Math 574
class #03
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An Abstract Problem and the Lax Milgram Lemma

Defn: Let V be a Banach space. The mapping $f : V \to \mathbb{R}$ is called linear iff

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$$

for all $\alpha, \beta \in \mathbb{R}$ and all $u, v \in V$. Such a mapping is called bounded iff there is a number $M > 0$ such that

$$|f(v)| \leq M \|v\|$$

for all $v \in V$. The set of all bounded linear mappings from V to $\mathbb{R}$ is denoted V' and is called the dual space of V. (f is called a functional.)

Defn: Let V be a Banach space. A mapping $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ is called a bilinear form iff

$$a(\alpha u + \beta v, w) = \alpha a(u, w) + \beta a(v, w)$$

and

$$a(u, \alpha v + \beta w) = \alpha a(u, v) + \beta a(u, w)$$

for all $\alpha, \beta \in \mathbb{R}$ and all $u, v, w \in V$.

$a(\cdot,\cdot)$ is called <u>coercive</u> iff $\exists\, \alpha > 0$ such that

$$\alpha \|v\|_V^2 \leq a(v,v),$$

for all $v \in V$ and called <u>continuous</u> iff $\exists\, \gamma > 0$ such that

$$a(u,v) \leq \gamma \|u\|_V \|v\|_V$$

for all $u, v \in V$.

<u>The Abstract Galerkin Method</u>

Next, we want to generalize our model problem from earlier.

We have the following <u>abstract</u> <u>weak</u> <u>problem</u>.

Suppose that $a(\cdot,\cdot)$ is a continuous and coercive bilinear form and $f \in V'$ is given. Find $u \in V$ such that

$$a(u,v) = f(v) \qquad \forall\, v \in V. \tag{3.1}$$

Here V is a Banach space.

Let V_h be a finite dimensional subspace of V.

The <u>abstract</u> <u>Galerkin</u> <u>approx.</u> is defined as follows: find $u_h \in V_h$ such that

(3.2) $$a(u_h, v) = f(v) \qquad \forall v \in V_h$$

<u>Lax Milgram lemma</u>: Let V be a Hilbert space. Suppose $a(\cdot,\cdot)$ is a cont. and coercive (not necessarily symmetric) bilinear form on V. Let $f \in V'$ be given. Then, there exists a unique solution to (3.1).

<u>Proof</u>: For existence, see Brenner and Scott.

<u>Uniqueness</u>: Let $u_1, u_2 \in V$ be solutions to (3.1). Then

$$a(u_1 - u_2, v) = 0 \qquad \forall v \in V.$$

Hence

$$0 = a(u_1 - u_2, u_1 - u_2) \geq \alpha \|u_1 - u_2\|_V^2 \geq 0.$$

Thus, $u_1 = u_2$ ///

<u>Proposition</u>: With the hypothesis of the last Theorem (L-M Lemma), the solution $u \in V$ is stable in the sense that

(3.3) $$\|u\|_V \leq \frac{M}{\alpha}.$$

Proof: The solution $u \in V$ to (3.1) satisfies

$$a(u,v) = f(v) \qquad \forall\, v \in V$$

Thus

$$\begin{aligned} \alpha \|u\|_V^2 &\le a(u,u) \\ &= f(u) \\ &\le |f(u)| \\ &\le M \|u\|_V . \end{aligned}$$

If $u \neq 0$, then

$$\alpha \|u\|_V \le M .$$

If $u = 0$, then the ineq. is trivial. ///

Proposition: With the same hypotheses as for the last theorem (L-M lemma), if V_h is a finite dimensional subspace of V, then there is a unique solution $u_h \in V_h$ to (3.2). And, moreover, this solution is stable in the sense that

(3.4) $$\|u_h\|_V \le \frac{M}{\alpha} . \qquad \text{(stability)}$$

Proof: Exercise.

$$(V, (\cdot,\cdot)_V, \|\cdot\|_V) \qquad (V_h, (\cdot,\cdot)_V, \|\cdot\|_V)$$

(Abstract) Cea's Lemma (Non-symmetric case):
Suppose that $a(\cdot,\cdot)$ is a coercive, cont. bilinear form on V. Suppose $f \in V'$. Let V_h be a finite dimensional subspace of V, a Hilbert space. Let $u \in V$ and $u_h \in V_h$ be the solutions of (3.1) and (3.2), resp. Then,

(3.5) $$a(u-u_h, v) = 0, \qquad \forall\, v \in V_h, \qquad \text{(FGO)}$$

and

$$\|u-u_h\|_V \leq \frac{\gamma}{\alpha} \min_{v \in V_h} \|u-v\|_V .$$

Proof: Homework exercise. ///

The Symmetric Problem

Riesz Rep. Theorem: Let $(V,(\cdot,\cdot)_*)$ be a Hilbert space and let $f \in V'$. Then, $\exists!\, u = u_{f,*} \in V$ such that

$$f(v) = (u,v)_* \qquad \forall\, v \in V.$$

Proof: See Brenner and Scott. ///

Suppose $f \in V'$ is given and $a(\cdot,\cdot)$ is a symmetric, coercive and continuous bilinear form on V.

Consider the following problem: find $u \in V$ such that

(3.6) $$a(u,v) = f(v), \qquad \forall v \in V.$$

The <u>symmetric Galerkin approx</u> is as follows: find $u_h \in V_h$ such that

(3.7) $$a(u_h, v) = f(v), \qquad \forall v \in V_h.$$

<u>Proposition</u>: If $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a symmetric, coercive, and continuous bilinear form on V, then

1) $a(\cdot,\cdot)$ is an inner product on V

2) $\|v\|_E := \sqrt{a(v,v)}$ is a norm on V and is equivalent to the base norm, $\|\cdot\|_V$, ie, $\exists\, C_1, C_2 > 0$ such that

$$C_1 \|v\|_E \leq \|v\|_V \leq C_2 \|v\|_E$$

for all $v \in V$.

3) $(V, a(\cdot,\cdot))$ and $(V_h, a(\cdot,\cdot))$ are Hilbert Spaces, the latter being finite dim., topologically equivalent to $(V, (\cdot,\cdot)_V)$ and $(V_h, (\cdot,\cdot)_V)$.

Proof: Exercise. ///

Proposition: If $a(\cdot,\cdot)$ is a symmetric, coercive, continuous bilinear form on a Hilbert space $(V, (\cdot,\cdot)_V)$ and $f \in V'$, then problems (3.6) and (3.7) are uniquely solvable.

Proof: This follows from the RRT, since $(V, a(\cdot,\cdot))$ and $(V_h, a(\cdot,\cdot))$ are Hilbert spaces.

$$\exists!\ u \in V \text{ st } f(v) = a(u,v), \quad \forall v \in V,$$

$$\exists!\ u_h \in V_h \text{ st } f(v) = a(u_h,v), \quad \forall v \in V_h. \qquad ///$$

Cea's Lemma (Symmetric Case): With the same hypotheses as in the last Prop.,

(FGO) $$a(u-u_h, v) = 0, \qquad \forall v \in V_h,$$

and

$$\|u-u_h\|_E = \min_{v \in V_h} \|u-v\|_E.$$

Recall

$$\|v\|_E := \sqrt{a(v,v)} \qquad \forall v \in V.$$

Proposition: With the same hypotheses again, define

$$G(v) := \frac{1}{2} a(v,v) - f(v) \qquad \forall v \in V$$

Thus, the problems,

$$u = \operatorname*{argmin}_{v \in V} G(v)$$

$$u_h = \operatorname*{argmin}_{v \in V_h} G(v)$$

are equivalent to problems (3.6) and (3.7), resp. Thus these problems are uniquely solvable.

Proof: A fun exercise. ///

Multi Indices

Defn: A multi-index of dimension d is a d-tuple

$$\alpha = (\alpha_1, \dots, \alpha_d)$$

with

$$\alpha_i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

We write

$$\alpha \in \mathbb{N}_0^d.$$

The quantity

$$|\alpha| := \sum_{i=1}^{d} \alpha_i$$

is called the <u>order</u> of α.

<u>Example</u>: With $\vec{x} \in \mathbb{R}^d$ we use the notation

$$\vec{x}^{\alpha} := \prod_{i=1}^{d} x_i^{\alpha_i}$$

<u>Example</u>: let $p: \mathbb{R}^d \to \mathbb{R}$ be a polynomial of degree $n \in \mathbb{N}_0$ or less. We can write

$$p(\vec{x}) = \sum_{|\alpha| \leq n} a_\alpha \vec{x}^{\alpha}$$

where $a_\alpha \in \mathbb{R}$ is called the coefficient of multi index α.

<u>Example</u>: Suppose $p: \mathbb{R}^2 \to \mathbb{R}$ is a poly. of degree $n = 3$.

$$\begin{aligned}
p(\vec{x}) &= \sum_{|\alpha| \leq 3} a_\alpha x^{\alpha} \\
&= a_{00} \\
&\quad + a_{10} x_1 + a_{01} x_2 \\
&\quad + a_{20} x_1^2 + a_{11} x_1 x_2 + a_{02} x_2^2 \\
&\quad + a_{30} x_1^3 + a_{21} x_1^2 x_2 + a_{12} x_1 x_2^2 + a_{03} x_2^3
\end{aligned}$$