Math 574
class # 05
09/02/2025

Sobolev Spaces

<u>Defn</u> (Sobolev Spaces): Let $k \in \mathbb{N}_0$, $p \in [1,\infty]$. Suppose $\Omega \subseteq \mathbb{R}^d$ is open. The set

$$W^{k,p}(\Omega) := \{ v \in L^p(\Omega) \mid \partial_w^\alpha v \in L^p(\Omega),\ 1 \le |\alpha| \le k \}$$

is called the <u>k,p Sobolev Space.</u> The norm on the space is

$$\|v\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \le k} \|\partial^\alpha v\|^p_{L^p(\Omega)} \right)^{1/p} & 1 \le p < \infty \\ \max_{|\alpha| \le k} \|\partial^\alpha v\|_{L^\infty(\Omega)} & p = \infty \end{cases} \tag{5.1}$$

When $p = 2$, we write $H^k(\Omega) = W^{k,2}(\Omega)$.

The latter has the inner product

$$(u, v)_{H^k(\Omega)} := \sum_{|\alpha| \le k} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)}.$$

Clearly,

$$\|u\|_{H^k(\Omega)} = \sqrt{(u,u)_{H^k(\Omega)}}.$$

<u>Theorem</u>: Let $\Omega \subseteq \mathbb{R}^d$ be open. The Sobolev space $W^{k,p}(\Omega)$ is a Banach space.

<u>Proof</u>: Let $\{v_n\}$ be a Cauchy sequence in $W^{k,p}(\Omega)$. Then, for any $\alpha \in \mathbb{N}_0^d$, $|\alpha| \le k$, $\{\partial^\alpha v_n\}$ is a Cauchy sequence in $L^p(\Omega)$.

But $L^p(\Omega)$ is a Banach space, as we well know. There is $v_\alpha \in L^p(\Omega)$ such that

(5.2) $$\partial^\alpha v_n \xrightarrow{n\to\infty} v_\alpha \quad \text{in } L^p(\Omega),$$

that is,

$$\| \partial^\alpha v_n - v_\alpha \|_{L^p(\Omega)} \xrightarrow{n\to\infty} 0$$

We want to show that $\partial^\alpha v$ exists and

$$\partial^\alpha v = v_\alpha \quad \text{a.e. in } \Omega,$$

where

$$v_n \xrightarrow{n\to\infty} v \quad \text{in } L^p(\Omega).$$

For all $\phi \in C_0^\infty(\Omega)$,

(5.3) $$\int_\Omega v_n \partial^\alpha \phi \, d\vec{x} = (-1)^{|\alpha|} \int_\Omega \partial^\alpha v_n \, \phi \, d\vec{x},$$

Since $v_n \in W^{k,p}$. But

$$\left| \int_\Omega v_n \partial^\alpha \phi \, d\vec{x} - \int_\Omega v \, \partial^\alpha \phi \, d\vec{x} \right|$$

(5.4) $$= \left| \int_\Omega (v_n - v) \partial^\alpha \phi \, d\vec{x} \right|$$

$$\leq \| v_n - v \|_{L^p} \| \partial^\alpha \phi \|_{L^q} \qquad \text{(Hölder's ineq)}$$

$$\xrightarrow{n\to\infty} 0$$

likewise

$$\left| \int_\Omega \partial^\alpha v_n \phi \, d\vec{x} - \int v_\alpha \phi \, d\vec{x} \right|$$

(5.5)

$$\leq \| \partial^\alpha v_n - v_\alpha \|_{L^p} \| \phi \|_{L^q}$$

$$\xrightarrow{n \to \infty} 0$$

From (5.3), (5.4) and (5.5), we have

(5.6) $$\int_\Omega v \partial^\alpha \phi \, d\vec{x} = (-1)^{|\alpha|} \int_\Omega v_\alpha \phi \, d\vec{x},$$

Therefore, for all α, $1 \leq |\alpha| \leq k$,

$$v_\alpha = \partial^\alpha v \quad \text{in } L^p(\Omega).$$

Hence $v_n \to v$ in $W^{k,p}(\Omega)$. ///

Corollary: $H^k(\Omega) = W^{k,2}(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)}$$

for all $u, v \in H^k(\Omega)$.

Defn: Suppose $\Omega \subseteq \mathbb{R}^d$ is an open set. $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. We set

$$H_0^k(\Omega) = W_0^{k,2}(\Omega).$$

Defn: Let X be a normed linear space and $Y \subset X$. Y is said to be <u>dense in X</u> iff for any $x \in X$ and any $\varepsilon > 0$, there is a point $y \in Y$ such that

$$\| x - y \|_X \leq \varepsilon .$$

<u>Example</u>: By Weierstrass' Theorem the polynomials are dense in $(C([a,b]), \| \cdot \|_\infty)$.

Defn: Let $(X, \| \cdot \|_X)$ be a normed linear space. Define $\overline{X}$ to be the space obtained by adding the limit points of every Cauchy sequence in X. If X is a Banach space, then $X = \overline{X}$. Otherwise $X \subset \overline{X}$. $\overline{X}$ is called the <u>completion</u> of X. X is a dense subset of $\overline{X}$.

Atkinson and Han (2008)

<u>Examples</u>: $(L^p(a,b), \| \cdot \|_{L^p})$ is the completion of $(C([a,b]), \| \cdot \|_{L^p})$, $1 \leq p < \infty$.

?

Theorem: Assume that $\Omega \subseteq \mathbb{R}^d$ is open and $v \in W^{k,p}(\Omega)$, $1 \leq p < \infty$. There exists a sequence $\{v_n\} \subset C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that

$$\| v_n - v \|_{k,p} \longrightarrow 0 \qquad \text{as } n \to \infty$$

We say that $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

Theorem: Assume that $\Omega \subseteq \mathbb{R}^d$ is a bounded open set with a Lipschitz smooth boundary $\partial\Omega$. Suppose that $v \in W^{k,p}(\Omega)$. There exists a sequence $\{v_n\} \subset C^\infty(\bar{\Omega})$ such that

$$\| v_n - v \|_{k,p} \to 0 \quad \text{as} \quad n \to \infty.$$

Corollary: With the same hypotheses, $W^{k,p}(\Omega)$ is the completion of $C^\infty(\bar{\Omega})$ wrt $\| \cdot \|_{k,p}$.

Ω is open

Theorem: For any $v \in W_0^{k,p}(\Omega)$, there is a sequence $\{v_n\} \subset C_0^\infty(\Omega)$ such that

$$\| v_n - v \|_{k,p} \to 0 \quad \text{as} \quad n \to \infty.$$

Sobolev Embedding Theorems

Defn: Let V and W be two Banach spaces with $V \subset W$. We say that the space V is <u>continuously embedded</u> in W and write

$$V \hookrightarrow W$$

iff there is a constant $C > 0$ such that

$$\| v \|_W \leq C \| v \|_V$$

for all $v \in V$.

We say that V is <u>compactly embedded</u> in W and write

$$V \hookrightarrow\hookrightarrow W$$

iff (i) there is a constant $C > 0$ such that

$$\|v\|_W \leq C\|v\|_V$$

for all $v \in V$ and (ii) for every $\{v_n\} \subset V$ that is bounded, there is a subsequence

$$\{v_{n_k}\} \subseteq \{v_n\}$$

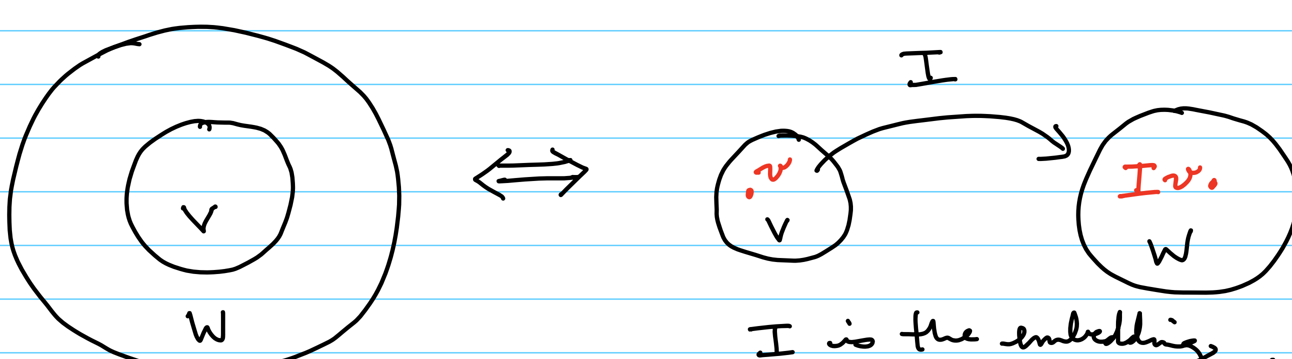
and a point $v \in W$ such that

$$\|v_{n_k} - v\|_W \longrightarrow 0 \quad \text{as} \quad k \to \infty.$$

$V \hookrightarrow W$ iff the embedding operator is continuous

$V \hookrightarrow\hookrightarrow W$ iff the embedding operator is compact.

How to remember the definitions?



I is the embedding operator. (The identity.)

For the map to be cont (bounded), I must be an $C > 0$ such that

$$\|v\|_W = \|Iv\|_W \leq C \|v\|_V$$

Compact operators have the additional bit about sequences...

<u>Theorem</u>: Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz boundary. Then the following are valid.

1) If $k < d/p$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \leq p^*$, where

$$\frac{1}{p^*} = \frac{1}{p} - k/d \qquad \text{(Conjugate } \overset{p^*}{\text{Sobolev}} \text{ Exp)}$$

2) If $k = d/p$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, for any $q < \infty$.

3) If $k > d/p$, then

$$W^{k,p}(\Omega) \hookrightarrow C^{k-[d/p]-1,\beta}(\overline{\Omega})$$

where

$$\beta = \begin{cases} [d/p] + 1 - d/p & d/p \notin \mathbb{Z} \\ \text{any pos. real} < 1 & d/p \in \mathbb{Z} \end{cases}$$

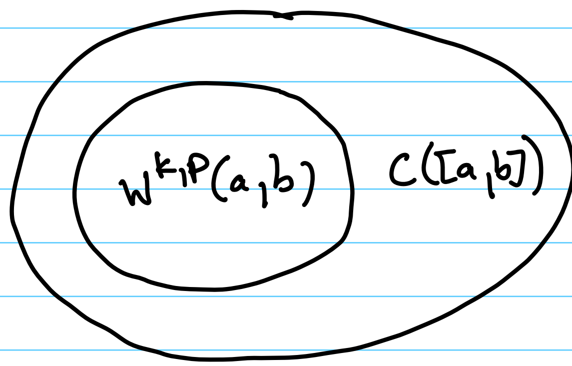
Here $[x]$ is the largest integer less than or equal to x.

<u>Example</u>: Suppose $d=1$. For any $k \geq 1$ and $p>1$,

$$k > \frac{d}{p} = \frac{1}{p} \iff kp > d = 1.$$

Consequently, for $\Omega = (a,b)$,

$$W^{k,p}(a,b) \hookrightarrow C([a,b])$$



$\|v\|_\infty \leq C \|v\|_{k,p}$

This takes a bit of interpretation.