Math 574
class #04
8/27/2025

Weak Derivatives

Let $v \in C^r(\Omega;\mathbb{R})$, $\Omega \subseteq \mathbb{R}^d$ open. Then, if $|\alpha| \le r$,

$$\partial^\alpha v := \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}} \in C(\Omega)$$

Example: $d = 3$ $\quad |\alpha| = 4$ $\quad \alpha = (2,1,1)$.

$$\partial^\alpha v = \frac{\partial^4 v}{\partial x_1^2 \partial x_2 \partial x_3}$$

Clearly, we have

$$C^m(\Omega) = \{ v \in C(\Omega) \mid \partial^\alpha v \in C(\Omega),\ 1 \le |\alpha| \le m \}$$
$$C^m(\overline{\Omega}) = \{ v \in C(\overline{\Omega}) \mid \partial^\alpha v \in C(\overline{\Omega}),\ 1 \le |\alpha| \le m \}$$

$$C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega) \qquad C^\infty(\overline{\Omega}) = \bigcap_{m=0}^{\infty} C^m(\overline{\Omega})$$

Defn: Let $\Omega \subseteq \mathbb{R}^d$ be an open set. Suppose $f \in C(\Omega)$. Then,

$$\operatorname{supp}(f) := \overline{\{ \vec{x} \in \Omega \mid f(\vec{x}) \neq \vec{0} \}}$$

supp(f) is called the <u>support</u> of f.

We say that f has <u>compact support</u> iff supp(f) is bounded and

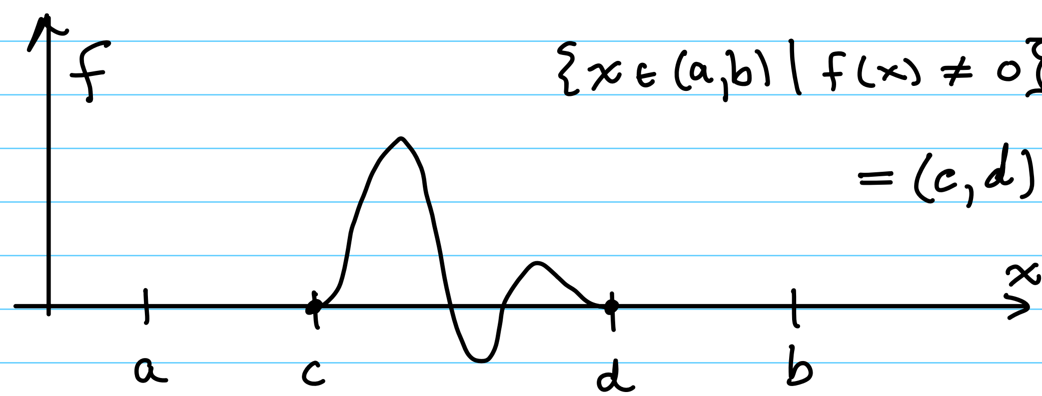
$$\mathrm{supp}(f) \subset \Omega$$

proper subset.

<u>Example</u>: $\Omega = (a, b)$ and

$$-\infty < a < c < d < b < \infty$$

Suppose f has the graph



$$\{x \in (a,b) \mid f(x) \neq 0\} = (c,d)$$

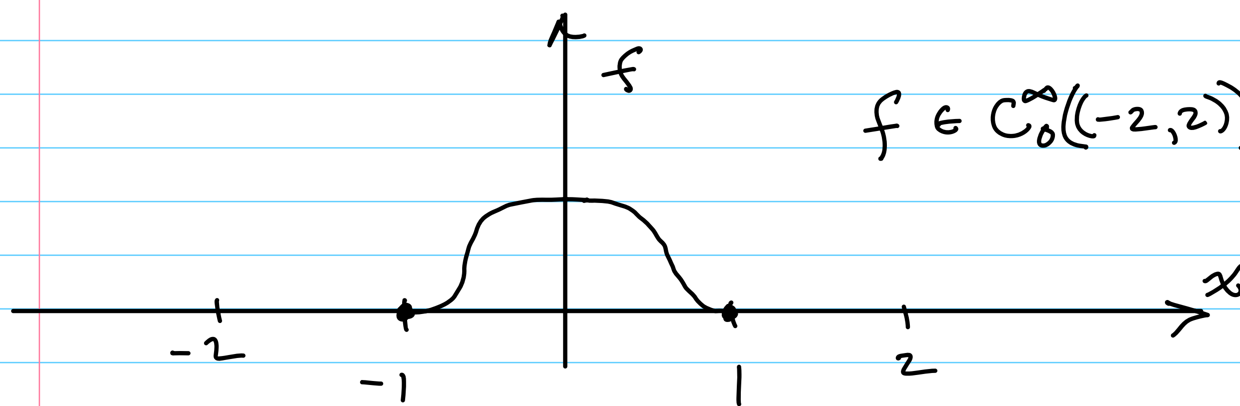
$$\mathrm{supp}(f) = [c,d] \subset (a,b)$$

<u>Defn</u>: Suppose $\Omega \subseteq \mathbb{R}^d$ is open (and, often, bounded).

$$C_0^\infty(\Omega) := \{v \in C^\infty \mid v \text{ has compact support}\}$$

Example: Suppose $\Omega = (-2,2)$.

$$f(x) = \begin{cases} \exp\left(\frac{1}{x^2-1}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$



$f \in C_0^\infty((-2,2))$

Defn: Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}$ open.

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \,\middle|\, f \text{ is meas and } \int_\Omega |f|^p \, d\vec{x} < \infty \right\}$$

Theorem: L^p-spaces, $1 \leq p < \infty$, are Banach Spaces with the norms

$$\|f\|_{L^p} := \left(\int_\Omega |f|^p \, d\vec{x} \right)^{1/p}$$

L^2 is a Hilbert space with the inner product

$$(f,g)_{L^2} = \int_\Omega f(\vec{x}) g(\vec{x}) \, d\vec{x}$$

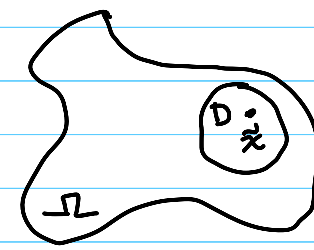
Usually Ω is open and bounded.

Defn: Let $1 \le p < \infty$. A function $v : \Omega \subseteq \mathbb{R}^d \to \mathbb{R}$ is called <u>locally p-integrable</u>, $v \in L^p_{loc}(\Omega)$, iff, for every $\vec{x} \in \Omega$, there is an open set D such that $\vec{x} \in D$,

$$\bar{D} \subset \Omega,$$

and

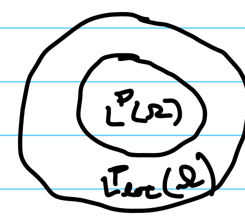
$$v \in L^p(D).$$



Remark: L^p_{loc} functions can behave badly near a point, or boundary, et cetera.

$$L^p(\Omega) \subset L^p_{loc}(\Omega)$$

$$L^p_{loc}(\Omega) \not\subset L^p(\Omega)$$



Lemma (4.1): Let $v \in L^1_{loc}(\Omega)$, where $\Omega \subseteq \mathbb{R}^d$ is a nontrivial open set. If

$$\int_\Omega v(\vec{x})\, \phi(\vec{x})\, d\vec{x} = 0, \quad \forall \phi \in C_0^\infty(\Omega),$$

then

$$v = 0 \text{ a.e. in } \Omega.$$

Defn: Let Ω be a nontrivial open set in $\mathbb{R}^d$ and assume $v, w \in L^1_{loc}(\Omega)$. Let $\alpha \in \mathbb{N}_0^d$ be a multi-index. w is called the <u>weak α partial derivative</u> of v iff

$$\int_\Omega v\, \partial^\alpha \phi\, d\vec{x} = (-1)^{|\alpha|} \int_\Omega w\, \phi\, d\vec{x} \tag{4.1}$$

for all $\phi \in C_0^\infty(\Omega)$. We write

$$\partial_w^\alpha v = w.$$

<u>Lemma</u> (4.2): A weak derivative, if it exists, is unique up to a set of measure zero.

<u>Proof</u>: Suppose $v \in L^1_{loc}(\Omega)$ has two weak α partial derivatives

$$\partial_w^\alpha v = w_1$$

$$\partial_w^\alpha v = w_2$$

Then, it follows that

$$\int_\Omega [w_1 - w_2]\, \phi(\vec{x})\, d\vec{x} = 0 \qquad \forall\, \phi \in C_0^\infty(\Omega).$$

By lemma (4.1), $w_1 = w_2$ a.e. in Ω. ///

<u>Remark</u>: If $v \in C^m(\Omega)$, then for each $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le m$, the classical partial derivative $\partial^\alpha v$ is also the weak partial derivative $\partial_w^\alpha v$.

<u>Example</u>: $d = 2$ $\Omega \subseteq \mathbb{R}^2$ is open and bounded.

$$\int_\Omega v \frac{\partial \phi}{\partial x_j}\, dx_1 dx_2 = \cancelto{0}{\int_{\partial\Omega} n_j v \phi\, ds} - \int_\Omega \frac{\partial v}{\partial x_j} \phi\, dx_1 dx_2.$$

Proposition: Suppose that $v \in C^0([a,b])$. Let P be a partition of $[a,b]$, i.e.,

$$P = \{x_0, x_1, \dots, x_n\}$$

where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Assume that, for all $k = 1, \dots, n$,

$$v \in C^1([x_{k-1}, x_k]).$$

Then v is weakly differentiable. The first order weak derivative is

$$w_*(x) = \begin{cases} v'(x), & x \in \bigcup_{k=1}^{n} (x_{k-1}, x_k) \\ 1 & x \in P \end{cases}$$

Proof: Let us establish this for a simple case. Suppose $[a,b] = [0,1]$ and $P = \{0, 1/2, 1\}$.

$$\int_0^1 v(x)\,\phi'(x)\,dx = \int_0^{1/2} v(x)\,\phi'(x)\,dx + \int_{1/2}^{1} v(x)\,\phi'(x)\,dx$$

$$\overset{\text{IBP}}{=} v\phi\Big|_0^{1/2} - \int_0^{1/2} v'(x)\,\phi(x)\,dx$$

$$+ v\phi\Big|_{1/2}^{1} - \int_{1/2}^{1} v'(x)\phi(x)\,dx$$

$$= v(1/2)\phi(1/2) - \int_0^{1/2} v'(x)\phi(x)\,dx$$

$$- v(1/2)\phi(1/2) - \int_{1/2}^{1} v'(x)\phi(x)\,dx$$

$$= -\int_0^1 w_*(x)\phi(x)\,dx.$$

Since w_* is $L^1_{loc}(0,1)$, the result is confirmed ///

Proposition: Suppose $\Omega \subseteq \mathbb{R}^d$ is open and $u, v \in L^1_{loc}(\Omega)$. If, for $\alpha \in \mathbb{N}_0^d$, $\partial_w^\alpha u$ and $\partial_w^\alpha v$ exist, then, for any $\alpha, \beta \in \mathbb{R}$, $\partial_w^\alpha(\alpha u + \beta v)$ exists and

$$\partial_w^\alpha(\alpha u + \beta v) = \alpha \partial_w^\alpha u + \beta \partial_w^\alpha v$$

Proof: Exercise.

Proposition: Let $p, q \in (1, \infty)$, related by

$$1/p + 1/q = 1$$

Assume $u, \partial_w^\alpha u \in L^p_{loc}(\Omega)$ and $v, \partial_w^\alpha v \in L^q_{loc}(\Omega)$ with $|\alpha| = 1$. Then $\partial_w^\alpha(uv)$ exists and

$$\partial_w^\alpha(uv) = \partial_w^\alpha u \cdot v + u\,\partial_w^\alpha v.$$