Math 574
class # 08
09/11/2025

1D Finite Elements (Cont.)

In the special case that

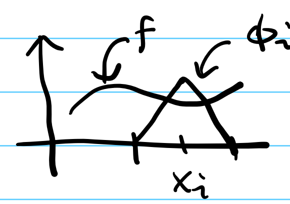
$$h_i = h > 0, \quad \forall\, i \in \{1, \cdots, M+1\},$$

$$a(\phi_j, \phi_i) = a_{i,j} = [A]_{i,j} = \begin{cases} 2/h, & i = j \\ -1/h, & i = j-1 \neq M \\ -1/h, & i = j+1 \neq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$A = \begin{bmatrix} 2/h & -1/h & & & 0 \\ -1/h & 2/h & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1/h \\ 0 & & & -1/h & 2/h \end{bmatrix} \in \mathbb{R}^{M \times M}_{sym}$$

The Forcing Vector

For $1 \leq i \leq M$,



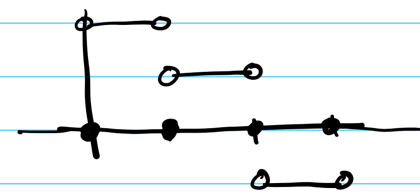
$$\begin{aligned} f_i = [\vec{f}]_i &= (f, \phi_i)_{L^2} \\ &= \int_0^1 f(x)\, \phi_i(x)\, dx \\ &= \int_{x_{i-1}}^{x_i} f(x) \frac{x - x_{i-1}}{h_i}\, dx \\ &\quad + \int_{x_i}^{x_{i+1}} f(x) \frac{(x_{i+1} - x)}{h_{i+1}}\, dx \end{aligned}$$

If $h_i = h$, $1 \le i \le M+1$ (uniform mesh), then

$$f_i = \frac{1}{h}\left\{ \int_{x_{i-1}}^{x_i} f(x)(x - x_{i-1})\,dx + \int_{x_i}^{x_{i+1}} f(x)(x_{i+1} - x)\,dx \right\}$$

Suppose that f is constant on each triangle K_i

$$f(x) = \sum_{i=1}^{M+1} \chi_i \tilde{f}_i$$



where

$$\chi_i = \chi_{K_i} \qquad 1 \le i \le M+1$$

Then

$$f_i = \frac{1}{h}\left\{ \int_{x_{i-1}}^{x_i} \tilde{f}_i (x - x_{i-1})\,dx + \int_{x_i}^{x_{i+1}} \tilde{f}_{i+1} (x_{i+1} - x)\,dx \right\}$$

$$= \frac{1}{h}\left\{ \tilde{f}_i\, h^2/2 + \tilde{f}_{i+1}\, h^2/2 \right\}$$

$$= \frac{h}{2}\left(\tilde{f}_i + \tilde{f}_{i+1}\right), \quad 1 \le i \le M.$$

Recall that, the Galerkin approx $u_h \in \mathcal{M}_{0,1}$ satisfies

$$a(u_h, \phi_i) = f(\phi_i), \qquad i = 1, \dots, M.$$

Since $B_{0,1}$ is a basis for $\mathcal{M}_{0,1}$, we know that

$$u_h = \sum_{i=1}^{M} u_i \phi_i$$

and

$$A\vec{u} = \vec{f}$$

where

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \end{bmatrix} \in \mathbb{R}^M$$

is the displacement vector.

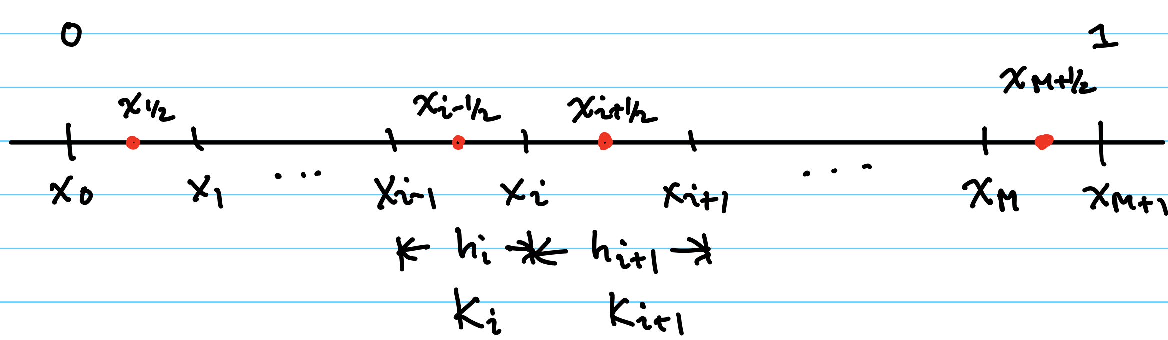
We showed previously that A is SPD.

In this case, A is also sparse (mostly zeros) and tridiagonal.

Quadratic Elements in 1D

Suppose $\Omega = (0,1)$ and $P = \{x_i\}_{i=0}^{M+1}$ is a partition. Define

$$x_{i+1/2} := \frac{x_{i+1} + x_i}{2}, \quad 0 \le i \le M.$$



$$h_i := x_i - x_{i-1}, \quad 1 \le i \le M+1$$

$$K_i := (x_{i-1}, x_i), \quad 1 \le i \le M+1$$

Defn: The <u>quadratic nodal set</u> is the set

$$\mathcal{N}_2 := \{x_0, x_{1/2}, x_1, x_{3/2}, \dots, x_M, x_{M+1/2}, x_M\}$$
$$= \{x_i\}_{i=0}^{M+1} \cup \{x_{i+1/2}\}_{i=0}^{M}$$
$$= P \cup \{x_{i+1/2}\}_{i=0}^{M}$$

<u>Midpoint Nodes</u>

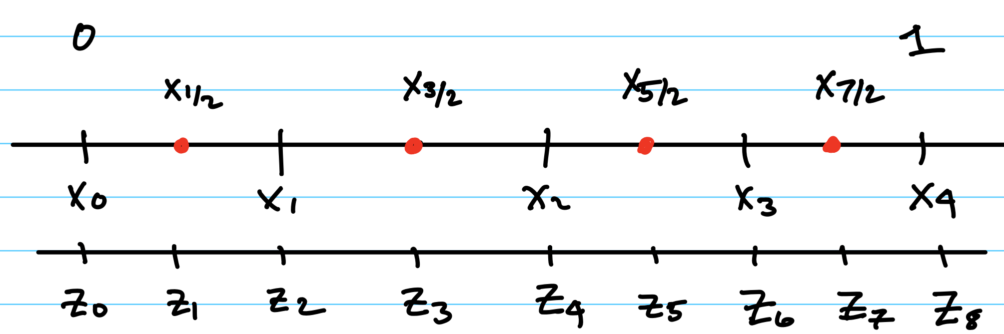
Define
$$\mathcal{N}_{0,2} := \{x_i\}_{i=1}^{M} \cup \{x_{i+1/2}\}_{i=0}^{M} = \mathcal{N}_2 \setminus \{x_0, x_{M+1}\}.$$

The set $\mathcal{N}_{0,2}$ does not contain the endpoint.

We will usually write
$$\mathcal{N}_2 = \{z_i\}_{i=0}^{N+1}, \quad 0 = z_0 < z_1 < \dots < z_{N+1} = 1,$$
where
$$N = 2M+1,$$
and
$$\mathcal{N}_{0,2} = \{z_i\}_{i=1}^{N}.$$

Example: $M = 3$. Then $N = 7$.

$$\#(\mathcal{N}_2) = 9 \qquad \#(\mathcal{N}_{0,2}) = 7$$



Defn: Let $\phi_i \in \mathcal{M}_2$ be defined via the rule

$$\phi_i(z_j) = \delta_{i,j} \qquad 0 \le i \le N \tag{8.1}$$

where

$$\mathcal{N}_2 = \{z_j\}_{j=0}^{N+1}, \quad 0 = z_0 < z_1 < \cdots < z_{M+1} = 1$$

is the quadratic nodal sets.

Proposition: The functions ϕ_i above are uniquely defined by (8.1). Furthermore, the sets

$$\mathcal{B}_2 = \{\phi_i\}_{i=0}^{N+1=2M+2}$$

and

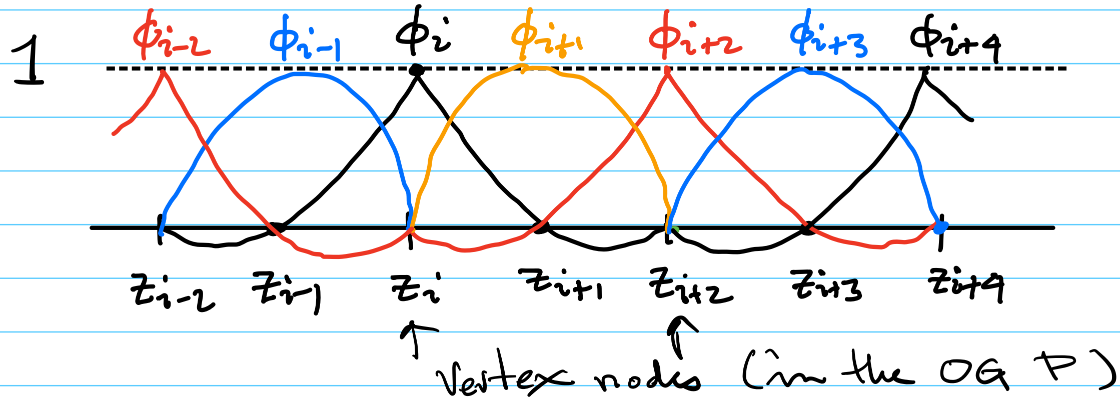
$$\mathcal{B}_{0,2} = \{\phi_i\}_{i=1}^{N=2M+1}$$

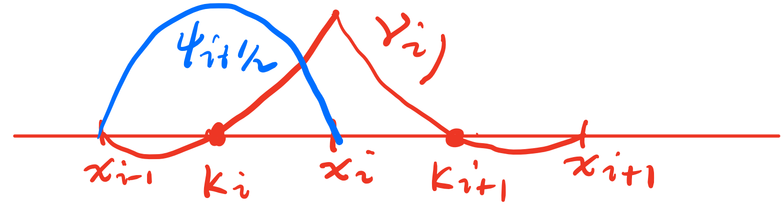
are bases for the spaces $\mathcal{M}_2 \overset{ss}{\subseteq} H^1(0,1)$ and $\mathcal{M}_{0,2} \overset{ss}{\subseteq} H^1_0(0,1)$, respectively. Thus,

$$\dim(\mathcal{M}_2) = 2M+3 = N+2, \quad \dim(\mathcal{M}_{0,2}) = 2M+1 = N$$

Proof: Exercise. ///

What do these basis look like?





It is often easier to break these into two separate function categories.

$$\boxed{\begin{array}{l} K_0 := \emptyset \\ K_{M+2} := \emptyset \end{array}}$$

$$\nu_i(x) = \begin{cases} 2(x - x_{i-1})(x - x_{i-1/2})\frac{1}{h_i^2}, & x \in \overline{K_i} \\ 2(x_{i+1} - x)(x_{i+1/2} - x)\frac{1}{h_i^2}, & x \in \overline{K_{i+1}} \\ 0 & \text{otherwise} \end{cases}$$

for $i = 0, 1, \cdots, M+1$, and

@ midpoints

$$\psi_{i+1/2}(x) = \begin{cases} 4(x_i - x)(x - x_{i-1})\frac{1}{h_i^2} & x \in K_i \\ 0 & \text{otherwise} \end{cases}$$

for $i = 0, 1, \cdots, M$.

Then, we have the correspondence

$$\nu_i = \phi_{2i} \qquad i = 0, \cdots, M+1$$

and

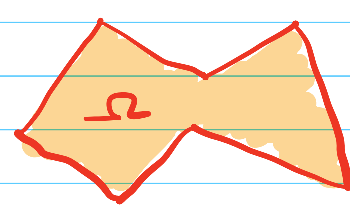
$$\psi_{i+1/2} = \phi_{2i+1} \qquad i = 0, \cdots, M$$

These functions $\psi_{i+1/2}$ are called <u>bubble functions</u>.

<u>FEM in 2D</u>

<u>Defn</u>: Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, polygonal domain. $\mathcal{T}_h = \{K\}$ is a <u>triangulation</u> of Ω iff

1) $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} \overline{K}$



2) Each $K \subseteq \Omega$ is an open, non-degenerate triangle

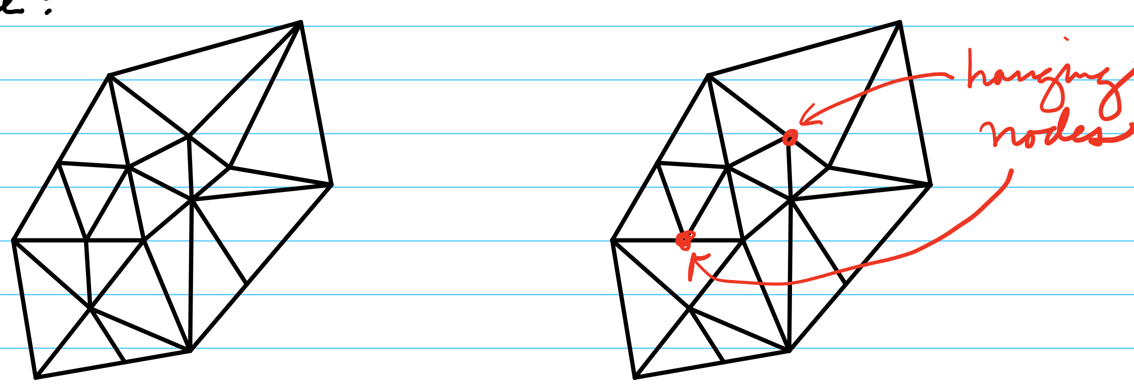
3) For distinct triangles $K_i, K_j \in \mathcal{T}_h$,

$$K_i \cap K_j = \emptyset \qquad \text{(disjoint)}$$

and

$$\overline{K}_i \cap \overline{K}_j = \begin{cases} \emptyset \\ \quad \text{or} \\ \text{a common vertex (a point)} \\ \quad \text{or} \\ \text{a common full edge of } \overline{K}_i \text{ and } \overline{K}_j \end{cases}$$

Example:



yes! no!

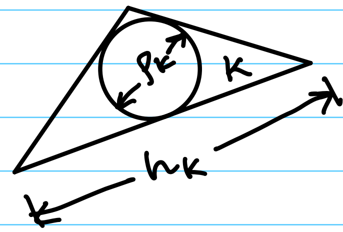
<u>Def</u>: Let $\mathcal{T}_h = \{K\}$ be a triangulation of a polygonal domain $\Omega \subset \mathbb{R}^2$. Define

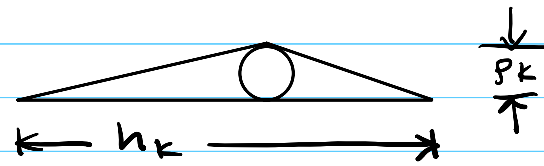
$$h_K := \max\{\|\vec{x} - \vec{y}\|_2 \mid \vec{x}, \vec{y} \in \overline{K}\}$$

and

$$h := \max_{K \in \mathcal{T}_h} h_K$$

The number $\rho_K > 0$ is the diameter of the largest inscribed circle in K





The ratio $\frac{h_K}{\rho_K} > 1$ is called the <u>chunkiness parameter</u> and will be important for our analysis later.