Math 574
class #17
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Argyris Error Estimates for Biharmonic Problem

Interpolation Estimate:

$$\|u - \Pi_h u\|_{H^m} \le C h^{6-m} |u|_{H^6(\Omega)}$$

for any $u \in H^6(\Omega)$, and $m \le 5$. For the problem at hand, $m = 2$. Thus,

$$\|u - \Pi_h u\|_{H^2} \le C h^4 |u|_{H^6(\Omega)}.$$

Cea's lemma guarantees that

$$\|u - u_h\|_{H_0^2(\Omega)} = \min_{v \in V_h} \|u - v\|_{H_0^2(\Omega)}$$

Now

$$H^6(\Omega) \hookrightarrow C^{6-1-1}(\overline{\Omega})$$

$$= C^4(\overline{\Omega})$$

since

$$6 = k > \frac{d}{2} = \frac{2}{2}.$$

Thus $\Pi_h u$ is well-defined for $u \in H^6$. So

$$\|u - u_h\|_{H_0^2(\Omega)} = \min_{v \in V_h} \|u - v\|_{H_0^2(\Omega)}$$

$$\le \|u - \Pi_h u\|_{H_0^2(\Omega)}$$

Combining with interpolation error estimates,

$$\|u-u_h\|_{H_0^2(\Omega)} \leq C h^4 |u|_{H^6(\Omega)} .$$

Assuming elliptic regularity, we can use a Nitsche-like trick to get

$$\|u-u_h\|_{L^2(\Omega)} \leq C h^6 |u|_{H^6(\Omega)} \quad ///$$

FEM for Parabolic Problems

<u>Diffusion Problem</u>: Find $u \in C^2(\bar{\Omega} \times [0,T])$ such that

(17.1) $$\begin{cases} \partial_t u - \Delta u = f(t), & \text{in } \Omega, \ 0<t\leq T, \\ u = 0, & \text{on } \partial\Omega \ 0<t\leq T, \\ u(\cdot,0) = v, & \text{in } \Omega. \end{cases}$$

<u>Definition</u> (17.1): The dual space of $H_0^1(\Omega)$ is denoted $H^{-1}(\Omega)$ and we write

$$H^{-1}(\Omega) := \left(H_0^1(\Omega)\right)'$$

Recall that the dual space is the space of all bounded linear functionals acting on a Hilbert (Banach) space

One can show that the dual space is also a Hilbert (Banach) space.

Usually the dual space is equipped with the operator norm:

$$\|f\|_{H'} = \sup_{\substack{\phi \in H \\ \phi \neq 0}} \frac{|f(\phi)|}{\|\phi\|_H}$$

For the present case

$$\|f\|_{H^{-1}} = \sup_{0 \neq \phi \in H_0^1} \frac{|f(\phi)|}{|\phi|_{H^1}}$$

Defn (17.2): Let H be a Hilbert space and let $f \in H'$ be arbitrary the notation

$$\langle \cdot, \cdot \rangle : H' \times H \to \mathbb{R}$$

is called the dual pairing and is defined as

$$\langle f, \phi \rangle := f(\phi) \qquad \forall \phi \in H_0$$

Let us write out the weak formulation of the diffusion problem (17.1).

Defn (17.3): Let $f \in L^2(0,T; L^2)$ and $v \in L^2(\Omega)$ be given. u is called a weak solution to the diffusion problem iff

1) $u \in L^2(0,T; H_0^1)$

2) u is weakly differentiable in time and

$$\partial_t u \in L^2(0,T; H^{-1})$$

3) For almost every $t \in [0,T]$ and for every $\phi \in H_0^1(\Omega)$

(17.2) $$\langle \partial_t u, \phi \rangle + (\nabla u, \nabla v) = (f, \phi)$$

What are these spaces

$$L^2(0,T; X)?$$

Defn (17.4): Let X be a Banach space with norm $\|\cdot\|_X$. The function space

$$L^p(0,T; X)$$

consists of all "strongly measurable" functions

$$\phi : [0,T] \longrightarrow X$$

such that

$$\|\phi\|_{L^p(0,T;X)} := \left[\int_0^T \|\phi(t)\|_X^p \, dt \right]^{1/p} < \infty.$$

Theorem (17.5) Suppose that u is a weak solution to the diffusion problem. Then

$$u \in C([0,T]; L^2)$$

and

$$\max_{0\le t\le T} \|u(t)\|_{L^2}^2 \le \|v\|_{L^2}^2 + C\int_0^T \|f\|_{L^2}^2\, dt$$

and

$$\int_0^T \|u(t)\|_{H_0^1}^2\, dt \le \|v\|_{L^2}^2 + C\int_0^T \|f\|_{L^2}^2\, dt.$$

If u has additional regularities

$$u \in L^2(0,T; H_0^1 \cap H^2) \qquad \partial_t u \in L^2(0,T; L^2)$$

with $v \in H_0^1(\Omega)$, then

$$u \in C([0,T]; H_0^1)$$

and

$$\max_{0\le t\le T} \|u(t)\|_{H_0^1}^2 \le \|v\|_{H_0^1}^2 + \int_0^T \|f\|_{L^2}^2\, dt$$

$$\int_0^T \|\Delta u\|_{L^2}^2\, dt \le \|v\|_{H_0^1}^2 + \int_0^T \|f\|_{L^2}^2\, dt$$

Proof: See Evans (2010) book on PDE for the proof that $u \in C([0,T]; L^2)$.

Now, in the weak formulation, set $\phi = u$. Then

$$\langle \partial_t u, u\rangle + (\nabla u, \nabla u) = (f, u).$$

From Evans (2010) we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 = \langle \partial_t u, u\rangle$$

Since

$$u \in L^2(0,T;H_0^1), \quad \partial_t u \in L^2(0,T;H^{-1}).$$

Thus

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|u\|_{H_0^1}^2 = (f,w)$$

AGMI

$$\boxed{\begin{array}{c} ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2 \\ ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2 \\ \forall \varepsilon > 0 \end{array}}$$

$$\overset{C.S.}{\le} \|f\|_{L^2}\|u\|_{L^2}$$

$$\overset{\text{Poincare}}{\le} C\|f\|_{L^2}\|u\|_{H_0^1}^2$$

$$\overset{AGMI}{\le} C\left(\frac{C}{2}\|f\|_{L^2}^2 + \frac{1}{2C}\|u\|_{H_0^1}^2\right)$$

$$= \frac{C}{2}\|f\|_{L^2}^2 + \frac{1}{2}\|u\|_{H_0^1}^2$$

Rearranging terms and multiplying by 2, we get

$$\frac{d}{dt}\|u\|_{L^2}^2 + \|u\|_{H_0^1}^2 \le C\|f\|_{L^2}^2$$

Integrating in t from 0 to $s \in [0,T]$, we have

$$\|u(s)\|_{L^2}^2 - \|u(0)\|_{L^2}^2 + \int_0^s \|u(t)\|_{H_0^1}^2\,dt$$

$$\le C\int_0^s \|f(t)\|_{L^2}^2\,dt$$

Thus

$$\|u(s)\|_{L^2}^2 + \int_0^s \|u(t)\|_{H_0^1}^2\,dt \le \|v\|_{L^2}^2 + C\int_0^s \|f(t)\|_{L^2}^2\,dt.$$

This implies that

$$\max_{0\le t\le T} \|u(t)\|_{L^2}^2 + \int_0^T \|u(t)\|_{H_0^1}^2\,dt$$
$$\le \|v\|_{L^2}^2 + C\int_0^T \|f(t)\|_{L^2}^2\,dt .$$

Again, we skip the proof that $u\in C([0,T];H_0^1)$ from its assumed better regularity. We refer again to Evans (2010).

Since $u\in H_0^1\cap H^2$, we have, for all $\phi\in H_0^1$,

$$(\partial_t u,\phi) - (\Delta u,\phi) = (f,\phi).$$

Set $\phi = -\Delta u$:

$$-(\partial_t u,\Delta u) + (\Delta u,\Delta u) = -(f,\Delta u)$$

Integrating by parts again,

$$(\partial_t \nabla u,\nabla u) + \|\Delta u\|_{L^2}^2 = -(f,\Delta u)$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 = -(f,\Delta u)$$
$$\overset{C.S.}{\le} \|f\|_{L^2}\|\Delta u\|_{L^2}$$
$$\overset{AGMI}{\le} \frac{1}{2}\|f\|_{L^2}^2 + \frac{1}{2}\|\Delta u\|_{L^2}^2$$

Thus

$$\frac{d}{dt}\|u\|_{H_0^1}^2 + \|\Delta u\|_{L^2}^2 \le \|f\|_{L^2}^2$$

Integrating in t we get the other two estimates as well. ///

Special Case

Suppose that $f \equiv 0$. Then

$$\max_{0 \le t \le T} \|u(t)\|_{L^2}^2 \le \|v\|_{L^2}^2$$

$$\max_{0 \le t \le T} \|u(t)\|_{H_0^1}^2 \le \|v\|_{L^2}^2$$

In this case, it is easy to show that

$$\|u(t_2)\|_{L^2}^2 \le \|u(t_1)\|_{L^2}^2$$

$$\|u(t_2)\|_{H_0^1}^2 \le \|u(t_1)\|_{L^2}^2$$

for all $0 \le t_1 \le t_2 \le T$.

Continuous Dependence on Initial Data

Suppose that u_1, u_2 solve

$$\partial_t u_i - \Delta u_i = f \quad \text{in } \Omega \quad 0 < t \le T$$

$$u_i = 0 \quad \text{on } \partial\Omega \quad 0 < t \le T$$

$$u_i(0) = v_i \quad \text{in } \Omega$$

Suppose that both satisfy the diffusion problem in the weak sense. Then, $e := u_1 - u_2$ is a weak solution to

$$\partial_t e - \Delta e = 0, \quad \text{in } \Omega \quad 0 < t \leq T,$$

$$e = 0, \quad \text{on } \partial\Omega \quad 0 < t \leq T,$$

$$e(0) = v_2 - v_1, \quad \text{in } \Omega.$$

and

$$\max_{0 \leq t \leq T} \|e\|_{L^2}^2 = \max_{0 \leq t \leq T} \|u_2(t) - u_1(t)\|_{L^2}^2$$

$$\leq \|v_2 - v_1^0\|_{L^2}^2.$$

If additional regularities hold,

$$\max_{0 \leq t \leq T} \|e\|_{H_0^1}^2 = \max_{0 \leq t \leq T} \|u_2(t) - u_1(t)\|_{H_0^1}^2$$

$$\leq \|v_2 - v_1^0\|_{H_0^1}^2.$$

Semi-Discrete FEM for Diffusion Problem

Suppose that Ω is a polygonal domain in $\mathbb{R}^2$. Let

$$V_h = \mathcal{M}_{r,0} \subset H_0^1(\Omega),$$

with basis

$$B = \{\phi_1, \cdots, \phi_{N_r^0}\}$$

Defn (17.6): The Galerkin approximation of the of the weak form of the diffusion problem is defined as follows:

Find $u_h : [0,T] \longrightarrow V_h = \mathcal{M}_{0,r}$, in the form

$$u_h(\cdot, t) = \sum_{i=1}^{N_r^0} u_i(t)\,\phi_i$$

such that

(17.3) $$(\partial_t u_h, \psi)_{L^2} + (\nabla u_h, \nabla \psi) = (f, \psi)$$

for all $\psi \in V_h$, for all $t \in (0,T]$, with

(17.4) $$u_h(0) = v_h := P_h v \in V_h.$$

Here P_h is some projection operator from L^2 to V_h, ie,

$$P_h : L^2 \longrightarrow V_h$$

and

$$P_h(P_h v) = P_h v \qquad \forall\, v \in L^2.$$

We will discuss some possibilities.

The defn above is equivalent to

(17.5) $$(\partial_t u_h(t), \phi_i) + a(u_h(t), \phi_i) = (f(t), \phi_i),$$

for $i = 1, \cdots, N_r^0$, for all $t \in (0,T]$.

Proposition (17.7): Define

$$G = [g_{i,j}]_{i,j=1}^{N_r^o} \qquad A = [a_{i,j}]_{i,j=1}^{N_r^o}$$

via

$$g_{i,j} := (\phi_j, \phi_i)_{L^2}$$

and

$$a_{i,j} := a(\phi_j, \phi_i)_{L^2}$$

for all $1 \le i,j \le N_r^o$. Define $\vec{f} : [0,T] \to \mathbb{R}^{N_r^o}$ via

$$\vec{f}(t) = [f_i(t)]_{i=1}^{N_r^o}, \quad f_i(t) := (f(t), \phi_i)_{L^2}.$$

Finally, define $\vec{v} = [v_i] \in \mathbb{R}^{N_r^o}$ via

$$v_i := (v_h, \phi_i)_{L^2}.$$

Let $u_h : [0,T] \to M_{0,r}$ be a strong ODE solution to (17.3) subject to the initial conditions (17.4), then

(17.6) $$G \vec{u}'(t) + A \vec{u}(t) = \vec{f}(t)$$

where

(17.7) $$u_h(\cdot, t) = \sum_{i=1}^{N_r^o} u_i(t) \phi_i$$

and

(17.8) $$\vec{u}(t) = [u_i(t)] \in \mathbb{R}^{N_r^o},$$

subject to the initial condition

(17.9) $$G \vec{u}(0) = \vec{v} \in \mathbb{R}^{N_r^o}.$$

Proof: Let us start with (17.5). Inserting (17.7) into (17.5), we get, for $i = 1, \dots, N_r^o$.

$$\sum_{j=1}^{N_r^0} (\phi_j, \phi_i) \frac{du_j}{dt}(t) + \sum_{j=1}^{N_r^0} a(\phi_j, \phi_i) u_j(t) = f_i(t).$$

In other words,

$$\sum_{j=1}^{N_r^0} g_{j,i}\, u_j'(t) + \sum_{j=1}^{N_r^0} a_{j,i} u_j(t) = f_i(t),$$

which is the component form of (17.6).

Finally, using (17.4), for $1 \le i \le N_r^0$,

$$(u_h(0), \phi_i)_{L^2} = (v_h, \phi_i)_{L^2}$$

This implies that

$$\begin{aligned}(v_h, \phi_i)_{L^2} &= \Big(\sum_{j=1}^{N_r^0} u_j(0)\, \phi_j, \phi_i \Big)_{L^2} \\ &= \sum_{j=1}^{N_r^0} g_{i,j}\, u_j(0) \\ &= \left[G\, \vec{u}(0) \right]_i .\end{aligned}$$

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