Math 673/4

Multigrid Methods: A Mostly Matrix-Based Approach

Chapter 07: Cell-Centered Finite Difference Methods and Multigrid

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Chapter 07, Part 3 of 3

Cell-Centered Finite Difference Methods and Multigrid

The Variable-Diffusivity Case

Variable-Diffusion Coefficient

In this section we will develop a cell-centered finite difference method and a multigrid solver for a one-dimensional problem with a variable diffusion coefficient. To be specific, let $f \in C(0,1)$, and $D \in C^1(0,1) \cap C([0,1])$ be given. We assume that there are constants $0 < D_{\min} \leq D_{\max}$ for which

$$D_{\min} \leq D(x) \leq D_{\max}, \qquad \forall\, x \in [0,1].$$

The problem we shall be concerned with is to find $u : [0,1] \to \mathbb{R}$ that solves

$$\begin{cases} u(0) = 0, \\ -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(D \dfrac{\mathrm{d}u}{\mathrm{d}x} \right) = f, \quad \text{in } (0,1), \\ u(1) = 0. \end{cases} \tag{1}$$

We will operate in the framework of classical solutions. In other words, a solution to (1) is understood as a function $u \in C^2(0,1) \cap C([0,1])$ that satisfies (1) for every point in $[0,1]$.

Discrete minus Laplacian Operator

We now introduce the cell-centered finite difference discretization of (1). We first define the operator $-\Delta_\ell^D : \mathcal{C}_{\ell,0} \to \mathcal{C}_\ell^\circ$ as follows: given $v_\ell \in \mathcal{C}_{\ell,0}$, for $1 \le i \le n_\ell$, we have

$$\begin{aligned}\left(-\Delta_\ell^D v_\ell\right)_i &:= -\Delta_\ell^D v_{\ell,i} \\ &:= \frac{-D_{\ell,i+1/2}\left(v_{\ell,i+1} - v_{\ell,i}\right) + D_{\ell,i-1/2}\left(v_{\ell,i} - v_{\ell,i-1}\right)}{h_\ell^2},\end{aligned} \tag{2}$$

where

$$D_{\ell,i+1/2} := D(i \cdot h_\ell), \qquad 0 \le i \le n_\ell.$$

Cell-Centered Finite Difference Approximation

The cell-centered finite difference approximation of (1) is to find $u_\ell \in \mathcal{C}_{\ell,0}$ that satisfies, for $1 \leq i \leq n_\ell$,

$$-\Delta_\ell^D u_\ell = f_\ell, \tag{3}$$

where $f_\ell \in \mathcal{C}_\ell^\circ$ is defined, for $1 \leq i \leq n_\ell$, as $f_{\ell,i} := f(x_{\ell,i})$.

The reader may verify that this is a formally second order scheme.

The stiffness matrix can be computed as usual, i.e., by considering the n_ℓ equations that are given in (3), and the fact that $u_\ell \in \mathcal{C}_{\ell,0}$.

Cell-Centered Finite Difference Approximation

To obtain the first equation, we set $i = 1$ in (3) to see that

$$\frac{1}{h_\ell^2}\left[-D_{\ell,1/2}u_{\ell,0} + \left(D_{\ell,1/2} + D_{\ell,3/2}\right)u_{\ell,1} - D_{\ell,3/2}u_{\ell,2}\right] = f_{\ell,1}.$$

In addition, since $u_\ell \in \mathcal{C}_{\ell,0}$, we must have

$$u_{\ell,0} = -u_{\ell,1}.$$

Combining these last two identities, the first equation is

$$\frac{1}{h_\ell^2}\left[\left(2D_{\ell,1/2} + D_{\ell,3/2}\right)u_{\ell,1} - D_{\ell,3/2}u_{\ell,2}\right] = f_{\ell,1}.$$

For $2 \leq i \leq n_\ell - 1$ the equation is (3). Using the right hand boundary condition,

$$u_{\ell,n_\ell+1} = -u_{\ell,n_\ell}.$$

the last equation is

$$\frac{1}{h_\ell^2}\left[-D_{\ell,n_\ell-1/2}u_{\ell,n_\ell-1} + \left(D_{\ell,n_\ell-1/2} + 2D_{\ell,n_\ell+1/2}\right)u_{\ell,n_\ell}\right] = f_{\ell,n_\ell}.$$

Cell-Centered Finite Difference Approximation

Thus, the cell-centered finite difference approximation for the non-constant diffusion coefficient can be represented as

$$\mathsf{A}_\ell \boldsymbol{u}_\ell = \boldsymbol{f}_\ell,$$

where the level-ℓ stiffness matrix is the following tridiagonal matrix:

$$\mathsf{A}_\ell = \frac{1}{h_\ell^2} \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & a_3 & \ddots & \\ & & \ddots & \ddots & b_{n_\ell - 1} \\ & & & b_{n_\ell - 1} & a_{n_\ell} \end{bmatrix} \in \mathbb{R}^{n_\ell \times n_\ell}, \qquad (4)$$

where

$$\begin{aligned} a_1 &= 2D_{\ell,1/2} + D_{\ell,3/2}, \\ a_i &= D_{\ell,i-1/2} + D_{\ell,i+1/2}, \quad 2 \le i \le n_\ell - 1, \\ a_{n_\ell} &= D_{\ell,n_\ell-1/2} + 2D_{\ell,n_\ell+1/2}, \\ b_i &= -D_{\ell,i+1/2}, \quad 1 \le i \le n_\ell - 1. \end{aligned}$$

Remark

In what follows we shall denote by $-\Delta_\ell^D : \mathcal{C}_{\ell,0} \to \mathcal{C}_\ell^\circ$ *the operator representation of the matrix* A_ℓ*, and vice versa. In this representation, the boundary conditions are built into the space. We could equally well use the notation* $-\Delta_\ell^{D,\circ} : \mathcal{C}_\ell^\circ \to \mathcal{C}_\ell^\circ$ *as the operator representation of* A_ℓ*, where in this latter representation it is understood that the boundary conditions are built directly into the definition of the operator* $-\Delta_\ell^{D,\circ}$*.*

Proposition (Condition (G3))

Let $1 \leq \ell \leq L$. Assume that the stiffness matrices are defined as in (4) *and the prolongation and restriction matrices are as in Definition* **??**. *Then, the imbalanced Galerkin condition, that is, Assumption (G3) holds:*

$$\mathsf{R}_{\ell-1}\mathsf{A}_{\ell}\mathsf{P}_{\ell-1} = 2\mathsf{A}_{\ell-1}.$$

Proof.

To simplify the arguments we introduce some notation. We let $\boldsymbol{v}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}$ be arbitrary. Define $\boldsymbol{v}_{\ell} := \mathsf{P}_{\ell-1}\boldsymbol{v}_{\ell-1}$. Using the canonical identifications, we have the grid functions

$$\boldsymbol{v}_{\ell-1} \xleftrightarrow{\mathcal{G}_{\ell-1}} v_{\ell-1} \in \mathcal{C}^{\circ}_{\ell-1}$$
$$\boldsymbol{v}_{\ell} \xleftrightarrow{\mathcal{G}_{\ell}} v_{\ell} \in \mathcal{C}^{\circ}_{\ell}.$$

The index on the level-$(\ell-1)$ grid is the symbol I, so that $1 \leq I \leq n_{\ell-1}$, and the index on the level-ℓ grid is the symbol i, so that $1 \leq i \leq n_{\ell}$. See the figure on the next slide.

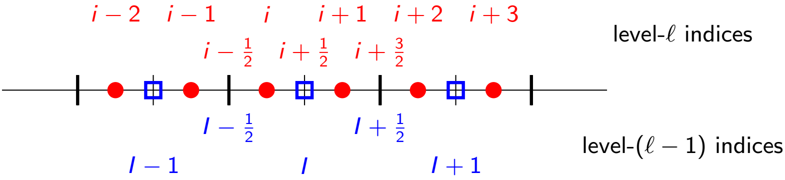


Figure: Notation for the proof of Proposition 1.2. Empty blue squares denote the cell-centered grid points for the level-$(\ell-1)$ mesh, whereas filled red circles are the level-ℓ cell-centered grid points.

Proof (Cont.)

Define

$$\alpha_I := \left[\mathsf{R}_{\ell-1}\mathsf{A}_\ell\mathsf{P}_{\ell-1}\boldsymbol{v}_{\ell-1}\right]_I .$$

Then,

$$\begin{aligned}
\alpha_I &= \frac{1}{2h_\ell^2}\left[-D_{\ell,i+1/2}\left(v_{\ell,i+1}-v_{\ell,i}\right)+D_{\ell,i-1/2}\left(v_{\ell,i}-v_{\ell,i-1}\right)\right] \\
&\quad+\frac{1}{2h_\ell^2}\left[-D_{\ell,i+3/2}\left(v_{\ell,i+2}-v_{\ell,i+1}\right)+D_{\ell,i+1/2}\left(v_{\ell,i+1}-v_{\ell,i}\right)\right] \\
&= \frac{1}{2h_\ell^2}\left[-D_{\ell,i+1/2}\left(v_{\ell-1,I}-v_{\ell-1,I}\right)+D_{\ell,i-1/2}\left(v_{\ell-1,I}-v_{\ell-1,I-1}\right)\right] \\
&\quad+\frac{1}{2h_\ell^2}\left[-D_{\ell,i+3/2}\left(v_{\ell-1,I+1}-v_{\ell-1,I}\right)+D_{\ell,i+1/2}\left(v_{\ell-1,I}-v_{\ell-1,I}\right)\right] .
\end{aligned}$$

Proof (Cont.)

Canceling flux terms of opposite sign, we obtain

$$
\begin{aligned}
\alpha_I &= \frac{1}{2}\left[\frac{-D_{\ell,i+3/2}\left(v_{\ell-1,I+1}-v_{\ell-1,I}\right)}{(\frac{1}{2}h_{\ell-1})^2}\right] + \frac{1}{2}\left[\frac{D_{\ell,i-1/2}\left(v_{\ell-1,I}-v_{\ell-1,I-1}\right)}{(\frac{1}{2}h_{\ell-1})^2}\right] \\
&= 2\frac{-D_{\ell,i+3/2}\left(v_{\ell-1,I+1}-v_{\ell-1,I}\right)+D_{\ell,i-1/2}\left(v_{\ell-1,I}-v_{\ell,I-1}\right)}{h_{\ell-1}^2} \\
&= 2\frac{-D_{\ell-1,I+1/2}\left(v_{\ell-1,I+1}-v_{\ell-1,I}\right)+D_{\ell-1,I-1/2}\left(v_{\ell-1,I}-v_{\ell,I-1}\right)}{h_{\ell-1}^2} \\
&= 2[\mathrm{A}_{\ell-1}\boldsymbol{v}_{\ell-1}]_I .
\end{aligned}
$$

The argument must change slightly when the $I = 1$ and $I = n_{\ell-1}$. We leave it to the reader to check those cases separately. After those details are confirmed, we can conclude that $\mathrm{R}_{\ell-1}\mathrm{A}_\ell\mathrm{P}_{\ell-1} = 2\mathrm{A}_{\ell-1}$, as claimed. □

Remark

We have shown that the imbalanced Galerkin condition, that is, Assumption (G3), holds. There are two missing pieces that would be required to establish convergence of our multigrid method. We need, in particular, a smoothing property, using, for example, a Richardson-type smoother, and an approximation property. Neither of these two pieces is trivial in the variable diffusivity context.

Cell-Centered Multigrid in Two Dimensions

The Two-Dimensional Problem

In this section, we describe the cell-centered multigrid method in two space dimensions. For brevity, however, we do not provide a detailed account of the theory, since, at least for the constant-diffusivity case, this only involves minor changes from the one-dimensional case.

We let $\Omega = (0,1)^2$, and assume that $f \in C(\Omega)$, and $D \in C(\overline{\Omega}) \cap C^1(\Omega)$ are given. We assume that there are constants $0 < D_{\min} \leq D_{\max}$ for which

$$D_{\min} \leq D(\boldsymbol{x}) \leq D_{\max}, \qquad \forall\, \boldsymbol{x} \in \overline{\Omega}.$$

We will refer to the function D as the diffusivity coefficient. Presently, let us operate in the framework of classical solutions. In other words, we seek after a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ that, at every point of $\overline{\Omega}$ satisfies

$$\begin{cases} -\nabla \cdot (D \nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Grid Parameters

The approximation shall be carried out via the cell-centered finite difference method. We begin by defining a two-dimensional grid of cell-centered points covering the domain Ω. Similar to the one-dimensional case, for each level $\ell = 0, 1, \ldots, L$, we set the number of grid points in each coordinate direction and the grid size to be, respectively,

$$m_\ell := q \cdot 2^\ell, \quad \text{and} \quad h_\ell := \frac{1}{m_\ell},$$

where $q \in \mathbb{N}$ is fixed.

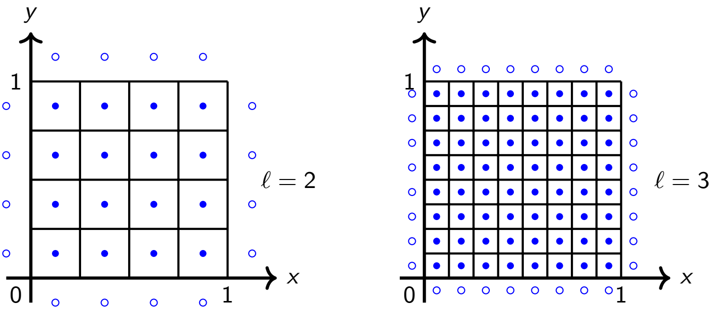


Figure: Example of two-dimensional cell-centered grids and grid points for $\ell = 2$ and $\ell = 3$. The ghost points are indicated by empty circles.

Definition (Cell-Centered Grid)

Let $0 \leq \ell \leq L$. The **cell-centered grid points** are defined via

$$\boldsymbol{x}_{\ell,i,j} := (\zeta_{\ell,i}, \zeta_{\ell,j}), \qquad i,j = 0, \ldots, m_\ell + 1,$$

where

$$\zeta_{\ell,k} := \left(k - \frac{1}{2}\right) h_\ell, \qquad k = 0, \ldots, m_\ell + 1.$$

The **set of cell-centered grid points** is

$$C_\ell := \{\boldsymbol{x}_{\ell,i,j} \mid 0 \leq i,j \leq m_\ell + 1\} \setminus \{\boldsymbol{x}_{\ell,0,0}, \boldsymbol{x}_{\ell,0,m_\ell+1}, \boldsymbol{x}_{\ell,m_\ell+1,0}, \boldsymbol{x}_{\ell,m_\ell+1,m_\ell+1}\}.$$

The grid points that do not belong to Ω are called **ghost points**. The **set of interior cell-centered grid points** is

$$C_\ell^\circ := \{\boldsymbol{x}_{\ell,i,j} \mid 1 \leq i,j \leq m_\ell\} = C_\ell \cap \Omega.$$

Finally, the **grid cells** are

$$K_{\ell,i,j} := (h_\ell(i-1), h_\ell i] \times (h_\ell(j-1), h_\ell j], \qquad i,j = 1, \ldots, m_\ell.$$

Definition (Grid Functions)

For every $0 \leq \ell \leq L$, we define the space of **piecewise constant functions** as

$$V_\ell := \left\{ v : (0,1]^2 \to \mathbb{R} \;\middle|\; v|_{K_{\ell,i,j}} \in \mathbb{P}_0,\ 1 \leq i,j \leq m_\ell \right\}.$$

The spaces of **cell-centered grid functions** and **interior cell-centered grid functions** are defined, respectively, as

$$\mathcal{C}_\ell := \{ v_\ell : C_\ell \to \mathbb{R} \} \quad \text{and} \quad \mathcal{C}_\ell^\circ := \{ v_\ell : C_\ell^\circ \to \mathbb{R} \}.$$

Definition (Grid Functions (Cont.))

The subspace of **homogeneous Dirichlet cell-centered grid functions** is denoted

$$\mathcal{C}_{\ell,0} := \left\{ u_\ell \in \mathcal{C}_\ell \mid u_\ell = 0 \text{ on } \partial\Omega \right\},$$

where by $u_\ell = 0$ on $\partial\Omega$ we mean that

$$\begin{aligned} \frac{u_{\ell,0,j} + u_{\ell,1,j}}{2} &= 0, \quad 1 \le j \le m_\ell, \\ \frac{u_{\ell,n_\ell,j} + u_{\ell,n_\ell+1,j}}{2} &= 0, \quad 1 \le j \le m_\ell, \\ \frac{u_{\ell,i,0} + u_{\ell,i,1}}{2} &= 0, \quad 1 \le i \le m_\ell, \\ \frac{u_{\ell,i,n_\ell} + u_{\ell,i,n_\ell+1}}{2} &= 0, \quad 1 \le i \le m_\ell. \end{aligned} \tag{6}$$

We say that the **discrete homogeneous Dirichlet boundary conditions** hold for $u_\ell \in \mathcal{C}_\ell$, and we write $u_\ell = 0$ on $\partial\Omega$, iff (6) holds.

The Lexicographical Correspondence

As in the one dimensional case, there is a canonical way to relate functions in V_ℓ, $\mathcal{C}_{\ell,0}$, and $\mathcal{C}_\ell^\circ$ with vectors in $\mathbb{R}^{m_\ell^2}$. In particular, this can be accomplished using the so-called lexicographical correspondence:

$$v_\ell \in \mathcal{C}_\ell^\circ \stackrel{\text{lex}}{\longleftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^{m_\ell^2}, \tag{7}$$

where, for $1 \leq i, j \leq m_\ell$,

$$[\boldsymbol{v}_\ell]_{i+(j-1)m_\ell} = v_{\ell,i,j}.$$

This identification clearly shows that $\dim(\mathcal{C}_\ell^\circ) = m_\ell^2$. To keep our notation standardized, we define

$$n_\ell := m_\ell^2.$$

And thus, because of the mutual identifications between V_ℓ, $\mathcal{C}_{\ell,0}$, $\mathcal{C}_\ell^\circ$, and $\mathbb{R}^{m_\ell^2}$, we are justified in writing

$$\dim(\mathcal{C}_\ell^\circ) = \dim(\mathcal{C}_{\ell,0}) = \dim(V_\ell) = n_\ell, \quad 0 \leq \ell \leq L.$$

Constant Diffusivity

Let us first consider the case that

$$D \equiv 1 \quad \text{on} \quad \overline{\Omega},$$

the so-called constant diffusivity case. Thus, we need an approximation of the $-\Delta$ operator on Ω. For this purpose, for $0 \leq \ell \leq L$, we construct the cell-centered finite difference minus Laplacian, which is an operator $-\Delta_\ell : \mathcal{C}_{\ell,0} \to \mathcal{C}_\ell$ defined as follows: given a grid function $v_\ell \in \mathcal{C}_{\ell,0}$ then, for $1 \leq i,j \leq m_\ell$,

$$(-\Delta_\ell v_\ell)_{i,j} := -\Delta_\ell v_{\ell,i,j} := \frac{-v_{\ell,i+1,j} - v_{\ell,i-1,j} - v_{\ell,i,j+1} - v_{\ell,i,j-1} + 4v_{\ell,i,j}}{h_\ell^2}.$$

The Standard Stencil

The grid operator $-\Delta_\ell$ can be expressed as a stencil in the usual way. For $1 \leq i, j \leq m_\ell$,

$$[-\Delta_\ell]_{i,j} = \frac{1}{h_\ell^2} \begin{array}{cc} \left[\begin{array}{ccc} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{array}\right] & \begin{array}{c} j+1 \\ j \\ j-1 \end{array} \\ \begin{array}{ccc} i-1 & i & i+1 \end{array} & \end{array}$$

Cell-Centered Finite Difference Approximation

We are now ready to describe our finite difference scheme. We first discretize the right hand side f. We define, for each $\ell \geq 0$, the grid function $f_\ell \in \mathcal{C}_\ell^\circ$ via

$$f_{\ell,i,j} := f\left(\boldsymbol{x}_{\ell,i,j}\right), \quad 1 \leq i,j \leq m_\ell. \tag{8}$$

We must then find the grid function $u_\ell \in \mathcal{C}_{\ell,0}$ such that

$$\left(-\Delta_\ell u_\ell\right)_{i,j} = f_{\ell,i,j} \quad 1 \leq i,j \leq m_\ell, \tag{9}$$

with the discrete homogeneous Dirichlet boundary conditions as in (6).

Modified Stencils Near Boundaries

We can incorporate the boundary conditions into the definition of the minus discrete laplacian operator by modifying the stencil of the grid operator $-\Delta_\ell$ near the boundaries. This gives rise to the so-called **interior minus discrete Laplacian** $-\Delta_\ell^\circ : \mathcal{C}_\ell \to \mathcal{C}_\ell$. For $2 \leq i, j \leq m_\ell - 1$ the stencils of $-\Delta_\ell$ and $-\Delta_\ell^\circ$ coincide. On the other hand, at the left boundary, we have, for $2 \leq j \leq m_\ell - 1$,

$$[-\Delta_\ell^\circ]_{1,j} = \frac{1}{h_\ell^2} \begin{array}{cc} \begin{bmatrix} & -1 & \\ & 5 & -1 \\ & -1 & \end{bmatrix} & \begin{array}{c} j+1 \\ j \\ j-1 \end{array} \\ \begin{array}{ccc} 0 & \quad 1 & \quad 2 \end{array} & \end{array}$$

Modified Stencils Near Boundaries

Similarly, at the bottom boundary, the stencil of Δ_ℓ is, for $2 \leq i \leq m_\ell - 1$,

$$[-\Delta_\ell^\circ]_{i,1} = \frac{1}{h_\ell^2} \begin{array}{cc} \left[\begin{array}{ccc} & -1 & \\ -1 & 5 & -1 \\ & & \end{array}\right] & \begin{array}{c} 2 \\ 1 \\ 0 \end{array} \\ \begin{array}{ccc} i-1 & i & i+1 \end{array} & \end{array}$$

Modified Stencils Near Boundaries

Lastly, at the bottom left corner, we have the stencil

$$[-\Delta_\ell^\circ]_{1,1} = \frac{1}{h_\ell^2} \begin{array}{c c} \left[\begin{array}{ccc} & -1 & \\ & 6 & -1 \\ & & \end{array}\right] & \begin{array}{c} 2 \\ 1 \\ 0 \end{array} \\ \begin{array}{ccc} 0 & 1 & 2 \end{array} & \end{array}$$

The remaining edge/corner cases can be easily inferred from those presented.

Equivalent Formulations

It should be clear that the following operator representations are equivalent:

$$-\Delta_\ell^\circ : \mathcal{C}_\ell^\circ \to \mathcal{C}_\ell^\circ$$
$$-\Delta_\ell : \mathcal{C}_{\ell,0} \to \mathcal{C}_\ell^\circ.$$

As a result, the following problem statements are equivalent: (P°): given $f_\ell \in \mathcal{C}_\ell^\circ$ find $u_\ell \in \mathcal{C}_\ell^\circ$ satisfying

$$-\Delta_\ell^\circ u_\ell = f_\ell.$$

(P_0): given $f_\ell \in \mathcal{C}_\ell^\circ$, find $u_\ell \in \mathcal{C}_{\ell,0}$ satisfying

$$-\Delta_\ell u_\ell = f_\ell.$$

Matrix-Vector Formulation

We now use the lexicographic correspondence to realize that the finite difference problem is equivalent to the following matrix problem: (M°): given

$$\boldsymbol{f}_\ell \in \mathbb{R}^{n_\ell} \xleftrightarrow{\text{lex}} f_\ell \in \mathcal{C}_\ell^\circ,$$

find $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$ such that

$$\mathsf{A}_\ell \boldsymbol{u}_\ell = \boldsymbol{f}_\ell,$$

where the stiffness matrix, A_ℓ, is defined as

$$\mathsf{A}_\ell = \frac{1}{h_\ell^2} \begin{bmatrix} \mathsf{B}_\ell & -\mathsf{I}_\ell & & & \\ -\mathsf{I}_\ell & \mathsf{C}_\ell & -\mathsf{I}_\ell & & \\ & -\mathsf{I}_\ell & \ddots & \ddots & \\ & & \ddots & \mathsf{C}_\ell & -\mathsf{I}_\ell \\ & & & -\mathsf{I}_\ell & \mathsf{B}_\ell \end{bmatrix} \in \mathbb{R}^{n_\ell \times n_\ell}.$$

Matrix-Vector Formulation

Here, $I_\ell \in \mathbb{R}^{m_\ell \times m_\ell}$ is the identity matrix,

$$\mathsf{B}_\ell = \begin{bmatrix} 6 & -1 & & & & \\ -1 & 5 & -1 & & & \\ & -1 & 5 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 5 & -1 \\ & & & & -1 & 6 \end{bmatrix} \in \mathbb{R}^{m_\ell \times m_\ell},$$

and

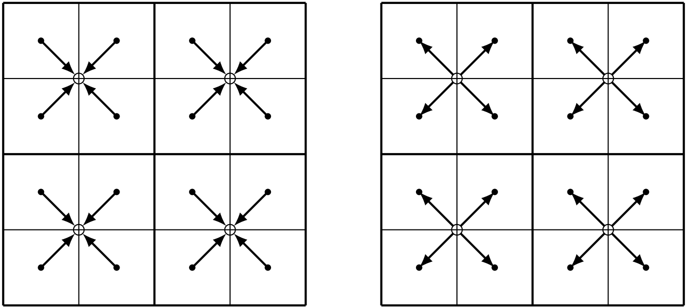
$$\mathsf{C}_\ell = \begin{bmatrix} 5 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 5 \end{bmatrix} \in \mathbb{R}^{m_\ell \times m_\ell}.$$

Restriction Operator

We now define the prolongation and restriction matrices via their grid operator representations.

The restriction matrix $\mathsf{R}_{\ell-1} : \mathbb{R}^{n_\ell} \to \mathbb{R}^{n_{\ell-1}}$ can be understood as a grid operator, $R_{\ell-1} : \mathcal{C}_\ell^\circ \to \mathcal{C}_{\ell-1}^\circ$, which, given the fine-level grid function $v_\ell \in \mathcal{C}_{\ell,0}$, produces the coarse-level grid function $v_{\ell-1} = R_{\ell-1} v_\ell \in \mathcal{C}_{\ell-1}^\circ$ that is defined, for $1 \le i,j \le m_{\ell-1}$, as

$$v_{\ell-1,i,j} := \frac{v_{\ell,2i,2j} + v_{\ell,2i-1,2j} + v_{\ell,2i,2j-1} + v_{\ell,2i-1,2j-1}}{4}. \tag{10}$$



(a) Restriction (b) Prolongation

Figure: A graphical illustration of the (a) restriction and (b) prolongation operations. In restriction, the four values at surrounding fine-grid points (filled circles) are averaged to give the value at the coarse grid point (unfilled circle). In prolongation, the value at the coarse grid point is copied into the values of the fine grid function at the surrounding four fine grid cell-centered points.

Prolongation Operator

Analogously, the prolongation grid operator is denoted $P_{\ell-1} : \mathcal{C}^{\circ}_{\ell-1} \to \mathcal{C}^{\circ}_{\ell}$. For the coarse-grid function $v_{\ell-1} \in \mathcal{C}^{\circ}_{\ell-1}$, it produces $v_\ell = P_{\ell-1} v_{\ell-1} \in \mathcal{C}^{\circ}_{\ell}$ such that, for $1 \leq i, j \leq m_{\ell-1}$,

$$
\begin{aligned}
v_{\ell,2i,2j} &= v_{\ell-1,i,j}, \\
v_{\ell,2i-1,2j} &= v_{\ell-1,i,j}, \\
v_{\ell,2i,2j-1} &= v_{\ell-1,i,j}, \\
v_{\ell,2i-1,2j-1} &= v_{\ell-1,i,j}.
\end{aligned}
\tag{11}
$$

Again, there is an associated prolongation matrix, $\mathsf{P}_{\ell-1} \in \mathbb{R}^{n_\ell \times n_{\ell-1}}$, whose entries can be determined via the lexicographic correspondence.

The following result shows that, as in the one-dimensional case, condition (G3) holds.

Proposition (Condition (G3))

Let $1 \leq \ell \leq L$, and assume the restriction and prolongation operators are defined via (10) *and* (11)*, respectively. Then, their respective matrix representations,* $\mathsf{R}_{\ell-1} \in \mathbb{R}^{n_{\ell-1} \times n_\ell}$ *and* $\mathsf{P}_{\ell-1} \in \mathbb{R}^{n_\ell \times n_{\ell-1}}$*, satisfy*

$$\mathsf{R}_{\ell-1}\mathsf{P}_{\ell-1} = \mathsf{I}_{\ell-1}, \qquad \mathsf{R}_{\ell-1} = \frac{1}{4}\mathsf{P}_{\ell-1}^{\top}, \qquad 2\mathsf{A}_{\ell-1} = \mathsf{R}_{\ell-1}\mathsf{A}_\ell\mathsf{P}_{\ell-1}.$$

In other words, our multigrid procedure satisfies the imbalanced Galerkin conditions, that is, Assumption (G3), with $r = \frac{1}{4}$.

Proof.

Exercise. □

Eigenvalues and Eigenvectors

The eigenvalues and eigenvectors of the stiffness matrix can be found by examination of the one-dimensional case. In fact, it is sufficient to note that the two-dimensional discrete Laplacian operator $-\Delta_\ell$ can be decomposed into the sum of two one-dimensional discrete Laplacian operators, one for each coordinate direction. Indeed, if $w_\ell \in \mathcal{C}_{\ell,0}$, then, for $1 \leq i, j \leq m_\ell$,

$$-\Delta_\ell w_{\ell,i,j} = \frac{-w_{\ell,i-1,j} + 2w_{\ell,i,j} - w_{\ell,i+1,j}}{h_\ell^2} + \frac{-w_{\ell,i,j-1} + 2w_{\ell,i,j} - w_{\ell,i,j+1}}{h_\ell^2}.$$

Eigenvalues and Eigenvectors

Define, for $1 \leq s, t \leq m_\ell$, the grid functions $w_\ell^{(s,t)} \in \mathcal{C}_{\ell,0}$ via

$$w_{\ell,i,j}^{(s,t)} = \sin\left(s\pi\zeta_{\ell,i}\right)\sin\left(t\pi\zeta_{\ell,j}\right), \qquad 1 \leq i, j \leq m_\ell.$$

Then, it is not difficult to show that

$$-\Delta_\ell w_{\ell,i,j}^{(s,t)} = \lambda_\ell^{(s,t)} w_{\ell,i,j}^{(s,t)},$$

where

$$\lambda_\ell^{(s,t)} = \lambda_\ell^{(s)} + \lambda_\ell^{(t)} = \frac{4}{h_\ell^2}\left(\sin^2\left(\frac{s\pi h_\ell}{2}\right) + \sin^2\left(\frac{t\pi h_\ell}{2}\right)\right).$$

Clearly, the largest eigenvalue is given by

$$\lambda_\ell^{(m_\ell,m_\ell)} = \lambda_\ell^{(m_\ell)} + \lambda_\ell^{(m_\ell)} = 2\lambda_\ell^{(m_\ell)} = \frac{8}{h_\ell^2}.$$

Thus, we have just shown that the spectral radius of the two-dimensional cell-centered stiffness matrix is

$$\rho\left(\mathsf{A}_\ell\right) = \frac{8}{h_\ell^2}.$$

Remark

All the ingredients are in place to implement our standard multigrid algorithms. The next steps are to establish a smoothing property and the α-weak approximation property (A2) holds. The smoothing property is straightforward. The proof of the α-weak approximation property follows what we did in the one-dimensional case.

Once those steps are taken, we can conclude that the various versions of the W-cycle multigrid method converge with contraction factors independent of the number of levels L.

Variable Diffusivity

To conclude this section, we describe how to proceed in the case of a problem with a variable diffusivity coefficient.

A formally second order cell-centered finite difference method is to find $u_\ell \in \mathcal{C}_{\ell,0}$ such that

$$-\Delta_\ell^D u_{\ell,i,j} = f_{\ell,i,j}, \qquad 1 \le i,j \le m_\ell,$$

where, as before, the function $f_\ell \in \mathcal{C}_\ell^\circ$ is defined by $f_{\ell,i,j} = f(\boldsymbol{x}_{\ell,i,j})$. We define the operator $-\Delta_\ell^D : \mathcal{C}_{\ell,0} \to \mathcal{C}_\ell^\circ$, for $1 \le i,j \le m_\ell$, via

$$\begin{aligned}
\left(-\Delta_\ell^D v_\ell\right)_{i,j} &:= -\Delta_\ell^D v_{\ell,i,j} \\
&:= \frac{-D_{\ell,i+1/2,j}\left(v_{\ell,i+1,j} - v_{\ell,i,j}\right) + D_{\ell,i-1/2,j}\left(v_{\ell,i,j} - v_{\ell,i-1,j}\right)}{h_\ell^2} \\
&\quad + \frac{-D_{\ell,i,j+1/2}\left(v_{\ell,i,j+1} - v_{\ell,i,j}\right) + D_{\ell,i,j-1/2}\left(v_{\ell,i,j} - v_{\ell,i,j-1}\right)}{h_\ell^2},
\end{aligned}$$

for every $v_\ell \in \mathcal{C}_{\ell,0}$.

Recursive, Top-Down Variable Diffusivity

The discrete diffusivity values are computed recursively as follows: at the finest level, $\ell = L$,

$$D_{L,i+1/2,j} := D\left(ih_L, \left(j - \frac{1}{2}\right) h_L\right), \quad 0 \leq i, \leq m_L, \quad 1 \leq j, \leq m_L,$$

$$D_{L,i,j+1/2} := D\left(\left(i - \frac{1}{2}\right) h_L, jh_L\right) \quad 1 \leq i, \leq m_L, \quad 0 \leq j, \leq m_L.$$

For $1 \leq \ell \leq L$, we define

$$D_{\ell-1,I+1/2,J} := \frac{D_{\ell,2I+1/2,2J-1} + D_{\ell,2I+1/2,2J}}{2}, \quad \begin{array}{l} 0 \leq I \leq m_{\ell-1}, \\ 1 \leq J \leq m_{\ell-1}, \end{array}$$

$$D_{\ell-1,I,J+1/2} := \frac{D_{\ell,2I-1,2J+1/2} + D_{\ell,2I,2J+1/2}}{2}, \quad \begin{array}{l} 0 \leq I \leq m_{\ell-1}, \\ 1 \leq J \leq m_{\ell-1} \end{array}.$$

Referring to the particular indexing in the figure on the next slide, observe that

$$
\begin{aligned}
D_{\ell-1,I+1/2,J} &:= \frac{D_{\ell,i+3/2,j} + D_{\ell,i+3/2,j+1}}{2}, \\
D_{\ell-1,I-1/2,J} &:= \frac{D_{\ell,i-1/2,j} + D_{\ell,i-1/2,j+1}}{2}, \\
D_{\ell-1,I,J+1/2} &:= \frac{D_{\ell,i,j+3/2} + D_{\ell,i+1,j+3/2}}{2}, \\
D_{\ell-1,I,J-1/2} &:= \frac{D_{\ell,i,j-1/2} + D_{\ell,i+1,j-1/2}}{2}.
\end{aligned}
\qquad (12)
$$

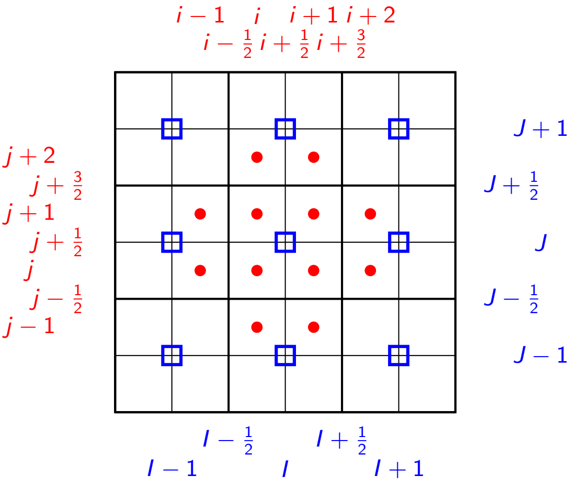


Figure: The level-$(\ell-1)$ grid points are denoted by empty blue squares, whereas filled red circles denote the level-ℓ grid points.

Equivalent Grid Operators

As usual, we can use the discrete homogeneous Dirichlet boundary conditions, $u_\ell = 0$ on $\partial\Omega$, defined as in (6), to eliminate degrees of freedom at the ghost cells. This allows us to define the equivalent grid operator

$$-\Delta_\ell^{D,\circ} : \mathcal{C}_\ell^\circ \to \mathcal{C}_\ell^\circ .$$

The definition of the level-ℓ stiffness matrix A_ℓ expediently follows.

As in the variable-coefficient one-dimensional case, or the two-dimensional constant-coefficient case, we aim to verify condition (G3). This verification requires the very specific construction of the discrete diffusivity functions, as we now show.

Proposition (Condition (G3))

Let $0 \le \ell \le L$. Assume that $\mathsf{A}_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$ is defined in accordance with $-\Delta_\ell^{D,\circ}$. Then, the imbalanced Galerkin condition, that is Assumption (G3), holds: for $1 \le \ell \le L$,

$$\mathsf{R}_{\ell-1}\mathsf{A}_\ell \mathsf{P}_{\ell-1} = 2\mathsf{A}_{\ell-1},$$

or, in terms of grid operators

$$-R_{\ell-1}\Delta_\ell^{D,\circ} P_{\ell-1} = -2\Delta_{\ell-1}^{D,\circ}. \tag{13}$$

Proof.

The proof is a long exercise in juggling notation. We let $v_{\ell-1} \in \mathcal{C}_{\ell-1}^{\circ}$ be arbitrary. Define

$$\alpha_{I,J} := -4h_\ell^2 \left(R_{\ell-1}\Delta_\ell^{D,\circ} P_{\ell-1} v_{\ell-1} \right)_{I,J},$$

for any $2 \le I, J \le m_{\ell-1} - 1$. We will stay away from the grid boundaries, for simplicity and brevity. The reader should complete the proof by adapting the calculation appropriately at the boundaries.

Proof (Cont.)

Define

$$v_\ell = P_{\ell-1} v_{\ell-1} \in \mathcal{C}_\ell^\circ.$$

Using the definition of the prolongation operator we observe, for example,

$$v_{\ell,i+2,j} = v_{\ell-1,I+1,J}, \quad v_{\ell,i+1,j} = v_{\ell-1,I,J},$$

et cetera, referring to the figure on the next slide. Essentially, we replace the value of v_ℓ at every red point with the value of $v_{\ell-1}$ at the nearest blue square (coarse-grid) point.

To further lessen our writing, let us introduce the difference operators

$$\delta^x v_{\ell,i+1/2,j} := v_{\ell,i+1,j} - v_{\ell,i,j}, \quad 0 \le i, \le m_\ell, \quad 1 \le j, \le m_\ell,$$

and

$$\delta^y v_{\ell,i,j+1/2} := v_{\ell,i,j+1} - v_{\ell,i,j}, \quad 1 \le i, \le m_\ell, \quad 0 \le j, \le m_\ell.$$

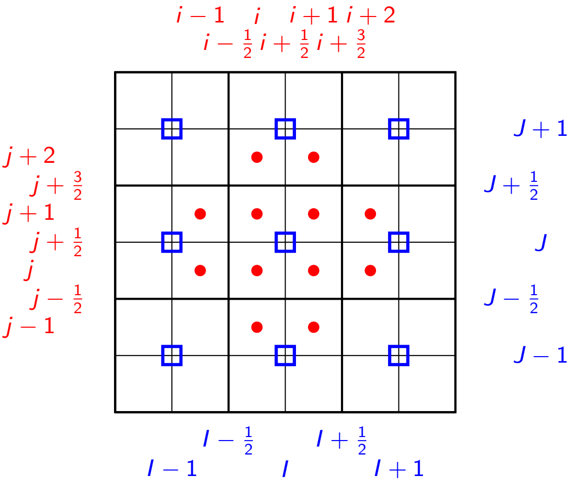


Figure: The level-$(\ell-1)$ grid points are denoted by empty blue squares, whereas filled red circles denote the level-ℓ grid points.

Proof (Cont.)

Then,

$$\begin{aligned}
\alpha_{I,J} = & -D_{\ell,i+1/2,j}\delta^x v_{\ell,i+1/2,j} + D_{\ell,i-1/2,j}\delta^x v_{\ell,i-1/2,j} \\
& - D_{\ell,i,j+1/2}\delta^y v_{\ell,i,j+1/2} + D_{\ell,i,j-1/2}\delta^y v_{\ell,i,j-1/2} \\
& - D_{\ell,i+3/2,j}\delta^x v_{\ell,i+3/2,j} + D_{\ell,i+1/2,j}\delta^x v_{\ell,i+1/2,j} \\
& - D_{\ell,i+1,j+1/2}\delta^y v_{\ell,i+1,j+1/2} + D_{\ell,i+1,j-1/2}\delta^y v_{\ell,i+1,j-1/2} \\
& - D_{\ell,i+1/2,j+1}\delta^x v_{\ell,i+1/2,j+1} + D_{\ell,i-1/2,j+1}\delta^x v_{\ell,i-1/2,j+1} \\
& - D_{\ell,i,j+3/2}\delta^y v_{\ell,i,j+3/2} + D_{\ell,i,j+1/2}\delta^y v_{\ell,i,j+1/2} \\
& - D_{\ell,i+3/2,j+1}\delta^x v_{\ell,i+3/2,j+1} + D_{\ell,i+1/2,j+1}\delta^x v_{\ell,i+1/2,j+1} \\
& - D_{\ell,i+1,j+3/2}\delta^y v_{\ell,i+1,j+3/2} + D_{\ell,i+1,j+1/2}\delta^y v_{\ell,i+1,j+1/2}.
\end{aligned}$$

Proof (Cont.)

Cancelling (flux) terms that have opposite signs, we find

$$\begin{aligned}
\alpha_{I,J} = {} & D_{\ell,i-1/2,j}\left(v_{\ell,i,j} - v_{\ell,i-1,j}\right) \\
& + D_{\ell,i,j-1/2}\left(v_{\ell,i,j} - v_{\ell,i,j-1}\right) \\
& - D_{\ell,i+3/2,j}\left(v_{\ell,i+2,j} - v_{\ell,i+1,j}\right) \\
& + D_{\ell,i+1,j-1/2}\left(v_{\ell,i+1,j} - v_{\ell,i+1,j-1}\right) \\
& + D_{\ell,i-1/2,j+1}\left(v_{\ell,i,j+1} - v_{\ell,i-1,j+1}\right) \\
& - D_{\ell,i,j+3/2}\left(v_{\ell,i,j+2} - v_{\ell,i,j+1}\right) \\
& - D_{\ell,i+3/2,j+1}\left(v_{\ell,i+2,j+1} - v_{\ell,i+1,j+1}\right) \\
& - D_{\ell,i+1,j+3/2}\left(v_{\ell,i+1,j+2} - v_{\ell,i+1,j+1}\right).
\end{aligned}$$

Proof (Cont.)

Using the definition of the prolongation operator, referring to the particular 2-level indexing in the figure, and collecting like terms, we have

$$\begin{aligned}
\alpha_{I,J} = & -\left(D_{\ell,i+3/2,j} + D_{\ell,i+3/2,j+1}\right)\left(v_{\ell-1,I+1,J} - v_{\ell-1,I,J}\right) \\
& + \left(D_{\ell,i-1/2,j} + D_{\ell,i-1/2,j+1}\right)\left(v_{\ell-1,I,J} - v_{\ell-1,I-1,J}\right) \\
& - \left(D_{\ell,i,j+3/2} + D_{\ell,i+1,j+3/2}\right)\left(v_{\ell-1,I,J+1} - v_{\ell-1,I,J}\right) \\
& + \left(D_{\ell,i,j-1/2} + D_{\ell,i+1,j-1/2}\right)\left(v_{\ell-1,I,J} - v_{\ell-1,I,J-1}\right).
\end{aligned}$$

Proof (Cont.)

Defining the discrete coefficients at the coarse-grid level via (12), we find

$$\begin{aligned}
\alpha_{I,J} = 2[&-D_{\ell-1,I+1/2,J}\left(v_{\ell-1,I+1,J} - v_{\ell-1,I,J}\right) \\
&+ D_{\ell-1,I-1/2,J}\left(v_{\ell-1,I,J} - v_{\ell-1,I-1,J}\right) \\
&- D_{\ell-1,I,J+1/2}\left(v_{\ell-1,I,J+1} - v_{\ell-1,I,J}\right) \\
&+ D_{\ell-1,I,J-1/2}\left(v_{\ell-1,I,J} - v_{\ell-1,I,J-1}\right)] \\
= -2h_{\ell-1}^2 &\left(\Delta_{\ell-1}^{D,\circ} v_{\ell-1}\right)_{I,J}.
\end{aligned}$$

This yields (13). □

Remark

We just proven that Assumption (G3) holds. To establish convergence of the various multigrid methods, we need to establish a smoothing property and an approximation property.

However, the proof of a result analogous to the α-weak approximation property for the one-dimensional, constant-diffusivity case, is a non-trivial undertaking and beyond the scope of our little book.

Cell-Centered Smoothers

To conclude this section, let us describe two simple smoothing strategies that work well in the cell-centered finite difference setting, the damped quasi-Jacobi smoother smoother and the damped quasi-Gauss-Seidel smoother. To keep the discussion general, we will consider the variable diffusivity case.

Damped Quasi-Jacobi Smoother

Let us first describe a damped quasi-Jacobi method. Let $u_\ell^k \in \mathcal{C}_\ell^\circ$ be given. It is an easy task to find a unique function $u_\ell^k \in \mathcal{C}_{\ell,0}$ – for which we use the same symbol, in an obvious abuse of notation – that agrees with the first at every interior cell-centered grid point but now has discrete homogeneous Dirichlet boundary conditions. Define the grid function $z_\ell \in \mathcal{C}_\ell^\circ$ as follows: for each $1 \le i, j \le m_\ell$,

$$
\begin{aligned}
f_{\ell,i,j} = {} & \frac{-D_{\ell,i+1/2,j}\left(u^k_{\ell,i+1,j} - z_{\ell,i,j}\right) + D_{\ell,i-1/2,j}\left(z_{\ell,i,j} - u^k_{\ell,i-1,j}\right)}{h_\ell^2} \\
& + \frac{-D_{\ell,i,j+1/2}\left(u^k_{\ell,i,j+1} - z_{\ell,i,j}\right) + D_{\ell,i,j-1/2}\left(z_{\ell,i,j} - u^k_{\ell,i,j-1}\right)}{h_\ell^2}.
\end{aligned}
$$

Damped Quasi-Jacobi Smoother

We can explicitly solve for $z_{\ell,i,j}$ at each index i and j, obtaining

$$z_{\ell,i,j} = \frac{h_\ell^2 f_{\ell,i,j} + D_{\ell,i+1/2,j} u_{\ell,i+1,j}^k + D_{\ell,i-1/2,j} u_{\ell,i-1,j}^k}{D_{\ell,i+1/2,j} + D_{\ell,i-1/2,j} + D_{\ell,i,j+1/2} + D_{\ell,i,j-1/2}} + \frac{D_{\ell,i,j+1/2} u_{\ell,i,j+1}^k + D_{\ell,i,j-1/2} u_{\ell,i,j-1}^k}{D_{\ell,i+1/2,j} + D_{\ell,i-1/2,j} + D_{\ell,i,j+1/2} + D_{\ell,i,j-1/2}}.$$

Observe that it does not matter in what order we compute the entries of the grid function z_ℓ. Once all the entries of z_ℓ are found, we compute the update, $u_\ell^{k+1} \in \mathcal{C}_\ell^\circ$, via

$$u_\ell^{k+1} = \omega z_\ell + (1-\omega) u_\ell^k \in \mathcal{C}_\ell^\circ,$$

where $\omega \in (0,1]$ is a damping parameter.

Damped Quasi-Jacobi Smoother

This method is called *quasi-Jacobi*, because, unlike the true Jacobi method, we do not modify the stencil near the boundary to take into account the discrete homogeneous Dirichlet boundary conditions. In particular, we always divide by

$$D_{\ell,i+1/2,j} + D_{\ell,i-1/2,j} + D_{\ell,i,j+1/2} + D_{\ell,i,j-1/2}$$

in quasi-Jacobi, regardless of the indices, whereas in true damped Jacobi the denominator would be modified at boundary and corner cells.

It is well known that using $\omega = 1$, which corresponds to no damping, results in a method that does not adequately smooth high frequency errors, and the method fails to converge. Choosing a value in the range $\omega \in [1/2, 2/3]$ usually provides good multigrid convergence, as we will see.

Definition

Suppose that $1 \leq i, j \leq m$. We say that the indices i and j are **traversed in forward lexicographic order** iff the indices are put into one-to-one correspondence with the integers $\{1, 2, \ldots, m^2\}$, that is to say, the indices are counted, via the mapping

$$(i, j) \mapsto F(i, j) := i + (j - 1)n.$$

The function F is called the **forward enumeration function**. We say that the indices i and j are **traversed in backward lexicographic order** iff the indices are counted via the mapping

$$(i, j) \mapsto B(i, j) := n - i + 1 + (n - j)n.$$

The function B is called the **backward enumeration function**.

An example of the forward and backward lexicographic enumerations of a cell-centered 4-by-4 grid is illustrated in the figure on the next slide.

13	14	15	16
9	10	11	12
5	6	7	8
1	2	3	4

(a) Forward Lexicographic

4	3	2	1
8	7	6	5
12	11	10	9
16	15	14	13

(b) Backward Lexicographic

Figure: (a) Forward lexicographic and (b) backward lexicographic enumeration of the cells of a 4×4 grid.

Forward Damped Quasi-Gauss-Seidel Smoother

Let $u_\ell^k \in \mathcal{C}_\ell^\circ$. Denote by the same symbol, $u_\ell^k \in \mathcal{C}_{\ell,0}$, the unique homogeneous Dirichlet cell-centered grid function that agrees with the first at every interior cell-centered grid point. Suppose that the indices $i, j \in \{1, \ldots, m_\ell\}$ are traversed in *forward lexicographic order*. Then, for each $1 \leq i, j \leq m_\ell$, define $\alpha_{F,i,j}$ via

$$f_{\ell,i,j} = \frac{-D_{\ell,i+1/2,j}\left(u_{\ell,i+1,j}^k - \alpha_{F,i,j}\right) + D_{\ell,i-1/2,j}\left(\alpha_{F,i,j} - u_{\ell,i-1,j}^{k+1}\right)}{h_\ell^2} + \frac{-D_{\ell,i,j+1/2}\left(u_{\ell,i,j+1}^k - \alpha_{F,i,j}\right) + D_{\ell,i,j-1/2}\left(\alpha_{F,i,j} - u_{\ell,i,j-1}^{k+1}\right)}{h_\ell^2}.$$

Forward Damped Quasi-Gauss-Seidel Smoother

Solving for $\alpha_{F,i,j}$, we have

$$\alpha_{F,i,j} = \frac{h_\ell^2 f_{\ell,i,j} + D_{\ell,i+1/2,j} u^k_{\ell,i+1,j} + D_{\ell,i-1/2,j} u^{k+1}_{\ell,i-1,j}}{D_{\ell,i+1/2,j} + D_{\ell,i-1/2,j} + D_{\ell,i,j+1/2} + D_{\ell,i,j-1/2}} + \frac{D_{\ell,i,j+1/2} u^k_{\ell,i,j+1} + D_{\ell,i,j-1/2} u^{k+1}_{\ell,i,j-1}}{D_{\ell,i+1/2,j} + D_{\ell,i-1/2,j} + D_{\ell,i,j+1/2} + D_{\ell,i,j-1/2}}.$$

Before moving to the next index in the forward lexicographic order, we update via the formula

$$u^{k+1}_{\ell,i,j} = \omega \alpha_{F,i,j} + (1-\omega) u^k_{\ell,i,j},$$

where $\omega \in (0,1]$ is a fixed damping parameter. Observe that in the forward Gauss-Seidel framework, as soon as an updated value of $u^{k+1}_{\ell,i,j}$ is obtained at (i,j), it is used in the calculations of $u^{k+1}_{\ell,i',j'}$, where

$$F(i,j) < F(i',j').$$

The name *quasi-Gauss-Seidel* is used because we do not modify the stencil at boundary and corner cells, as would be done by true Gauss-Seidel method.

Backward Damped Quasi-Gauss-Seidel Smoother

The idea for the backward smoother is analogous to the forward version, but we traverse the indices in the opposite order. Let $u_\ell^k \in \mathcal{C}_\ell^\circ$. Denote by the same symbol, $u_\ell^k \in \mathcal{C}_{\ell,0}$, the unique homogeneous Dirichlet cell-centered grid function that agrees with the first at every interior cell-centered grid point. Suppose that the indices $i, j \in \{1, \ldots, m_\ell\}$ are traversed in *backward lexicographic order*. Then, for each $1 \leq i, j \leq m_\ell$, define $\alpha_{B,i,j}$ via

$$\begin{aligned} f_{\ell,i,j} = {} & \frac{-D_{\ell,i+1/2,j}\left(u_{\ell,i+1,j}^{k+1} - \alpha_{B,i,j}\right) + D_{\ell,i-1/2,j}\left(\alpha_{B,i,j} - u_{\ell,i-1,j}^{k}\right)}{h_\ell^2} \\ & + \frac{-D_{\ell,i,j+1/2}\left(u_{\ell,i,j+1}^{k+1} - \alpha_{B,i,j}\right) + D_{\ell,i,j-1/2}\left(\alpha_{B,i,j} - u_{\ell,i,j-1}^{k}\right)}{h_\ell^2}. \end{aligned}$$

Backward Damped Quasi-Gauss-Seidel Smoother

Solving for $\alpha_{B,i,j}$, we have

$$\begin{aligned}\alpha_{B,i,j} = & \frac{h_\ell^2 f_{\ell,i,j} + D_{\ell,i+1/2,j} u_{\ell,i+1,j}^{k+1} + D_{\ell,i-1/2,j} u_{\ell,i-1,j}^{k}}{D_{\ell,i+1/2,j} + D_{\ell,i-1/2,j} + D_{\ell,i,j+1/2} + D_{\ell,i,j-1/2}} \\ & + \frac{D_{\ell,i,j+1/2} u_{\ell,i,j+1}^{k+1} + D_{\ell,i,j-1/2} u_{\ell,i,j-1}^{k}}{D_{\ell,i+1/2,j} + D_{\ell,i-1/2,j} + D_{\ell,i,j+1/2} + D_{\ell,i,j-1/2}}.\end{aligned}$$

Before moving to the next index in the backward lexicographic order, we update via the formula

$$u_{\ell,i,j}^{k+1} = \omega \alpha_{B,i,j} + (1-\omega) u_{\ell,i,j}^{k},$$

where $\omega \in (0,1]$ is a fixed damping parameter. In the backward Gauss-Seidel framework, as soon an updated value of $u_{\ell,i,j}^{k+1}$ is obtained at (i,j), it is used in the calculations of all future values, that is, $u_{\ell,i',j'}^{k+1}$, where

$$B(i,j) < B(i',j').$$