Math 574
class #16
10/16/2025

Condition Number of A

Consider the Galerkin approx of the model problem: find $u_h \in V_h = \mathcal{M}_{0,r}$ such that

$$a(u_h, v) = (f, v)_{L^2} \qquad \forall\, v \in V_h$$

where

$$a(u,v) := (\nabla u, \nabla v)_{L^2}$$

$$= \sum_{|\alpha|=1} (\partial^\alpha u, \partial^\alpha v)_{L^2}$$

Define

$$A = [a_{i,j}]_{i,j=1}^{N_r^0}$$

where

$$N_r^0 = \dim(\mathcal{M}_{0,r}),$$

$$a_{i,j} = a(\phi_j, \phi_i), \qquad 1 \le i, j \le N_r^0$$

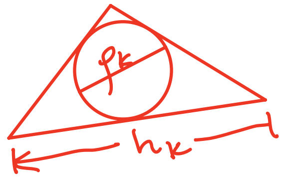
and

$$\mathcal{B} = \{\phi_1, \dots, \phi_{N_r^0}\} \subset \mathcal{M}_{0,r}$$

is a basis for $V_h = \mathcal{M}_{0,r}$.

Theorem (16.1): Suppose that $\{\mathcal{T}_h\}_h$ is a quasi-uniform, shape regular family of triangulation of the 2D polygonal domain Ω, i.e., there are constants $\beta_1, \beta_2 > 0$, independent of h, such that

(16.1) $$1 \geq \frac{h_K}{h} \geq \beta_1, \quad \forall\, K \in \mathcal{T}_h, \quad \forall\, h > 0 \qquad \text{(quasi-uniform)}$$



(16.2) $$1 > \frac{\rho_K}{h_K} \geq \beta_2, \quad \forall K \in \mathcal{T}_h, \quad \forall h > 0 \quad \text{(shape regular)}.$$

Then, there are constants $C_1, C_2 > 0$, independent of h, that such

$$C_1 h^{-2} \leq \kappa_2(A) \leq C_2 h^{-2},$$

for all $h > 0$ sufficiently small.

Proof: We will only prove the upper bound. The lower bound is a bit harder. See Brenner (2007).

Recall that A is SPD. Therefore

$$\kappa_2(A) := \|A\|_2 \|A^{-1}\|_2 = \frac{\lambda_{max}}{\lambda_{min}}.$$

Now, let $v \in V_h$. $\exists !\ v_1, \dots, v_{N_I^0} \in \mathbb{R}$ such that

$$v = \sum_{i=1}^{N_I^0} v_i \phi_i$$

$v_1, \dots, v_{N_I^0}$ are called the coordinates of v wrt the basis B. We write

$$v \in V_h \overset{B}{\longleftrightarrow} \vec{v} \in \mathbb{R}^{N_I^0}, \quad \vec{v} := (v_1, \dots, v_{N_I^0})$$

From homework, recall that, there exist constants

(16.3) $$C_3 h^2 \|\vec{v}\|_2^2 \leq \|v\|_{L^2(\Omega)}^2 \leq C_4 h^2 \|\vec{v}\|_2^2$$

for all $v \in V_h$. C_3 and C_4 are independent of h.

Furthermore there is a constant $C_5 > 0$ such that, for all $v \in V_h$,

(16.4) $$a(v,v) = |v|_{H^1}^2 \leq C_5 h^{-2} \|v\|_{L^2}^2 .$$

C_5 is independent of h.

Then, for all $\vec{v} \in \mathbb{R}^{N_r^\circ}$,

$$R(\vec{v}) = \frac{\vec{v}^T A \vec{v}}{\vec{v}^T \vec{v}}$$

$$= \frac{a(v,v)}{\|\vec{v}\|_2^2}$$

$v \in V_h \xleftrightarrow{B} \vec{v} \in \mathbb{R}^{N_r^\circ}$

$$\overset{(16.4)}{\leq} \frac{C_5 h^{-2} \|v\|_{L^2}^2}{\|\vec{v}\|_2^2}$$

$$\overset{(16.3)}{\leq} \frac{C_5 h^{-2} C_4 h^2 \|\vec{v}\|_2^2}{\|\vec{v}\|_2^2}$$

$$= C_4 C_5 .$$

Similarly,

$$R(\vec{v}) = \frac{\vec{v}^T A \vec{v}}{\|\vec{v}\|_2^2}$$

$$= \frac{a(v,v)}{\|\vec{v}\|_2^2}$$

$v \in V_h \xleftrightarrow{B} \vec{v} \in \mathbb{R}^{N_r^\circ}$

$$\overset{\text{Poincare}}{\geq} \frac{C_P \|v\|_{L^2}^2}{\|\vec{v}\|_2^2}$$

$$\overset{(16.3)}{\geq} \frac{C_P C_3 h^2 \|\vec{v}\|_2^2}{\|\vec{v}\|_2^2}$$

$$= C_3 C_p h^2$$

Thus,

$$R(\vec{w}_{max}) = \lambda_{max} \leq C_4 C_5$$

and

$$R(\vec{w}_{min}) = \lambda_{min} \geq C_3 C_p h^2.$$

Therefore,

$$\kappa_2(A) = \frac{\lambda_{max}}{\lambda_{min}} \leq \frac{C_4 C_5}{C_3 C_p} h^{-2}. \quad ///$$

Remark: This suggests that the stiffness matrix is ill-conditioned

$$\kappa_2(A) \to \infty \quad \text{as} \quad h \to 0.$$

The Biharmonic Problem

The Biharmonic Problem is given as follows: given $f \in C^0(\Omega)$, find

$$u \in C^4(\Omega) \cap C^1(\bar{\Omega})$$

such that

$$(16.5) \quad \begin{cases} \Delta^2 u = \Delta(\Delta u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \hat{n} \cdot \nabla u =: \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

The boundary conditions are said to be of <u>clamped type</u>.

Define

$$V = H_0^2(\Omega) := \{ v \in H^2(\Omega) \mid v = 0 = \frac{\partial v}{\partial n} \text{ on } \partial\Omega \}$$

This space is a Hilbert space. It is a topologically-closed, proper subspace of $H^2(\Omega)$.

$$H_0^2(\Omega) = \text{closure}\left(C_0^\infty(\Omega) \right)$$

with respect to the $\|\cdot\|_{H^2}$-norm.

Multiply by a test function $v \in V = H_0^2(\Omega)$ and use Green's 2^{nd} Id., i.e.,

$$(\Delta w, v)_{L^2(\Omega)} = (w, \Delta v)_{L^2(\Omega)} + \cancelto{0}{\left(\frac{\partial w}{\partial n}, v\right)_{L^2(\partial\Omega)}}$$

$$- \cancelto{0}{\left(w, \frac{\partial v}{\partial n}\right)_{L^2(\partial\Omega)}},$$

to get

$$(\Delta u, \Delta v)_{L^2} = (f, v)_{L^2} \quad \forall v \in V = H_0^2(\Omega)$$

setting $w = \Delta u$.

<u>Defn</u> (16.2): The <u>weak formulation</u> of the Biharmonic Problem (with clamped boundary conditions) is defined as follows: Given $f \in L^2(\Omega)$, find $u \in V = H_0^2(\Omega)$ such that

(16.6) $$a(u, v) = (f, v)_{L^2} \quad \forall v \in V = H_0^2(\Omega)$$

where

$$a(u,v) = (\Delta u, \Delta v)_{L^2}, \quad \forall\, u,v \in V = H_0^2(\Omega) \tag{16.7}$$

Let $V_h \subset V = H_0^2(\Omega)$ be a finite dimensional subspace. The <u>Galerkin approximation</u> of the Biharmonic Problem is defined as usual: Find $u_h \in V_h$ such that

$$a(u_h, v) = (f, v)_{L^2}, \quad \forall\, v \in V_h.$$

<u>Theorem</u> (16.3): Let $\Omega \subset \mathbb{R}^d$ be bounded and open. There exists a constant $C > 0$ such that, for all $v \in H_0^2(\Omega)$,

Poincare-type ineq.

$$\|u\|^2_{H^2(\Omega)} \le C\,(\Delta u, \Delta u)_{L^2(\Omega)} = C\,\|\Delta u\|^2_{L^2}$$

<u>Proof</u>: Homework exercise. ///

<u>Proposition</u> (16.4): The bilinear form

$$a(u,v) = (\Delta u, \Delta v)_{L^2}$$

is coercive and continuous of the Hilbert space $H_0^2(\Omega)$, with respect to the norm $\|\cdot\|_{H^2}$.

<u>Proof</u>: Homework exercise. ///

Corollary (16.5) The bilinear form

$$a(u,v) = (u,v)_{H_0^2} := (\Delta u, \Delta v)_{L^2} \qquad \forall\, u,v \in H^2(\Omega)$$

is an inner product on $H_0^2(\Omega)$. $\left(H_0^2(\Omega); (\cdot,\cdot)_{H_0^2}\right)$ is a Hilbert space with norm

$$\|u\|_{H_0^2} = \sqrt{(u,u)_{H_0^2}}$$
$$= \sqrt{(\Delta u, \Delta u)_{L^2}}.$$

Remark: Note that

$$\|u\|_{H_0^2} \neq |u|_{H^2} \qquad \left(\|u\|_{H_0^1} = |u|_{H^1}\right)$$

Why? Homework exercise.

Proposition (16.6): Both the weak formulation of the Biharmonic Problem and its Galerkin approx problem have unique solutions.

Proof: Use either RRT or Lax-Milgram Lemma. The details are left for a homework exercise. ///

But, we have a slight problem.

Theorem: (16.7) Let $\mathcal{T}_h$ be a conforming triangulation of the polygonal domain $\Omega \subset \mathbb{R}^2$. Suppose that V_h is a linear subspace of $L^2(\Omega)$ with the property that

$$v|_K \in \mathbb{P}_r(K), \quad \forall\, K \in \mathcal{T}_h, \quad \forall\, v \in V_h.$$

Then
$$V_h \subset H^k(\Omega) \iff V_h \subset C^{k-1}(\bar{\Omega})$$

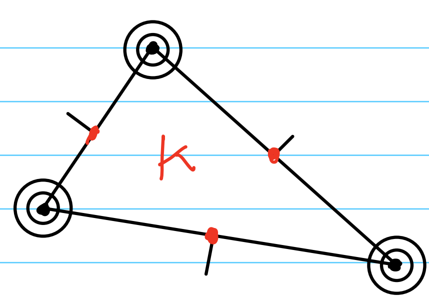
If $V_h \subset C^{k-1}(\bar{\Omega})$ its dimension is finite.

Proof: See, for example, Braess (2007).

Remark: For the biharmonic problem, we need $V_h \subset H^2(\Omega)$. Therefore, we require that
$$V_h \subset C^1(\bar{\Omega})$$

It turns out, it is not an easy task to construct a piecewise polynomial space to satisfy $V_h \subset C^1(\bar{\Omega})$.

The Argyris Triangle



$v|_K \in \mathbb{P}_5(K)$

21 degrees of freedom (DOF)

$V_h \subset C^1(\bar{\Omega})$

	DOF
$\partial^\alpha(\vec{a}_i)$, $\lvert\alpha\rvert \le 2$, $i = 1, 2, 3$,	$3(1+2+3) = 18$
$\frac{\partial v}{\partial n}(\vec{a}_{1,2})$, $\frac{\partial v}{\partial n}(\vec{a}_{2,3})$, $\frac{\partial v}{\partial n}(\vec{a}_{3,1})$	3

<u>Proposition</u> (16.8): Let $v \in \mathbb{P}_5(K)$, where $K \subset \mathbb{R}^2$ is a non-degenerate triangle. Then v is uniquely determined by the 21 degrees of freedom

	DOF
$\partial^\alpha(\vec{a}_i)$, $\lvert\alpha\rvert \le 2$, $i = 1,2,3$,	$3(1+2+3) = 18$
$\frac{\partial v}{\partial n}(\vec{a}_{1,2})$, $\frac{\partial v}{\partial n}(\vec{a}_{2,3})$, $\frac{\partial v}{\partial n}(\vec{a}_{3,1})$	3

<u>Proof</u>: See Brenner and Scott (2008). ///

<u>Remark</u>: A piecewise quintic polynomial space can be constructed using the Argyris triangle.

Now, let us construct an interpolant.

<u>Defn</u> (16.9): Let $\{\mathcal{T}_h\}_h$ be a family of shape regular triangulations of the polygonal domain $\Omega \subset \mathbb{R}^2$.

Suppose

$$V_h = \{ v \in C^1(\bar{\Omega}) \mid v|_K \in \mathbb{P}_5(K), \ \forall K \in \mathcal{T}_h \}$$

Define the <u>C^1 interpolant</u> $\Pi_h : C^2(\bar{\Omega}) \to V_h$, for each $v \in C^2(\bar{\Omega})$, via

$$\partial^\alpha \Pi_h v = \partial^\alpha v \qquad \lvert\alpha\rvert \le 2 \qquad \text{(18 DOF)}$$

at vertex nodes, and

$$\frac{\partial}{\partial n} \Pi_h v = \frac{\partial v}{\partial n} \qquad \text{(3 DOF)}$$

at edge midpoint nodes.

Argyris Error Estimates for Biharmonic Problem

Interpolation Estimate:

$$\|u - \Pi_h u\|_{H^m} \le C h^{6-m} |u|_{H^6(\Omega)}$$

for any $u \in H^6(\Omega)$, and $m \le 5$. For the problem at hand, $m = 2$. Thus,

$$\|u - \Pi_h u\|_{H^2} \le C h^4 |u|_{H^6(\Omega)}.$$

Cea's lemma guarantees that

$$\|u - u_h\|_{H_0^2(\Omega)} = \min_{v \in V_h} \|u - v\|_{H_0^2(\Omega)}$$

Now

$$H^6(\Omega) \hookrightarrow C^{6-1-1}(\bar{\Omega})$$

$$= C^4(\bar{\Omega})$$

since

$$6 = k > \frac{d}{2} = \frac{2}{2}.$$

Thus $\Pi_h u$ is well-defined for $u \in H^6$. So

$$\|u - u_h\|_{H_0^2(\Omega)} = \min_{v \in V_h} \|u - v\|_{H_0^2(\Omega)}$$

$$\le \|u - \Pi_h u\|_{H_0^2(\Omega)}$$

Combining with interpolation error estimates,

$$\|u-u_h\|_{H_0^2(\Omega)} \leq C h^4 |u|_{H^6(\Omega)} .$$

Assuming elliptic regularity, we can use a Nitsche-like trick to get

$$\|u-u_h\|_{L^2(\Omega)} \leq C h^6 |u|_{H^6(\Omega)} \quad ///$$

at edge midpoint nodes.

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