Math 673

# Multigrid Methods: A Mostly Matrix-Based Approach

## Chapter 09: Additive Preconditioners Based on Subspace Decompositions

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# Chapter 09, Part 2 of 2

# Additive Preconditioners Based on Subspace Decompositions

# Hierarchical Basis Preconditioner

Now, we need to connect the spaces $W_j$ to $V_\ell$ where $0 \leq j \leq \ell$. In so doing, we will have the tools to build a preconditioner based on the hierarchical basis. Be careful, the number of indices in this section can get a little overwhelming.

## Proposition

*Let $\mathcal{B}_j^W = \{\phi_{j,i}\}_{i=1}^{m_j}$ and $\mathcal{B}_\ell^V = \{\psi_{\ell,i}\}_{i=1}^{n_\ell}$ be the usual bases for $W_j$ and $V_\ell$, respectively. For each $0 \leq j \leq \ell$, there are unique numbers*

$$q_{j,k,i}^{\ell} \in \mathbb{R}, \quad 1 \leq k \leq n_\ell, \quad 1 \leq i \leq m_j,$$

*such that*

$$\phi_{j,i} = \sum_{k=1}^{n_\ell} q_{j,k,i}^{\ell} \psi_{\ell,k}. \tag{1}$$

## Proof.

Exercise. □

## Definition (Hierarchical Prolongation Matrix)

Define the matrix $\mathsf{Q}_j^\ell \in \mathbb{R}^{n_\ell \times m_j}$ via

$$\left[\mathsf{Q}_j^\ell\right]_{i,k} := q_{j,k,i}^\ell, \quad 1 \le k \le n_\ell, \quad 1 \le i \le m_j.$$

$\mathsf{Q}_j^\ell$ is called a **hierarchical prolongation matrix**.

## Lemma

*Suppose that* $\mathrm{Q}_j^\ell$ *is a hierarchical prolongation matrix and* $\boldsymbol{w}_j \in \mathbb{R}^{m_j}$ *is the coordinate vector of the function* $w_j \in W_j$ *with respect to the basis* $\mathcal{B}_j^W$*. Then,*

$$\operatorname{rank}(\mathrm{Q}_j^\ell) = m_j,$$

*and the coordinate vector of* $\boldsymbol{w}_j \in V_\ell$ *in the basis* $\mathcal{B}_\ell^V$ *is simply*

$$\mathrm{Q}_j^\ell \boldsymbol{w}_j \in \mathbb{R}^{n_\ell}.$$

## Proof.

Exercise. □

## Remark

*Note that the family of spaces $W_j$ are hierarchical, but are not nested*

$$W_0 \not\subset W_1 \not\subset W_2 \cdots .$$

*Furthermore, it makes no sense to stack the prolongation matrices as we did in the past:*

$$\mathsf{Q}_j^\ell \neq \mathsf{Q}_k^\ell \mathsf{Q}_j^k,$$

*for $j < k < \ell$. In fact, the product on the right hand side is not usually defined.*

## Definition

Define the operator $B_j : W_j \to W_j'$ via

$$B_j[w_j](v_j) := \sum_{r=1}^{m_j} w_j\left(\boldsymbol{N}_{j,r}^{W}\right) v_j\left(\boldsymbol{N}_{j,r}^{W}\right), \quad \forall\ v_j \in W_j.$$

Here, $W_j'$ means the dual space of $W_j$. Let $\mathcal{B}_j^W = \{\phi_{j,i}\}_{i=1}^{m_j}$ be the usual basis for $W_j$. Define the matrix $\mathsf{B}_j \in \mathbb{R}^{m_j \times m_j}$ via

$$\begin{aligned}
[\mathsf{B}_j]_{i,k} &:= B_j[\phi_{j,i}](\phi_{j,k}) \\
&= \sum_{r=1}^{m_j} \phi_{j,i}\left(\boldsymbol{N}_{j,r}^{W}\right) \phi_{j,k}\left(\boldsymbol{N}_{j,r}^{W}\right) \\
&= \sum_{r=1}^{m_j} \delta_{ir}\delta_{rk} = \delta_{ik}. \qquad (2)
\end{aligned}$$

## Definition (Hierarchical Basis Preconditioner)

Suppose that $\mathcal{B}_\ell^V = \{\psi_{\ell,i}\}_{i=1}^{n_\ell}$ is the usual basis for the finite element space $V_\ell$. Let $\mathsf{A}_L \in \mathbb{R}^{n_L \times n_L}$ be the SPD matrix defined via

$$[\mathsf{A}_L]_{i,j} = a(\psi_{L,j}, \psi_{L,i}), \quad 1 \le i,j \le n_L,$$

where

$$a(u,v) = (\nabla u, \nabla v)_{L^2}, \quad \forall\ u,v \in H_0^1(\Omega).$$

The **hierarchical basis preconditioner for** $\mathsf{A}_L$ is defined as

$$\mathsf{B}_H = \sum_{\ell=0}^{L} \mathsf{Q}_\ell^L \mathsf{B}_\ell^{-1} \mathsf{Z}_\ell^L = \sum_{\ell=0}^{L} \mathsf{Q}_\ell^L \mathsf{Z}_\ell^L, \tag{3}$$

where $\mathsf{B}_\ell$ is as in (2), $\mathsf{Q}_\ell^L \in \mathbb{R}^{n_L \times m_\ell}$ is the hierarchical prolongation matrix from the Definition 1 and

$$\mathsf{Z}_\ell^L = \left(\mathsf{Q}_\ell^L\right)^T.$$

## Lemma

*Assumption (SS1) holds for the hierarchical basis decomposition. In particular, for any object*

$$\boldsymbol{u}_L \in \mathbb{R}^{n_L} \overset{\mathcal{B}_L^V}{\leftrightarrow} u_L \in V_L$$

*there exist unique objects*

$$\boldsymbol{w}_\ell \in \mathbb{R}^{m_\ell} \overset{\mathcal{B}_\ell^W}{\leftrightarrow} w_\ell \in W_\ell, \quad 0 \leq \ell \leq L,$$

*such that*

$$\boldsymbol{u} = \sum_{\ell=0}^{L} \mathsf{Q}_\ell^L \boldsymbol{w}_\ell \in \mathbb{R}^{n_L} \overset{\mathcal{B}_L^V}{\leftrightarrow} u_L = \sum_{\ell=0}^{L} w_\ell \in V_L.$$

*Furthermore, the hierarchical basis preconditioner,* $\mathsf{B}_H$*, defined in* (3)*, is SPD.*

## Proof.

This follows from the lemmas on the last slide deck. □

## Remark

*Our goal is now to show that*

$$\lambda_{\min}(\mathsf{B}_H\mathsf{A}_L) \geq C_1\left(1 + |\log(h_L)|^2\right)^{-1},$$

*and*

$$\lambda_{\max}(\mathsf{B}_H\mathsf{A}_L) \leq C_2,$$

*where* $C_1, C_2 > 0$ *are independent of* $L$*, using Theorem on the eigenvalues of the* CA *matrix in the last slide deck. If this is the case*

$$\frac{\lambda_{\max}}{\lambda_{\min}} =: \kappa(\mathsf{B}_H\mathsf{A}_L) \leq \frac{C_2}{C_1}\left(1 + |\log(h_L)|^2\right).$$

*This estimate is quite useful, since the logarithmic dependence on* $h_L$ *is so weak. For example, suppose*

$$h_L = (1/2)^L,$$

*which is entirely reasonable. Then*

$$|\log(h_L)|^2 = L^2\,|\log(1/2)|^2.$$

*Our analysis that follows will only work for* $d = 2$*. In three space dimensions we lose the nice logarithmic dependence in the lower bound.*

Now, we need some technical lemmas. For more details, see chapter 7 of Brenner's book.

## Lemma (Inverse inequality)

*Suppose that $\Omega$ is an open polygonal domain in $\mathbb{R}^d$, $\mathcal{T}_\ell$, $0 \le \ell \le L$ is a nested family of triangulations of $\Omega$, and $V_\ell$, $0 \le \ell \le L$ are the associated piecewise-linear finite element spaces. Assume that $1 \le q \le \infty$. There exists a $C > 0$, independent of $\ell$ such that*

$$\|v\|_{H^1(K)} \le C h_\ell^{-1+d/2-d/q} \|v\|_{L^q(K)}, \tag{4}$$

*for all $K \in \mathcal{T}_\ell$ and all $v \in V_\ell$.*

## Proof.

See section 5.3 of Brenner's book. □

In two space dimensions $H^1 \hookrightarrow L^p$, for any $1 \leq p < \infty$. We cannot quite get control for $p = \infty$. But, if the function space is finite dimensional we can almost get control of the $p = \infty$ case. Here is the result.

## Lemma

*Suppose that $\Omega$ is an open polygonal domain in $\mathbb{R}^2$, $\mathcal{T}_\ell$, $0 \leq \ell \leq L$ is a nested family of triangulations of $\Omega$, and $V_\ell$, $0 \leq \ell \leq L$ are the associated piecewise-linear finite element spaces. There exists a $C > 0$, independent of $\ell$, such that*

$$\|v_\ell\|_{L^\infty(\Omega)} \leq C(1 + \log(h_\ell)) \left|v_\ell\right|_{H^1(\Omega)},$$

*for any $v_\ell \in V_\ell$.*

## Proof.

See section 4.9 of Brenner's book. □

## Lemma

*Suppose that $\mathcal{I}_\ell : C(\overline{\Omega}) \to V_\ell$, $0 \le \ell \le L$, is the Lagrange nodal interpolation operator, and $\mathcal{I}_{-1} \equiv 0$. There exists a constant $C > 0$, independent of $\ell$, such that*

$$\left\| \mathcal{I}_\ell u_L - \mathcal{I}_{\ell-1} u_L \right\|_{L^2(\Omega)} \le C h_\ell \left( 1 + \sqrt{L - \ell} \right) |u_L|_{H^1(\Omega)} . \tag{5}$$

*for all $u_L \in V_L$, where $\Omega \subset \mathbb{R}^2$ (i.e. d=2).*

## Proof.

Define the piecewise constant function $\bar{u}_L^\ell$ such that

$$\bar{u}_L^\ell|_K := \frac{1}{|K|}\int_K u_L(\boldsymbol{x})d\boldsymbol{x}, \quad \forall\ K \in \mathcal{T}_\ell.$$

Then

$$\begin{aligned}
\left\|\mathcal{I}_\ell u_L - \mathcal{I}_{\ell-1} u_L\right\|_{L^2(\Omega)}^2 &= \left\|\mathcal{I}_\ell u_L - \mathcal{I}_{\ell-1}\left[\mathcal{I}_\ell[u_L]\right]\right\|_{L^2(\Omega)}^2 \\
&= Ch_\ell^2 \left|\mathcal{I}_\ell[u_L]\right|_{H^1(\Omega)} \\
&= Ch_\ell^2 \sum_{K\in\mathcal{T}_\ell} \left|\mathcal{I}_\ell[u_L]\right|_{H^1(K)}^2 \\
&= Ch_\ell^2 \sum_{K\in\mathcal{T}_\ell} \left|\mathcal{I}_\ell u_L - \bar{u}_L^\ell\right|_{H^1(K)}^2 \\
&\overset{(4)}{\leq} Ch_\ell^2 \sum_{K\in\mathcal{T}_\ell} \left\|\mathcal{I}_\ell u_L - \bar{u}_L^\ell\right\|_{L^\infty(K)}^2 \\
&\leq Ch_\ell^2 \sum_{K\in\mathcal{T}_\ell} \left\|u_L - \bar{u}_L^\ell\right\|_{L^\infty(K)}^2 .
\end{aligned}$$

## Proof (Cont.)

Now notice that

$$h_\ell = h_0 2^{-\ell} \quad 1 \leq \ell \leq L.$$

So,

$$\log(h_\ell / h_L) = \log(2^{L-\ell}) = (L-\ell)\log(2).$$

The result follows. □

## Lemma

*There is some constant $C_1 > 0$ such that*

$$\lambda_{\min}(\mathsf{B}_H \mathsf{A}_L) \geq \frac{C_1}{1 + |\log(h_L)|^2}. \tag{6}$$

## Proof.

By definition for any $w_{\ell,1}, w_{\ell,2} \in W_\ell$

$$C_\ell[w_{\ell,1}](w_{\ell,2}) = \sum_{i=1}^{m_\ell} w_{\ell,1}(\boldsymbol{N}^W_{\ell,i}) w_{\ell,2}(\boldsymbol{N}^W_{\ell,i}).$$

Let

$$\boldsymbol{w}_{\ell,\alpha} \in \mathbb{R}^{m_\ell} \overset{\mathcal{B}^W_\ell}{\leftrightarrow} w_{\ell,\alpha}, \quad \alpha = 1, 2.$$

Then

$$\begin{aligned}
(\mathsf{C}_\ell \boldsymbol{w}_{\ell,1}, \boldsymbol{w}_{\ell,2})_\ell &= \sum_{i=1}^{m_\ell} [\boldsymbol{w}_{\ell,1}]_i [\boldsymbol{w}_{\ell,2}]_i \\
&= \sum_{i=1}^{m_\ell} w_{\ell,1}(\boldsymbol{N}^W_{\ell,i}) w_{\ell,2}(\boldsymbol{N}^W_{\ell,i}) \\
&= C_\ell [w_{\ell,1}] (w_{\ell,2}) \\
&= C_\ell [w_{\ell,2}] (w_{\ell,1}) \\
&=: \langle w_{\ell,1}, w_{\ell,2} \rangle_{\mathsf{C}_\ell} .
\end{aligned}$$

## Proof (Cont.)

This is like a mass-lumping inner product. All that is missing is a factor of ${h_\ell}^2$. Using a previous lemma, there are constant $\tilde{C}_1, \tilde{C}_2 > 0$ such that, for all $0 \leq \ell \leq L$,

$$\tilde{C}_1 {h_\ell}^2 \left\langle w_{\ell,\alpha}, w_{\ell,\alpha} \right\rangle_{\mathsf{C}_\ell} \leq \|w_{\ell,\alpha}\|^2_{L^2(\Omega)} \leq \tilde{C}_2 {h_\ell}^2 \left\langle w_{\ell,\alpha}, w_{\ell,\alpha} \right\rangle_{\mathsf{C}_\ell} \tag{7}$$

Therefore, for any $w_\ell \in W_\ell \overset{\mathcal{B}_\ell^W}{\leftrightarrow} \boldsymbol{w}_\ell \in \mathbb{R}^{m_\ell}$,

$$\begin{aligned} \left(\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell\right)_\ell &= {h_\ell}^{-2} {h_\ell}^2 \left\langle w_\ell, w_\ell \right\rangle_{\mathsf{C}_\ell} \\ &\overset{(7)}{\leq} \tilde{\tilde{C}}_1 {h_\ell}^{-2} \|w_\ell\|^2_{L^2(\Omega)} \\ &\overset{\mathcal{I}_{-1}\equiv 0}{=} \tilde{\tilde{C}}_1 {h_\ell}^{-2} \|w_\ell - \mathcal{I}_{\ell-1} w_\ell\|^2_{L^2(\Omega)} \\ &\leq \tilde{\tilde{C}}_2 \left|w_\ell\right|^2_{H^1(\Omega)} \quad \text{(interp. error)} \\ &\leq \tilde{\tilde{C}}_3 {h_\ell}^{-2} \|w_\ell\|^2_{L^2(\Omega)} \quad \text{(inverse ineq.)} \\ &\overset{(7)}{\leq} \tilde{\tilde{C}}_4 \left(\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell\right)_\ell . \end{aligned} \tag{8}$$

## Proof (Cont.)

Therefore, there are constants $\tilde{\tilde{C}}_5, \tilde{\tilde{C}}_6 > 0$ such that we have the equivalence

$$\tilde{\tilde{C}}_5 \sum_{\ell=0}^{L} |w_\ell|^2_{H^1(\Omega)} \leq \sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell \leq \tilde{\tilde{C}}_6 \sum_{\ell=0}^{L} |w_\ell|^2_{H^1(\Omega)}, \tag{9}$$

for any $w_\ell \in W_\ell$, in general. Now, let $u_L \in V_L$ be given and

$$u_L = \sum_{\ell=0}^{L} w_\ell, \quad \exists!\, w_\ell \in W_\ell, \quad 0 \leq \ell \leq L.$$

Recall that

$$w_\ell = \mathcal{I}_\ell u_L - \mathcal{I}_{\ell-1} u_L, \quad 1 \leq \ell \leq L,$$

and

$$w_0 = \mathcal{I}_0 u_L.$$

## Proof (Cont.)

Then, from (8)

$$\begin{aligned}
\sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell &\leq \tilde{\tilde{C}}_1 \sum_{\ell=0}^{L} {h_\ell}^{-2} \|w_\ell\|^2_{L^2(\Omega)} \\
&\overset{(5)}{\leq} C \sum_{\ell=0}^{L} \left(1 + \sqrt{L-\ell}\right)^2 |u_L|^2_{H^1(\Omega)} \\
&\leq C \sum_{\ell=0}^{L} (1 + L - \ell) \, |u_L|^2_{H^1(\Omega)} \\
&\leq C \left(1 + L + L^2\right) |u_L|^2_{H^1(\Omega)} \\
&\overset{L\geq 1}{\leq} CL^2 \, |u_L|^2_{H^1(\Omega)} .
\end{aligned}$$

## Proof (Cont.)

But

$$\begin{aligned} |u_L|^2_{H^1(\Omega)} &= (\nabla u_L, \nabla u_L) \\ &= a(u_L, u_L) \\ &= (A_L \vec{u}_L, \vec{u}_L). \end{aligned}$$

And

$$\begin{aligned} |\log(h_L)|^2 &= \left|\log(h_0 2^{-L})\right|^2 \\ &= |\log(h_0) - L\log(2)|^2 \\ &= \log^2(h_0) - 2\log(h_0) L \log(2) + L^2 \log^2(2). \end{aligned}$$

So

$$L^2 \leq C\left(1 + |\log(h_L)|^2\right), \quad \exists\, C > 0.$$

## Proof (Cont.)

Thus,

$$\sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell) \leq C \left(1 + |\log(h_L)|^2\right) (A\vec{u}_L, \vec{u}_L),$$

and it follows from the theorem on the eigenvalues of CA that

$$\lambda_{\min}(\mathsf{B}_H \mathsf{A}_L) \geq C_1 \left(1 + |\log(h_L)|^2\right)^{-1}.$$

□

Next, we need a little technical lemma, a kind of convolution result.

## Lemma

*Let* $a_j, b_j \geq 0, -\infty < j < \infty$, *with*

$$s_1 := \sum_{j=-\infty}^{\infty} a_j \leq \infty,$$

*and*

$$s_2 := \sum_{j=-\infty}^{\infty} b_j \leq \infty.$$

*Then*

$$\sum_{j=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} a_{j-k} b_k \right)^2 \leq s_1^2 s_2. \tag{10}$$

## Proof.

Exercise. □

## Lemma

*For any* $v_\ell \in V_\ell$ *and* $v_k \in V_k$, $0 \leq \ell \leq k \leq L$, *and d=2, there is a constant* $C > 0$ *such that*

$$\int_\Omega \nabla v_\ell \cdot \nabla v_k \boldsymbol{dx} \leq 2^{(\ell-k)/2} C \left| v_\ell \right|_{H^1(\Omega)} \left( h_k^{-1} \left\| v_k \right\|_{L^2(\Omega)} \right). \tag{11}$$

## Proof.

For any $K \in \mathcal{T}_h$, since $\Delta v_\ell|_K \equiv 0$,

$$
\begin{aligned}
\int_K \nabla v_\ell \cdot \nabla v_k d\mathbf{x} &= \int_{\partial K} \frac{\partial v_\ell}{\partial n} v_k ds \\
&\leq C h_\ell^{-1} |v_\ell|_{H^1(K)} \int_{\partial K} |v_k| ds \\
&\leq \left( C h_\ell^{-1} |v_\ell|_{H^1(K)} \right) \left( h_k \sum_{\mathbf{N}_k \in \partial K} |v_k(\mathbf{N}_k)| \right) \\
&\overset{\text{C.S.}}{\leq} \left( C h_\ell^{-1} |v_\ell|_{H^1(K)} \right) \left( h_k \left( \frac{h_\ell}{h_k} \right)^{1/2} \left( \sum_{\mathbf{N}_k \in \partial K} |v_k(\mathbf{N}_k)|^2 \right)^{1/2} \right) \\
&\overset{(??)}{\leq} C \left( \frac{h_k}{h_\ell} \right)^{1/2} |v_\ell|_{H^1(K)} h_k^{-1} \|v_k\|_{L^2(K)} .
\end{aligned}
$$

## Proof (Cont.)

Thus

$$
\begin{aligned}
\int_\Omega \nabla v_\ell \cdot \nabla v_k \boldsymbol{dx} &= \sum_{K \in \mathcal{T}_h} \int_K \nabla v_\ell \cdot \nabla v_k \boldsymbol{dx} \\
&\leq C 2^{(\ell-k)/2} \sum_{K \in \mathcal{T}_h} |v_\ell|_{H^1(K)} h_k^{-1} \|v_k\|_{L^2(K)} \\
&\overset{\text{C.S.}}{\leq} C 2^{(\ell-k)/2} |v_\ell|_{H^1(\Omega)} h_k^{-1} \|v_k\|_{L^2(\Omega)} .
\end{aligned}
$$

□

## Lemma (Strengthened Cauchy-Schwarz Inequality)

*For any* $w_\ell \in W_\ell$ *and* $w_k \in W_k$, $0 \leq \ell \leq k \leq L$, *there is a constant* $C > 0$ *such that*

$$\int_\Omega \nabla w_\ell \cdot \nabla w_k d\mathbf{x} \leq 2^{(\ell-k)/2} C \left| w_\ell \right|_{H^1(\Omega)} \left| w_k \right|_{H^1(\Omega)} . \tag{12}$$

## Proof.

Observe that

$$w_k = w_k - \mathcal{I}_{\ell-1}(w_k).$$

We use the interpolation error estimate

$$\left\| w_k - \mathcal{I}_{k-1}(w_k) \right\|_{L^2(\Omega)} \leq C h_k \left| w_k \right|_{H^1(\Omega)},$$

to conclude that

$$\left\| w_k \right\|_{L^2(\Omega)} \leq C h_k \left| w_k \right|_{H^1(\Omega)}.$$

Now, we use the last result. Since $w_\ell \in V_\ell$ and $w_k \in V_k$,

$$\begin{aligned} \int_\Omega \nabla w_\ell \cdot \nabla w_k \boldsymbol{dx} &\leq C 2^{(\ell-k)/2} \left| w_\ell \right|_{H^1(\Omega)} h_k^{-1} \left\| w_k \right\|_{L^2(\Omega)} \\ &\leq 2^{(\ell-k)/2} C \left| w_\ell \right|_{H^1(\Omega)} \left| w_k \right|_{H^1(\Omega)} \end{aligned}$$

$\square$

## Lemma

*There is a constant $C_2 > 0$ such that*

$$\lambda_{\max}(\mathsf{B}_H \mathsf{A}_L) \leq C_2,$$

*independent of $L$.*

## Proof.

Let $v_L \in V_L$ be arbitrary.

$$v_L \in V_L \overset{\mathcal{B}_L}{\leftrightarrow} \boldsymbol{v} \in \mathbb{R}^n.$$

There exist unique $w_\ell \in W_\ell \overset{\mathcal{B}^W_\ell}{\leftrightarrow} \boldsymbol{w}_\ell \in \mathbb{R}^{m_\ell}$ such that

$$v_L = \sum_{\ell=0}^{L} w_\ell \leftrightarrow \boldsymbol{v} = \sum_{\ell=0}^{L} Q_\ell^L \boldsymbol{w}_\ell.$$

Then

$$\begin{aligned}
(\boldsymbol{v}, \boldsymbol{v})_{\mathsf{A}_L} &= (\boldsymbol{v}, A_L \boldsymbol{v}) \\
&= a(\boldsymbol{v}, \boldsymbol{v}) \\
&= a\left(\sum_{\ell=0}^{L} w_\ell, \sum_{k=0}^{L} w_k\right) \\
&= \int_\Omega \left(\nabla \sum_{\ell=0}^{L} w_\ell\right) \cdot \left(\nabla \sum_{k=0}^{L} w_k\right) d\boldsymbol{x}
\end{aligned}$$

## Proof (Cont.)

$$
\begin{aligned}
&= \sum_{\ell,k=0}^{L} \int_\Omega \nabla w_\ell \cdot \nabla w_k \, d\boldsymbol{x} \\
&\overset{(12)}{\leq} C \sum_{\ell,k=0}^{L} 2^{-|\ell-k|/2} \left| w_\ell \right|_{H^1(\Omega)} \left| w_k \right|_{H^1(\Omega)} \\
&\leq C \sum_{\ell=0}^{L} \left( \sum_{k=0}^{L} 2^{-|\ell-k|/2} \left| w_k \right|_{H^1(\Omega)} \right) \left| w_\ell \right|_{H^1(\Omega)} \qquad (13) \\
&\overset{\text{C.S.}}{\leq} C \left\{ \sum_{\ell=0}^{L} \left( \sum_{k=0}^{L} 2^{-|\ell-k|/2} \left| w_k \right|_{H^1(\Omega)} \right)^2 \right\}^{1/2} \left\{ \sum_{\ell=0}^{L} \left| w_\ell \right|_{H^1(\Omega)}^2 \right\}^{1/2} \\
&\overset{(10)}{\leq} C \left\{ \sum_{\ell=0}^{L} \left| w_\ell \right|_{H^1(\Omega)}^2 \right\}^{1/2} \left\{ \sum_{\ell=0}^{L} \left| w_\ell \right|_{H^1(\Omega)}^2 \right\}^{1/2} \\
&= C \sum_{\ell=0}^{L} \left| w_\ell \right|_{H^1(\Omega)}^2 \overset{(9)}{\leq} C_2 \sum_{\ell=0}^{L} (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{C_\ell} .
\end{aligned}
$$

## Proof (Cont.)

Recall that, since decomposition are unique

$$\begin{aligned}
\lambda_{\max}(\mathsf{B}_H \mathsf{A}_L) &\overset{(??)}{=} \max_{\boldsymbol{u}\in\mathbb{R}^n_*} \frac{(\boldsymbol{u},\boldsymbol{u})_{\mathsf{A}_L}}{\sum_{\ell=0}^{L}(\boldsymbol{w}_\ell,\boldsymbol{w}_\ell)_{\mathsf{C}_\ell}} \\
&\overset{(13)}{=} \max_{\boldsymbol{u}\in\mathbb{R}^n_*} \frac{\mathbb{C}_2\sum_{\ell=0}^{L}(\boldsymbol{w}_\ell,\boldsymbol{w}_\ell)_{\mathsf{C}_\ell}}{\sum_{\ell=0}^{L}(\boldsymbol{w}_\ell,\boldsymbol{w}_\ell)_{\mathsf{C}_\ell}} \\
&\leq C_2.
\end{aligned}$$

$\square$

## Theorem

*There is a constant $C > 0$ independent of $L$, such that*

$$\kappa(\mathsf{B}_H\mathsf{A}_L) = \frac{\lambda_{\max}(\mathsf{B}_H\mathsf{A}_L)}{\lambda_{\min}(\mathsf{B}_H\mathsf{A}_L)} \leq C\left(1 + |\log(h_L)|^2\right). \tag{14}$$

*independent of $L$.*

## Proof.

The result follows from Lemma 9 and 12 $\square$

# The BPX Preconditioner

# The BPX Preconditioner

The BPX preconditioner has a slightly better perforance than the hierarchical basis preconditioner, in the sense that the logarithmic dependence on $h_L$ can be removed. For this method we choose

$$W_\ell := V_\ell, \quad 0 \le \ell \le L.$$

Thus $W_L = V_L$ and

$$m_\ell = n_\ell, \quad 0 \le \ell \le L.$$

## Definition

Define the operator $C_\ell : V_\ell \to V_\ell'$ via

$$\mathcal{C}_\ell \left[v_{\ell,1}\right] \left(v_{\ell,2}\right) = \sum_{i=1}^{n_\ell} v_{\ell,1}(\boldsymbol{N}_{\ell,i}^W) v_{\ell,2}(\boldsymbol{N}_{\ell,i}^W).$$

The matrix $\mathsf{C}_\ell \in \mathbb{R}^{m_\ell \times m_\ell}$ is defined as

$$\left[\mathsf{C}_\ell\right]_{j,k} = \mathcal{C}_\ell \left[\phi_{\ell,j}\right] \left(\phi_{\ell,k}\right) = \delta_{j,k}, \quad 1 \leq j, k \leq n_\ell,$$

where $\mathcal{B}_\ell = \{\phi_{\ell,j}\}_{j=1}^{n_\ell}$ is the Lagrange nodal basis for the piecewise linear FE space $V_\ell, 0 \leq \ell \leq L$. The BPX preconditioner is

$$\mathcal{C}_{BPX} := \sum_{\ell=0}^{L} \mathsf{P}_\ell^L \mathsf{C}_\ell^{-1} \mathcal{R}_\ell^L = \sum_{\ell=0}^{L} \mathsf{P}_\ell^L \mathcal{R}_\ell^L, \tag{15}$$

where $\mathsf{P}_\ell^L \in \mathbb{R}^{n \times n_\ell}$ is the standard prolongation matrix from Chapter 6 and $\mathcal{R}_\ell^L = \left(\mathsf{P}_\ell^L\right)^T$.

## Lemma

*Assumption (SS1) holds for the BPX framework, i.e., for every $u_L \in V_L$, there exists $v_\ell \in V_\ell, 0 \leq \ell \leq L$, such that*

$$u_L = \sum_{\ell=0}^{L} v_\ell,$$

*or, equivalently*

$$\boldsymbol{u} = \sum_{\ell=0}^{L} \mathsf{P}_\ell^I \boldsymbol{v}_\ell,$$

*with*

$$V_\ell \ni v_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^n,$$

*and*

$$V_L \ni u_L \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{u} \in \mathbb{R}^n.$$

**Proof.**

This is trivial because of the nestedness of the the spaces

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{L-1} \subset V_l.$$

□

**Remark**

*Note that the decomposition is no longer unique.*

## Lemma

*For any $v_j \in V_j, v_\ell \in V_\ell$,*

$$\int_\Omega \nabla v_j \cdot \nabla v_\ell dx \leq C 2^{-|j-\ell|/2} \left( h_j^{-1} \|v_j\|_{L^2(\Omega)} \right) \left( h_\ell{}^{-1} \|v_\ell\|_{L^2(\Omega)} \right), \qquad (16)$$

*for some $C > 0$.*

## Proof.

This is follows from (11) and the inverse inequality

$$|v_j|_{H^1(\Omega)} \leq c h_j^{-1} \|v_j\|_{L^2(\Omega)}.$$

□

## Lemma

*For some $C_2 > 0$ that is independent of $L$,*

$$\lambda_{\max}\left(B_{BPX}A_L\right) \leq C_2.$$

*for some $C > 0$.*

**Proof.**

Let $u_L \in V_L$ be arbitrary. There exists $v_\ell \in V_\ell, 0 \leq \ell \leq L$, such that

$$u_L = \sum_{\ell=0}^{L} v_\ell,$$

or

$$\boldsymbol{u} = \sum_{\ell=0}^{L} \mathsf{P}_\ell^I \boldsymbol{v}_\ell.$$

## Proof (Cont.)

The decomposition is not unique, however. Then

$$
\begin{aligned}
(\boldsymbol{u}, \boldsymbol{u})_{A_L} &= (\boldsymbol{u}, A_L \boldsymbol{u}) \\
&= a(\boldsymbol{u}, \boldsymbol{u}) \\
&= a\left(\sum_{j=0}^{L} v_j, \sum_{\ell=0}^{L} v_\ell\right) \\
&= \sum_{\ell,j=0}^{L} a(v_j, v_\ell) \\
&\overset{(16)}{\leq} C \sum_{\ell,j=0}^{L} 2^{-|j-\ell|/2} h_j^{-1} \|v_\ell\|_{L^2(\Omega)} h_\ell \|v_k\|_{L^2(\Omega)} \\
&\overset{(10)}{\leq} C \sum_{j=0}^{L} h_j^{-2} \|v_j\|_{L^2(\Omega)} \\
&\overset{\text{MG Norm Equiv.}}{\leq} C_2 \sum_{j=0}^{L} (\boldsymbol{v}_j, \boldsymbol{v}_j)_{C_j} = C_2 \sum_{j=0}^{L} (C_j \boldsymbol{v}_j, \boldsymbol{v}_j)_j
\end{aligned}
$$

## Proof (Cont.)

Now,

$$
\begin{aligned}
\lambda_{\max}(C_{BPX}A_L) &\overset{\text{Eigenvalues of CA}}{=} \max_{\boldsymbol{u}\in\mathbb{R}^n_*} \frac{(\boldsymbol{u},\boldsymbol{u})_{\mathsf{A}_L}}{\displaystyle\min_{\boldsymbol{u}=\sum_{\ell=0}^{L}\mathsf{P}_\ell^L\boldsymbol{v}'_\ell} \sum_{\ell=0}^{L}(\boldsymbol{u}'_\ell,\boldsymbol{u}'_\ell)_{\mathsf{C}_\ell}} \\
&\leq \max_{\boldsymbol{u}\in\mathbb{R}^n_*} \frac{C_2\sum_{\ell=0}^{L}(\mathsf{C}_\ell\boldsymbol{w}_\ell,\boldsymbol{w}_\ell)_\ell}{\displaystyle\min_{\boldsymbol{v}'_\ell}\sum_{\ell=0}^{L}(\mathsf{C}_\ell\boldsymbol{w}_\ell,\boldsymbol{w}_\ell)} \\
&\leq C_2.
\end{aligned}
$$

Recall that the minimum was achievable, so we could take $\boldsymbol{v}_\ell = \boldsymbol{v}'_\ell$. □

## Lemma

*There is a constant $C_1 > 0$ that is independent of $L$, such that*

$$\lambda_{\min}\left(B_{BPX}A_L\right) \geq C_1.$$

*for some $C > 0$.*

## Proof.

Let $u_L \in V_L$ be arbitrary. Set

$$v_\ell =: \mathcal{R}_\ell u_L - R_{\ell-1} u_L, \quad 0 \leq \ell \leq L,$$

where $\mathcal{R}_\ell : H_0^1(\Omega) \to V_\ell$ is the Ritz projection for $0 \leq \ell \leq L$ and $R_{-1} \equiv 0$. Since

$$\mathcal{R}_\ell u_L = u_L,$$

it follows that

$$u_L = \sum_{\ell=0}^{L} v_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{u}_\ell = \sum_{\ell=0}^{L} \mathrm{P}_\ell^L v_\ell.$$

Moreover,

$$a(v_j, v_\ell) = 0, \quad 0 \leq j \neq \ell \leq L. \tag{17}$$

To see this, recall that, in general,

$$a(R_j u_L, v_j') = a(u_L, v_j'), \quad \forall v_j' \in V_j.$$

## Proof (Cont.)

Suppose $j < \ell$, for definiteness. Then

$$a(R_j u_L, v'_\ell) = a(u_L, v'_\ell), \quad \forall\, v'_\ell \in V_\ell.$$

In particular, since

$$v_j := R_j u_L - R_{j-1} u_L \in V_j \subset V_\ell,$$

and

$$a(\mathcal{R}_\ell u_L, v_j) = a(u_L, v_j),$$

likewise

$$a(R_{\ell-1} u_L, v_j) = a(u_L, v_j),$$

Subtracting, we have

$$a(\mathcal{R}_\ell u_L - R_{\ell-1} u_L, v_j) = 0$$

## Proof (Cont.)

To make further progress, let us assume that $\Omega$ is convex. Then the standard regularity condition holds. And, for $1 \leq \ell \leq L$,

$$
\begin{aligned}
h_\ell^{-2} \left\| v_\ell \right\|_{L^2(\Omega)}^2 &= h_\ell^{-2} \left\| \mathcal{R}_\ell u_L - R_{\ell-1} u_L \right\|_{L^2(\Omega)}^2 \\
&= h_\ell^{-2} \left\| \mathcal{R}_\ell u_L - R_{\ell-1} \mathcal{R}_\ell u_L \right\|_{L^2(\Omega)}^2 \\
&\overset{(??)}{\leq} C h_\ell^{-2} h_\ell^{2} \left| \mathcal{R}_\ell u_L - R_{\ell-1} \mathcal{R}_\ell u_L \right|_{H^1(\Omega)}^2 \qquad (18) \\
&= C \left| \mathcal{R}_\ell u_L - R_{\ell-1} \mathcal{R}_\ell u_L \right|_{H^1(\Omega)}^2 \\
&= C \left| v_\ell \right|_{H^1(\Omega)}^2 .
\end{aligned}
$$

To see that $R_{\ell-1} = R_{\ell-1}\mathcal{R}_\ell$, let $u \in H_0^1(\Omega)$ be arbitrary. Then

$$a(R_{\ell-1}(\mathcal{R}_\ell u), v'_{\ell-1}) = a(\mathcal{R}_\ell u, v'_{\ell-1}), \quad \forall v'_{\ell-1} \in V_{\ell-1}.$$

But,

$$a(\mathcal{R}_\ell u, v'_{\ell-1}) = a(u, v'_{\ell-1}), \quad \forall v'_{\ell-1} \in V_{\ell-1}.$$

## Proof (Cont.)

Since

$$a(\mathcal{R}_\ell u, v'_\ell) = a(u, v'_\ell), \quad \forall v'_\ell \in V_\ell,$$

and

$$V_{\ell-1} \subset V_\ell.$$

But

$$a(R_{\ell-1} u, v'_{\ell-1}) = a(u, v'_{\ell-1}), \quad \forall v'_{\ell-1} \in V_{\ell-1}.$$

Hence

$$a(R_{\ell-1}(\mathcal{R}_\ell u), v'_{\ell-1}) = a(\mathrm{R}_{\ell-1} u, v'_{\ell-1}), \quad \forall v'_{\ell-1} \in V_{\ell-1}.$$

And we conclude that $R_{\ell-1} = R_{\ell-1}\mathcal{R}_\ell$ since

$$R_{\ell-1}(\mathcal{R}_\ell u), R_{\ell-1} u \in V_{\ell-1}.$$

Estimate (17) holds trivially for $\ell = 0$.

## Proof (Cont.)

Finally,

$$\begin{aligned}\sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_\ell &\overset{\text{MG Norm Equiv.}}{\leq} C \sum_{\ell=0}^{L} h_\ell{}^{-2} \|v_\ell\|_{L^2(\Omega)}^2 \\ &\overset{(18)}{\leq} C_1^{-1} \sum_{\ell=0}^{L} |v_\ell|_{H^1(\Omega)}^2 \\ &\overset{(17)}{=} C_1^{-1} |u_L|_{H^1(\Omega)}^2 .\end{aligned} \tag{19}$$

## Proof (Cont.)

Also,

$$
\begin{aligned}
\lambda_{\min}(C_{BPX}A_L) &= \min_{\boldsymbol{u}\in\mathbb{R}^n_*} \frac{(\boldsymbol{u},\boldsymbol{u})_{\mathsf{A}_L}}{\displaystyle\min_{\boldsymbol{u}=\sum_{\ell=0}^{L}\mathsf{P}_\ell^L\boldsymbol{v}'_\ell}\sum_{\ell=0}^{L}\left(\boldsymbol{u}'_\ell,\boldsymbol{u}'_\ell\right)_{\mathsf{C}_\ell}} \\
&\geq \min_{\boldsymbol{u}\in\mathbb{R}^n_*} \frac{(\mathsf{A}_L\boldsymbol{u},\boldsymbol{u})_L}{\displaystyle\min_{\boldsymbol{v}'_\ell}\sum_{\ell=0}^{L}(\mathsf{C}_\ell\boldsymbol{v}_\ell,\boldsymbol{v}_\ell)} \\
&\geq \min_{\boldsymbol{u}\in\mathbb{R}^n_*} \frac{(\mathsf{A}_L\boldsymbol{u},\boldsymbol{u})_L}{C_1^{-1}\left|u_L\right|_{H^1(\Omega)}} \\
&= C_1.
\end{aligned}
$$

$\square$

## Theorem

$$\kappa\left(B_{BPX}A_L\right) = \frac{\lambda_{\max}\left(B_{BPX}A_L\right)}{\lambda_{\min}\left(B_{BPX}A_L\right)} \leq \frac{C_2}{C_1}.$$

## Proof.

Follows from Lemma 18 and 19. □