must bound $\|\theta(t)\|_{L^2}$. To do this, we must form an error equation for $\theta(t)$.

Start with Equation (20.2):

$$(\partial_t u, \psi) + (\nabla u, \nabla\psi) = (f, \psi), \qquad \forall \psi \in H_0^1 \supset V_h.$$

By the defn of the Ritz projection

$$(\partial_t u - \partial_t R_h u + \partial_t R_h u, \psi) + (\nabla R_h u, \nabla \psi) = (f, \psi), \qquad \forall \psi \in V_h.$$

Hence, for all $\psi \in V_h$,

(20.6)
$$(\partial_t R_h u, \psi) + (\nabla R_h u, \nabla\psi) = (f, \psi) - (\partial_t \rho, \psi)$$

Next, subtract (20.6) from the Galerkin approx (20.3) to get

$$(\partial_t \theta, \psi) + (\nabla\theta, \nabla\psi) = (\partial_t \rho, \psi), \qquad \forall \psi \in V_h.$$

Now, since $\theta \in V_h$, set $\psi = \theta$ to get

$$(\partial_t \theta, \theta) + |\theta|_{H^1}^2 = (\partial_t \rho, \theta)$$

for all $t \in (0,T]$.

Observe that

$$(\partial_t \theta, \theta) = \frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 = \|\theta\|_{L^2}\frac{d}{dt}\|\theta\|_{L^2}$$

This is a neat little trick!

$$\|\theta\|_{L^2}\frac{d}{dt}\|\theta\|_{L^2} \leq (\partial_t \theta, \theta) + |\theta|_{H^1}^2$$

$$= (\partial_t \rho, \theta)$$

$$\overset{C.S.}{\leq} \|\partial_t \rho\|_{L^2}\|\theta\|_{L^2}$$

Assuming that $\theta(t) \not\equiv 0$, for all $t \in [0,T]$,

(20.7) $$\frac{d}{dt}\|\theta(t)\|_{L^2} \leq \|\partial_t \rho(t)\|_{L^2},$$

for all $t \in (0,T]$. Integrating,

(20.8) $$\|\theta(t)\|_{L^2} \leq \|\theta(0)\|_{L^2} + \int_0^t \|\partial_t \rho\|_{L^2}\, ds,$$

for any $t \in (0,T]$. But

$$\|\theta(0)\|_{L^2} = \|R_h v - v_h\|_{L^2}$$

$$= \|R_h v - v + v - v_h\|_{L^2}$$

$$\leq \|R_h v - v\|_{L^2} + \|v - P_h v\|_{L^2}$$

(20.9) $$\leq C h^{r+1} \|v\|_{H^{r+1}} + \|v - v_h\|_{L^2}$$

Likewise,

$$\|\partial_t \rho\|_{L^2} = \|\partial_t u - \partial_t R_h u\|_{L^2}$$

Here we have made the reasonable assumption that $R_h u$ is time-differentiable. This can be proven, for the assumed regularities above.

We also assume that

$$\partial_t R_h u = R_h \partial_t u.$$

This can also be shown for functions with the assumed regularities. See Thomée (2006).

Thus, from the properties of the Ritz projection,

(20.10) $$\begin{aligned} \|\partial_t \rho\|_{L^2} &= \|\partial_t u - R_h \partial_t u\|_{L^2} \\ &\leq C h^{r+1} \|\partial_t u\|_{H^{r+1}} \end{aligned}$$

Combining (20.8), (20.9), (20.10), we have

(20.11) $$\begin{aligned} \|\theta(t)\|_{L^2} \leq\ & C h^{r+1} \|v\|_{H^{r+1}} + \|v - v_h\|_{L^2} \\ & + C h^{r+1} \int_0^t \|\partial_t u\|_{H^{r+1}} \, ds \end{aligned}$$

Recall (20.6):

$$\begin{aligned} \|u - u_h(t)\|_{L^2} \leq\ & C h^{r+1} \left(\|v\|_{H^{r+1}} + \int_0^t \|\partial_t u\|_{H^{r+1}} \, ds \right) \\ & + \|\theta(t)\|_{L^2} \end{aligned}$$

Using (20.11), we have

$$\|u - u_h(t)\|_{L^2} \leq C h^{r+1} \left(\|v\|_{H^{r+1}} + \int_0^t \|\partial_t u\|_{H^{r+1}}\, ds \right) + \|v - v_h\|_{L^2},$$

which is the desired result (20.4). ///

Theorem (20.4): Suppose the hypotheses of Theorem (20.2) are satisfied. Assume that u is a solution to

$$(\partial_t u, \psi) + (\nabla u, \nabla \psi) = (f, \psi),$$

for all $\psi \in H_0^1(\Omega)$, for all $t \in (0,T]$, where $u(\cdot, 0) = v$, and u has the regularities

$$u \in L^2(0,T; H_0 \cap H^{r+1}) \cap H^1(0,T; H^{r+1}) \cap H^2(0,T; L^2) \implies \boxed{\int_0^T \|\partial_{tt} u\|_{L^2}^2 < \infty}$$

and

$$v \in H_0^1 \cap H^{r+1}.$$

Suppose $u_h^k \in V_h = \mathcal{M}_{0,r}$, $k = 0, \dots, K$, satisfies

(20.12) $$(\delta_s u_h^k, \psi) + (\nabla u_h^k, \nabla \psi) = (f, \psi)$$

for all $\psi \in V_h$, with

$$u_h^0 = v_h = P_h v \in V_h,$$
$$\delta_s u_h^k := \frac{u_h^k - u_h^{k-1}}{s},$$
$$s = \frac{T}{K}.$$

Suppose that

$$\|v - v_h\|_{L^2} \le C h^{r+1} \|v\|_{H^{r+1}}.$$

Then

(20.13)
$$\|u_h^k - u(t_k)\|_{L^2} \le C h^{r+1}\Big(\|v\|_{H^{r+1}} + \int_0^{t_k} \|\partial_t u\|_{H^{r+1}}\, d\tau\Big) + s \int_0^{t_k} \|\partial_t^2 u\|_{L^2}\, d\tau.$$

Proof: As before, set

$$\rho(t) := u(t) - R_h u(t)$$

$$\theta^k = R_h u(t_k) - u_h^k \qquad k = 0, 1, \dots, K,$$

where

$$t_k = s \cdot k, \quad k = 0, 1, \dots, K.$$

Then

$$u(t_k) - u_h^k = \rho(t_k) + \theta^k$$
$$= \rho^k + \theta^k,$$

where

$$\rho^k := \rho(t_k).$$

Note that θ^k is defined only at t_k, that is, at discrete times, whereas ρ is defined at all times.

As before

$$\|\rho(t_k)\|_{L^2} \leq C h^{r+1}\left(\|v\|_{H^{r+1}} + \int_0^{t_k} \|\partial_t u(\tau)\|_{H^{r+1}}\, d\tau\right),$$

for $k = 0, 1, \ldots, K$. By the defn of R_h,

$$(\partial_t u, \psi) + (\nabla R_h u, \nabla \psi) = (f, \psi), \quad \forall \psi \in V_h.$$

For $k = 1, \ldots, K$, and for all $\psi \in V_h$,

(20.14) $$(R_h(\delta_s u(t_k)), \psi) + (\nabla R_h u(t_k), \nabla \psi) = (f(t_k), \psi) + (w^k, \psi)$$

where

$$\delta_s u(t) := \frac{u(t) - u(t-s)}{s}, \quad t \in [s, T],$$

and

$$w(t) := R_h(\delta_s u(t)) - \partial_t u(t), \quad w^k := w(t_k).$$

Subtracting (20.12) (the Backward Euler-Galerkin scheme) from (20.14) to get

$$(\delta_s \theta^k, \psi) + (\nabla \theta^k, \nabla \psi) = (w^k, \psi), \quad \forall \psi \in V_h.$$

Next, set $\psi = \theta^k$, to get

$$(\theta^k - \theta^{k-1}, \theta^k) + s\,|\theta^k|_{H^1}^2 = s\,(w_k, \theta^k)$$

Then

$$\|\theta^k\|_{L^2}^2 - (\theta^{k-1}, \theta^k) + s\,|\theta^k|_{H^1}^2 = s\,(w^k, \theta^k)$$

$$\overset{C.S}{\leq} s\,\|w^k\|_{L^2}\|\theta^k\|_{L^2}$$

This implies

$$\|\theta^k\|_{L^2}^2 - \|\theta^{k-1}\|_{L^2}\|\theta^k\|_{L^2} \leq s\|w^k\|_{L^2}\|\theta^k\|_{L^2},$$

which, in turn, implies

$$\|\theta^k\|_{L^2} - \|\theta^{k-1}\|_{L^2} \leq s\|w^k\|_{L^2}$$

for $k = 1, 2, \cdots, K$. Summing from $k=1$ to $k = \ell$

(20.15) $$\|\theta^\ell\|_{L^2} \leq \|\theta^0\|_{L^2} + s\sum_{k=1}^{\ell}\|w^k\|_{L^2}$$

for any $1 \leq \ell \leq K$. Define

$$w_1^k := (R_h - I)\,\delta_s u(t_k),$$

$$w_2^k := \delta_s u(t_k) - \partial_t u(t_k).$$

Then

$$w^k = w_1^k + w_2^k.$$

It follows that

$$w_1^k = \frac{1}{s}(R_h - I)\int_{t_{k-1}}^{t_k}\partial_t u(\tau)\,d\tau$$

$$= \frac{1}{s}\int_{t_{k-1}}^{t_k}(R_h - I)\,\partial_t u(\tau)\,d\tau$$

Then

$$\|w_1^k\|_{L^2} \leq \frac{1}{s}\int_{t_{k-1}}^{t_k}\|R_h\partial_t u(\tau) - \partial_t u(\tau)\|_{L^2}\,d\tau$$

$$\leq \frac{1}{s}Ch^{r+1}\int_{t_{k-1}}^{t_k}\|\partial_t u(\tau)\|_{H^{r+1}}\,d\tau.$$

So

(20.16) $$\delta \sum_{k=1}^{\ell} \|w_1^k\|_{L^2} \leq C h^{r+1} \int_0^{t_\ell} \|\partial_t u(\tau)\|_{L^2}\, d\tau$$

To estimate w_2^k, note

$$\begin{aligned} \delta w_2^k &= u(t_k) - u(t_{k-1}) - \delta \partial_t u(t_k) \\ &= -\int_{t_{k-1}}^{t_k} (\tau - t_{k-1})\, \partial_{tt} u(\tau)\, d\tau, \end{aligned}$$

where we have used Taylor's Theorem with integral remainder. (Remember that?)

Thus,

$$\begin{aligned} \delta \sum_{k=1}^{\ell} \|w_2^k\|_{L^2} &\leq \sum_{k=1}^{\ell} \int_{t_{k-1}}^{t_k} |\tau - t_{k-1}| \cdot \|\partial_{tt} u(\tau)\|_{L^2}\, d\tau \\ &\leq \delta \int_0^{t_\ell} \|\partial_{tt} u(\tau)\|_{L^2}\, d\tau. \end{aligned}$$ (20.17)

Also, observe

$$\begin{aligned} \|\theta^0\|_{L^2} &= \|R_h v - U_h^0\|_{L^2} \\ &\leq \|R_h v - v\|_{L^2} + \|v - v_h\|_{L^2} \\ &\leq C h^{r+1} \|v\|_{H^{r+1}} \end{aligned}$$ (20.18)

Combining (20.15) — (20.18)

$$\|\theta^\ell\|_{L^2} \le \|\theta^0\|_{L^2} + s\sum_{k=1}^{\ell}\|w_1^k\|_{L^2} + s\sum_{k=1}^{\ell}\|w_2^k\|_{L^2}$$

$$\le C h^{r+1}\|v\|_{H^{r+1}} + C h^{r+1}\int_0^{t_\ell}\|\partial_t u(\tau)\|_{H^{r+1}}\,d\tau$$

$$+ s\int_0^{t_\ell}\|\partial_{tt} u(\tau)\|_{L^2}\,d\tau$$

Since

$$\|\rho^\ell\|_{L^2} \le C h^{r+1}\left(\|v\|_{H^{r+1}} + \int_0^{t_\ell}\|\partial_t u\|_{H^{r+1}}\,d\tau\right),$$

it follows that

$$\|u(t_k) - u_h^k\|_{L^2}$$

$$\le \|\theta^k\|_{L^2} + \|\rho^k\|_{L^2}$$

$$\le C h^{r+1}\left(\|v\|_{H^{r+1}} + \int_0^{t_k}\|\partial_t u(\tau)\|_{H^{r+1}}\,d\tau\right)$$

$$+ s\int_0^{t_k}\|\partial_{tt} u(\tau)\|_{L^2}\,d\tau \qquad ///$$