Math 673

Multigrid Methods: A Mostly Matrix-Based Approach

Chapter 06: Multigrid and the Conforming Finite Element Method

Abner J. Salgado and Steven M. Wise

asalgad1@utk.edu swise1@.utk.edu
University of Tennessee

Fall 2024

Chapter 06, Part 2 of 2

Multigrid and the Conforming Finite Element Method

Strong Approximation Property

Strong Approximation Property

Now, we want to show that the strong approximation property, Assumption (A3), holds for the present framework. In particular, we will prove that there is some constant $C_{\mathrm{A3}} > 0$, independent of ℓ, such that

$$\left\| \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell \right) \boldsymbol{u}_\ell \right\|_\ell^2 \leq C_{\mathrm{A3}}^2 \rho_\ell^{-1} \left\| \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell \right) \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell}^2 , \tag{1}$$

for all $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$. We need a bit more PDE and FE theory first.

Let $f \in H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)'$ be given. The weak form of the model problem can be stated as follows: find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = \langle f, v \rangle , \quad \forall\, v \in H_0^1(\Omega).$$

It is well-known that a unique solution $u \in H_0^1(\Omega)$ exists. In fact, this can be proven using either the Lax-Milgram theorem or Riesz Representation theorem.

Strong Approximation Property

A conforming FE approximation of the problem may be written as follows: find $u_\ell \in V_\ell$ such that

$$a(u_\ell, v_\ell) = \langle f, v \rangle\,, \quad \forall\, v_\ell \in V_\ell, \tag{2}$$

where V_ℓ is the family of nested, conforming finite element subspaces of $H_0^1(\Omega)$ that we constructed earlier. It is easy to show that, also, that a unique finite element approximation $u_\ell \in V_\ell$ always exists.

Observe that every $f \in L^2(\Omega)$ gives rise to an $L_f \in H^{-1}$ in a natural way:

$$\langle L_f, v \rangle := L_f(v) = (f, v)_{L^2(\Omega)}\,, \quad \forall\, v \in H_0^1(\Omega).$$

Definition

We say that the model problem satisfies the **standard regularity condition** iff when $f \in L^2(\Omega) \cap H^{-1}(\Omega)$, then $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and

$$|u|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}, \tag{3}$$

for some universal (regularity) constant $C > 0$, which only depends upon the domain Ω.

Theorem (Convexity implies Standard Regularity)

If Ω is convex and polyhedral, then the standard regularity condition holds.

Theorem (Galerkin Orthogonality and Cea's Lemma)

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, or 3, be an open polyhedral domain and suppose $\mathcal{T}_h$ is a family of triangulations of Ω parameterized by

$$h := \max_{K \in \mathcal{T}_h} \operatorname{diam}(K),$$

and

$$V_h := \left\{ v \in C^0(\overline{\Omega}) \;\middle|\; v|_K \in \mathbb{P}_1(K),\ \forall\, K \in \mathcal{T}_h, v|_{\partial\Omega} \equiv 0 \right\}.$$

Suppose that $f \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$ is the unique solution to

$$a(u, v) = (f, v), \quad \forall\, v \in H_0^1(\Omega). \tag{4}$$

Assume that $u_h \in V_h$ is the unique solution to

$$a(u_h, v) = (f, v), \quad \forall\, v \in V_h. \tag{5}$$

Then,

$$a(u - u_h, v) = 0, \quad \forall\, v \in V_h. \tag{6}$$

Theorem (Galerkin Orthogonality and Cea's Lemma (Cont.))

Furthermore,

$$\|u - u_h\|_{H_0^1(\Omega)} = \min_{w \in V_h} \|u - w\|_{H_0^1(\Omega)}, \tag{7}$$

where

$$\|w\|_{H_0^1(\Omega)} := \sqrt{a(w, w)}, \quad \forall\, w \in H_0^1(\Omega).$$

Proof.

Since (4) holds for all $v \in H_0^1(\Omega)$ and $V_h \subset H_0^1(\Omega)$,

$$a(u, v) = (f, v), \quad \forall\, v \in V_h. \tag{8}$$

Subtracting (5) from (8), we immediately get (6).
Next, for any $w \in V_h$,

$$\begin{aligned}
\|u - u_h\|^2_{H_0^1(\Omega)} &= a(u - u_h, u - u_h) \\
&= a(u - u_h, u - u_h) + a(u - u_h, u_h - w) \\
&= a(u - u_h, u - w) \\
&\overset{\text{C.S.}}{\leq} \|u - u_h\|_{H_0^1(\Omega)} \|u - w\|_{H_0^1(\Omega)} .
\end{aligned}$$

Thus,

$$\|u - u_h\|_{H_0^1(\Omega)} \leq \|u - w\|_{H_0^1(\Omega)} ,$$

and

$$\|u - u_h\|_{H_0^1(\Omega)} \leq \inf_{w \in V_h} \|u - w\|_{H_0^1(\Omega)} .$$

Consequently, (7) holds. □

Definition (Piecewise Linear Lagrange Nodal Interpolation Operator)

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ or 3, be an open polyhedral domain and suppose $\mathcal{T}_h$ and V_h are as above. Suppose that $\{\boldsymbol{N}_{h,j}\}_{i=1}^{n_h}$ is the set of interior vertices of V_h and

$$\mathcal{B}_h = \{\psi_{h,i}\}_{i=1}^{n_h}$$

is the Lagrange nodal basis for V_h, where the hat functions satisfy

$$\psi_{h,i}(\boldsymbol{N}_{h,j}) = \delta_{i,j}, \quad 1 \leq i, j \leq n.$$

The **piecewise linear Lagrange nodal interpolation operator**, denoted $\mathcal{I}_h : C(\overline{\Omega}) \cap H_0^1(\Omega) \to V_h$, is defined as follows: for any $u \in C(\overline{\Omega}) \cap H_0^1(\Omega)$,

$$\mathcal{I}_h u := \sum_{i=1}^{n_h} u(\boldsymbol{N}_{h,i}) \psi_{h,i}.$$

Remark

In the case that the spaces are nested and indexed by ℓ, we replace the subscripts h by ℓ.

Next, we need some approximation theory.

Theorem

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ or 3, be an open polyhedral domain. Suppose $\mathcal{T}_h$ and V_h are as defined above and $\mathcal{I}_h : C(\overline{\Omega}) \cap H_0^1(\Omega) \to V_h$ is the Lagrange nodal interpolation operator. Assume that $\mathcal{T}_h$ is a shape regular family of triangulations. Then, there exists a constant $C > 0$, independent of h, but, perhaps, dependent upon s, such that

$$\|u - \mathcal{I}_h u\|_{H^s(\Omega)} \leq C h^{2-s} |u|_{H^2(\Omega)}, \quad s = 0, 1,$$

for all $u \in C(\overline{\Omega}) \cap H_0^1(\Omega) \cap H^2(\Omega)$.

Combining Cea's lemma and the last result, we immediately obtain the following:

Theorem

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, or 3, be an open polyhedral domain and suppose $\mathcal{T}_h$ and V_h are as above. Assume that $\mathcal{T}_h$ is a shape regular family of triangulations. Suppose that $f \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega) \cap H^2(\Omega)$ is the unique solution to (4). Assume that $u_h \in V_h$ is the unique solution to (5). There exists a constant $C > 0$, independent of h, such that

$$\|u - u_h\|_{H_0^1(\Omega)} \leq Ch\,|u|_{H^2(\Omega)}\,.$$

To get an estimate of the error in the L^2 norm, we need a trick.

Theorem (Nitsche's Trick)

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2,$ or 3, be an open polyhedral domain and suppose $\mathcal{T}_h$ and V_h are as above. Assume that $\mathcal{T}_h$ is a shape regular family of triangulations. Suppose that $f \in H^{-1}(\Omega)$, $u \in H_0^1(\Omega)$ is the unique solution to (4), and $u_h \in V_h$ is the unique solution to (5). Then, if Ω is convex (so that the standard regularity condition holds) there is a constant $C > 0$, independent of h, such that

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch\,|u - u_h|_{H^1(\Omega)}\,, \tag{9}$$

If, in addition, it is known that $f \in L^2(\Omega)$, so that $u \in H_0^1(\Omega) \cap H^2(\Omega)$, then

$$|u - u_h|_{H^1(\Omega)} \leq Ch\,|u|_{H^2(\Omega)}\,, \tag{10}$$

for some $C > 0$. All together,

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^2\,|u|_{H^2(\Omega)}\,. \tag{11}$$

Proof.

Set $e = u - u_h \in H_0^1(\Omega)$. Let $z_e \in H_0^1(\Omega)$ be the unique solution of dual problem

$$a(v, z_e) = (e, v)_{L^2(\Omega)}, \quad \forall\, v \in H_0^1(\Omega).$$

Notice that, since $a(\cdot\,,\cdot)$ is symmetric, the dual problem is equivalent to the original problem. Since Ω is assumed to be convex polyhedral, by the elliptic regularity result of Theorem 2, we find that, $z_e \in H_0^1(\Omega) \cap H^2(\Omega)$ with

$$|z_e|_{H^2(\Omega)} \leq C \|e\|_{L^2(\Omega)}.$$

Now, suppose that $v_h \in V_h$ is arbitrary and set $v = e$ in the dual problem. Using Galerkin orthogonality and the Cauchy-Schwartz inequality, we have

$$\|e\|_{L^2(\Omega)}^2 = a(e, z_e) = a(e, z_e - v_h) \leq \|e\|_{H_0^1(\Omega)} \|z_e - v_h\|_{H_0^1(\Omega)}.$$

Proof (Cont.)

Let us choose $v_h = \mathcal{I}_h z_e$, where $\mathcal{I}_h : C(\overline{\Omega}) \cap H_0^1(\Omega) \to V_h$ is the piecewise linear Lagrange nodal interpolation operator. By Theorem 6,

$$
\begin{aligned}
\|e\|_{L^2(\Omega)}^2 &\leq \|e\|_{H_0^1(\Omega)} \|z_e - \mathcal{I}_h z_e\|_{H_0^1(\Omega)} \\
&\leq Ch^{2-1} \|e\|_{H_0^1(\Omega)} |z_e|_{H^2(\Omega)} \\
&\leq Ch \|e\|_{H_0^1(\Omega)} \|e\|_{L^2(\Omega)} .
\end{aligned}
$$

Therefore,

$$
\|e\|_{L^2(\Omega)} \leq Ch \|e\|_{H_0^1(\Omega)} ,
$$

and the result follows. □

Definition (Ritz Projection)

Let $\mathcal{T}_h$ and V_h be as in the last theorem. Let $u \in H_0^1(\Omega)$ be arbitrary. Define the **Ritz projection**, $\mathcal{R}_h : H_0^1(\Omega) \to V_h$, as follows: $\mathcal{R}_h u \in V_h$ is the unique solution to

$$a\left(\mathcal{R}_h u, v_h\right) = a\left(u, v_h\right), \quad \forall\, v_h \in V_h.$$

In the case that $V_h = V_\ell$ and $\mathcal{T}_h = \mathcal{T}_\ell$, we write $\mathcal{R}_h =: \mathcal{R}_\ell$ and

$$a\left(\mathcal{R}_\ell u, v_\ell\right) = a\left(u, v_\ell\right), \quad \forall\, v_\ell \in V_\ell.$$

Remark

It should be clear that $\mathcal{R}_h u \in V_h$ is just the finite element approximation of u.

Lemma

Let $\mathcal{T}_\ell$ and V_ℓ be as usual, and suppose $u_\ell \in V_\ell$ is given. Then, if Ω is convex,

$$\left\| u_\ell - \mathcal{R}_{\ell-1} u_\ell \right\|_{L^2(\Omega)} \le C h_\ell \left| u_\ell - \mathcal{R}_{\ell-1} u_\ell \right|_{H^1(\Omega)}, \tag{12}$$

for some constant $C > 0$ that is independent of $\ell \ge 1$.

Proof.

Observe that $u_\ell \in V_\ell \subset H_0^1(\Omega)$. But $u_\ell \notin H^2(\Omega)$. u_ℓ plays the role of the exact PDE solution in Theorem 8, but it is not H^2-regular. But this does not matter. We may still apply (9), since Ω is convex, to conclude

$$\left\| u_\ell - \mathcal{R}_{\ell-1} u_\ell \right\|_{L^2(\Omega)} \leq C h_{\ell-1} \left| u_\ell - \mathcal{R}_{\ell-1} u_\ell \right|_{H^1(\Omega)}$$

for some $C > 0$ that is independent of ℓ. Now, note that

$$h_{\ell-1} = 2h_\ell,$$

and the result follows. $\square$

Remark

We again point out that for nested triangulations, $\mathcal{T}_\ell$, we do not need to assume separately that the family is shape regular. It is by construction.

Theorem

Let $\mathcal{T}_\ell$ and V_ℓ be as usual, and suppose that Ω is convex polyhedral. Then the strong approximation property is satisfied. In particular, there is some $C_{\mathrm{A3}} > 0$, independent of ℓ, such that

$$\left\| \boldsymbol{u}_\ell - \tilde{\Pi}_\ell \boldsymbol{u}_\ell \right\|_\ell^2 \leq C_{\mathrm{A3}}^2 \rho_\ell^{-1} \left\| \boldsymbol{u}_\ell - \tilde{\Pi}_\ell \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell}^2 \tag{13}$$

for all $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$.

Proof.

Let $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$ be arbitrary. Suppose $u_\ell \in V_\ell$ is the unique function whose coordinate vector is $\boldsymbol{u}_\ell$ with basis $\mathcal{B}_\ell$, that is,

$$u_\ell \in V_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}.$$

Referring to (12),

$$|u_\ell - \mathcal{R}_{\ell-1} u_\ell|^2_{H^1(\Omega)} = a\left(u_\ell - \mathcal{R}_{\ell-1} u_\ell, u_\ell - \mathcal{R}_{\ell-1} u_\ell\right).$$

Let $\boldsymbol{w}_\ell \in \mathbb{R}^{n_\ell}$ be the unique coordinate vector of

$$u_\ell - \mathcal{R}_{\ell-1} u_\ell \in V_\ell$$

with respect to the Lagrange nodal basis $\mathcal{B}_\ell$. We want to show that

$$\boldsymbol{w}_\ell = \boldsymbol{u}_\ell - \tilde{\Pi}_\ell \boldsymbol{u}_\ell = \boldsymbol{u}_\ell - \mathsf{P}_{\ell-1}\mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell \boldsymbol{u}_\ell.$$

We begin with the definition of $\mathcal{R}_{\ell-1}$:

$$a\left(\mathcal{R}_{\ell-1} u_\ell, v_{\ell-1}\right) = a\left(u_\ell, v_{\ell-1}\right), \quad \forall\, v_{\ell-1} \in V_{\ell-1}.$$

Proof (Cont.)

Set $u'_{\ell-1} := \mathcal{R}_{\ell-1} u_\ell \in V_{\ell-1}$ and use the correspondences

$$\boldsymbol{u}'_{\ell-1} \in \mathbb{R}^{n_{\ell-1}} \overset{\mathcal{B}_{\ell-1}}{\leftrightarrow} u'_{\ell-1} \in V_{\ell-1}$$

and

$$\boldsymbol{v}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}} \overset{\mathcal{B}_{\ell-1}}{\leftrightarrow} v_{\ell-1} \in V_{\ell-1}.$$

Then,

$$\begin{aligned} a\left(\mathcal{R}_{\ell-1} u_\ell, v_{\ell-1}\right) &= \left(\boldsymbol{u}'_{\ell-1}, \boldsymbol{v}_{\ell-1}\right)_{\mathsf{A}_{\ell-1}} \\ &= \left(\mathsf{A}_{\ell-1}\boldsymbol{u}'_{\ell-1}, \boldsymbol{v}_{\ell-1}\right)_{\ell-1}, \end{aligned}$$

and

$$\begin{aligned} a\left(u_\ell, v_{\ell-1}\right) &= \left(\boldsymbol{u}_\ell, \mathsf{P}_{\ell-1}\boldsymbol{v}_{\ell-1}\right)_{\mathsf{A}_\ell} \\ &= \left(\mathsf{A}_\ell\boldsymbol{u}_\ell, \mathsf{P}_{\ell-1}\boldsymbol{v}_{\ell-1}\right)_\ell \\ &= \left(\mathsf{R}_{\ell-1}\mathsf{A}_\ell\boldsymbol{u}_\ell, \boldsymbol{v}_{\ell-1}\right)_{\ell-1}. \end{aligned}$$

Proof (Cont.)

So, it follows that

$$\mathsf{A}_{\ell-1}\boldsymbol{u}'_{\ell-1} = \mathsf{R}_{\ell-1}\mathsf{A}_\ell \boldsymbol{u}_\ell,$$

and

$$\boldsymbol{u}'_{\ell-1} = \mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell \boldsymbol{u}_\ell = \Pi_{\ell-1}\boldsymbol{u}_\ell.$$

Therefore,

$$\begin{aligned} \boldsymbol{w}_\ell &= \boldsymbol{u}_\ell - \mathsf{P}_{\ell-1}\boldsymbol{u}'_{\ell-1} \\ &= \boldsymbol{u}_\ell - \mathsf{P}_{\ell-1}\Pi_{\ell-1}\boldsymbol{u}_\ell \\ &= \boldsymbol{u}_\ell - \tilde{\Pi}_\ell \boldsymbol{u}_\ell. \end{aligned}$$

It follows that

$$\begin{aligned} \left|u_\ell - \mathcal{R}_{\ell-1}u_\ell\right|^2_{H^1(\Omega)} &= (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{A}_\ell} \\ &= \|\boldsymbol{w}_\ell\|^2_{\mathsf{A}_\ell} \\ &= \left\|\boldsymbol{u}_\ell - \tilde{\Pi}_\ell \boldsymbol{u}_\ell\right\|^2_{\mathsf{A}_\ell}. \end{aligned}$$

Proof (Cont.)

Recall we have shown the norm equivalence

$$C_1 h_\ell^d \left\| \boldsymbol{v}_\ell \right\|_\ell^2 \leq \left\| v_\ell \right\|_{L^2(\Omega)}^2 \leq C_2 h_\ell^d \left\| \boldsymbol{v}_\ell \right\|_\ell^2 . \tag{14}$$

Finally, using the norm equivalence in (14)

$$\begin{aligned} C_1 h_\ell^d \left\| \boldsymbol{u}_\ell - \tilde{\Pi}_\ell \boldsymbol{u}_\ell \right\|_\ell^2 &\overset{(14)}{\leq} \left\| u_\ell - \mathcal{R}_{\ell-1} u_\ell \right\|_{L^2(\Omega)}^2 \\ &\overset{(12)}{\leq} C h_\ell^2 \left| u_\ell - \mathcal{R}_{\ell-1} u_\ell \right|_{H^1(\Omega)}^2 \\ &= C h_\ell^2 \left\| \boldsymbol{u}_\ell - \tilde{\Pi}_\ell \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell}^2 . \end{aligned}$$

In the proof of the theorem in the last slide deck, we showed that

$$C_6^{(n_\ell)} h_\ell^{d-2} \leq \rho_\ell \leq C_7^{(n_\ell)} h_\ell^{d-2}.$$

Combining this with the last estimate gives the desired result (13). □

Corollary

Let $\mathcal{T}_\ell$ and V_ℓ be defined as usual with A_ℓ the standard stiffness matrix for the model problem. Then, the weak approximation property, Assumption (A4) holds: there exists a constant $C_{\mathrm{A4}} > 0$, independent of ℓ, such that

$$\left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right) \boldsymbol{u}_\ell, \boldsymbol{u}_\ell\right)_{\mathsf{A}_\ell} \leq C_{\mathrm{A4}}^2 \rho_\ell^{-1} \left\|\mathsf{A}_\ell \boldsymbol{u}_\ell\right\|_\ell^2, \tag{15}$$

for all $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$.

Proof.

Since the Galerkin condition (A0) and the strong approximation property hold, the result follows immediately from the fact that (A3) implies (A4). □

Remark

Therefore, using Richardson's smoother, the W-cycle and V-cycle algorithms defined in Chapter 4 converge. There is nothing more to do!

The Full Multigrid Algorithm

The Full Multigrid Algorithm

Most of our readers have heard it said that multigrid is an optimal-order method. What does this mean? Well, it really means that a good enough approximation to the finite element approximation can be found by some multigrid algorithm in $\mathcal{O}(n_L)$ operations, where n_L is the number of unknowns (degrees of freedom) in our finite element solution. By contrast, if one were to use Gaussian elimination to find the solution, $\mathcal{O}(n_L^3)$ operations would be required. But we need to be precise about which multigrid algorithm we use. In particular, we need another multigrid operator, which we now define.

Definition (Full Multigrid Operator)

Suppose that the multigrid operator, MG, is as in Definition **??**, $r \in \mathbb{N}$, and $1 \leq s \leq L$. Assume that $f \in L^2(\Omega)$, and define

$$\boldsymbol{f}_\ell := \begin{bmatrix} (f, \phi_{\ell,1})_{L^2(\Omega)} \\ (f, \phi_{\ell,2})_{L^2(\Omega)} \\ \vdots \\ (f, \phi_{\ell,n_\ell})_{L^2(\Omega)} \end{bmatrix} \in \mathbb{R}^{n_\ell}, \quad 0 \leq \ell \leq L.$$

The **full multigrid operator**,

$$\hat{\boldsymbol{u}}_s := \mathrm{FMG}\,(s)\,, \tag{16}$$

is defined as follows:

- $$\hat{\boldsymbol{u}}_0 := \mathsf{A}_0^{-1} \boldsymbol{f}_0.$$

Definition (Full Multigrid Operator (Cont.))

- **for** $\ell = 1 : s$
  - $$\boldsymbol{u}_\ell^{(0)} := \mathsf{P}_{\ell-1} \hat{\boldsymbol{u}}_{\ell-1};$$
  - $$\boldsymbol{u}_\ell^{(\sigma+1)} := \mathrm{MG}\left(\boldsymbol{f}_\ell, \ell, \boldsymbol{u}_\ell^{(\sigma)} \right), \quad 0 \leq \sigma \leq r-1;$$
  - $$\hat{\boldsymbol{u}}_\ell := \boldsymbol{u}_\ell^{(r)}.$$

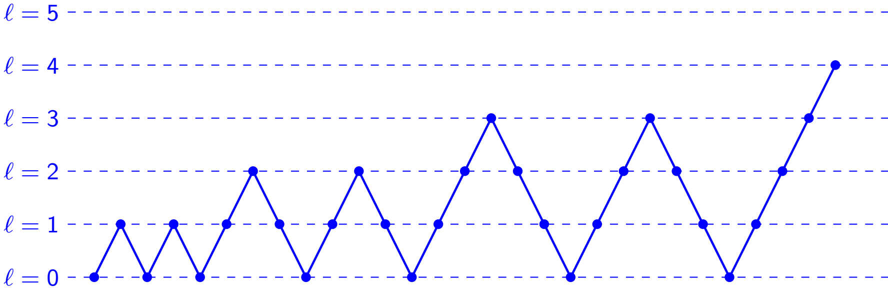

Figure: The shape of the full multigrid algorithm assuming $r = 2$ and $p = 1$.

Theorem

Suppose that, in general, for all $\boldsymbol{u}_\ell^{(0)}$

$$\left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \mathrm{MG}\left(\boldsymbol{g}_\ell, \ell, \boldsymbol{u}_\ell^{(0)}\right) \right\|_{\mathsf{A}_\ell} \leq \gamma \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right\|_{\mathsf{A}_\ell}, \tag{17}$$

where $0 < \gamma < 1$ *is independent of* ℓ *and*

$$\boldsymbol{u}_\ell^{\mathrm{E}} := \mathsf{A}_\ell^{-1} \boldsymbol{g}_\ell.$$

Assume that $f \in L^2(\Omega)$ *and* r *in the full multigrid algorithm satisfies*

$$\gamma^r < \frac{1}{2}.$$

Suppose that

$$\hat{u}_\ell \in V_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \hat{\boldsymbol{u}}_\ell := \mathrm{FMG}(\ell) \in \mathbb{R}^{n_\ell}.$$

Theorem (Cont.)

Then, there exists a constant, $C > 0$, independent of ℓ, such that

$$|u_\ell - \hat{u}_\ell|_{H^1(\Omega)} = \|u_\ell - \hat{u}_\ell\|_{H_0^1(\Omega)} \le Ch_\ell \, |u|_{H^2(\Omega)} \,, \tag{18}$$

where $u_\ell \in V_\ell$ is the finite element approximation satisfying

$$a\,(u_\ell, v_\ell) = (f, v_\ell)\,, \quad \forall\, v_\ell \in V_\ell,$$

and $u \in H_0^1(\Omega) \cap H^2(\Omega)$ is the solution to

$$a\,(u, v) = (f, v)\,, \quad \forall\, v \in H_0^1(\Omega).$$

Proof.

Define

$$\hat{e}_\ell := u_\ell - \hat{u}_\ell \in V_\ell.$$

This is the algebraic error in computing the finite element approximation. Clearly $\hat{e}_0 \equiv 0$. In general,

$$|\hat{e}_\ell|^2_{H^1(\Omega)} = a\left(\hat{e}_\ell, \hat{e}_\ell\right) = \|\hat{\boldsymbol{e}}_\ell\|^2_{\mathsf{A}_\ell},$$

where

$$\hat{\boldsymbol{e}}_\ell = \boldsymbol{u}_\ell - \hat{\boldsymbol{u}}_\ell \in \mathbb{R}^{n_\ell} \overset{\mathcal{B}_\ell}{\leftrightarrow} \hat{e}_\ell = u_\ell - \hat{u}_\ell \in V_\ell.$$

Proof (Cont.)

Then

$$\begin{aligned}
|\hat{e}_\ell|_{H^1(\Omega)} &= \|\boldsymbol{u}_\ell - \hat{\boldsymbol{u}}_\ell\|_{\mathsf{A}_\ell} \\
&\overset{(17)}{\leq} \gamma^r \|\boldsymbol{u}_\ell - \mathsf{P}_{\ell-1}\hat{\boldsymbol{u}}_{\ell-1}\|_{\mathsf{A}_\ell} \\
&= \gamma^r |u_\ell - \hat{u}_{\ell-1}|_{H^1(\Omega)} \\
&\leq \gamma^r \left\{ |u_\ell - u|_{H^1(\Omega)} + |u - u_{\ell-1}|_{H^1(\Omega)} + |u_{\ell-1} - \hat{u}_{\ell-1}|_{H^1(\Omega)} \right\} \\
&\overset{(10)}{\leq} \gamma^r \left\{ Ch_\ell |u|_{H^2(\Omega)} + 2Ch_\ell |u|_{H^2(\Omega)} + |\hat{e}_{\ell-1}|_{H^1(\Omega)} \right\} \\
&= C\gamma^r h_\ell |u|_{H^2(\Omega)} + \gamma^r |\hat{e}_{\ell-1}|_{H^1(\Omega)} .
\end{aligned} \tag{19}$$

By the same reasoning,

$$|\hat{e}_{\ell-1}|_{H^1(\Omega)} \leq C\gamma^r h_{\ell-1} |u|_{H^2(\Omega)} + \gamma^r |\hat{e}_{\ell-2}|_{H^1(\Omega)} . \tag{20}$$

Combining (19) and (20), we have

$$|\hat{e}_\ell|_{H^1(\Omega)} \leq C\gamma^r h_\ell |u|_{H^2(\Omega)} + C\gamma^{2r} h_{\ell-1} |u|_{H^2(\Omega)} + \gamma^{2r} |\hat{e}_{\ell-2}|_{H^1(\Omega)} .$$

Proof (Cont.)

Continuing in this fashion and using $\hat{e}_0 \equiv 0$, we have

$$
\begin{aligned}
|\hat{e}_\ell|_{H^1(\Omega)} &\le \left\{ Ch_\ell \gamma^r + Ch_{\ell-1}\gamma^{2r} + Ch_{\ell-2}\gamma^{3r} + \cdots + Ch_1\gamma^{\ell r} \right\} |u|_{H^2(\Omega)} \\
&= \left\{ Ch_\ell \gamma^r + Ch_\ell 2\gamma^{2r} + Ch_\ell 2^2\gamma^{3r} + \cdots + Ch_\ell 2^{\ell-1}\gamma^{\ell r} \right\} \\
&= \frac{Ch_\ell}{2} \left\{ 2\gamma^r + 2^2\gamma^{2r} + 2^3\gamma^{3r} + \cdots + 2^\ell \gamma^{\ell r} \right\} |u|_{H^2(\Omega)} \\
&\le \frac{C\gamma^r}{1-2\gamma^r} h_\ell \, |u|_{H^2(\Omega)} .
\end{aligned}
$$

The theorem is proven. $\square$

Remark

Let us think about what the last result tells us. Using the triangle inequality,

$$\begin{aligned}
\|u - \hat{u}_\ell\|_{H_0^1(\Omega)} &\leq \|u - u_\ell\|_{H_0^1(\Omega)} + \|u_\ell - \hat{u}_\ell\|_{H_0^1(\Omega)} \\
&\leq Ch_\ell \, |u|_{H^2(\Omega)} + Ch_\ell \, |u|_{H^2(\Omega)} \\
&= Ch_\ell \, |u|_{H^2(\Omega)} \, .
\end{aligned}$$

In other words, the solution that we compute using the full multigrid operator, provided r is sufficiently large, is just as good as the finite element approximation. Why go any further? The next result shows that the cost of the full multigrid operator is optimal.

Proposition (Work Estimate for Full Multigrid)

Suppose $1 \le \ell \le L$, *and, as usual,* $n_\ell = \dim(V_\ell)$. *Assume that*

$$C_1 2^{d\cdot\ell} \le n_\ell \le C_2 2^{d\cdot\ell}, \quad 0 \le \ell \le L,$$

for some $C_2 \ge C_1 > 0$ *that are independent of* ℓ, *where* $d = 1, 2$ *or* 3 *is the dimension of space. If*

$$p < 2^d,$$

then the amount of work, W_s, *for the full multigrid operator* $\mathrm{FMG}(s)$ *satisfies*

$$W_s \le C n_s,$$

where $C > 0$ *is a constant that is independent of* s.

Proof.

By w_ℓ let us denote the work required for computing the output of the multigrid operator, $\mathrm{MG}(\cdot, \ell, \cdot)$, for $1 \leq \ell \leq L$. Then, assuming that the work is dominated by smoothing,

$$w_\ell \leq C(m_1 + m_2)n_\ell + p w_{\ell-1},$$

where $C > 0$ is independent of ℓ. Similarly,

$$w_{\ell-1} \leq C(m_1 + m_2)n_{\ell-1} + p w_{\ell-2}.$$

Combining the last two inequalities gives

$$w_\ell \leq C(m_1 + m_2)n_\ell + pC(m_1 + m_2)n_{\ell-1} + p^2 w_{\ell-2}.$$

Proof (Cont.)

Continuing in this fashion, we obtain

$$\begin{aligned} w_\ell &\leq C(m_1+m_2)\left\{n_\ell + p n_{\ell-1} + p^2 n_{\ell-2} + \cdots + p^\ell n_0\right\} \\ &\leq CC_2(m_1+m_2)2^{d\cdot\ell}\left\{1+\frac{p}{2^d}+\left(\frac{p}{2^d}\right)^2+\cdots+\left(\frac{p}{2^d}\right)^\ell\right\} \\ &\leq \frac{CC_2(m_1+m_2)2^{d\cdot\ell}}{1-\frac{p}{2^d}} \\ &\leq \frac{CC_2(m_1+m_2)}{C_1\left(1-\frac{p}{2^d}\right)}n_\ell \\ &= Cn_\ell. \end{aligned}$$

Finally, neglecting the cost of the prolongation step, we have

$$W_s = W_{s-1} + rw_s \leq W_{s-1} + rCn_s.$$

Likewise, at level $s-1$,

$$W_{s-1} \leq W_{s-2} + rCn_{s-1}.$$

Proof (Cont.)

Consequently,

$$W_s \leq rCn_s + rCn_{s-1} + W_{s-2}.$$

Continuing in this fashion,

$$\begin{aligned} W_s &\leq rCn_s + rCn_{s-1} + \cdots + rCn_0 \\ &\leq rCC_2 \left(2^{d\cdot s} + 2^{d\cdot(s-1)} + \cdots + 1 \right) \\ &= rCC_2 2^{d\cdot s} \left(1 + \frac{1}{2^d} + \left(\frac{1}{2^d}\right)^2 + \cdots + \left(\frac{1}{2^d}\right)^s \right) \\ &\leq \frac{rCC_2}{1 - \frac{1}{2^d}} 2^{d\cdot s} \\ &\leq \frac{rCC_2}{C_1\left(1 - \frac{1}{2^d}\right)} n_s \\ &\leq Cn_s. \end{aligned}$$

$\square$

Some Computational Experiments

To conclude this chapter, and as a complement to Section **??**, let us perform some computational experiments to confirm the predicted convergence results of our various multigrid algorithms. In particular, let us use the algorithms to approximate the solution of

$$\mathsf{A}_L \boldsymbol{u}_L^{\mathrm{E}} = \boldsymbol{f}_L,$$

where A_L is the standard finite element stiffness matrix for the 1D model problem. In our experiments, we specify the exact solution:

$$\left[\boldsymbol{u}_L^{\mathrm{E}}\right]_i = u_{L,i}^{\mathrm{E}} = \exp(\sin(3.0\pi * x_{L,i})) - 1.0.$$

Observe that, as in the experiment results of chapter 3, we are using a uniform mesh in one space dimension. The force vector is manufactured by setting $\boldsymbol{f}_L := \mathsf{A}_L \boldsymbol{u}_L^{\mathrm{E}}$. We report on several computational experiments in Table 1. The initial approximation, $\boldsymbol{u}_L^{(0)}$, is chosen via pseudorandom number selection. The main Matlab codes implementing the multigrid algorithm are given in the listings of this chapter. The error reduction for the multigrid V-cycle algorithm ($p = 1$), using the parameters $\omega = 2/3$, $m_1 = m_2 = 3$, $n_L = 255$, and $L = 7$ is shown in the figure on the next slide.

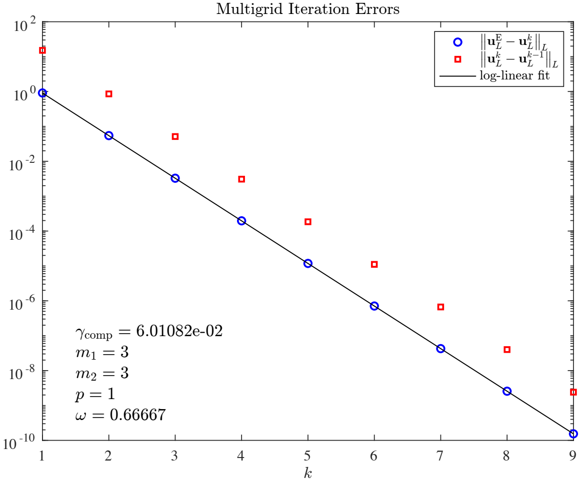

Figure: The error reduction for the multigrid V-cycle algorithm, using the parameters $\omega = 2/3$, $m_1 = m_2 = 3$, $n_L = 255$, and $L = 7$. The factor γ_{comp} is computed using a log-linear fit of the last four error values. Note that $\left\|\boldsymbol{u}_L^k - \boldsymbol{u}_L^{k-1}\right\|_L$ is a good indicator of the error.

n_L	L	ω	m_1	m_2	p	γ_{comp}
63	5	2/3	3	3	1	5.93×10^{-02}
127	6	2/3	3	3	1	5.99×10^{-02}
255	7	2/3	3	3	1	6.01×10^{-02}
511	8	2/3	3	3	1	6.01×10^{-02}
1023	9	2/3	3	3	1	6.01×10^{-02}
127	6	2/3	4	4	1	4.65×10^{-02}
127	6	2/3	5	5	1	3.81×10^{-02}
127	6	2/3	6	6	1	3.21×10^{-02}
127	6	0.50	3	3	1	8.06×10^{-02}
127	6	0.55	3	3	1	7.27×10^{-02}
127	6	0.60	3	3	1	6.66×10^{-02}
127	6	0.65	3	3	1	6.14×10^{-02}
127	6	0.70	3	3	1	5.71×10^{-02}
127	6	0.75	3	3	1	5.39×10^{-02}
127	6	0.80	3	3	1	5.91×10^{-02}
127	6	0.50	3	3	2	5.44×10^{-02}
127	6	0.50	3	3	3	5.42×10^{-02}

Table: Computed multigrid convergence factors, γ_{comp}, for various parameter choices. The factor γ_{comp} is computed using a log-linear fit of the last four error values $\left\| \boldsymbol{u}_L^{\mathrm{E}} - \boldsymbol{u}_L^k \right\|_L$.