Math 574
class #15
10/14/2025

A Non-Symmetric Problem

Let's do a quick review...

Given $f \in C^0(\Omega)$, $\vec{b} \in \mathbb{R}^d$ and $c > 0$. Find $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that

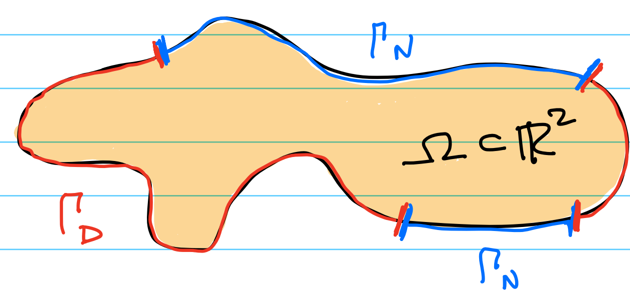
(14.11) $$\begin{cases} -\Delta u + \vec{b} \cdot \nabla u + cu = f & \text{in } \Omega \subset \mathbb{R}^d \\ u = 0 & \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_N \end{cases}$$

Here

$$\Gamma_D \cup \Gamma_N = \partial\Omega,$$

$$\Gamma_D \cap \Gamma_N = \emptyset,$$

$$\mu_{d-1}(\Gamma_D), \mu_{d-1}(\Gamma_N) > 0$$



Let us formulate a weak version of the problem. Let

$$v \in V := \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D \}$$

be arbitrary. Then, using integration-by-parts,

$$(f,v)_{L^2} - c(u,v)_{L^2} - (\vec{b}\nabla u, v)_{L^2}$$

$$= -(\Delta u, v)_{L^2}$$

$$= (\nabla u, \nabla v)_{L^2} - \int_{\partial\Omega} \overbrace{\hat{n}\cdot\nabla u}^{\frac{\partial u}{\partial n}}\, v \, dS$$

$$= (\nabla u, \nabla v)_{L^2} - \int_{\Gamma_N} \frac{\partial u}{\partial n} v \, ds$$

$$- \int_{\Gamma_D} \frac{\partial u}{\partial n} \cancelto{0}{v} \, ds$$

$$= (\nabla u, \nabla v)_{L^2} - \int_{\Gamma_N} g v \, ds$$

Thus,

(14.12) $$a(u,v) = L(v), \quad \forall\, v \in V$$

where

(14.13) $$\begin{cases} a(u,v) := (\nabla u, \nabla v)_{L^2} + (\vec{b}\cdot\nabla u, v)_{L^2} + c(u,v)_{L^2}, \\ L(v) := (f,v)_{L^2} + (g,v)_{L^2(\Gamma_N)}. \end{cases}$$

<u>Definition</u> (14.4): The following is a weak formulation of the non-symmetric problem (14.11). Given $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$, $c > 0$, and $\vec{b} \in \mathbb{R}^d$, find $u \in V$, where

(14.14) $$V := \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D \}$$

such that (14.12) holds. We call such a $u \in V$, if it exists, a <u>weak solution to (14.11)</u>.

<u>Proposition</u> (15.1): The space defined in (14.4) is a Hilbert space. The functional $L: V \to \mathbb{R}$ is linear and bounded. The bilinear form

$$a : V \times V \to \mathbb{R}$$

is, in general, non-symmetric, but is unconditionally continuous. In particular,

$$|a(u,v)| \leq \gamma \|u\|_{H^1} \|v\|_{H^1}, \quad \forall\, u,v \in V,$$

where

$$\gamma = 1 + \|\vec{b}\|_2 + C.$$

a is coercive, provided that

(15.1) $$\beta := \min\{1, C\} > \frac{\|\vec{b}\|_2}{2}.$$

In particular, in this case,

$$\alpha \|u\|_{H^1} \leq a(u,u), \quad \forall\, u \in V,$$

where

$$\alpha = \beta - \frac{\|\vec{b}\|_2}{2} > 0.$$

Under condition (14.15) there exists a unique weak solution $u \in V$ to (14.11).

Proof: The proof that V is a Hilbert space is an exercise. The conclusion follows from the Lax-Milgram lemma, provided we can prove the other points.

1) <u>L is a bnded linear operator on V.</u>

L is clearly linear. To show that it is bounded, let $v \in V$ be arbitrary. Then

$$|L(v)| \leq \left|\int_\Omega f\, v\, d\vec{x}\right| + \left|\int_{\Gamma_N} g\, v\, d\vec{x}\right|$$

$$\overset{C.S}{\leq} \|f\|_{L^2}\|v\|_{L^2} + \|g\|_{L^2(\Gamma_N)}\|v\|_{L^2(\Gamma_N)}$$

Using the Trace Theorem (Lecture 06)

$$\|v\|_{L^2(\Gamma_N)} \leq C_T\|v\|_{H^1(\Omega)}$$

we have

$$|L(v)| \leq \left(\|f\|_{L^2} + C_T\|g\|_{L^2(\Gamma_N)}\right)\|v\|_{H^1}$$

$$= C\|v\|_{H^1}. \quad //$$

(To use the Trace Theorem we need to assume that the boundary Γ_N is sufficiently regular. Let us assume that $\partial\Omega$ is Lipschitz cont. This should be enough.)

2) <u>a is continuous</u>: Let $u, v \in V$ be arbitrary. Then

$$|a(u,v)| \leq |(\nabla u, \nabla v)_{L^2}| + |(\vec{b}\cdot\nabla u, c)_{L^2}| + c|(u,v)_{L^2}|$$

$$\leq |u|_{H^1}|v|_{H^1} + \|\vec{b}\|_2|u|_{H^1}\|v\|_{L^2}$$

$$+ \, c \, \|u\|_{L^2} \|v\|_{L^2}$$

$$\leq \left(1 + \|\vec{b}\|_2 + c\right) \|u\|_{H^1} \|v\|_{H^1}$$

$$= \gamma \, \|u\|_{H^1} \|v\|_{H^1} \quad //$$

3| <u>a is coercive</u>:

$$a(u,u) = |u|_{H^1}^2 + (\vec{b} \cdot \nabla u, u) + c\|u\|_{L^2}^2$$

For non-symmetric terms,

$$-(\vec{b} \cdot \nabla \vec{u}, u) \leq |(\vec{b} \cdot \nabla u, u)|$$

$$\leq \|\vec{b}\|_2 \, |u|_{H^1} \|u\|_{L^2}$$

$$\leq \|\vec{b}\|_2 \|u\|_{H^1}$$

This estimate is not optimal. We can do just a bit better by using the AGMI:

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

$$-(\vec{b} \cdot \nabla \vec{u}, u) \leq |(\vec{b} \cdot \nabla u, u)|$$

$$\leq \|\vec{b}\|_2 \, |u|_{H^1} \|u\|_{L^2}$$

$$= \left(\|\vec{b}\|_2^{1/2} |u|_{H^1}\right) \cdot \left(\|\vec{b}\|_2^{1/2} \|u\|_{L^2}\right)$$

$$\overset{AGMI}{\leq} \frac{\|\vec{b}\|_2}{2} |u|_{H^1}^2 + \frac{\|\vec{b}\|_2}{2} \|u\|_{L^2}^2$$

$$= \frac{\|\vec{b}\|_2}{2} \left(\|u\|_{L^2}^2 + |u|_{H^1}^2\right)$$

$$= \frac{\|\vec{b}\|_2}{2} \|u\|_{H^1}^2 .$$

Thus,

$$(\vec{b}\cdot\nabla u, u) \geq -\frac{\|\vec{b}\|_2}{2}\|u\|_{H^1}^2$$

so that

$$\begin{aligned} a(u,u) &\geq |u|_{H^1}^2 - \frac{\|\vec{b}\|_2}{2}\|u\|_{H^1}^2 + c\|u\|_{L^2}^2 \\ &\geq \beta\left(\|u\|_{L^2}^2 + |u|_{H^1}^2\right) - \frac{\|\vec{b}\|_2}{2}\|u\|_{H^1}^2 \\ &= \alpha\|u\|_{H^1}^2 \end{aligned}$$

where

$$\alpha = \beta - \frac{\|\vec{b}\|_2}{2} > 0 . \quad //$$

Using the Lax-Milgram lemma, a unique solution $u \in V$ to (14.12) exists. ///

<u>Proposition</u> (15.2): Under assumption (15.1), that is,

(15.1) $$\beta := \min\{1, c\} > \frac{\|\vec{b}\|_2}{2} .$$

the Galerkin approximation

$$u_h \in V_h := \{ v \in M_r \mid v|_{\Gamma_D} = 0 \}$$

which satisfies

(15.2) $$a(u_h, v) = L(v) \qquad \forall v \in V_h$$

where a and L are defined as in (14.13), exists and is unique. Furthermore,

(15.3) $$\|u-u_h\|_{H^1} \leq C \inf_{v\in V_h} \|u-v\|_{H^1}.$$

If the mesh (which is used in the construction of M_r and V_h) is shape regular and $u \in H^{r+1}(\Omega)$, then

(15.4) $$\|u-u_h\|_{H^1} \leq C h^r |u|_{H^{r+1}}.$$

Proof: Existence and Uniqueness follows from the Lax-Milgram lemma since V_h is a Hilbert space. (15.3) is a direct consequence of Cea's lemma. (15.4) follows from interpolation error estimates on shape regular mesh families. ///

In this case, since the bilinear form is non-symmetric, we cannot apply Nitsche's trick. We need some more flexible theory.

Theorem (15.3) [Aubin-Nitsche lemma]: Suppose that $a: V\times V \to \mathbb{R}$ is a coercive, continuous bilinear form on V, a closed subspace of $H^1(\Omega)$. Let $V_h \subset V$ be finite dimensional, with $V_h \subseteq M_r$. Suppose that $f\in L^2(\Omega)$ and

$$u\in V \qquad a(u,v) = (f,v)_{L^2} \qquad \forall\, v\in V$$

$$u_h\in V_h \qquad a(u_h,v) = (f,v)_{L^2} \qquad \forall\, v\in V_h$$

Then

(15.5) $$\|u-u_h\|_{L^2} \le C\|u-u_h\|_{H^1} \times \sup_{g\in L^2}\left\{\frac{1}{\|g\|_{L^2}}\inf_{v\in V_h}\|\phi_g-v\|_{H^1}\right\}$$

where $\phi_g \in V$ is the unique solution to the adjoint problem

$$a(v,\phi_g) = (g,v)_{L^2} \quad \forall\, v\in V.$$

Proof: Recall FGO:

$$a(e,v)=0 \quad \forall\, v\in V_h$$

with

$$e := u-u_h.$$

For any $g\in L^2(\Omega)$,

$$a(e,\phi_g) = (g,e)_{L^2}$$

Using FGO,

$$a(e,\phi_g-v) = (g,e)_{L^2} \quad \forall\, v\in V_h$$

Now,

$$|(g,e)_{L^2}| = |a(e,\phi_g-v)|$$
$$\le \gamma\,\|e\|_{H^1}\cdot\|\phi_g-v\|_{H^1} \quad \forall\, v\in V_h$$

Thus,

(15.6) $$|(g,e)_{L^2}| \le \gamma\,\|e\|_{H^1}\inf_{v\in V_h}\|\phi_g-v\|_{H^1},$$

given $g\in L^2(\Omega)$.

It is a simple exercise to show that

$$\|u\|_{L^2} = \sup_{v \in L^2} \frac{|(u,v)_{L^2}|}{\|v\|_{L^2}}.$$

Using this identity, we have

$$\|e\|_{L^2} = \sup_{g \in L^2} \frac{|(e,g)_{L^2}|}{\|g\|_{L^2}}$$

$$\overset{(15.6)}{\leq} \sup_{g \in L^2} \left\{ \frac{\gamma \|e\|_{H^1} \inf_{v \in V_h} \|\phi_g - v\|_{H^1}}{\|g\|_{L^2}} \right\}$$

$$= \gamma \|e\|_{H^1} \sup_{g \in L^2} \left\{ \frac{1}{\|g\|_{L^2}} \inf_{v \in V_h} \|\phi_g - v\|_{H^1} \right\}$$

///

<u>Corollary</u> (15.4): Suppose that $\phi_g \in V \subseteq H^1(\Omega)$ satisfies the adjoint problem

$$a(v, \phi_g) = (g, v)_{L^2} \quad \forall v \in V.$$

Assume that elliptic regularity holds: there exists a $C_R > 0$, such that for every $g \in L^2(\Omega)$, it follows that $\phi_g \in H^1(\Omega) \cap H^2(\Omega)$ and

$$\|\phi_g\|_{H^2} \leq C_R \|g\|_{L^2(\Omega)}.$$

Then

$$\|u - u_h\|_{L^2} \leq C\, h\, \|u - u_h\|_{H^1}$$

Proof: Since $\phi_g \in H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$,

$$\inf_{v \in V_h} \|\phi_g - v\|_{H^1} \leq \|\phi_g - \tilde{\Pi}_h \phi_g\|_{H^1}$$
$$\leq C h |\phi_g|_{H^2}$$

where

$$\tilde{\Pi}_h : V \cap H^2(\Omega) \longrightarrow V_h \cap \mathcal{M}_1$$

is an appropriate piecewise linear interpolant.

Using elliptic regularity,

$$\inf_{v \in V_h} \|\phi_g - v\| \leq C h |\phi_g|_{H^2}$$
$$\leq C C_R h \|g\|_{L^2}$$

Thus,

$$\sup_{g \in L^2} \left\{ \frac{1}{\|g\|_{L^2}} \inf_{v \in V_h} \|\phi_g - v\|_{H^1} \right\} \leq C h \qquad ///$$

Question: Can we apply this to get an L^2 error estimate for the non-symmetric problem.

Answer: No, unless $g \equiv 0$. Recall that our problem was as follows: find $u \in V$ such that

$$a(u,v) = L(v), \quad \forall v \in V,$$

where

$$L(v) := (f,v)_{L^2} + (g,v)_{L^2(\Gamma_N)}.$$

If $g \equiv 0$, then we get an L^2 estimate as before:

$$\|u - u_h\|_{L^2} \leq C h^2 |u|_{H^2}.$$

Of course, we must be able to assume that elliptic regularity holds.