Math 673

Multigrid Methods: A Mostly Matrix-Based Approach

Chapter 03: Fourier Analysis of the Two-Grid Algorithm

Abner J. Salgado and Steven M. Wise

asalgad1@utk.edu — swise1@.utk.edu
University of Tennessee

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Chapter 03, Part 1 of 3

Fourier Analysis of the Two-Grid Algorithm

The Model Problem

In this chapter, we will consider the model elliptic problem in 1D:

$$\begin{cases} -u'' = f, & \text{in} \quad \Omega = (0,1), \\ u = 0, & \text{on} \quad \partial\Omega = \{0,1\}. \end{cases} \tag{1}$$

We will examine two kinds of discretization for the problem, and, then, we will apply the two-grid algorithm to solve the resulting linear systems of equations. We will use a Fourier-type analysis to show that the algorithm converges.

The inspiration for this chapter is a short subsection of Chapter 8 in the classic numerical analysis text by Stoer and Bulirsch. We have greatly expanded the content and placed it in the larger context of the general two-grid and multigrid algorithms. However, the simplicity and brevity of the presentation in Stoer and Bulirsch is always inspiring and certainly worth a read.

Discretization of the Model Problem in 1D

The Finite Difference Method

Let us create a grid for the finite difference scheme with uniform spacing

$$h_1 := \frac{1}{n_1+1}, \quad n_1 \in \mathbb{N}.$$

The grid points are

$$x_{1,i} = i \cdot h_1, \quad i = 0, 1, 2, \ldots, n_1+1. \tag{2}$$

The finite difference problem is as follows: find the grid function

$$u_1 = \left(u_{1,0}, u_{1,1}, u_{1,2}, \ldots, u_{1,n_1}, u_{1,n_1+1}\right),$$

such that

$$\begin{cases} \frac{-u_{1,i-1}+2u_{1,i}-u_{1,i+1}}{h_1^2} & = & f(x_{1,i}) =: \tilde{f}_{1,i}, \\ u_{1,0} = u_{1,n_1+1} & = & 0. \end{cases} \tag{3}$$

It should be obvious that $u_{1,i} \approx u(x_{1,i})$. In fact, one can show, under certain reasonable assumptions, that

$$\max_{1 \le i \le n_1} \left| u(x_{1,i}) - u_{1,i} \right| \le C h_1^2,$$

where $C > 0$ is a constant that is independent of h_1.

The Finite Difference Method

Now, let us set

$$\boldsymbol{u}_1^{\mathrm{FD}} = \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ \vdots \\ u_{1,n_1-1} \\ u_{1,n_1} \end{bmatrix}, \qquad \boldsymbol{f}_1^{\mathrm{FD}} = \begin{bmatrix} h_1 \tilde{f}_{1,1} \\ h_1 \tilde{f}_{1,2} \\ \vdots \\ h_1 \tilde{f}_{1,n_1-1} \\ h_1 \tilde{f}_{1,n_1} \end{bmatrix} \in \mathbb{R}^{n_1},$$

and

$$\mathsf{A}_1 := \begin{bmatrix} \frac{2}{h_1} & -\frac{1}{h_1} & & & \\ -\frac{1}{h_1} & \frac{2}{h_1} & -\frac{1}{h_1} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{h_1} & \frac{2}{h_1} & -\frac{1}{h_1} \\ & & & -\frac{1}{h_1} & \frac{2}{h_1} \end{bmatrix} \in \mathbb{R}^{n_1 \times n_1}.$$

Then in matrix form, the finite difference approximation is as follows: find $\boldsymbol{u}_1^{\mathrm{FD}} \in \mathbb{R}^{n_1}$, such that

$$\mathsf{A}_1 \boldsymbol{u}_1^{\mathrm{FD}} = \boldsymbol{f}_1^{\mathrm{FD}}. \tag{4}$$

The Finite Difference Stiffness Matrix

Here A_1 is our finite difference fine grid stiffness matrix. We will show momentarily that it is SPD. For now, observe that we have deliberately chosen to multiply the finite difference equation by h_1 before defining the stiffness matrix, giving the strange inclusion of the factor $\frac{1}{h_1}$ in its construction. This is a deliberate choice so that we obtain the same scaling as that of the finite element case, which we define next.

$$\mathsf{A}_1 := \begin{bmatrix} \frac{2}{h_1} & -\frac{1}{h_1} & & & \\ -\frac{1}{h_1} & \frac{2}{h_1} & -\frac{1}{h_1} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{h_1} & \frac{2}{h_1} & -\frac{1}{h_1} \\ & & & -\frac{1}{h_1} & \frac{2}{h_1} \end{bmatrix} \in \mathbb{R}^{n_1 \times n_1}.$$

The Finite Element Method

Now, let us use the piecewise linear finite element method to approximate the solution to the model problem. The finite element method is based on the weak formulation of the model problem: find $u \in H_0^1(0,1)$, such that

$$\left(\frac{du}{dx}, \frac{dv}{dx}\right)_{L^2(0,1)} = (f, v)_{L^2(0,1)}, \quad \forall\, v \in H_0^1(0,1).$$

Next we need to define a finite dimensional subspace of $H_0^1(0,1)$. Consider

$$V_1 := \left\{ v \in C^0([0,1]) \;\middle|\; v(0) = v(1) = 0,\ v|_{K_{1,i}} \in \mathbb{P}_1(K_{1,i}),\ 1 \le i \le n_1 \right\}, \quad (5)$$

where the grid is comprised of n_1 equally sized intervals

$$K_{1,i} := (x_{1,i-1}, x_{1,i}), \quad i = 1, \ldots, n_1,$$

and the grid point set, $\{x_{1,i}\}_{i=0}^{n_1+1}$, is as defined in (2).

Definition (Hat Function)

For $i = 1, \ldots, n_1$, define $\psi_{1,i} \in V_1$ via

$$\psi_{1,i}(x_{1,j}) = \delta_{i,j}, \quad 1 \leq j \leq n_1.$$

$\psi_{1,i}$ is called a **level-1 hat function**.

For example, here are the three hat functions for $n_1 = 3$.

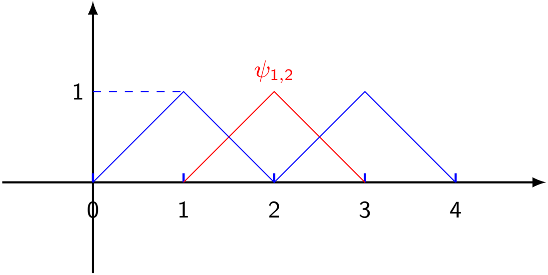

Figure: Piecewise linear basis (hat) functions in 1D.

Proposition

V_1 is a vector subspace of $H_0^1(0,1)$, and the set $B_1 = \{\psi_{1,i}\}_{i=1}^{n_1}$ is a basis for V_1. Thus, the dimension of V_1 is n_1.

Proof.

Exercise. $\square$

The Finite Element Method

The finite element approximation of the model problem is as follows: find $u_1 \in V_1$, such that

$$\left(u_1', \psi_{1,i}'\right)_{L^2(0,1)} = \left(f, \psi_{1,i}\right)_{L^2(0,1)}, \tag{6}$$

for all $i = 1, 2, \ldots, n_1$.

Now set

$$\boldsymbol{f}_1^{\mathrm{FE}} = \begin{bmatrix} \left(f, \psi_{1,1}\right)_{L^2(0,1)} \\ \left(f, \psi_{1,2}\right)_{L^2(0,1)} \\ \vdots \\ \left(f, \psi_{1,n_1-1}\right)_{L^2(0,1)} \\ \left(f, \psi_{1,n_1}\right)_{L^2(0,1)} \end{bmatrix} \in \mathbb{R}^{n_1}.$$

We expand u_1 in the basis of hat functions:

$$u_1 = \sum_{j=1}^{n_1} u_{1,j}\psi_{1,j} \in V_1. \tag{7}$$

The Finite Element Method

Let us set

$$\boldsymbol{u}_1^{\mathrm{FE}} := \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ \vdots \\ u_{1,n_1-1} \\ u_{1,n_1} \end{bmatrix} \in \mathbb{R}^{n_1}.$$

This represents the coordinate vector of u_1 in the basis of hat functions, B_1. Now, plugging (7) into (6), we get

$$\begin{aligned} (f, \psi_{1,i})_{L^2(0,1)} &= \left(u_1', \psi_{1,i}'\right)_{L^2(0,1)} \\ &= \left(\sum_{j=1}^{n_1} u_{1,j}\psi_{1,j}', \psi_{1,i}'\right)_{L^2(0,1)} \\ &= \sum_{j=1}^{n_1} \left(\psi_{1,j}', \psi_{1,i}'\right)_{L^2(0,1)} u_{1,j}. \end{aligned} \tag{8}$$

The Finite Element Method

Define the matrix $\mathsf{A}_1 = [a_{1,i,j}] \in \mathbb{R}^{n_1 \times n_1}$ via

$$a_{1,i,j} = \left(\psi'_{1,j}, \psi'_{1,i}\right)_{L^2(0,1)} = \left(\psi'_{1,i}, \psi'_{1,j}\right)_{L^2(0,1)}. \tag{9}$$

Then (8) is just

$$\left[\mathsf{A}_1 \boldsymbol{u}_1^{\mathrm{FE}}\right]_i = \left[\boldsymbol{f}_1^{\mathrm{FE}}\right]_i, \quad 1 \le i \le n_1,$$

or, equivalently,

$$\mathsf{A}_1 \boldsymbol{u}_1^{\mathrm{FE}} = \boldsymbol{f}_1^{\mathrm{FE}}. \tag{10}$$

The Finite Element Method

Lastly, let us calculate the elements of the matrix A_1. Some entries of A_1 are obvious. Notice that if $|i-j| \geq 2$ then the supports of the two hat functions do not overlap at more than a single point, and, thus, do not interact. Consequently,

$$a_{1,i,j} = 0.$$

Otherwise, the two hat functions will have interactions and thus their inner product may not be zero. See the figure below.

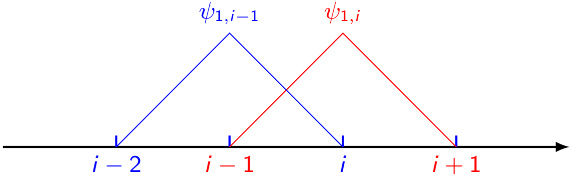

Figure: Interacting hat functions in 1D.

The Finite Element Method

Observe that, for $2 \leq i \leq n_1$,

$$\begin{aligned} a_{1,i,i-1} &= \left(\psi'_{1,i-1}, \psi'_{1,i}\right)_{L^2(0,1)} \\ &= \left(\psi'_{1,i-1}, \psi'_{1,i}\right)_{L^2(T_{1,i})} \\ &= \left(-\frac{1}{h_1}, \frac{1}{h_1}\right)_{L^2(T_{1,i})} \\ &= -\frac{1}{h_1}. \end{aligned}$$

Likewise, for $1 \leq i \leq n_1 - 1$,

$$a_{1,i-1,i} = -\frac{1}{h_1}.$$

Similarly, for $1 \leq i \leq n_1$,

$$a_{1,i,i} = \left(\psi'_{1,i}, \psi'_{1,i}\right)_{L^2(0,1)} = \frac{2}{h_1}.$$

The Finite Element Stiffness Matrix

Thus, we have, as before,

$$\mathsf{A}_1 := \begin{bmatrix} \frac{2}{h_1} & -\frac{1}{h_1} & & & \\ -\frac{1}{h_1} & \frac{2}{h_1} & -\frac{1}{h_1} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{h_1} & \frac{2}{h_1} & -\frac{1}{h_1} \\ & & & -\frac{1}{h_1} & \frac{2}{h_1} \end{bmatrix} \in \mathbb{R}^{n_1 \times n_1}. \tag{11}$$

Definition (Stiffness Matrix)

The matrix $\mathsf{A}_1 \in \mathbb{R}^{n_1 \times n_1}$ defined in (11) is called the **fine level stiffness matrix**, or the **level-1 stiffness matrix**.

Properties of the Stiffness Matrix

Properties of the Stiffness Matrix

We have shown that the determination of the finite element approximation is reduced to the following matrix problem: find $\boldsymbol{u}_1^{\mathrm{FE}} \in \mathbb{R}^{n_1}$, such that

$$\mathsf{A}_1 \boldsymbol{u}_1^{\mathrm{FE}} = \boldsymbol{f}_1^{\mathrm{FE}}. \tag{12}$$

Thus, we should investigate the properties of the stiffness matrix A_1 to establish the existence, uniqueness of the problem, and the difficulty to invert A_1 by studying the condition number.

The next result shows that A_1 is always invertible.

Theorem (Stiffness Matrix is SPD)

The level-1 stiffness matrix, $\mathsf{A}_1 \in \mathbb{R}^{n_1 \times n_1}$*, is SPD. Its eigenvalues are*

$$\lambda_1^{(k)} = \frac{4}{h_1} \sin^2\left(\frac{k\pi h_1}{2}\right) = \frac{2}{h_1}\left(1 - \cos(k\pi h_1)\right). \tag{13}$$

$k = 1, 2, \ldots, n_1$*, and the corresponding eigenvectors are*

$$\left[\boldsymbol{v}_1^{(k)}\right]_i = v_{1,i}^{(k)} = \sin\left(k\pi x_{1,i}\right), \quad 1 \leq i \leq n_1. \tag{14}$$

Proof.

Clearly, A_1 is symmetric. We will show that it is positive definite by showing that all of its eigenvalues are positive. For $1 \le i \le n_1$,

$$\begin{aligned} \left[\mathsf{A}_1 \boldsymbol{v}_1^{(k)}\right]_i &= -\frac{1}{h_1}\sin(k\pi x_{1,i-1}) + \frac{2}{h_1}\sin(k\pi x_{1,i}) - \frac{1}{h_1}\sin(k\pi x_{1,i+1}) \\ &= \frac{2}{h_1}\left[1 - \cos(k\pi h_1)\right]\sin(k\pi x_{1,i}). \end{aligned}$$

Thus

$$\left[\mathsf{A}_1 \boldsymbol{v}_1^{(k)}\right]_i = \lambda_1^{(k)} v_{1,i}^{(k)},$$

for $1 \le i \le n_1$. Since the eigenvalues are positive , A_1 is SPD. □

See the figure on the next slide showing the eigenvalues of A_1.

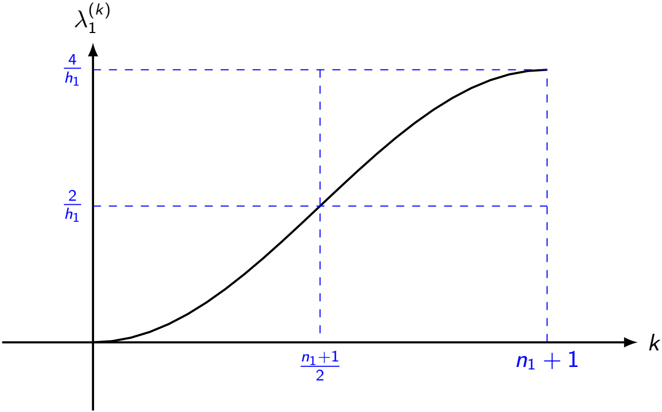

Figure: Eigenvalues of the level-1 stiffness matrix A_1.

Before we get into the next result concerned with approximating the condition number, it will be useful to have a technical lemma.

Lemma

The following quadratic estimates hold

$$1 - \frac{x^2}{2} < \cos(x), \quad 0 < x \leq \frac{\pi}{2}, \tag{15}$$

and

$$\cos(x) < 1 - \frac{x^2}{3}, \quad 0 < x \leq \frac{\pi}{2}. \tag{16}$$

The lower and upper bounds we want may be viewed graphically in the figure on the next slide.

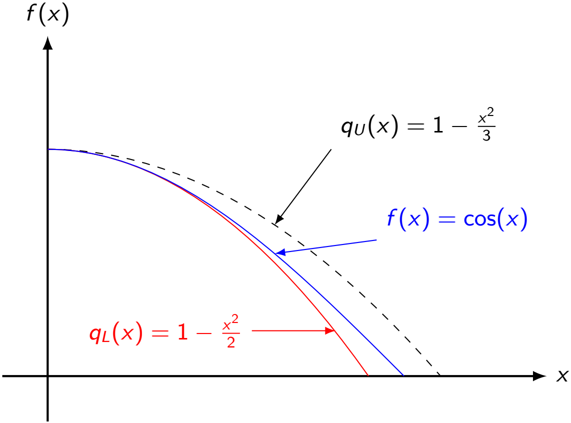

Figure: Upper and lower quadratic bounds of $\cos(x)$ near $x = 0$.

Proof.

(Lower bound): By Taylor's Theorem, for $0 < x \leq \frac{\pi}{2}$,

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}\cos(\xi),$$

for some ξ, with

$$0 < \xi < x \leq \frac{\pi}{2}.$$

Hence

$$0 < \xi < \frac{\pi}{2},$$

and, therefore,

$$0 < \cos(\xi) < 1.$$

It follows that

$$0 < \frac{x^4}{4!}\cos(\xi) = \cos(x) - 1 + \frac{x^2}{2}, \quad 0 < x \leq \frac{\pi}{2},$$

which yields the lower bound (15).

Proof (Cont.)

(Upper bound): Again by Taylor's Theorem, we have, for $0 < x \leq \frac{\pi}{2}$,

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{6!}\cos(\xi),$$

for some $0 < \xi < \frac{\pi}{2}$. So

$$\cos(x) - 1 + \frac{x^2}{2} - \frac{x^4}{24} = -\frac{x^6}{6!}\cos(\xi) < 0,$$

and, therefore,

$$\frac{x^2}{2} - \frac{x^4}{24} < 1 - \cos(x), \quad 0 < x \leq \frac{\pi}{2}.$$

But

$$\frac{x^2}{2} - \frac{x^4}{24} \geq \frac{x^2}{3}, \quad 0 < x \leq 2,$$

as may be easily discovered. Thus,

$$\frac{x^2}{3} < 1 - \cos(x), \quad 0 < x \leq \frac{\pi}{2}.$$

which implies the upper bound (16). $\square$

Theorem (Stiffness Matrix Condition Number)

The spectral condition number of the level-1 stiffness matrix, A_1, *that is,*

$$\kappa_2(\mathsf{A}_1) := \|\mathsf{A}_1\|_2 \left\|\mathsf{A}_1^{-1}\right\|_2 = \frac{\lambda_1^{(n_1)}}{\lambda_1^{(1)}},$$

satisfies the estimates

$$C_1 h_1^{-2} \le \kappa_2(\mathsf{A}_1) \le C_2 h_1^{-2},$$

for some constants $0 < C_1 \le C_2$.

Proof.

Since A_1 is SPD, it follows that

$$\kappa_2(\mathsf{A}_1) = \frac{\lambda_1^{(n_1)}}{\lambda_1^{(1)}} = \frac{1 - \cos(n_1 \pi h_1)}{1 - \cos(\pi h_1)}.$$

Proof (Cont.)

Consider the function $f(x) = 1 - \cos(\pi x)$, as shown in the figure on the next slide. Observe that, since $n_1 \geq 1$, it follows that

$$0 < h_1 \leq \frac{1}{2} \Leftrightarrow 0 < \pi h_1 \leq \frac{\pi}{2}.$$

Furthermore,

$$\frac{1}{2} \leq n_1 h_1 < 1 \Leftrightarrow \frac{\pi}{2} \leq n_1 \pi h_1 < \pi,$$

and, we find, referring to the figure on the next slide, that

$$1 \leq 1 - \cos(n_1 \pi h_1) < 2.$$

From this it follows that

$$\frac{1}{1 - \cos(\pi h_1)} \leq \kappa_2(\mathsf{A}_1) \leq \frac{2}{1 - \cos(\pi h_1)}.$$

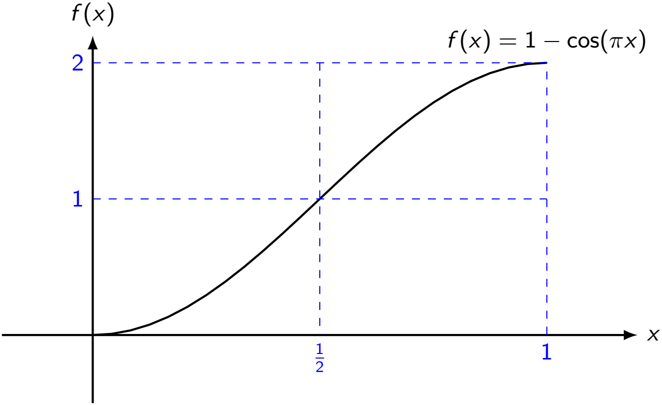

Figure: A plot of the function $f(x) = 1 - \cos(\pi x)$.

Proof (Cont.)

Next, using the quadratic estimate (15), we have

$$1 - \cos(\pi h_1) < \frac{\pi^2}{2} h_1^2, \quad 0 < \pi h_1 \leq \frac{\pi}{2},$$

and we conclude the desired lower bound on the condition number:

$$\frac{2}{\pi^2 h_1^2} \leq \frac{1}{1 - \cos(\pi h_1)}, \quad 0 < h_1 \leq \frac{1}{2}.$$

Similarly, using the quadratic estimate (16), we have

$$\frac{\pi^2 h_1^2}{3} < 1 - \cos(\pi h_1), \quad 0 < \pi h_1 \leq \frac{\pi}{2},$$

which implies

$$\frac{1}{1 - \cos(\pi h_1)} < \frac{3}{\pi^2 h_1^2}, \quad 0 < h_1 \leq \frac{1}{2},$$

and

$$\frac{2}{\pi^2 h_1^2} \leq \kappa_2(\mathsf{A}_1) \leq \frac{6}{\pi^2 h_1^2}.$$

□

Definition (Big Theta)

Suppose that $f : [0,1] \to \mathbb{R}$ is a continuous function. If there exist constants $0 < C_1 \leq C_2$ and numbers $x_0 \in (0,1)$, $r \in \mathbb{R}$, such that, for all $x \in (0, x_0)$,

$$C_1 x^r \leq f(x) \leq C_2 x^r,$$

we write

$$f(x) = \Theta(x^r), \quad \text{as} \quad x \to 0,$$

and we say that $f(x)$ **is big theta of** x^r**, as** $x \to 0$.

Remark

In light of our last definition, our last theorem says that

$$\kappa_2(\mathsf{A}_1) = \Theta(h_1^{-2}), \quad \textit{as} \quad h_1 \to 0.$$

In other words, the spectral condition number blows up as $h_1 \to 0$*. This usually means that the performance of any classical GLIS is degraded for small grid sizes, as we shall see. Two-grid and multigrid algorithms are designed to overcome this degradation.*

The Damped Jacobi Smoother

The Damped Jacobi Smoother

Recall an obvious fact: for both the finite difference method and the finite element method, we eventually need to solve some kind of linear system as follows:

$$\mathsf{A}_1 \boldsymbol{u}_1^{\square} = \boldsymbol{f}_1^{\square} \in \mathbb{R}^{n_1} \tag{17}$$

Here, the $\square$ supercript can be "FD" or "FE." Let us omit this superscript for simplicity and brevity going forward.

To approximate the solution of (17), let us apply the damped Jacobi method, which is again a splitting method.

The Damped Jacobi Smoother

This requires a splitting of A_1, that is,

$$\mathsf{A}_1 = \mathsf{D} - \mathsf{U} - \mathsf{L}. \tag{18}$$

where

$$\mathsf{D} = \begin{bmatrix} \frac{2}{h_1} & & & & \\ & \frac{2}{h_1} & & & \\ & & \ddots & & \\ & & & \frac{2}{h_1} & \\ & & & & \frac{2}{h_1} \end{bmatrix} \in \mathbb{R}^{n_1 \times n_1}, \tag{19}$$

$$\mathsf{U} = \begin{bmatrix} 0 & \frac{1}{h_1} & & & \\ & 0 & \frac{1}{h_1} & & \\ & & \ddots & \ddots & \\ & & & 0 & \frac{1}{h_1} \\ & & & & 0 \end{bmatrix} \in \mathbb{R}^{n_1 \times n_1}, \tag{20}$$

and, of course, $\mathsf{L} = \mathsf{U}^\top$.

The Damped Jacobi Smoother

The damped Jacobi method reads

$$\begin{aligned} \boldsymbol{z}_1 &= \mathsf{D}^{-1}\left(\mathsf{U}+\mathsf{U}^\top\right)\boldsymbol{u}_1^{(\sigma)} + \mathsf{D}^{-1}\boldsymbol{f}_1^{\square}, \\ \boldsymbol{u}_1^{(\sigma+1)} &= \omega\boldsymbol{z}_1 + (1-\omega)\boldsymbol{u}_1^{(\sigma)}, \end{aligned} \tag{21}$$

where $0 < \omega \leq 1$. Eliminating $\boldsymbol{z}_1$, we have the equivalent version

$$\boldsymbol{u}_1^{(\sigma+1)} = \boldsymbol{u}_1^{(\sigma)} + \omega\mathsf{D}^{-1}\left(\boldsymbol{f}_1^{\square} - \mathsf{A}_1\boldsymbol{u}_1^{(\sigma)}\right). \tag{22}$$

In our two-grid terminology,

$$\mathsf{S}_1 = \omega\mathsf{D}^{-1}$$

and

$$\begin{aligned} \mathsf{K}_1 &= \mathsf{I}_1 - \mathsf{S}_1\mathsf{A}_1 \\ &= \mathsf{I}_1 - \omega\mathsf{D}^{-1}\mathsf{A}_1 \\ &= \mathsf{I}_1 - \omega\frac{h_1}{2}\mathsf{A}_1. \end{aligned}$$

Since $\mathsf{S}_1 = \mathsf{S}_1^\top$, $\mathsf{K}_1 = \mathsf{K}_1^*$.

The Damped Jacobi Smoother

The damped Jacobi method may be written in component form as

$$\begin{aligned} z_{1,i} &= \frac{h_1}{2}\left\{ \frac{1}{h_1} u_{1,i-1}^{(\sigma)} + \frac{1}{h_1} u_{1,i+1}^{(\sigma)} \right\} + \frac{h_1}{2} f_{1,i}^{\square}, \\ u_{1,i}^{(\sigma+1)} &= \omega z_{1,i} + (1-\omega) u_{1,i}^{(\sigma)}. \end{aligned}$$

Unlike the Gauss-Seidel method, we do not use component-wise updates immediately after they are generated. Therefore, the order in which we pass through the components is irrelevant.

Theorem (Eigen-Pairs of Damped Jacobi)

Let $\mathsf{K}_1 = \mathsf{I}_1 - \omega \mathsf{D}^{-1}\mathsf{A}_1$ *be the error transfer matrix for the damped Jacobi method applied to the model problem* (17)*. The eigenvectors of* K_1 *are the same as those for the level-1 stiffness matrix,* A_1*, that is,*

$$\left[\boldsymbol{v}_1^{(k)}\right]_i = v_{1,i}^{(k)} = \sin(k\pi x_{1,i}), \quad 1 \le i \le n_1,$$

for $k = 1, \ldots, n_1$*. The eigenvalues of* K_1 *are*

$$\begin{aligned} \mu_1^{(k)}(\omega) &= \omega\cos(k\pi h_1) + 1 - \omega \\ &= 1 - 2\omega\sin^2\left(\frac{k\pi h_1}{2}\right), \quad 1 \le k \le n_1. \end{aligned} \tag{23}$$

Proof.

$$
\begin{aligned}
\mathsf{K}_1 \boldsymbol{v}_1^{(k)} &= \boldsymbol{v}_1^{(k)} - \omega \mathsf{D}^{-1}\mathsf{A}_1 \boldsymbol{v}_1^{(k)} \\
&= \boldsymbol{v}_1^{(k)} - \omega \frac{h_1}{2}\lambda_1^{(k)} \boldsymbol{v}_1^{(k)} \\
&= \left(1 - \omega \frac{h_1}{2}\frac{2}{h_1}\left(1 - \cos(k\pi h_1)\right)\right) \boldsymbol{v}_1^{(k)} \\
&= \left(1 - \omega\left(1 - \cos(k\pi h_1)\right)\right) \boldsymbol{v}_1^{(k)} \\
&= \left(1 - \omega 2 \sin^2\left(\frac{k\pi h_1}{2}\right)\right) \boldsymbol{v}_1^{(k)}.
\end{aligned}
$$

So,

$$
\mu_1^{(k)}(\omega) = 1 - \omega + \omega\cos(k\pi h_1) = 1 - 2\omega\sin^2\left(\frac{k\pi h_1}{2}\right).
$$

$\square$

The next result shows why the damped Jacobi method's performance, as a standalone solver, is degraded as $h_1 \to 0$.

Theorem (Spectral Radius of Damped Jacobi)

Let $\mathsf{K}_1 = \mathsf{I}_1 - \omega \mathsf{D}^{-1}\mathsf{A}_1$ *be the error transfer matrix for the damped Jacobi method applied to the model problem* (17)*. Then*

$$\rho(\mathsf{K}_1) = \mu_1^{(1)}(\omega) = 1 - \omega\Theta(h_1^2), \quad \textit{as} \quad h_1 \to 0,$$

for all $0 < \omega \leq 1$*, that is, there exist constants* $0 < C_1 \leq C_2$*, independent of* h_1 *and* ω*, such that*

$$0 \leq C_1 \omega h_1^2 \leq 1 - \rho(\mathsf{K}_1) \leq C_2 \omega h_1^2.$$

Proof.

One can see clearly from the figure on the next page that $\rho(\mathsf{K}_1) = \mu_1^{(1)}(\omega)$, where

$$\mu_1^{(1)}(\omega) = \omega \cos(\pi h_1) + 1 - \omega.$$

We proved earlier (estimates (15) and (16)) that, for $0 < x \leq \frac{\pi}{2}$,

$$1 - \frac{x^2}{2} < \cos(x) < 1 - \frac{x^2}{3}.$$

Thus, for all $0 < \omega \leq 1$, and all $0 < x \leq \frac{\pi}{2}$,

$$\omega \frac{x^2}{2} > \omega\,(1 - \cos(x)) > \omega \frac{x^2}{3}.$$

Consequently, since $0 < \pi h_1 \leq \frac{\pi}{2}$,

$$\frac{\omega\pi^2}{2} h_1^2 \geq 1 - \rho(\mathsf{K}_1) \geq \frac{\omega\pi^2}{3} h_1^2.$$

Proof (Cont.)

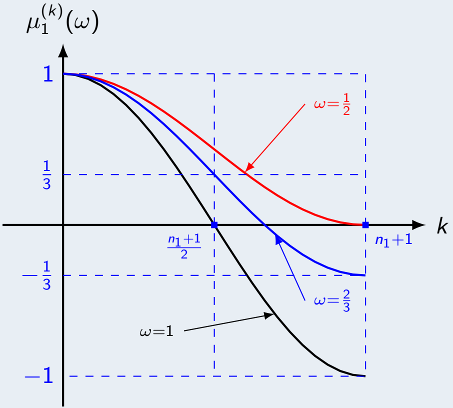

Figure: Plots of the eigenvalues of K_1, $\mu_1^{(k)}(\omega) = 1 - 2\omega \sin^2\left(\frac{k\pi h_1}{2}\right)$, as functions of k, for various values of $\omega \in (0, 1]$.

□

Remark (Smoothing Effect)

In the multigrid setting, we want the Jacobi method (the smoother) to have a "smoothing" effect on the error. In other words, we want to dampen high-frequency modes of the error faster than low-frequency modes. Recall, for the damped Jacobi method

$$\boldsymbol{e}_1^{(\sigma+1)} = \mathsf{K}_1 \boldsymbol{e}_1^{(\sigma)},$$

where

$$\mathsf{K}_1 = \mathsf{I}_1 - \frac{\omega h_1}{2} \mathsf{A}_1.$$

Now, expand $\boldsymbol{e}_1^{(\sigma)}$ *in the basis of eigenvectors* $\left\{ \boldsymbol{v}_1^{(k)} \right\}_{k=1}^{n_1}$*: there exist unique numbers*

$$\epsilon_k^{(\sigma)} \in \mathbb{R}, \quad k = 1, 2, \ldots, n_1.$$

such that

$$\boldsymbol{e}_1^{(\sigma)} = \sum_{k=1}^{n_1} \epsilon_k^{(\sigma)} \boldsymbol{v}_1^{(k)}.$$

Remark (Cont.)

Then

$$\boldsymbol{e}_1^{(\sigma+1)} = \sum_{k=1}^{n_1} \mu_1^{(k)}(\omega)\epsilon_k^{(\sigma)}\boldsymbol{v}_1^{(k)}. \tag{24}$$

By choosing ω judiciously, we can bias the smoothing process to favor the dampening of "high frequency" error components.