Math 574
class #22
11/13/2025

## About Bochner Spaces

These are spaces of the form

$$L^p(0,T;\, L^q(\Omega))$$

and

$$H^1(0,T;\, L^2(\Omega))$$

Recall, for example,

$$f \in L^p(0,T;\, L^q(\Omega)) \implies \int_0^T \|f(t)\|_{L^q}^p \, dt < \infty$$

$$f \in H^1(0,T;\, L^2(\Omega)) \implies f \in L^2(0,T;\, L^2) \text{ and } \partial_t f \in L^2(0,T;\, L^2)$$

$$\implies \int_0^T \|\partial_t f(t)\|_{L^2}^2 \, dt < \infty$$

<u>Proposition</u>: If $f \in L^2(0,T;\, L^2(\Omega))$ then

$$\int_0^T \|f(t)\|_{L^2}^1 \, dt < \infty$$

<u>Proof</u>: Using Cauchy-Schwartz in time

$$\int_0^T \|f(t)\|_{L^2}^1 \, dt = \int_0^T \|f(t)\|_{L^2} \cdot 1 \, dt$$

$$\overset{C.S.}{\leq} \sqrt{\int_0^T \|f(t)\|_{L^2}^2 \, dt} \sqrt{\int_0^T 1^2 \, dt} < \infty$$

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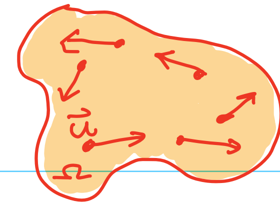
$$-\Delta \vec{u} = \begin{bmatrix} -\Delta u_1 \\ -\Delta u_2 \\ \vdots \\ -\Delta u_d \end{bmatrix}$$




# Stokes Equation

Suppose that $\Omega \subseteq \mathbb{R}^d$, $d=2$ or $3$. Find $\vec{u} : \Omega \to \mathbb{R}^d$ and $p : \Omega \to \mathbb{R}$, such that

(22.1)
$$\begin{cases} -\Delta \vec{u} + \nabla p = \vec{f} & \text{in } \Omega \\ \nabla \cdot \vec{u} = 0 & \text{in } \Omega \\ \vec{u} = \vec{0} & \text{on } \partial\Omega \end{cases}$$

This is called Stokes' problem.

Definition (22.1): $(\vec{u}, p)$ is called a <u>classical solution</u> to Stokes's problem iff

$$\vec{u} \in \left[C^2(\Omega) \cap C^0(\bar{\Omega})\right]^d$$

and

$$p \in C^1(\Omega)$$

and these functions satisfy (22.1) pointwise.

$(\vec{u}, p)$ is called a <u>weak solution</u> to Stokes' problem iff

$$\vec{u} \in \left[H_0^1(\Omega)\right]^d$$

and

$$p \in L_0^2(\Omega) := \left\{ v \in L^2(\Omega) \;\middle|\; \int_\Omega v \, d\vec{x} = 0 \right\}$$

and

(22.2)
$$\begin{cases} (\nabla \vec{u}, \nabla \vec{v}) - (p, \nabla \cdot \vec{v}) = (\vec{f}, \vec{v}) & \forall \vec{v} \in \left[H_0^1\right]^d \\ (q, \nabla \cdot \vec{u}) = 0 & \forall q \in L_0^2(\Omega). \end{cases}$$

where
$$\vec{f} \in \left[L^2(\Omega)\right]^d$$
is given.

Here we define the objects

$$(\nabla\vec{u}, \nabla\vec{v})_{L^2} = \sum_{i=1}^{d}\sum_{j=1}^{d}\left(\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j}\right)_{L^2}$$

and
$$\|\nabla\vec{u}\|_{L^2} = \sqrt{(\nabla\vec{u}, \nabla\vec{u})_{L^2}}$$
$$= \sqrt{\sum_{i=1}^{d}\sum_{j=1}^{d}\left(\frac{\partial u_i}{\partial x_j}, \frac{\partial u_i}{\partial x_j}\right)_{L^2}}$$

Clearly, using integration by parts, if $(\vec{u}, p)$ is a classical solution it is also a weak solution.

We will follow the book by William Layton for this material:

W. Layton, <u>Introduction to the Numerical Analysis of Incompressible Viscous Flows</u>, (2008) SIAM.

<u>Defn</u> (22.2): The subspace

$$\nabla\cdot\vec{v} = \sum_{j=1}^{d}\frac{\partial v_i}{\partial x_i} \in L^2(\Omega)$$

(22.3) $$V := \left\{\vec{v} \in \left[H_0^1(\Omega)\right]^d \,\middle|\, (q, \nabla\cdot\vec{v}) = 0 \quad \forall q \in L_0^2(\Omega)\right\}$$

is called the <u>weakly divergence-free subspace</u>.

**Lemma** (22.3) The object $b : \left[H_0^1(\Omega)\right]^d \times L_0^2(\Omega) \to \mathbb{R}$, defined by

$$b(\vec{v}, q) := (q, \nabla\cdot\vec{v}),$$

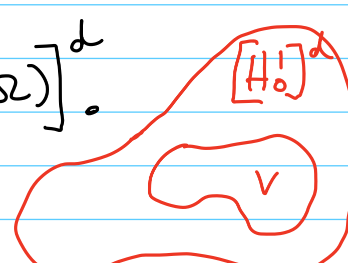
is a bilinear form on $\left[H_0^1(\Omega)\right]^d \times L_0^2(\Omega)$ that is continuous. As a consequence of this, $V$ is a closed subspace of $\left[H_0^1(\Omega)\right]^d$ and is, therefore, a Hilbert space.

**Proof**: Continuity of $b(\cdot,\cdot)$:

$$\begin{aligned}
|b(\vec{v}, q)| &= (q, \nabla\cdot\vec{v}) \\
&\overset{C.S.}{\leq} \|q\|_{L^2} \|\nabla\cdot\vec{v}\|_{L^2} \\
&\overset{\Delta\text{-ineq}}{\leq} \|q\|_{L^2} \sum_{i=1}^{d} \left\|\frac{\partial v_i}{\partial x_i}\right\|_{L^2} \\
&\leq \|q\|_{L^2} \sum_{i=1}^{d}\sum_{j=1}^{d} \left\|\frac{\partial v_i}{\partial x_j}\right\|_{L^2} \\
&\overset{C.S.}{\leq} \|q\|_{L^2} \sqrt{\sum_{i=1}^{d}\sum_{j=1}^{d} \left\|\frac{\partial v_i}{\partial x_j}\right\|_{L^2}^2} \\
&\qquad\qquad \times \sqrt{\sum_{i=1}^{d}\sum_{j=1}^{d} 1^2} \\
&= d\,\|q\|_{L^2} \|\nabla\vec{v}\|_{L^2}
\end{aligned}$$

Now, suppose that

$$\{\vec{v}_n\} \subset V \quad \text{and} \quad \vec{v}_n \to \vec{v} \quad \text{in} \quad \left[H_0^1(\Omega)\right]^d.$$




Then, for any $q \in L_0^2(\Omega)$,

$$|b(\vec{v},q)| = \Big|b(\vec{v}-\vec{v}_n,q) + \overbrace{b(\vec{v}_n,q)}^{=0}\Big|$$

$$\leq d\|q\|_{L^2}\,\|\nabla(\vec{v}-\vec{v}_n)\|_{L^2}.$$

The only possibility is that $b(\vec{v},q)=0$. This implies $\vec{v}\in V$, which implies $V$ is closed.

Any closed proper subspace of a Hilbert space is itself a Hilbert space. ///

<u>Theorem</u> (22.4): There is a constant $C>0$ such that

$$\|\vec{u}\|_{L^2} \leq C\,\|\nabla\vec{u}\|_{L^2},$$

for all $\vec{u}\in[H_0^1(\Omega)]^d$, where

$$(\vec{u},\vec{v})_{L^2} = \sum_{i=1}^{d}(u_i,v_i), \quad \forall\,\vec{u},\vec{v}\in[L^2(\Omega)]^d,$$

and

$$\|\vec{u}\|_{L^2}^2 = \sum_{i=1}^{d}(u_i,u_i) = \sum_{i=1}^{d}\|u_i\|_{L^2}^2, \quad \forall\,\vec{u}\in[L^2(\Omega)]^d.$$

Proof: Suppose that $\vec{u}\in[H_0^1(\Omega)]^d$. Then $u_i\in H_0^1$, $i=1,\dots,d$, and

$$\|\vec{u}\|_{L^2}^2 = \sum_{i=1}^{d}\|u_i\|_{L^2}^2$$

$$\overset{\text{Poincare}}{\leq} C\sum_{i=1}^{d}\|\nabla u_i\|_{L^2}^2$$

$$= C\sum_{i=1}^{d}\sum_{j=1}^{d}\left\|\frac{\partial u_i}{\partial x_j}\right\|_{L^2}^2$$

$$= C\|\nabla\vec{u}\|_{L^2}^2. \quad ///$$

$$\boxed{\|\nabla\vec{u}\|_{L^2}^2 \le \|\vec{u}\|_{H^1}^2 = \|\vec{u}\|_{L^2}^2 + \|\nabla\vec{u}\|_{L^2}^2 \le (1+C)\|\nabla\vec{u}\|_{L^2}^2}$$

Define, for any $\vec{u},\vec{v} \in [H_0^1(\Omega)]^d$

$$a(\vec{u},\vec{v}) := (\nabla\vec{u}, \nabla\vec{v})_{L^2} = \sum_{i=1}^{d}\sum_{j=1}^{d}\left(\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j}\right)_{L^2}$$

and

$$b(\vec{v},q) := (q, \nabla\cdot\vec{v})_{L^2}$$

for any $\vec{v} \in [H_0^1(\Omega)]^d$, $q \in L_0^2(\Omega)$.

Then, if $(\vec{u},p) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$ is a weak solution to the Stokes' problem, it follows that

$$\begin{aligned} a(\vec{u},\vec{v}) - b(\vec{v},p) &= (\vec{f},\vec{v}), \quad \forall \vec{v} \in [H_0^1]^d \\ -b(\vec{u},\vec{q}) &= 0, \quad \forall q \in L_0^2. \end{aligned}$$

This has the form of a so-called <u>saddle point</u> problem.

<u>Definition</u> (22.5): Suppose that

$$a : X \times X \to \mathbb{R}$$

and

$$b : X \times Q \to \mathbb{R}$$

are bilinear forms. Assume that

$$f : X \to \mathbb{R}$$

and

$$g : Q \to \mathbb{R}$$

are bounded linear functionals. Then the problem is called a <u>saddle-point</u> problem iff it is of the following form: Find $u \in X$ and $p \in Q$ such that

$$a(u,v) - b(v,p) = f(v), \quad \forall v \in X,$$
$$-b(u,q) = g(q), \quad \forall q \in Q.$$

Now, if $(u,p) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$ is a weak solution to Stokes problem, that is, it solves (22.2), then

$$\vec{u} \in V := \{ \vec{v} \in [H_0^1(\Omega)]^d \mid (q, \nabla\cdot\vec{v}) = 0, \ \forall q \in L_0^2(\Omega) \}$$

and

(22.4) $$(\nabla\vec{u}, \nabla\vec{v}) = (f, \vec{v}), \qquad \forall \vec{v} \in V.$$

<u>Theorem</u> (22.6): Suppose that $f \in V'$. Then, the problem

(22.5) $$(\nabla\vec{u}, \nabla\vec{v}) = f(\vec{v}), \qquad \forall \vec{v} \in V,$$

has a unique solution $\vec{u} \in V$, and

$$\|\nabla\vec{u}\|_{L^2} =: |\vec{u}|_{H^1} = \|f\|_{V'} = \sup_{\substack{\vec{w} \in V \\ \vec{w} \neq 0}} \frac{|f(\vec{w})|}{|\vec{w}|_{H^1}}.$$

**Proof:** By the vector Poincaré ineq

$$a(\vec{u},\vec{v}) := (\nabla\vec{u},\nabla\vec{v})$$

is coercive on $V$ and is, therefore, an inner product on $V$.

The result now follows by the RRT. ///

The reformulation of the problem in (22.4) eliminates the pressure. To solve the original problem, with the pressure, we need the so-called inf-sup condition.

We need some tools to introduce the inf-sup condition.

__Definition__ (22.7): Suppose that $V$ is as in (22.3). We define

$$V^\perp := \left\{ \vec{v} \in \left[H_0^1(\Omega)\right]^d \;\middle|\; a(\vec{v},\vec{u}) = 0 \quad \forall\, \vec{u} \in V \right\},$$

the orthogonal complement of $V$ with respect to $a(\cdot,\cdot)$.

__Theorem__ (22.8): Given any $q \in L_0^2(\Omega)$, there exists a unique $\vec{v} \in V^\perp$ such that

$$\nabla\cdot\vec{v} = q \quad \left(\text{in } L^2(\Omega)\right).$$

In other words, the divergence operator is an isomorphism from $V^\perp$ onto $L^2_0(\Omega)$. Moreover, there is a constant $C>0$ such that

$$|\vec{v}|_{H^1} \leq C \|q\|_{L^2}$$

Proof: See Girault and Raviart (1986). ///