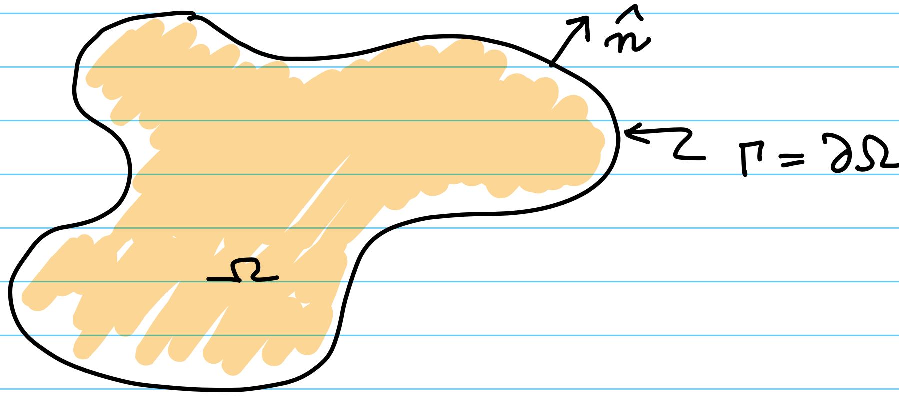
Math 574
class #01
08/19/2025

Finite Element Methods

Consider a bounded, open set $\Omega \subset \mathbb{R}^2$.



Model Problem: Find $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that

(1.1) $$\begin{cases} -\Delta u := -\nabla \cdot (\nabla u) = f & \text{in } \Omega \\ u \equiv 0 & \text{on } \partial\Omega = \Gamma \end{cases}$$

where $f \in C^0(\Omega)$ is a given function.

This is known as Poisson's problem. The equation

$$-\Delta u = f$$

is called the Poisson equation.

Now, we need an identity.

Theorem (Green's 1^{st} ID) Let $v, w : \bar{\Omega} \to \mathbb{R}$ with

$$v \in C^1(\Omega) \cap C^0(\bar{\Omega})$$

and

$$w \in C^2(\Omega) \cap C^1(\bar{\Omega}).$$

Then

$$(1.2) \qquad \int_\Omega \nabla v \cdot \nabla w \, d\vec{x} = \int_\Gamma v \frac{\partial w}{\partial n} \, ds - \int_\Omega v \, \Delta w \, d\vec{x}$$

where

$$\frac{\partial w}{\partial n} := \hat{n} \cdot \nabla w.$$

Let us multiply Poisson's equation by v and integrate. We have

$$-\int_\Omega \Delta u \, v \, d\vec{x} \overset{(1.2)}{=} \int_\Omega \nabla u \cdot \nabla v \, d\vec{x} - \int_\Gamma v \frac{\partial u}{\partial n} \, ds$$

$$\overset{(1.1.1)}{=} \int_\Omega f v \, d\vec{x},$$

where we have assumed that

$$u \in C^2(\Omega) \cap C^1(\bar{\Omega}) \qquad \text{(smoother)}$$

and

$$v \in C^1(\Omega) \cap C^0(\bar{\Omega}).$$

Now, suppose that $v \equiv 0$ on Γ, the same as the solution u. Then

$$\int_\Omega \nabla u \cdot \nabla v \, d\vec{x} = \int_\Omega f v \, d\vec{x}$$

for all $v \in C^1(\Omega) \cap C^0(\bar{\Omega})$ with $v \equiv 0$ on Γ.

Let's define a space. Define

(1.3) $$V := \{ v \in C^1(\Omega) \cap C^0(\bar{\Omega}) \mid v \equiv 0 \text{ on } \Gamma = \partial\Omega \}$$

This is a bona fide linear (vector) space!

Let us also define a bilinear form. Define, for all $u, v \in V$

(1.4) $$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, d\vec{x}.$$

Finally, we define a functional. Define

(1.5) $$F(v) := \int_\Omega f v \, d\vec{x}$$

for all $v \in V$.

We now have the following <u>weak formulation</u> of the model problem: Find $u \in V$ such that

(1.6) $$a(u,v) = F(v) \qquad \forall \; v \in V.$$

<u>Proposition</u>: If u solves problem (1.1) and

$$u \in C^2(\Omega) \cap C^1(\bar{\Omega})$$

then u solves problem (1.6).

Proof: Apply Green's 1^{st} Identity. ///

What about the converse?

Next, let us consider a <u>variational formulation</u> of the model problem: Find $u \in V$ such that

(1.7) $$u = \operatorname*{argmin}_{v \in V} G(v)$$

where

(1.8) $$G(v) := \frac{1}{2} a(v,v) - F(v)$$

for all $v \in V$.

<u>Proposition</u> u solves (1.7) iff u solves (1.6)

Proof: ($\Leftarrow$): Suppose that u solves (1.6). Let $w \in V$ be arbitrary.

$$G(u+w) = \frac{1}{2} a(u,u) + a(u,w) + \frac{1}{2} a(w,w)$$

$$- F(u) - F(w)$$

$$= G(u) + \overbrace{a(u,w) - F(w)}^{=0}$$

$$+ \frac{1}{2} a(w,w)$$

$$\overset{(1.6)}{=} G(u) + \frac{1}{2} a(w,w)$$

Suppose that $w \not\equiv 0$, then we can show that

$$a(w,w) > 0.$$

More on this latter. Further,

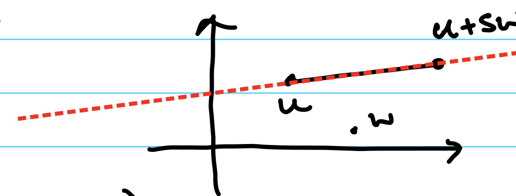
$$a(w,w) = 0 \iff w \equiv 0 \text{ in } V$$

Hence

$$G(u+w) \geq G(u) \qquad \forall\, w \in V.$$

And, the inequality is strict iff $w \not\equiv 0$. Thus $u \in V$ solves (1.7). //

($\Rightarrow$): Suppose that $u \in V$ solves (1.7). Suppose $w \in V$, $w \not\equiv 0$, $s \in \mathbb{R}$. Consider



$$g(s) := G(u+sw)$$

$$= G(u) + s\big(a(u,w) - F(w)\big) + \frac{s^2}{2}\, a(w,w).$$

For fixed u and w, this is a simple quadratic polynomial in s. It is convex since

$$a(w,w) > 0$$

It must be true that $g(s)$ has a global min at $s=0$. Why?

If so

$$\left.\frac{dg(s)}{ds}\right|_{s=0} = 0$$

But observe that

$$\frac{d}{ds} g(s)\Big|_{s=0} = a(u,w) - F(w)$$

So

$$a(u,w) = F(w).$$

Now let $w \in V$, $w \not\equiv 0$ be arbitrary. ///

What about $w \equiv 0$?

Now, it turns out that the weak formulation above is not weak enough for our purposes.

Later, we are going to replace V with something weaker, namely, we will set

$$V = H_0^1 := \{ v \in L^2(\Omega) \mid \partial_x v, \partial_y v \in L^2(\Omega) \text{ and } v \equiv 0 \text{ on } \partial\Omega \}$$

For now we will not worry too much about this.

Galerkin Methods

Let V_h be a finite dimensional subspace of V with basis

$$B = \{\phi_1, \dots, \phi_M\} \qquad \phi_j \in V, \ j = 1, \dots, M$$

(Think $h = \frac{1}{M}$.) Then, of course,

$$\dim(V_h) = M$$

<u>Galerkin Method</u>: Find $u_h \in V_h$ such that

(1.9) $$a(u_h, v) = F(v) \qquad \forall\, v \in V_h.$$

<u>Ritz - Galerkin Method</u>: Find $u_h \in V_h$ such that

(1.10) $$u_h = \underset{v \in V_h}{\operatorname{argmin}}\, G(v)$$

Recall our definitions:

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, d\vec{x}$$

$$F(v) := \int_\Omega f v \, d\vec{x}$$

$$G(v) = \frac{1}{2} a(u,w) - F(v)$$

Defn: For all $v \in V$, define

$$\|v\|_E := \sqrt{a(v,v)}.$$

This object is called the <u>energy norm</u>.

Is this a norm?

Proposition: $a : V \times V \to \mathbb{R}$ is an inner product on

$$V = \{ v \in C^1(\Omega) \cap C^0(\bar{\Omega}) \mid v \equiv 0 \text{ on } \partial\Omega \}$$

and $\|\cdot\|_E : V \to \mathbb{R}$ is its induced norm.

Proof: The only difficult point is

$$a(u,u) = 0 \iff u \equiv 0 \in V$$

We shall address this in more detail later. ///

Theorem (Cauchy-Schwartz): Let $(\cdot,\cdot) : W \times W \to \mathbb{R}$ be an inner product on the real vector space W. Suppose that $\|\cdot\| : W \to \mathbb{R}$ is the induced norm on W, ie,

$$\|v\| = \sqrt{(v,v)} \quad \forall v \in W.$$

Then

(1.11) $$|(u,v)| \leq \|u\| \cdot \|v\|$$

for all $u, v \in W$, with equality iff u and v are linearly dependent.

Proof: If either $u \equiv 0$ or $v \equiv 0$, (1.11) is trivial. Assume not and define, for all $t \in \mathbb{R}$,

$$p(t) = (tu + v, tu + v).$$

Then

$$p(t) = t^2 \|u\|^2 + 2t(u,v) + \|v\|^2.$$

p is a quadratic, unless $u \equiv 0$, and its is convex.

Now

$$p(t) > 0 \qquad \forall\, t \in \mathbb{R}$$

unless

$$tu + v = 0 \qquad \exists\, t \in \mathbb{R}$$

Let us assume that this is not the case. Then, for all $t \in \mathbb{R}$,

$$p(t) = t^2 \|u\|^2 + 2t\,(u,v) + \|v\|^2 > 0.$$

Therefore,

$$\Delta = 4\left((u,v) - \|u\|^2 \|v\|^2\right) < 0,$$

as the polynomial has only complex conjugate roots.

Now if $t_* u + v = 0$, for some $t_* \in \mathbb{R}$, then

$$p(t_*) = 0$$

and

$$\Delta = 0,$$

that is, p has a double roots at $t = t_*$. ///