Math 574
class #06
09/04/2025

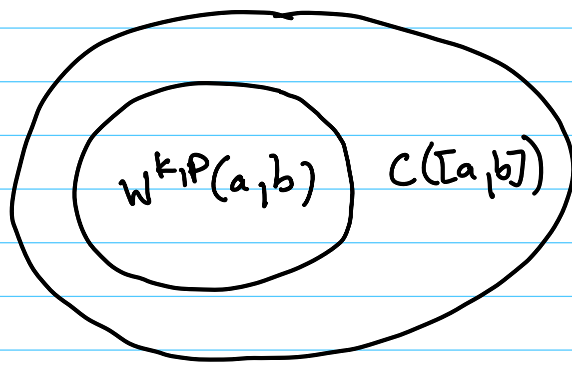
Sobolev Embeddings

Example: Suppose $d=1$. For any $k \geq 1$ and $p > 1$,

$$k > \frac{d}{p} = \frac{1}{p} \iff kp > d = 1.$$

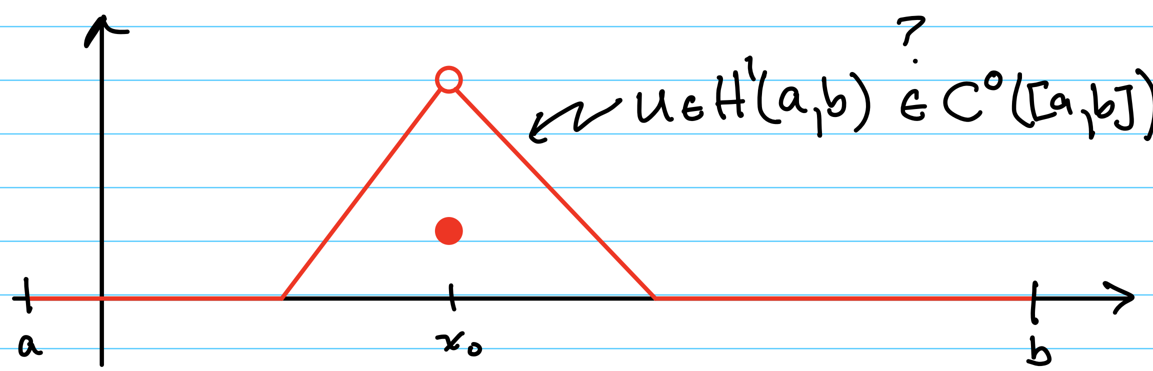
Consequently, for $\Omega = (a,b)$,

$$W^{k,p}(a,b) \hookrightarrow C([a,b])$$



$$\|v\|_\infty \leq C\|v\|_{k,p}$$

This takes a bit of interpretation.



Example: Suppose $k=2$, $p=2$, $d=1$

$$4 = kp > d = 1$$

$$W^{2,2}(a,b) = H^2(a,b) \hookrightarrow C^{\ell,\beta}([a,b])$$

where

$$\ell = k - \left[{}^{d}/_{p}\right] - 1 = 2 - \left[\tfrac{1}{2}\right] - 1 = 1$$

$$\beta = \left[{}^{d}/_{p}\right] + 1 - {}^{d}/_{p} = \left[\tfrac{1}{2}\right] + 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

$$H^2(a,b) \hookrightarrow C^{1,1/2}([a,b]) \hookrightarrow C^1([a,b])$$

You drop the Hölder exponent.

<u>Example</u>: Suppose $1 \le d \le 3$, $p = 2$, $k = 2$. Then

$$W^{2,2}(\Omega) = H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$$

This is an important result for FEM error estimates.

<u>Example</u>: Suppose $k = 1$, $p = 2$, $d = 3$. Then

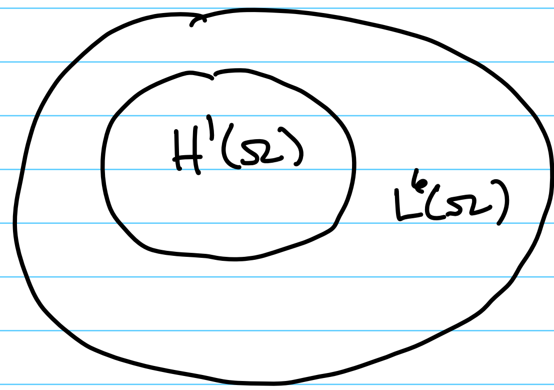
$$1 = k < {}^{d}/_{p} = {}^{3}/_{2} \iff 2 = kp < d = 3$$

$$W^{k,p}(\Omega) = W^{1,2}(\Omega) = H^1(\Omega) \hookrightarrow L^q(\Omega)$$

for all

$$1 \le q \le p^* = 6$$

$$\frac{1}{p^*} = \frac{1}{p} - \frac{k}{d} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$



$\exists\, C > 0$

$\|v\|_{L^2} \le C \|v\|_{H^1}$

$\forall\, v \in H^1(\Omega)$

Example: $k = 1$, $p = 2$, $d = 2$. Then,

$$1 = k = d/p = 1 \iff 2 = kp = d = 2.$$

$$W^{k,p}(\Omega) = H^k(\Omega) \hookrightarrow L^q(\Omega)$$

for all $q < \infty$. This is the critical case.

Theorem: Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz boundary. Then the following are valid.

1) If $k < d/p$, then $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for any $q \le p^*$, where

$$\frac{1}{p^*} = \frac{1}{p} - k/d$$

2) If $k = d/p$, then $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, for any $q < \infty$.

3) If $k > d/p$, then

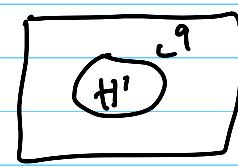
$$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow C^{k-[d/p]-1,\beta}(\overline{\Omega})$$

where

$$\beta \in [0, [d/p]+1-d/p).$$

<u>Example</u>: $d=3$, $k=1$, $p=2$.

$$2 = k\cdot p < d = 3$$



$$H^1(\Omega)=W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$$

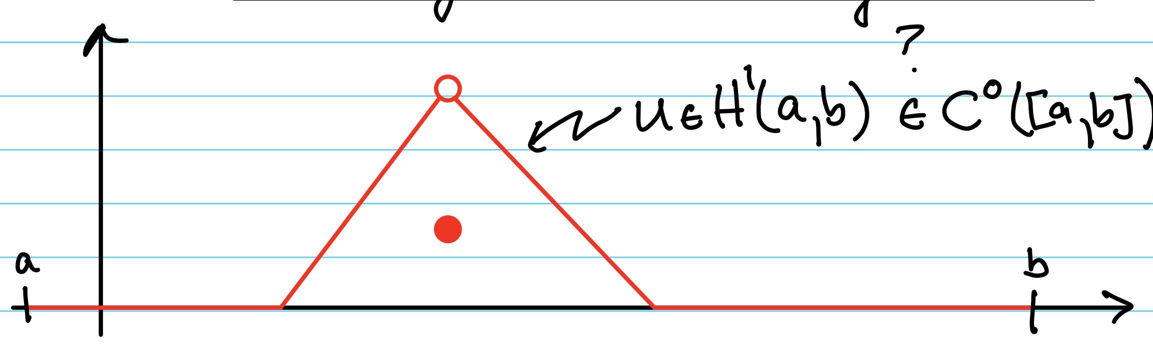
for all $q\in[1,6]$. Let $\{v_n\}\subset H^1(\Omega)$, such that $\exists M>0$ such that

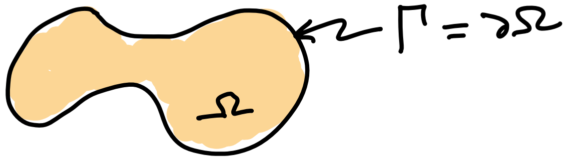
$$\|v_n\|_{H^1(\Omega)} \leq M \quad \forall\, n\in\mathbb{N}.$$

$\exists\ \{v_{n_k}\}\subseteq\{v_n\}$ and a point $v\in L^q(\Omega)$ such that

$$\|v_{n_k}-v\|_{L^q} \xrightarrow{k\to\infty} 0. \quad \left(v_{n_k} \xrightarrow{L^q} v\right)$$

<u>Boundary Traces (Boundary Values)</u>





How can we get boundary values from a function whose values a certain points (in a set of measure zero) can be changed?

<u>Theorem</u>: Suppose that $\Omega \subset \mathbb{R}^d$ is an open bounded domain with Lipschitz boundary

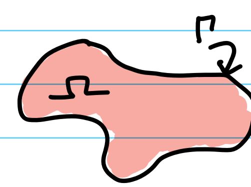
$$\Gamma = \partial\Omega.$$

For any $p \in [1,\infty)$ there is a continuous linear operator $\gamma: W^{1,p}(\Omega) \to L^p(\Gamma)$ with the following properties

1) $\gamma v = v\big|_{\Gamma} \;\; \forall \; v \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$

2) There is a constant $C > 0$ such that

$$\|\gamma v\|_{L^p(\Gamma)} \leq C \|v\|_{W^{1,p}(\Omega)}$$



for all $v \in W^{1,p}(\Omega)$.

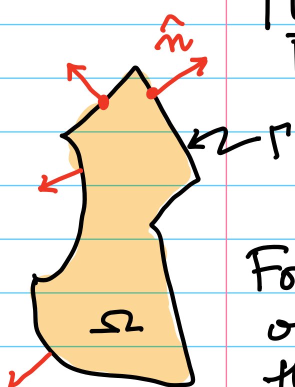
3) The mapping $\gamma: W^{1,p}(\Omega) \to L^p(\Gamma)$ is compact, i.e., for any bounded sequence $\{v_n\} \subset W^{1,p}(\Omega)$ there is a subsequence $\{v_{n_k}\} \subseteq \{v_n\}$ and a point $v \in L^p(\Omega)$, such that

$$\gamma v_n \to v \quad \text{in } L^p(\Gamma)$$

Defn: The operator $\gamma: W^{1,p}(\Omega) \to L^p(\Gamma)$, whose existence is guaranteed by the last theorem is called the <u>trace operator</u>.

Integration-By-Parts Formulae

Recall a well-known result.



<u>Theorem</u>: Suppose that $\Omega \subseteq \mathbb{R}^d$ is a bounded open domain with Lipschitz boundary

$$\Gamma = \partial\Omega.$$

For almost every point $\vec{x} \in \Gamma$ there is an outward-pointing unit normal vector, $\hat{n}(\vec{x})$, that is perpendicular to the surface Γ at $\vec{x}$.

Furthermore, for all $u, v \in C^1(\bar{\Omega})$, for any $j \in \{1, \cdots, d\}$,

$$\int_\Omega \frac{\partial u}{\partial x_j} v \, d\vec{x} = \int_\Gamma u v n_j \, ds - \int_\Omega u \frac{\partial v}{\partial x_j} d\vec{x}$$

where

$$[\hat{n}(\vec{x})]_j = n_j(\vec{x}).$$

<u>Theorem</u>: Suppose that $\Omega \subset \mathbb{R}^d$ is an open bounded set with Lipschitz boundary

$$\Gamma = \partial\Omega$$

Then,

$$\int_\Omega \frac{\partial u}{\partial x_j} v \, d\vec{x} = \int_\Gamma u v n_j \, ds - \int_\Omega u \frac{\partial v}{\partial x_j} d\vec{x} \tag{6.1}$$

for all $u, v \in H^1(\Omega)$.

Proof: $C^1(\bar{\Omega})$ is dense in $H^1(\Omega)$, it is possible to show. There a sequences $\{u_n\}, \{v_n\} \subset C^1(\bar{\Omega})$, such that

$$\|u_n - u\|_{H^1}, \|v_n - v\|_{H^1} \to 0$$

Since $u_n, v_n \in C^1(\bar{\Omega})$, for each n.

$$\int_\Omega \frac{\partial u_n}{\partial x_j} v_n \, d\vec{x} = \int_\Gamma u_n v_n n_j \, ds - \int_\Omega u_n \frac{\partial v_n}{\partial x_j} d\vec{x} \tag{6.2}$$

Using Cauchy - Schwartz

$$\left| \int_\Omega \frac{\partial u_n}{\partial x_j} v_n \, d\vec{x} - \int_\Omega \frac{\partial u}{\partial x_j} v \, d\vec{x} \right|$$

$$= \left| \int_\Omega \frac{\partial u_n}{\partial x_j} v \, d\vec{x} - \int_\Omega \frac{\partial u_n}{\partial x_j} v \, dx + \int_\Omega \frac{\partial u_n}{\partial x_j} v_n \, d\vec{x} \right.$$

$$\left. - \int_\Omega \frac{\partial u}{\partial x_j} v \, d\vec{x} \right|$$

$$\leq \left| \int_\Omega \frac{\partial u_n}{\partial x_j} (v - v_n) \, d\vec{x} \right| + \left| \int_\Omega \left(\frac{\partial u_n}{\partial x_j} - \frac{\partial u}{\partial x_j} \right) v \, d\vec{x} \right|$$

$$\boxed{\begin{aligned}&\forall f \in H^1(\Omega)\\ &\|f\|_{L^2} \le \|f\|_{H^1}\\ &\left\|\frac{\partial f}{\partial x_j}\right\|_{L^2} \le \|f\|_{H^1}\end{aligned}}$$

C.S.

$$\le \left\|\frac{\partial u_n}{\partial x_j}\right\|_{L^2} \|v - v_n\|_{L^2} + \|v\|_{L^2} \left\|\frac{\partial u_n}{\partial x_j} - \frac{\partial u}{\partial x_j}\right\|_{L^2}$$

$$\le \|u_n\|_{H^1} \|v - v_n\|_{H^1} + \|v\|_{H^1} \|u_n - u\|_{H^1}$$

By the Squeeze Theorem

$$\int_\Omega \frac{\partial u_n}{\partial x_j} v_n \, d\vec{x} \xrightarrow{n \to \infty} \int_\Omega \frac{\partial u}{\partial x_j} v \, d\vec{x}$$

By a similar analysis

$$\int_\Omega u_n \frac{\partial v_n}{\partial x_j} \, d\vec{x} \xrightarrow{n \to \infty} \int_\Omega u \frac{\partial v}{\partial x_j} \, d\vec{x}$$

Using the continuity of the trace operator

$$\|u_n - u\|_{L^2(\Gamma)} \le C \|u_n - u\|_{H^1(\Omega)} \xrightarrow{n \to \infty} 0$$

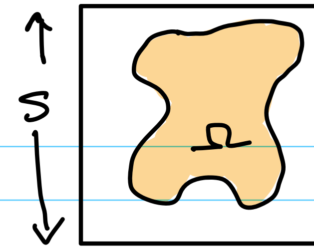
and

$$\|v_n - v\|_{L^2(\Gamma)} \le C \|v_n - v\|_{H^1(\Omega)} \xrightarrow{n \to \infty} 0$$

It follows that (exercise)

$$\int_\Gamma u_n v_n n_j \, dS \to \int_\Gamma u v n_j \, dS.$$

Passing to the limit in (6.2) we get (6.1). ///



Poincaré Ineq

Theorem: Suppose $\Omega \subset \mathbb{R}^d$ is an open bounded domain contained in a d-dimensional hyper-cube of side length s. Then,

$$\|v\|_{L^2(\Omega)} \le s\,|v|_{H^1(\Omega)}$$

(6.3)

$$= s\sqrt{\sum_{|\alpha|=1} \|\partial^\alpha v\|_{L^2}^2}$$

for all $v \in H_0^1(\Omega)$.

Proof: It is an exercise to show that (6.3) holds for all $v \in C_0^\infty(\Omega)$.

Let $v \in H_0^1(\Omega)$ be arbitrary. There is a sequence $\{v_n\} \subset C_0^\infty(\Omega)$ such that

$$v_n \to v \text{ in } H^1(\Omega).$$

$$
\begin{aligned}
\|v\|_{L^2} &= \|v + v_n - v_n\|_{L^2} \\
&\overset{\Delta\text{-ineq}}{\le} \|v - v_n\|_{L^2} + \|v_n\|_{L^2} \\
&\overset{\text{Poincare}}{\le} \|v - v_n\|_{L^2} + s\,|v_n|_{H^1} \\
&= \|v - v_n\|_{L^2} + s\,|v + v_n - v|_{H^1} \\
&\overset{\Delta\text{-ineq}}{\le} s\,|v|_{H^1} + \|v - v_n\|_{L^2} + s\,|v - v_n|_{H^1}
\end{aligned}
$$

$$\leq S|v|_{H^1} + (1+S)\|v - v_n\|_{H^1}$$

Let $\varepsilon > 0$ be arbitrary. There is an N such that for all $n \geq N$,

$$\|v - v_n\|_{H^1} \leq \frac{\varepsilon}{1+S}.$$

Then

$$\|v\|_{L^2} \leq S|v|_{H^1} + \varepsilon$$

for $\varepsilon > 0$, no matter how small. (6.3) must be true. ///