Math 673

Multigrid Methods: A Mostly Matrix-Based Approach

Chapter 08: Multigrid as a Multipilcative Process

Abner J. Salgado and Steven M. Wise

asalgad1@utk.edu swise1@.utk.edu
University of Tennessee

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Chapter 08

Multigrid as a Multipilcative Process

Introduction

In this chapter, we reformulate some of our multigrid algorithms using objects called T matrices. We use the same finite element setting as in the last chapter. However, the ideas can be generalized. This reformulation will make it obvious that the standard multigrid methods are multiplicative GLIS methods.

Multilevel Matrices

Definition (Multilevel Prolongation Matrix)

Suppose $0 \le j < \ell$. Define the **multilevel prolongation matrix**, $\mathsf{P}_{j,\ell}$, via

$$\mathsf{P}_{j,\ell} := \mathsf{P}_{\ell-1}\mathsf{P}_{\ell-2}\cdots\mathsf{P}_j \in \mathbb{R}^{n_\ell \times n_j}.$$

In particular,

$$\mathsf{P}_{\ell-1,\ell} = \mathsf{P}_{\ell-1} \in \mathbb{R}^{n_\ell \times n_{\ell-1}}.$$

Lemma

Suppose $v_j \in V_j$ for some $0 \leq j < \ell$, where $V_0 \subset V_1 \subset \cdots \subset V_\ell$ are the usual nested finite element spaces. Let $\boldsymbol{v}_j \in \mathbb{R}^{n_j}$ be the coordinate vector of v_j with respect to the basis $\mathcal{B}_j$. Then, the unique coordinate vector of $v_j \in V_\ell$ in the basis $\mathcal{B}_\ell$ is

$$\mathsf{P}_{j,\ell}\boldsymbol{v}_j \in \mathbb{R}^{n_\ell}.$$

Proof.

Simple exercise. □

Definition (Multilevel Restriction Matrix)

Define $\mathsf{R}_{j,\ell} \in \mathbb{R}^{n_j \times n_\ell}$, for $0 \leq j < \ell$, via

$$\mathsf{R}_{j,\ell} = \mathsf{P}_{j,\ell}{}^{T}.$$

$\mathsf{R}_{j,\ell}$ is called the **multilevel restriction matrix**.

Lemma

With the usual construction for the conforming finite element method, we have, for any $0 \leq j < \ell$,

$$\mathsf{A}_j = \mathsf{R}_{j,\ell}\mathsf{A}_\ell\mathsf{P}_{j,\ell} \in \mathbb{R}^{n_j \times n_j}.$$

Proof.

This follows because the Galerkin condition holds:

$$
\begin{aligned}
\mathsf{A}_j &= \mathsf{R}_j \mathsf{A}_{j+1} \mathsf{P}_j \\
&= \mathsf{R}_j \mathsf{R}_{j+1} \mathsf{A}_{j+2} \mathsf{P}_{j+1} \mathsf{P}_j \\
&\vdots \\
&= \mathsf{R}_j \cdots \mathsf{R}_{\ell-1} \mathsf{A}_\ell \mathsf{P}_{\ell-1} \cdots \mathsf{P}_j \\
&= \mathsf{R}_{j,\ell} \mathsf{A}_\ell \mathsf{P}_{j,\ell}.
\end{aligned}
$$

□

Definition

For any $0 \leq j < \ell$, define the matrix

$$\Pi_{j,\ell} := \mathsf{A}_j^{-1} \mathsf{R}_{j,\ell} \mathsf{A}_\ell \in \mathbb{R}^{n_j \times n_\ell}.$$

Lemma

We have, for $0 \leq j < \ell$,

$$\Pi_{j,\ell} := \Pi_j \Pi_{j+1} \cdots \Pi_{\ell-1}.$$

Proof.

The matrix product on the right hand side is

$$\begin{aligned}
\Pi_j \cdots \Pi_{\ell-1} &= A_j^{-1} R_j A_{j+1} A_{j+1}^{-1} R_{j+1} A_{j+2} \cdots A_{\ell-1}^{-1} R_{\ell-1} A_\ell \\
&= A_j^{-1} R_j R_{j+1} \cdots R_{\ell-1} A_\ell \\
&= \Pi_{j,\ell}.
\end{aligned}$$

$\square$

Definition (Multilevel Ritz Projection Matrix)

Define, for any $0 \leq j < \ell$, the **multilevel Ritz projection matrix** via

$$\tilde{\Pi}_{j,\ell} := \mathsf{P}_{j,\ell}\Pi_{j,\ell} \in \mathbb{R}^{n_\ell \times n_\ell}.$$

Observe that

$$\tilde{\Pi}_{\ell-1,\ell} = \tilde{\Pi}_{\ell} \in \mathbb{R}^{n_\ell \times n_\ell}.$$

Theorem

Let $a(\cdot,\cdot)$ and V_ℓ be defined as usual for the conforming finite method. Let $0 \leq j < \ell$ and $u_\ell \in V_\ell$ be arbitrary. Set

$$u_j' = \mathcal{R}_j u_\ell \in V_j \overset{\mathcal{B}_j}{\leftrightarrow} \boldsymbol{u}_j' \in \mathbb{R}^{n_j}.$$

Then, if $\boldsymbol{u}_\ell$ is the coordinate vector of $u_\ell \in V_\ell$ with respect to the basis $\mathcal{B}_\ell$, it follows that the unique representation of $\mathcal{R}_j u_\ell \in V_j$ in the basis $\mathcal{B}_j$ is precisely

$$\boldsymbol{u}_j' = \Pi_{j,\ell} \boldsymbol{u}_\ell \in \mathbb{R}^{n_j}.$$

Further, the unique representation of $\mathcal{R}_j u_\ell \in V_\ell$ in the basis $\mathcal{B}_\ell$ is precisely

$$\tilde{\Pi}_{j,\ell} \boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}.$$

Proof.

Let $u_\ell \in V_\ell$ be given. $\mathcal{R}_j u_\ell$ is defined as the unique solution to

$$a\left(\mathcal{R}_j u_\ell, v_j\right) = a\left(u_\ell, v_j\right), \quad \forall \ v_j \in V_j.$$

Then

$$a\left(\mathcal{R}_j u_\ell, v_j\right) = \left(\boldsymbol{u}_j', \boldsymbol{v}_j\right)_{\mathsf{A}_j}.$$

On the other hand

$$a\left(u_\ell, v_j\right) = \left(\boldsymbol{u}_\ell, P_j^\ell \boldsymbol{v}_j\right)_{\mathsf{A}_\ell},$$

where

$$\boldsymbol{v}_j \in \mathbb{R}^{n_j} \overset{\mathcal{B}_\ell}{\leftrightarrow} v_j \in V_j,$$

as usual.

Proof (Cont.)

Going further, we have

$$\begin{aligned} a(u_\ell, v_j) &= (\mathsf{A}_\ell \boldsymbol{u}_\ell, \mathsf{P}_{j,\ell}\boldsymbol{v}_j)_\ell \\ &= (\mathsf{R}_{j,\ell}\mathsf{A}_\ell \boldsymbol{u}_\ell, \boldsymbol{v}_j)_j\,, \end{aligned}$$

and

$$a(\mathcal{R}_j u_\ell, v_j) = \left(\mathsf{A}_j \boldsymbol{u}_j', \boldsymbol{v}_j\right)_j.$$

Therefore,

$$\mathsf{A}_j \boldsymbol{u}_j' = \mathsf{R}_{j,\ell}\mathsf{A}_\ell \boldsymbol{u}_\ell,$$

or

$$\boldsymbol{u}_j' = \mathsf{A}_j^{-1}\mathsf{R}_{j,\ell}\mathsf{A}_\ell \boldsymbol{u}_\ell = \Pi_{j,\ell}\boldsymbol{u}_\ell.$$

The second part follows from Lemma 2. □

Definition (Multilevel T-matrix)

Define, for any $0 \leq j < \ell$,

$$\mathsf{T}_{j,\ell}(m) := \Pi_{j,\ell} - \mathsf{K}_j^m \Pi_{j,\ell} \in \mathbb{R}^{n_j \times n_\ell},$$

where m is a non-negative integer exponent. Define

$$\tilde{\mathsf{T}}_{j,\ell}(m) = \mathsf{P}_{j,\ell} \mathsf{T}_{j,\ell} \in \mathbb{R}^{n_\ell \times n_\ell},$$

The square matrix $\tilde{\mathsf{T}}_{j,\ell}$ is called a **multilevel** T-**matrix**.

Remark

Whenever the number of smoothing steps m is understood, we write $\mathsf{T}_{j,\ell}$ in stead of $\mathsf{T}_{j,\ell}(m)$ and $\tilde{\mathsf{T}}_{j,\ell}$ instead of $\tilde{\mathsf{T}}_{j,\ell}(m)$. Of course, $\tilde{\mathsf{T}}_{j,\ell}(0) = \mathsf{O}$.

Further Properties of the Multilevel Matrices

Now, let us review some properties of the object that we have created.

Lemma

Let $0 \leq j < \ell$. *Then*

$$\Pi_{j,\ell} \mathsf{P}_{j,\ell} = \mathsf{I}_j, \tag{1}$$

and

$$\tilde{\Pi}_{j,\ell}^2 = \tilde{\Pi}_{j,\ell}.$$

Proof.

The Galerkin condition holds in the sense that

$$\mathsf{A}_j = \mathsf{R}_{j,\ell} A_\ell \mathsf{P}_{j,\ell}. \tag{2}$$

By definition

$$\Pi_{j,\ell} = \mathsf{A}_j^{-1} \mathsf{R}_{j,\ell} \mathsf{A}_\ell,$$

so that

$$\begin{aligned} \Pi_{j,\ell} \mathsf{P}_{j,\ell} &= \mathsf{A}_j^{-1} \mathsf{R}_{j,\ell} \mathsf{A}_\ell \mathsf{P}_{j,\ell} \\ &= \mathsf{A}_j^{-1} \mathsf{A}_j \\ &= \mathsf{I}_j. \end{aligned}$$

Now,

$$\tilde{\Pi}_{j,\ell}^2 = \mathsf{P}_{j,\ell} \Pi_{j,\ell} \mathsf{P}_{j,\ell} \Pi_{j,\ell} = \mathsf{P}_{j,\ell} \Pi_{j,\ell} = \tilde{\Pi}_{j,\ell}.$$

□

Definition

Let $0 \leq j < \ell$. Define

$$\mathsf{T}'_{j,\ell} := \Pi_{j,\ell} - (\mathsf{K}_j^*)^m \Pi_{j,\ell}$$

and

$$\tilde{\mathsf{T}}'_{j,\ell} := \mathsf{P}_{j,\ell} \mathsf{T}'_{j,\ell},$$

where

$$\mathsf{K}_j^* = \mathsf{I}_j - \mathsf{S}_j^T \mathsf{A}_j,$$

as usual.

Lemma

Let $0 \leq j < \ell$. *Then*

$$\tilde{\Pi}_{j,\ell}^{*} = \tilde{\Pi}_{j,\ell}, \tag{3}$$

and

$$\tilde{\mathsf{T}}_{j,\ell}^{*} = \tilde{\mathsf{T}}_{j,\ell}'.$$

Proof.

Recall

$$\left(\tilde{\Pi}_{j,\ell}\boldsymbol{u}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} = \left(\boldsymbol{u}_\ell, \tilde{\Pi}_{j,\ell}^{*}\boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell},$$

for all $\boldsymbol{u}_\ell, \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$. Then

$$\begin{aligned}
\left(\tilde{\Pi}_{j,\ell}\boldsymbol{u}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} &= \left(\mathsf{P}_{j,\ell}\Pi_{j,\ell}\boldsymbol{u}_\ell, \mathsf{A}_\ell\boldsymbol{v}_\ell\right)_\ell \\
&= \left(\Pi_{j,\ell}\boldsymbol{u}_\ell, \mathsf{R}_{j,\ell}\mathsf{A}_\ell\boldsymbol{v}_\ell\right)_j \\
&= \left(\mathsf{A}_j^{-1}\mathsf{R}_{j,\ell}\mathsf{A}_\ell\boldsymbol{u}_\ell, \mathsf{R}_{j,\ell}\mathsf{A}_\ell\boldsymbol{v}_\ell\right)_j \\
&= \left(\mathsf{R}_{j,\ell}\mathsf{A}_\ell\boldsymbol{u}_\ell, \mathsf{A}_j^{-1}\mathsf{R}_{j,\ell}\mathsf{A}_\ell\boldsymbol{v}_\ell\right)_j \\
&= \left(\mathsf{A}_\ell\boldsymbol{u}_\ell, \tilde{\Pi}_{j,\ell}\boldsymbol{v}_\ell\right)_\ell \\
&= \left(\boldsymbol{u}_\ell, \tilde{\Pi}_{j,\ell}^{*}\boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell}.
\end{aligned}$$

Now

$$\tilde{\mathsf{T}}_{j,\ell} = \tilde{\Pi}_{j,\ell} - \mathsf{P}_{j,\ell}\mathsf{K}_j^m\Pi_{j,\ell}.$$

Proof (Cont.)

Therefore,

$$
\begin{aligned}
\left(\tilde{\mathsf{T}}_{j,\ell}\boldsymbol{u}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} &= \left(\tilde{\Pi}_{j,\ell}\boldsymbol{u}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} - \left(\mathsf{P}_{j,\ell}\mathsf{K}_j^m\Pi_{j,\ell}\boldsymbol{u}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} \\
&= \left(\boldsymbol{u}_\ell, \tilde{\Pi}_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} - \left(\mathsf{P}_{j,\ell}\mathsf{K}_j^m\Pi_{j,\ell}\boldsymbol{u}_\ell, \mathsf{A}_\ell\boldsymbol{v}_\ell\right)_\ell \\
&= \left(\boldsymbol{u}_\ell, \tilde{\Pi}_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} - \left(\mathsf{K}_j^m\Pi_{j,\ell}\boldsymbol{u}_\ell, \mathsf{R}_{j,\ell}\mathsf{A}_\ell\boldsymbol{v}_\ell\right)_j \\
&= \left(\boldsymbol{u}_\ell, \tilde{\Pi}_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} - \left(\mathsf{K}_j^m\Pi_{j,\ell}\boldsymbol{u}_\ell, \mathsf{A}_j\mathsf{A}_j^{-1}\mathsf{R}_{j,\ell}\mathsf{A}_\ell\boldsymbol{v}_\ell\right)_j \\
&= \left(\boldsymbol{u}_\ell, \tilde{\Pi}_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} - \left(\mathsf{K}_j^m\Pi_{j,\ell}\boldsymbol{u}_\ell, \mathsf{A}_j^{-1}\mathsf{R}_{j,\ell}\mathsf{A}_\ell\boldsymbol{v}_\ell\right)_{\mathsf{A}_j} \\
&= \left(\boldsymbol{u}_\ell, \tilde{\Pi}_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} - \left(\Pi_{j,\ell}\boldsymbol{u}_\ell, \left(\mathsf{K}_j^m\right)^*\Pi_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_j} \\
&= \left(\boldsymbol{u}_\ell, \tilde{\Pi}_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} - \left(\Pi_{j,\ell}\boldsymbol{u}_\ell, \left(\mathsf{K}_j^*\right)^m\Pi_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_j} \\
&= \left(\boldsymbol{u}_\ell, \tilde{\Pi}_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} - \left(\Pi_{j,\ell}\boldsymbol{u}_\ell, \left(\mathsf{K}_j^*\right)^m\Pi_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_j}
\end{aligned}
$$

□

Proof (Cont.)

$$
\begin{aligned}
&= \left(\boldsymbol{u}_\ell, \tilde{\Pi}_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} - \left(\mathsf{A}_j^{-1}\mathsf{R}_{j,\ell}\mathsf{A}_\ell\boldsymbol{u}_\ell, \left(\mathsf{K}_j^*\right)^m \Pi_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_j} \\
&= \left(\boldsymbol{u}_\ell, \tilde{\Pi}_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} - \left(\mathsf{R}_{j,\ell}\mathsf{A}_\ell\boldsymbol{u}_\ell, \left(\mathsf{K}_j^*\right)^m \Pi_{j,\ell}\boldsymbol{v}_\ell\right)_j \\
&= \left(\boldsymbol{u}_\ell, \tilde{\Pi}_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} - \left(\mathsf{A}_\ell\boldsymbol{u}_\ell, \mathsf{P}_{j,\ell}\left(\mathsf{K}_j^*\right)^m \Pi_{j,\ell}\boldsymbol{v}_\ell\right)_\ell \\
&= \left(\boldsymbol{u}_\ell, \tilde{\Pi}_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} - \left(\boldsymbol{u}_\ell, \mathsf{P}_{j,\ell}\left(\mathsf{K}_j^*\right)^m \Pi_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} \\
&= \left(\boldsymbol{u}_\ell, \tilde{\mathsf{T}}'_{j,\ell}\boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell}.
\end{aligned}
$$

So,

$$\tilde{\Pi}_{j,\ell}^* = \tilde{\Pi}_{j,\ell},$$

and

$$\tilde{\mathsf{T}}_{j,\ell}^* = \tilde{\mathsf{T}}'_{j,\ell}.$$

□

Remark

We note that, in general

$$\tilde{\mathsf{T}}_{j,\ell}^2 \neq \tilde{\mathsf{T}}_{j,\ell}.$$

In other words, the T *matrix,* $\tilde{\mathsf{T}}_{j,\ell}$*, is not a projection matrix.*

Theorem

Let $0 \leq j < \ell$. *Then*

$$\left(\mathsf{I}_\ell - \tilde{\mathsf{\Pi}}_{j,\ell}\right)\left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{j,\ell}\right) = \mathsf{I}_\ell - \tilde{\mathsf{\Pi}}_{j,\ell}, \tag{4}$$

and

$$\left(\mathsf{I}_\ell - \tilde{\mathsf{T}}^*_{j,\ell}\right)\left(\mathsf{I}_\ell - \tilde{\mathsf{\Pi}}_{j,\ell}\right) = \mathsf{I}_\ell - \tilde{\mathsf{\Pi}}_{j,\ell}. \tag{5}$$

Proof.

The left hand side of (4) is

$$\mathsf{M}_\ell := \mathsf{I}_\ell - \tilde{\Pi}_{j,\ell} - \tilde{\mathsf{T}}_{j,\ell} + \tilde{\Pi}_{j,\ell}\tilde{\mathsf{T}}_{j,\ell}.$$

By definition,

$$\begin{aligned} \tilde{\Pi}_{j,\ell}\tilde{\mathsf{T}}_{j,\ell} &= \mathsf{P}_{j,\ell}\Pi_{j,\ell}\mathsf{P}_{j,\ell}\mathsf{T}_{j,\ell} \\ &\overset{(1)}{=} \mathsf{P}_{j,\ell}\mathsf{I}_j\mathsf{T}_{j,\ell} \\ &= \tilde{\mathsf{T}}_{j,\ell}. \end{aligned}$$

So

$$\mathsf{M}_\ell := \mathsf{I}_\ell - \tilde{\Pi}_{j,\ell} - \tilde{\mathsf{T}}_{j,\ell} + \tilde{\mathsf{T}}_{j,\ell} = \mathsf{I}_\ell - \tilde{\Pi}_{j,\ell}.$$

The left hand side of (5) is

$$\mathsf{M}'_\ell = \mathsf{I}_\ell - \tilde{\mathsf{T}}^*_{j,\ell} - \tilde{\Pi}_{j,\ell} + \tilde{\mathsf{T}}^*_{j,\ell}\tilde{\Pi}_{j,\ell}.$$

Proof (Cont.)

Then

$$\begin{aligned}
\tilde{\mathsf{T}}^{*}_{j,\ell}\tilde{\Pi}_{j,\ell} &= \mathsf{P}_{j,\ell}\, T_j^{\prime\,\ell}\, \mathsf{P}_{j,\ell}\Pi_{j,\ell} \\
&= \mathsf{P}_{j,\ell}\left(\Pi_{j,\ell} - (\mathsf{K}_j^{*})^{m}\Pi_{j,\ell}\right)\mathsf{P}_{j,\ell}\Pi_{j,\ell} \\
&\overset{(1)}{=} \mathsf{P}_{j,\ell}\left(\mathsf{I}_{j,\ell} - (\mathsf{K}_j^{*})^{m}\right)\Pi_{j,\ell} \\
&= \tilde{\mathsf{T}}^{*}_{j,\ell}.
\end{aligned}$$

So

$$\mathsf{M}'_{\ell} = \mathsf{I}_{\ell} - \tilde{\Pi}^{*}_{j,\ell},$$

as desired. □

Lemma

Let $0 \leq j < \ell$. *Then*

$$\mathsf{I}_\ell - \tilde{\Pi}_{j,\ell} = \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{j,\ell}^*\right)\left(\mathsf{I}_\ell - \tilde{\Pi}_{j,\ell}\right)\left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{j,\ell}\right). \tag{6}$$

Proof.

Since the Galerkin condition holds, $\tilde{\Pi}_{j,\ell}$ is a bona fide projection matrix (Lemma 12):

$$\tilde{\Pi}_{j,\ell}^2 = \tilde{\Pi}_{j,\ell},$$

and

$$\left(\mathsf{I}_\ell - \tilde{\Pi}_{j,\ell}\right)^2 = \mathsf{I}_\ell - \tilde{\Pi}_{j,\ell}$$

is a direct consequence. By the last result

$$\begin{aligned}
\mathsf{I}_\ell - \tilde{\Pi}_{j,\ell} &= \left(\mathsf{I}_\ell - \tilde{\Pi}_{j,\ell}\right)\left(\mathsf{I}_\ell - \tilde{\Pi}_{j,\ell}\right) \\
&\overset{(4)}{\underset{(5)}{=}} \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{j,\ell}^*\right)\left(\mathsf{I}_\ell - \tilde{\Pi}_{j,\ell}\right)\left(\mathsf{I}_\ell - \tilde{\Pi}_{j,\ell}\right)\left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{j,\ell}\right) \\
&= \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{j,\ell}^*\right)\left(\mathsf{I}_\ell - \tilde{\Pi}_{j,\ell}\right)\left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{j,\ell}\right).
\end{aligned}$$

□

Lemma

Let $0 \leq i < j < \ell$. *Then*

$$\mathsf{P}_{j,\ell}\left(\mathsf{I}_j - \tilde{\mathsf{T}}_{i,j}\right) = \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{i,\ell}\right)\mathsf{P}_{j,\ell}. \tag{7}$$

Proof.

$$
\begin{aligned}
\mathsf{P}_{j,\ell}\left(\mathsf{I}_j - \tilde{\mathsf{T}}_{i,j}\right) &\overset{(1)}{=} \mathsf{P}_{j,\ell}\left(\mathsf{I}_j - \tilde{\mathsf{T}}_{i,j}\right)\Pi_{j,\ell}\mathsf{P}_{j,\ell} \\
&= \left\{\mathsf{P}_{j,\ell}\Pi_{j,\ell} - \mathsf{P}_{j,\ell}\mathsf{P}_{i,j}\mathsf{T}_{i,j}\Pi_{j,\ell}\right\}\mathsf{P}_{j,\ell} \\
&= \left\{\tilde{\Pi}_{j,\ell} - \mathsf{P}_{i,\ell}\left(\Pi_{i,j} - \mathsf{K}_i^m\Pi_{i,j}\right)\Pi_{j,\ell}\right\}\mathsf{P}_{j,\ell} \\
&= \left\{\tilde{\Pi}_{j,\ell} - \mathsf{P}_{i,\ell}\Pi_{i,j} + \mathsf{P}_{i,\ell}\mathsf{K}_i^m\Pi_{i,\ell}\right\}\mathsf{P}_{j,\ell} \\
&= \left\{\tilde{\Pi}_{j,\ell} - \tilde{\mathsf{T}}_{i,\ell}\right\}\mathsf{P}_{j,\ell} \\
&= \mathsf{P}_{j,\ell}\Pi_{j,\ell}\mathsf{P}_{j,\ell} - \tilde{\mathsf{T}}_{i,\ell}\mathsf{P}_{j,\ell} \\
&\overset{(1)}{=} \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{i,\ell}\right)\mathsf{P}_{j,\ell}.
\end{aligned}
$$

□

Corollary

Let $0 \le i < j < \ell$. *Then*

$$\mathsf{P}_{j,\ell}\left(\mathsf{I}_j - \tilde{\Pi}_{i,j}\right) = \left(\mathsf{I}_\ell - \tilde{\Pi}_{i,\ell}\right)\mathsf{P}_{j,\ell}. \tag{8}$$

Multigrid Error Transfer Matrices in Multiplicative Forms

Now, using the definitions and properties of the multilevel matrices, we can rewrite the error transfer matrices of some of the common multigrid algorithms.

Theorem

Let V_ℓ, $\mathcal{T}_\ell$, and $a(\cdot,\cdot)$ be defined as usual. Consider the symmetric V-cycle algorithm: $m = m_1 = m_2$ and $p = 1$. The error transfer matrix can be expressed as

$$\begin{aligned} \mathsf{E}_\ell \quad = \quad & (\mathsf{K}_\ell^*)^m \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{\ell-1,\ell}^*\right) \times \cdots \times \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{1,\ell}^*\right) \left(\mathsf{I}_\ell - \tilde{\Pi}_{0,\ell}^*\right) \\ & \times \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{1,\ell}\right) \times \cdots \times \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{\ell-1,\ell}\right) (\mathsf{K}_\ell)^m, \end{aligned} \tag{9}$$

for all $\ell \geq 1$.

Proof.

Define the quantity

$$\mathsf{M}_{j,\ell} := \mathsf{I}_\ell - \tilde{\Pi}_{j,\ell} + \mathsf{P}_{j,\ell}\mathsf{E}_j\Pi_{j,\ell},$$

for any $0 \le j < \ell$. Observe that, when $j = 0$,

$$\mathsf{M}_{0,\ell} = \mathsf{I}_\ell - \tilde{\Pi}_{\ell,0},$$

since $\mathsf{E}_0 = 0$. Now, by Theorem **??**,

$$\mathsf{M}_{j,\ell} = \mathsf{I}_\ell - \tilde{\Pi}_{j,\ell} + \mathsf{P}_{j,\ell}\left(\mathsf{K}_j^*\right)^m\left(\mathsf{I}_j - \tilde{\Pi}_{j-1,j} + \mathsf{P}_{j-1,j}\mathsf{E}_{j-1}\Pi_{j-1,j}\right)\mathsf{K}_j^m\Pi_{j,\ell}. \quad (10)$$

In other words,

$$\mathsf{M}_{j,\ell} = \mathsf{I}_\ell - \tilde{\Pi}_{j,\ell} + \mathsf{P}_{j,\ell}\left(\mathsf{K}_j^*\right)^m\mathsf{M}_{j-1,j}\mathsf{K}_j^m\Pi_{j,\ell}.$$

Proof (Cont.)

Now, observe that

$$
\begin{aligned}
\mathsf{P}_{j,\ell}\left(\mathsf{K}_j^*\right)^m &\overset{(1)}{=} \mathsf{P}_{j,\ell}\left(\mathsf{K}_j^*\right)^m \Pi_{j,\ell}\mathsf{P}_{j,\ell} \\
&= \left(\mathsf{P}_{j,\ell}\Pi_{j,\ell} - \mathsf{P}_{j,\ell}\Pi_{j,\ell} + \mathsf{P}_{j,\ell}\left(\mathsf{K}_j^*\right)^m \Pi_{j,\ell}\right)\mathsf{P}_{j,\ell} \\
&= \left(\mathsf{P}_{j,\ell}\Pi_{j,\ell} - \tilde{\mathsf{T}}_{j,\ell}^*\right)\mathsf{P}_{j,\ell} \\
&= \mathsf{P}_{j,\ell}\Pi_{j,\ell}\mathsf{P}_{j,\ell} - \tilde{\mathsf{T}}_{j,\ell}^*\mathsf{P}_{j,\ell} \\
&\overset{(1)}{=} \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{j,\ell}^*\right)\mathsf{P}_{j,\ell}. \qquad (11)
\end{aligned}
$$

Proof (Cont.)

Similarly,

$$\begin{aligned}
\mathsf{K}_j^m \Pi_{j,\ell} &= \Pi_{j,\ell} - \Pi_{j,\ell} + \mathsf{K}_j^m \Pi_{j,\ell} \\
&= \Pi_{j,\ell} - \mathsf{T}_{j,\ell} \\
&\overset{(1)}{=} \Pi_{j,\ell} \mathsf{P}_{j,\ell} \left(\Pi_{j,\ell} - \mathsf{T}_{j,\ell} \right) \\
&= \Pi_{j,\ell} \left(\mathsf{P}_{j,\ell} \Pi_{j,\ell} - \tilde{\mathsf{T}}_{j,\ell} \right) \\
&= \Pi_{j,\ell} \mathsf{P}_{j,\ell} \Pi_{j,\ell} - \Pi_{j,\ell} \tilde{\mathsf{T}}_{j,\ell} \\
&\overset{(1)}{=} \Pi_{j,\ell} \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{j,\ell} \right) .
\end{aligned} \tag{12}$$

Proof (Cont.)

Putting (10) – (12) together, we have

$$
\begin{aligned}
\mathsf{M}_{j,\ell} &= \mathsf{I}_\ell - \tilde{\Pi}_{j,\ell} + \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}^*_{j,\ell}\right) \mathsf{P}_{j,\ell} \left\{\mathsf{I}_j - \tilde{\Pi}_{j-1,j} + \mathsf{P}_{j-1,j}\mathsf{E}_{j-1}\Pi_{j-1,j}\right\} \\
&\quad \times \Pi_{j,\ell}\left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{j,\ell}\right) \\
&\overset{(6)}{=} \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}^*_{j,\ell}\right)\left\{\mathsf{I}_\ell - \tilde{\Pi}_{j,\ell} + \mathsf{P}_{j,\ell}\left(\mathsf{I}_j - \tilde{\Pi}_{j-1,j} + \mathsf{P}_{j-1,j}\mathsf{E}_{j-1}\Pi_{j-1,j}\right)\Pi_{j,\ell}\right\} \\
&\quad \times \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{j,\ell}\right) \\
&= \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}^*_{j,\ell}\right)\left\{\mathsf{I}_\ell - \tilde{\Pi}_{j,\ell} + \tilde{\Pi}_{j,\ell} - \tilde{\Pi}_{j-1,\ell} + \mathsf{P}_{j-1,\ell}\mathsf{E}_{j-1}\Pi_{j-1,\ell}\right\} \\
&\quad \times \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{j,\ell}\right).
\end{aligned}
$$

Proof (Cont.)

Or, equivalently,

$$\mathsf{M}_{j,\ell} = \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}^*_{j,\ell}\right) \mathsf{M}_{j-1,\ell} \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{j,\ell}\right).$$

Therefore

$$\begin{aligned}
\mathsf{M}_{\ell-1,\ell} &= \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}^*_{\ell-1,\ell}\right) \mathsf{M}_{\ell-2,\ell} \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{\ell-1,\ell}\right) \\
&= \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}^*_{\ell-1,\ell}\right) \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}^*_{\ell-2,\ell}\right) \mathsf{M}_{\ell-3,\ell} \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{\ell-3,\ell}\right) \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{\ell-1,\ell}\right) \\
&\ \vdots \\
&= \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}^*_{\ell-1,\ell}\right) \times \cdots \times \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}^*_{1,\ell}\right) \left(\mathsf{I}_\ell - \tilde{\Pi}_{\ell,0}\right) \\
&\quad \times \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{1,\ell}\right) \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{2,\ell}\right) \times \cdots \times \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{\ell-1,\ell}\right)
\end{aligned}$$

But recall that

$$\mathsf{E}_\ell = \left(\mathsf{K}^*_\ell\right)^m M_{\ell-1,\ell} \mathsf{K}^m_\ell.$$

The result is proven. □

Corollary

For the one-sided V-cycle with only pre-smoothing ($p = 1, m := m_1 > 0$ and $m_2 = 0$), we have

$$\mathsf{E}_\ell^{\mathrm{pre}} = \left(\mathsf{I}_\ell - \tilde{\Pi}_{\ell,0}\right)\left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{1,\ell}\right)\left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{2,\ell}\right) \times \cdots \times \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{\ell-1,\ell}\right)\mathsf{K}_\ell^m .$$

For the algorithm with only post-smoothing ($p = 1, m := m_2 > 0$ and $m_1 = 0$), we have

$$\mathsf{E}_\ell^{\mathrm{post}} = (\mathsf{K}_\ell^*)^m \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{\ell-1,\ell}^*\right) \times \cdots \times \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{1,\ell}^*\right)\left(\mathsf{I}_\ell - \tilde{\Pi}_{\ell,0}\right).$$

Therefore, for the symmetric V-cycle,

$$\mathsf{E}_\ell = \mathsf{E}_\ell^{\mathrm{post}} \times \mathsf{E}_\ell^{\mathrm{pre}}.$$

Furthermore,

$$\left(\mathsf{E}_\ell^{\mathrm{post}}\right)^* = \mathsf{E}_\ell^{\mathrm{pre}}.$$

Clearly,

$$\mathsf{E}_\ell^* = \mathsf{E}_\ell = \left(\mathsf{E}_\ell^{\mathrm{pre}}\right)^* \mathsf{E}_\ell^{\mathrm{pre}}$$

is SPSD.

Theorem

Both of the one-sided V-cycle methods converge, for any $m > 0$, provided Richardson's method is used for smoothing.

Proof.

We have shown, in Theorem **??**, that there is some C_0 such that

$$\left\| \mathsf{E}_\ell \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell} \le \frac{C_0}{m + C_0} \left\| \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell},$$

for all $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$. We wish the prove that

$$\left\| \mathsf{E}_\ell^{\mathrm{pre}} \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell} \le \gamma,$$

for some $0 \le \gamma < 1$. Observe that

$$\begin{aligned}
\left\| \mathsf{E}_\ell^{\mathrm{pre}} \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell}^2 &= \left(\mathsf{E}_\ell^{\mathrm{pre}} \boldsymbol{u}_\ell, \mathsf{E}_\ell^{\mathrm{pre}} \boldsymbol{u}_\ell \right)_{\mathsf{A}_\ell} \\
&= \left(\boldsymbol{u}_\ell, \left(\mathsf{E}_\ell^{\mathrm{pre}}\right)^* \mathsf{E}_\ell^{\mathrm{pre}} \boldsymbol{u}_\ell \right)_{\mathsf{A}_\ell} \\
&= \left(\boldsymbol{u}_\ell, \mathsf{E}_\ell \boldsymbol{u}_\ell \right)_{\mathsf{A}_\ell} \\
&\overset{\text{C.S.}}{\le} \left\| \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell} \left\| \mathsf{E}_\ell \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell} \\
&\le \left\| \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell} \frac{C_0}{m + C_0} \left\| \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell} \\
&\le \frac{C_0}{m + C_0} \left\| \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell}^2 .
\end{aligned}$$

Proof (Cont.)

Thus

$$\|\mathsf{E}_\ell^{\mathrm{pre}} \boldsymbol{u}_\ell\|_{\mathsf{A}_\ell} \leq \sqrt{\frac{C_0}{m + C_0}} \, \|\boldsymbol{u}_\ell\|_{\mathsf{A}_\ell} .$$

□

Theorem

For the W-cycle algorithm with only pre-smoothing ($m := m_1 > 0, m_2 = 0$, $p = 2$), the error transfer matrix may be expressed as

$$\mathsf{E}_\ell^{w,\mathrm{pre}} = \mathsf{F}_\ell \mathsf{E}_\ell^{\mathrm{pre}},$$

where $\mathsf{E}_\ell^{\mathrm{pre}}$ is defined above, and $\mathsf{F}_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$ is a matrix with

$$\|\mathsf{F}_\ell\|_{\mathsf{A}_\ell} \leq 1.$$

consequently the one-sided W-cycle method with pre-smoothing converges for any $m > 0$.

Proof.

Exercise. □

Theorem

For the symmetric W-cycle algorithm ($p = 2$ and $m := m_1 = m_2$) the error transfer matrix is

$$\mathsf{E}_\ell^W = \left(\mathsf{E}_\ell^{\mathrm{pre}}\right)^* \mathsf{D}_\ell \mathsf{E}_\ell^{\mathrm{pre}}$$

where

$$\mathsf{E}_\ell^{\mathrm{pre}} = \left(\mathsf{I}_\ell - \tilde{\Pi}_{\ell,0}\right)\left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{1,\ell}\right)\left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{2,\ell}\right) \times \cdots \times \left(\mathsf{I}_\ell - \tilde{\mathsf{T}}_{\ell-1,\ell}\right) \mathsf{K}_\ell^m.$$

and

$$\left\|\mathsf{D}_\ell\right\|_{\mathsf{A}_\ell} \le 1, \quad \forall\, \ell \ge 1.$$

The algorithm converges if the symmetric V-cycle algorithm converges with the uniform contraction $0 < \gamma < 1$, i.e.

$$\left\|\mathsf{E}_\ell \boldsymbol{u}_\ell\right\|_{\mathsf{A}_\ell} \le \gamma \left\|\boldsymbol{u}_\ell\right\|_{\mathsf{A}_\ell},$$

for all $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$, with $\mathsf{E}_\ell = \left(\mathsf{E}_\ell^{\mathrm{pre}}\right)^* \mathsf{E}_\ell^{\mathrm{pre}}$. Here γ may (and usually does) depend upon m.

Proof.

Exercise. □