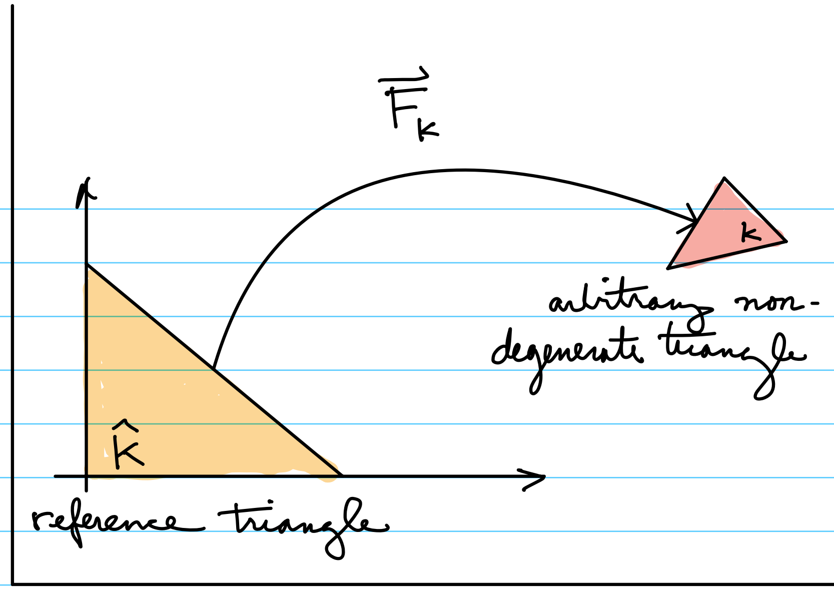
Math 574
class #18
10/28/2025



Let's switch gears and have a look at a homework problem. Let's prove an inverse ineq.:

$$|v|_{H^1} \le \frac{C}{h} \|v\|_{L^2}, \quad \forall\, v \in V_h = \mathcal{M}_r.$$

Proof: On the reference triangle all norms are equivalent: $\exists$ $C_1, C_2 > 0$ such that

$$C_1 \|\hat{v}\|_{L^2(\hat{K})} \le \|\hat{v}\|_{H^1(\hat{K})} \le C_2 \|\hat{v}\|_{L^2(\hat{K})}$$

for all $\hat{v} \in \mathbb{P}_r(\hat{K})$, since $\mathbb{P}_r(\hat{K})$ is finite dimensional.

Thus,

$$\|\hat{v}\|^2_{H^1(\hat{K})} \le C_2^2 \|\hat{v}\|^2_{L^2(\hat{K})}$$

which implies

$$|\hat{v}|^2_{H^1(\hat{K})} \le \|\hat{v}\|^2_{L(\hat{K})} + |\hat{v}|^2_{H^1(\hat{K})} \le C_2^2 \|\hat{v}\|^2_{L^2(\hat{K})}$$

The constant C_2 depends upon r and $\hat{K}$, but is independent of $\hat{v}$:

$$|\hat{v}|^2_{H^1(\hat{K})} \le C_2^2 \|\hat{v}\|^2_{L^2(\hat{K})}, \quad \forall\, \hat{v} \in \mathbb{P}_r(\hat{K}).$$

Now, let $v \in \mathbb{P}_r(K)$ be arbitrary. Then

$$\hat{v}(\cdot) := v(\vec{F}_K(\cdot)) \in \mathbb{P}_r(\hat{K}).$$

We know that

$$\int_K f(\vec{x})\,d\vec{x} = 2|K| \int_{\hat{K}} f(\vec{F}_K(\hat{\vec{x}}))\,d\hat{\vec{x}}$$

in general. So

$$\|v\|^2_{L^2(K)} = \int_K v^2(\vec{x})\,d\vec{x}$$

$$= 2|K| \int_{\hat{K}} \hat{v}^2(\hat{\vec{x}})\,d\hat{\vec{x}}$$

$$= 2|K|\,\|\hat{v}\|^2_{L^2(\hat{K})}$$

We need to transform

$$|v|^2_{H^1(K)} = \sum_{|\alpha|=1} \int_K \left(\partial^\alpha v(\vec{x})\right)^2 d\vec{x}$$

It suffices to use an estimate: $\forall\, v \in H^m(K)$,

$$|v|_{H^m(K)} \le C\,\|B_K^{-1}\|_2^m\,|\det B_K|^{1/2}\,|\hat{v}|_{H^m(\hat{K})}.$$

We have then, for all $v \in \mathbb{P}_r(K)$, $(m=1)$

$$|v|_{H^1(K)} \le C\,\|B_K^{-1}\|_2\,|\det B_K|^{1/2}\,|\hat{v}|_{H^1(\hat{K})}$$

$$\leq C C_2 \|B_K^{-1}\|_2 |\det(B_K)|^{1/2} \|\hat{v}\|_{L^2(\hat{K})}$$

$$= \frac{C C_2 \|B_K^{-1}\|_2 |\det(B_K)|^{1/2}}{\sqrt{2|K|}} \|v\|_{L^2(K)}$$

Recall that

$$\det(B_K) = 2|K|$$

and

$$\|B_K^{-1}\|_2 \leq \frac{\hat{h}}{\rho_K} = \frac{\sqrt{2}}{\rho_K}$$

Hence

$$|v|_{H^1(K)} \leq \frac{C}{\rho_K} \|v\|_{L^2(K)}, \quad \forall v \in \mathbb{P}_r(K).$$

or

$$|v|^2_{H^1(K)} \leq \frac{C}{\rho_K^2} \|v\|^2_{L^2(K)}, \quad \forall v \in \mathbb{P}_r(K).$$

Let $\mathcal{T}_h$ be shape regular and quasi-uniform.
Then

$$\text{(shape reg)} \quad 1 \leq \frac{h_K}{\rho_K} \leq \sigma, \quad \forall K \in \mathcal{T}_h, \quad \forall h.$$

$$\text{(quasi-uni)} \quad 0 < C_q \leq \frac{h_K}{h} \leq 1, \quad \forall K \in \mathcal{T}_h, \quad \forall h.$$

Suppose $v \in V_h = \mathcal{M}_r$ is arbitrary. Then

$$|v|^2_{H^1(\Omega)} = \sum_{K \in \mathcal{T}_h} |v|^2_{H^1(K)} \qquad \left(\int_\Omega [\cdot] \, d\vec{x} = \sum_{K \in \mathcal{T}_h} \int_K [\cdot] \, d\vec{x} \right)$$

$$\boxed{0 < C_q h \leq h_K \Rightarrow \frac{1}{h_K} \leq \frac{1}{C_q h}}$$

$$\leq \sum_{K \in \mathcal{T}_h} \frac{C}{\rho_K^2} \|v\|_{L^2(K)}^2$$

$$= \sum_{K \in \mathcal{T}_h} C \frac{h_K^2}{\rho_K^2} h_K^{-2} \|v\|_{L^2(K)}^2$$

$$\leq \sum_{K \in \mathcal{T}_h} C \sigma^2 \frac{1}{C_q^2 h^2} \|v\|_{L^2(K)}^2$$

$$= \frac{C}{h^2} \|v\|_{L^2(\Omega)}^2$$

Thus, there is some $C > 0$, such that

$$|v|_{H^1(\Omega)} \leq \frac{C}{h} \|v\|_{L^2(\Omega)}$$

for all $v \in V_h = \mathcal{M}_r$. ///

A Quick Review

FEM for Parabolic Problems

Diffusion Problem: Find $u \in C^2(\bar{\Omega} \times [0,T])$ such that

(17.1)
$$\begin{cases} \partial_t u - \Delta u = f(\cdot, t) & \text{in } \Omega, \ 0 < t \leq T, \\ u = 0, & \text{on } \partial\Omega \ 0 < t \leq T, \\ u(\cdot, 0) = v, & \text{in } \Omega. \end{cases}$$

Defn (17.3): Let $f \in L^2(0,T; L^2)$ and $v \in L^2(\Omega)$ be given. u is called a weak solution to the diffusion problem iff

1) $u \in L^2(0,T; H^1_0)$

2) u is weakly differentiable in time and
$$\partial_t u \in L^2(0,T; H^{-1})$$

3) For almost every $t \in [0,T]$ and for every $\phi \in H^1_0(\Omega)$

(17.2)
$$\langle \partial_t u, \phi \rangle + (\nabla u, \nabla v) = (f, \phi) := (f, \phi)_{L^2}$$

4) Finally
$$\lim_{t \to 0} \| u(t) - v \|_{L^2} = 0.$$

Theorem (17.5) Suppose that u is a weak solution to the diffusion problem. Then

$$u \in C([0,T]; L^2) \quad \text{(Sobolev-like)}$$

and

$$\max_{0 \le t \le T} \|u(t)\|_{L^2}^2 \le \|v\|_{L^2}^2 + C\int_0^T \|f\|_{L^2}^2 \, dt$$

and

$$\int_0^T \|u(t)\|_{H_0^1}^2 \, dt \le \|v\|_{L^2}^2 + C\int_0^T \|f\|_{L^2}^2 \, dt.$$

If u has additional regularities

$$u \in L^2(0,T; H_0^1 \cap H^2) \qquad \partial_t u \in L^2(0,T; L^2)$$

with $v \in H_0^1(\Omega)$, then

$$u \in C([0,T]; H_0^1)$$

and

$$\max_{0 \le t \le T} \|u(t)\|_{H_0^1}^2 \le \|v\|_{H_0^1}^2 + \int_0^T \|f\|_{L^2}^2 \, dt$$

$$\int_0^T \|\Delta u\|_{L^2}^2 \, dt \le \|v\|_{H_0^1}^2 + \int_0^T \|f\|_{L^2}^2 \, dt$$

Proof: Last time we gave a rigorous proof. This time we give a more intuitive, but formal, argument.

Let us start with the strong form:

$$\partial_t u - \Delta u = f(x,t)$$

Multiply by u and integrate:

$$\int_\Omega \partial_t u\, u\, d\vec{x} - \int_\Omega \Delta u\, u\, d\vec{x} = \int_\Omega f\, u\, d\vec{x}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\, d\vec{x} + \int_\Omega |\nabla u|^2\, d\vec{x} = \int_\Omega f\, u\, d\vec{x}$$

or

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + |u|_{H^1}^2 &= (f,u)\\ &\overset{\text{C.S.}}{\leq} \|f\|_{L^2}\|u\|_{L^2}\\ &\overset{\text{Poincare}}{\leq} C\|f\|_{L^2}|u|_{H^1}\\ &\overset{\text{AGMI}}{\leq} \frac{Cc}{2}\|f\|_{L^2}^2 + \frac{C}{2c}|u|_{H^1}^2\end{aligned}$$

Hence,

$$\frac{d}{dt}\|u\|_{L^2}^2 + |u|_{H^1}^2 \leq C\|f\|_{L^2}^2.$$

Now, we integrate in time $t=0$ to $t=s\in(0,T]$:

$$\int_0^s \frac{d}{dt}\|u(t)\|_{L^2}^2\, dt + \int_0^s |u(t)|_{H^1}^2\, dt \leq C\int_0^s \|f(t)\|_{L^2}^2\, dt$$

$$\begin{aligned}\|u(s)\|_{L^2}^2 - \|v\|_{L^2}^2 + \int_0^s |u(t)|_{H^1}^2\, dt &\leq C\int_0^s \|f(t)\|_{L^2}^2\, dt\\ &\leq C\int_0^T \|f(t)\|_{L^2}^2\, dt\end{aligned}$$

It follows that, for any $s\in(0,T]$,

$$\|u(s)\|_{L^2}^2 + \int_0^s |u(t)|_{H^1}^2\,dt \le \underbrace{\|v\|_{L^2}^2 + \int_0^T \|f(t)\|_{L^2}^2\,dt}_{\text{independent of } s}$$

From the last estimate it follows that, for every $s \in [0,T]$,

$$\|u(s)\|_{L^2}^2 \le \|v\|_{L^2}^2 + \int_0^T \|f(t)\|_{L^2}^2\,dt$$

Thus,

$$\max_{0 \le s \le T} \|u(s)\|_{L^2}^2 \le \|v\|_{L^2}^2 + \int_0^T \|f(t)\|_{L^2}^2\,dt.$$

By a similar argument

$$\int_0^T |u(t)|_{H^1}^2\,dt \le \|v\|_{L^2}^2 + \int_0^T \|f(t)\|_{L^2}^2\,dt.$$

It follows that

$$\max_{0 \le s \le T} \|u(s)\|_{L^2}^2 + \int_0^T |u(t)|_{H^1}^2\,dt \le 2\left\{\|v\|_{L^2}^2 + \int_0^T \|f(t)\|_{L^2}^2\,dt\right\}.$$

Again, starting with the strong form, multiply by $-\Delta u$ and integrate:

$$-\int_\Omega \partial_t u\,\Delta u\,dx + \int_\Omega (\Delta u)^2\,d\vec{x} = -\int_\Omega f\,\Delta u\,d\vec{x}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx + \|\Delta u\|_{L^2}^2 = -(f, \Delta u)$$

$$\overset{CS}{\leq} \|f\|_{L^2} \|\Delta u\|_{L^2}$$

$$\overset{AGMI}{\leq} \frac{1}{2}\|f\|_{L^2}^2 + \frac{1}{2}\|\Delta u\|_{L^2}^2$$

Thus,

$$\frac{d}{dt}|u|_{H^1}^2 + \|\Delta u\|_{L^2}^2 \leq \|f\|_{L^2}^2$$

Integrating in time from $t=0$ to $t=s\in(0,T]$,

$$|u(s)|_{H^1}^2 - |v|_{H^1}^2 + \int_0^s \|\Delta u(t)\|_{L^2}^2 dt$$

$$\leq \int_0^s \|f(t)\|_{L^2}^2 dt$$

$$\leq \int_0^T \|f(t)\|_{L^2}^2 dt.$$

We get

$$|u(s)|_{H^1}^2 + \int_0^s \|\Delta u(t)\|_{L^2}^2 dt \leq |v|_{H^1}^2 + \int_0^T \|f(t)\|_{L^2}^2 dt,$$

from which follows

$$\max_{0\leq s\leq T} |u(s)|_{H^1}^2 \leq |v|_{H^1}^2 + \int_0^T \|f(t)\|_{L^2}^2 dt$$

and

$$\int_0^T \|\Delta u(t)\|_{L^2}^2 dt \leq |v|_{H^1}^2 + \int_0^T \|f(t)\|_{L^2}^2 dt.$$

Special Case

Suppose that $f \equiv 0$. Then we can show that

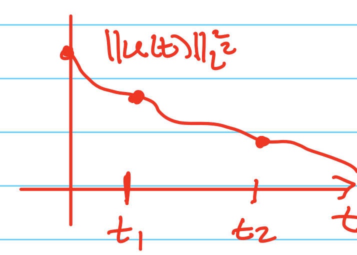
$$\max_{0 \le t \le T} \|u(t)\|_{L^2}^2 \le \|v\|_{L^2}^2$$

$$\max_{0 \le t \le T} \|u(t)\|_{H_0^1}^2 \le \|v\|_{H_0^1}^2$$

In this case, it is easy to show that

$$\boxed{\begin{aligned} \|u(t_2)\|_{L^2}^2 &\le \|u(t_1)\|_{L^2}^2 \\ \|u(t_2)\|_{H_0^1}^2 &\le \|u(t_1)\|_{H_0^1}^2 \end{aligned}}$$

exercise



for all $0 \le t_1 \le t_2 \le T$.

Continuous Dependence on Initial Data

Suppose that u_1, u_2 solve

$$\partial_t u_i - \Delta u_i = f \quad \text{in } \Omega \quad 0 < t \le T$$

$$u_i = 0 \quad \text{on } \partial\Omega \quad 0 < t \le T$$

$$u_i(0) = v_i \quad \text{in } \Omega$$

Suppose that both satisfy the diffusion problem in the weak sense. Then, $e := u_2 - u_1$ is a weak solution to

$$\partial_t e - \Delta e = 0, \quad \text{in } \Omega \quad 0 < t \le T,$$
$$e = 0, \quad \text{on } \partial\Omega \quad 0 < t \le T,$$
$$e(0) = v_2 - v_1, \quad \text{in } \Omega.$$

and

$$\max_{0 \le t \le T} \| e(t) \|_{L^2}^2 = \max_{0 \le t \le T} \| u_2(t) - u_1(t) \|_{L^2}^2$$
$$\le \| v_2 - v_1 \|_{L^2}^2 .$$

If additional regularities hold,

$$\max_{0 \le t \le T} \| e \|_{H_0^1}^2 = \max_{0 \le t \le T} \| u_2(t) - u_1(t) \|_{H_0^1}^2$$
$$\le \| v_2 - v_1 \|_{H_0^1}^2 .$$

The theory above guarantees that weak solns are unique!

Semi-Discrete FEM for Diffusion Problem

Suppose that Ω is a polygonal domain in $\mathbb{R}^2$.
Let

$$V_h = \mathcal{M}_{0,r} \subset H_0^1(\Omega),$$

with basis

$$\mathcal{B} = \{ \phi_1, \cdots, \phi_{N_r^0} \}$$

Defn (18.1): The Galerkin approximation of the of the weak form of the diffusion problem is defined as follows:

Find $u_h : [0,T] \to V_h = \mathcal{M}_{0,r}$, in the form

$$u_h(\cdot,t) = \sum_{i=1}^{N_r^0} u_i(t)\,\phi_i$$

such that

(18.1) $$(\partial_t u_h, \psi)_{L^2} + (\nabla u_h, \nabla \psi) = (f, \psi)$$

for all $\psi \in V_h$, for all $t \in (0,T]$, with

(18.2) $$u_h(0) = v_h := P_h v \in V_h.$$

Here P_h is some projection operator from L^2 to V_h, ie,

$$P_h : L^2 \longrightarrow V_h$$

and

$$P_h(P_h v) = P_h v \qquad \forall\, v \in L^2.$$

We will discuss some possibilities.

The defn above is equivalent to

(18.3) $$(\partial_t u_h(t), \phi_i) + a(u_h(t), \phi_i) = (f(t), \phi_i),$$

for $i = 1, \cdots, N_r^0$, for all $t \in (0,T]$.

Proposition (18.2): Define

$$G = [g_{i,j}]_{i,j=1}^{N_r^0} \qquad A = [a_{i,j}]_{i,j=1}^{N_r^0}$$

via

$$g_{i,j} := (\phi_j, \phi_i)_{L^2} \qquad \text{(mass matrix)}$$

and

$$a_{i,j} := a(\phi_j, \phi_i)_{L^2} \qquad \text{(stiffness matrix)}$$

for all $1 \le i,j \le N_r^0$. Define $\vec{f} : [0,T] \to \mathbb{R}^{N_r^0}$ via

$$\vec{f}(t) = [f_i(t)]_{i=1}^{N_r^0}, \qquad f_i(t) := (f(t), \phi_i)_{L^2}.$$

Finally, define $\vec{v} = [v_i] \in \mathbb{R}^{N_r^0}$ via

$$v_i := (v_h, \phi_i)_{L^2}.$$

Let $u_h : [0,T] \to M_{0,r}$ be a strong ODE solution to (18.1) subject to the initial conditions (18.2), then

(18.4) $$G\vec{u}'(t) + A\vec{u}(t) = \vec{f}(t)$$

where

(18.5) $$u_h(\cdot, t) = \sum_{i=1}^{N_r^0} u_i(t)\, \phi_i$$

and

(18.6) $$\vec{u}(t) = [u_i(t)] \in \mathbb{R}^{N_r^0},$$

subject to the initial condition

(18.7) $$G\vec{u}(0) = \vec{v} \in \mathbb{R}^{N_r^0}.$$

Proof: Let us start with (18.3). Inserting (18.5) into (18.3), we get, for $i = 1, \dots, N_r^0$.

$$\sum_{j=1}^{N_r^0} (\phi_j, \phi_i) \frac{du_j}{dt}(t) + \sum_{j=1}^{N_r^0} a(\phi_j, \phi_i) u_j(t) = f_i(t).$$

In other words,

$$\sum_{j=1}^{N_r^0} g_{j,i} \, u_j'(t) + \sum_{j=1}^{N_r^0} a_{j,i} u_j(t) = f_i(t),$$

which is the component form of (18.4).

Finally, using (18.2), for $1 \le i \le N_r^0$,

$$(u_h(0), \phi_i)_{L^2} = (v_h, \phi_i)_{L^2}$$

This implies that

$$\begin{aligned}
(v_h, \phi_i)_{L^2} &= \Big(\sum_{j=1}^{N_r^0} u_j(0) \phi_j , \phi_i \Big)_{L^2} \\
&= \sum_{j=1}^{N_r^0} g_{i,j} \, u_j(0) \\
&= \left[G \vec{u}(0) \right]_i .
\end{aligned}$$

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