Chen - Shen FCH Scheme

$$\dot{\phi} = M\Delta\mu$$

$$\mu = \varepsilon^2 \Delta w - w F''(\phi) + \eta_1 w$$
$$+ (\eta_1 - \eta_2) F'(\phi)$$

$$\omega = \varepsilon^2 \Delta\phi - F'(\phi)$$

The first-order scheme is

$$\phi^{k+1} - SM\Delta\mu^{k+1} = \phi^k$$

$$\mu^{k+1} = \varepsilon^2 \Delta w^{k+1} - (\sigma_1 - \eta_1)\omega^{k+1}$$
$$+ (\sigma_1 - F''(\phi^k))\,\tilde{\omega}^k$$
$$+ (\eta_1 - \eta_2) F'(\phi^k)$$

$$\omega^{k+1} = \varepsilon^2 \Delta\phi^{k+1} - \sigma_2\phi^{k+1} - F'(\phi^k) + \sigma_2\phi^k$$

$$\tilde{\omega}^k := \varepsilon^2 \Delta\phi^k - F'(\phi^k)$$

where

$$\sigma_1, \sigma_2 \geq 0$$

Define

$$\boxed{q_1 := (\sigma_1 - F''(\phi^k))\,\tilde{\omega}^k + (\eta_1 - \eta_2)\,F'(\phi^k)}$$

$$\boxed{q_2 := \sigma_2 \phi^k - F'(\phi^k)}$$

Then

$$\phi^{k+1} - SM\Delta\mu^{k+1} = \phi^k$$

$$\mu^{k+1} = \varepsilon^2 \Delta \omega^{k+1} - (\sigma_1 - \eta_1)\omega^{k+1} + q_1$$

$$\omega^{k+1} = \varepsilon^2 \Delta \phi^{k+1} - \sigma_2 \phi^{k+1} + q_2$$

Eliminate ω^{k+1}:

$$\begin{aligned}
\mu^{k+1} &= \varepsilon^2 \Delta \left(\varepsilon^2 \Delta \phi^{k+1} - \sigma_2 \phi^{k+1} + q_2 \right) \\
&\quad - (\sigma_1 - \eta_1)\left(\varepsilon^2 \Delta \phi^{k+1} - \sigma_2 \phi^{k+1} + q_2 \right) + q_1 \\
&= \varepsilon^4 \Delta^2 \phi^{k+1} - \varepsilon^2 \sigma_2 \Delta \phi^{k+1} + \varepsilon^2 \Delta q_2 \\
&\quad - (\sigma_1 - \eta_1)\varepsilon^2 \Delta \phi^{k+1} + \sigma_2 (\sigma_1 - \eta_1) \phi^{k+1} \\
&\quad - (\sigma_1 - \eta_1) q_2 + q_1
\end{aligned}$$

$$\boxed{\mu^{k+1} = \varepsilon^4 \Delta^2 \phi^{k+1} - \left[\varepsilon^2 \sigma_2 + (\sigma_1 - \eta_1)\varepsilon^2\right] \Delta \phi^{k+1} + \sigma_2(\sigma_1 - \eta_1)\phi^{k+1} + q_3}$$

where

$$\boxed{q_3 := \varepsilon^2 \Delta q_2 - (\sigma_1 - \eta_1) q_2 + q_1}$$

Eliminate μ^{k+1}:

$$\phi^{k+1} - sM\Delta \left\{ \varepsilon^4 \Delta^2 \phi^{k+1} - \left[\sigma_2 + (\sigma_1 - \eta_1)\right]\varepsilon^2 \Delta \phi^{k+1} + \sigma_2(\sigma_1 - \eta_1)\phi^{k+1} + q_3 \right\} = \phi^k$$

$$\phi^{k+1} - sM\varepsilon^4 \Delta^3 \phi^{k+1} + sM\varepsilon^2\left(\sigma_2 + (\sigma_1 - \eta_1)\right)\Delta^2 \phi^{k+1} - sM\sigma_2(\sigma_1 - \eta_1)\Delta\phi^{k+1} = \phi^k + sM\Delta q_3$$

$$\boxed{\phi^{k+1} - sM\sigma_2(\sigma_1 - \eta_1)\Delta\phi^{k+1} + sM\varepsilon^2(\sigma_2 + \sigma_1 - \eta_1)\Delta^2\phi^{k+1} - sM\varepsilon^4\Delta^3\phi^{k+1} = q_4}$$

$$\boxed{q_4 := \phi^k + sM\Delta q_3}$$