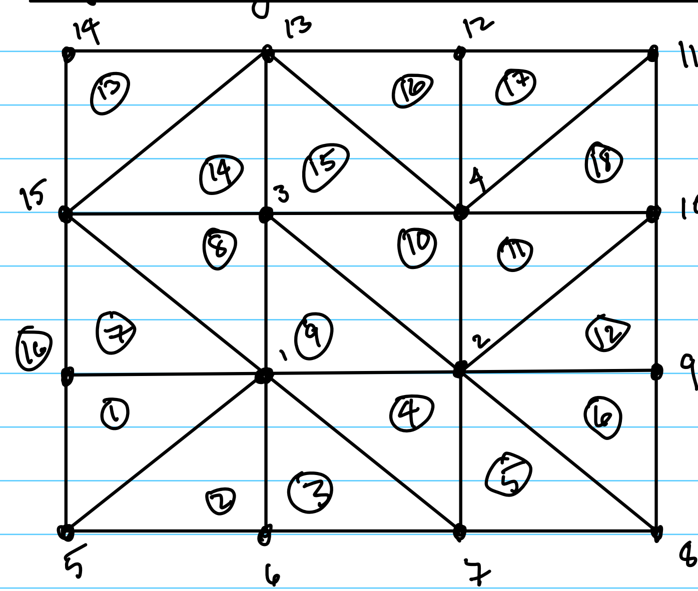
Math 574
class #10
09/18/2025

Representing The Mesh with Arrays



$$\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} \bar{K}$$

coordinate array

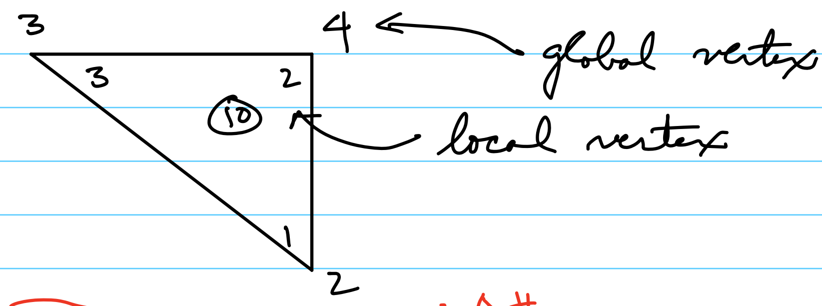
$$Z = \underbrace{\begin{bmatrix} x_{1,1} & x_{2,1} & x_{3,1} & x_{4,1} \\ x_{1,2} & x_{2,2} & x_{3,2} & x_{4,2} \end{bmatrix}}_{\text{interior vertex coordinates}} \underbrace{\begin{bmatrix} x_{5,1} & \cdots & x_{16,1} \\ x_{5,2} & & x_{16,2} \end{bmatrix}}_{\text{bndry vertex coordinates}}$$

local-to-global node mapping

$$T = \begin{array}{c} \left[\begin{array}{cccccccccccccccccc} 5 & 6 & 6 & 1 & 7 & 8 & 1 & 1 & 1 & 2 & 2 & 9 & 15 & 15 & 3 & 4 & 4 & 4 \\ 1 & 1 & 7 & 7 & 8 & 9 & 15 & 3 & 2 & 4 & 10 & 10 & 13 & 3 & 4 & 12 & 11 & 10 \\ 16 & 5 & 1 & 2 & 2 & 2 & 16 & 15 & 3 & 3 & 4 & 2 & 14 & 13 & 13 & 13 & 12 & 11 \end{array}\right] \\ \begin{array}{cccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \end{array} \end{array}$$

T gives the local to global mapping of vertices.

look at triangle 10, K_{10}:



local node number

$\Delta\#$

$$T(2,10) = 4$$

global node number

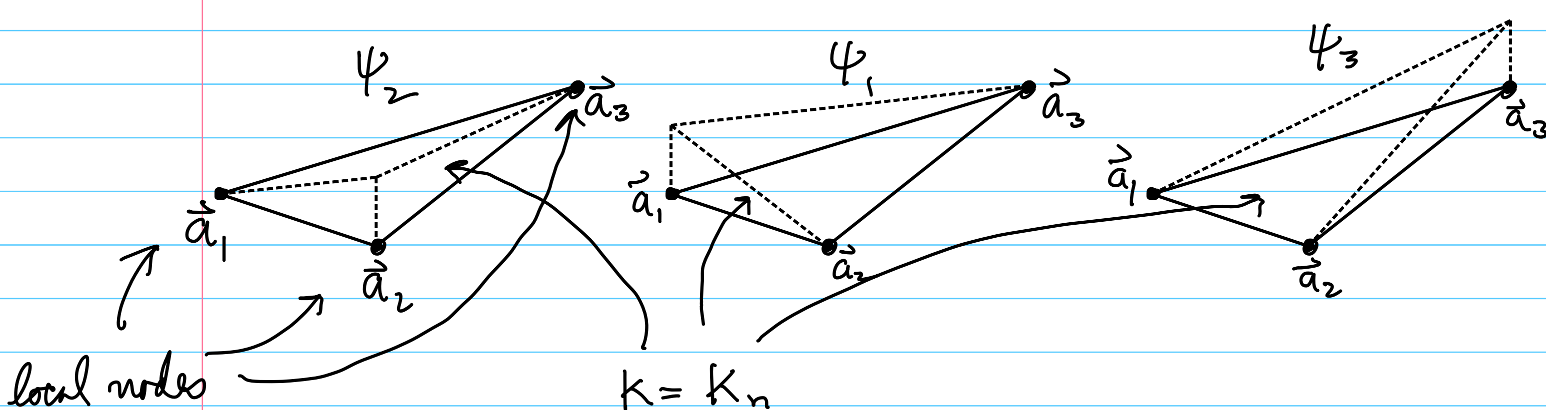
In words, the global node number of local node 2 in triangle 10 is 4.

The Local Stiffness Matrix and Load Vector

Piecewise Linear Case $(r=1)$

Let us index the triangles of the triangulation

$$\mathcal{T}_h = \{K_i\}_{i=1}^{M}$$



ψ_1, ψ_2, ψ_3 are called the <u>local</u> <u>Lagrange</u> <u>nodal</u> <u>basis</u> <u>functions</u>. They satisfy

$$\psi_i(\vec{a}_j) = \delta_{i,j} \qquad i,j \in \{1,2,3\}$$

Define, for $n = 1, \cdots, M$ (for each triangle),

$$A^{(n)} := \left[a^{(n)}_{\alpha,\beta}\right]_{\alpha,\beta=1}^{3} \in \mathbb{R}^{3\times3}_{sym}$$

where

$$a^{n}_{\alpha,\beta} = \int_{K_n} \nabla\psi_\beta \cdot \nabla\psi_\alpha \, d\vec{x}.$$

$A^{(n)}$ is called the <u>local stiffness matrix</u>.

Define, for $n = 1, \cdots, M$,

$$\vec{f}^{(n)} = \left[f^{(n)}_\alpha\right]_{\alpha=1}^{3} \in \mathbb{R}^3$$

where

$$f^{(n)}_\alpha = \int_{K_n} f\,\psi_\alpha \, d\vec{x}.$$

This is called the <u>local load/force vector</u>.

We will discuss practical ways to compute these later using the reference triangle and affine mappings.

For now, let us assume that $A^{(n)}$ and $\vec{f}^{(n)}$ can be easily obtained.

Assembly of A and $\vec{f}$ $(r=1)$

We assemble A and f, the global stiffness matrix and global force vector, resp, by looping over triangles of the mesh, rather than the nodes of the mesh.

Recall that, for the model problem

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

we have

$$A \in \mathbb{R}^{N_1^0 \times N_1^0}_{sym} \qquad \vec{f} \in \mathbb{R}^{N_1^0}$$

$$\boxed{[A]_{28,15} = \int_{S_{28} \cap S_{15}} \nabla\phi_{15} \cdot \nabla\phi_{28}\, dx \quad \text{example}}$$

Initialize: Set

$\boxed{\text{Matlab-Style Array Sections}}$

$$A(1:N_1^0, 1:N_1^0) = 0$$

$$f(1:N_1^0) = 0$$

Triangle loop: do $n = 1:M$

alpha loop: do $\alpha = 1:3$

if $T(\alpha, n) > N_1^0$ index alpha loop

beta loop: do $\beta = 1:3$

if $T(\beta, n) > N_1^0$ index beta loop

$$A(T(\alpha,n), T(\beta,n)) = A(T(\alpha,n), T(\beta,n)) + a^{(n)}_{\alpha\beta}$$

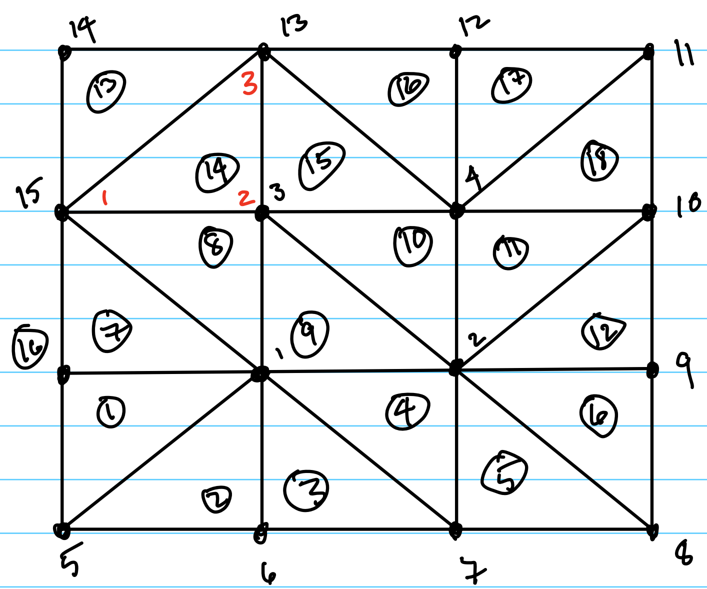
end do beta loop

$$f(T(\alpha,n)) = f(T(\alpha,n)) + f_\alpha^{(n)}$$

end do alpha loop

end do triangle loop

Example:



$M = 18$

$N_1^0 = 4$

$N_1 = 16$

Boundary nodes are indexed after interior nodes!

$$T = \begin{array}{cccccccccc} 5 & 6 & 6 & 1 & 7 & 8 & 1 & 1 & 1 & 2 \\ 1 & 1 & 7 & 7 & 8 & 9 & 15 & 3 & 2 & 4 \\ 16 & 5 & 1 & 2 & 2 & 2 & 16 & 15 & 3 & 3 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

$$\begin{array}{cccccccc} 2 & 9 & 15 & 15 & 3 & 4 & 4 & 4 \\ 10 & 10 & 13 & 3 & 4 & 12 & 11 & 10 \\ 4 & 2 & 14 & 13 & 13 & 13 & 12 & 11 \\ \hline 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \end{array}$$

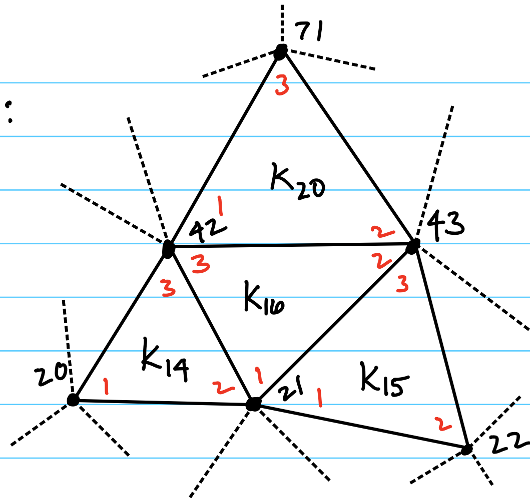
11

M

What is the meaning of $T(2,14)$?

Example :

$(r=1)$



The red numbers are the local node numbers.

$$T = \begin{bmatrix} \cdots & 20 & 21 & 21 & \cdots \\ & 21 & 22 & 43 & \\ & 42 & 43 & 42 & \end{bmatrix}$$
$$\begin{array}{ccc} 14 & 15 & 16 \end{array}$$

Let's look at the assembly of A.

$n = 16 \qquad \alpha = 1 \qquad \beta = 1$

$$T(\alpha, n) = 21 \qquad T(\beta, n) = 21$$

$$A(21,21) = A(21,21) + a^{(16)}_{1,1}$$

$n = 16 \qquad \alpha = 1 \qquad \beta = 2$

$$T(\alpha, n) = 21 \qquad T(\beta, n) = 43$$

$$A(21,43) = A(21,43) + a^{(16)}_{1,2}$$

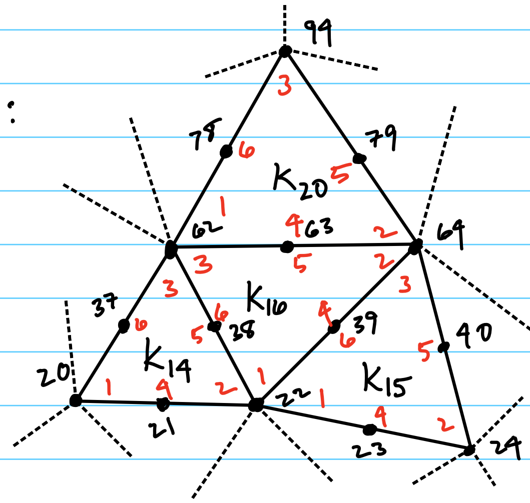
$n = 16 \qquad \alpha = 1 \qquad \beta = 3$

$$T(\alpha, n) = 21 \qquad T(\beta, n) = 42$$

$$A(21,42) = A(21,42) + a^{(16)}_{1,3}$$

$$\vdots$$

Example :

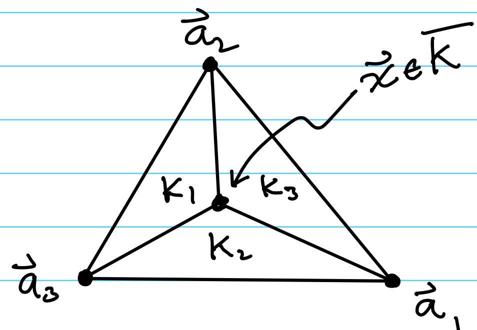


For piecewise quadratic FEM. We need to add edge midpoint nodes to the lists.

$$T = \begin{array}{c} \left[\begin{array}{ccccc} \cdots & 22 & 22 & \cdots \\ & 24 & 64 & \\ & 64 & 62 & \\ & 23 & 39 & \\ & 40 & 63 & \\ & 39 & 38 & \end{array}\right] \\ \begin{array}{cc} 15 & 16 \end{array} \end{array}$$

Barycentric Coordinates

Suppose that $K \in \mathcal{T}_h$ is arbitrary.



$$\lambda_i(\vec{x}) := \frac{|K_i|}{|K|} \qquad i = 1,2,3$$

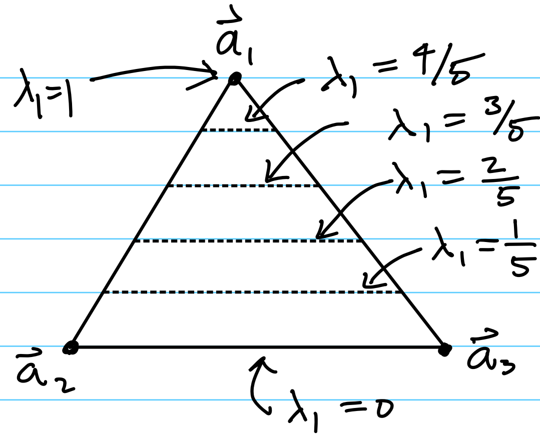
$\vec{a}_i$ are the vertices.

Clearly

$$\lambda_1(\vec{a}_1) = 1 \qquad \lambda_1(\vec{a}_2) = 0 \qquad \lambda_1(\vec{a}_3) = 0$$

In general

$$\lambda_i(\vec{a}_j) = \delta_{i,j} \qquad i,j = 1,2,3.$$



$\lambda_1 \in P_1(K)$

Note that $\lambda_1, \lambda_2, \lambda_3$ are not independent:

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 = \frac{|K_1| + |K_2| + |K_3|}{|K|}$$

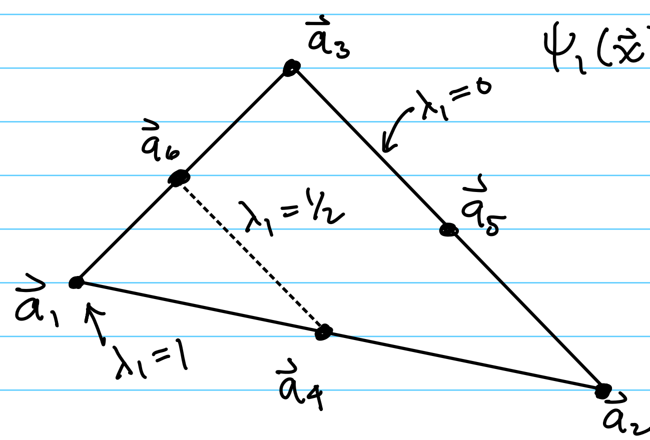
Observe that

$$0 \le \lambda_i(\vec{x}) \le 1, \quad \vec{x} \in \overline{K}, \quad i = 1, 2, 3.$$

Finally, observe that

$$\psi_i(\vec{x}) = \lambda_i(x), \qquad i = 1, 2, 3$$

Quadratic Case



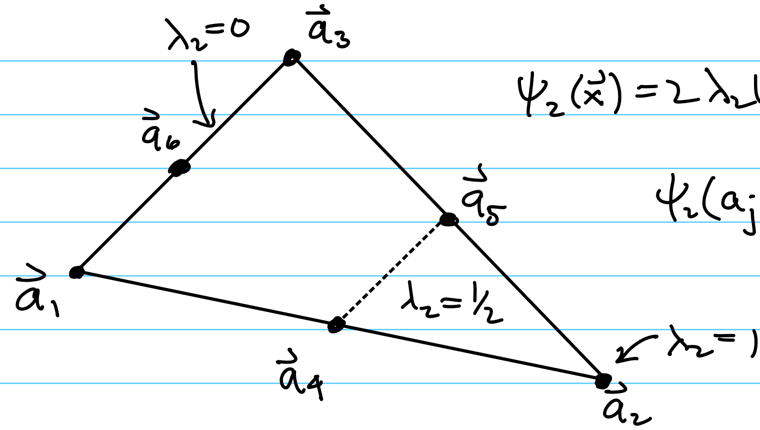
$$\psi_1(\vec{x}) = 2\lambda_1(\vec{x})\left(\lambda_1(\vec{x}) - 1/2\right)$$

$$\psi_1(\vec{a}_1) = 1$$

$$\psi_1(\vec{a}_6) = 0$$

$$\psi(\vec{a}_4) = 0$$

$$\psi_1(\vec{a}_2) = \psi_1(\vec{a}_5) = \psi_1(\vec{a}_3) = 0$$



$$\varphi_2(\vec{x}) = 2\lambda_2(\vec{x})\left(\lambda_2(\vec{x}) - 1/2\right)$$

$$\varphi_2(a_j) = \delta_{2j}$$

Likewise

$$\varphi_3(\vec{x}) = 2\lambda_3(\vec{x})\left(\lambda_3(\vec{x}) - 1/2\right)$$

For the bubble functions

$$\varphi_4(\vec{x}) = 4\lambda_1(\vec{x})\lambda_2(\vec{x}),$$

$$\varphi_5(\vec{x}) = 4\lambda_2(\vec{x})\lambda_3(\vec{x}),$$

$$\varphi_6(x) = 4\lambda_1(\vec{x})\lambda_3(\vec{x}).$$

The reader can confirm that

$$\varphi_i(\vec{a}_j) = \delta_{ij} \qquad i,j = 1,2,\cdots,6$$

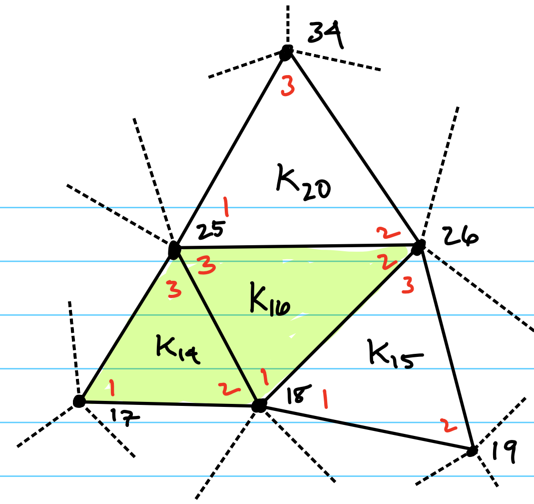
This is a local basis: For any $v \in \mathbb{P}_2(K)$, $\exists!$ $c_1, \cdots, c_6$ such that

$$\vec{v}(\vec{x}) = \sum_{j=1}^{6} c_j \varphi_j(\vec{x}).$$

Clearly

$$c_j = v(\vec{a}_j), \quad j = 1,2,\cdots,6.$$

Example: $(r=1)$



global stiffness matrix

$$a_{25,18} = \int_{\Omega} \nabla\phi_{18} \cdot \nabla\phi_{25} \, d\vec{x}$$

$$= \int_{S_{18} \cap S_{25}} \nabla\phi_{18} \cdot \nabla\phi_{25} \, d\vec{x}$$

$$= \int_{K_{14}} \nabla\phi_{18} \cdot \nabla\phi_{25} \, d\vec{x} + \int_{K_{16}} \nabla\phi_{18} \cdot \nabla\phi_{25} \, d\vec{x}$$

$$= \int_{K_{14}} \nabla\psi_{2} \cdot \nabla\psi_{3} \, d\vec{x} + \int_{K_{16}} \nabla\psi_{1} \cdot \nabla\psi_{3} \, d\vec{x}$$

$$= a^{(14)}_{3,2} + a^{(16)}_{3,1}$$

local stiffness matrices for triangles 14 and 16.