Math 673/4

Multigrid Methods:
A Mostly Matrix-Based Approach

Chapter 09: Additive Preconditioners Based on Subspace Decompositions

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Chapter 09, Part 2 of 2

Additive Preconditioners Based on Subspace Decompositions

Hierarchical Basis Preconditioner

Now, we need to connect the spaces W_j to V_ℓ where $0 \leq j \leq \ell$. In so doing, we will have the tools to build a preconditioner based on the hierarchical basis. Be careful, the number of indices in this section can get a little overwhelming.

Proposition

Let $\mathcal{B}_j^W = \{\phi_{j,i}\}_{i=1}^{m_j}$ and $\mathcal{B}_\ell^V = \{\psi_{\ell,i}\}_{i=1}^{n_\ell}$ be the usual bases for W_j and V_ℓ, respectively. For each $0 \leq j \leq \ell$, there are unique numbers

$$q_{j,k,i}^\ell \in \mathbb{R}, \quad 1 \leq k \leq n_\ell, \quad 1 \leq i \leq m_j,$$

such that

$$\phi_{j,i} = \sum_{k=1}^{n_\ell} q_{j,k,i}^\ell \psi_{\ell,k}. \tag{1}$$

Proof.

Exercise. □

Definition (Hierarchical Prolongation Matrix)

Define the matrix $Q_j^\ell \in \mathbb{R}^{n_\ell \times m_j}$ via

$$\left[Q_j^\ell\right]_{k,i} := q_{j,k,i}^\ell, \quad 1 \le k \le n_\ell, \quad 1 \le i \le m_j.$$

Q_j^ℓ is called a **hierarchical prolongation matrix**.

Lemma

Suppose that Q_j^ℓ is a hierarchical prolongation matrix and $\boldsymbol{w}_j \in \mathbb{R}^{m_j}$ is the coordinate vector of the function $w_j \in W_j$ with respect to the basis $\mathcal{B}_j^W$. Then,

$$\operatorname{rank}(Q_j^\ell) = m_j,$$

and the coordinate vector of $w_j \in V_\ell$ in the basis $\mathcal{B}_\ell^V$ is simply

$$Q_j^\ell \boldsymbol{w}_j \in \mathbb{R}^{n_\ell}.$$

Proof.

Exercise. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark

Note that the family of spaces W_j are hierarchical, but are not nested

$$W_0 \not\subset W_1 \not\subset W_2 \cdots .$$

Furthermore, it makes no sense to stack the prolongation matrices as we did in the past:

$$Q_j^\ell \neq Q_k^\ell Q_j^k,$$

for $j < k < \ell$. In fact, the product on the right hand side is not usually defined.

Definition

Define the bilinear form $C_j : W_j \times W_j \to \mathbb{R}$ via

$$C_j(w_j, v_j) := \sum_{r=1}^{m_j} w_j\left(\boldsymbol{N}_{j,r}^W\right) v_j\left(\boldsymbol{N}_{j,r}^W\right), \quad \forall \ w_j, v_j \in W_j.$$

Let $\mathcal{B}_j^W = \{\phi_{j,i}\}_{i=1}^{m_j}$ be the usual basis for W_j. Define the matrix $\mathsf{C}_j \in \mathbb{R}^{m_j \times m_j}$ via

$$
\begin{aligned}
\left[\mathsf{C}_j\right]_{i,k} &:= C_j\left(\phi_{j,i}, \phi_{j,k}\right) \\
&= \sum_{r=1}^{m_j} \phi_{j,i}\left(\boldsymbol{N}_{j,r}^W\right) \phi_{j,k}\left(\boldsymbol{N}_{j,r}^W\right) \\
&= \sum_{r=1}^{m_j} \delta_{ir}\delta_{kr} \\
&= \delta_{ik}.
\end{aligned}
\tag{2}
$$

Definition (Hierarchical Basis Preconditioner)

Suppose that $\mathcal{B}_\ell^V = \{\psi_{\ell,i}\}_{i=1}^{n_\ell}$ is the usual basis for the finite element space V_ℓ. Let $A_L \in \mathbb{R}^{n_L \times n_L}$ be the SPD matrix defined via

$$[A_L]_{i,j} = a(\psi_{L,j}, \psi_{L,i}), \quad 1 \leq i, j \leq n_L,$$

where

$$a(u, v) = (\nabla u, \nabla v)_{L^2}, \quad \forall \ u, v \in H_0^1(\Omega).$$

The **hierarchical basis preconditioner for** A_L is defined as

$$C_{\mathrm{H}} = \sum_{\ell=0}^{L} Q_\ell^L C_\ell^{-1} Z_\ell^L = \sum_{\ell=0}^{L} Q_\ell^L Z_\ell^L, \tag{3}$$

where C_ℓ is as in (2), $Q_\ell^L \in \mathbb{R}^{n_L \times m_\ell}$ is the hierarchical prolongation matrix from a previous definition and

$$Z_\ell^L = \left(Q_\ell^L\right)^\top.$$

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Lemma

Assumption (SS1) holds for the hierarchical basis decomposition. In particular, for any object

$$\boldsymbol{u}_L \in \mathbb{R}^{n_L} \overset{\mathcal{B}_L^V}{\leftrightarrow} u_L \in V_L$$

there exist unique objects

$$\boldsymbol{w}_\ell \in \mathbb{R}^{m_\ell} \overset{\mathcal{B}_\ell^W}{\leftrightarrow} w_\ell \in W_\ell, \quad 0 \leq \ell \leq L,$$

such that

$$\boldsymbol{u}_L = \sum_{\ell=0}^{L} \mathsf{Q}_\ell^L \boldsymbol{w}_\ell \in \mathbb{R}^{n_L} \overset{\mathcal{B}_L^V}{\leftrightarrow} u_L = \sum_{\ell=0}^{L} w_\ell \in V_L.$$

Furthermore, the hierarchical basis preconditioner, C_H, defined in (3), is SPD.

Proof.

This follows from the lemmas on the last slide deck. The details are left for an exercise. $\qquad \square$

Remark

Our goal is now to show that

$$\lambda_{\min}(C_H A_L) \geq C \left(1 + |\log(h_L)|^2\right)^{-1},$$

and

$$\lambda_{\max}(C_H A_L) \leq C,$$

where these constants are positive and independent of L. If this is the case,

$$\frac{\lambda_{\max}}{\lambda_{\min}} =: \kappa(C_H A_L) \leq C \left(1 + |\log(h_L)|^2\right).$$

This estimate is quite useful, since the logarithmic dependence on h_L is quite weak. For example, suppose

$$h_L = \frac{1}{2^L},$$

which is entirely reasonable. Then

$$|\log(h_L)|^2 = L^2 |\log(1/2)|^2.$$

Our analysis that follows will only work for $d = 2$.

Now, we need some technical lemmas. For more details, see the book by Brenner and Scott.

Theorem (Mean-Zero Poincaré)

Suppose that Ω is an open polyhedral set in $\mathbb{R}^d$. Then, for every $u \in H^1(\Omega)$,

$$\|u - \bar{u}\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}, \tag{4}$$

for some constant $C > 0$ that is independent of u by dependent upon Ω, where $\bar{u}$ is the average of u:

$$\bar{u} := \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx.$$

As a consequence, for every $u \in H^1(\Omega)$,

$$\|u - \bar{u}\|_{H^1(\Omega)} \leq C \, |u - \bar{u}|_{H^1(\Omega)} = C \, |u|_{H^1(\Omega)}, \tag{5}$$

for some constant $C > 0$ that is independent of u by dependent upon Ω.

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Theorem (Inverse Inequality)

Suppose that Ω is an open polyhedral domain in $\mathbb{R}^d$, $\mathcal{T}_j$, $0 \leq j \leq L$ is a nested family of triangulations of Ω, and S_j, $0 \leq j \leq L$, are the associated piecewise-linear finite element spaces. Assume that $1 \leq p, q \leq \infty$ and $0 \leq m \leq \ell$. Then, for all $v \in S_j$ and all $K \in \mathcal{T}_j$,

$$\|v\|_{W^{p,\ell}(K)} \leq C h_j^{m-\ell+d/p-d/q} \|v\|_{W^{q,m}(K)}, \qquad (6)$$

for some constant $C > 0$ depends on the shape of the triangle K. Here $W^{p,\ell}$ is the Sobolev space of derivative-order ℓ.

Proof.

See Brenner and Scott (2008), Section 5.3.   □

The following result is known as a Trace Estimate and allows one to control the values of a function on a boundary by its function values and derivatives in the interior of the domain.

Theorem (Trace Estimate)

Suppose that Ω is an open polyhedral domain in $\mathbb{R}^d$ and $1 \leq p \leq \infty$. Then, there is a constant $C > 0$, which depends upon the shape of Ω, such that

$$\|v\|_{L^p(\partial\Omega)} \leq C \|v\|_{L^p(\Omega)}^{1-1/p} \|v\|_{W^{1,p}(\Omega)}^{1/p}, \tag{7}$$

for all $v \in W^{1,p}(\Omega)$.

Proof.

See Brenner and Scott (2008), Section 1.6 for a proof in a simplified setting and for a discussion about the more general case here. □

Theorem

Suppose that Ω is an open polyhedral domain in $\mathbb{R}^d$, $\mathcal{T}_\ell$, $0 \leq \ell \leq L$ is a nested family of triangulations of Ω, and S_ℓ, $0 \leq \ell \leq L$, are the associated piecewise-linear finite element spaces. Then, for all $v_\ell \in S_\ell$, $\ell \geq 1$,

$$\|v_\ell - \mathcal{I}_{\ell-1} v_\ell\|_{L^2(\Omega)} \leq Ch_\ell \, |v_\ell|_{H^1(\Omega)} \,, \tag{8}$$

for some constant $C > 0$ that is independent of ℓ.

Proof.

Note that the previous interpolation error estimate from Chapter 6 does not cover this case, since v_ℓ is not in $H^2(\Omega)$. However, the stated result still holds since we are interpolating a very specific class of functions. The proof of the one dimensional case is left as an exercise. □

In two space dimensions $H^1 \hookrightarrow L^p$, for any $1 \leq p < \infty$. We cannot quite get control for $p = \infty$. But, if the function space is finite dimensional we can get control of the $p = \infty$ case, at the cost of an h-dependence. Here is the result from Section 4.9 in the book by Brenner and Scott.

Theorem

Suppose that Ω is an open polygonal domain in $\mathbb{R}^2$ and $\mathcal{T}_\ell$, $0 \leq \ell \leq L$ is a nested family of triangulations of Ω. Then, for any $v_\ell \in V_\ell$,

$$\|v_\ell\|_{L^\infty(\Omega)} \leq C \sqrt{1 + |\log(h_\ell)|}\, |v_\ell|_{H^1(\Omega)},$$

for some constant $C > 0$ that is independent of ℓ but depends upon the shape of Ω. Further, for all $v_\ell \in S_\ell$ and any $K \in \mathcal{T}_\ell$,

$$\|v_\ell - \bar{v}_\ell\|_{L^\infty(K)} \leq C \sqrt{1 + |\log(h_\ell)|}\, |v_\ell|_{H^1(K)},$$

for some constant $C > 0$ that is independent of ℓ but depends upon the shape of the triangle $K \in \mathcal{T}_\ell$, where

$$\bar{v}_\ell = \frac{1}{|K|} \int_K v_\ell(x)\, dx.$$

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Lemma

Suppose that $0 \leq j < \ell$. For any $v_\ell \in S_\ell$,

$$\|v_\ell - \bar{v}_{j,\ell}\|_{L^\infty(K_j)} \leq C \sqrt{1 + \left|\log\left(\frac{h_j}{h_\ell}\right)\right|} \, |v_\ell|_{H^1(K_j)}, \tag{9}$$

for some constant $C > 0$ that is independent of j and ℓ but depends upon the shape of the triangle $K_j \in \mathcal{T}_j$, where

$$\bar{v}_{j,\ell} = \frac{1}{|K_j|} \int_{K_j} v_\ell(\boldsymbol{x}) \, d\boldsymbol{x}.$$

Proof.

Exercise. □

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Lemma

Assume that $\Omega \subset \mathbb{R}^2$ is a polygonal domain. Suppose that $\mathcal{I}_\ell : C(\overline{\Omega}) \to V_\ell$, $0 \le \ell \le L$, is the Lagrange nodal interpolation operator, and $\mathcal{I}_{-1} \equiv 0$. Then, for all $u_L \in V_L$,

$$\|\mathcal{I}_\ell u_L - \mathcal{I}_{\ell-1} u_L\|_{L^2(\Omega)} \le Ch_\ell \left(1 + \sqrt{L-\ell}\right) |u_L|_{H^1(\Omega)}. \qquad (10)$$

for some constant $C > 0$ that is independent of but depends upon the shape of Ω.

Proof.

Define the piecewise constant function $\bar{u}_L^\ell$ such that

$$\bar{u}_L^\ell|_K := \frac{1}{|K|} \int_K u_L(x) \, dx, \quad \forall K \in \mathcal{T}_\ell.$$

Proof (Cont.)

Then,

$$
\begin{aligned}
\|\mathcal{I}_\ell u_L - \mathcal{I}_{\ell-1} u_L\|_{L^2(\Omega)}^2 &= \|\mathcal{I}_\ell u_L - \mathcal{I}_{\ell-1}\left[\mathcal{I}_\ell[u_L]\right]\|_{L^2(\Omega)}^2 \\
&\overset{(8)}{\leq} Ch_\ell^2 \sum_{K \in \mathcal{T}_\ell} |\mathcal{I}_\ell[u_L]|_{H^1(K)}^2 \\
&= Ch_\ell^2 \sum_{K \in \mathcal{T}_\ell} \left|\mathcal{I}_\ell u_L - \bar{u}_L^\ell\right|_{H^1(K)}^2 \\
&\overset{(6)}{\leq} Ch_\ell^2 \sum_{K \in \mathcal{T}_\ell} \left\|\mathcal{I}_\ell u_L - \bar{u}_L^\ell\right\|_{L^\infty(K)}^2 \\
&\leq Ch_\ell^2 \sum_{K \in \mathcal{T}_\ell} \left\|u_L - \bar{u}_L^\ell\right\|_{L^\infty(K)}^2 \\
&\overset{(9)}{\leq} Ch_\ell^2 \sum_{K \in \mathcal{T}_\ell} \left(1 + \left|\log\left(\frac{h_\ell}{h_L}\right)\right|\right) |u_L|_{H^1(K)}^2 \\
&= Ch_\ell^2 \left(1 + \left|\log\left(\frac{h_\ell}{h_L}\right)\right|\right) |u_L|_{H^1(\Omega)}^2 .
\end{aligned}
$$

Proof (Cont.)

Now, notice that

$$h_\ell = h_0 2^{-\ell} \quad 1 \le \ell \le L.$$

So,

$$\log(h_\ell/h_L) = \log(2^{L-\ell}) = (L - \ell)\log(2).$$

The result follows. $\square$

Lemma

There is some constant $C_7 > 0$, independent of L, such that

$$\lambda_{\min}(C_H A_L) \geq \frac{C_7}{1 + |\log(h_L)|^2}. \tag{11}$$

Proof.

By definition, for any $w_{\ell,1}, w_{\ell,2} \in W_\ell$

$$C_\ell\left(w_{\ell,1}, w_{\ell,2}\right) = \sum_{i=1}^{m_\ell} w_{\ell,1}(\boldsymbol{N}_{\ell,i}^W) w_{\ell,2}(\boldsymbol{N}_{\ell,i}^W).$$

Proof (Cont.)

Let

$$\boldsymbol{w}_{\ell,\alpha} \in \mathbb{R}^{m_\ell} \overset{\mathcal{B}_\ell^W}{\leftrightarrow} w_{\ell,\alpha} \in W_\ell, \quad \alpha = 1, 2.$$

Then,

$$\begin{aligned}
(\mathsf{C}_\ell \boldsymbol{w}_{\ell,1}, \boldsymbol{w}_{\ell,2})_\ell &= \sum_{i=1}^{m_\ell} [\boldsymbol{w}_{\ell,1}]_i \, [\boldsymbol{w}_{\ell,2}]_i \\
&= \sum_{i=1}^{m_\ell} w_{\ell,1}(\boldsymbol{N}_{\ell,i}^W) w_{\ell,2}(\boldsymbol{N}_{\ell,i}^W) \\
&= C_\ell \, (w_{\ell,1}, w_{\ell,2}) \, .
\end{aligned}$$

This last object is like a mass-lumping inner product. All that is missing is a factor of h_ℓ^2.

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Proof (Cont.)

There are constants $C_3 > 0$, $C_4 > 0$ such that, for all $0 \leq \ell \leq L$,

$$C_3 h_\ell^2 C_\ell (w_\ell, w_\ell) \leq \|w_\ell\|_{L^2(\Omega)}^2 \leq C_4 h_\ell^2 C_\ell (w_\ell, w_\ell), \tag{12}$$

for all $w_\ell \in W_\ell$. Therefore, for any $w_\ell \in W_\ell \overset{\mathcal{B}_\ell^W}{\leftrightarrow} \boldsymbol{w}_\ell \in \mathbb{R}^{m_\ell}$,

$$
\begin{aligned}
(C_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell &= h_\ell^{-2} h_\ell^2 C_\ell (w_\ell, w_\ell) \\
&\overset{(12)}{\leq} C_3^{-1} h_\ell^{-2} \|w_\ell\|_{L^2(\Omega)}^2 \\
&= C_3^{-1} h_\ell^{-2} \|w_\ell - \mathcal{I}_{\ell-1} w_\ell\|_{L^2(\Omega)}^2 \\
&\overset{(??)}{\leq} C_3^{-1} C |w_\ell|_{H^1(\Omega)}^2 \\
&\overset{(6)}{\leq} C_3^{-1} C h_\ell^{-2} \|w_\ell\|_{L^2(\Omega)}^2 \\
&\overset{(12)}{\leq} C_3^{-1} C C_4 (C_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell .
\end{aligned}
\tag{13}
$$

Proof (Cont.)

Therefore, there are constants $C_5 > 0$, $C_6 > 0$, such that we have the equivalence

$$C_5 \sum_{\ell=0}^{L} |w_\ell|^2_{H^1(\Omega)} \leq \sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell \leq C_6 \sum_{\ell=0}^{L} |w_\ell|^2_{H^1(\Omega)}, \tag{14}$$

for any collection (w_ℓ), with $w_\ell \in W_\ell$, in general. Now, let $u_L \in V_L$ be given and

$$u_L = \sum_{\ell=0}^{L} w_\ell, \quad \exists! \, w_\ell \in W_\ell, \quad 0 \leq \ell \leq L.$$

Recall that

$$w_\ell = \mathcal{I}_\ell u_L - \mathcal{I}_{\ell-1} u_L, \quad 1 \leq \ell \leq L,$$

and

$$w_0 = \mathcal{I}_0 u_L.$$

Proof (Cont.)

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We make the usual identification $w_\ell \in W_\ell \overset{\mathcal{B}_\ell^W}{\leftrightarrow} \boldsymbol{w}_\ell \in \mathbb{R}^{m_\ell}$, and we observe that

$$(\boldsymbol{w}_\ell)_{\ell=0}^L \in \mathsf{Q}[\boldsymbol{u}_L],$$

with respect to the hierarchical prolongation matrices from Definition 1. Then, from (13)

$$
\begin{aligned}
\sum_{\ell=0}^L \left(\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell \right)_\ell
&\leq C_3^{-1} C \sum_{\ell=0}^L h_\ell^{-2} \left\| w_\ell \right\|_{L^2(\Omega)}^2 \\
&\overset{(10)}{\leq} C \sum_{\ell=0}^L \left(1 + \sqrt{L-\ell} \right)^2 \left| u_L \right|_{H^1(\Omega)}^2 \\
&\leq C \sum_{\ell=0}^L (1 + L - \ell) \left| u_L \right|_{H^1(\Omega)}^2 \\
&\leq C \left(1 + L + L^2 \right) \left| u_L \right|_{H^1(\Omega)}^2 \\
&\overset{L \geq 1}{\leq} C L^2 \left| u_L \right|_{H^1(\Omega)}^2 .
\end{aligned}
$$

Proof (Cont.)

But
$$|u_L|^2_{H^1(\Omega)} = a(u_L, u_L) = (\mathsf{A}_L \boldsymbol{u}_L, \boldsymbol{u}_L)_L,$$

and

$$\begin{aligned}
|\log(h_L)|^2 &= \left|\log(h_0 2^{-L})\right|^2 \\
&= |\log(h_0) - L\log(2)|^2 \\
&= \log^2(h_0) - 2\log(h_0)L\log(2) + L^2\log^2(2).
\end{aligned}$$

So,
$$L^2 \le C\left(1 + |\log(h_L)|^2\right), \quad \exists\, C > 0.$$

Thus,
$$\sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell) \le C\left(1 + |\log(h_L)|^2\right)(\mathsf{A}_L \boldsymbol{u}_L, \boldsymbol{u}_L)_L,$$

and
$$\lambda_{\min}(\mathsf{C}_{\mathrm{H}} \mathsf{A}_L) \ge C_7 \left(1 + |\log(h_L)|^2\right)^{-1}.$$

$\square$

In the last line we use the "big" theorem from the first slide deck:

Theorem (Eigenvalues of CA)

Suppose that Assumption (SS1) holds for the set of prolongation matrices $\{Q_j\}_{j=0}^{L}$ and C is an additive subspace preconditioner with respect to $\{Q_j\}_{j=0}^{L}$. The eigenvalues of CA are positive, provided A is SPD with respect to $(\,\cdot\,,\cdot\,)$. Moreover

$$\lambda_{\max}(CA) = \max_{\boldsymbol{u} \in \mathbb{R}^n_\star} \frac{(A\boldsymbol{u}, \boldsymbol{u})}{\displaystyle \min_{(\boldsymbol{w}_\ell) \in \mathbb{Q}[\boldsymbol{u}]} \sum_{\ell=0}^{L} (C_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell}, \qquad (15)$$

$$\lambda_{\min}(CA) = \min_{\boldsymbol{u} \in \mathbb{R}^n_\star} \frac{(A\boldsymbol{u}, \boldsymbol{u})}{\displaystyle \min_{(\boldsymbol{w}_\ell) \in \mathbb{Q}[\boldsymbol{u}]} \sum_{\ell=0}^{L} (C_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell}. \qquad (16)$$

Lemma (Trace-Type Estimate)

Suppose that Ω is an open polygonal domain in $\mathbb{R}^2$ and $\mathcal{T}_\ell$, $0 \le \ell \le L$ is a nested family of triangulations of Ω. Assume that S_ℓ is the usual family of finite element spaces subordinate to $\mathcal{T}_\ell$. Then, for every $K \in \mathcal{T}_\ell$ and every $v_\ell, w_\ell \in S_\ell$, there is a constant $C > 0$, which depends upon the shape of K, such that

$$\int_{\partial K} \frac{\partial v_\ell}{\partial n} w_\ell \, \mathrm{d}s \le C h_\ell^{-1} |v_\ell|_{H^1(K)} \|w_\ell\|_{L^1(\partial K)}. \tag{17}$$

Proof.

$$\int_{\partial K} \frac{\partial v_\ell}{\partial n} w_\ell \, \mathrm{d}s \overset{\text{Young Ineq.}}{\le} \left\| \frac{\partial v_\ell}{\partial n} \right\|_{L^\infty(\partial K)} \|w_\ell\|_{L^1(\partial k)}$$

$$\overset{(7)}{\le} C |v_\ell|_{W^{1,\infty}(K)} \|w_\ell\|_{L^1(\partial k)}$$

$$\overset{(6)}{\le} C h_\ell^{-1} |v_\ell|_{H^1(K)} \|w_\ell\|_{L^1(\partial k)}.$$

$\square$

Lemma (Strengthened Cauchy-Schwarz Inequality I)

For any $v_\ell \in V_\ell$ and $v_k \in V_k$, $0 \leq \ell \leq k \leq L$, and d=2, there is a constant $C > 0$ such that

$$\int_\Omega \nabla v_\ell \cdot \nabla v_k \, \mathrm{d}x \leq 2^{(\ell-k)/2} C \, |v_\ell|_{H^1(\Omega)} \left(h_k^{-1} \, \|v_k\|_{L^2(\Omega)} \right). \qquad (18)$$

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Proof.

For any $K \in \mathcal{T}_\ell$, since $\Delta v_\ell|_K \equiv 0$,

$$
\begin{aligned}
\int_K \nabla v_\ell \cdot \nabla v_k \, \mathrm{d}\boldsymbol{x} &= \int_{\partial K} \frac{\partial v_\ell}{\partial \boldsymbol{n}} v_k \, \mathrm{d}\boldsymbol{s} \\
&\overset{(17)}{\leq} Ch_\ell^{-1} |v_\ell|_{H^1(K)} \int_{\partial K} |v_k| \, \mathrm{d}\boldsymbol{s} \\
&\leq Ch_\ell^{-1} |v_\ell|_{H^1(K)} \left(h_k \sum_{\boldsymbol{N}_k \in \partial K} |v_k(\boldsymbol{N}_k)| \right) \\
&\overset{\text{C.S.}}{\leq} Ch_\ell^{-1} |v_\ell|_{H^1(K)} h_k \left(\frac{h_\ell}{h_k} \right)^{1/2} \left(\sum_{\boldsymbol{N}_k \in \partial K} |v_k(\boldsymbol{N}_k)|^2 \right)^{1/2} \\
&\leq C \left(\frac{h_k}{h_\ell} \right)^{1/2} |v_\ell|_{H^1(K)} h_k^{-1} \|v_k\|_{L^2(K)} .
\end{aligned}
$$

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Proof (Cont.)

Thus,

$$
\begin{aligned}
\int_{\Omega} \nabla v_{\ell} \cdot \nabla v_k \, \mathrm{d}x &= \sum_{K \in \mathcal{T}_{\ell}} \int_{K} \nabla v_{\ell} \cdot \nabla v_k \, \mathrm{d}x \\
&\leq C 2^{(\ell-k)/2} \sum_{K \in \mathcal{T}_{\ell}} |v_{\ell}|_{H^1(K)} \, h_k^{-1} \|v_k\|_{L^2(K)} \\
&\overset{\text{C.S.}}{\leq} C 2^{(\ell-k)/2} |v_{\ell}|_{H^1(\Omega)} \, h_k^{-1} \|v_k\|_{L^2(\Omega)} .
\end{aligned}
$$

□

Lemma (Strengthened Cauchy-Schwarz Inequality II)

For any $w_\ell \in W_\ell$ and $w_k \in W_k$, $0 \leq \ell \leq k \leq L$, there is a constant $C > 0$ such that

$$\int_\Omega \nabla w_\ell \cdot \nabla w_k \, d\boldsymbol{x} \leq 2^{(\ell-k)/2} C \, |w_\ell|_{H^1(\Omega)} \, |w_k|_{H^1(\Omega)} \,. \tag{19}$$

Proof.

Observe that

$$w_k = w_k - \mathcal{I}_{k-1} w_k.$$

We use the interpolation error estimate

$$\|w_k - \mathcal{I}_{k-1} w_k\|_{L^2(\Omega)} \leq C h_k \, |w_k|_{H^1(\Omega)} \,,$$

to conclude that

$$\|w_k\|_{L^2(\Omega)} \leq C h_k \, |w_k|_{H^1(\Omega)} \,.$$

Proof (Cont.)

Now, we use the last result. Since $w_\ell \in V_\ell$ and $w_k \in V_k$,

$$
\begin{aligned}
\int_\Omega \nabla w_\ell \cdot \nabla w_k \, dx \;\; &\leq \;\; C 2^{(\ell-k)/2} \, |w_\ell|_{H^1(\Omega)} \, h_k^{-1} \, \|w_k\|_{L^2(\Omega)} \\
&\leq \;\; 2^{(\ell-k)/2} C \, |w_\ell|_{H^1(\Omega)} \, |w_k|_{H^1(\Omega)} \, .
\end{aligned}
$$

$\square$

Next, we need a little technical lemma, a kind of convolution result.

Lemma (Discrete Convolution Estimate)

Suppose that $\{a_j\}, \{b_j\} \subset \mathbb{R}$ are sequences satisfying $a_j, b_j \geq 0$, for $-\infty < j < \infty$, with

$$s_1 := \sum_{j=-\infty}^{\infty} a_j < \infty,$$

and

$$s_2 := \sum_{j=-\infty}^{\infty} b_j^2 < \infty.$$

Then,

$$\sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} a_{j-k} b_k \right)^2 \leq s_1^2 s_2. \tag{20}$$

Proof.

Exercise. □

Lemma

There is a constant $C_8 > 0$, independent of L, such that

$$\lambda_{\max}(C_H A_L) \leq C_8.$$

Proof.

Let $v_L \in V_L$ be arbitrary.

$$v_L \in V_L \overset{\mathcal{B}_L}{\leftrightarrow} \mathbf{v}_L \in \mathbb{R}^{n_L}.$$

There exist unique $w_\ell \in W_\ell \overset{\mathcal{B}_\ell^W}{\leftrightarrow} \mathbf{w}_\ell \in \mathbb{R}^{m_\ell}$, $\ell = 0, \ldots, L$, such that

$$v_L = \sum_{\ell=0}^{L} w_\ell \overset{\mathcal{B}_L}{\leftrightarrow} \mathbf{v}_L = \sum_{\ell=0}^{L} Q_\ell^L \mathbf{w}_\ell.$$

Proof (Cont.)

Then,

$$
\begin{aligned}
(\mathbf{v}_L, \mathbf{v}_L)_{A_L} &= a(v_L, v_L) \\
&= a\left(\sum_{\ell=0}^{L} w_\ell, \sum_{k=0}^{L} w_k\right) \\
&= \int_\Omega \left(\nabla \sum_{\ell=0}^{L} w_\ell\right) \cdot \left(\nabla \sum_{k=0}^{L} w_k\right) d\mathbf{x} \\
&= \sum_{\ell,k=0}^{L} \int_\Omega \nabla w_\ell \cdot \nabla w_k \, d\mathbf{x} \\
&\overset{(19)}{\leq} C \sum_{\ell,k=0}^{L} 2^{-|\ell-k|/2} |w_\ell|_{H^1(\Omega)} |w_k|_{H^1(\Omega)} \\
&\leq C \sum_{\ell=0}^{L} \left(\sum_{k=0}^{L} 2^{-|\ell-k|/2} |w_k|_{H^1(\Omega)}\right) |w_\ell|_{H^1(\Omega)} .
\end{aligned}
$$

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Proof (Cont.)

Continuing with the estimate,

$$
\begin{aligned}
(\boldsymbol{v}_L, \boldsymbol{v}_L)_{A_L} \;\overset{\text{C.S.}}{\leq}\; & C \left\{ \sum_{\ell=0}^{L} \left(\sum_{k=0}^{L} 2^{-|\ell-k|/2} \, |w_k|_{H^1(\Omega)} \right)^2 \right\}^{1/2} \left\{ \sum_{\ell=0}^{L} |w_\ell|^2_{H^1(\Omega)} \right\}^{1/2} \\
\overset{(20)}{\leq}\; & C \left\{ \sum_{\ell=0}^{L} |w_\ell|^2_{H^1(\Omega)} \right\}^{1/2} \left\{ \sum_{\ell=0}^{L} |w_\ell|^2_{H^1(\Omega)} \right\}^{1/2} \\
=\; & C \sum_{\ell=0}^{L} |w_\ell|^2_{H^1(\Omega)} \\
\overset{(14)}{\leq}\; & \frac{C}{C_5} \sum_{\ell=0}^{L} (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{C_\ell} .
\end{aligned}
$$

Proof (Cont.)

Recall that, since decompositions are unique,

$$
\lambda_{\max}(\mathsf{C}_{\mathrm{H}}\mathsf{A}_L) \overset{(15)}{=} \max_{\boldsymbol{v}_L \in \mathbb{R}^{n_L}_\star} \frac{(\boldsymbol{v}_L, \boldsymbol{v}_L)_{\mathsf{A}_L}}{\sum_{\ell=0}^{L}(\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{C}_\ell}}
$$

$$
= \max_{\boldsymbol{v}_L \in \mathbb{R}^{n_L}_\star} \frac{\frac{C}{C_5}\sum_{\ell=0}^{L}(\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{C}_\ell}}{\sum_{\ell=0}^{L}(\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{C}_\ell}}
$$

$$
\leq C_8,
$$

using the estimate on the previous slide. □

Theorem

$$\kappa(\mathsf{C}_{\mathrm{H}}\mathsf{A}_L) = \frac{\lambda_{\max}(\mathsf{C}_{\mathrm{H}}\mathsf{A}_L)}{\lambda_{\min}(\mathsf{C}_{\mathrm{H}}\mathsf{A}_L)} \le \frac{C_8}{C_7}\left(1 + |\log(h_L)|^2\right). \tag{21}$$

Proof.

The result follows from the last few lemmas. $\qquad\square$

The BPX Preconditioner

The BPX Preconditioner

The BPX preconditioner has a slightly better performance than the hierarchical basis preconditioner, in the sense that the logarithmic dependence on h_L can be removed.

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Definition (BPX Preconditioner)

Define the bilinear form $C_\ell : V_\ell \times V_\ell \to \mathbb{R}$ via

$$C_\ell(w_\ell, v_\ell) = \sum_{i=1}^{n_\ell} w_\ell(\boldsymbol{N}_{\ell,i}) v_\ell(\boldsymbol{N}_{\ell,i}), \quad \forall\, w_\ell, v_\ell \in V_\ell.$$

The associated matrix $C_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$ is defined as

$$[C_\ell]_{j,k} = C_\ell(\psi_{\ell,j}, \psi_{\ell,k}) = \delta_{j,k}, \quad 1 \le j, k \le n_\ell,$$

where $\mathcal{B}_\ell = \{\psi_{\ell,j}\}_{j=1}^{n_\ell}$ is the standard Lagrange nodal basis for the piecewise linear finite element space V_ℓ, $0 \le \ell \le L$. The **BPX preconditioner** is precisely

$$C_{\mathrm{BPX}} := \sum_{\ell=0}^{L} P_{\ell,L} C_\ell^{-1} R_{\ell,L} = \sum_{\ell=0}^{L} P_{\ell,L} R_{\ell,L}, \tag{22}$$

where $P_{\ell,L} \in \mathbb{R}^{n_L \times n_\ell}$ is the standard multilevel prolongation matrix and $R_{\ell,L} = P_{\ell,L}^\top$.

Remark

Note that we have dropped, and will continue to drop, the superscripted V and just write $\mathcal{B}_\ell = \{\psi_{\ell,j}\}_{j=1}^{n_\ell}$ for the standard Lagrange nodal basis for the piecewise linear finite element space V_ℓ, $0 \le \ell \le L$.

Remark

In the BPX framework, we effectively are taking

$$Q_\ell^L = P_{\ell,L}.$$

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Lemma

Assumption (SS1) holds for the BPX framework, that is, for every $u_L \in V_L$, there exists $v_\ell \in V_\ell$, $0 \leq \ell \leq L$, such that

$$u_L = \sum_{\ell=0}^{L} v_\ell,$$

or, equivalently

$$\boldsymbol{u}_L = \sum_{\ell=0}^{L} \mathsf{P}_{\ell,L} \boldsymbol{v}_\ell,$$

with

$$V_\ell \ni v_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell},$$

and

$$V_L \ni u_L \overset{\mathcal{B}_L}{\leftrightarrow} \boldsymbol{u}_L \in \mathbb{R}^{n_L}.$$

This decomposition is not unique.

Proof.

Exercise. □

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Lemma (Strengthened Cauchy-Schwarz III)

Let $0 \leq j \leq \ell$. For any $v_j \in V_j$ and $v_\ell \in V_\ell$,

$$\int_\Omega \nabla v_j \cdot \nabla v_\ell \, dx \leq C 2^{-|j-\ell|/2} \frac{\|v_j\|_{L^2(\Omega)}}{h_j} \frac{\|v_\ell\|_{L^2(\Omega)}}{h_\ell}, \qquad (23)$$

for some $C > 0$.

Proof.

This is follows from (18) and the inverse inequality

$$|v_j|_{H^1(\Omega)} \leq c h_j^{-1} \|v_j\|_{L^2(\Omega)}.$$

□

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Lemma

For some $C_9 > 0$ that is independent of L,

$$\lambda_{\max}\left(C_{\mathrm{BPX}}A_L\right) \leq C_9.$$

for some $C_9 > 0$ that is independent of L.

Proof.

Let $u_L \in V_L$ be arbitrary. There exist $v_\ell \in V_\ell$, $0 \leq \ell \leq L$, such that

$$u_L = \sum_{\ell=0}^{L} v_\ell,$$

or

$$\boldsymbol{u}_L = \sum_{\ell=0}^{L} P_{\ell,L}\boldsymbol{v}_\ell, \quad V_\ell \ni v_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}.$$

As usual, we write

$$(\boldsymbol{v}_\ell) \in Q[\boldsymbol{u}_L],$$

though the decomposition is not unique.

Proof (Cont.)

Then,

$$
\begin{aligned}
(\boldsymbol{u}_L, \boldsymbol{u}_L)_{A_L} &= \sum_{\ell=0}^{L} \sum_{j=0}^{L} a(v_j, v_\ell) \\
&\overset{(23)}{\leq} C \sum_{\ell=0}^{L} \sum_{j=0}^{L} 2^{-|j-\ell|/2} h_j^{-1} \|v_j\|_{L^2(\Omega)} h_\ell^{-1} \|v_\ell\|_{L^2(\Omega)} \\
&\overset{\text{C.S.}}{\leq} C \left\{ \sum_{\ell=0}^{L} \left(\sum_{j=0}^{L} 2^{-\frac{|\ell-j|}{2}} \frac{\|v_j\|_{L^2(\Omega)}}{h_j} \right)^2 \right\}^{\frac{1}{2}} \left\{ \sum_{\ell=0}^{L} \left(\frac{\|v_\ell\|_{L^2(\Omega)}}{h_\ell} \right)^2 \right\}^{\frac{1}{2}} \\
&\overset{(20)}{\leq} C \sum_{j=0}^{L} h_j^{-2} \|v_j\|_{L^2(\Omega)}^2 \\
&\leq C \sum_{j=0}^{L} \left(C_j \boldsymbol{v}_j, \boldsymbol{v}_j \right)_j .
\end{aligned}
$$

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Proof (Cont.)

Now, for $(\boldsymbol{v}_\ell) \in Q[\boldsymbol{u}_L]$, as above,

$$
\lambda_{\max}(C_{\mathrm{BPX}}A_L) \stackrel{(15)}{=} \max_{\boldsymbol{u}_L \in \mathbb{R}_\star^{n_L}} \frac{(\boldsymbol{u}_L, \boldsymbol{u}_L)_{A_L}}{\min_{(\boldsymbol{w}_\ell) \in Q[\boldsymbol{u}_L]} \sum_{\ell=0}^{L} (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{C_\ell}}
$$

$$
\leq \max_{\boldsymbol{u}_L \in \mathbb{R}_\star^{n_L}} \frac{C \sum_{\ell=0}^{L} (C_\ell \boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_\ell}{\min_{(\boldsymbol{w}_\ell) \in Q[\boldsymbol{u}_L]} \sum_{\ell=0}^{L} (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{C_\ell}}
$$

$$
\leq C_9.
$$

Recall that the minimum in the denominator is achievable for some $(\boldsymbol{w}_\ell) \in Q[\boldsymbol{u}_L]$, so we have taken $\boldsymbol{v}_\ell = \boldsymbol{w}_\ell$ in the third step to conclude the upper bound. □

Before we establish a lower for $\lambda_{\min}(C_{\mathrm{BPX}} A_L)$ we need another technical lemma.

Lemma

Let $u \in H_0^1(\Omega)$ be arbitrary. Then, for any $1 \leq \ell \leq L$,

$$\mathcal{R}_{\ell-1} u = \mathcal{R}_{\ell-1}(\mathcal{R}_\ell u), \tag{24}$$

where $\mathcal{R}_\ell : H_0^1(\Omega) \to V_\ell$ is the Ritz projection, for $0 \leq \ell \leq L$. In other words, $\mathcal{R}_{\ell-1} = \mathcal{R}_{\ell-1} \mathcal{R}_\ell$.

Proof.

By definition,

$$a(\mathcal{R}_{\ell-1}(\mathcal{R}_\ell u), w_{\ell-1}) = a(\mathcal{R}_\ell u, w_{\ell-1}), \quad \forall w_{\ell-1} \in V_{\ell-1}.$$

But, since $V_{\ell-1} \subset V_\ell$, we also have

$$a(\mathcal{R}_\ell u, w_{\ell-1}) = a(u, w_{\ell-1}), \quad \forall w_{\ell-1} \in V_{\ell-1}.$$

Proof (Cont.)

Also observe that

$$a(\mathcal{R}_{\ell-1}u, w_{\ell-1}) = a(u, w_{\ell-1}), \quad \forall\, w_{\ell-1} \in V_{\ell-1}.$$

Hence,

$$a(\mathcal{R}_{\ell-1}(\mathcal{R}_\ell u), w_{\ell-1}) = a(\mathcal{R}_{\ell-1}u, w_{\ell-1}), \quad \forall\, w_{\ell-1} \in V_{\ell-1}.$$

And we conclude that $\mathcal{R}_{\ell-1} = \mathcal{R}_{\ell-1}\mathcal{R}_\ell$ since

$$\mathcal{R}_{\ell-1}(\mathcal{R}_\ell u), \mathcal{R}_{\ell-1}u \in V_{\ell-1}.$$

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Lemma

There is a constant $C_{10} > 0$ that is independent of L, such that

$$\lambda_{\min}\left(C_{\mathrm{BPX}}A_L\right) \geq C_{10}.$$

for some $C_{10} > 0$ that is independent of L.

Proof.

Let $u_L \in V_L$ be arbitrary. Set

$$v_\ell := \mathcal{R}_\ell u_L - \mathcal{R}_{\ell-1}u_L, \quad 0 \leq \ell \leq L,$$

where $\mathcal{R}_\ell : H_0^1(\Omega) \to V_\ell$ is the Ritz projection, for $0 \leq \ell \leq L$, and $R_{-1} \equiv 0$. Since

$$\mathcal{R}_L u_L = u_L,$$

it follows that

$$u_L = \sum_{\ell=0}^{L} v_\ell \in V_L \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{u}_L = \sum_{\ell=0}^{L} \mathsf{P}_{\ell,L}\boldsymbol{v}_\ell \in \mathbb{R}^{n_L}, \quad v_\ell \in V_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}.$$

Proof (Cont.)

Moreover,
$$a(v_j, v_\ell) = 0, \quad 0 \le j \ne \ell \le L. \tag{25}$$

To see this, recall that, in general,

$$a(\mathcal{R}_\ell u_L, w_\ell) = a(u_L, w_\ell), \quad \forall \, w_\ell \in V_\ell.$$

Suppose $j < \ell$, for definiteness. Then, since $V_j \subset V_\ell$,

$$a(\mathcal{R}_\ell u_L, w_j) = a(u_L, w_j), \quad \forall \, w_j \in V_j.$$

In particular, since

$$v_j \coloneqq \mathcal{R}_j u_L - \mathcal{R}_{j-1} u_L \in V_j \subset V_\ell,$$

it follows that

$$a(\mathcal{R}_\ell u_L, v_j) = a(u_L, v_j).$$

Likewise,

$$a(\mathcal{R}_{\ell-1} u_L, v_j) = a(u_L, v_j).$$

Subtracting, we have

$$a(\mathcal{R}_\ell u_L - \mathcal{R}_{\ell-1} u_L, v_j) = 0.$$

Proof (Cont.)

To make further progress, let us assume that Ω is convex. Then the standard regularity condition holds. And, for $1 \leq \ell \leq L$,

$$
\begin{aligned}
h_\ell^{-2} \left\| v_\ell \right\|_{L^2(\Omega)}^2
&= && h_\ell^{-2} \left\| \mathcal{R}_\ell u_L - \mathcal{R}_{\ell-1} u_L \right\|_{L^2(\Omega)}^2 \\
&\overset{(24)}{=} && h_\ell^{-2} \left\| \mathcal{R}_\ell u_L - \mathcal{R}_{\ell-1}(\mathcal{R}_\ell u_L) \right\|_{L^2(\Omega)}^2 \\
&\overset{\text{Nitsche}}{\leq} && C h_\ell^{-2} h_\ell^2 \left| \mathcal{R}_\ell u_L - \mathcal{R}_{\ell-1}(\mathcal{R}_\ell u_L) \right|_{H^1(\Omega)}^2 \\
&= && C \left| \mathcal{R}_\ell u_L - \mathcal{R}_{\ell-1}(\mathcal{R}_\ell u_L) \right|_{H^1(\Omega)}^2 \\
&= && C \left| v_\ell \right|_{H^1(\Omega)}^2 . && (26)
\end{aligned}
$$

Estimate (26) holds trivially for $\ell = 0$.

Proof (Cont.)

Therefore,

$$
\begin{aligned}
\sum_{\ell=0}^{L} \left(\mathbf{C}_\ell \mathbf{v}_\ell, \mathbf{v}_\ell \right)_\ell
&\leq C \sum_{\ell=0}^{L} h_\ell^{-2} \left\| v_\ell \right\|_{L^2(\Omega)}^2 \\
&\overset{(26)}{\leq} C \sum_{\ell=0}^{L} \left| v_\ell \right|_{H^1(\Omega)}^2 \\
&\overset{(25)}{=} C \left| u_L \right|_{H^1(\Omega)}^2 .
\end{aligned}
\qquad (27)
$$

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Proof (Cont.)

So, finally,

$$
\begin{aligned}
\lambda_{\min}(\mathrm{C}_{\mathrm{BPX}}\mathrm{A}_L) &= \min_{\boldsymbol{u}_L \in \mathbb{R}^{n_L}_\star} \frac{(\boldsymbol{u}_L, \boldsymbol{u}_L)_{\mathrm{A}_L}}{\min\limits_{(\boldsymbol{w}_\ell) \in \mathsf{Q}[\boldsymbol{u}_L]} \sum\limits_{\ell=0}^{L} (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathrm{C}_\ell}} \\
&\overset{(27)}{\geq} \min_{\boldsymbol{u}_L \in \mathbb{R}^{n_L}_\star} \frac{(\mathrm{A}_L \boldsymbol{u}_L, \boldsymbol{u}_L)_L}{C\, |u_L|_{H^1(\Omega)}} \\
&= C_{10}.
\end{aligned}
$$

□

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Theorem

$$\kappa \left(\mathrm{C_{BPX}} \mathrm{A}_L \right) = \frac{\lambda_{\mathsf{max}} \left(\mathrm{C_{BPX}} \mathrm{A}_L \right)}{\lambda_{\mathsf{min}} \left(\mathrm{C_{BPX}} \mathrm{A}_L \right)} \leq \frac{C_9}{C_{10}}. \tag{28}$$

Proof.

Follows from the previous lemmas.   □