Math 673/4

# Multigrid Methods: A Mostly Matrix-Based Approach

## Chapter 05: Multigrid: Algorithms and Abstract Convergence Theory

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# Chapter 05, Part 2 of 3

# Multigrid: Algorithms and Abstract Convergence Theory

# Convergence of the Two-Grid Method Revisited

## Theorem (Convergence of the Two-Grid Method)

*Suppose that $L = 1$ (two-grid) $m_1 = m \geq 1$ and $m_2 = 0$ (one-sided). Suppose that Assumptions (G0, strong Galerkin condition), (A0, strong approximation property), and (S1, first smoothing property) all hold. Then*

$$\left\| \boldsymbol{u}_1^{\mathrm{E}} - \mathrm{TG}\left(\boldsymbol{f}_1, \boldsymbol{u}_1^{(0)}\right) \right\|_{\mathsf{A}_1} \leq C_{\mathrm{A0}} C_{\mathrm{S1}} m^{-1/2} \left\| \boldsymbol{u}_1^{\mathrm{E}} - \boldsymbol{u}_1^{(0)} \right\|_{\mathsf{A}_1},$$

*where*

$$\mathsf{A}_1 \boldsymbol{u}_1^{\mathrm{E}} = \boldsymbol{f}_1.$$

*Written another way,*

$$\left\| \boldsymbol{e}_1^{k+1} \right\|_{\mathsf{A}_1} \leq C_{\mathrm{A0}} C_{\mathrm{S1}} m^{-1/2} \left\| \boldsymbol{e}_1^{k} \right\|_{\mathsf{A}_1},$$

*where $\boldsymbol{e}_1^k = \boldsymbol{u}_1^{\mathrm{E}} - \boldsymbol{u}_1^{(0)}$.*

## Proof.

Recall that, in the present case,

$$\mathsf{E}_1 = \left(\mathsf{I}_1 - \tilde{\Pi}_1\right) \mathsf{K}_1^m,$$

and

$$\boldsymbol{e}_1^{k+1} = \mathsf{E}_1 \boldsymbol{e}_1^k,$$

or, equivalently

$$\boldsymbol{u}_1^{\mathrm{E}} - \mathrm{TG}\left(\boldsymbol{f}_1, \boldsymbol{u}_1^{(0)}\right) = \mathsf{E}_1 \left(\boldsymbol{u}_1^{\mathrm{E}} - \boldsymbol{u}_1^{(0)}\right).$$

When we prove (G0) and (A0) imply (A1) in the last slide deck, we also see that

$$\left\| \left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right) \boldsymbol{u}_\ell \right\|_{\mathsf{A}_\ell} \leq C_{\mathrm{A}0} \rho_\ell^{-1/2} \left\| \mathsf{A}_\ell \boldsymbol{u}_\ell \right\|_\ell, \tag{1}$$

for any $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$.

## Proof (Cont.)

Applying (1) (with $\ell = 1$), and using Assumption (S1), we have

$$\begin{aligned}
\left\| \boldsymbol{e}_1^{k+1} \right\|_{\mathsf{A}_1} &= \left\| \left( \mathsf{I}_1 - \tilde{\Pi}_1 \right) \mathsf{K}_1^m \boldsymbol{e}_1^k \right\|_{\mathsf{A}_1} \\
&\overset{(1)}{\leq} C_{\mathrm{A}0} \rho_1^{-1/2} \left\| \mathsf{A}_1 \mathsf{K}_1^m \boldsymbol{e}_1^k \right\|_1 \\
&\overset{(\mathrm{S1})}{\leq} C_{\mathrm{A}0} \rho_1^{-1/2} C_{\mathrm{S}1} \rho_1^{1/2} m^{-1/2} \left\| \boldsymbol{e}_1^k \right\|_{\mathsf{A}_1} \\
&= C_{\mathrm{A}0} C_{\mathrm{S}1} m^{-1/2} \left\| \boldsymbol{e}_1^k \right\|_{\mathsf{A}_1} .
\end{aligned}$$

□

Next, we prove the two-grid method converges even with a significantly weakened approximation property. Before we get to that result, we need a technical lemma.

## Lemma

*For Richardson's smoother we have the following stabilities:*

$$\begin{aligned} \left\|\mathsf{K}_\ell \boldsymbol{v}_\ell\right\|_{\mathsf{A}_\ell} &\leq \left\|\boldsymbol{v}_\ell\right\|_{\mathsf{A}_\ell}, \qquad (2) \\ \left(\mathsf{K}_\ell \boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_\ell &\leq \left(\boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_\ell, \qquad (3) \end{aligned}$$

*for all* $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}, \ell \geq 0$

## Proof.

Let $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$ be arbitrary. Suppose that $B_\ell := \left\{\boldsymbol{w}_\ell^{(1)}, \boldsymbol{w}_\ell^{(2)}, \cdots, \boldsymbol{w}_\ell^{(n_\ell)}\right\}$ is an orthonormal basis of eigenvectors of $\mathsf{A}_\ell$ with respect to $(\,\cdot\,,\,\cdot\,)_\ell$. Then, there exist unique constants $\alpha_1, \alpha_2, \cdots, \alpha_{n_\ell} \in \mathbb{R}$, such that

$$\boldsymbol{v}_\ell = \sum_{k=1}^{n_\ell} \alpha_k \boldsymbol{w}_\ell^{(k)}.$$

Recall

$$\mathsf{K}_\ell = \mathsf{I}_\ell - \Lambda_\ell^{-1}\mathsf{A}_\ell,$$

with

$$\rho(\mathsf{A}_\ell) =: \rho_\ell \le \Lambda_\ell \le C_{\mathrm{R}}\rho_\ell,$$

where $C_{\mathrm{R}} \ge 1$ is independent of $\ell$. Then

$$\mathsf{K}_\ell \boldsymbol{w}_\ell^{(k)} = \mu_\ell^{(k)} \boldsymbol{w}_\ell^{(k)},$$

where

$$\mu_\ell^{(k)} := \left(1 - \frac{\lambda_\ell^{(k)}}{\Lambda_\ell}\right).$$

## Proof (Cont.)

The $\lambda_\ell^{(k)}$ are the positive eigenvalues for the SPD matrix $\mathsf{A}_\ell$, and the $\mu_\ell^{(k)}$ are the eigenvalues for $\mathsf{K}_\ell$, $k = 1, \ldots, n_\ell$. Thus

$$\begin{aligned} \|\mathsf{K}_\ell \boldsymbol{v}_\ell\|_{\mathsf{A}_\ell}^2 &= (\mathsf{K}_\ell \boldsymbol{v}_\ell, \mathsf{A}_\ell \mathsf{K}_\ell \boldsymbol{v}_\ell)_\ell \\ &= \sum_{k=1}^{n_\ell} \left(\mu_\ell^{(k)}\right)^2 \lambda_\ell^{(k)} \alpha_k^2. \end{aligned}$$

Recall for the Richardson's smoother, we have

$$\Lambda_\ell \geq \rho_\ell = \rho(\mathsf{A}_\ell), \quad 1 \leq \ell \leq L, \tag{4}$$

and thus,

$$0 \leq \mu_\ell^{(k)} = 1 - \frac{\lambda_\ell^{(k)}}{\Lambda_\ell} \leq 1, \tag{5}$$

and we have

$$\|\mathsf{K}_\ell \boldsymbol{v}_\ell\|_{\mathsf{A}_\ell}^2 \leq \|\boldsymbol{v}_\ell\|_{\mathsf{A}_\ell}^2 .$$

Hence,

$$\|\mathsf{K}_\ell \boldsymbol{v}_\ell\|_{\mathsf{A}_\ell} \leq \|\boldsymbol{v}_\ell\|_{\mathsf{A}_\ell} .$$

## Proof (Cont.)

For the second estimate,

$$
\begin{aligned}
(\mathsf{K}_\ell \boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_\ell &= \left( \mathsf{K}_\ell \sum_{k=1}^{n_\ell} \alpha_k \boldsymbol{w}_\ell^{(k)}, \sum_{k=1}^{n_\ell} \alpha_k \boldsymbol{w}_\ell^{(k)} \right)_\ell \\
&= \left( \sum_{k=1}^{n_\ell} \alpha_k \mathsf{K}_\ell \boldsymbol{w}_\ell^{(k)}, \sum_{k=1}^{n_\ell} \alpha_k \boldsymbol{w}_\ell^{(k)} \right)_\ell \\
&= \left( \sum_{k=1}^{n_\ell} \alpha_k \mu_\ell^{(k)} \boldsymbol{w}_\ell^{(k)}, \sum_{k=1}^{n_\ell} \alpha_k \boldsymbol{w}_\ell^{(k)} \right)_\ell \\
&= \sum_{k=1}^{n_\ell} \alpha_k^2 \mu_\ell^{(k)} \\
&\overset{(5)}{\leq} \sum_{k=1}^{n_\ell} \alpha_k^2 \\
&= (\boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_\ell .
\end{aligned}
$$

□

## Theorem ($\alpha$–Weak Convergence of the Two-Grid Method)

*Suppose that $L = 1$ (two-grid) $m_1 = m \geq 1$ and $m_2 = 0$ (one-sided). Assume that Assumptions (G0, strong Galerkin condition) and (A2, $\alpha$–weak approximation property), hold. Suppose that smoothing is performed with the Richardson method. Then*

$$\left\| \boldsymbol{u}_1^{\mathrm{E}} - \mathrm{TG}\left(\boldsymbol{f}_1, \boldsymbol{u}_1^{(0)}\right) \right\|_{\mathsf{A}_1} \leq \left( \frac{C_{\mathrm{A2}} C_{\mathrm{S1}}}{m^{1/2}} \right)^{\alpha} \left\| \boldsymbol{u}_1^{\mathrm{E}} - \boldsymbol{u}_1^{(0)} \right\|_{\mathsf{A}_1}, \tag{6}$$

*where*

$$\mathsf{A}_1 \boldsymbol{u}_1^{\mathrm{E}} = \boldsymbol{f}_1.$$

*Written another way,*

$$\left\| \boldsymbol{e}_1^{k+1} \right\|_{\mathsf{A}_1} \leq \left( \frac{C_{\mathrm{A2}} C_{\mathrm{S1}}}{m^{1/2}} \right)^{\alpha} \left\| \boldsymbol{e}_1^{k} \right\|_{\mathsf{A}_1}, \tag{7}$$

*where $\boldsymbol{e}_1^k = \boldsymbol{u}_1^{\mathrm{E}} - \boldsymbol{u}_1^{(0)}$ and $\boldsymbol{e}_1^{k+1} = \boldsymbol{u}_1^{\mathrm{E}} - \boldsymbol{u}_1^{(3)}$.*

## Proof.

Again

$$\mathsf{E}_1 = \left(\mathsf{I}_1 - \tilde{\Pi}_1\right) \mathsf{K}_1^m,$$

and

$$\boldsymbol{e}_1^{k+1} = \mathsf{E}_1 \boldsymbol{e}_1^k,$$

or, equivalently

$$\boldsymbol{u}_1^{\mathrm{E}} - \mathrm{TG}\left(\boldsymbol{f}_1, \boldsymbol{u}_1^{(0)}\right) = \mathsf{E}_1 \left(\boldsymbol{u}_1^{\mathrm{E}} - \boldsymbol{u}_1^{(0)}\right).$$

Since the strong Galerkin condition holds,

$$\left(\mathsf{I}_1 - \tilde{\Pi}_1\right)^2 = \mathsf{I}_1 - \tilde{\Pi}_1,$$

and it is always true that

$$\left(\mathsf{I}_1 - \tilde{\Pi}_1\right)^* = \mathsf{I}_1 - \tilde{\Pi}_1.$$

## Proof (Cont.)

Thus, the $\alpha$–weak approximation can be expressed as follows: for all $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$ and $1 \leq \ell \leq L$,

$$\left\|(\mathsf{I}_1 - \tilde{\Pi}_1)\boldsymbol{u}_1\right\|_{\mathsf{A}_1}^2 = \left(\left(\mathsf{I}_1 - \tilde{\Pi}_1\right)\boldsymbol{u}_1, \boldsymbol{u}_1\right)_{\mathsf{A}_\ell}$$
$$\leq \frac{C_{\mathrm{A2}}^{2\alpha}}{\rho_1^{\alpha}} \left\|\mathsf{A}_1\boldsymbol{u}_1\right\|_1^{2\alpha} \left\|\boldsymbol{u}_1\right\|_{\mathsf{A}_1}^{2(1-\alpha)},$$

for some $C_{\mathrm{A2}} > 0$.

## Proof (Cont.)

Next, recall that Richardson's method satisfies Assumption (S1). Using this fact, the stability of Richardson's method, and the $\alpha$–weak approximation property (A2), we have

$$\begin{aligned}
\left\| \boldsymbol{e}_1^{k+1} \right\|_{\mathsf{A}_1} &= \left\| \left( \mathsf{I}_1 - \tilde{\Pi}_1 \right) \mathsf{K}_1^m \boldsymbol{e}_1^k \right\|_{\mathsf{A}_1} \\
&\leq \frac{C_{\mathrm{A2}}^{\alpha}}{\rho_1^{\alpha/2}} \left\| \mathsf{A}_1 \mathsf{K}_1^m \boldsymbol{e}_1^k \right\|_1^{\alpha} \left\| \mathsf{K}_1^m \boldsymbol{e}_1^k \right\|_{\mathsf{A}_1}^{(1-\alpha)} \\
&\overset{(\mathit{Stability})}{\leq} \frac{C_{\mathrm{A2}}^{\alpha}}{\rho_1^{\alpha/2}} \left\| \mathsf{A}_1 \mathsf{K}_1^m \boldsymbol{e}_1^k \right\|_1^{\alpha} \left\| \boldsymbol{e}_1^k \right\|_{\mathsf{A}_1}^{(1-\alpha)} \\
&\overset{(\mathrm{S1})}{\leq} \frac{C_{\mathrm{A2}}^{\alpha}}{\rho_1^{\alpha/2}} C_{\mathrm{S1}}^{\alpha} \rho_1^{\alpha/2} m^{-\alpha/2} \left\| \boldsymbol{e}_1^k \right\|_{\mathsf{A}_1} \\
&= \left( \frac{C_{\mathrm{A2}} C_{\mathrm{S1}}}{m^{1/2}} \right)^{\alpha} \left\| \boldsymbol{e}_1^k \right\|_{\mathsf{A}_1} .
\end{aligned}$$

□

# Convergence of the W-Cycle Algorithm

In this section, we will prove that the W-cycle converges, provided that we perform enough smoothing iterations per cycle. The proof of the convergence of the W-cycle algorithm uses a technique called a perturbation argument. Basically, we will show that the error is equal to the error in the two-grid method plus a perturbation that we can control.

### Theorem (Convergence of the One-Sided W-Cycle)

*Suppose that* $p \geq 2$, $m_1 = m \geq 1$, *and* $m_2 = 0$ *(one-sided). Suppose, further, that Assumptions (G0, Galerkin condition) and (A0, strong approximation property) hold and the smoothing is done by Richardson's smoother. Then for any* $0 < \gamma < 1$, $m$ *can be chosen large enough so that*

$$\left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \mathrm{MG}\left( \boldsymbol{g}_\ell, \ell, \boldsymbol{u}_\ell^{(0)} \right) \right\|_{\mathsf{A}_\ell} \leq \gamma \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right\|_{\mathsf{A}_\ell},$$

*for any* $\ell \geq 0$, *where*

$$\mathsf{A}_\ell \boldsymbol{u}_\ell^{\mathrm{E}} = \boldsymbol{g}_\ell.$$

**Proof.**

The proof is by induction.

(Base cases): The cases $\ell = 0$, and $\ell = 1$ (which is two-grid) are clearly true.

(Induction hypothesis): Assume

$$\left\| \mathsf{E}_{\ell-1} \boldsymbol{w}_{\ell-1} \right\|_{\mathsf{A}_{\ell-1}} \leq \gamma \left\| \boldsymbol{w}_{\ell-1} \right\|_{\mathsf{A}_{\ell-1}}$$

is true for any $\boldsymbol{w}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}$.

## Proof (Cont.)

(General case): Suppose that $\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})}, \boldsymbol{r}_{\ell-1}^{(1)} \in \mathbb{R}^{n_{\ell-1}}$ satisfy

$$\mathsf{A}_{\ell-1}\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} = \boldsymbol{r}_{\ell-1}^{(1)}.$$

Recall, $\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})}$ is the *exact* coarse grid correction. Then

$$\begin{aligned}
\boldsymbol{u}_\ell^{\mathrm{E}} - \mathrm{MG}\left(\boldsymbol{g}_\ell, \ell, \boldsymbol{u}_\ell^{(0)}\right) &= \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(2)} \\
&= \boldsymbol{u}_\ell^{\mathrm{E}} - \left\{\boldsymbol{u}_\ell^{(1)} + \mathsf{P}_{\ell-1}\boldsymbol{q}_{\ell-1}^{(1)}\right\} \\
&= \boldsymbol{u}_\ell^{\mathrm{E}} - \left(\boldsymbol{u}_\ell^{(1)} + \mathsf{P}_{\ell-1}\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})}\right) \\
&\quad + \mathsf{P}_{\ell-1}\left(\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \boldsymbol{q}_{\ell-1}^{(1)}\right) \\
&= \boldsymbol{u}_\ell^{\mathrm{E}} - \mathrm{TG}\left(\boldsymbol{g}_\ell, \boldsymbol{u}_\ell^{(0)}\right) + \mathsf{P}_{\ell-1}\left(\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \boldsymbol{q}_{\ell-1}^{(1)}\right).
\end{aligned}$$

## Proof (Cont.)

Suppose that $m \in \mathbb{N}$ satisfies

$$0 < \left( \frac{C_{\mathrm{A0}} C_{\mathrm{S1}}}{\gamma - \gamma^p} \right)^2 \leq m. \tag{8}$$

We have proved Richardson's smoother satisfies Assumption (S1) in the last slide deck. Thus,

$$\begin{aligned} \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \mathrm{MG}\left( \boldsymbol{g}_\ell, \ell, \boldsymbol{u}_\ell^{(0)} \right) \right\|_{\mathsf{A}_\ell} &\leq \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \mathrm{TG}\left( \boldsymbol{g}_\ell, \boldsymbol{u}_\ell^{(0)} \right) \right\|_{\mathsf{A}_\ell} \\ &\quad + \left\| \mathsf{P}_{\ell-1} \left( \boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \boldsymbol{q}_{\ell-1}^{(1)} \right) \right\|_{\mathsf{A}_\ell} \\ &\overset{\text{(TG Conv.)}}{\leq} C_{\mathrm{A0}} C_{\mathrm{S1}} m^{-1/2} \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right\|_{\mathsf{A}_\ell} \\ &\quad + \left\| \mathsf{P}_{\ell-1} \left( \boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \boldsymbol{q}_{\ell-1}^{(1)} \right) \right\|_{\mathsf{A}_\ell} . \end{aligned} \tag{9}$$

## Proof (Cont.)

Now, observe that, for any $\boldsymbol{w}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}$,

$$
\begin{aligned}
\left\|\mathsf{P}_{\ell-1}\boldsymbol{w}_{\ell-1}\right\|_{\mathsf{A}_\ell}^2 &= \left(\mathsf{P}_{\ell-1}\boldsymbol{w}_{\ell-1}, \mathsf{P}_{\ell-1}\boldsymbol{w}_{\ell-1}\right)_{\mathsf{A}_\ell} \\
&= \left(\mathsf{P}_{\ell-1}\boldsymbol{w}_{\ell-1}, \mathsf{A}_\ell\mathsf{P}_{\ell-1}\boldsymbol{w}_{\ell-1}\right)_\ell \\
&= \left(\boldsymbol{w}_{\ell-1}, \mathsf{P}_{\ell-1}^\top\mathsf{A}_\ell\mathsf{P}_{\ell-1}\boldsymbol{w}_{\ell-1}\right)_{\ell-1} \\
&= \left(\boldsymbol{w}_{\ell-1}, \mathsf{R}_{\ell-1}\mathsf{A}_\ell\mathsf{P}_{\ell-1}\boldsymbol{w}_{\ell-1}\right)_{\ell-1} \\
&\overset{(G0)}{=} \left(\boldsymbol{w}_{\ell-1}, \mathsf{A}_{\ell-1}\boldsymbol{w}_{\ell-1}\right)_{\ell-1} \\
&= \left(\boldsymbol{w}_{\ell-1}, \boldsymbol{w}_{\ell-1}\right)_{\mathsf{A}_{\ell-1}} \\
&= \left\|\boldsymbol{w}_{\ell-1}\right\|_{\mathsf{A}_{\ell-1}}^2 .
\end{aligned}
$$

## Proof (Cont.)

In the proof of the Multigrid error relation theorem in the last slide deck, we showed that

$$\begin{aligned}
\boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \boldsymbol{q}_{\ell-1}^{(1)} &= \mathsf{E}_{\ell-1}^{p} \boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} \\
&\overset{\text{(MG Err. Rel.)}}{=} \mathsf{E}_{\ell-1}^{p} \Pi_{\ell-1} \left( \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(1)} \right) \\
&= \mathsf{E}_{\ell-1}^{p} \Pi_{\ell-1} \mathsf{K}_\ell^{m} \left( \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right).
\end{aligned}$$

Using the induction hypothesis,

$$\begin{aligned}
\left\| \mathsf{P}_{\ell-1} \left( \boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \boldsymbol{q}_{\ell-1}^{(1)} \right) \right\|_{\mathsf{A}_\ell} &= \left\| \boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \boldsymbol{q}_{\ell-1}^{(1)} \right\|_{\mathsf{A}_{\ell-1}} \\
&= \left\| \mathsf{E}_{\ell-1}^{p} \Pi_{\ell-1} \mathsf{K}_\ell^{m} \left( \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right) \right\|_{\mathsf{A}_{\ell-1}} \\
&\overset{\text{Ind. Hyp.}}{\leq} \gamma^p \left\| \Pi_{\ell-1} \mathsf{K}_\ell^{m} \left( \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right) \right\|_{\mathsf{A}_{\ell-1}}.
\end{aligned}$$

## Proof (Cont.)

Since we are assuming the Galerkin condition (Assumption (G0)) holds, it follows that

$$\|\Pi_{\ell-1}\boldsymbol{w}_\ell\|_{\mathsf{A}_{\ell-1}} = \left\|\tilde{\Pi}_\ell \boldsymbol{w}_\ell\right\|_{\mathsf{A}_\ell}.$$

Furthermore,

$$\begin{aligned}
\left\|\tilde{\Pi}_\ell \boldsymbol{w}_\ell\right\|_{\mathsf{A}_\ell}^2 &= \left(\tilde{\Pi}_\ell \boldsymbol{w}_\ell, \tilde{\Pi}_\ell \boldsymbol{w}_\ell\right)_{\mathsf{A}_\ell} \\
&= \left(\tilde{\Pi}_\ell^2 \boldsymbol{w}_\ell, \boldsymbol{w}_\ell\right)_{\mathsf{A}_\ell} \\
&= \left(\tilde{\Pi}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell\right)_{\mathsf{A}_\ell} \\
&\overset{\text{C.S.}}{\leq} \left\|\tilde{\Pi}_\ell \boldsymbol{w}_\ell\right\|_{\mathsf{A}_\ell} \|\boldsymbol{w}_\ell\|_{\mathsf{A}_\ell}.
\end{aligned}$$

So, we have the stability

$$\left\|\tilde{\Pi}_\ell \boldsymbol{w}_\ell\right\|_{\mathsf{A}_\ell} \leq \|\boldsymbol{w}_\ell\|_{\mathsf{A}_\ell}. \tag{10}$$

## Proof (Cont.)

Therefore,

$$
\begin{aligned}
\left\| \mathsf{P}_{\ell-1} \left( \boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \boldsymbol{q}_{\ell-1}^{(1)} \right) \right\|_{\mathsf{A}_\ell} &\leq \gamma^p \left\| \Pi_{\ell-1} \mathsf{K}_\ell^m \left( \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right) \right\|_{\mathsf{A}_{\ell-1}} \\
&= \gamma^p \left\| \tilde{\Pi}_\ell \mathsf{K}_\ell^m \left( \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right) \right\|_{\mathsf{A}_\ell} \\
&\overset{(10)}{\leq} \gamma^p \left\| \mathsf{K}_\ell^m \left( \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right) \right\|_{\mathsf{A}_\ell} \\
&\overset{(\text{Stability})}{\leq} \gamma^p \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right\|_{\mathsf{A}_\ell} .
\end{aligned}
$$

## Proof (Cont.)

Combining this with estimate (9), we have

$$
\begin{aligned}
&\left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \mathrm{MG}\left(\boldsymbol{g}_\ell, \ell, \boldsymbol{u}_\ell^{(0)}\right) \right\|_{\mathsf{A}_\ell} \\
&\qquad \le \quad C_{\mathrm{A}0} C_{\mathrm{S}1} m^{-1/2} \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right\|_{\mathsf{A}_\ell} + \left\| \mathsf{P}_{\ell-1}\left( \boldsymbol{q}_{\ell-1}^{(1,\mathrm{E})} - \boldsymbol{q}_{\ell-1}^{(1)} \right) \right\|_{\mathsf{A}_\ell} \\
&\qquad \le \quad C_{\mathrm{A}0} C_{\mathrm{S}1} m^{-1/2} \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right\|_{\mathsf{A}_\ell} + \gamma^p \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right\|_{\mathsf{A}_\ell} \\
&\qquad \le \quad \left( C_{\mathrm{A}0} C_{\mathrm{S}1} m^{-1/2} + \gamma^p \right) \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right\|_{\mathsf{A}_\ell} \\
&\qquad \overset{(8)}{\le} \quad \left( C_{\mathrm{A}0} C_{\mathrm{S}1} \left( \left( \frac{C_{\mathrm{A}0} C_{\mathrm{S}1}}{\gamma - \gamma^p} \right)^2 \right)^{-1/2} + \gamma^p \right) \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right\|_{\mathsf{A}_\ell} \\
&\qquad = \quad \gamma \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right\|_{\mathsf{A}_\ell}.
\end{aligned}
$$

□

### Remark

*Notice that we need $p > 1$ for this argument to work. Otherwise $\gamma - \gamma^p$ is zero and $m$ would need to be infinitely large to get convergence.*

We can weaken our approximation property assumptions and still achieve convergence.

## Theorem ($\alpha$–Weak Convergence of the One-Sided W-Cycle)

*Suppose that* $p \geq 2$, $m_1 = m \geq 1$, *and* $m_2 = 0$ *(one-sided). Suppose, further, that Assumptions (G0, strong Galerkin condition) and (A2,* $\alpha$*–weak approximation property) hold and the smoothing is done by Richardson's smoother. Then for any* $0 < \gamma < 1$, $m$ *can be chosen large enough so that*

$$\left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \mathrm{MG}\left(\boldsymbol{g}_\ell, \ell, \boldsymbol{u}_\ell^{(0)}\right) \right\|_{\mathsf{A}_\ell} \leq \gamma \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right\|_{\mathsf{A}_\ell},$$

*for any* $\ell \geq 0$, *where*

$$\mathrm{A}_\ell \boldsymbol{u}_\ell^{\mathrm{E}} = \boldsymbol{g}_\ell.$$

*In particular, it suffices to choose* $m \in \mathbb{N}$ *such that*

$$\left( \frac{C_{\mathrm{A2}} C_{\mathrm{S1}}}{(\gamma - \gamma^p)^{1/\alpha}} \right) 2 \leq m$$

*to achieve the desired contraction* $\gamma \in (0,1)$.

**Proof.**

The proof is quite similar to the previous case and is based on a perturbation argument. But, one needs to use the $\alpha$–weak convergence of the two-grid method. □

# Convergence of the Simple Symmetric V-Cycle

In this section, we will prove that the simple symmetric V-cycle algorithm ($p = 1$ and $m_1 = m_2 = 1$) converges. First we need a new, and useful, smoothing assumption.

## Definition (Assumption (S2))

We say that the multigrid algorithm satisfies the **second smoothing property**, equivalently, **Assumption (S2)**, iff there is some $C_{\mathrm{S2}} > 0$ such that

$$\|\boldsymbol{v}_\ell\|_\ell^2 \le \rho_\ell C_{\mathrm{S2}}^2 \left(\overline{\mathsf{K}}_\ell \boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_\ell, \qquad (11)$$

for all $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$ and $\ell \ge 1$, where

$$\overline{\mathsf{K}}_\ell := (\mathsf{I}_\ell - \mathsf{K}_\ell^* \mathsf{K}_\ell)\, \mathsf{A}_\ell^{-1}.$$

**Lemma**

*Richardson's smoother satisfies Assumption (S2) with* $\mathsf{S}_\ell = \Lambda_\ell^{-1} \mathsf{I}_\ell = \mathsf{S}_\ell^\top$.

## Proof.

Recall

$$\rho_\ell \le \Lambda_\ell \le C_{\mathrm{R}} \rho_\ell,$$

for some $C_{\mathrm{R}} \ge 1$ that is independent of $\ell$. Then

$$\begin{aligned}
\overline{\mathsf{K}}_\ell &= (\mathsf{I}_\ell - \mathsf{K}_\ell^* \mathsf{K}_\ell)\mathsf{A}_\ell^{-1} \\
&= \left\{ \mathsf{I}_\ell - \left( \mathsf{I}_\ell - \Lambda_\ell^{-1} \mathsf{A}_\ell \right) \left( \mathsf{I}_\ell - \Lambda_\ell^{-1} \mathsf{A}_\ell \right) \right\} \mathsf{A}_\ell^{-1} \\
&= \left( \mathsf{I}_\ell - \left\{ \mathsf{I}_\ell - 2\Lambda_\ell^{-1} \mathsf{A}_\ell + \Lambda_\ell^{-2} \mathsf{A}_\ell^2 \right\} \right) \mathsf{A}_\ell^{-1} \\
&= 2\Lambda_\ell^{-1} \mathsf{I}_\ell - \Lambda_\ell^{-2} \mathsf{A}_\ell .
\end{aligned}$$

Define

$$\mathsf{J}_\ell := \rho_\ell C_{\mathrm{R}} \overline{\mathsf{K}}_\ell - \mathsf{I}_\ell .$$

If we can show that $\mathsf{J}_\ell$ is SPSD with respect to $(\,\cdot\,,\,\cdot\,)_\ell$ then we get (S2) with $C_{\mathrm{S2}}^2 = C_{\mathrm{R}}$.

## Proof (Cont.)

$\mathsf{J}_\ell$ is clearly symmetric with respect to $(\cdot,\cdot)_\ell$. Now let $\left\{\boldsymbol{v}_\ell^{(1)}, \boldsymbol{v}_\ell^{(2)}, \cdots, \boldsymbol{v}_\ell^{(n_\ell)}\right\}$ be the orthonormal basis of eigenvectors of $\mathsf{A}_\ell$ with respect to $(\cdot,\cdot)_\ell$. Then

$$
\begin{aligned}
\mathsf{J}_\ell \boldsymbol{v}_\ell^{(k)} &= \rho_\ell C_{\mathrm{R}} \overline{\mathsf{K}}_\ell \boldsymbol{v}_\ell^{(k)} - \boldsymbol{v}_\ell^{(k)} \\
&= \rho_\ell C_{\mathrm{R}} \left(2\Lambda_\ell^{-1}\mathsf{I}_\ell - \Lambda_\ell^{-2}\mathsf{A}_\ell\right) \boldsymbol{v}_\ell^{(k)} - \boldsymbol{v}_\ell^{(k)} \\
&= 2\rho_\ell C_{\mathrm{R}} \Lambda_\ell^{-1} \boldsymbol{v}_\ell^{(k)} - \rho C_{\mathrm{R}} \Lambda_\ell^{-2} \lambda_\ell^{(k)} \boldsymbol{v}_\ell^{(k)} - \boldsymbol{v}_\ell^{(k)} \\
&= \left(2\rho_\ell C_{\mathrm{R}} \Lambda_\ell^{-1} - \rho_\ell C_{\mathrm{R}} \Lambda_\ell^{-2} \lambda_\ell^{(k)} - 1\right) \boldsymbol{v}_\ell^{(k)}.
\end{aligned}
$$

Set

$$
\eta_\ell^{(k)} := 2\rho_\ell C_{\mathrm{R}} \Lambda_\ell^{-1} - \rho_\ell C_{\mathrm{R}} \Lambda_\ell^{-2} \lambda_\ell^{(k)} - 1.
$$

## Proof (Cont.)

We want to show that $\eta_\ell^{(k)} \geq 0$ for all $1 \leq k \leq n_\ell$.

$$\begin{aligned}
\eta_\ell^{(k)} &= 2C_{\mathrm{R}}\frac{\rho_\ell}{\Lambda_\ell} - C_{\mathrm{R}}\frac{\rho_\ell \lambda_\ell^{(k)}}{\Lambda_\ell^2} - 1 \\
&\geq 2C_{\mathrm{R}}\frac{\rho_\ell}{\Lambda_\ell} - C_{\mathrm{R}}\frac{\rho_\ell}{\Lambda_\ell} - 1 \qquad \left(\text{since } -\lambda_\ell^{(k)} \geq -\Lambda_\ell\right) \\
&= C_{\mathrm{R}}\frac{\rho_\ell}{\Lambda_\ell} - 1 \\
&\geq 1 - 1 \qquad \left(\text{since } C_{\mathrm{R}}\rho_\ell \geq \Lambda_\ell\right) \\
&= 0.
\end{aligned}$$

Thus the eigenvalues of $\mathsf{J}_\ell$, $\eta_\ell^{(k)}$, are all non-negative and $\mathsf{J}_\ell$ is SPSD. This implies

$$0 \leq (\mathsf{J}_\ell \boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_\ell = \rho_\ell C_{\mathrm{R}} \left(\overline{\mathsf{K}}_\ell \boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_\ell - (\boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_\ell \,,$$

and (S2) follows with $C_{\mathrm{S2}}^2 = C_{\mathrm{R}}$. □

Next, we need two more technical lemmas.

## Lemma

*Let* $\mathsf{J}_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$ *and* $\mathsf{J}_\ell = \mathsf{J}_\ell^*$. *Then*

$$\left(\mathsf{J}_\ell \boldsymbol{v}_\ell, \mathsf{J}_\ell \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} - \left(\mathsf{J}_\ell^2 \boldsymbol{v}_\ell, \mathsf{J}_\ell^2 \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} \leq \left(\boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} - \left(\mathsf{J}_\ell \boldsymbol{v}_\ell, \mathsf{J}_\ell \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell}, \tag{12}$$

*for any* $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$

## Proof.

Since $A_\ell$ is SPD.

$$\begin{aligned}
0 &\leq \left\| \left( I_\ell - J_\ell^2 \right) \boldsymbol{v}_\ell \right\|_{A_\ell}^2 \\
&= \left( \left( I_\ell - J_\ell^2 \right) \boldsymbol{v}_\ell, \left( I_\ell - J_\ell^2 \right) \boldsymbol{v}_\ell \right)_{A_\ell} \\
&= \left( I_\ell \boldsymbol{v}_\ell - J_\ell^2 \boldsymbol{v}_\ell, I_\ell \boldsymbol{v}_\ell - J_\ell^2 \boldsymbol{v}_\ell \right)_{A_\ell} \\
&= (\boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_{A_\ell} - \left( J_\ell^2 \boldsymbol{v}_\ell, \boldsymbol{v}_\ell \right)_{A_\ell} - \left( \boldsymbol{v}_\ell, J_\ell^2 \boldsymbol{v}_\ell \right)_{A_\ell} + \left( J_\ell^2 \boldsymbol{v}_\ell, J_\ell^2 \boldsymbol{v}_\ell \right)_{A_\ell} \\
&= (\boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_{A_\ell} - (J_\ell \boldsymbol{v}_\ell, J_\ell \boldsymbol{v}_\ell)_{A_\ell} - (J_\ell \boldsymbol{v}_\ell, J_\ell \boldsymbol{v}_\ell)_{A_\ell} + \left( J_\ell^2 \boldsymbol{v}_\ell, J_\ell^2 \boldsymbol{v}_\ell \right)_{A_\ell} \\
&= (\boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_{A_\ell} - 2 (J_\ell \boldsymbol{v}_\ell, J_\ell \boldsymbol{v}_\ell)_{A_\ell} + \left( J_\ell^2 \boldsymbol{v}_\ell, J_\ell^2 \boldsymbol{v}_\ell \right)_{A_\ell} .
\end{aligned}$$

So

$$(J_\ell \boldsymbol{v}_\ell, J_\ell \boldsymbol{v}_\ell)_{A_\ell} - \left( J_\ell^2 \boldsymbol{v}_\ell, J_\ell^2 \boldsymbol{v}_\ell \right)_{A_\ell} \leq (\boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_{A_\ell} - (J_\ell \boldsymbol{v}_\ell, J_\ell \boldsymbol{v}_\ell)_{A_\ell} .$$

$\square$

## Lemma

*For any* $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$

$$\left(\Pi_{\ell-1}\boldsymbol{v}_\ell, \Pi_{\ell-1}\boldsymbol{v}_\ell\right)_{\mathsf{A}_{\ell-1}} = \left(\boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} - \left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell}. \tag{13}$$

## Proof.

Recall that we always have

$$\mathsf{R}_{\ell-1}\mathsf{A}_\ell = \mathsf{A}_{\ell-1}\Pi_{\ell-1},$$

and

$$\tilde{\Pi}_\ell = \mathsf{P}_{\ell-1}\mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell = \mathsf{P}_{\ell-1}\Pi_{\ell-1}.$$

Then

$$\begin{aligned}
(\Pi_{\ell-1}\boldsymbol{v}_\ell, \Pi_{\ell-1}\boldsymbol{v}_\ell)_{\mathsf{A}_{\ell-1}} &= (\Pi_{\ell-1}\boldsymbol{v}_\ell, \mathsf{A}_{\ell-1}\Pi_{\ell-1}\boldsymbol{v}_\ell)_{\ell-1} \\
&= (\Pi_{\ell-1}\boldsymbol{v}_\ell, \mathsf{R}_{\ell-1}\mathsf{A}_\ell\boldsymbol{v}_\ell)_{\ell-1} \\
&= \left(\mathsf{R}_{\ell-1}^\top\Pi_{\ell-1}\boldsymbol{v}_\ell, \mathsf{A}_\ell\boldsymbol{v}_\ell\right)_\ell \\
&= (\mathsf{P}_{\ell-1}\Pi_{\ell-1}\boldsymbol{v}_\ell, \mathsf{A}_\ell\boldsymbol{v}_\ell)_\ell \\
&= \left(\tilde{\Pi}_\ell\boldsymbol{v}_\ell, \mathsf{A}_\ell\boldsymbol{v}_\ell\right)_\ell \\
&= \left(\tilde{\Pi}_\ell\boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} \\
&= (\boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_{\mathsf{A}_\ell} - \left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell}.
\end{aligned}$$

The simple symmetric V-cycle method is gotten by setting $m_1 = m_2 = 1$. It is somewhat surprising that the method converges, because only one pre-smoothing and one post-smoothing iteration is preformed.

### Theorem

*Suppose that Assumptions (G1, weak Galerkin condition), (A1, weak approximation property), and (S2, second smoothing property) all hold. Suppose that $p = 1$, $m_1 = m_2 = m = 1$, and $\mathsf{S}_\ell = \mathsf{S}_\ell^\top$. Then*

$$0 \leq (\mathsf{E}_\ell \boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathrm{A}_\ell} \leq \frac{C_{\mathrm{A1}}^2 C_{\mathrm{S2}}^2}{C_{\mathrm{A1}}^2 C_{\mathrm{S2}}^2 + 1} (\boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathrm{A}_\ell} ,$$

*for all $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$.*

## Proof.

Recall, since $p = 1$, $m_1 = m_2 = m = 1$, and $\mathsf{S}_\ell = \mathsf{S}_\ell^\top$,

$$\mathsf{E}_\ell = \mathsf{K}_\ell \left( \mathsf{I}_\ell - \tilde{\Pi}_\ell \right) \mathsf{K}_\ell + \mathsf{K}_\ell \mathsf{P}_{\ell-1} \mathsf{E}_{\ell-1} \Pi_{\ell-1} \mathsf{K}_\ell .$$

In particular, notice that

$$\mathsf{K}_\ell^* = \mathsf{I}_\ell - \mathsf{S}_\ell^\top \mathsf{A}_\ell = \mathsf{I}_\ell - \mathsf{S}_\ell \mathsf{A}_\ell = \mathsf{K}_\ell .$$

Now, set

$$T_1 := \left( \left( \mathsf{I}_\ell - \tilde{\Pi}_\ell \right) \boldsymbol{w}_\ell, \boldsymbol{w}_\ell \right)_{\mathsf{A}_\ell} ,$$

and

$$T_2 := \left( \mathsf{P}_{\ell-1} \mathsf{E}_{\ell-1} \Pi_{\ell-1} \boldsymbol{w}_\ell, \boldsymbol{w}_\ell \right)_{\mathsf{A}_\ell} ,$$

where

$$\boldsymbol{w}_\ell = \mathsf{K}_\ell \boldsymbol{u}_\ell .$$

Then

$$\left( \mathsf{E}_\ell \boldsymbol{u}_\ell, \boldsymbol{u}_\ell \right)_{\mathsf{A}_\ell} = T_1 + T_2 .$$

## Proof (Cont.)

Let us first consider $T_1$:

$$\begin{aligned}
T_1 &= \left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right) \boldsymbol{w}_\ell, \boldsymbol{w}_\ell\right)_{\mathsf{A}_\ell} \\
&\overset{(A1)}{\leq} C_{A1}^2 \rho_\ell^{-1} \left\| \mathsf{A}_\ell \boldsymbol{w}_\ell \right\|_\ell^2 \\
&= C_{A1}^2 \rho_\ell^{-1} \left\| \mathsf{A}_\ell \mathsf{K}_\ell \boldsymbol{u}_\ell \right\|_\ell^2 \\
&\overset{(S2)}{\leq} C_{A1}^2 \rho_\ell^{-1} C_{S2}^2 \rho_\ell \left(\overline{\mathsf{K}}_\ell \mathsf{A}_\ell \mathsf{K}_\ell \boldsymbol{u}_\ell, \mathsf{A}_\ell \mathsf{K}_\ell \boldsymbol{u}_\ell\right)_\ell \\
&= C_{A1}^2 C_{S2}^2 \left(\left(\mathsf{I}_\ell - \mathsf{K}_\ell^* \mathsf{K}_\ell\right) \mathsf{A}_\ell^{-1} \mathsf{A}_\ell \mathsf{K}_\ell \boldsymbol{u}_\ell, \mathsf{A}_\ell \mathsf{K}_\ell \boldsymbol{u}_\ell\right)_\ell \\
&= C_{A1}^2 C_{S2}^2 \left(\left(\mathsf{I}_\ell - \mathsf{K}_\ell^* \mathsf{K}_\ell\right) \mathsf{K}_\ell \boldsymbol{u}_\ell, \mathsf{A}_\ell \mathsf{K}_\ell \boldsymbol{u}_\ell\right)_\ell \\
&= C_{A1}^2 C_{S2}^2 \left\{ \left(\mathsf{K}_\ell \boldsymbol{u}_\ell, \mathsf{A}_\ell \mathsf{K}_\ell \boldsymbol{u}_\ell\right)_\ell - \left(\mathsf{K}_\ell^* \mathsf{K}_\ell \mathsf{K}_\ell \boldsymbol{u}_\ell, \mathsf{A}_\ell \mathsf{K}_\ell \boldsymbol{u}_\ell\right)_\ell \right\} \\
&= C_{A1}^2 C_{S2}^2 \left\{ \left(\mathsf{K}_\ell \boldsymbol{u}_\ell, \mathsf{K}_\ell \boldsymbol{u}_\ell\right)_{\mathsf{A}_\ell} - \left(\mathsf{K}_\ell^* \mathsf{K}_\ell \mathsf{K}_\ell \boldsymbol{u}_\ell, \mathsf{K}_\ell \boldsymbol{u}_\ell\right)_{\mathsf{A}_\ell} \right\} \\
&= C_{A1}^2 C_{S2}^2 \left\{ \left(\mathsf{K}_\ell \boldsymbol{u}_\ell, \mathsf{K}_\ell \boldsymbol{u}_\ell\right)_{\mathsf{A}_\ell} - \left(\mathsf{K}_\ell^2 \boldsymbol{u}_\ell, \mathsf{K}_\ell^2 \boldsymbol{u}_\ell\right)_{\mathsf{A}_\ell} \right\} \\
&\overset{(12)}{\leq} C_{A1}^2 C_{S2}^2 \left\{ \left(\boldsymbol{u}_\ell, \boldsymbol{u}_\ell\right)_{\mathsf{A}_\ell} - \left(\mathsf{K}_\ell \boldsymbol{u}_\ell, \mathsf{K}_\ell \boldsymbol{u}_\ell\right)_{\mathsf{A}_\ell} \right\}. \qquad (14)
\end{aligned}$$

## Proof (Cont.)

The proof proceeds by induction. The base case is trivial, and we skip that.

(Induction hypothesis): Assume that, for any $\boldsymbol{w}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}$,

$$\left(\mathsf{E}_{\ell-1}\boldsymbol{w}_{\ell-1}, \boldsymbol{w}_{\ell-1}\right)_{\mathsf{A}_{\ell-1}} \leq \gamma \left(\boldsymbol{w}_{\ell-1}, \boldsymbol{w}_{\ell-1}\right)_{\mathsf{A}_{\ell-1}}, \quad \gamma := \frac{C_{\mathrm{A1}}^2 C_{\mathrm{S2}}^2}{C_{\mathrm{A1}}^2 C_{\mathrm{S2}}^2 + 1}.$$

## Proof (Cont.)

(General case): Now, we turn to the bound for $T_2$. First, note that

$$\begin{aligned} T_2 &= \left(\mathsf{E}_{\ell-1}\Pi_{\ell-1}\boldsymbol{w}_\ell, \mathsf{R}_{\ell-1}\mathsf{A}_\ell\boldsymbol{w}_\ell\right)_{\ell-1} \\ &= \left(\mathsf{E}_{\ell-1}\Pi_{\ell-1}\boldsymbol{w}_\ell, \mathsf{A}_{\ell-1}\Pi_{\ell-1}\boldsymbol{w}_\ell\right)_{\ell-1} \\ &= \left(\mathsf{E}_{\ell-1}\Pi_{\ell-1}\boldsymbol{w}_\ell, \Pi_{\ell-1}\boldsymbol{w}_\ell\right)_{\mathsf{A}_{\ell-1}}. \end{aligned}$$

Then

$$\begin{aligned} T_2 &= \left(\mathsf{E}_{\ell-1}\Pi_{\ell-1}\boldsymbol{w}_\ell, \Pi_{\ell-1}\boldsymbol{w}_\ell\right)_{\mathsf{A}_{\ell-1}} \\ &\overset{\text{ind. hyp.}}{\leq} \gamma\left(\Pi_{\ell-1}\boldsymbol{w}_\ell, \Pi_{\ell-1}\boldsymbol{w}_\ell\right)_{\mathsf{A}_{\ell-1}} \\ &\overset{(13)}{=} \gamma\left\{\left(\boldsymbol{w}_\ell, \boldsymbol{w}_\ell\right)_{\mathsf{A}_\ell} - \left(\left(\mathsf{I}_\ell - \tilde{\Pi}_\ell\right)\boldsymbol{w}_\ell, \boldsymbol{w}_\ell\right)_{\mathsf{A}_\ell}\right\} \\ &= \gamma\left(\boldsymbol{w}_\ell, \boldsymbol{w}_\ell\right)_{\mathsf{A}_\ell} - \gamma T_1. \end{aligned} \tag{15}$$

## Proof (Cont.)

To finish up,

$$
\begin{aligned}
(\mathsf{E}_\ell \boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathsf{A}_\ell} &= T_1 + T_2 \\
&= (1-\gamma) T_1 + \gamma T_1 + T_2 \\
&\overset{(15)}{\leq} (1-\gamma) T_1 + \gamma T_1 + \gamma (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{A}_\ell} - \gamma T_1 \\
&= (1-\gamma) T_1 + \gamma (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{A}_\ell} \\
&\overset{(14)}{\leq} (1-\gamma) C_{\mathrm{A}1}^2 C_{\mathrm{S}2}^2 \left\{ (\boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathsf{A}_\ell} - (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{A}_\ell} \right\} + \gamma (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{A}_\ell} \\
&= \left( 1 - \frac{C_{\mathrm{A}1}^2 C_{\mathrm{S}2}^2}{C_{\mathrm{A}1}^2 C_{\mathrm{S}2}^2 + 1} \right) C_{\mathrm{A}1}^2 C_{\mathrm{S}2}^2 \left\{ (\boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathsf{A}_\ell} - (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{A}_\ell} \right\} \\
&\quad + \frac{C_{\mathrm{A}1}^2 C_{\mathrm{S}2}^2}{C_{\mathrm{A}1}^2 C_{\mathrm{S}2}^2 + 1} (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{A}_\ell}
\end{aligned}
$$

## Proof (Cont.)

$$
\begin{aligned}
&= \frac{C_{\mathrm{A1}}^2 C_{\mathrm{S2}}^2}{C_{\mathrm{A1}}^2 C_{\mathrm{S2}}^2 + 1} \left\{ (\boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathrm{A}_\ell} - (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathrm{A}_\ell} \right\} \\
&\quad + \frac{C_{\mathrm{A1}}^2 C_{\mathrm{S2}}^2}{C_{\mathrm{A1}}^2 C_{\mathrm{S2}}^2 + 1} (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathrm{A}_\ell} \\
&= \gamma \left\{ (\boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathrm{A}_\ell} - (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathrm{A}_\ell} \right\} + \gamma (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathrm{A}_\ell} \\
&= \gamma (\boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathrm{A}_\ell} .
\end{aligned}
$$

$\square$

## Corollary (Convergence of Simple Symmetric V-Cycle)

*Suppose that hypotheses of the last theorem hold and* $\boldsymbol{u}_\ell^{\mathrm{E}}, \boldsymbol{g}_\ell \in \mathbb{R}^{n_\ell}$ *satisfy*

$$\mathsf{A}_\ell \boldsymbol{u}_\ell^{\mathrm{E}} = \boldsymbol{g}_\ell.$$

*Then, given any* $\boldsymbol{u}_\ell^{(0)} \in \mathbb{R}^{n_\ell}$,

$$\begin{aligned} \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(3)} \right\|_{\mathsf{A}_\ell} &= \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \mathrm{MG}\left( \boldsymbol{g}_\ell, \ell, \boldsymbol{u}_\ell^{(0)} \right) \right\|_{\mathsf{A}_\ell} \\ &\le \frac{M}{M+m} \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right\|_{\mathsf{A}_\ell}, \end{aligned}$$

*where*

$$M = C_{\mathrm{A1}}^2 C_{\mathrm{S2}}^2 \quad \textit{and} \quad m = 1.$$

## Proof.

We need only to show that

$$\left\|\mathsf{E}_\ell \boldsymbol{v}_\ell\right\|_{\mathsf{A}_\ell} \leq \frac{M}{M+m} \left\|\boldsymbol{v}_\ell\right\|_{\mathsf{A}_\ell},$$

is true for any $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$. Since $\mathsf{E}_\ell$ is SPSD w.r.t. $(\cdot,\cdot)_{\mathsf{A}_\ell}$, for $\ell \geq 1$, there is a basis of eigenvectors of $\mathsf{E}_\ell$, $\left\{\boldsymbol{w}_\ell^{(1)}, \cdots, \boldsymbol{w}_\ell^{(n_\ell)}\right\}$, such that

$$\mathsf{E}_\ell \boldsymbol{w}_\ell^{(j)} = \epsilon_\ell^{(j)} \boldsymbol{w}_\ell^{(j)},$$

$$\left(\boldsymbol{w}_\ell^{(i)}, \boldsymbol{w}_\ell^{(j)}\right)_{\mathsf{A}_\ell} = \delta_{ij},$$

and

$$0 \leq \epsilon_\ell^{(1)} \leq \epsilon_\ell^{(2)} \leq \cdots \leq \epsilon_\ell^{(n_\ell)}.$$

## Proof (Cont.)

Suppose

$$\boldsymbol{v}_\ell = \sum_{k=1}^{n_\ell} c_k \boldsymbol{w}_\ell^{(k)}.$$

Then

$$(\mathsf{E}_\ell \boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_{\mathsf{A}_\ell} = \sum_{k=1}^{n_\ell} c_k^2 \epsilon_\ell^{(k)}$$

and

$$(\boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_{\mathsf{A}_\ell} = \sum_{k=1}^{n_\ell} c_k^2.$$

The last theorem guarantees that

$$\sum_{k=1}^{n_\ell} c_k^2 \epsilon_\ell^{(k)} \leq \frac{M}{M+m} \sum_{k=1}^{n_\ell} c_k^2,$$

for any $c_1, \cdots, c_{n_\ell} \in \mathbb{R}$. This implies that

$$0 \leq \epsilon_\ell^{(k)} \leq \frac{M}{M+m}, \quad 1 \leq k \leq n_\ell.$$

## Proof (Cont.)

Therefore

$$\begin{aligned}\|\mathsf{E}_\ell \boldsymbol{v}_\ell\|_{\mathsf{A}_\ell}^2 &= (\mathsf{E}_\ell \boldsymbol{v}_\ell, \mathsf{E}_\ell \boldsymbol{v}_\ell)_{\mathsf{A}_\ell} \\ &= \sum_{k=1}^{n_\ell} c_k^2 \left(\epsilon_\ell^{(k)}\right)^2 \\ &\leq \left(\frac{M}{M+m}\right)^2 \sum_{k=1}^{n_\ell} c_k^2 \\ &= \left(\frac{M}{M+m}\right)^2 \|\boldsymbol{v}_\ell\|_{\mathsf{A}_\ell}^2 .\end{aligned}$$

□

# Convergence of the General Symmetric V-Cycle

We now consider the general symmetric V-cycle. Our goal is to show that the convergence rate can be improved if more smoothing steps are performed. We need a technical lemma first.

## Lemma (Richardson's Smoother)

*Suppose that smoothing is done with Richardson's smoother, that is,*

$$\mathsf{S}_\ell = \Lambda_\ell^{-1} \mathsf{I}_\ell,$$

*where*

$$\rho_\ell \le \Lambda_\ell \le C_{\mathrm{R}} \rho_\ell,$$

*for some $C_{\mathrm{R}} \ge 1$ that is independent of $\ell$. Then, for any $m \ge 1$, $\ell \ge 1$, and all $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$, we have*

$$\left( (\mathsf{I}_\ell - \mathsf{K}_\ell) \mathsf{K}_\ell^{2m} \boldsymbol{v}_\ell, \boldsymbol{v}_\ell \right)_{\mathsf{A}_\ell} \le \frac{1}{2m} \left( \left( \mathsf{I}_\ell - \mathsf{K}_\ell^{2m} \right) \boldsymbol{v}_\ell, \boldsymbol{v}_\ell \right)_{\mathsf{A}_\ell}. \tag{16}$$

*Consequently,*

$$\rho_\ell^{-1} \left\| \mathsf{A}_\ell \mathsf{K}_\ell^m \boldsymbol{v}_\ell \right\|_\ell^2 \le \frac{C_{\mathrm{R}}}{2m} \left( \|\boldsymbol{v}_\ell\|_{\mathsf{A}_\ell}^2 - \|\mathsf{K}_\ell^m \boldsymbol{v}_\ell\|_{\mathsf{A}_\ell}^2 \right). \tag{17}$$

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## Proof.

Let $i, j \in \mathbb{Z}$, with $0 \leq j \leq i$. Then

$$\begin{aligned}
\left( (\mathsf{I}_\ell - \mathsf{K}_\ell)\, \mathsf{K}_\ell^i \boldsymbol{v}_\ell, \boldsymbol{v}_\ell \right)_{\mathsf{A}_\ell} &= \left( \mathsf{A}_\ell (\mathsf{I}_\ell - \mathsf{K}_\ell) \mathsf{K}_\ell^i \boldsymbol{v}_\ell, \boldsymbol{v}_\ell \right)_\ell \\
&= \Lambda_\ell^{-1} \left( \mathsf{A}_\ell^2 \mathsf{K}_\ell^i \boldsymbol{v}_\ell, \boldsymbol{v}_\ell \right)_\ell \\
&= \Lambda_\ell^{-1} \left( \mathsf{K}_\ell^i \mathsf{A}_\ell \boldsymbol{v}_\ell, \mathsf{A}_\ell \boldsymbol{v}_\ell \right)_\ell \\
&\overset{(3)}{\leq} \Lambda_\ell^{-1} \left( \mathsf{K}_\ell^j \mathsf{A}_\ell \boldsymbol{v}_\ell, \mathsf{A}_\ell \boldsymbol{v}_\ell \right)_\ell \\
&= \left( (\mathsf{I}_\ell - \mathsf{K}_\ell) \mathsf{K}_\ell^j \boldsymbol{v}_\ell, \boldsymbol{v}_\ell \right)_{\mathsf{A}_\ell} .
\end{aligned} \tag{18}$$

## Proof (Cont.)

Therefore,

$$
\begin{aligned}
2m\left((\mathsf{I}_\ell - \mathsf{K}_\ell)\mathsf{K}_\ell^{2m}\boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} &= \sum_{j=0}^{2m-1}\left((\mathsf{I}_\ell - \mathsf{K}_\ell)\mathsf{K}_\ell^{2m}\boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} \\
&\overset{(18)}{\leq} \sum_{j=0}^{2m-1}\left((\mathsf{I}_\ell - \mathsf{K}_\ell)\mathsf{K}_\ell^{j}\boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell} \\
&= \left((\mathsf{I}_\ell - \mathsf{K}_\ell^{2m})\boldsymbol{v}_\ell, \boldsymbol{v}_\ell\right)_{\mathsf{A}_\ell},
\end{aligned}
$$

where the last equality follows since the sum telescopes.

## Proof (Cont.)

To prove the second estimate, we use the first,

$$
\begin{aligned}
\rho_\ell^{-1} \left\| \mathsf{A}_\ell \mathsf{K}_\ell^m \boldsymbol{v}_\ell \right\|_\ell^2 &= \rho_\ell^{-1} \left( \mathsf{A}_\ell \mathsf{K}_\ell^m \boldsymbol{v}_\ell, \mathsf{A}_\ell \mathsf{K}_\ell^m \boldsymbol{v}_\ell \right)_\ell \\
&= \rho_\ell^{-1} \left( \mathsf{A}_\ell \mathsf{K}_\ell^m \boldsymbol{v}_\ell, \mathsf{K}_\ell^m \boldsymbol{v}_\ell \right)_{\mathsf{A}_\ell} \\
&= \rho_\ell^{-1} \Lambda_\ell \left( (\mathsf{I}_\ell - \mathsf{K}_\ell) \mathsf{K}_\ell^m \boldsymbol{u}_\ell, \mathsf{K}_\ell^m \boldsymbol{u}_\ell \right)_{\mathsf{A}_\ell} \\
&= \rho_\ell^{-1} \Lambda_\ell \left( (\mathsf{I}_\ell - \mathsf{K}_\ell) \mathsf{K}_\ell^{2m} \boldsymbol{v}_\ell, \boldsymbol{v}_\ell \right)_{\mathsf{A}_\ell} \\
&\overset{(16)}{\leq} \frac{\Lambda_\ell}{2m\rho_\ell} \left( (\mathsf{I}_\ell - \mathsf{K}_\ell^{2m}) \boldsymbol{v}_\ell, \boldsymbol{v}_\ell \right)_{\mathsf{A}_\ell} \\
&\leq \frac{C_{\mathrm{R}}}{2m} \left( (\mathsf{I}_\ell - \mathsf{K}_\ell^{2m}) \boldsymbol{v}_\ell, \boldsymbol{v}_\ell \right)_{\mathsf{A}_\ell} \\
&= \frac{C_{\mathrm{R}}}{2m} \left\{ (\boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_{\mathsf{A}_\ell} - (\mathsf{K}_\ell^m \boldsymbol{v}_\ell, \mathsf{K}_\ell^m \boldsymbol{v}_\ell)_{\mathsf{A}_\ell} \right\}. \qquad \square
\end{aligned}
$$

We are now in a position to give the famous result of Braess and Hackbusch from 1983, which provided the first uniform convergence estimate for the V-Cycle algorithm. The proof given here is a simplified, streamlined version of the original based on the presentation in Brenner and Scott (2008).

## Theorem (Braess–Hackbusch)

*Let $p = 1$, $m_1 = m_2 = m \geq 1$. Assume that assumptions (G1) and (A1) hold. If the smoothing is done with Richardson's smoother, then*

$$0 \leq (\mathsf{E}_\ell \boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathsf{A}_\ell} \leq \gamma \left(\boldsymbol{u}_\ell, \boldsymbol{u}_\ell\right)_{\mathsf{A}_\ell},$$

*for all $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$, where*

$$\gamma := \frac{M}{M+m}, \qquad M := \frac{C_{\mathrm{A1}}^2 C_{\mathrm{R}}}{2}.$$

## Proof.

As before, we begin with an expression for the error propagation matrix

$$\mathsf{E}_\ell = \mathsf{K}_\ell^m(\mathsf{I}_\ell - \tilde{\Pi}_\ell)\mathsf{K}_\ell^m + \mathsf{K}_\ell^m \mathsf{P}_{\ell-1}\mathsf{E}_{\ell-1}\Pi_{\ell-1}\mathsf{K}_\ell^m,$$

where

$$\mathsf{K}_\ell = \mathsf{I}_\ell - \Lambda_\ell^{-1}\mathsf{A}_\ell = \mathsf{K}_\ell^*,$$

and, for some $C_{\mathrm{R}} \geq 1$,

$$\rho_\ell \leq \Lambda_\ell \leq C_{\mathrm{R}}\rho_\ell.$$

As before, given $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$, set

$$\begin{aligned}
\boldsymbol{w}_\ell &:= \mathsf{K}_\ell^m \boldsymbol{u}_\ell,\\
T_1 &:= \left((\mathsf{I}_\ell - \tilde{\Pi}_\ell)\boldsymbol{w}_\ell, \boldsymbol{w}_\ell\right)_{\mathsf{A}_\ell},\\
T_2 &:= \left(\mathsf{P}_{\ell-1}\mathsf{E}_{\ell-1}\Pi_{\ell-1}\boldsymbol{w}_\ell, \boldsymbol{w}_\ell\right)_{\mathsf{A}_\ell}.
\end{aligned}$$

Then

$$(\mathsf{E}_\ell \boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathsf{A}_\ell} = T_1 + T_2.$$

## Proof (Cont.)

We first estimate $T_1$.

$$\begin{aligned} T_1 &= \left( (\mathsf{I}_\ell - \tilde{\Pi}_\ell)\boldsymbol{w}_\ell, \boldsymbol{w}_\ell \right)_{\mathsf{A}_\ell} \\ &\overset{(A1)}{\leq} C_{\mathrm{A}1}^2 \rho_\ell^{-1} \left\| \mathsf{A}_\ell \boldsymbol{w}_\ell \right\|_\ell^2 \\ &= C_{\mathrm{A}1}^2 \rho_\ell^{-1} \left\| \mathsf{A}_\ell \mathsf{K}_\ell^m \boldsymbol{u}_\ell \right\|_\ell^2 \\ &\overset{(17)}{\leq} \frac{M}{m} \left\{ (\boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathsf{A}_\ell} - (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{A}_\ell} \right\}. \end{aligned} \tag{19}$$

An induction argument yields

$$T_2 \leq \gamma \left( \boldsymbol{w}_\ell, \boldsymbol{w}_\ell \right)_{\mathsf{A}_\ell} - \gamma T_1. \tag{20}$$

## Proof (Cont.)

Therefore,

$$
\begin{aligned}
(\mathsf{E}_\ell \boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathsf{A}_\ell} &= T_1 + T_2 \\
&\overset{(20)}{\leq} (1-\gamma) T_1 + \gamma (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{A}_\ell} \\
&\overset{(19)}{\leq} (1-\gamma) \frac{M}{m} \left\{ (\boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathsf{A}_\ell} - (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{A}_\ell} \right\} + \gamma (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{A}_\ell} \\
&= \gamma (\boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathsf{A}_\ell} .
\end{aligned}
$$

$\square$

## Corollary (Convergence of General Symmetric V-Cycle)

*Suppose that hypotheses of the last theorem hold and* $\boldsymbol{u}_\ell^{\mathrm{E}}, \boldsymbol{g}_\ell \in \mathbb{R}^{n_\ell}$ *satisfy*

$$\mathsf{A}_\ell \boldsymbol{u}_\ell^{\mathrm{E}} = \boldsymbol{g}_\ell .$$

*Then, given any* $\boldsymbol{u}_\ell^{(0)} \in \mathbb{R}^{n_\ell}$,

$$\begin{aligned} \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(3)} \right\|_{\mathsf{A}_\ell} &= \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \mathrm{MG}\left( \boldsymbol{g}_\ell, \ell, \boldsymbol{u}_\ell^{(0)} \right) \right\|_{\mathsf{A}_\ell} \\ &\le \frac{M}{M+m} \left\| \boldsymbol{u}_\ell^{\mathrm{E}} - \boldsymbol{u}_\ell^{(0)} \right\|_{\mathsf{A}_\ell}, \end{aligned}$$

*where*

$$M = \frac{C_{\mathrm{A1}}^2 C_{\mathrm{R}}}{2} \quad \text{and} \quad m \ge 1.$$

## Proof.

Exercise. □