Math 574
class #19
10/30/25

The semi-discrete Galerkin approx for the diffusion problem may be expressed as

$$G\vec{u}'(t) + A\vec{u}(t) = \vec{f}(t)$$

with

$$G\vec{u}(0) = \vec{v}$$

<u>Proposition</u> (19.1): The mass matrix

$$G = [g_{i,j}]_{i,j=1}^{N_r^0}$$

where

$$g_{i,j} = (\phi_j, \phi_i)_{L^2}$$

is SPD.

<u>Proof</u>: let

$$u_n = \sum_{j=1}^{N_r^0} u_j \phi_j \quad \overset{B=\{\phi_j\}}{\longleftrightarrow} \quad \vec{u} \in \mathbb{R}^{N_r^0}$$

be arbitrary. Then

$$\begin{aligned}
\vec{u}^T G \vec{u} &= \sum_{i=1}^{N_r^0} u_i [G\vec{u}]_i \\
&= \sum_{i=1}^{N_r^0} \sum_{j=1}^{N_r^0} u_i g_{ij} u_j \\
&= (u_n, u_n)_{L^2} \\
&\geq 0
\end{aligned}$$

Observe that $\vec{u}^T G \vec{u} = 0$ iff $u_n = 0$ iff $\vec{u} = \vec{0}$. ///

Thus,

$$\vec{u}'(t) + G^{-1} A \vec{u}(t) = G^{-1} \vec{f}(t)$$

$$\vec{u}(0) = G^{-1} \vec{v}$$

By the standard theory of (linear) ODE, this system has a unique solution on $[0,T]$, for some T, provided f is "nice" on $[0,T]$.

Stability

Recall the semi-discrete Galerkin approximation:
Find $u_h : [0,T] \to V_h = M_{0,r}$ such that

(19.1) $$(u_h'(t), \psi)_{L^2} + a(u_h(t), \psi) = (f(t), \psi)_{L^2}$$

for all $\psi \in V_h$, for all $t \in [0,T]$, with

(19.2) $$u_h(0) = v_h = P_h v \in V_h .$$

Theorem (19.2): Suppose that, for all $h > 0$,

$$\|v_h\|_{L^2}^2 \le \|v\|_{L^2}^2 + C \qquad \left(\text{or } \|v_h\|_{L^2} \le C \|v\|_{L^2}\right)$$

for some $C \geq 0$ independent of h. Then, the semi-discrete approx is stable in the following sense:

(19.3) $$\max_{0 \le t \le T} \|u_h(t)\|_{L^2}^2 \le C_0$$

and

(19.4) $$\int_0^T \|u_h(t)\|_{H^1}^2 \, dt \le C_0$$

where

$$C_0 := \|v\|_{L^2}^2 + C \int_0^T \|f\|_{L^2}^2 \, dt + C .$$

Proof: In (19.1) set $\psi = u_n(t)$. Then

$$(u_n'(t), u_n(t)) + |u_n(t)|_{H^1}^2 = (f(t), u_n(t))$$

This implies

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{L^2}^2 + |u_n(t)|_{H^1}^2 = (f(t), u_n(t))$$

$$\overset{C.S}{\leq} \|f(t)\|_{L^2}\|u_n(t)\|_{L^2}$$

$$\overset{\text{Poincare}}{\leq} c\|f(t)\|_{L^2}|u_n(t)|_{H^1}$$

$$\overset{AGMI}{\leq} \frac{c^2}{2}\|f(t)\|_{L^2}^2 + \frac{1}{2}|u_n(t)|_{H^1}^2$$

$$\frac{d}{dt}\|u_n(t)\|_{L^2}^2 + |u_n(t)|_{H^1}^2 \leq c^2\|f(t)\|_{L^2}^2$$

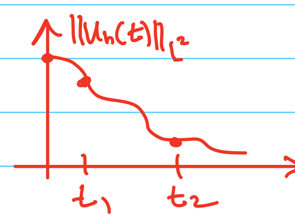
Integrating, from $t=0$ to $t=s$,

$$\|u_n(s)\|_{L^2}^2 - \|v_n\|_{L^2}^2 + \int_0^s |u_n(t)|_{H^1}^2\,dt \leq c^2\int_0^s \|f(t)\|_{L^2}^2\,dt$$

$$\leq c^2\int_0^T \|f(t)\|_{L^2}^2\,dt$$

$$\|u_n(s)\|_{L^2}^2 + \int_0^s |u_n(t)|_{H^1}^2\,dt \leq \|v\|_{L^2}^2 + c^2\int_0^T \|f(t)\|_{L^2}^2\,dt + C$$

The result follows. ///



Corollary (19.3): If $f \equiv 0$. Then,

$$\|u_n(t_2)\|_{L^2}^2 \leq \|u_n(t_1)\|_{L^2}^2 \leq \|v\|_{L^2}^2 + C$$

for all $0 \leq t_1 \leq t_2 \leq T$.

Backward-Euler-Galerkin Methods

Let $K \in \mathbb{N}$ be given. Set

$$s = \frac{T}{K}$$

Find

$$u_h^1, u_h^2, u_h^3, \dots, u_h^k, \dots, u_h^K \in V_h = \mathcal{M}_{0,r}$$

such that $\quad (u_h^k \approx u(t_k))$

(19.5) $$\left(\frac{u_h^{k+1} - u_h^k}{s}, \psi\right)_{L^2} + a(u_h^{k+1}, \psi) = (f(t_{k+1}), \psi)_{L^2}$$

for all $\psi \in V_h$, $k = 0, 1, \dots, K-1$, with

(19.6) $$u_h^0 = v_h = P_h v \in V_h$$

where

$$t_k := s \cdot k, \quad k = 0, 1, \dots, K.$$

This is equivalent to

(19.7) $$G\vec{u}^{k+1} + sA\vec{u}^{k+1} = s\vec{f}^{k+1} + G\vec{u}^k$$

with

(19.8) $$G\vec{u}^0 = \vec{v} \in \mathbb{R}^{N_r^0}.$$

Here we define

$$\vec{u}^k = [u_i^k]_{i=1}^{N_r^0}, \quad \vec{f}^k = [f_i^k]_{i=1}^{N_r^0}, \quad \vec{v} = [v_i]_{i=1}^{N_r^0}$$

and

$$u_h^k = \sum_{j=1}^{N_r^0} u_j^k \phi_j, \quad f_i^k = (f(t_k), \phi_i)_{L^2}$$

$$v_h = \sum_{i=1}^{N_r^0} v_i \phi_i.$$

The scheme is equivalent to

$$(G+sA)\vec{u}^{k+1} = s\vec{f}^{k+1} + G\vec{u}^{k}.$$

Since $G+sA$ is SPD, the scheme is well-defined.

Stability

Theorem (19.4): Suppose that

$$\|v_h\|_{L^2}^2 \leq \|v\|_{L^2}^2 + C$$

and

$$s\sum_{k=1}^{K} \|f(t_k)\|_{L^2}^2 \leq \int_0^T \|f(t)\|_{L^2}^2\,dt + C$$

where $C \geq 0$ is independent of h and s. Then

$$\max_{0\leq k\leq K} \|u_h^k\|_{L^2}^2 + s\sum_{k=1}^{K} |u_h^k|_{H^1}^2 \leq C_0$$

where $C_0 \geq 0$ is independent of h and s.

Proof: In (19.5), set $\psi = u_h^{k+1} \in V_h$. Then

$$(u_h^{k+1} - u_h^k, u_h^{k+1})_{L^2} + s\,|u_h^{k+1}|_{H^1}^2 = s\,(f(t_{k+1}), u_h^{k+1})_{L^2}$$

Observe that

$$(u_h^{k+1} - u_h^k, u_h^{k+1})_{L^2} = \frac{1}{2}\|u_h^{k+1}\|_{L^2}^2 - \frac{1}{2}\|u_h^k\|_{L^2}^2 + \frac{1}{2}\|u_h^{k+1} - u_h^k\|_{L^2}^2$$

This is called the polarization identity.

(follows from $(a-b)a = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2$)

Thus,

$$\frac{1}{2}\|u_h^{k+1}\|_{L^2}^2 - \frac{1}{2}\|u_h^k\|_{L^2}^2 + \frac{1}{2}\|u_h^{k+1}-u_h^k\|_{L^2}^2 + s|u_h^{k+1}|_{H^1}^2$$

$$= s\left(f(t_{k+1}), u_h^{k+1}\right)_{L^2}$$

$$\overset{C.S.}{\leq} s\,\|f(t_{k+1})\|_{L^2}\,\|u_h^{k+1}\|_{L^2}$$

$$\overset{\text{Poincare}}{\leq} s\,C\,\|f(t_{k+1})\|_{L^2}\,|u_h^{k+1}|_{H^1}$$

$$\overset{AGMI}{\leq} \frac{C^2 s}{2}\|f(t_{k+1})\|_{L^2}^2 + \frac{s}{2}|u_h^{k+1}|_{H^1}^2$$

This implies

$$\|u_h^{k+1}\|_{L^2} - \|u_h^k\|_{L^2} + s|u_h^{k+1}|_{H^1}^2 \leq C^2 s\,\|f(t_{k+1})\|_{L^2}$$

Now, we sum from $k=0$ to $k=\ell-1$, $1\leq \ell\leq K$.

$$\|u_h^\ell\|_{L^2}^2 - \|u_h^0\|_{L^2}^2 + s\sum_{k=1}^{\ell}|u_h^k|_{H^1}^2 \leq C^2 s\sum_{k=1}^{\ell}\|f(t_k)\|_{L^2}^2$$

Thus,

$$\|u_h^\ell\|_{L^2}^2 + s\sum_{k=1}^{\ell}|u_h^k|_{H^1}^2 \leq \|v_h\|_{L^2}^2 + C^2 s\sum_{k=1}^{\ell}\|f(t_k)\|_{L^2}^2.$$

$$\leq \|v\|_{L^2}^2 + C^2\int_0^T\|f(t)\|_{L^2}^2\,dt$$

$$+ C.$$

The result now follows. ///