Math 574
class # 21
11/11/2025

Convergence of Implicit-Euler-Galerkin Method

Theorem (21.1): Suppose the hypotheses of Theorem (20.2) are satisfied. Assume that u is a solution to

(21.1) $$(\partial_t u, \psi) + (\nabla u, \nabla \psi) = (f, \psi),$$

for all $\psi \in H_0^1(\Omega)$, for all $t \in (0, T]$, where $u(\cdot, 0) = v$, and u has the regularities

$$u \in L^2(0,T; H_0^1 \cap H^{r+1}) \cap H^1(0,T; H^{r+1}) \cap H^2(0,T; L^2) \Rightarrow \boxed{\int_0^T \|\partial_{tt} u\|_{L^2}^2 dt < \infty}$$

and

$$v \in H_0^1 \cap H^{r+1}.$$

Suppose $u_h^k \in V_h = \mathcal{M}_{0,r}$, $k = 0, \dots, K$, satisfies

(21.2) $$(\delta_s u_h^k, \psi) + (\nabla u_h^k, \nabla \psi) = (f, \psi)$$

for all $\psi \in V_h$, with

$$u_h^0 = v_h = P_h v \in V_h,$$

$$\delta_s u_h^k := \frac{u_h^k - u_h^{k-1}}{s},$$

$$s := \frac{T}{K}.$$

Suppose that

$$\|v - v_h\|_{L^2} \leq C h^{r+1} \|v\|_{H^{r+1}}.$$

Then

(21.3)
$$\|u_h^k - u(t_k)\|_{L^2} \leq C h^{r+1} \left(\|v\|_{H^{r+1}} + \int_0^{t_k} \|\partial_t u\|_{H^{r+1}} \, d\tau \right) + s \int_0^{t_k} \|\partial_{tt} u\|_{L^2} \, d\tau .$$

Proof: As before, set

$$\rho(t) := u(t) - R_h u(t)$$

$$\theta^k := R_h u(t_k) - u_h^k \qquad k = 0, 1, \cdots, K,$$

where

$$t_k = s \cdot k, \quad k = 0, 1, \cdots, K.$$

Then

$$u(t_k) - u_h^k = \rho(t_k) + \theta^k$$
$$= \rho^k + \theta^k,$$

where

$$\rho^k := \rho(t_k).$$

Note that θ^k is defined only at t_k, that is, at discrete times, whereas ρ is defined at all times.

As before

$$\|\rho(t_k)\|_{L^2} \le C h^{r+1}\left(\|v\|_{H^{r+1}} + \int_0^{t_k} \|\partial_t u(\tau)\|_{H^{r+1}}\, d\tau\right),$$

for $k = 0, 1, \dots, K$. By the defn of R_h,

$$(\partial_t u, \psi) + (\nabla R_h u, \nabla \psi) = (f, \psi), \quad \forall \psi \in V_h.$$

For $k = 1, \dots, K$, and for all $\psi \in V_h$,

(21.4) $$\left(R_h(\delta_s u(t_k)), \psi\right) + \left(\nabla R_h u(t_k), \nabla \psi\right) = \left(f(t_k), \psi\right) + (w^k, \psi)$$

where

$$\delta_s u(t) := \frac{u(t) - u(t-s)}{s}, \quad t \in [s, T],$$

and

$$w(t) := R_h(\delta_s u(t)) - \partial_t u(t), \quad w^k := w(t_k).$$

Subtracting (21.2) (the Backward Euler-Galerkin scheme) from (21.4) to get

$$(\delta_s \theta^k, \psi) + (\nabla \theta^k, \nabla \psi) = (w^k, \psi), \quad \forall \psi \in V_h.$$

Next, set $\psi = \theta^k$, to get

$$(\theta^k - \theta^{k-1}, \theta^k) + s\,|\theta^k|_{H^1}^2 = s\,(w_k, \theta^k)$$

Then

$$\|\theta^k\|_{L^2}^2 - (\theta^{k-1}, \theta^k) + s\,|\theta^k|_{H^1}^2 = s\,(w^k, \theta^k) \overset{C.S}{\le} s\,\|w^k\|_{L^2} \|\theta^k\|_{L^2}$$

$$\boxed{(\theta^{k-1},\theta^k) \le \|\theta^{k-1}\|_{L^2}\|\theta^k\|_{L^2} \implies -\|\theta^{k-1}\|_{L^2}\|\theta^k\|_{L^2} \le -(\theta^{k-1},\theta^k)}$$

This implies

$$\|\theta^k\|_{L^2}^2 - \|\theta^{k-1}\|_{L^2}\|\theta^k\|_{L^2} \le s\|w^k\|_{L^2}\|\theta^k\|_{L^2},$$

which, in turn, implies

$$\|\theta^k\|_{L^2} - \|\theta^{k-1}\|_{L^2} \le s\|w^k\|_{L^2}$$

for $k=1,2,\dots,K$. Summing from $k=1$ to $k=\ell$

(21.5) $$\|\theta^\ell\|_{L^2} \le \|\theta^0\|_{L^2} + s\sum_{k=1}^{\ell}\|w^k\|_{L^2}$$

for any $1\le \ell\le K$. Define

$$w_1^k := (R_h - I)\,\delta_s u(t_k),$$

$$w_2^k := \delta_s u(t_k) - \partial_t u(t_k).$$

Then

$$w^k = w_1^k + w_2^k.$$

It follows that

$$w_1^k = \frac{1}{s}(R_h - I)\int_{t_{k-1}}^{t_k}\partial_t u(\tau)\,d\tau$$

$$= \frac{1}{s}\int_{t_{k-1}}^{t_k}(R_h - I)\,\partial_t u(\tau)\,d\tau$$

$$\boxed{\left\|\int_{t_a}^{t_b} u(\tau)\,d\tau\right\| \le \int_{t_a}^{t_b}\|u(\tau)\|\,d\tau}$$

Then

$$\|w_1^k\|_{L^2} \le \frac{1}{s}\int_{t_{k-1}}^{t_k}\|R_h\partial_t u(\tau) - \partial_t u(\tau)\|_{L^2}\,d\tau$$

$$\le \frac{1}{s}Ch^{r+1}\int_{t_{k-1}}^{t_k}\|\partial_t u(\tau)\|_{H^{r+1}}\,d\tau.$$

Taylor's Theorem:

$$u(t_{k-1}) = u(t_k) + \partial_t u(t_k)(t_{k-1} - t_k) + \tilde{R}_2[u](t_k)$$

So

(21.6) $$\delta \sum_{k=1}^{\ell} \| W_1^k \|_{L^2} \le C h^{r+1} \int_0^{t_\ell} \| \partial_t u(\tau) \|_{L^2} \, d\tau$$

To estimate W_2^k, note

$$\begin{aligned} \delta W_2^k &= u(t_k) - u(t_{k-1}) - \delta \partial_t u(t_k) \\ &= -\int_{t_{k-1}}^{t_k} (\tau - t_{k-1}) \partial_{tt} u(\tau) \, d\tau , \end{aligned}$$

where we have used Taylor's Theorem with integral remainder. (Remember that?)

Thus,

$$\begin{aligned} \delta \sum_{k=1}^{\ell} \| W_2^k \|_{L^2} &\le \sum_{k=1}^{\ell} \int_{t_{k-1}}^{t_k} | \tau - t_{k-1} | \cdot \| \partial_{tt} u(\tau) \|_{L^2} \, d\tau \\ &\le \delta \int_0^{t_\ell} \| \partial_{tt} u(\tau) \|_{L^2} \, d\tau . \end{aligned}$$ (21.7)

Also, observe

$$\begin{aligned} \| \theta^0 \|_{L^2} &= \| R_h v - U_h^0 \|_{L^2} \\ &\le \| R_h v - v \|_{L^2} + \| v - v_h \|_{L^2} \\ &\le C h^{r+1} \| v \|_{H^{r+1}} \end{aligned}$$ (21.8)

Combining (21.5) – (21.8)

$$\|\theta^\ell\|_{L^2} \le \|\theta^0\|_{L^2} + s\sum_{k=1}^{\ell}\|w_1^k\|_{L^2} + s\sum_{k=1}^{\ell}\|w_2^k\|_{L^2}$$

$$\le C h^{r+1}\|v\|_{H^{r+1}} + C h^{r+1}\int_0^{t_\ell}\|\partial_t u(\tau)\|_{H^{r+1}}\,d\tau + s\int_0^{t_\ell}\|\partial_{tt}u(\tau)\|_{L^2}\,d\tau$$

Since

$$\|\rho^\ell\|_{L^2} \le C h^{r+1}\left(\|v\|_{H^{r+1}} + \int_0^{t_\ell}\|\partial_t u\|_{H^{r+1}}\,d\tau\right),$$

it follows that

$$\|u(t_k) - u_h^k\|_{L^2}$$

$$\le \|\theta^k\|_{L^2} + \|\rho^k\|_{L^2}$$

$$\le C h^{r+1}\left(\|v\|_{H^{r+1}} + \int_0^{t_k}\|\partial_t u(\tau)\|_{H^{r+1}}\,d\tau\right) + s\int_0^{t_k}\|\partial_{tt}u(\tau)\|_{L^2}\,d\tau \qquad ///$$

Crank – Nicolson – Galerkin Method

Given $u_h^{k-1} \in V_h \subset H_0^1(\Omega)$, find $u_h^k \in V_h$, such that

(21.9) $$\left(\frac{u_h^k - u_h^{k-1}}{s}, \psi\right) + \frac{1}{2}\left(\nabla(u_h^k + u_h^{k-1}), \nabla\psi\right) = \left(f(t_{k-1/2}), \psi\right)$$

for all $\psi \in V_h$, for $k = 1, 2, \cdots, K = T/s$, where

$$u_h^0 = v_h = P_h v \in V_h$$

<u>Theorem</u> (21.2): Suppose that either

$$\|v_h\|_{L^2}^2 \leq C + \|v\|_{L^2}^2$$

where $C>0$ is independent of h and s. Then

$$\max_{0\leq k\leq K} \|u_h^k\|_{L^2}^2 \leq C$$

and

$$s\sum_{k=1}^{K} \left|\tfrac{1}{2}(u_h^k + u_h^{k-1})\right|_{H^1}^2 \leq C$$

for some $C>0$ that is independent of s and h.

<u>Proof</u>: In (21.9), set

$$\varphi = u_h^{k-1/2} := \tfrac{1}{2}\left(u_h^k + u_h^{k-1}\right) \in V_h$$

Then, for $k=1,\ldots,K$,

$$\begin{aligned}
\frac{1}{2s}\|u_h^k\|_{L^2}^2 - \frac{1}{2s}\|u_h^{k-1}\|_{L^2}^2 + |u_h^{k-1/2}|_{H^1}^2 &= \left(f(t_{k-1/2}), u_h^{k-1/2}\right) \\
&\overset{\text{C.S.}}{\leq} \|f(t_{k-1/2})\|_{L^2}\|u_h^{k-1/2}\|_{L^2} \\
&\overset{\text{Poincare}}{\leq} \|f(t_{k-1/2})\|_{L^2}\, C|u_h^{k-1/2}|_{H^1} \\
&\overset{\text{AGMI}}{\leq} \frac{C^2}{2}\|f(t_{k-1/2})\|_{L^2}^2 + \frac{1}{2}|u_h^{k-1/2}|_{H^1}^2
\end{aligned}$$

Thus,

$$\frac{1}{2s}\|u_h^k\|_{L^2}^2 - \frac{1}{2s}\|u_h^{k-1}\|_{L^2}^2 + \frac{1}{2}|u_h^{k-1/2}|_{H^1}^2 \leq \frac{C^2}{2}\|f(t_{k-1/2})\|_{L^2}$$

which implies

$$\|u_h^k\|_{L^2}^2 - \|u_h^{k-1}\|_{L^2}^2 + s|u_h^{k-1/2}|_{H^1}^2 \leq sC^2\|f(t_{k-1/2})\|_{L^2}.$$

Summing from $k=1$ to $k=\ell$, $1\leq \ell \leq K$, we get

$$\begin{aligned}
\|u_h^\ell\|_{L^2}^2 + s\sum_{k=1}^{\ell}|u_h^{k-1/2}|_{H^1}^2 &\leq \|v_h\|_{L^2}^2 \\
&\quad + sC^2\sum_{k=1}^{\ell}\|f(t_{k-1/2})\|_{L^2}^2 \\
&\leq C + \|v\|_{L^2}^2 \\
&\quad + sC^2\sum_{k=1}^{K}\|f(t_{k-1/2})\|_{L^2}^2 \\
&\leq C + \|v\|_{L^2}^2 \\
&\quad + C + C^2\int_0^T\|f(t)\|_{L^2}^2\,dt \\
&= C,
\end{aligned}$$

and $C>0$ is independent of h and s. ///