Math 574
class #13
10/02/2025

A Quotient Space Estimate

We will need the following estimate for our error analysis of Lagrange nodal interpolation.

Theorem (13.1): Let Ω be an open, bounded, Lipschitz domain in $\mathbb{R}^d$. There is a constant $C = C(\Omega, k) > 0$, such that

(13.1) $$\inf_{p \in \mathbb{P}_k(\Omega)} \|v + p\|_{H^{k+1}(\Omega)} \le C \, |v|_{H^{k+1}(\Omega)}$$

for all $v \in H^{k+1}(\Omega)$.

Recall that

$$\|\vec{v}\|_{H^{k+1}(\Omega)} = \sqrt{\sum_{|\alpha| \le k+1} (\partial^\alpha v, \partial^\alpha v)_{L^2}}$$

$$|\vec{v}|_{H^{k+1}(\Omega)} = \sqrt{\sum_{|\alpha| = k+1} (\partial^\alpha v, \partial^\alpha v)_{L^2}}$$

Proposition: Let $\Omega \subset \mathbb{R}^d$ be an open, bounded, Lipschitz domain the object

$$\inf_{p \in \mathbb{P}_k(\Omega)} \|v + p\|_{H^{k+1}(\Omega)},$$

the so called quotient norm, is a bona fide norm on the space

$$V = H^{k+1}(\Omega) / \mathbb{P}_k(\Omega)$$

$$= \{ [v] \mid [v] = \{ v + q \mid q \in \mathbb{P}_k(\Omega) \}, v \in H^{k+1}(\Omega) \}$$

which is called the <u>quotient space</u>.

<u>Proof</u>: See the book by Atkinson and Han. ///

<u>Theorem</u> (13.2): Let $r, m \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, with

$$r \geq 1, \quad r+1 \geq m.$$

Suppose that $\hat{\Pi} : C(\bar{\hat{K}}) \to \mathbb{P}_r(\hat{K})$ is the nodal interpolant on the reference triangle. Then, there is a constant $C > 0$ such that

(13.2) $$|\hat{v} - \hat{\Pi}\hat{v}|_{H^m(\hat{K})} \leq C |\hat{v}|_{H^{r+1}(\hat{K})}$$

for all $\hat{v} \in H^{r+1}(\hat{K})$.

<u>Proof</u>: If $r \geq 1$, $H^{r+1}(\hat{K}) \hookrightarrow C(\bar{\hat{K}})$. Thus, if $\hat{v} \in H^{r+1}(\hat{K})$, $\hat{v} \in C(\bar{\hat{K}})$, and $\hat{\Pi}\hat{v}$ is well-defined. Now,

$$\|\hat{\Pi}\hat{v}\|_{H^m(\hat{K})} = \left\| \sum_{i=1}^{d} \hat{v}(\hat{a}_i) \hat{\varphi}_i(\cdot) \right\|_{H^m(\hat{K})}$$

$$\overset{\Delta\text{-ineq}}{\leq} \sum_{i=1}^{d} |v(\hat{a}_i)| \cdot \|\hat{\varphi}_i\|_{H^m(\hat{K})}$$

$$\|\hat{v}\|_{C^0(\bar{K})} = \max_{\hat{x}\in\bar{K}} |\hat{v}(\hat{x})|$$

$$\leq C \|\hat{v}\|_{C(\bar{K})}$$ (C depends upon m and r)

$$\overset{\text{Sobolev}}{\leq} C \|\hat{v}\|_{H^{r+1}(\Omega)}$$

$\mathcal{L} : V \to W$

$\|\mathcal{L}v\|_W \leq C\|v\|_V$

bounded op!

Thus

$$\hat{\Pi} : H^{r+1}(\hat{K}) \longrightarrow H^m(\hat{K})$$

is a bounded linear operator.

Recall that

$$\hat{\Pi}\hat{v} = \hat{v} \quad \forall\, \hat{v} \in \mathbb{P}_r(\hat{K})$$

This is called polynomial invariance.

Now, for any $\hat{p} \in \mathbb{P}_r(\hat{K})$ (arbitrary)

$$
\begin{aligned}
|\hat{v} - \hat{\Pi}\hat{v}|_{H^m(\hat{K})} &\leq \|\hat{v} - \hat{\Pi}\hat{v}\|_{H^m(\hat{K})} \\
&= \|\hat{v} - \hat{\Pi}\hat{v} + \hat{p} - \hat{\Pi}\hat{p}\|_{H^m(\hat{K})} \\
&\overset{\Delta\text{-ineq}}{\leq} \|\hat{v} + \hat{p}\|_{H^m(\hat{K})} + \|\hat{\Pi}(\hat{v} + \hat{p})\|_{H^m(\hat{K})} \\
&\overset{\text{bnded. op.}}{\leq} \|\hat{v} + \hat{p}\|_{H^{r+1}(\hat{K})} + C\|\hat{v} + \hat{p}\|_{H^{r+1}(\hat{K})} \\
&= (1 + C)\, \|\hat{v} + \hat{p}\|_{H^{r+1}(\hat{K})}
\end{aligned}
$$

Since $\hat{p} \in \mathbb{P}_r(\hat{K})$ is arbitrary

$$|\hat{v} - \hat{\Pi}\hat{v}|_{H^m(\hat{K})} \leq C \inf_{\hat{p}\in\mathbb{P}_r(\hat{K})} \|\hat{v} + \hat{p}\|_{H^{r+1}(\hat{K})}$$

$$\overset{(13.1)}{\leq} \quad C\,|\hat{v}|_{H^{r+1}(\hat{K})}.$$

Theorem (13.3): Assume that $\vec{F}_K : \hat{K} \to K$ is the canonical affine mapping:

$$\vec{x} = \vec{F}(\hat{\vec{x}}) = B_K \hat{\vec{x}} + \vec{a}_{K,1}$$

For any function $v : K \to \mathbb{R}$,

$$\hat{v}(\hat{\vec{x}}) := v(\vec{F}_K(\hat{\vec{x}})) \qquad \forall\, \hat{\vec{x}} \in \hat{K}$$

Then $v \in H^m(K)$ iff $\hat{v} \in H^m(\hat{K})$. There exists a constant $C > 0$, independent of $\hat{K}$ and K, such that

(13.3) $$|\hat{v}|_{H^m(\hat{K})} \leq C\,\|B_K\|_2^m\,|\det B_K|^{-1/2}\,|v|_{H^m(K)}$$

and

(13.4) $$|v|_{H^m(K)} \leq C\,\|B_K^{-1}\|_2^m\,|\det B_K|^{1/2}\,|\hat{v}|_{H^m(\hat{K})}$$

Proof: Homework exercise. ///

Theorem (13.4): Let $r, m \in \mathbb{N}_0$, $r \geq 1$, $r+1 \geq m$. Suppose

$$\Pi_K : C(\overline{K}) \to \mathbb{P}_r(K)$$

is the Lagrange nodal interpolant on K. There exists a $C > 0$, which is independent of the shape and size of K, such that

(13.5) $$|v - \Pi_K v|_{H^m(K)} \leq C\,\frac{h_K^{r+1}}{\rho_K^m}\,|v|_{H^{r+1}(K)},$$

for all $v \in H^{r+1}(K)$.

Proof: First observe that, for all $\hat{\vec{x}} \in \hat{K}$,

$$(\hat{v} - \hat{\Pi}\hat{v})(\hat{\vec{x}}) = (v - \Pi_K v)(\vec{F}_K(\hat{\vec{x}}))$$

By Theorem 13.3,

$$|v - \Pi_K v|_{H^m(K)} \overset{(13.4)}{\leq} C \|B_K^{-1}\|_2^m |\det B_K|^{1/2} \times |\hat{v} - \hat{\Pi}\hat{v}|_{H^m(\hat{K})}$$

$$\overset{(13.2)}{\leq} C \|B_K^{-1}\|_2^m |\det B_K|^{1/2} \times |\hat{v}|_{H^{r+1}(\hat{K})}$$

$$\overset{(13.3)}{\leq} C \|B_K^{-1}\|_2^m \cancel{|\det B_K|^{1/2}} \times C \|B_K\|_2^{r+1} \cancel{|\det B_K|^{-1/2}} \times |v|_{H^{r+1}(K)}.$$

Thus,

$$|v - \Pi_K v|_{H^m(K)} \leq C \|B_K^{-1}\|_2^m \|B_K\|_2^{r+1} |v|_{H^{r+1}(K)}$$

$$\leq C \left(\frac{\hat{h}}{\rho_K}\right)^m \left(\frac{h_K}{\hat{\rho}}\right)^{r+1} |v|_{H^{r+1}(K)}$$

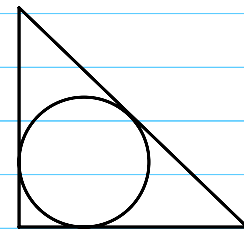
where we have used a lemma from lecture 12:

$$\|B_K\|_2 \leq \frac{h_K}{\hat{\rho}} \qquad \|B_K^{-1}\|_2 \leq \frac{\hat{h}}{\rho_K}.$$

Note that

$$\hat{h} = \sqrt{2}$$

$$\hat{\rho} \approx \frac{1}{2}$$



Thus,

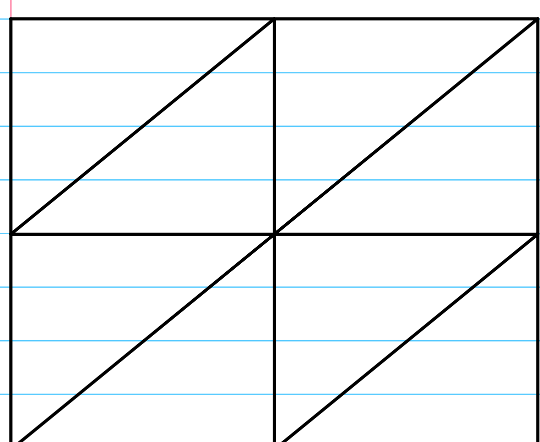
$$|v - \Pi_K v|_{H^m(K)} \leq C \left(\frac{\hat{h}}{\rho_K}\right)^m \left(\frac{h_K}{\hat{\rho}}\right)^{r+1} |v|_{H^{r+1}(K)}$$

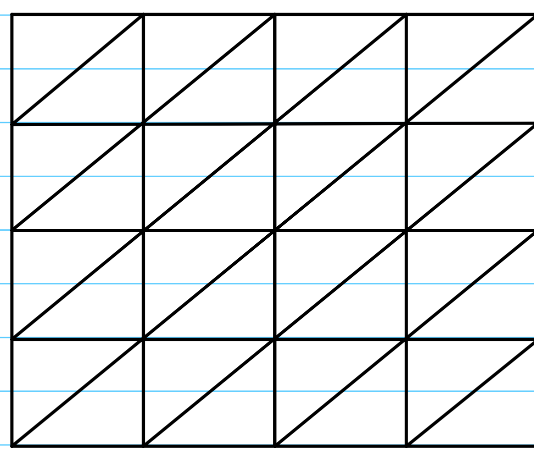
$$= C \frac{h_K^{r+1}}{\rho_K^m} |v|_{H^{r+1}(K)} \quad ///$$

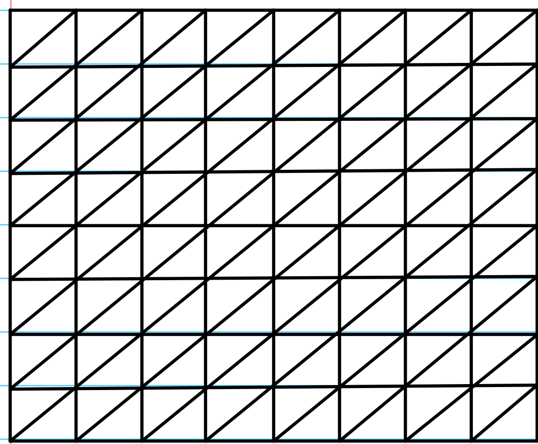
<u>Defn</u> (13.5): A family of triangulations, $\{\mathcal{T}_h\}_h$, of the polygonal domain $\Omega \subset \mathbb{R}^2$ is said to be <u>shape regular</u> iff, $\exists\, \sigma \geq 1$, independent of h, such that

$$1 \leq \frac{h_K}{\rho_K} \leq \sigma \qquad \forall\, K \in \mathcal{T}_h \quad \forall\, h$$

<u>Example</u>:

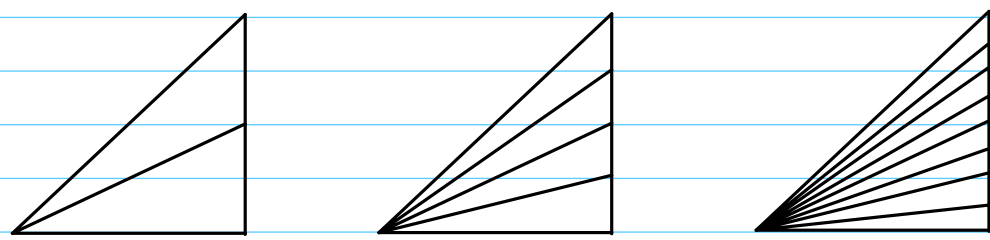






This family of triangulations is shape regular

Example:



$\frac{h_K}{\rho_K} \nearrow \infty$

This family is not shape regular

Corollary (13.6): If $\{\tau_h\}_h$ is a shape regular family, $\exists$ $C>0$, independent of the shape and size of K, such that

$$\|v-\Pi_K v\|_{H^m(K)} \leq C\, h_K^{r+1-m} |v|_{H^{r+1}(K)}$$

for all $v \in H^{r+1}(K)$, $\forall K \in \tau_h$.

Proof:

$$|v-\Pi_K v|_{H^m(K)} \leq C \frac{h_K^{r+1}}{\rho_K^m} |v|_{H^{r+1}(K)}$$

$$= C \left(\frac{h_K}{\rho_K}\right)^m h_K^{r+1-m} |v|_{H^{r+1}(K)}$$

$$\leq C \sigma^m h_K^{r+1-m} |v|_{H^{r+1}(K)} .$$

To finish the proof, write

$$\|v - \Pi_K v\|_{H^m(K)}^2 = \sum_{t=0}^{m} |v - \Pi_K v|_{H^t(K)}^2 . \quad ///$$