Math 574
class #23
11/18/2025

Stokes Problem

$$-\Delta \vec{u} + \nabla p = \vec{f} \quad \text{in } \Omega$$
$$\nabla \cdot \vec{u} = 0 \quad \text{in } \Omega$$
$$\vec{u} = \vec{0} \quad \text{on } \partial\Omega$$

Multiply by a test function and integrate by parts to obtain the weak solution.

$$\int_\Omega -\Delta \vec{u} \cdot \vec{v} \, d\vec{x} = -\sum_{i=1}^{d} \int_\Omega \Delta u_i v_i \, d\vec{x}$$

$$= \sum_{i=1}^{d} \left\{ \int_\Omega \nabla u_i \cdot \nabla v_i \, d\vec{x} - \int_{\partial\Omega} \partial_n u_i \cancelto{0}{v_i} \, d\vec{x} \right\}$$

for all $\vec{v} \in [H_0^1(\Omega)]^d$. Define

$$a(\vec{u}, \vec{v}) := \sum_{i=1}^{d} \int_\Omega \nabla u_i \cdot \nabla v_i \, d\vec{x}$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{d} \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, d\vec{x}$$

$$= \int_\Omega \nabla \vec{u} : \nabla \vec{v} \, d\vec{x}.$$

The weak form of the Stokes Problem can be written as follows: find $\vec{u} \in [H_0^1(\Omega)]^d$

and $p \in L_0^2(\Omega)$, such that

$$a(\vec{u},\vec{v}) - b(\vec{v},p) = (\vec{f},\vec{v}), \quad \forall \vec{v} \in X := [H_0^1]^d,$$
$$-b(\vec{u},q) = 0, \quad \forall q \in Q := L_0^2,$$

where

$$b(\vec{v},q) := (q, \nabla\cdot\vec{v}),$$

for all $\vec{v} \in [H_0^1(\Omega)]^d$ and all $q \in L_0^2(\Omega)$.

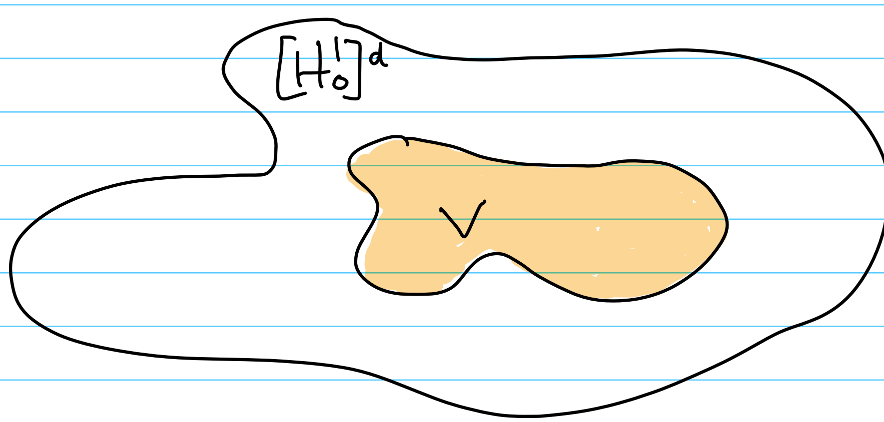
Recall that

(22.3) $$V := \{\vec{v} \in [H_0^1]^d \mid b(\vec{v},q) = 0 \ \forall q \in L_0^2\}.$$

$(\vec{u},p)$ is a weak solution to the Stokes Problem implies that $\vec{u} \in V$ is a solution to

$$a(\vec{u},\vec{v}) = (f,\vec{v}) \quad \forall \vec{v} \in V.$$

V is a closed proper subspace of the Hilbert space $[H_0^1]^d$ and is, therefore, a Hilbert space.



The Inf-Sup Condition

Let's briefly review a couple of facts from last time.

Definition (22.7): Suppose that V is as in (22.3). We define

$$V^{\perp} := \{ \vec{v} \in [H_0^1(\Omega)]^d \mid a(\vec{v},\vec{u}) = 0 \quad \forall\, \vec{u} \in V \},$$

the orthogonal complement of V with respect to $a(\cdot,\cdot)$.

Theorem (22.8): Given any $q \in L_0^2(\Omega)$, there exists a unique $\vec{v} \in V^{\perp}$ such that

$$\nabla \cdot \vec{v} = q \quad (\text{in } L^2(\Omega)).$$

In other words, the divergence operator is an isomorphism from $V^{\perp}$ onto $L_0^2(\Omega)$. Moreover, there is a constant $C > 0$ such that

$$|\vec{v}|_{H^1} \leq C \|q\|_{L^2}$$

Proof: See Girault and Raviart (1986). ///

Theorem (23.1): There exists a constant $\beta > 0$ such that

(23.1) $$\sup_{\substack{\vec{v} \in [H_0^1]^d \\ \vec{v} \neq 0}} \frac{(q, \nabla \cdot \vec{v})}{|\vec{v}|_{H^1}} \geq \beta \|q\|_{L^2}$$

for all $q \in L^2_0(\Omega)$.

<u>Proof</u>: let $q \in L^2_0$, $q \neq 0$, be arbitrary. There is a unique $\vec{v}_q \in V^\perp$ such that

$$\nabla \cdot \vec{v}_q = q .$$

This implies

$$(q, \nabla \cdot \vec{v}_q) = \|q\|^2_{L^2}$$

and

$$|\vec{v}_q|_{H^1} \leq C \|q\|_{L^2} \quad \left(\Longleftrightarrow \quad \frac{1}{|\vec{v}_q|_{H^1}} \geq \frac{1}{C\|q\|_{L^2}} \right)$$

for some $C > 0$ that is independent of q. Then

$$\frac{(q, \nabla \cdot \vec{v}_q)}{|\vec{v}_q|_{H^1}} = \frac{\|q\|^2_{L^2}}{|\vec{v}_q|_{H^1}} \geq \frac{1}{C} \frac{\|q\|^2_{L^2}}{\|q\|_{L^2}} = \frac{\|q\|_{L^2}}{C} .$$

Next, observe that

$$\frac{(q, \nabla \cdot \vec{v}_q)}{|\vec{v}_q|_{H^1}} \leq \sup_{\substack{\vec{v} \in [H^1_0]^d \\ \vec{v} \neq 0}} \frac{(q, \nabla \cdot \vec{v})}{|\vec{v}|_{H^1}} .$$

Therefore, setting $\beta = \frac{1}{C}$ we get the result. ///

Estimate (23.1) is equivalent to the inf-sup condition, which is

(23.2) $$\boxed{\inf_{\substack{q \in L^2_0(\Omega) \\ q \neq 0}} \sup_{\substack{\vec{v} \in [H^1_0]^d \\ \vec{v} \neq 0}} \frac{(q, \nabla \cdot \vec{v})}{\|q\|_{L^2} |\vec{v}|_{H^1}} \geq \beta .}$$

We say the pair $X = [H^1_0]^d$ and $Q = L^2_0$ satisfy the inf-sup condition relative to $b(\cdot, \cdot)$.

Recall from last time that there is a unique solution to the following: $\exists!\ \vec{u} \in V$ such that

$$a(\vec{u},\vec{v}) = (\vec{f},\vec{v}), \qquad \forall\ \vec{v} \in V,$$

where

$$V = \{\vec{u} \in [H_0^1]^d \mid (q, \nabla\cdot\vec{u}) = 0 \quad \forall q \in L_0^2\}.$$

Therefore we can argue that $p \in L_0^2$ from the Stokes problem must solve

(23.3) $$-b(\vec{v},p) = (\vec{f},\vec{v}) - a(\vec{u},\vec{v}), \quad \forall\ \vec{v} \in [H_0^1]^d.$$

once $\vec{u} \in V$ is determined. To solve (23.3), we need an abstract version of the Lax-Milgram lemma.

Theorem (23.2) (Generalized Lax-Milgram lemma): Let U and V be real Hilbert spaces. Suppose

$$\tilde{a} : U \times V \to \mathbb{R}$$

is a continuous bilinear form and $l \in V'$. Assume further that

(23.4) $$\sup_{\substack{v \in V \\ v \neq 0}} \frac{\tilde{a}(u,v)}{\|v\|_V} \geq \alpha \|u\|_U, \quad \forall u \in U,$$

and

(23.5) $$\sup_{u \in U} \tilde{a}(u,v) > 0 \quad \forall v \in V,\ v \neq 0.$$

Then, there is a unique solution $u \in U$

that satisfies

$$\tilde{a}(u,v) = \ell(v), \qquad \forall\, v \in V.$$

Moreover

$$\|u\|_U \le \frac{\|\ell\|_{V'}}{\alpha}.$$

Proof: See Atkinson and Han (2008). ///

Let us interpret our pressure problem (23.3) in the context of the last result.

$$(q, \nabla\cdot\vec{v}) =: b(\vec{v}, q) = \tilde{a}(q, \vec{v})$$

$$U = L^2_0(\Omega), \quad V = [H^1_0]^d$$

$$\ell(\vec{v}) = a(\vec{u}, \vec{v}) - (\vec{f}, \vec{v}), \quad \forall\, \vec{v} \in [H^1_0]^d$$

Condition (23.4) is satisfied because of the inf-sup condition (23.2). The only condition left to check is (23.5).

In the language of the Stokes Problem, we must show that

$$\sup_{q \in L^2_0} (q, \nabla\cdot\vec{v}) > 0 \qquad \forall\, \vec{v} \in [H^1_0]^d$$

Setting $q = \nabla\cdot\vec{v}$, it is clear that

$$(q, \nabla\cdot\vec{v}) = \|\nabla\cdot\vec{v}\|^2_{L^2} \geq 0.$$

Theorem (23.2): There is a unique weak solution $(\vec{u}, p) \in [H_0^1]^d \times L_0^2$ to the Stokes' problem. Additionally, the pressure solves

$$(p, \nabla \cdot \vec{v}) = -(\vec{f}, \vec{v})$$

for all $\vec{v} \in V^\perp$.

Proof: Use the Generalized Lax-Milgram lemma and the inf-sup condition. ///

Theorem (23.3): Let $(\vec{u}, p) \in [H_0^1]^d \times L_0^2$ be the unique solution to (22.2), the Stokes' problem. Then,

(23.6) $$|\vec{u}|_{H^1} + \|p\|_{L^2} \leq \left(1 + \frac{2}{\beta}\right) \|\vec{f}\|_{H^{-1}}.$$

Proof: Recall

$$|\vec{u}|_{H^1} \overset{RRT}{=} \|\vec{f}\|_{V'} \leq \|\vec{f}\|_{H^{-1}} =: \sup_{\substack{\vec{v} \in [H_0^1]^d \\ \vec{v} \neq 0}} \frac{|(\vec{f}, \vec{v})|}{|\vec{v}|_{H^1}}$$

Using the inf-sup condition, for any $\vec{v} \in [H_0^1]^d$

$$\begin{aligned} b(\vec{v}, p) = (p, \nabla \cdot \vec{v}) &= (\nabla \vec{u}, \nabla \vec{v}) - (\vec{f}, \vec{v}) \\ &\overset{C.S.}{\leq} \|\nabla \vec{u}\|_{L^2} \|\nabla \vec{v}\|_{L^2} + \|\vec{f}\|_{H^{-1}} \|\nabla \vec{v}\|_{L^2} \\ &= \left(\|\nabla \vec{u}\|_{L^2} + \|\vec{f}\|_{H^{-1}}\right) \|\nabla \vec{v}\|_{L^2} \\ &\leq 2 \|\vec{f}\|_{H^{-1}} \|\nabla \vec{v}\|_{L^2}. \end{aligned}$$

Hence,

$$\frac{(p, \nabla\cdot\vec{v})}{|\vec{v}|_{H^1}} \leq 2\,\|\vec{f}\|_{H^{-1}}, \qquad \forall\, v \in [H_0^1]^d.$$

Thus

$$2\,\|\vec{f}\|_{H^{-1}} \geq \sup_{\substack{\vec{v}\in[H_0^1]^d \\ \vec{v}\neq 0}} \frac{(p, \nabla\cdot\vec{v})}{|\vec{v}|_{H^1}} \overset{(23.1)}{\geq} \beta\,\|p\|_{L^2}.$$

Putting the estimates together,

$$\begin{aligned}|\vec{u}|_{H^1} + \|p\|_{L^2} &\leq \|\vec{f}\|_{H^{-1}} + \frac{2}{\beta}\|\vec{f}\|_{H^{-1}} \\ &= \left(1+\frac{2}{\beta}\right)\|\vec{f}\|_{H^{-1}} \quad ///\end{aligned}$$

This is the basic stability of the Stokes' Problem.

Mixed FEM for Stokes Problem

let

$$X_h \subset X := [H_0^1(\Omega)]^d, \qquad \dim(X_h) = M$$

$$Q_h \subset Q := L_0^2(\Omega), \qquad \dim(Q_h) = N.$$

Typically $M > N$. Suppose that we have the respective bases

$$B_X = \{\vec{\phi}_1, \dots, \vec{\phi}_M\} \subset X_h$$

$$B_Q = \{\psi_1, \dots, \psi_N\} \subset Q_h$$

The <u>Galerkin</u> <u>approximation</u> is given as usual:

Find $\vec{u}_h \in X_h$ and $p_h \in Q_h$ such that

$$a(\vec{u}_h, \vec{v}_h) - b(\vec{v}_h, p_h) = (\vec{f}, \vec{v}_h), \quad \forall \vec{v}_h \in X_h,$$
$$-b(\vec{u}_h, q_h) = 0, \quad \forall q_h \in Q_h.$$

where

$$a(\vec{u}, \vec{v}) = (\nabla \vec{u}, \nabla \vec{v})_{L^2}, \quad \forall \vec{u}, \vec{v} \in X$$
$$b(\vec{v}, q) = (q, \nabla \cdot \vec{v})_{L^2}, \quad \forall \vec{v} \in X, q \in Q.$$

Now, set

$$\vec{u}_h = \sum_{j=1}^{M} u_j \phi_j,$$

$$p_h = \sum_{j=1}^{N} p_j \psi_j.$$

Define

$$A = [a_{i,j}] \in \mathbb{R}^{M \times M}, \quad a_{i,j} = a(\vec{\phi}_j, \vec{\phi}_i),$$
$$B = [b_{i,j}] \in \mathbb{R}^{M \times N}, \quad b_{i,j} = b(\vec{\phi}_i, \psi_j).$$

Define

$$\vec{u} = [u_i], \quad i = 1, \cdots, M,$$

$$\vec{p} = [p_i], \quad i = 1, \cdots, N.$$

Finally, define

$$\vec{f} = [f_i], \qquad f_i := (\vec{f}, \vec{\phi}_i), \quad i = 1, \dots, M.$$

Then,

$$\begin{aligned} A\vec{u} - B\vec{p} &= \vec{f} \\ -B^T\vec{u} \quad &= \vec{0} \end{aligned}$$

Typically, as we have indicated, $M > N$. We will show this shortly. Therefore, in block form,

$$\begin{bmatrix} A & -B \\ -B^T & 0 \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{p} \end{bmatrix} = \begin{bmatrix} \vec{f} \\ \vec{0} \end{bmatrix}$$

The coefficient matrix

$$C = \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix}$$

is symmetric, but not SPD. Why?

In the next lectures, we will find sufficient conditions to guarantee that C is invertible.