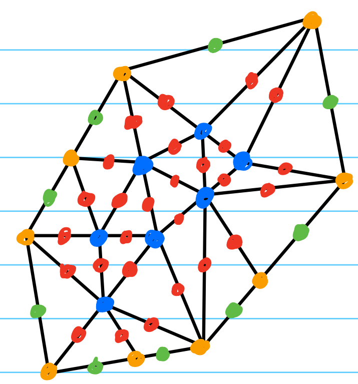
Math 574
class #09
09/16/2025

Nodal Points in 2D FEM



Defn: Let $\mathcal{T}_h = \{K\}$ be a triangulation of the polygonal domain $\Omega \subset \mathbb{R}^2$. Define

$$\mathcal{V} := \{\vec{x} \in \overline{\Omega} \mid \vec{x} \text{ is a vertex of } K, \ \exists K \in \mathcal{T}_h\}$$

This is called the vertex set. Define

$$\mathcal{V}^\circ := \mathcal{V} \cap \Omega.$$

This is the interior vertex set. The elements of $\mathcal{V}^\circ$ are called interior vertices. Define

$$\partial\mathcal{V} := \mathcal{V} \setminus \mathcal{V}^\circ$$

This is the set of boundary vertices.

Now, define the set of edge midpoint nodes:

$$\mathcal{E} := \left\{\vec{x} = \frac{\vec{x}_1+\vec{x}_2}{2} \ \middle|\ \vec{x}_1 \neq \vec{x}_2, \ \vec{x}_1, \vec{x}_2 \in \mathcal{V}\cap K, \ \exists K \in \mathcal{T}_h\right\}$$

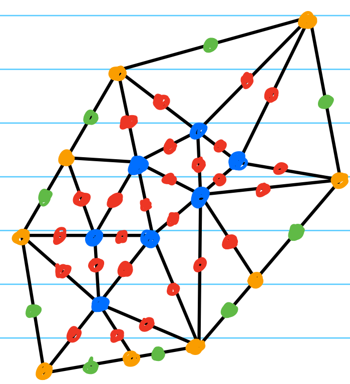
We define also,

$\bullet$ $\quad \mathcal{E}^\circ := \mathcal{E} \cap \Omega$ (interior edge midpoint nodes)

and

$\bullet$ $\quad \partial\mathcal{E} := \mathcal{E} \setminus \mathcal{E}^\circ$ (boundary edge midpoint nodes.)

Example:



$\mathcal{V}^\circ = \{\bullet\}$

$\partial\mathcal{V} = \{\bullet\}$

$\mathcal{E}^\circ = \{\bullet\}$

$\partial\mathcal{E} = \{\bullet\}$

Defn: Let $\mathcal{T}_h = \{K\}$ be a triangulation of the polygonal open set $\Omega \subset \mathbb{R}^2$. We define the piecewise polynomial finite element spaces, for $r \in \mathbb{N}$, via

$$\mathcal{M}_r := \{ v \in C^0(\bar{\Omega}) \mid v|_K \in \mathbb{P}_r(K) \ \forall K \in \mathcal{T}_h \} \overset{ss}{\subseteq} H^1(\Omega)$$

and

$$\mathcal{M}_{0,r} := \{ v \in \mathcal{M}_r \mid v|_{\partial\Omega} \equiv 0 \} \overset{ss}{\subseteq} H^1_0(\Omega)$$

Theorem: Let $\mathcal{T}_h = \{K\}$ be a triangulation of $\Omega \subset \mathbb{R}^2$. Suppose

$$v|_{\bar{K}} \in C^\infty(\bar{K}) \quad \forall K \in \mathcal{T}_h$$

Then $v \in H^l(\Omega)$ iff $v \in C^{l-1}(\bar{\Omega})$.

Proof: Exercise. See also Atkinson and Han ///

Key Concept

Remark: $\mathcal{M}_r$ ($\mathcal{M}_{0,r}$) is a non-trivial, proper subspace of $H^1(\Omega)$ ($H^1_0(\Omega)$) by the last theorem.

Lagrange Nodal Bases

Let's build bases for $\mathcal{M}_1$ ($\mathcal{M}_{0,1}$) and $\mathcal{M}_2$ ($\mathcal{M}_{0,2}$).

Defn: Let $\mathcal{T}_h = \{K\}$ be a triangulation of the open polygonal domain $\Omega \subset \mathbb{R}^2$. Define

$$\mathcal{N}_1 := \mathcal{V}, \quad N_1 := \#(\mathcal{N}_1);$$

$$\mathcal{N}_2 := \mathcal{V} \cup \mathcal{E}, \quad N_2 := \#(\mathcal{N}_2);$$

$$\left.\begin{aligned} \mathcal{N}_{0,1} &:= \mathcal{V}^\circ, \quad N_1^\circ := \#(\mathcal{N}_{0,1}); \\ \mathcal{N}_{0,2} &:= \mathcal{V}^\circ \cup \mathcal{E}^\circ, \quad N_2^\circ := \#(\mathcal{N}_{0,2}). \end{aligned}\right\} \text{(Model Prob)}$$

Define $\phi_{r,i} \in \mathcal{M}_r$, $1 \le i \le N_r$, $r = 1, 2$,

$$\phi_{r,i}(\vec{z}_j) = \delta_{i,j}, \quad 1 \le i, j \le N_r, \quad \mathcal{N}_r = \{\vec{z}_j\}_{j=1}^{N_r}$$

Define $\phi^\circ_{r,i} \in \mathcal{M}_{0,r}$, $1 \le i \le N_r$, $r = 1, 2$,

(Model Prob.)

$$\phi^\circ_{r,i}(\vec{z}_j) = \delta_{i,j}, \quad 1 \le i, j \le N_r^\circ, \quad \mathcal{N}_{0,r} = \{\vec{z}_j\}_{j=1}^{N_r^\circ}$$

The functions $\phi_{r,i}$, $\phi^\circ_{r,i}$ are called <u>Lagrange nodal basis functions</u>.

Proposition: Let $\mathcal{T}_h = \{K\}$ be a triangulation of the open, polygonal set $\Omega \subseteq \mathbb{R}^2$. The set

$$B_r = \{\phi_{r,i}\}_{i=1}^{N_r}, \quad r = 1, 2,$$

is a basis for M_r and the set

$$B_{0,r} = \{\phi^0_{r,i}\}_{i=1}^{N^0_r}, \quad r = 1, 2$$

is a basis for $M_{0,r}$.

Proof: Exercise ///

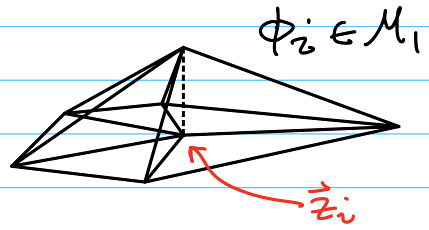
Corollary:

$$\dim(M_r) = N_r = \# \text{ of nodes}$$

$$\dim(M_{0,r}) = N^0_r = \# \text{ of interior nodes.}$$

What do the basis functions look like?

r = 1 :



$\boxed{\text{piecewise linear basis elements}}$

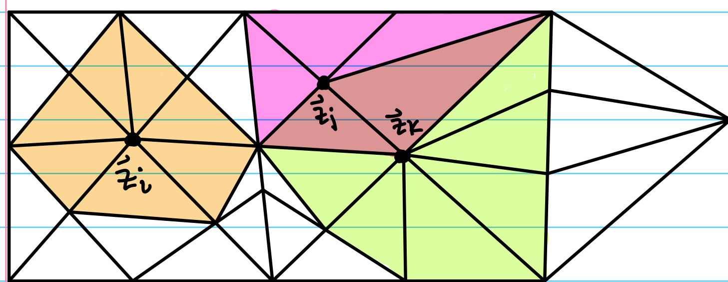
Observe that

$$\operatorname{supp}(\phi_i) = \bigcup_{\substack{K \in \mathcal{T}_h \\ \vec{z}_i \in \bar{K}}} \bar{K} = \text{union of the triangles that have } \vec{z}_i \text{ as a vertex.}$$

Recall

$$a_{i,j} = [A]_{i,j} = a(\phi_j, \phi_i)$$

Consider the triangulation



$$\text{meas}(\text{supp}(\phi_i) \cap \text{supp}(\phi_j)) = 0$$

2D measure

$$\Rightarrow \quad a(\phi_j, \phi_i) = 0 \quad \text{(generically)}$$

$$\text{meas}(\text{supp}(\phi_j) \cap \text{supp}(\phi_K)) \neq 0$$

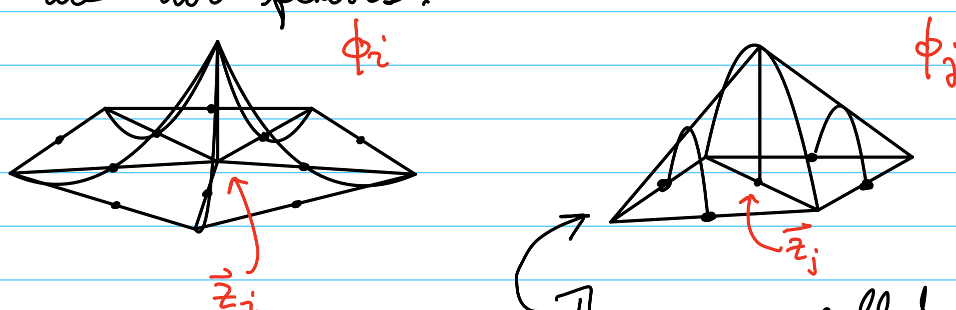
$$\Rightarrow \quad a(\phi_K, \phi_j) \text{ not generically zero}$$

(this does not guarantee it is nonzero)

What do the piecewise quadratic basis functions look like?

There are two flavors!

$r = 2$:



These are called bubble functions.

__Defn:__ let

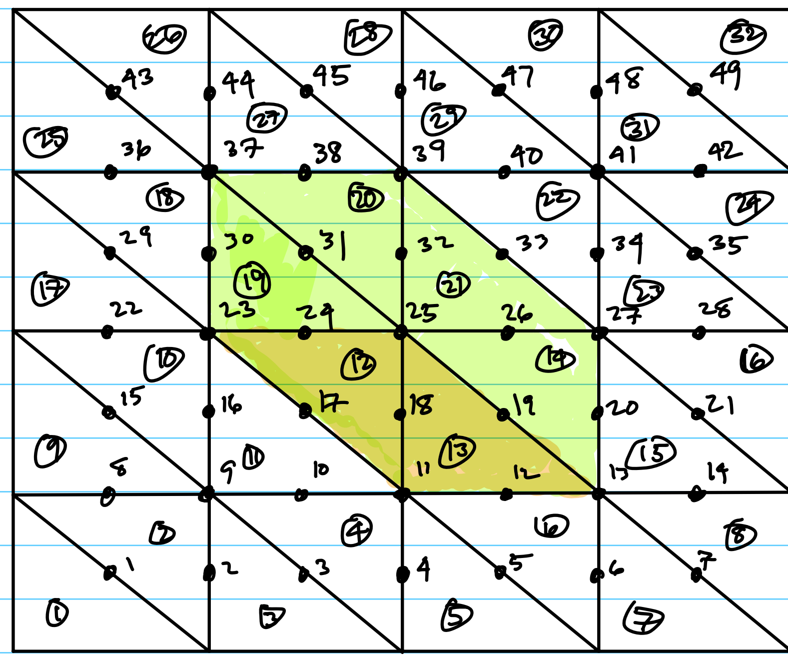
$$\{\vec{z}_j\}_{j=1}^{N_2^\circ} = \mathcal{V}^\circ \cup \mathcal{E}^\circ$$

The coordination number $cn(\vec{z}_j)$ is the number of triangles of $\mathcal{T}_h$ that have $\vec{z}_j$ as a point

$$cn(\vec{z}_j) = \#(\{K \in \mathcal{T}_h \mid \vec{z}_j \in \overline{K}\})$$

Model prob in 2D, with $r=2$.

$\dim(\mathcal{M}_{0,2}) = 49$



$N_2^\circ = 49$

$\#(\mathcal{T}_h) = 32$

$cn(\vec{z}_{25}) = 6$

$cn(\vec{z}_{18}) = 2$

$$\text{meas}(S_{25} \cap S_{18}) = \text{meas}(\overline{K}_{12} \cup \overline{K}_{13}) > 0$$

$$a(\phi_{25}, \phi_{18}) \neq 0 \text{ (generally)}$$

But

$$a(\phi_{16}, \phi_{25}) = 0 \text{ (generally)}$$

A General Procedure for FEM Computing

(a) Input data: f and the geometry of Ω

(b) Construct $\mathcal{T}_h$ for Ω

(c) Compute the local (or elemental) stiffness matrices and local (or elemental) force vectors for each $K \in \mathcal{T}_h$. (I need to explain this.)

(d) Assemble the global stiffness matrix A and global force vector $\vec{f}$.

(e) Solve $A\vec{u} = \vec{f}$ (sparse matrix prob.)

(f) Post process.

Local Stiffness matrices

$$[A]_{i,j} = a(\phi_j, \phi_i) = \int_\Omega \nabla\phi_j \cdot \nabla\phi_i \, d\vec{x}$$

global stiffness matrix

$$= \sum_{K \in \mathcal{T}_h} \int_K \nabla\phi_j \cdot \nabla\phi_i \, d\vec{x}$$

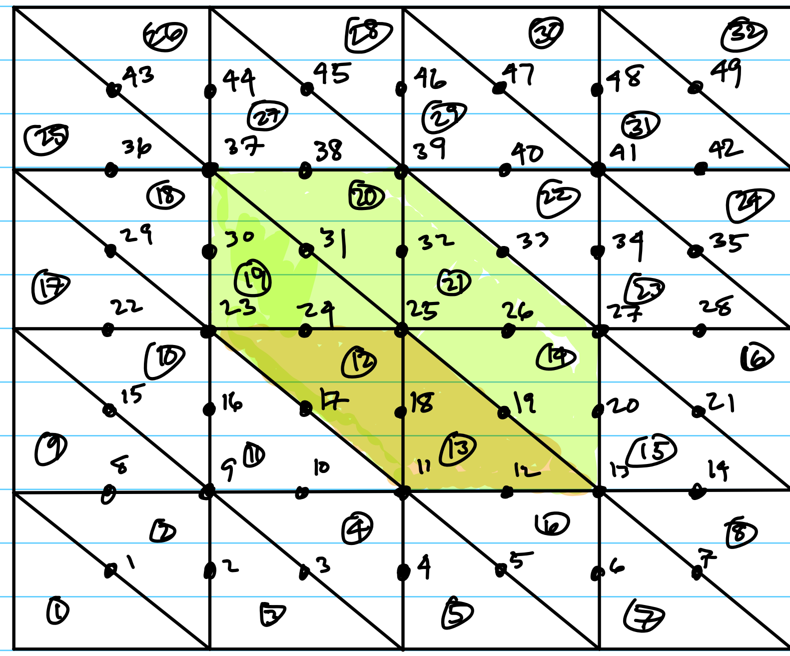
$$= \sum_{K \subset S_i \cap S_j} \int_K \nabla\phi_j \cdot \nabla\phi_i \, d\vec{x} \qquad S_i := \mathrm{Spp}(\phi_i)$$

Example: let us compute

$$[A]_{25,18} = a(\phi_{18}, \phi_{25})$$

$$= \int_\Omega \nabla\phi_{18} \cdot \nabla\phi_{25} \, d\vec{x}$$

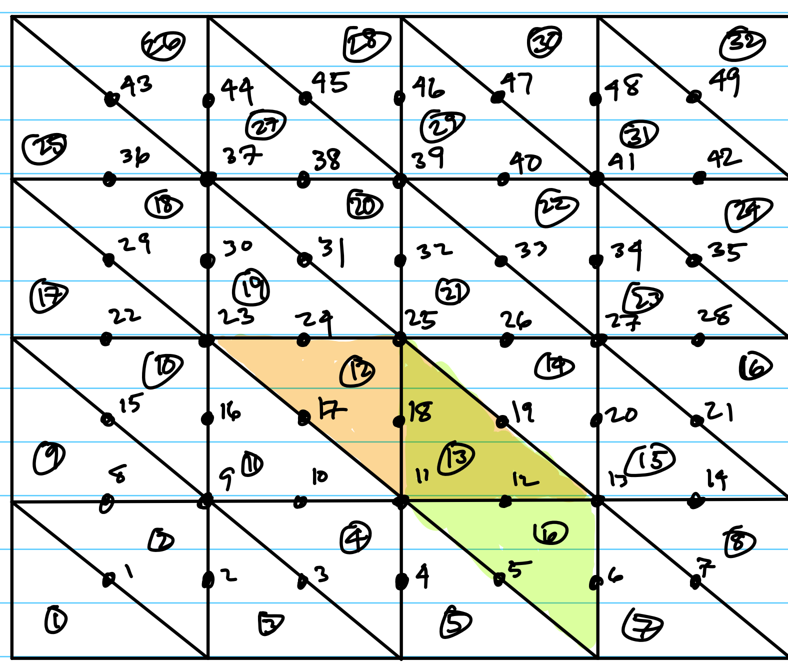
$$= \int_{K_{12}} \nabla\phi_{18} \cdot \nabla\phi_{25} \, d\vec{x} + \int_{K_{13}} \nabla\phi_{18} \cdot \nabla\phi_{25} \, d\vec{x}$$



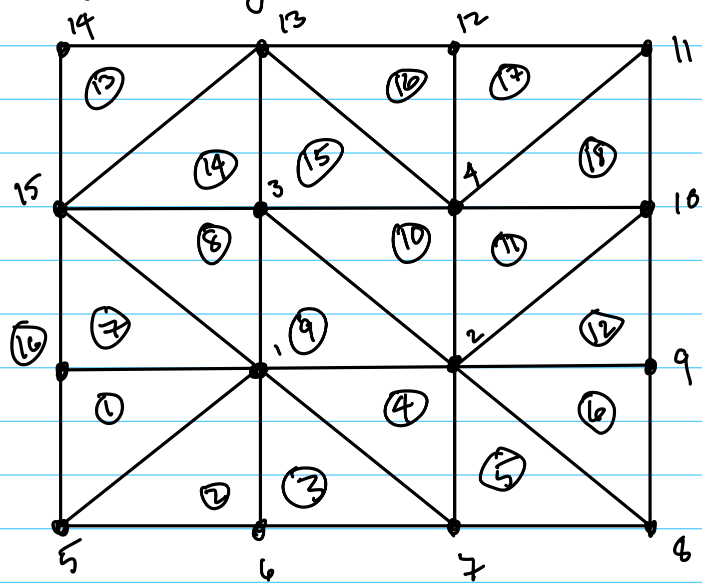
Example:

$$[A]_{18,12} = \int_{\Omega} \nabla\phi_{12} \cdot \nabla\phi_{18} \, d\vec{x}$$

$$= \int_{K_{13}} \nabla\phi_{12} \cdot \nabla\phi_{18} \, d\vec{x}$$



Representing The Mesh with Arrays



$$\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} \bar{K}$$

$$Z = \begin{bmatrix} x_{1,1} & x_{2,1} & x_{3,1} & x_{4,1} & x_{5,1} & \cdots & x_{16,1} \\ x_{1,2} & x_{2,2} & x_{3,2} & x_{4,2} & x_{5,2} & & x_{16,2} \end{bmatrix}$$

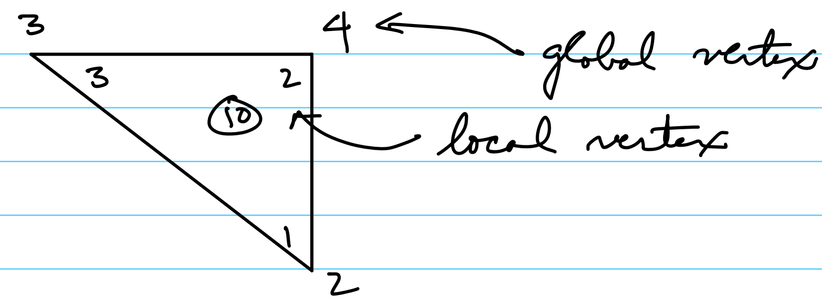
interior vertex coordinates — bndry vertex coordinates

$$T = \begin{array}{c} \left[\begin{array}{cccccccccc} 5 & 6 & 6 & 1 & 7 & 8 & 1 & 1 & 1 & 2 \\ 1 & 1 & 7 & 7 & 8 & 9 & 15 & 3 & 2 & 4 \\ 16 & 5 & 1 & 2 & 2 & 2 & 16 & 15 & 3 & 3 \end{array}\right. \\ \hline \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \end{array}$$

$$\begin{array}{c} \left.\begin{array}{cccccccc} 2 & 9 & 15 & 15 & 3 & 4 & 4 & 4 \\ 10 & 10 & 13 & 3 & 4 & 12 & 11 & 10 \\ 4 & 2 & 14 & 13 & 13 & 13 & 12 & 11 \end{array}\right] \\ \hline \begin{array}{cccccccc} 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \end{array} \end{array}$$

T gives the local to global mapping of vertices.

look at triangle 10, K_{10}:



$$T(2,10) = 4$$

In words, the global node number of local node 2 in triangle 10 is 4.