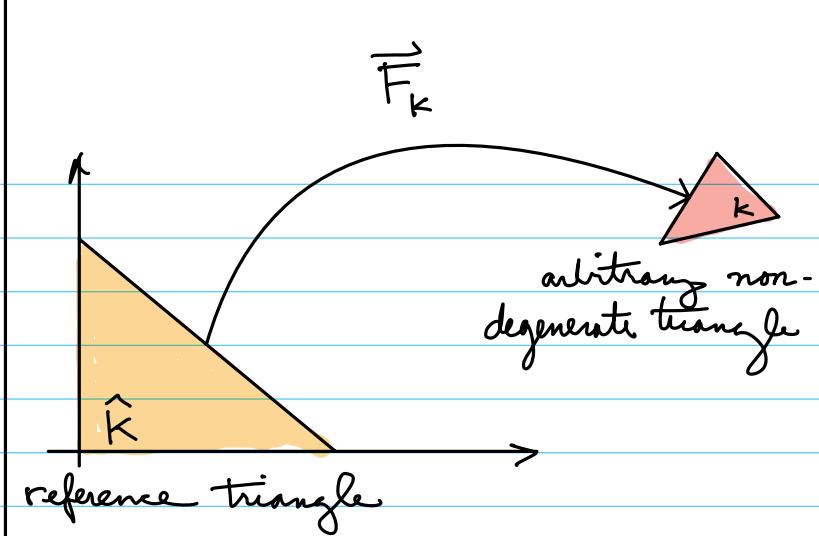


Math 574
class #18
10/28/2025

lets' switch gears
and have a look
at a homework
problem. lets prove
an inverse ineq.:



$$\|v\|_{H^1} \leq \frac{C}{r} \|v\|_{L^2}, \quad \forall v \in V_r = M_r.$$

Proof: On the reference triangle all norms are equivalent: $\exists C_1, C_2 > 0$ such that

$$C_1 \|\hat{v}\|_{L^2(\hat{K})} \leq \|\hat{v}\|_{H^1(\hat{K})} \leq C_2 \|\hat{v}\|_{L^2(\hat{K})}$$

for all $\hat{v} \in \hat{V}_r(\hat{K})$, since $\hat{V}_r(\hat{K})$ is finite dimensional.

Thus,

$$\|\hat{v}\|_{H^1(\hat{K})}^2 \leq C_2^2 \|\hat{v}\|_{L^2(\hat{K})}^2$$

which implies

$$\|\hat{v}\|_{H^1(\hat{K})}^2 \leq \|\hat{v}\|_{L^2(\hat{K})}^2 + \|\hat{v}\|_{H^1(\hat{K})}^2 \leq C_2^2 \|\hat{v}\|_{L^2(\hat{K})}^2$$

The constant C_2 depends upon r and \hat{K} , but is independent of \hat{v} :

$$\|\hat{v}\|_{H^1(\hat{K})}^2 \leq C_2^2 \|\hat{v}\|_{L^2(\hat{K})}^2, \quad \forall \hat{v} \in \hat{V}_r(\hat{K}).$$

Now, let $v \in V_r(K)$ be arbitrary. Then

$$\hat{v}(\cdot) := v(F_K(\cdot)) \in \hat{V}_r(\hat{K}).$$

We know that

$$\int_K f(\vec{x}) d\vec{x} = 2|K| \int_{\hat{K}} f(\vec{F}_K(\hat{\vec{x}})) d\hat{\vec{x}}$$

in general. So

$$\begin{aligned} \|v\|_{L^2(K)}^2 &= \int_K v^2(\vec{x}) d\vec{x} \\ &= 2|K| \int_{\hat{K}} \hat{v}^2(\hat{\vec{x}}) d\hat{\vec{x}} \\ &= 2|K| \|\hat{v}\|_{L^2(\hat{K})}^2 \end{aligned}$$

We need to transform

$$\|v\|_{H^1(K)}^2 = \sum_{|\alpha|=1} \int_K (\partial^\alpha v(\vec{x}))^2 d\vec{x}$$

It suffices to use an estimate: $\forall v \in H^m(K)$,

$$\|v\|_{H^m(K)} \leq C \|\mathcal{B}_K^{-1}\|_2^m |\det \mathcal{B}_K|^{1/2} \|\hat{v}\|_{H^m(\hat{K})}.$$

We have then, for all $v \in P_r(K)$, ($m=1$)

$$\|v\|_{H^1(K)} \leq C \|\mathcal{B}_K^{-1}\|_2 |\det \mathcal{B}_K|^{1/2} \|\hat{v}\|_{H^1(\hat{K})}$$

$$\leq C C_2 \|B_K^{-1}\|_2 |\det(B_K)|^{1/2} \|\hat{v}\|_{L^2(K)}$$

$$= \frac{C C_2 \|B_K^{-1}\|_2 |\det(B_K)|^{1/2}}{\sqrt{2|K|}} \|v\|_{L^2(K)}$$

Recall that

$$\det(B_K) = 2|K|$$

and

$$\|B_K^{-1}\|_2 \leq \frac{\hat{h}_K}{f_K} = \frac{\sqrt{2}}{f_K}$$

Hence

$$|v|_{H^1(K)} \leq \frac{C}{f_K} \|v\|_{L^2(K)}, \quad \forall v \in P_r(K).$$

or

$$|v|_{H^1(K)}^2 \leq \frac{C}{f_K^2} \|v\|_{L^2(K)}^2, \quad \forall v \in P_r(K).$$

let γ_h be shape regular and quasi-uniform.
Then

$$(\text{shape reg}) \quad 1 \leq \frac{h_K}{f_K} \leq \Gamma, \quad \forall K \in \gamma_h, \quad \forall h.$$

$$(\text{quasi-uni}) \quad 0 < C_q \leq \frac{h_K}{h} \leq 1, \quad \forall K \in \gamma_h, \quad \forall h.$$

Suppose $v \in V_h = M_r$ is arbitrary. Then

$$|v|_{H^1(\Omega)}^2 = \sum_{K \in \gamma_h} |v|_{H^1(K)}^2$$

$$\left(\int_{\Omega} [\cdot] d\bar{x} = \sum_{K \in \gamma_h} \int_K [\cdot] d\bar{x} \right)$$

$$0 < C_g h \leq h_K \Rightarrow \frac{1}{h_K} \leq \frac{1}{C_g h}$$

$$\leq \sum_{K \in \mathcal{V}_h} \frac{C}{\beta_K^2} \|v\|_{L^2(K)}^2$$

$$= \sum_{K \in \mathcal{V}_h} C \frac{h_K^2}{\beta_K^2} h_K^{-2} \|v\|_{L^2(K)}^2$$

$$\leq \sum_{K \in \mathcal{V}_h} C \sigma^2 \frac{1}{C_g^2 h^2} \|v\|_{L^2(K)}^2$$

$$= \frac{C}{h^2} \|v\|_{L^2(\Omega)}^2$$

Thus, there is some $C > 0$, such that

$$\|v\|_{H^1(\Omega)} \leq \frac{C}{h} \|v\|_{L^2(\Omega)}$$

for all $v \in V_h = M_T$. //

A Quick Review

FEM for Parabolic Problems

Diffusion Problem: Find $u \in C^2(\bar{\Omega} \times [0, T])$ such that

$$(17.1) \quad \left\{ \begin{array}{l} \partial_t u - \Delta u = f(\cdot, t) \quad \text{in } \Omega, \quad 0 < t \leq T, \\ u = 0, \quad \text{on } \partial\Omega \quad 0 < t \leq T, \\ u(\cdot, 0) = v, \quad \text{in } \Omega. \end{array} \right.$$

Defn (17.3): let $f \in L^2(0, T; L^2)$ and $v \in L^2(\Omega)$ be given. u is called a weak solution to the diffusion problem iff

$$1) \quad u \in L^2(0, T; H_0)$$

2) u is weakly differentiable in time and

$$\partial_t u \in L^2(0, T; H^{-1})$$

3) For almost every $t \in [0, T]$ and for every $\phi \in H_0(\Omega)$

$$(17.2) \quad \langle \partial_t u, \phi \rangle + (\nabla u, \nabla \phi) = (f, \phi) := (f, \phi)_{L^2}$$

4) Finally

$$\lim_{t \rightarrow 0} \|u(t) - v\|_{L^2} = 0.$$

Theorem (17.5) Suppose that u is a weak solution to the diffusion problem. Then

$$u \in C([0, T]; L^2)$$

(Sobolev-liku)

and

$$\max_{0 \leq t \leq T} \|u(t)\|_{L^2}^2 \leq \|v\|_{L^2}^2 + C \int_0^T \|f\|_{L^2}^2 dt$$

and

$$\int_0^T \|u(t)\|_{H^1}^2 dt \leq \|v\|_{L^2}^2 + C \int_0^T \|f\|_{L^2}^2 dt.$$

If u has additional regularities

$$u \in L^2(0, T; H_0^1 \cap H^2) \quad \partial_t u \in L^2(0, T; L^2)$$

with $v \in H_0^1(\Omega)$, then

$$u \in C([0, T]; H_0^1)$$

and

$$\max_{0 \leq t \leq T} \|u(t)\|_{H_0^1}^2 \leq \|v\|_{H_0^1}^2 + \int_0^T \|f\|_{L^2}^2 dt$$

$$\int_0^T \|\Delta u\|_{L^2}^2 dt \leq \|v\|_{H_0^1}^2 + \int_0^T \|f\|_{L^2}^2 dt$$

Proof: last time we gave a rigorous proof. This time we give a more intuitive, but formal, argument.

Let us start with the strong form:

$$\partial_t u - \Delta u = f(x, t)$$

Multiply by u and integrate:

$$\int_{\Omega} \partial_t u u \, dx - \int_{\Omega} \Delta u u \, dx = \int_{\Omega} f u \, dx$$

It follows that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx = \int_{\Omega} f u \, dx$$

or

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \|u\|_{H^1}^2 = (f, u)$$

$$\stackrel{\text{C.S.}}{\leq} \|f\|_{L^2} \|u\|_{L^2}$$

$$\stackrel{\text{Poincaré}}{\leq} C \|f\|_{L^2} \|u\|_{H^1}$$

$$\stackrel{\text{AGMII}}{\leq} \frac{cc}{2} \|f\|_{L^2}^2 + \frac{c}{2c} \|u\|_{H^1}^2$$

Hence,

$$\frac{d}{dt} \|u\|_{L^2}^2 + \|u\|_{H^1}^2 \leq c \|f\|_{L^2}^2.$$

Now, we integrate in time $t=0$ to $t=s \in (0, T]$:

$$\int_0^s \frac{d}{dt} \|u(t)\|_{L^2}^2 dt + \int_0^s \|u(t)\|_{H^1}^2 dt \leq c \int_0^s \|f(t)\|_{L^2}^2 dt$$

$$\|u(s)\|_{L^2}^2 - \|u(0)\|_{L^2}^2 + \int_0^s \|u(t)\|_{H^1}^2 dt \leq c \int_0^s \|f(t)\|_{L^2}^2 dt$$

$$\leq c \int_0^T \|f(t)\|_{L^2}^2 dt$$

It follows that, for any $s \in (0, T]$,

$$\|u(s)\|_{L^2}^2 + \int_0^s \|u(t)\|_{H^1}^2 dt \leq \|v\|_{L^2}^2 + \underbrace{\int_0^T \|f(t)\|_{L^2}^2 dt}_{\text{independent of } s}$$

From the last estimate it follows that, for every $s \in (0, T]$,

$$\|u(s)\|_{L^2}^2 \leq \|v\|_{L^2}^2 + \int_0^T \|f(t)\|_{L^2}^2 dt$$

Thus,

$$\max_{0 \leq s \leq T} \|u(s)\|_{L^2}^2 \leq \|v\|_{L^2}^2 + \int_0^T \|f(t)\|_{L^2}^2 dt.$$

By a similar argument

$$\int_0^T \|u(t)\|_{H^1}^2 dt \leq \|v\|_{L^2}^2 + \int_0^T \|f(t)\|_{L^2}^2 dt.$$

It follows that

$$\max_{0 \leq s \leq T} \|u(s)\|_{L^2}^2 + \int_0^T \|u(t)\|_{H^1}^2 dt \leq 2 \left\{ \|v\|_{L^2}^2 + \int_0^T \|f(t)\|_{L^2}^2 dt \right\}.$$

Again, starting with the strong form, multiply by $-\Delta u$ and integrate:

$$-\int_{\Omega} \partial_t u \Delta u dx + \int_{\Omega} (\Delta u)^2 dx = -\int_{\Omega} f \Delta u dx$$

It follows that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 dx + \|\Delta u\|_{L^2}^2 &= -(f, \Delta u) \\ &\stackrel{\text{CS}}{\leq} \|f\|_{L^2} \|\Delta u\|_{L^2} \\ &\stackrel{\text{AGM}}{\leq} \frac{1}{2} \|f\|_{L^2}^2 + \frac{1}{2} \|\Delta u\|_{L^2}^2 \end{aligned}$$

Thus,

$$\frac{d}{dt} \|u\|_{H^1}^2 + \|\Delta u\|_{L^2}^2 \leq \|f\|_{L^2}^2$$

Integrating in time from $t=0$ to $t=s \in (0, T]$,

$$\begin{aligned} \|u(s)\|_{H^1}^2 - \|v\|_{H^1}^2 + \int_0^s \|\Delta u(t)\|_{L^2}^2 dt \\ \leq \int_0^s \|f(t)\|_{L^2}^2 dt \\ \leq \int_0^T \|f(t)\|_{L^2}^2 dt. \end{aligned}$$

We get

$$\|u(s)\|_{H^1}^2 + \int_0^s \|\Delta u(t)\|_{L^2}^2 dt \leq \|v\|_{H^1}^2 + \int_0^T \|f(t)\|_{L^2}^2 dt,$$

from which follows

$$\max_{0 \leq s \leq T} \|u(s)\|_{H^1}^2 \leq \|v\|_{H^1}^2 + \int_0^T \|f(t)\|_{L^2}^2 dt$$

and

$$\int_0^T \|\Delta u(t)\|_{L^2}^2 dt \leq \|v\|_{H^1}^2 + \int_0^T \|f(t)\|_{L^2}^2 dt.$$

Special Case

Suppose that $f = 0$. Then we can show that

$$\max_{0 \leq t \leq T} \|u(t)\|_{L^2}^2 \leq \|v\|_{L^2}^2$$

$$\max_{0 \leq t \leq T} \|u(t)\|_{H_0^1}^2 \leq \|v\|_{H_0^1}^2$$

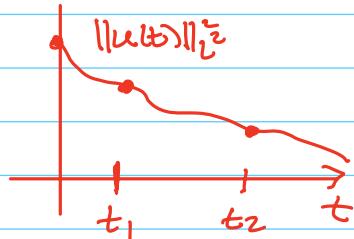
In this case, it is easy to show that

$$\|u(t_2)\|_{L^2}^2 \leq \|u(t_1)\|_{L^2}^2$$

$$\|u(t_2)\|_{H_0^1}^2 \leq \|u(t_1)\|_{H_0^1}^2$$

for all $0 \leq t_1 \leq t_2 \leq T$.

Exercise



Continuous Dependence on Initial Data

Suppose that u_1, u_2 solve

$$\partial_t u_i - \Delta u_i = f \quad \text{in } \Omega \quad 0 < t \leq T$$

$$u_i = 0 \quad \text{on } \partial\Omega \quad 0 < t \leq T$$

$$u_i(0) = v_i \quad \text{in } \Omega$$

Suppose that both satisfy the diffusion problem in the weak sense. Then, $e := u_2 - u_1$ is a weak solution to

$$\partial_t e - \Delta e = 0, \quad \text{in } \Omega \quad 0 < t \leq T,$$

$$e = 0, \quad \text{on } \partial\Omega \quad 0 < t \leq T,$$

$$e(0) = v_2 - v_1, \quad \text{in } \Omega.$$

and

$$\begin{aligned} \max_{0 \leq t \leq T} \|e(t)\|_{L^2}^2 &= \max_{0 \leq t \leq T} \|u_2(t) - u_1(t)\|_{L^2}^2 \\ &\leq \|v_2 - v_1\|_{L^2}^2. \end{aligned}$$

If additional regularities hold,

$$\begin{aligned} \max_{0 \leq t \leq T} \|e\|_{H^1}^2 &= \max_{0 \leq t \leq T} \|u_2(t) - u_1(t)\|_{H^1}^2 \\ &\leq \|v_2 - v_1\|_{H^1}^2. \end{aligned}$$

The theory above guarantees that weak solns are unique!

Semi-Discrete FEM for Diffusion Problem

Suppose that Ω is a polygonal domain in \mathbb{R}^2 .
Let

$$V_h = M_{0,r} \subset H^1(\Omega),$$

with basis

$$B = \{\phi_1, \dots, \phi_{N_r^0}\}$$

Defn (18.1): The Galerkin approximation of the of the weak form of the diffusion problem is defined as follows:

Find $u_h: [0, T] \rightarrow V_h = M_{\text{gr}}$, in the form

$$u_h(\cdot, t) = \sum_{i=1}^{N_r^0} u_i(t) \phi_i$$

such that

$$(18.1) \quad (\partial_t u_h, \psi)_{L^2} + (\nabla u_h, \nabla \psi) = (f, \psi)$$

for all $\psi \in V_h$, for all $t \in (0, T]$, with

$$(18.2) \quad u_h(0) = v_h := P_h v \in V_h.$$

Here P_h is some projection operator from L^2 to V_h , i.e.,

$$P_h: L^2 \rightarrow V_h$$

and

$$P_h(P_h v) = P_h v \quad \forall v \in L^2.$$

We will discuss some possibilities.

The defn above is equivalent to

$$(18.3) \quad (\partial_t u_h(t), \phi_i) + a(u_h(t), \phi_i) = (f(t), \phi_i),$$

for $i = 1, \dots, N_r^0$, for all $t \in (0, T]$.

Proposition (18.2): Define

$$\text{via } G = [g_{i,j}]_{i,j=1}^{N_r^0} \quad A = [a_{i,j}]_{i,j=1}^{N_r^0}$$

$$g_{i,j} := (\phi_j, \phi_i)_{L^2} \quad (\text{mass matrix})$$

and

$$a_{i,j} := a(\phi_j, \phi_i)_{L^2} \quad (\text{stiffness matrix})$$

for all $1 \leq i, j \leq N_r^0$. Define $\vec{f}: [0, T] \rightarrow \mathbb{R}^{N_r^0}$ via

$$\vec{f}(t) = [f_i(t)]_{i=1}^{N_r^0}, \quad f_i(t) := (f(t), \phi_i)_{L^2}.$$

Finally, define $\vec{v} = [v_i] \in \mathbb{R}^{N_r^0}$ via

$$v_i := (v_h, \phi_i)_{L^2}.$$

let $u_h: [0, T] \rightarrow M_{0,r}$ be a strong ODE solution to (18.1) subject to the initial conditions (18.2), then

$$(18.4) \quad G \vec{u}'(t) + A \vec{u}(t) = \vec{f}(t)$$

where

$$(18.5) \quad u_h(\cdot, t) = \sum_{i=1}^{N_r^0} u_i(t) \phi_i$$

$$(18.6) \quad \vec{u}(t) = [u_i(t)] \in \mathbb{R}^{N_r^0},$$

subject to the initial condition

$$(18.7) \quad G \vec{u}(0) = \vec{v} \in \mathbb{R}^{N_r^0}.$$

Proof: Let us start with (18.3). Inserting (18.5) into (18.3), we get, for $i=1, \dots, N_r^0$.

$$\sum_{j=1}^{N_r^0} (\phi_j, \phi_i) \frac{du_j}{dt}(t) + \sum_{j=1}^{N_r^0} a(\phi_j, \phi_i) u_j(t) = f_i(t).$$

In other words,

$$\sum_{j=1}^{N_r^0} g_{j,i} u'_j(t) + \sum_{j=1}^{N_r^0} a_{j,i} u_j(t) = f_i(t),$$

which is the component form of (18.4).

Finally, using (18.2), for $1 \leq i \leq N_r^0$,

$$(u_n(0), \phi_i)_{L^2} = (v_n, \phi_i)_{L^2}$$

This implies that

$$\begin{aligned} (v_n, \phi_i)_{L^2} &= \left(\sum_{j=1}^{N_r^0} u_j(0) \phi_j, \phi_i \right)_{L^2} \\ &= \sum_{j=1}^{N_r^0} g_{j,i} u_j(0) \\ &= [G \vec{u}(0)]_i. \end{aligned}$$

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