Math 574
class #17
10/21/2025

Argyris Error Estimates for Biharmonic Problem

Interpolation Estimate:

$$\|u - \Pi_h u\|_{H^m} \leq C h^{6-m} |u|_{H^6(\Omega)}$$

for any $u \in H^6(\Omega)$, and $m \leq 5$. For the problem at hand, $m = 2$. Thus,

$$\|u - \Pi_h u\|_{H^2} \leq C h^4 |u|_{H^6(\Omega)}.$$

Cea's lemma guarantees that

$$\|u - u_h\|_{H_0^2(\Omega)} = \min_{v \in V_h} \|u - v\|_{H_0^2(\Omega)}$$

Now

$$H^6(\Omega) \hookrightarrow C^{6-1-1}(\overline{\Omega}) \qquad \text{(Sobolev Embedding)}$$

$$= C^4(\overline{\Omega})$$

since

$$6 = k > \frac{d}{2} = \frac{2}{2}.$$

Thus $\Pi_h u$ is well-defined for $u \in H^6$. So

$$\|u - u_h\|_{H_0^2(\Omega)} = \min_{v \in V_h} \|u - v\|_{H_0^2(\Omega)}$$

$$\leq \|u - \Pi_h u\|_{H_0^2(\Omega)}$$

Combining with interpolation error estimates,

$$\|u-u_h\|_{H_0^2(\Omega)} \leq C h^4 |u|_{H^6(\Omega)}.$$

Assuming elliptic regularity, we can use a Nitsche-like trick to get

$$\|u-u_h\|_{L^2(\Omega)} \leq C h^6 |u|_{H^6(\Omega)} \quad ///$$

FEM for Parabolic Problems

__Diffusion Problem__: Find $u \in C^2(\bar{\Omega}\times[0,T])$ such that

(17.1)
$$\begin{cases} \partial_t u - \Delta u = f(\cdot,t) & \text{in } \Omega,\ 0<t\leq T, \\ u(\cdot,t) = 0, & \text{on } \partial\Omega\ \ 0<t\leq T, \\ u(\cdot,0) = v, & \text{in } \Omega. \end{cases}$$

__Definition__ (17.1): The dual space of $H_0^1(\Omega)$ is denoted $H^{-1}(\Omega)$ and we write

$$H^{-1}(\Omega) := \left(H_0^1(\Omega)\right)'$$

Recall that the dual space is the space of all bounded linear functionals acting on a Hilbert (Banach) space.

One can show that the dual space is also a Hilbert (Banach) space.

Usually the dual space is equipped with the operator norm:

$$\|f\|_{H'} = \sup_{\substack{\phi \in H \\ \phi \neq 0}} \frac{|f(\phi)|}{\|\phi\|_H}$$

For the present case

$$\|f\|_{H^{-1}} = \sup_{0 \neq \phi \in H_0^1} \frac{|f(\phi)|}{|\phi|_{H^1}}$$

<u>Defn</u> (17.2): Let H be a Hilbert space and let $f \in H'$ be arbitrary the notation

$$\langle \cdot , \cdot \rangle : H' \times H \to \mathbb{R}$$

is called the <u>dual pairing</u> and is defined as

$$\langle f, \phi \rangle := f(\phi) \qquad \forall \, \phi \in H_0$$

Let us write out the weak formulation of the diffusion problem (17.1).

<u>Defn</u> (17.3): Let $f \in L^2(0,T; L^2)$ and $v \in L^2(\Omega)$ be given. u is called a <u>weak solution</u> <u>to the diffusion problem</u> iff

1) $u \in L^2(0,T; H_0^1)$

2) u is weakly differentiable in time and

$$\partial_t u \in L^2(0,T;H^{-1})$$

3) For almost every $t \in [0,T]$ and for every $\phi \in H_0^1(\Omega)$

(17.2) $$\langle \partial_t u, \phi \rangle + (\nabla u, \nabla v) = (f, \phi) := (f, \phi)_{L^2}$$

4) Finally,

$$\lim_{t \searrow 0} \|u(\cdot,t) - v\|_{L^2} = 0.$$

What are these spaces $L^2(0,T;X)$?

Defn (17.4): Let X be a Banach space with norm $\|\cdot\|_X$. The function space

$$L^p(0,T;X) \qquad (1 \le p < \infty)$$

consists of all "strongly measurable" functions

$$\phi : [0,T] \longrightarrow X$$

such that

$$\|\phi\|_{L^p(0,T;X)} := \left[\int_0^T \|\phi(t)\|_X^p \, dt\right]^{1/p} < \infty.$$

Theorem (17.5) Suppose that u is a weak solution to the diffusion problem. Then

$u : [0,T] \longrightarrow L^2$

this is continuous

$$u \in C([0,T]; L^2) \qquad \text{(Sobolev-like)}$$

and

$$\max_{0 \le t \le T} \|u(t)\|_{L^2}^2 \le \|v\|_{L^2}^2 + C\int_0^T \|f\|_{L^2}^2 \, dt$$

and

$$\int_0^T \|u(t)\|_{H_0^1}^2 \, dt \le \|v\|_{L^2}^2 + C\int_0^T \|f\|_{L^2}^2 \, dt.$$

If u has additional regularities

$$u \in L^2(0,T; H_0^1 \cap H^2) \qquad \partial_t u \in L^2(0,T; L^2)$$

with $v \in H_0^1(\Omega)$, then

$$u \in C([0,T]; H_0^1)$$

and

$$\max_{0 \le t \le T} \|u(t)\|_{H_0^1}^2 \le \|v\|_{H_0^1}^2 + \int_0^T \|f\|_{L^2}^2 \, dt$$

$$\int_0^T \|\Delta u\|_{L^2}^2 \, dt \le \|v\|_{H_0^1}^2 + \int_0^T \|f\|_{L^2}^2 \, dt$$

Proof: See Evans (2010) book on PDE for the proof that $u \in C([0,T]; L^2)$.

Now, in the weak formulation, set $\phi = u$. Then

$$\langle \partial_t u, u \rangle + (\nabla u, \nabla u) = (f, u).$$

From Evans (2010) we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 = \langle \partial_t u, u \rangle$$

Since

$$u \in L^2(0,T;H_0^1), \quad \partial_t u \in L^2(0,T;H^{-1}).$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|u\|_{H_0^1}^2 = (f, w)$$

$$\overset{\text{C.S.}}{\leq} \|f\|_{L^2}\|u\|_{L^2}$$

$$\overset{\text{Poincare}}{\leq} C\|f\|_{L^2}\|u\|_{H_0^1}$$

$$\overset{\text{AGMI}}{\leq} C\left(\frac{C}{2}\|f\|_{L^2}^2 + \frac{1}{2C}\|u\|_{H_0^1}^2\right)$$

$$= \frac{C}{2}\|f\|_{L^2}^2 + \frac{1}{2}\|u\|_{H_0^1}^2$$

AGMI

$$\boxed{\begin{gathered} ab \leq \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 \\ ab \leq \tfrac{\varepsilon}{2}a^2 + \tfrac{1}{2\varepsilon}b^2 \\ \forall \varepsilon > 0 \end{gathered}}$$

Rearranging terms and multiplying by 2, we get

$$\frac{d}{dt}\|u\|_{L^2}^2 + \|u\|_{H_0^1}^2 \leq C\|f\|_{L^2}^2$$

Integrating in t from 0 to $s \in [0,T]$, we have

$$\|u(s)\|_{L^2}^2 - \|u(0)\|_{L^2}^2 + \int_0^s \|u(t)\|_{H_0^1}^2\, dt$$

$$\leq C\int_0^s \|f(t)\|_{L^2}^2\, dt$$

Thus

$$\|u(s)\|_{L^2}^2 + \int_0^s \|u(t)\|_{H_0^1}^2\, dt \leq \|v\|_{L^2}^2 + C\int_0^s \|f(t)\|_{L^2}^2\, dt.$$

This implies that

$$\max_{0\le t\le T} \|u(t)\|_{L^2}^2 + \int_0^T \|u(t)\|_{H_0^1}^2\, dt$$

$$\le \|v\|_{L^2}^2 + C\int_0^T \|f(t)\|_{L^2}^2\, dt .$$

Again, we skip the proof that $u\in C([0,T];H_0^1)$ from its assumed better regularities. We refer again to Evans (2010).

Since $u(t)\in H_0^1\cap H^2$, $\forall t\in[0,T]$, we have, for all $\phi\in H_0^1$,

$$(\partial_t u,\phi) - (\Delta u,\phi) = (f,\phi).$$

Set $\phi = -\Delta u(t)$

$$-(\partial_t u,\Delta u) + (\Delta u,\Delta u) = -(f,\Delta u)$$

Integrating by parts again,

$$(\partial_t \nabla u,\nabla u) + \|\Delta u\|_{L^2}^2 = -(f,\Delta u)$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 = -(f,\Delta u)$$

$$\overset{C.S.}{\le} \|f\|_{L^2}\|\Delta u\|_{L^2}$$

$$\overset{AGMI}{\le} \frac{1}{2}\|f\|_{L^2}^2 + \frac{1}{2}\|\Delta u\|_{L^2}^2$$

Thus

$$\frac{d}{dt}\|u\|_{H_0^1}^2 + \|\Delta u\|_{L^2}^2 \le \|f\|_{L^2}^2$$

Integrating in t we get the other two estimates as well. ///

Special Case

Suppose that $f \equiv 0$. Then

$$\max_{0 \le t \le T} \|u(t)\|_{L^2}^2 \le \|v\|_{L^2}^2$$

$$\max_{0 \le t \le T} \|u(t)\|_{H_0^1}^2 \le \|v\|_{L^2}^2$$

In this case, it is easy to show that

$$\|u(t_2)\|_{L^2}^2 \le \|u(t_1)\|_{L^2}^2$$

$$\|u(t_2)\|_{H_0^1}^2 \le \|u(t_1)\|_{L^2}^2$$

for all $0 \le t_1 \le t_2 \le T$.