Math 673

Multigrid Methods: A Mostly Matrix-Based Approach

Chapter 05: Multigrid

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Chapter 05, Part 1 of 2
Multigrid

Introduction

The idea behind multigrid is to replace the exact solution in the coarse-grid correction by a recursive application of a "two-grid" method. In this chapter, we will learn how to extend the two-grid method to obtain various general multigrid algorithms.

This chapter borrows heavily from the elegant presentations in the books by Brenner and Scott (2008), and Braess (2007). The books by Hackbusch (1985) and Wesseling (2004) are also excellent resources and go into significantly greater depth.

Basic Building Blocks of Multigrid

Canonical Level Inner Products

Suppose that

$$1 \leq n_0 < n_1 < \cdots < n_\ell < \cdots < n_L \in \mathbb{N}.$$

We refer to $\mathbb{R}^{n_\ell}$ as the level ℓ space, or, sometimes more informally, as level ℓ.

Symmetry and transposition are understood with respect to the canonical product on level ℓ, $(\,\cdot\,,\,\cdot\,)_\ell : \mathbb{R}^{n_\ell} \times \mathbb{R}^{n_\ell} \to \mathbb{R}$, which is defined as

$$(\boldsymbol{u}_\ell, \boldsymbol{v}_\ell)_\ell := \boldsymbol{v}_\ell^\top \boldsymbol{u}_\ell = \sum_{j=1}^{n_\ell} u_{\ell,j} v_{\ell,j},$$

for all $\boldsymbol{u}_\ell, \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$.

We will need the following simple property in this and following chapters.

Proposition (Coarse-Fine Transposition)

Suppose that $1 \le n_{\ell-1} < n_\ell$ *and* $\mathrm{B} \in \mathbb{R}^{n_{\ell-1} \times n_\ell}$ *is arbitrary. Then, for any* $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$ *and any* $\boldsymbol{v}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}$*,*

$$(\mathrm{B}\boldsymbol{u}_\ell, \boldsymbol{v}_{\ell-1})_{\ell-1} = \left(\boldsymbol{u}_\ell, \mathrm{B}^\top \boldsymbol{v}_{\ell-1}\right)_\ell,$$

where $\mathrm{B}^\top \in \mathbb{R}^{n_\ell \times n_{\ell-1}}$ *is the usual matrix transpose of* B*.*

Proof.

Exercise. □

Prolongation and Restriction

We will assume in this chapter, and most everywhere else, unless otherwise noted, that the restriction matrices,

$$\mathsf{R}_{\ell-1} \in \mathbb{R}^{n_{\ell-1} \times n_\ell}, \quad 1 \le \ell \le L,$$

are full rank, i.e.,

$$\operatorname{rank}(\mathsf{R}_{\ell-1}) = n_{\ell-1}.$$

We assume that the prolongation matrices satisfy

$$\mathsf{P}_{\ell-1} = \mathsf{R}_{\ell-1}^\top \in \mathbb{R}^{n_\ell \times n_{\ell-1}}, \quad 1 \le \ell \le L.$$

Thus, the prolongation matrices are also of full rank and satisfy

$$(\mathsf{R}_{\ell-1}\boldsymbol{u}_\ell, \boldsymbol{v}_{\ell-1})_{\ell-1} = (\boldsymbol{u}_\ell, \mathsf{P}_{\ell-1}\boldsymbol{v}_{\ell-1})_\ell\,,$$

for any $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$ and any $\boldsymbol{v}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}$,

A Family of SPD Stiffness Matrices

We assume that we have in our possession a family of (square) SPD stiffness matrices

$$\mathsf{A}_\ell \in \mathbb{R}^{n_\ell \times n_\ell}, \quad 0 \leq \ell \leq L.$$

We define the energy inner product on level ℓ, $(\cdot,\cdot)_{\mathsf{A}_\ell}$, via

$$(\boldsymbol{u}_\ell, \boldsymbol{v}_\ell)_{\mathsf{A}_\ell} := (\mathsf{A}_\ell \boldsymbol{u}_\ell, \boldsymbol{v}_\ell)_\ell\,,$$

for all $\boldsymbol{u}_\ell, \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$.

Basic Building Blocks of Multigrid

Our goal is to build an efficient iterative solver for the equation

$$\mathsf{A}_L \boldsymbol{u}_L^{\mathrm{E}} = \boldsymbol{f}_L,$$

given $\boldsymbol{f}_L \in \mathbb{R}^{n_L}$. As usual, we define

$$\boldsymbol{e}_L^{\square} := \boldsymbol{u}_L^{\mathrm{E}} - \boldsymbol{u}_L^{\square},$$

where $\boldsymbol{u}_L^{\square} \in \mathbb{R}^{n_L}$ is an approximate solution. Similarly,

$$\boldsymbol{r}_L^{\square} := \boldsymbol{f}_L - \mathsf{A}_L \boldsymbol{u}_L^{\square} = \mathsf{A}_L \boldsymbol{e}_L^{\square}.$$

Definition (Multigrid Operator)

Suppose $\ell \in \{0, 1, \ldots, L\}$, $\boldsymbol{g}_\ell \in \mathbb{R}^{n_\ell}$, and $\boldsymbol{u}_\ell^{(0)} \in \mathbb{R}^{n_\ell}$ are given. The vector $\boldsymbol{u}_\ell^{(3)} \in \mathbb{R}^{n_\ell}$ is computed via the recursive **multigrid operator**,

$$\boldsymbol{u}_\ell^{(3)} := \mathrm{MG}\left(\boldsymbol{g}_\ell, \ell, \boldsymbol{u}_\ell^{(0)}\right), \tag{1}$$

as follows:

- If $\ell = 0$, then
$$\boldsymbol{u}_0^{(3)} := \boldsymbol{u}_0^{(1,\mathrm{E})} := \mathsf{A}_0^{-1}\boldsymbol{g}_0. \tag{2}$$

Definition (Multigrid Operator Cont.)

- Otherwise, if $1 \leq \ell \leq L$, then
  - pre-smoothing:
    - $\boldsymbol{u}_\ell^{(1,0)} := \boldsymbol{u}_\ell^{(0)}$;
    - $\boldsymbol{u}_\ell^{(1,\sigma+1)} := \boldsymbol{u}_\ell^{(1,\sigma)} + \mathsf{S}_\ell \left(\boldsymbol{g}_\ell - \mathsf{A}_\ell \boldsymbol{u}_\ell^{(1,\sigma)} \right), \quad 0 \leq \sigma \leq m_1 - 1$;
    - $\boldsymbol{u}_\ell^{(1)} := \boldsymbol{u}_\ell^{(1,m_1)}$;
  - coarse grid correction:
    - $\boldsymbol{r}_\ell^{(1)} := \boldsymbol{g}_\ell - \mathsf{A}_\ell \boldsymbol{u}_\ell^{(1)}$;
    - $\boldsymbol{r}_{\ell-1}^{(1)} := \mathsf{R}_{\ell-1} \boldsymbol{r}_\ell^{(1)}$;
    - $\boldsymbol{q}_{\ell-1}^{(1,0)} := \boldsymbol{0}$;
    - $\boldsymbol{q}_{\ell-1}^{(1,\sigma+1)} := \mathrm{MG}\left(\boldsymbol{r}_{\ell-1}^{(1)}, \ell - 1, \boldsymbol{q}_{\ell-1}^{(1,\sigma)} \right), \quad 0 \leq \sigma \leq p - 1$;
    - $\boldsymbol{q}_{\ell-1}^{(1)} := \boldsymbol{q}_{\ell-1}^{(1,p)}$;
    - $\boldsymbol{q}_\ell^{(1)} := \mathsf{P}_{\ell-1} \boldsymbol{q}_{\ell-1}^{(1)}$
    - $\boldsymbol{u}_\ell^{(2)} := \boldsymbol{u}_\ell^{(1)} + \boldsymbol{q}_\ell^{(1)}$;
  - post-smoothing:
    - $\boldsymbol{u}_\ell^{(3,0)} := \boldsymbol{u}_\ell^{(2)}$;
    - $\boldsymbol{u}_\ell^{(3,\sigma+1)} := \boldsymbol{u}_\ell^{(3,\sigma)} + \mathsf{S}_\ell^\top \left(\boldsymbol{g}_\ell - \mathsf{A}_\ell \boldsymbol{u}_\ell^{(3,\sigma)} \right), \quad 0 \leq \sigma \leq m_2 - 1$;
    - and, finally,

$$\boldsymbol{u}_\ell^{(3)} := \boldsymbol{u}_\ell^{(3,m_2)}. \tag{3}$$

Definition (Multigrid Operator Cont.)

The vector $\boldsymbol{q}_{\ell-1}^{(1)} \in \mathbb{R}^{n_{\ell-1}}$ is called the **multigrid coarse grid correction**. The vector

$$\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} := \mathsf{A}_{\ell-1}^{-1}\boldsymbol{r}_{\ell-1}^{(1)},$$

the **exact coarse grid correction**.

Remark

In the two-grid method, we found that $\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} = \boldsymbol{q}_{\ell-1}^{(1)}$. *In the multigrid method, this is no longer the case, and we can only expect that* $\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} \approx \boldsymbol{q}_{\ell-1}^{(1)}$. *Much of multigrid analysis revolves around estimating the difference*

$$\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \boldsymbol{q}_{\ell-1}^{(1)}.$$

Definition

Let m_1 and m_2 be nonnegative integers and p be a positive integer. Suppose that $\boldsymbol{u}_L^k \in \mathbb{R}^{n_L}$ is given. Then

$$\boldsymbol{u}_L^{k+1} = \mathrm{MG}\left(\boldsymbol{f}_L, L, \boldsymbol{u}_L^k\right)$$

defines the **generic multigrid algorithm** for solving

$$\mathrm{A}_L \boldsymbol{u}_L^{\mathrm{E}} = \boldsymbol{f}_L.$$

Definition (Variants of the Multigrid Method)

A multigrid algorithm is called **one-sided** iff $m_2 = 0$ and $m_1 \geq 1$.
The algorithm is called a **W-cycle** iff $p \geq 2$ and is called a **V-cycle** iff $p = 1$.
The algorithm is called **symmetrized** iff $m_1 = m_2 = m$. We say that the algorithm is a **simple symmetric V-cycle** iff $p = 1$ and $m_1 = m_2 = 1$.

See the figure on the next page to obtain a more geometric understand of the above definition.

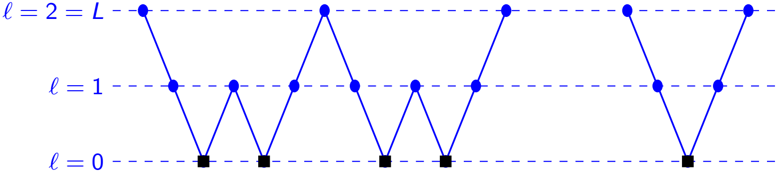

Figure: One full $p = 2$ W-cycle (left) and one full V-cycle (right) for three levels, $L = 2$.

Remark

In the case $L = 0$, we just do a direct solve. When $L = 1$, we recover the two-grid (two-level) method.

We now want to find the error transfer matrix for the general multigrid algorithm.

Definition

Let $\ell \geq 0$. Define, for $\ell = 0$,

$$\mathsf{E}_0 := \mathsf{O}_{n_0} \in \mathbb{R}^{n_0 \times n_0},$$

where $\mathsf{O}_{n_0} \in \mathbb{R}^{n_0 \times n_0}$ is the zero matrix, and, for $\ell \geq 1$,

$$\mathsf{E}_\ell := (\mathsf{K}_\ell^*)^{m_2} \left(\mathsf{I}_\ell - \mathsf{P}_{\ell-1} \left(\mathsf{I}_{\ell-1} - \mathsf{E}_{\ell-1}^p \right) \Pi_{\ell-1} \right) \mathsf{K}_\ell^{m_1} \in \mathbb{R}^{n_\ell \times n_\ell}, \tag{4}$$

where

$$\left.\begin{aligned} \mathsf{K}_\ell &:= \mathsf{I}_\ell - \mathsf{S}_\ell \mathsf{A}_\ell \\ \mathsf{K}_\ell^* &= \mathsf{I}_\ell - \mathsf{S}_\ell^\top \mathsf{A}_\ell \end{aligned}\right\} \in \mathbb{R}^{n_\ell \times n_\ell},$$

and

$$\Pi_{\ell-1} := \mathsf{A}_{\ell-1}^{-1} \mathsf{R}_{\ell-1} \mathsf{A}_\ell \in \mathbb{R}^{n_{\ell-1} \times n_\ell}.$$

For further use, let us also define

$$\tilde{\Pi}_\ell := \mathsf{P}_{\ell-1} \Pi_{\ell-1} \in \mathbb{R}^{n_\ell \times n_\ell}.$$

Remark

Observe that, for any $\boldsymbol{u}_\ell, \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$,

$$\left(\mathsf{K}_\ell \boldsymbol{u}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} = \left(\boldsymbol{u}_\ell, \mathsf{K}_\ell^* \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell}.$$

More generally, we use B^* *to denote the adjoint of* $\mathsf{B} \in \mathbb{R}^{n_\ell \times n_\ell}$ *with respect to the inner product* $\left(\,\cdot\,,\,\cdot\,\right)_{\mathsf{A}_\ell}$.

Theorem (Multigrid Error Relation)

Suppose that $\boldsymbol{u}_\ell^{\mathrm{E}}, \boldsymbol{g}_\ell \in \mathbb{R}^{n_\ell}$ *satisfy*

$$\mathsf{A}_\ell \boldsymbol{u}_\ell^{\mathrm{E}} = \boldsymbol{g}_\ell .$$

Then, given any $\boldsymbol{u}_\ell^{(0)} \in \mathbb{R}^{n_\ell}$,

$$\boldsymbol{u}_\ell^{\mathrm{E}} - \mathrm{MG}\left(\boldsymbol{g}_\ell, \ell, \boldsymbol{u}_\ell^{(0)}\right) = \mathsf{E}_\ell \left(\boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)}\right),$$

where E_ℓ *is the recursively-defined matrix in equation* (4). *In particular*

$$\boldsymbol{e}_L^{k+1} = \mathsf{E}_L \boldsymbol{e}_L^k .$$

Proof.

The proof is by induction. We will use the notation from the definition of the multigrid operator throughout.

(Base cases): Cases $\ell = 0$ and $\ell = 1$ (two-grid method) are clear.

(Induction hypothesis): Assume that the result is true for level $\ell - 1$. In other words, suppose that $\boldsymbol{u}_{\ell-1}^{\mathrm{E}}, \boldsymbol{g}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}$ satisfy

$$\mathsf{A}_{\ell-1}\boldsymbol{u}_{\ell-1}^{\mathrm{E}} = \boldsymbol{g}_{\ell-1}.$$

Then, given any initial vector $\boldsymbol{u}_{\ell-1}^{(0)} \in \mathbb{R}^{n_{\ell-1}}$,

$$\boldsymbol{u}_{\ell-1}^{\mathrm{E}} - \mathrm{MG}\left(\boldsymbol{g}_{\ell-1}, \ell - 1, \boldsymbol{u}_{\ell-1}^{(0)}\right) = \mathsf{E}_{\ell-1}\left(\boldsymbol{u}_{\ell-1}^{\mathrm{E}} - \boldsymbol{u}_{\ell-1}^{(0)}\right).$$

Proof (Cont.)

(Generic case): Suppose that $\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})}, \boldsymbol{r}_{\ell-1}^{(1)} \in \mathbb{R}^{n_{\ell-1}}$ satisfy

$$\mathsf{A}_{\ell-1}\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} = \boldsymbol{r}_{\ell-1}^{(1)}.$$

Then, using the induction hypothesis,

$$\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \mathrm{MG}\left(\boldsymbol{r}_{\ell-1}^{(1)}, \ell-1, \boldsymbol{0}\right) = \mathsf{E}_{\ell-1}\left(\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \boldsymbol{0}\right) = \mathsf{E}_{\ell-1}\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})}.$$

Written in the notation defining the multigrid operator,

$$\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \boldsymbol{q}_{\ell-1}^{(1,1)} = \mathsf{E}_{\ell-1}\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})}.$$

Applying the induction hypothesis again,

$$\begin{aligned} \boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \boldsymbol{q}_{\ell-1}^{(1,2)} &= \boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \mathrm{MG}\left(\boldsymbol{r}_{\ell-1}^{(1)}, \ell-1, \boldsymbol{q}_{\ell-1}^{(1,1)}\right) \\ &= \mathsf{E}_{\ell-1}\left(\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \boldsymbol{q}_{\ell-1}^{(1,1)}\right) \\ &= \mathsf{E}_{\ell-1}^2\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})}. \end{aligned}$$

Proof (Cont.)

Continuing in this fashion, for any $p \in \mathbb{N}$,

$$\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \boldsymbol{q}_{\ell-1}^{(1,p)} = \mathsf{E}_{\ell-1}^{p} \boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})}.$$

Consequently,

$$\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \boldsymbol{q}_{\ell-1}^{(1)} = \mathsf{E}_{\ell-1}^{p} \boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})}.$$

Solving for $\boldsymbol{q}_{\ell-1}^{(1)}$, we have

$$\boldsymbol{q}_{\ell-1}^{(1)} = \left(\mathsf{I}_{\ell-1} - \mathsf{E}_{\ell-1}^{p}\right) \boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})}.$$

Now,

$$\begin{aligned} \boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} &:= \mathsf{A}_{\ell-1}^{-1} \boldsymbol{r}_{\ell-1}^{(1)} \\ &= \mathsf{A}_{\ell-1}^{-1} \mathsf{R}_{\ell-1} \left(\boldsymbol{g}_{\ell} - \mathsf{A}_{\ell} \boldsymbol{u}_{\ell}^{(1)}\right) \\ &= \mathsf{A}_{\ell-1}^{-1} \mathsf{R}_{\ell-1} \mathsf{A}_{\ell} \left(\boldsymbol{u}_{\ell}^{\mathrm{E}} - \boldsymbol{u}_{\ell}^{(1)}\right) \\ &= \Pi_{\ell-1} \left(\boldsymbol{u}_{\ell}^{\mathrm{E}} - \boldsymbol{u}_{\ell}^{(1)}\right). \end{aligned} \tag{5}$$

Proof (Cont.)

Hence

$$\boldsymbol{q}_{\ell-1}^{(1)} = \left(\mathsf{I}_{\ell-1} - \mathsf{E}_{\ell-1}^{p}\right) \Pi_{\ell-1} \left(\boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(1)}\right).$$

For the error after post-smoothing,

$$\boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(3)} = (\mathsf{K}_\ell^*)^{m_2} \left(\boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(2)}\right);$$

for the error after coarse grid correction,

$$\begin{aligned} \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(2)} &= \boldsymbol{u}_\ell^{\mathrm{E}} - \left(\boldsymbol{u}_\ell^{(1)} + \mathsf{P}_{\ell-1}\boldsymbol{q}_{\ell-1}^{(1)}\right) \\ &= \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(1)} - \mathsf{P}_{\ell-1}\left(\mathsf{I}_{\ell-1} - \mathsf{E}_{\ell-1}^{p}\right) \Pi_{\ell-1} \left(\boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(1)}\right) \\ &= \left(\mathsf{I}_\ell - \mathsf{P}_{\ell-1}\left(\mathsf{I}_{\ell-1} - \mathsf{E}_{\ell-1}^{p}\right) \Pi_{\ell-1}\right) \left(\boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(1)}\right); \end{aligned}$$

and, for the error after pre-smoothing,

$$\boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(1)} = \mathsf{K}_\ell^{m_1} \left(\boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)}\right).$$

Proof (Cont.)

Putting it all together, we have

$$\boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(3)} \overset{(4)}{=} \mathsf{E}_\ell \left(\boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right).$$

□

Galerkin Conditions

Definition (Assumptions (A0) and (A1))

We say that the stiffness matrices satisfy the **strong Galerkin condition**, equivalently, **Assumption (A0)** holds, iff

$$\mathsf{A}_{\ell-1} = \mathsf{R}_{\ell-1}\mathsf{A}_{\ell}\mathsf{P}_{\ell-1}, \quad 1 \leq \ell \leq L. \tag{6}$$

We say that the **weak Galerkin condition** holds, equivalently, **Assumption (A1)** holds, iff

$$(\boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_{\mathsf{A}_\ell} \geq (\Pi_{\ell-1}\boldsymbol{v}_\ell, \Pi_{\ell-1}\boldsymbol{v}_\ell)_{\mathsf{A}_{\ell-1}}, \tag{7}$$

for all $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$ and all $1 \leq \ell \leq L$.

We will show that (A0) $\Longrightarrow$ (A1). To do this, we first need the following result.

Lemma

If Assumption (A0, strong Galerkin condition) holds then

$$\tilde{\Pi}_\ell = \tilde{\Pi}_\ell^2 \tag{8}$$

But $\tilde{\Pi}_\ell^* = \tilde{\Pi}_\ell$ holds even without Assumption (A0).

Proof.

If the Galerkin Condition holds then

$$\mathsf{A}_{\ell-1} = \mathsf{R}_{\ell-1}\mathsf{A}_\ell\mathsf{P}_{\ell-1}, \quad 1 \leq \ell \leq L.$$

Recall that

$$\tilde{\Pi}_\ell = \mathsf{P}_{\ell-1}\mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell, \quad 1 \leq \ell \leq L.$$

Consequently,

$$\begin{aligned}
\tilde{\Pi}_\ell^2 &= \mathsf{P}_{\ell-1}\mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell\mathsf{P}_{\ell-1}\mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell \\
&= \mathsf{P}_{\ell-1}\mathsf{A}_{\ell-1}^{-1}\left(\mathsf{R}_{\ell-1}\mathsf{A}_\ell\mathsf{P}_{\ell-1}\right)\mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell \\
&= \mathsf{P}_{\ell-1}\mathsf{A}_{\ell-1}^{-1}\mathsf{A}_{\ell-1}\mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell \\
&= \mathsf{P}_{\ell-1}\mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell \\
&= \tilde{\Pi}_\ell.
\end{aligned}$$

Proof (Cont.)

Next, let $\boldsymbol{u}_\ell, \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$ be arbitrary. Then using the definitions of $\Pi_{\ell-1}$ and $\tilde{\Pi}_\ell$,

$$\begin{aligned}
\left(\tilde{\Pi}_\ell \boldsymbol{u}_\ell, \boldsymbol{v}_\ell\right)_{A_\ell} &= \left(A_\ell \tilde{\Pi}_\ell \boldsymbol{u}_\ell, \boldsymbol{v}_\ell\right)_\ell \\
&= \left(A_\ell P_{\ell-1} A_{\ell-1}^{-1} R_{\ell-1} A_\ell \boldsymbol{u}_\ell, \boldsymbol{v}_\ell\right)_\ell \\
&= \left(\left(A_\ell P_{\ell-1} A_{\ell-1}^{-1} R_{\ell-1}\right) A_\ell \boldsymbol{u}_\ell, \boldsymbol{v}_\ell\right)_\ell \\
&= \left(A_\ell \boldsymbol{u}_\ell, \left(A_\ell P_{\ell-1} A_{\ell-1}^{-1} R_{\ell-1}\right)^\top \boldsymbol{v}_\ell\right)_\ell \\
&= \left(A_\ell \boldsymbol{u}_\ell, \tilde{\Pi}_\ell \boldsymbol{v}_\ell\right)_\ell \\
&= \left(\boldsymbol{u}_\ell, \tilde{\Pi}_\ell \boldsymbol{v}_\ell\right)_{A_\ell}.
\end{aligned}$$

So,

$$\tilde{\Pi}_\ell^* = \tilde{\Pi}_\ell.$$

□

Corollary

It always holds that

$$\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)^* = \mathsf{I}_\ell - \tilde{\Pi}_\ell. \tag{9}$$

If Assumption (A0) holds, then

$$\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)^2 = \mathsf{I}_\ell - \tilde{\Pi}_\ell. \tag{10}$$

Lemma ((A0) $\implies$ (A1))

Assumption (A0) implies Assumption (A1).

Proof.

First, for any $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$, consider

$$\begin{aligned}
(\Pi_{\ell-1}\boldsymbol{v}_\ell, \Pi_{\ell-1}\boldsymbol{v}_\ell)_{\mathsf{A}_{\ell-1}} &= (\Pi_{\ell-1}\boldsymbol{v}_\ell, \mathsf{A}_{\ell-1}\Pi_{\ell-1}\boldsymbol{v}_\ell)_{\ell-1} \\
&= \left(\Pi_{\ell-1}\boldsymbol{v}_\ell, \mathsf{A}_{\ell-1}\mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell\boldsymbol{v}_\ell\right)_{\ell-1} \\
&= (\Pi_{\ell-1}\boldsymbol{v}_\ell, \mathsf{R}_{\ell-1}\mathsf{A}_\ell\boldsymbol{v}_\ell)_{\ell-1} \\
&= \left(\mathsf{R}_{\ell-1}^\top\Pi_{\ell-1}\boldsymbol{v}_\ell, \mathsf{A}_\ell\boldsymbol{v}_\ell\right)_\ell \\
&= (\mathsf{P}_{\ell-1}\Pi_{\ell-1}\boldsymbol{v}_\ell, \mathsf{A}_\ell\boldsymbol{v}_\ell)_\ell \\
&= (\mathsf{P}_{\ell-1}\Pi_{\ell-1}\boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_{\mathsf{A}_\ell} \\
&= \left(\tilde{\Pi}_\ell\boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell}.
\end{aligned}$$

Proof (Cont.)

Using the last calculation

$$
\begin{aligned}
(\boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_{\mathsf{A}_\ell} - (\Pi_{\ell-1}\boldsymbol{v}_\ell, \Pi_{\ell-1}\boldsymbol{v}_\ell)_{\mathsf{A}_{\ell-1}} &= (\boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_{\mathsf{A}_\ell} - \left(\tilde{\Pi}_\ell \boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} \\
&= \left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} \\
&\overset{(10)}{=} \left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)^2 \boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} \\
&\overset{(9)}{=} \left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\boldsymbol{v}_\ell, \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} \\
&= \left\|\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\boldsymbol{v}_\ell\right\|_{\mathsf{A}_\ell}^2 \\
&\geq 0.
\end{aligned}
$$

Thus (A0) implies (A1). □

Definition (Assumption (A2))

We say that **Assumption (A2)** holds iff, for all $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$ and all $1 \le \ell \le L$,

$$\left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell \right) \boldsymbol{u}_\ell, \boldsymbol{u}_\ell \right)_{\mathsf{A}_\ell} \ge 0. \tag{11}$$

Corollary

Assumption (A1) is equivalent to Assumption (A2).

Proof.

We showed in the proof of the last lemma, using only the definitions of $\Pi_{\ell-1}$ and $\tilde{\Pi}_\ell$, that

$$(\boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathsf{A}_\ell} - (\Pi_{\ell-1}\boldsymbol{u}_\ell, \Pi_{\ell-1}\boldsymbol{u}_\ell)_{\mathsf{A}_{\ell-1}} = \left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\boldsymbol{u}_\ell, \boldsymbol{u}_\ell\right)_{\mathsf{A}_\ell}.$$

So (A1) holds iff (A2) holds. $\square$

Theorem

If $m_1 = m_2 = m$*, then, for all* $0 \leq \ell \leq L$

$$\mathsf{E}_\ell = \mathsf{E}_\ell^*.$$

If, in addition, Assumption (A1), holds, or, equivalently, Assumption (A2) holds, then

$$(\mathsf{E}_\ell \boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathsf{A}_\ell} \geq 0, \quad \forall\, \boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}.$$

Proof.

The proof is by induction.

(Base cases): The case $\ell = 0$ is trivial. For $\ell = 1$ (the two-grid algorithm) the proof follows as in a previous chapter.

(Induction hypothesis): Assume that

$$\left(\mathsf{E}_{\ell-1}\boldsymbol{u}_{\ell-1}, \boldsymbol{v}_{\ell-1}\right)_{\mathsf{A}_{\ell-1}} = \left(\boldsymbol{u}_{\ell-1}, \mathsf{E}_{\ell-1}\boldsymbol{v}_{\ell-1}\right)_{\mathsf{A}_{\ell-1}},$$

and

$$\left(\mathsf{E}_{\ell-1}\boldsymbol{u}_{\ell-1}, \boldsymbol{u}_{\ell-1}\right)_{\mathsf{A}_{\ell-1}} \geq 0$$

for all $\boldsymbol{u}_{\ell-1}, \boldsymbol{v}_{\ell-1} \in \mathbb{R}^{n_\ell}$.

Proof (Cont.)

(General case): We will make use of the definitions of $\Pi_{\ell-1}$ and $\tilde{\Pi}_\ell$, a number of times:

$$\Pi_{\ell-1} = \mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell.$$

So

$$\mathsf{A}_{\ell-1}\Pi_{\ell-1} = \mathsf{R}_{\ell-1}\mathsf{A}_\ell.$$

And

$$\tilde{\Pi}_\ell = \mathsf{P}_{\ell-1}\Pi_{\ell-1} = \mathsf{P}_{\ell-1}\mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell.$$

Proof (Cont.)

Thus, we have

$$
\begin{aligned}
&(\mathsf{E}_\ell \boldsymbol{u}_\ell, \boldsymbol{v}_\ell)_{\mathsf{A}_\ell} \\
&\quad = \left((\mathsf{K}_\ell^*)^m \left(\mathsf{I}_\ell - \mathsf{P}_{\ell-1}\left(\mathsf{I}_{\ell-1} - \mathsf{E}_{\ell-1}^p\right)\Pi_{\ell-1}\right)\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} \\
&\quad = \left(\left(\mathsf{I}_\ell - \mathsf{P}_{\ell-1}\left(\mathsf{I}_{\ell-1} - \mathsf{E}_{\ell-1}^p\right)\Pi_{\ell-1}\right)\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} \\
&\quad = \left(\left(\mathsf{I}_\ell - \mathsf{P}_{\ell-1}\Pi_{\ell-1} + \mathsf{P}_{\ell-1}\mathsf{E}_{\ell-1}^p\Pi_{\ell-1}\right)\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} \\
&\quad = \left((\mathsf{I}_\ell - \mathsf{P}_{\ell-1}\Pi_{\ell-1})\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} + \left(\mathsf{P}_{\ell-1}\mathsf{E}_{\ell-1}^p\Pi_{\ell-1}\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} \\
&\quad = \left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} + \left(\mathsf{P}_{\ell-1}\mathsf{E}_{\ell-1}^p\Pi_{\ell-1}\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} \\
&\quad = \left(\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} + \left(\mathsf{P}_{\ell-1}\mathsf{E}_{\ell-1}^p\Pi_{\ell-1}\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \mathsf{A}_\ell\mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_\ell \\
&\quad = \left(\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} + \left(\mathsf{E}_{\ell-1}^p\Pi_{\ell-1}\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \mathsf{R}_{\ell-1}\mathsf{A}_\ell\mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\ell-1} \\
&\quad = \left(\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} + \left(\mathsf{E}_{\ell-1}^p\Pi_{\ell-1}\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \mathsf{A}_{\ell-1}\Pi_{\ell-1}\mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\ell-1}
\end{aligned}
$$

Proof (Cont.)

$$
\begin{aligned}
&= \left(\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right) \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} + \left(\mathsf{E}_{\ell-1}^p \Pi_{\ell-1} \mathsf{K}_\ell^m \boldsymbol{u}_\ell, \Pi_{\ell-1} \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_{\ell-1}} \\
&= \left(\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right) \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} + \left(\Pi_{\ell-1} \mathsf{K}_\ell^m \boldsymbol{u}_\ell, \mathsf{E}_{\ell-1}^p \Pi_{\ell-1} \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_{\ell-1}} \\
&= \left(\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right) \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} + \left(\mathsf{A}_{\ell-1} \Pi_{\ell-1} \mathsf{K}_\ell^m \boldsymbol{u}_\ell, \mathsf{E}_{\ell-1}^p \Pi_{\ell-1} \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\ell-1} \\
&= \left(\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right) \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} + \left(\mathsf{R}_{\ell-1} \mathsf{A}_\ell \mathsf{K}_\ell^m \boldsymbol{u}_\ell, \mathsf{E}_{\ell-1}^p \Pi_{\ell-1} \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\ell-1} \\
&= \left(\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right) \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} + \left(\mathsf{A}_\ell \mathsf{K}_\ell^m \boldsymbol{u}_\ell, \mathsf{P}_{\ell-1} \mathsf{E}_{\ell-1}^p \Pi_{\ell-1} \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\ell} \\
&= \left(\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right) \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} + \left(\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \mathsf{P}_{\ell-1} \mathsf{E}_{\ell-1}^p \Pi_{\ell-1} \mathsf{K}_\ell^m \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} \\
&= \left(\boldsymbol{u}_\ell, \mathsf{E}_\ell \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} .
\end{aligned}
$$

Hence, symmetry is proven.

Proof (Cont.)

Notice we have not yet used Assumption (A2), only the definitions of $\Pi_{\ell-1}$ and $\tilde{\Pi}_\ell$, which are assumed to always hold.

Now, we setting $\boldsymbol{v}_\ell = \boldsymbol{u}_\ell$ in the last calculation, we have

$$\begin{aligned}
\left(\mathsf{E}_\ell \boldsymbol{u}_\ell, \boldsymbol{u}_\ell\right)_{\mathsf{A}_\ell} &= \underbrace{\left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \mathsf{K}_\ell^m \boldsymbol{u}_\ell\right)_{\mathsf{A}_\ell}}_{\geq 0,\ \text{Assumption (A2)}} \\
&\quad + \underbrace{\left(\mathsf{E}_{\ell-1}^p \Pi_{\ell-1}\mathsf{K}_\ell^m \boldsymbol{u}_\ell, \Pi_{\ell-1}\mathsf{K}_\ell^m \boldsymbol{u}_\ell\right)_{\mathsf{A}_{\ell-1}}}_{\geq 0,\ \text{induction hypothesis}} \\
&\geq 0,
\end{aligned}$$

for any $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$. The result is proven. $\square$

Remark

In the proof above we used the fact that any positive integer power of a self-adjoint positive semi-definite matrix is also positive semi-definite. This is easy to prove and is left as an exercise.

The Strong and Weak Approximation Properties

Definition (Assumptions (A3) and (A4))

We say that the multigrid algorithm satisfies the **strong approximation property**, equivalently, **Assumption (A3)**, iff, for all $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$ and $1 \le \ell \le L$,

$$\left\| \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell \right) \boldsymbol{u}_\ell \right\|_\ell^2 \le C_3^2 \rho_\ell^{-1} \left\| \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell \right) \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell}^2 , \tag{12}$$

for some $C_3 > 0$ that is independent of ℓ, where $\rho_\ell = \rho(\mathsf{A}_\ell)$. The multigrid algorithm satisfies the **weak approximation property**, equivalently, **Assumption (A4)**, iff for all $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$ and $1 \le \ell \le L$

$$\left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell \right) \boldsymbol{u}_\ell, \boldsymbol{u}_\ell \right)_{\mathsf{A}_\ell} \le C_4^2 \rho_\ell^{-1} \left\| \mathsf{A}_\ell \boldsymbol{u}_\ell \right\|_\ell^2 , \tag{13}$$

for some $C_4 > 0$ that is independent of ℓ.

Theorem

If the Galerkin condition, Assumption (A0), holds, then (A3) implies (A4).

Proof.

Since Assumption (A0) holds

$$\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)^2 \overset{(10)}{=} \mathsf{I}_\ell - \tilde{\Pi}_\ell.$$

Therefore,

$$\begin{aligned}
\left\|\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\boldsymbol{u}_\ell\right\|_{\mathsf{A}_\ell}^2 &= \left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\boldsymbol{u}_\ell, \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\boldsymbol{u}_\ell\right)_{\mathsf{A}_\ell} \\
&= \left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)^2\boldsymbol{u}_\ell, \boldsymbol{u}_\ell\right)_{\mathsf{A}_\ell} \\
&= \left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\boldsymbol{u}_\ell, \boldsymbol{u}_\ell\right)_{\mathsf{A}_\ell} \\
&= \left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\boldsymbol{u}_\ell, \mathsf{A}_\ell\boldsymbol{u}_\ell\right)_\ell \\
&\overset{\text{C.S.}}{\leq} \left\|\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\boldsymbol{u}_\ell\right\|_\ell \left\|\mathsf{A}_\ell\boldsymbol{u}_\ell\right\|_\ell \\
&\overset{(A3)}{\leq} C_3\rho_\ell^{-1/2}\left\|\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\boldsymbol{u}_\ell\right\|_{\mathsf{A}_\ell} \left\|\mathsf{A}_\ell\boldsymbol{u}_\ell\right\|_\ell .
\end{aligned}$$

Proof (Cont.)

Thus

$$\left\| \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell \right) \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell} \leq C_3 \rho_\ell^{-1/2} \left\| \mathsf{A}_\ell \boldsymbol{u}_\ell \right\|_\ell . \tag{14}$$

Squaring, we have

$$\left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell \right) \boldsymbol{u}_\ell, \boldsymbol{u}_\ell \right)_{\mathsf{A}_\ell} \leq C_3^2 \rho_\ell^{-1} \left\| \mathsf{A}_\ell \boldsymbol{u}_\ell \right\|_\ell^2 ,$$

which is the desired result with $C_4 = C_3$. □

The First Smoothing Property

Definition (Richardson's Smoother)

Suppose that $\boldsymbol{u}_\ell^{\mathrm{E}}, \boldsymbol{g}_\ell \in \mathbb{R}^{n_\ell}$ satisfy

$$\mathsf{A}_\ell \boldsymbol{u}_\ell^{\mathrm{E}} = \boldsymbol{g}_\ell.$$

Richardson's method for approximating $\boldsymbol{u}_\ell^{\mathrm{E}}$ is the GLIS defined via

$$\boldsymbol{u}_\ell^{(\sigma+1)} = \boldsymbol{u}_\ell^{(\sigma)} + \omega \left(\boldsymbol{g}_\ell - \mathsf{A}_\ell \boldsymbol{u}_\ell^{(\sigma)} \right),$$

where $\omega > 0$ is a parameter. Its error transfer matrix is

$$\mathsf{K}_\ell = \mathsf{I}_\ell - \omega \mathsf{A}_\ell = \mathsf{K}_\ell^*.$$

Definition (Richardson's Smoother Cont.)

Suppose that $\Lambda_\ell > 0$ is a number satisfying

$$C_s \rho_\ell \geq \Lambda_\ell \geq \rho_\ell = \rho(\mathsf{A}_\ell), \quad 1 \leq \ell \leq L, \tag{15}$$

for some $C_s > 1$ that is independent of ℓ, where $\rho(\mathsf{A}_\ell)$ is the spectral radius of A_ℓ. Choosing

$$\omega := \frac{1}{\Lambda_\ell},$$

we obtain **Richardson' s smoother**, and in this case the error transfer matrix is

$$\mathsf{K}_\ell = \mathsf{I}_\ell - \Lambda_\ell^{-1} \mathsf{A}_\ell = \mathsf{K}_\ell^*.$$

Definition (Assumption (A5))

We say that the multigrid algorithm satisfies the **first smoothing property**, equivalently, **Assumption (A5)**, iff

$$\left\| \mathsf{K}_\ell^m \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell^2} \leq C_5 \rho_\ell^{1/2} m^{-1/2} \left\| \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell}, \tag{16}$$

for all $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$ and $1 \leq \ell \leq L$, for some $C_5 > 0$ that is independent of ℓ.

Theorem

Richardson's smoother satisfies the first smoothing property, Assumption (A5).

Proof.

The proof is similar to the proof of the two-grid version. □