Math 673

Multigrid Methods:
A Mostly Matrix-Based Approach

Chapter 03: Fourier Analysis of the Two-Grid Algorithm

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Chapter 03, Part 2 of 3

Fourier Analysis of the Two-Grid Algorithm

The Gauss-Seidel Smoother

The Gauss-Seidel Smoother

To approximate the solution of

$$\mathsf{A}_1 \boldsymbol{u}_1 = \boldsymbol{f}_1 \in \mathbb{R}^{n_1} \tag{1}$$

we will next consider the forward Gauss-Seidel method. To define the method, we again split A_1 into

$$\mathsf{A}_1 = \mathsf{D} - \mathsf{U} - \mathsf{L}, \tag{2}$$

where $\mathsf{L} = \mathsf{U}^\top$, and

$$\mathsf{D} = \begin{bmatrix} \frac{2}{h_1} & & & & \\ & \frac{2}{h_1} & & & \\ & & \ddots & & \\ & & & \frac{2}{h_1} & \\ & & & & \frac{2}{h_1} \end{bmatrix} \in \mathbb{R}^{n_1 \times n_1}, \tag{3}$$

$$\mathsf{U} = \begin{bmatrix} 0 & \frac{1}{h_1} & & & \\ & 0 & \frac{1}{h_1} & & \\ & & \ddots & \ddots & \\ & & & 0 & \frac{1}{h_1} \\ & & & & 0 \end{bmatrix} \in \mathbb{R}^{n_1 \times n_1}. \tag{4}$$

The Gauss-Seidel Smoother

The Gauss-Seidel method can be expressed as

$$\boldsymbol{u}_1^{(\sigma+1)} = (\mathsf{D}-\mathsf{L})^{-1}\,\mathsf{U}\boldsymbol{u}_1^{(\sigma)} + (\mathsf{D}-\mathsf{L})^{-1}\,\boldsymbol{f}_1. \tag{5}$$

Equivalently,

$$\boldsymbol{u}_1^{(\sigma+1)} = \boldsymbol{u}_1^{(\sigma)} + (\mathsf{D}-\mathsf{L})^{-1}\left(\boldsymbol{f}_1 - \mathsf{A}_1\boldsymbol{u}_1^{(\sigma)}\right). \tag{6}$$

In our two-grid terminology,

$$\mathsf{S}_1 = (\mathsf{D}-\mathsf{L})^{-1}$$

and

$$\mathsf{K}_1 = \mathsf{I}_1 - \mathsf{S}_1\mathsf{A}_1 = \mathsf{I}_1 - (\mathsf{D}-\mathsf{L})^{-1}\,\mathsf{A}_1.$$

Clearly $\mathsf{S}_1 \neq \mathsf{S}_1^T$ and $\mathsf{K}_1 \neq \mathsf{K}_1^*$.

The Gauss-Seidel Smoother

For the particular application, the Gauss-Seidel method may be written in component form as

$$u_{1,i}^{(\sigma+1)} = \frac{h_1}{2}\left\{\frac{1}{h_1}u_{1,i-1}^{(\sigma+1)} + \frac{1}{h_1}u_{1,i+1}^{(\sigma)}\right\} + \frac{h_1}{2}f_{1,i}^{\square},$$

stepping through components from $i = 1$, in order, to $i = n_1$. We use component-wise updates immediately after they are generated. Therefore, the order in which we pass through the components is important.

Theorem (Eigen-Pairs of Gauss-Seidel)

Let $\mathsf{K}_1 = \mathsf{I}_1 - (\mathsf{D} - \mathsf{L})^{-1}\mathsf{A}_1$ *be the error transfer matrix for the Gauss-Seidel method applied to the model problem* (1)*. The eigenvectors of* K_1 *are*

$$\left[\boldsymbol{w}_1^{(k)}\right]_i = w_{1,i}^{(k)} = [\cos(k\pi h_1)]^i \sin(k\pi x_{1,i}), \quad 1 \le i \le n_1,$$

for $k = 1, \ldots, n_1$*. The eigenvalues of* K_1 *are*

$$\nu_1^{(k)} = \cos^2(k\pi h_1), \quad 1 \le k \le n_1. \tag{7}$$

Proof.

We begin by writing

$$\mathsf{K}_1 = (\mathsf{D} - \mathsf{L})^{-1}\mathsf{U}.$$

Then

$$\mathsf{K}_1\boldsymbol{w}_1^{(k)} = \nu_1^{(k)}\boldsymbol{w}_1^{(k)}$$

if and only if

$$\mathsf{U}\boldsymbol{w}_1^{(k)} = \nu_1^{(k)}(\mathsf{D} - \mathsf{L})\boldsymbol{w}_1^{(k)}.$$

The rest of the details are left to the reader. □

Remark

Notice that the eigenvectors of the error transfer matrix for the forward Gauss-Seidel method are different from those of the stiffness matrix. This complicates the analysis of the multigrid method when Gauss-Seidel smoothing is used. We will not consider this smoother further until we develop new analysis techniques based on subspace decompositions. We will talk about this again in a later chapter.

The Smoothing Effect

Low frequency modes are, essentially, those that make sense on the coarse grid. Let us make explicit now the structure of the coarse grid.

Definition (Nested and Uniform Grids)

We say that the **two-level grids are uniform and nested** iff $n_1 + 1 \geq 4$; $n_1 + 1$ is even; and

$$n_0 := \frac{n_1 + 1}{2} - 1.$$

In this case, we define the uniform grid sizes via

$$h_\ell = \frac{1}{n_\ell + 1}, \quad \ell = 0, 1,$$

and the grid points via,

$$x_{\ell,i} = i \cdot h_\ell, \quad \ell = 0, 1.$$

Example

For example, suppose $n_1 = 3$. A uniform and nested two-level grid is shown in the figure below.

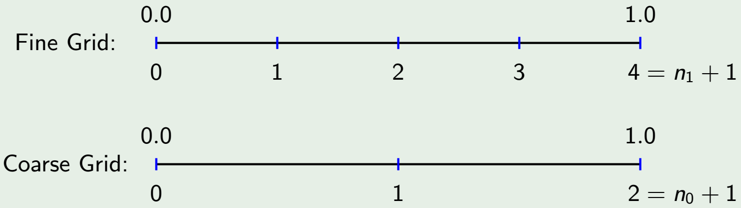

Figure: Fine and coarse grids for $n_1 = 3$, where the two-level grids are uniform and nested.

A quick recall of the property of the stiffness matrix.

Theorem (Stiffness Matrix is SPD)

The level-1 stiffness matrix, $\mathsf{A}_1 \in \mathbb{R}^{n_1 \times n_1}$, *is SPD. Its eigenvalues are*

$$\lambda_1^{(k)} = \frac{4}{h_1} \sin^2\left(\frac{k\pi h_1}{2}\right) = \frac{2}{h_1}\left(1 - \cos(k\pi h_1)\right). \tag{8}$$

$k = 1, 2, \ldots, n_1$, *and the corresponding eigenvectors are*

$$\left[\boldsymbol{v}_1^{(k)}\right]_i = v_{1,i}^{(k)} = \sin\left(k\pi x_{1,i}\right), \quad 1 \le i \le n_1. \tag{9}$$

Definition (High and Low Frequency Modes)

Suppose the two-level grids are uniform and nested. Consider an expansion of the form

$$\boldsymbol{v}_1 = \sum_{k=1}^{n_1} c_k \boldsymbol{v}_1^{(k)}, \tag{10}$$

where $\boldsymbol{v}_1^{(k)}$ is the k^{th} eigenvector defined in (9). We say that the k^{th} mode, $c_k \boldsymbol{v}_1^{(k)}$, is of **high frequency** iff

$$n_0 + 1 = \frac{n_1 + 1}{2} \leq k \leq n_1.$$

Otherwise, we say that the mode is of **low frequency**.

Another quick recall.

Theorem (Eigen-Pairs of Damped Jacobi)

Let $\mathsf{K}_1 = \mathsf{I}_1 - \omega \mathsf{D}^{-1}\mathsf{A}_1$ *be the error transfer matrix for the damped Jacobi method applied to the model problem* (1)*. The eigenvectors of* K_1 *are the same as those for the level-1 stiffness matrix,* A_1*, that is,*

$$\left[\boldsymbol{v}_1^{(k)}\right]_i = v_{1,i}^{(k)} = \sin(k\pi x_{1,i}), \quad 1 \le i \le n_1,$$

for $k = 1, \ldots, n_1$*. The eigenvalues of* K_1 *are*

$$\begin{aligned} \mu_1^{(k)}(\omega) &= \omega \cos(k\pi h_1) + 1 - \omega \\ &= 1 - 2\omega \sin^2\left(\frac{k\pi h_1}{2}\right), \quad 1 \le k \le n_1. \end{aligned} \tag{11}$$

Theorem

Suppose that $\mu_1^{(k)}(\omega)$ is the k^{th} eigenvalue (Equation (11)) of the error transfer matrix, K_1, of the damped Jacobi smoother and $0 < \omega \leq 1$. The quantity

$$S(\omega) = \max_{\frac{n_1+1}{2} \leq k \leq n_1} \left| \mu_1^{(k)}(\omega) \right|,$$

is minimized by

$$\omega = \omega_0 := \frac{2}{3},$$

in which case

$$\left| \mu_1^{(k)}(\omega_0) \right| \leq \frac{1}{3}, \tag{12}$$

for all $\frac{n_1+1}{2} \leq k \leq n_1$. More generally, if $0 < \omega \leq 1$, then

$$\left| \mu_1^{(k)}(\omega) \right| < 1. \tag{13}$$

Proof.

The proof follows by a careful examination of the plots of $\mu_1^{(k)}(\omega) = 1 - 2\omega \sin^2\left(\frac{k\pi h_1}{2}\right)$ for various values of ω. See the figure below.

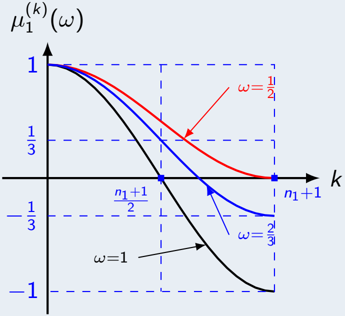

Figure: Plots of the eigenvalues of K_1, $\mu_1^{(k)}(\omega) = 1 - 2\omega \sin^2\left(\frac{k\pi h_1}{2}\right)$, as functions of k, for various values of $\omega \in (0, 1]$.

□

Remark

Recall, with $\omega = \omega_0$, we have

$$\boldsymbol{e}_1^{(\sigma+1)} = \sum_{k=1}^{n_1} \mu_1^{(k)}(\omega_0)\epsilon_k^{(\sigma)}\boldsymbol{v}_1^{(k)}$$

for the error after one smoothing iteration by the damped Jacobi method. High-frequency modes will be damped faster than those of low-frequency. In fact the modes $\frac{n_1+1}{2} \leq k \leq n_1$ will be reduced by at least $\frac{1}{3}$ after a single smoothing iteration.

The Two Multigrid Principles

1$^{\text{st}}$ MULTIGRID PRINCIPLE:
Many classical iterative methods have an error smoothing property – namely, high-frequency modes of the error are damped much more rapidly than those of low-frequency – but converge very slowly, especially as $h_1 \to 0$.

2$^{\text{nd}}$ MULTIGRID PRINCIPLE:
Low-frequency information is well approximated on a coarse grid.

Remark

We observed that a smoothed error is well-approximated on a coarse grid. By smoothed, we mean that the high frequency modes are greatly diminished. We can show that if the error is smooth, then the residual is almost as smooth. Let us explain.

Suppose that

$$\boldsymbol{e}_1^{(\sigma)} = \sum_{k=1}^{n_1} \epsilon_k^{(\sigma)} \boldsymbol{v}_1^{(k)}.$$

Since $\boldsymbol{r}_1^{(\sigma)} = \mathsf{A}_1 \boldsymbol{e}_1^{(\sigma)}$,

$$\begin{aligned} \boldsymbol{r}_1^{(\sigma)} &= \sum_{k=1}^{n_1} \epsilon_k^{(\sigma)} \mathsf{A}_1 \boldsymbol{v}_1^{(k)} \\ &= \sum_{k=1}^{n_1} \epsilon_k^{(\sigma)} \lambda_1^{(k)} \boldsymbol{v}_1^{(k)}. \end{aligned} \tag{14}$$

If the high frequency modes of the error are totally absent, then, of course, they will be absent from the residual as well.

Remark (Cont.)

However, we note that as $h_1 \to 0$, the high frequency eigenvalues can become large. See the figure below. Thus high frequency components of the error can be amplified in the residual somewhat.

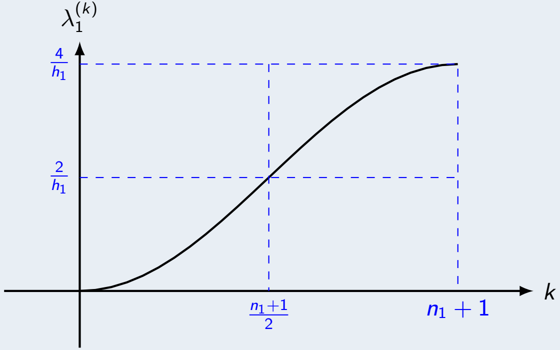

Figure: *Eigenvalues of the level-1 stiffness matrix* A_1.

Prolongation and Restriction Operators

Prolongation and Restriction Operators

We are going to approach the definition of the prolongation and the restriction matrices in what might seem like reverse order. We first define the prolongation matrix P_0, then we set $\mathsf{R}_0 = \mathsf{P}_0^\top$.

We will assume throughout that our two-level grids are uniform and nested. Our construction comes from the FEM point of view. Let us define some objects associated to the coarse grid. Set

$$V_0 := \left\{ v \in C^0([0,1]) \;\middle|\; v(0) = v(1) = 0,\ v|_{K_{0,i}} \in \mathbb{P}_1(K_{0,i}),\ 1 \le i \le n_0 \right\}, \tag{15}$$

where the coarse grid is comprised of n_0 equally sized intervals

$$K_{0,i} := (x_{0,i-1}, x_{0,i}), \quad i = 1, \ldots, n_0,$$

and the coarse grid point set, $\{x_{0,i}\}_{i=0}^{n_0+1}$, is as defined via

$$x_{0,i} = i \cdot h_0, \quad h_0 := \frac{1}{n_0 + 1}.$$

Similar to level-1 hat functions, we can define level-0 hat functions.

Definition (Hat Function)

For $i = 1, \ldots, n_0$, define $\psi_{0,i} \in V_0$ via

$$\psi_{0,i}(x_{0,j}) = \delta_{i,j}, \quad 1 \leq j \leq n_0.$$

$\psi_{0,i}$ is called a **level-0 hat function**.

The three hat functions for $n_1 = 3$ are shown in the figure on the next slide.

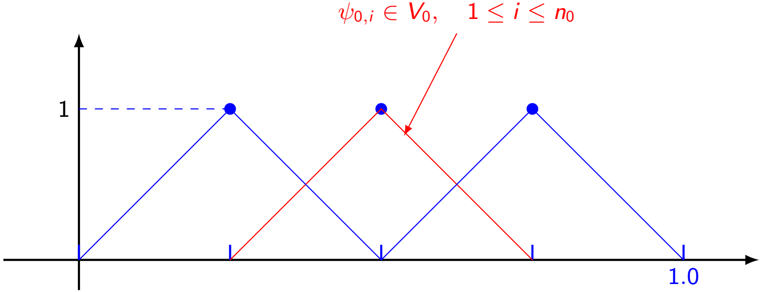

Figure: Level-0 hat function basis.

Example

Now, suppose $u_0 \in V_0$ is piecewise linear. See, for example, the figure below, where $n_0 = 3$. Recall, based on our uniform nested grids assumption

$$n_0 = \frac{n_1 + 1}{2} - 1 \Leftrightarrow n_1 = 2(n_0 + 1) - 1.$$

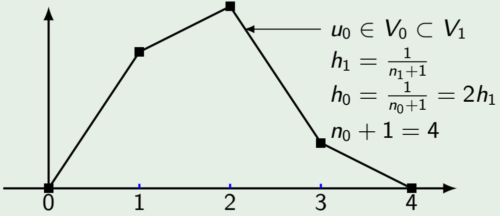

Figure: A piecewise linear function defined on the coarse grid, where $n_0 = 3$.

Example (Cont.)

Next consider the figure below, where $n_1 = 7$. The same function is shown below. Since, it turns out, $V_0 \subset V_1$, $u_0 \in V_1$. In other words, any coarse grid function can be trivially represented on the fine grid.

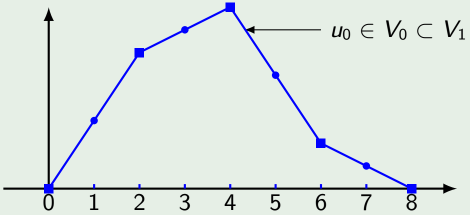

Figure: The coarse grid function from the previous figure as it would be represented on the fine grid.

Proposition

Suppose that our two level grids are uniform and nested. Then V_0, defined in (15)*, is a vector subspace of V_1, defined in* (**??**)*.*

Proof.

Exercise. □

Action of the Prolongation Matrix

The matrix $\mathsf{P}_0 \in \mathbb{R}^{n_1 \times n_0}$ will have the action

$$\mathsf{P}_0 \boldsymbol{u}_0 = \boldsymbol{u}_1 \in \mathbb{R}^{n_1},$$

where $\boldsymbol{u}_0 \in \mathbb{R}^{n_0}$ is coordinate representation of $u_0 \in V_0$ in the level-0 hat function basis and $\boldsymbol{u}_1 \in \mathbb{R}^{n_1}$ is the representation of $u_0 \in V_1$ in the level-1 hat function basis.

Example

Based on what we observed with the last example, each coarse grid function (function in V_0) must be a function in the piecewise linear space V_1. Our task in the present example is to find a matrix that maps the 3 degrees of freedom (DOFs) that uniquely characterize the course grid function into the 7 DOFs that uniquely characterize that function on the fine grid. For this example ($n_0 = 3, n_1 = 7$) the desired matrix is clearly

$$\mathsf{P}_0 = \begin{bmatrix} \frac{1}{2} & & \\ 1 & & \\ \frac{1}{2} & \frac{1}{2} & \\ & 1 & \\ & \frac{1}{2} & \frac{1}{2} \\ & & 1 \\ & & \frac{1}{2} \end{bmatrix} \in \mathbb{R}^{7\times 3} \implies \mathsf{R}_0 = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & & & & \\ & & \frac{1}{2} & 1 & \frac{1}{2} & & \\ & & & & \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix} \in \mathbb{R}^{3\times 7}.$$

Definition (Prolongation and Restriction)

Suppose that the positive integers n_0 and n_1 satisfy

$$n_1 = 2(n_0+1)-1.$$

The action of $\mathsf{P}_0 \in \mathbb{R}^{n_1 \times n_0}$ on the arbitrary vector $\boldsymbol{u}_0 \in \mathbb{R}^{n_0}$ is defined as follows

$$\begin{aligned}
[\mathsf{P}_0 \boldsymbol{u}_0]_1 : &= \frac{1}{2} u_{0,1}, \\
[\mathsf{P}_0 \boldsymbol{u}_0]_{n_1} : &= \frac{1}{2} u_{0,n_0}, \\
[\mathsf{P}_0 \boldsymbol{u}_0]_{2i} : &= u_{0,i}, \quad 1 \le i \le n_0, \\
[\mathsf{P}_0 \boldsymbol{u}_0]_{2i+1} : &= \frac{1}{2} \left(u_{0,i} + u_{0,i+1} \right), \quad 1 \le i \le n_0 - 1.
\end{aligned}$$

We define

$$\mathsf{R}_0 = \mathsf{P}_0^\top \in \mathbb{R}^{n_0 \times n_1}. \tag{16}$$

Theorem (Characterization of Prolongation Components)

Suppose that the positive integers n_0 and n_1 satisfy

$$n_1 = 2(n_0 + 1) - 1$$

and the prolongation operator, $\mathsf{P}_0 = [p_{0,i,j}] \in \mathbb{R}^{n_1 \times n_0}$, is as in the previous definition. Suppose that

$$B_\ell := \{\psi_{\ell,j}\}_{j=1}^{n_\ell}, \quad \ell = 0, 1.$$

are the hat function bases of V_ℓ, $\ell = 0, 1$, respectively. Then the numbers $p_{0,i,j}$, $1 \le i \le n_1$, $1 \le j \le n_0$ are the unique numbers satisfying

$$\psi_{0,j} = \sum_{i=1}^{n_1} p_{0,i,j} \psi_{1,i}. \tag{17}$$

Theorem (Characterization of Prolongation Components Cont.)

Thus, if $u_0 \in V_0 \subset V_1$ and

$$u_0 = \sum_{j=1}^{n_0} u_{0,j} \psi_{0,j},$$

then, the unique representation of u_0 in the basis B_1 is given by

$$\begin{aligned} u_0 &= \sum_{i=1}^{n_1} \sum_{j=1}^{n_0} p_{0,i,j} u_{0,j} \psi_{1,i} \\ &= \sum_{i=1}^{n_1} \left[\mathsf{P}_0 \boldsymbol{u}_0 \right]_i \psi_{1,i}. \end{aligned} \tag{18}$$

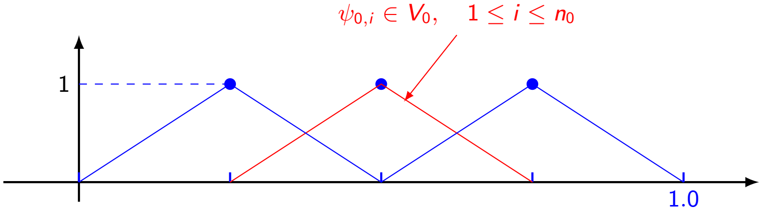

Figure: Level-0 hat function basis.

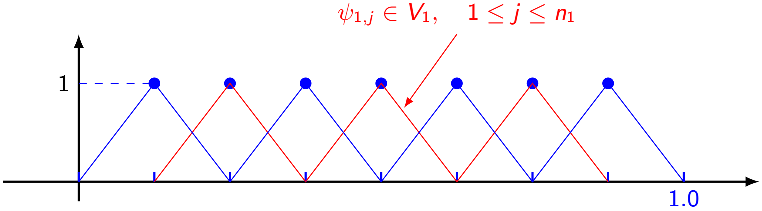

Figure: Level-1 hat function basis.

Proof.

Since $V_0 \subset V_1$, each level-0 hat basis function satisfies $\psi_{0,j} \in V_1$. Therefore, since $B_1 = \{\psi_{1,i}\}_{i=1}^{n_1}$ is a basis for V_1, there exist unique numbers, which we conveniently label $p_{0,i,j}$, $1 \leq i \leq n_1$, $1 \leq j \leq n_0$, such that

$$\psi_{0,j} = \sum_{i=1}^{n_1} p_{0,i,j} \psi_{1,i}, \quad \forall j = 1, \ldots, n_0.$$

We leave it as an exercise for the reader to prove that these are exactly the elements of the matrix P_0. This proves Equation (17).
Now, suppose that $u_0 \in V_0 \subset V_1$ is arbitrary and

$$u_0 = \sum_{j=1}^{n_0} u_{0,j} \psi_{0,j}.$$

Proof (Cont.)

The coordinate vector of this function in the hat-function basis $B_0 = \{\psi_{0,j}\}_{j=1}^{n_0}$ is precisely

$$\boldsymbol{u}_0 = \begin{bmatrix} u_{0,1} \\ u_{0,2} \\ \vdots \\ u_{0,n_0} \end{bmatrix}.$$

Proof (Cont.)

Using (17), we have

$$\begin{aligned} u_0 &= \sum_{j=1}^{n_0} u_{0,j}\psi_{0,j} \\ &= \sum_{j=1}^{n_0} u_{0,j} \sum_{j=1}^{n_0} u_{0,j}\psi_{0,j} \\ &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_0} p_{0,i,j}u_{0,j}\psi_{1,i} \\ &= \sum_{i=1}^{n_1} \left[\mathrm{P}_0 \boldsymbol{u}_0\right]_i \psi_{1,i}. \end{aligned}$$

The result is proven. $\square$

Corollary

Suppose that $\boldsymbol{u}_0 \in \mathbb{R}^{n_0}$ is the unique coordinate vector of the function $u_0 \in V_0$ with respect to the basis B_0. Then $\mathrm{P}_0\boldsymbol{u}_0$ is the unique coordinate vector of the function u_0 with respect to the basis B_1.

Theorem (Verification of the Galerkin Condition)

Suppose the two-level grids are uniform and nested. Let $\mathsf{P}_0, \mathsf{R}_0$ *be defined as above, with the level-1 stiffness matrix* A_1 *defined by*

$$a_{1,i,j} = \left(\psi_{1,j}', \psi_{1,i}'\right)_{L^2(0,1)} = \left(\psi_{1,i}', \psi_{1,j}'\right)_{L^2(0,1)}. \tag{19}$$

The matrix $\mathsf{A}_0 = [a_{0,i,j}] \in \mathbb{R}^{n_0 \times n_0}$ *satisfies the Galerkin condition, that is,*

$$\mathsf{A}_0 = \mathsf{R}_0 \mathsf{A}_1 \mathsf{P}_0,$$

iff

$$a_{0,i,j} = \left(\psi_{0,j}', \psi_{0,i}'\right)_{L^2(0,1)} = \left(\psi_{0,i}', \psi_{0,j}'\right)_{L^2(0,1)}.$$

Either way,

$$\mathsf{A}_0 = \begin{bmatrix} \frac{2}{h_0} & -\frac{1}{h_0} & & & \\ -\frac{1}{h_0} & \frac{2}{h_0} & -\frac{1}{h_0} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{h_0} & \frac{2}{h_0} & -\frac{1}{h_0} \\ & & & -\frac{1}{h_0} & \frac{2}{h_0} \end{bmatrix} \in \mathbb{R}^{n_0 \times n_0}.$$

Proof.

We prove only one direction. The other will be obvious.

($\Leftarrow$): Recall that, from (19), the level-1 stiffness matrix is defined via

$$[\mathsf{A}_1]_{i,j} := \left(\psi_{1,j}', \psi_{1,i}'\right)_{L^2(0,1)}.$$

Defining the level-0 stiffness matrix analogously and using (17), we have

$$\begin{aligned}
[\mathsf{A}_0]_{i,j} &= \left(\psi_{0,j}', \psi_{0,i}'\right)_{L^2(0,1)} \\
&= \left(\psi_{0,i}', \psi_{0,j}'\right)_{L^2(0,1)} \\
&\overset{(17)}{=} \left(\sum_{k=1}^{n_1} p_{0,k,i}\psi_{1,k}', \sum_{\ell=1}^{n_1} p_{0,\ell,j}\psi_{1,\ell}'\right)_{L^2(0,1)} \\
&= \sum_{k=1}^{n_1}\sum_{\ell=1}^{n_1} p_{0,k,i}\left(\psi_{1,k}', \psi_{1,\ell}'\right)_{L^2(0,1)} p_{0,\ell,j} \\
&= \sum_{k=1}^{n_1}\sum_{\ell=1}^{n_1} [\mathsf{R}_0]_{i,k}[\mathsf{A}_1]_{k,\ell}[\mathsf{P}_0]_{\ell,j} \\
&= [\mathsf{R}_0\mathsf{A}_1\mathsf{P}_0]_{i,j}. \qquad (20)
\end{aligned}$$

Proof (Cont.)

Finally, since

$$a_{0,i,j} = \left(\psi'_{0,j}, \psi'_{0,i}\right)_{L^2(0,1)} = \left(\psi'_{0,i}, \psi'_{0,j}\right)_{L^2(0,1)},$$

it is an easy exercise to show that

$$\mathsf{A}_0 = \begin{bmatrix} \frac{2}{h_0} & -\frac{1}{h_0} & & & \\ -\frac{1}{h_0} & \frac{2}{h_0} & -\frac{1}{h_0} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{h_0} & \frac{2}{h_0} & -\frac{1}{h_0} \\ & & & -\frac{1}{h_0} & \frac{2}{h_0} \end{bmatrix} \in \mathbb{R}^{n_0 \times n_0},$$

using the same techniques used to construct the level-1 stiffness matrix. □

Corollary

Suppose the two-level grids are uniform and nested and the level-0 stiffness matrix $\mathsf{A}_0 = [a_{0,i,j}] \in \mathbb{R}^{n_0 \times n_0}$ *satisfies the Galerkin condition. Then* A_0 *is clearly SPD and has the eigen-pairs*

$$\begin{aligned} \left[\boldsymbol{v}_0^{(k)}\right]_i &= \sin(k\pi x_{0,i}), \quad 1 \le i \le n_0, \\ \lambda_0^{(k)} &= \frac{2}{h_0}\left(1 - \cos(k\pi h_0)\right), \end{aligned} \tag{21}$$

for $k = 1, \ldots, n_0$*, where*

$$x_{0,i} = ih_0, \quad 0 \le i \le n_0 + 1.$$

and

$$h_0 = 2h_1 = \frac{1}{n_0 + 1}.$$