Math 673/4

Multigrid Methods: A Mostly Matrix-Based Approach

Chapter 09: Additive Preconditioners Based on Subspace Decompositions

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Chapter 09, Part 1 of 2

Additive Preconditioners Based on Subspace Decompositions

Introduction

In the last chapter we discovered that the common multigrid algorithms can be classified as multiplicative methods. In this chapter, we view an alternative to multiplicative methods, additive methods. These methods will be developed not as stand alone solvers but as preconditioners, or preconditioning strategies. The advantage over multiplicative methods is that these additive methods can be easily parallelized.

Now, when we design preconditioners, the idea is to use them in conjunction with the preconditioned conjugate gradient (CG) method. The main difficulty is in inverting the preconditioner. The two preconditioners devised in this chapter are straightforward to invert, though we will not delve into implementation issues.

Both preconditioners are based on the theory of subspace decomposition. Let us review that material first from a matrix–based point of view. The more abstract operator introduction can be found in the book by Brenner and Scott.

Subspace Decompositions

Definition

Suppose that $n \in \mathbb{Z}$ and $\{m_j\}_{j=0}^{L} \subset \mathbb{Z}$, with

$$0 < m_0 \leq m_1 \leq \cdots \leq m_j \leq \cdots \leq m_L \leq n.$$

The matrices

$$\mathsf{Q}_j \in \mathbb{R}^{n \times m_j},$$

are called **prolongation matrices** iff $\operatorname{rank}(\mathsf{Q}_j) = m_j$, for all $j \in \{0, 1, \ldots, L\}$. Let $\{\mathsf{Q}_j\}_{j=0}^{L}$ be a set of prolongation matrices. We say that **Assumption (SS1) holds for** $\{\mathsf{Q}_j\}_{j=0}^{L}$ or, equivalently, that the set $\{\mathsf{Q}_j\}_{j=0}^{L}$ **supports a subspace decomposition of** $\mathbb{R}^n$, iff for every $\boldsymbol{u} \in \mathbb{R}^n$, there exist vectors

$$\boldsymbol{w}_j \in \mathbb{R}^{m_j}, \quad 0 \leq j \leq L,$$

such that

$$\boldsymbol{u} = \sum_{j=0}^{L} \mathsf{Q}_j \boldsymbol{w}_j. \tag{1}$$

Example

The simplest example might also seem like the most obvious one. Consider $\mathbb{R}^n$, and set $L = n - 1$ and

$$m_0 = m_1 = \cdots = m_{n-1} = 1.$$

Define $\mathsf{Q}_j \in \mathbb{R}^{n\times 1}$ via

$$\mathsf{Q}_j = \hat{\boldsymbol{e}}_{j+1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow j + 1^{\mathrm{st}} \text{ entry.} \\ \\ \\ \\ \end{matrix}$$

Now, suppose that $\boldsymbol{u} \in \mathbb{R}^n$ is arbitrary. Then,

$$\boldsymbol{w}_j = [u_{j+1}] \in \mathbb{R}, \quad 0 \le j \le n,$$

where $[\boldsymbol{u}]_i = u_i$. It is easy to see that (1) holds.

Example

This example defines what is called a hierarchical basis decomposition, about which we will have more to say later. Suppose that $n = 2^3 - 1 = 7$. Set $L = 3 - 1 = 2$ and

$$m_0 = 1, \quad m_1 = 2, \quad m_2 = 4.$$

Note, in this case, that

$$\sum_{j=0}^{L} m_j = m_0 + m_1 + m_2 = 1 + 2 + 4 = 7 = n,$$

which is a typical feature of hierarchical basis decompositions. Define

$$\mathsf{Q}_0 = \begin{bmatrix} | \\ \hat{\boldsymbol{e}}_4 \\ | \end{bmatrix} \in \mathbb{R}^{7\times 1}, \quad \mathsf{Q}_1 = \begin{bmatrix} | & | \\ \hat{\boldsymbol{e}}_2 & \hat{\boldsymbol{e}}_6 \\ | & | \end{bmatrix} \in \mathbb{R}^{7\times 2},$$

and

$$\mathsf{Q}_2 = \begin{bmatrix} | & | & | & | \\ \hat{\boldsymbol{e}}_1 & \hat{\boldsymbol{e}}_3 & \hat{\boldsymbol{e}}_5 & \hat{\boldsymbol{e}}_7 \\ | & | & | & | \end{bmatrix} \in \mathbb{R}^{7\times 4}.$$

Example (Cont.)

For arbitrary $\boldsymbol{u} \in \mathbb{R}^7$, set

$$\boldsymbol{w}_0 = [u_4] \in \mathbb{R}^1, \quad \boldsymbol{w}_1 = \begin{bmatrix} u_2 \\ u_6 \end{bmatrix} \in \mathbb{R}^2, \quad \boldsymbol{w}_2 = \begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \end{bmatrix} \in \mathbb{R}^4.$$

Then, clearly,

$$\boldsymbol{u} = \sum_{j=0}^{2} \mathsf{Q}_j \boldsymbol{w}_j = \mathsf{Q}_0 \boldsymbol{w}_0 + \mathsf{Q}_1 \boldsymbol{w}_1 + \mathsf{Q}_2 \boldsymbol{w}_2.$$

Example

Here is another hierarchical basis decomposition. Can you can determine the general pattern from this example? Suppose that $n = 3 \cdot 2^3 - 1 = 23$. Set $L = 3$ and

$$m_0 = 2, \quad m_1 = 3, \quad m_2 = 6, \quad m_3 = 12.$$

Again,

$$\sum_{j=0}^{L} m_j = m_0 + m_1 + m_2 + m_3 = 2 + 3 + 6 + 12 = 23 = n.$$

Example (Cont.)

Define

$$Q_0 = \begin{bmatrix} | & | \\ \hat{\mathbf{e}}_8 & \hat{\mathbf{e}}_{16} \\ | & | \end{bmatrix} \in \mathbb{R}^{23\times 2},$$

$$Q_1 = \begin{bmatrix} | & | & | \\ \hat{\mathbf{e}}_4 & \hat{\mathbf{e}}_{12} & \hat{\mathbf{e}}_{20} \\ | & | & | \end{bmatrix} \in \mathbb{R}^{23\times 3},$$

$$Q_2 = \begin{bmatrix} | & | & | & | & | & | \\ \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_6 & \hat{\mathbf{e}}_{10} & \hat{\mathbf{e}}_{14} & \hat{\mathbf{e}}_{18} & \hat{\mathbf{e}}_{22} \\ | & | & | & | & | & | \end{bmatrix} \in \mathbb{R}^{23\times 6},$$

$$Q_3 = \begin{bmatrix} | & | & | & & | & | & | \\ \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_3 & \hat{\mathbf{e}}_5 & \cdots & \hat{\mathbf{e}}_{19} & \hat{\mathbf{e}}_{21} & \hat{\mathbf{e}}_{23} \\ | & | & | & & | & | & | \end{bmatrix} \in \mathbb{R}^{23\times 12}.$$

Example (Cont.)

For arbitrary $\boldsymbol{u} \in \mathbb{R}^{23}$, set

$$\boldsymbol{w}_0 = \begin{bmatrix} u_8 \\ u_{16} \end{bmatrix} \in \mathbb{R}^2, \quad \boldsymbol{w}_1 = \begin{bmatrix} u_4 \\ u_{12} \\ u_{20} \end{bmatrix} \in \mathbb{R}^3,$$

$$\boldsymbol{w}_2 = \begin{bmatrix} u_2 \\ u_6 \\ u_{10} \\ u_{14} \\ u_{18} \\ u_{22} \end{bmatrix} \in \mathbb{R}^6, \quad \boldsymbol{w}_3 = \begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ \vdots \\ u_{19} \\ u_{21} \\ u_{23} \end{bmatrix} \in \mathbb{R}^{12}.$$

Then,

$$\boldsymbol{u} = \sum_{j=0}^{3} \mathsf{Q}_j \boldsymbol{w}_j = \mathsf{Q}_0 \boldsymbol{w}_0 + \mathsf{Q}_1 \boldsymbol{w}_1 + \mathsf{Q}_2 \boldsymbol{w}_2 + \mathsf{Q}_3 \boldsymbol{w}_3.$$

Definition (Additive Subspace Preconditioner)

Suppose that Assumption (SS1) holds for the set of prolongation matrices $\{\mathsf{Q}_j\}_{j=0}^{L}$. The matrix $\mathsf{C} \in \mathbb{R}^{n\times n}$ is called an **additive subspace preconditioner with respect to** $\{\mathsf{Q}_j\}_{j=0}^{L}$ iff there are SPD matrices $\mathsf{C}_\ell \in \mathbb{R}^{m_\ell \times m_\ell}$, $0 \le \ell \le L$, such that

$$\mathsf{C} = \sum_{\ell=0}^{L} \mathsf{Q}_\ell \mathsf{C}_\ell^{-1} \mathsf{Z}_\ell, \tag{2}$$

where

$$\mathsf{Z}_\ell = \mathsf{Q}_\ell^\top \in \mathbb{R}^{m_\ell \times n}.$$

In other words,

$$(\mathsf{Q}_\ell \boldsymbol{u}_\ell, \boldsymbol{v}) = \left(\boldsymbol{u}_\ell, \mathsf{Q}_\ell^\top \boldsymbol{v}\right)_\ell = (\boldsymbol{u}_\ell, \mathsf{Z}_\ell \boldsymbol{v})_\ell \,,$$

for all $\boldsymbol{u}_\ell \in \mathbb{R}^{m_\ell}$ and $\boldsymbol{v} \in \mathbb{R}^n$.

Remark

In this last definition, and in what follows, $(\cdot\,,\,\cdot)$ *represents the standard Euclidean inner product on* $\mathbb{R}^n$, *and* $(\cdot\,,\,\cdot)_\ell$ *represents the standard Euclidean inner product on* $\mathbb{R}^{m_\ell}$, $0 \le \ell \le L$. *Similarly,* $\|\cdot\|$ *is the standard Euclidean norm on* $\mathbb{R}^n$, *whereas* $\|\cdot\|_\ell$ *is the standard Euclidean norm on* $\mathbb{R}^{m_\ell}$, $0 \le \ell \le L$.

Lemma

Suppose that Assumption (SS1) holds for the set of prolongation matrices $\{\mathsf{Q}_j\}_{j=0}^{L}$ *and* $\mathsf{C} \in \mathbb{R}^{n \times n}$ *is a preconditioner with respect to this family. Then* C *is SPD with respect to* $(\cdot\,,\,\cdot)$*, and, consequently, if* A *is also SPD with respect to* $(\cdot\,,\,\cdot)$*, then* CA *is SPD with respect to* $(\cdot\,,\,\cdot)_{\mathsf{C}^{-1}}$ *and* $(\cdot\,,\,\cdot)_{\mathsf{A}}$*.*

Proof.

C is clearly symmetric, since each C_ℓ^{-1} is symmetric. Now, let $\boldsymbol{u} \in \mathbb{R}^n$ be arbitrary. Then

$$
\begin{aligned}
(\boldsymbol{u}, \mathsf{C}\boldsymbol{u}) &\overset{(2)}{=} \left(\boldsymbol{u}, \sum_{\ell=0}^{L} \mathsf{Q}_\ell \mathsf{C}_\ell^{-1} \mathsf{Z}_\ell \boldsymbol{u}\right) \\
&= \sum_{\ell=0}^{L} \left(\boldsymbol{u}, \mathsf{Q}_\ell \mathsf{C}_\ell^{-1} \mathsf{Z}_\ell \boldsymbol{u}\right) \\
&= \sum_{\ell=0}^{L} \left(\mathsf{Z}_\ell \boldsymbol{u}, \mathsf{C}_\ell^{-1} \mathsf{Z}_\ell \boldsymbol{u}\right)_\ell \\
&\geq 0,
\end{aligned}
$$

since C_ℓ^{-1} is SPD with respect to $(\,\cdot\,,\,\cdot\,)_\ell$, $0 \leq \ell \leq L$.

Proof (Cont.)

Now, suppose that

$$\sum_{\ell=0}^{L} \left(\mathsf{Z}_\ell \boldsymbol{u}, \mathsf{C}_\ell^{-1} \mathsf{Z}_\ell \boldsymbol{u} \right)_\ell = 0.$$

Since, again, C_ℓ^{-1} is SPD, it must be true that

$$\mathsf{Z}_\ell \boldsymbol{u} = \mathbf{0}, \quad 0 \leq \ell \leq L. \tag{3}$$

Proof (Cont.)

In this case, since (SS1) holds, we have

$$
\begin{aligned}
\|\boldsymbol{u}\|^2 &= (\boldsymbol{u}, \boldsymbol{u}) \\
&\overset{(SS1)}{=} \left(\boldsymbol{u}, \sum_{\ell=0}^{L} \mathrm{Q}_\ell \boldsymbol{w}_\ell \right) \\
&= \sum_{\ell=0}^{L} (\boldsymbol{u}, \mathrm{Q}_\ell \boldsymbol{w}_\ell) \\
&= \sum_{\ell=0}^{L} (\mathrm{Z}_\ell \boldsymbol{u}, \boldsymbol{w}_\ell)_\ell \\
&\overset{(3)}{=} \sum_{\ell=0}^{L} (\boldsymbol{0}, \boldsymbol{w}_\ell)_\ell \\
&= 0.
\end{aligned}
$$

This implies that $\boldsymbol{u} = \boldsymbol{0}$, which shows that C is SPD with respect to $(\,\cdot\,,\,\cdot\,)$. The results concerning CA follow from a result in Chapter 01. □

Definition (Q Class)

Suppose that Assumption (SS1) holds for the set of prolongation matrices $\{Q_j\}_{j=0}^{L}$ and $\boldsymbol{u} \in \mathbb{R}^n$ is fixed. The set $Q[\boldsymbol{u}]$ is called the Q **class of** $\boldsymbol{u}$ iff

$$Q[\boldsymbol{u}] := \left\{ (\boldsymbol{w}_j)_{j=0}^{L} \;\middle|\; \boldsymbol{w}_j \in \mathbb{R}^{m_j},\ j = 0, \ldots, L,\ \boldsymbol{u} = \sum_{j=0}^{L} Q_j \boldsymbol{w}_j \right\}.$$

We write

$$(\boldsymbol{w}_j)_{j=0}^{L} \in Q[\boldsymbol{u}] \quad \text{or, simply,} \quad (\boldsymbol{w}_j) \in Q[\boldsymbol{u}],$$

and we remark that each $(\boldsymbol{w}_j)_{j=0}^{L}$ is an order tuple.

Theorem

Suppose that Assumption (SS1) holds for the set of prolongation matrices $\{\mathsf{Q}_j\}_{j=0}^{L}$ *and* $\mathsf{C} \in \mathbb{R}^{n\times n}$ *is a preconditioner with respect to this family. Then, for any* $\boldsymbol{u} \in \mathbb{R}^n$,

$$(\boldsymbol{u}, \boldsymbol{u})_{\mathsf{C}^{-1}} = \min_{(\boldsymbol{w}_\ell)\in \mathsf{Q}[\boldsymbol{u}]} \sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell \,. \tag{4}$$

Proof.

Since each $\mathsf{C}_\ell \in \mathbb{R}^{m_\ell \times m_\ell}$ is SPD with respect to $(\cdot\,,\,\cdot)_\ell$, $(\cdot\,,\,\cdot)_{\mathsf{C}_\ell^{-1}}$ is a bona fide inner product. Therefore

$$\begin{aligned} \left(\mathsf{C}_\ell^{-1}\boldsymbol{u}_\ell, \boldsymbol{v}_\ell\right)_\ell &= (\boldsymbol{u}_\ell, \boldsymbol{v}_\ell)_{\mathsf{C}_\ell^{-1}} \\ &\overset{\text{C.S.}}{\leq} \|\boldsymbol{u}_\ell\|_{\mathsf{C}_\ell^{-1}} \|\boldsymbol{v}_\ell\|_{\mathsf{C}_\ell^{-1}} \\ &= \sqrt{(\boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathsf{C}_\ell^{-1}}}\sqrt{(\boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_{\mathsf{C}_\ell^{-1}}}, \end{aligned}$$

for all $\boldsymbol{u}_\ell, \boldsymbol{v}_\ell \in \mathbb{R}^{m_\ell}$. Let $\boldsymbol{u} \in \mathbb{R}^n$ be arbitrary. Then

$$\boldsymbol{u} \overset{(SS1)}{=} \sum_{\ell=0}^{L} \mathsf{Q}_\ell \boldsymbol{w}_\ell, \quad \forall\, (\boldsymbol{w}_\ell) \in \mathsf{Q}[\boldsymbol{u}].$$

Proof (Cont.)

We have

$$
\begin{aligned}
(\boldsymbol{u}, \boldsymbol{u})_{\mathsf{C}^{-1}} &= \sum_{\ell=0}^{L} (\boldsymbol{u}, \mathsf{Q}_\ell \boldsymbol{w}_\ell)_{\mathsf{C}^{-1}} \\
&= \sum_{\ell=0}^{L} \left(\mathsf{Z}_\ell \mathsf{C}^{-1} \boldsymbol{u}, \boldsymbol{w}_\ell \right)_\ell \\
&= \sum_{\ell=0}^{L} \left(\mathsf{Z}_\ell \mathsf{C}^{-1} \boldsymbol{u}, \mathsf{C}_\ell \boldsymbol{w}_\ell \right)_{\mathsf{C}_\ell^{-1}} \\
&\overset{\text{C.S.}}{\leq} \sum_{\ell=0}^{L} \left\| \mathsf{Z}_\ell \mathsf{C}^{-1} \boldsymbol{u} \right\|_{\mathsf{C}_\ell^{-1}} \left\| \mathsf{C}_\ell \boldsymbol{w}_\ell \right\|_{\mathsf{C}_\ell^{-1}} \\
&\overset{\text{C.S.}}{\leq} \left(\sum_{\ell=0}^{L} \left\| \mathsf{Z}_\ell \mathsf{C}^{-1} \boldsymbol{u} \right\|_{\mathsf{C}_\ell^{-1}}^2 \right)^{1/2} \left(\sum_{\ell=0}^{L} \left\| \mathsf{C}_\ell \boldsymbol{w}_\ell \right\|_{\mathsf{C}_\ell^{-1}}^2 \right)^{1/2} \\
&= \left(\sum_{\ell=0}^{L} \left(\mathsf{Z}_\ell \mathsf{C}^{-1} \boldsymbol{u}, \mathsf{Z}_\ell \mathsf{C}^{-1} \boldsymbol{u} \right)_{\mathsf{C}_\ell^{-1}} \right)^{1/2} \left(\sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \mathsf{C}_\ell \boldsymbol{w}_\ell)_{\mathsf{C}_\ell^{-1}} \right)^{1/2}
\end{aligned}
$$

Proof (Cont.)

$$
\begin{aligned}
&= \left(\sum_{\ell=0}^{L} \left(\mathsf{Z}_\ell \mathsf{C}^{-1} \boldsymbol{u}, \mathsf{C}_\ell^{-1} \mathsf{Z}_\ell \mathsf{C}^{-1} \boldsymbol{u} \right)_\ell \right)^{1/2} \left(\sum_{\ell=0}^{L} \left(\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell \right)_\ell \right)^{1/2} \\
&\overset{(2)}{=} \left(\mathsf{C}^{-1} \boldsymbol{u}, \mathsf{C}\mathsf{C}^{-1} \boldsymbol{u} \right)^{1/2} \left(\sum_{\ell=0}^{L} \left(\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell \right)_\ell \right)^{1/2} \\
&= \|\boldsymbol{u}\|_{\mathsf{C}^{-1}} \left(\sum_{\ell=0}^{L} \left(\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell \right)_\ell \right)^{1/2} .
\end{aligned}
$$

So,

$$
(\boldsymbol{u}, \boldsymbol{u})_{\mathsf{C}^{-1}} \leq \sum_{\ell=0}^{L} \left(\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell \right)_\ell , \quad \forall \, (\boldsymbol{w}_\ell) \in \mathsf{Q}[\boldsymbol{u}].
$$

Proof (Cont.)

Now, for the particular choice

$$\boldsymbol{w}_\ell = \mathsf{C}_\ell^{-1}\mathsf{Z}_\ell\mathsf{C}^{-1}\boldsymbol{u} \in \mathbb{R}^{m_\ell}, \quad 0 \leq \ell \leq L,$$

we have

$$\boldsymbol{u} = \sum_{\ell=0}^{L} \mathsf{Q}_\ell \boldsymbol{w}_\ell,$$

and

$$\begin{aligned}
\sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell &= \sum_{\ell=0}^{L} \left(\mathsf{C}_\ell\mathsf{C}_\ell^{-1}\mathsf{Z}_\ell\mathsf{C}^{-1}\boldsymbol{u}, \mathsf{C}_\ell^{-1}\mathsf{Z}_\ell\mathsf{C}^{-1}\boldsymbol{u}\right)_\ell \\
&= \sum_{\ell=0}^{L} \left(\mathsf{C}^{-1}\boldsymbol{u}, \mathsf{Q}_\ell\mathsf{C}_\ell^{-1}\mathsf{Z}_\ell\mathsf{C}^{-1}\boldsymbol{u}\right) \\
&\overset{(2)}{=} \left(\mathsf{C}^{-1}\boldsymbol{u}, \mathsf{C}\mathsf{C}^{-1}\boldsymbol{u}\right) \\
&= (\boldsymbol{u}, \boldsymbol{u})_{\mathsf{C}^{-1}} .
\end{aligned}$$

□

Next, let us write what seem to be complicated formulas for the largest and smallest eigenvalues of CA. But, in light of the last result, the results are trivial.

Theorem (Eigenvalues of CA)

Suppose that Assumption (SS1) holds for the set of prolongation matrices $\{\mathsf{Q}_j\}_{j=0}^L$ *and* C *is defined as in Equation* (2)*. The eigenvalues of* CA *are positive, provided* A *is SPD with respect to* $(\cdot,\cdot)$*. Moreover*

$$\lambda_{\max}(\mathsf{CA}) = \max_{\boldsymbol{u}\in\mathbb{R}^n_\star} \frac{(\mathsf{A}\boldsymbol{u},\boldsymbol{u})}{\displaystyle\min_{(\boldsymbol{w}_\ell)\in\mathsf{Q}[\boldsymbol{u}]} \sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell}, \tag{5}$$

$$\lambda_{\min}(\mathsf{CA}) = \min_{\boldsymbol{u}\in\mathbb{R}^n_\star} \frac{(\mathsf{A}\boldsymbol{u},\boldsymbol{u})}{\displaystyle\min_{(\boldsymbol{w}_\ell)\in\mathsf{Q}[\boldsymbol{u}]} \sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell}. \tag{6}$$

Proof.

Recall that CA is SPD with respect to $(\cdot, \cdot)_{C^{-1}}$. Thus the eigenvalues are positive real and the corresponding eigenvectors may be chosen so that they form an orthonormal basis for $\mathbb{R}^n$ with respect to $(\cdot, \cdot)_{C^{-1}}$. Moreover the Rayleigh quotient formula holds

$$
\begin{aligned}
\lambda_{\max}(\mathsf{CA}) &= \max_{\boldsymbol{u} \in \mathbb{R}^n_\star} \frac{(\mathsf{CA}\boldsymbol{u}, \boldsymbol{u})_{\mathsf{C}^{-1}}}{(\boldsymbol{u}, \boldsymbol{u})_{\mathsf{C}^{-1}}} \\
&\overset{(4)}{=} \max_{\boldsymbol{u} \in \mathbb{R}^n_\star} \frac{(\mathsf{A}\boldsymbol{u}, \boldsymbol{u})}{\displaystyle\min_{(\boldsymbol{w}_\ell) \in \mathsf{Q}[\boldsymbol{u}]} \sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell}.
\end{aligned}
$$

The formula for $\lambda_{\min}(\mathsf{CA})$ is established similarly. $\square$

Hierarchical Bases

Hierarchical Bases

In this section, we will examine a particular subspace decomposition in the finite element setting, one based on a hierarchical decomposition of the finite element bases.

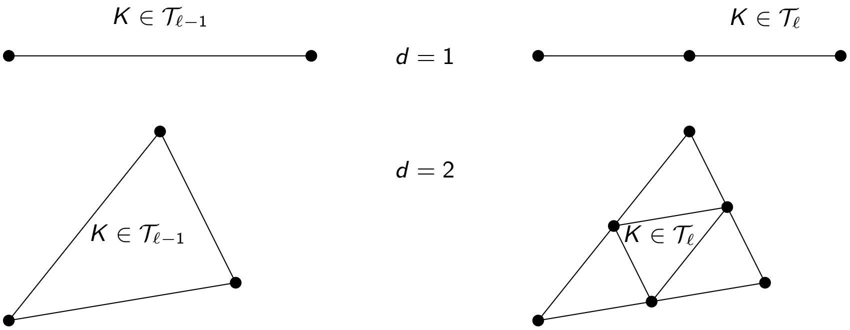

Figure: Bisection and quadrisection refinement in 1D and 2D, respectively.

Definition (Finite Element Spaces)

Suppose $d = 1$ or $d = 2$ and Ω is an open interval ($d = 1$) or an open convex polygonal domain ($d = 2$). Suppose $\mathcal{T}_0$ is an initial conforming partition ($d = 1$) or triangulation ($d = 2$) of Ω into subintervals ($d = 1$) or triangles ($d = 2$). Let $\mathcal{T}_\ell$ be the family of triangulations obtained by subdividing each interval K of $\mathcal{T}_{\ell-1}$ into 2 equal subintervals ($d = 1$) or each triangle K of $\mathcal{T}_{\ell-1}$ into 4 similar triangles by joining the edge midpoints ($d = 2$). See Figure 1.

Set

$$S_\ell := \left\{ v \in C^0(\overline{\Omega}) \;\middle|\; v|_K \in \mathbb{P}_1(K), \ \forall\, K \in \mathcal{T}_\ell \right\}, \tag{7}$$

and

$$V_\ell := \{ v \in S_\ell \mid v|_{\partial\Omega} \equiv 0 \} \tag{8}$$

for all $0 \le \ell \le L$. Define

$$n_\ell := \dim(V_\ell).$$

Set $W_0 := V_0$, and, for $1 \le \ell \le L$, define

$$W_\ell := \{ v \in V_\ell \mid v(\boldsymbol{N}_{\ell-1,j}) = 0, \ \forall 1 \le j \le n_{\ell-1} \}. \tag{9}$$

Recall that $\{\boldsymbol{N}_{\ell,j}\}_{j=1}^{n_\ell} \subset \Omega$ is set of interior vertices of $\mathcal{T}_\ell$.

Definition (Finite Element Spaces (Cont.))

Set

$$m_\ell := \dim(W_\ell).$$

By $\mathcal{B}_\ell^V$, $0 \leq \ell \leq L$, we denote the family of Lagrange nodal bases of V_ℓ

$$\mathcal{B}_\ell^V := \{\psi_{\ell,j}\}_{j=1}^{n_\ell},$$

with the property that

$$\psi_{\ell,j}(\boldsymbol{N}_{\ell,i}) = \delta_{i,j}, \quad 1 \leq i, j \leq n_\ell.$$

Hierarchical Bases

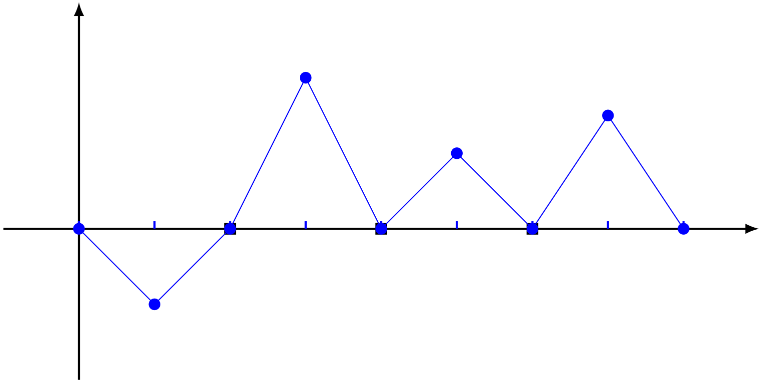

Figure: A sample function v from W_ℓ. Observe that $v(\boldsymbol{N}_{\ell-1,j}) = 0$, for all $1 \le j \le n_{\ell-1}$.

Lemma

For the spaces $W_\ell \subseteq V_\ell$ as in the last definition, we have

$$V_\ell = V_{\ell-1} \oplus W_\ell, \quad 1 \leq \ell \leq L, \tag{10}$$

and

$$V_L = W_0 \oplus W_1 \oplus \cdots \oplus W_L. \tag{11}$$

Proof.

Let $\mathcal{I}_\ell : C^0(\overline{\Omega}) \to V_\ell$ be the standard Lagrange linear nodal interpolation operator. It has the property that, for every $v \in C^0(\overline{\Omega})$,

$$\mathcal{I}_\ell(v)\left(\boldsymbol{N}_{\ell,i}\right) = v\left(\boldsymbol{N}_{\ell,i}\right), \quad 1 \leq i \leq n_\ell. \tag{12}$$

Let $v_\ell \in V_\ell$ be a given arbitrary function. Write

$$v_\ell = \mathcal{I}_{\ell-1}(v_\ell) + \left\{v_\ell - \mathcal{I}_{\ell-1}(v_\ell)\right\}.$$

Clearly

$$\mathcal{I}_{\ell-1}(v_\ell) \in V_{\ell-1},$$

and

$$v_\ell - \mathcal{I}_{\ell-1}(v_\ell) \in W_\ell.$$

Proof (Cont.)

Indeed,

$$v_\ell\left(\boldsymbol{N}_{\ell-1,j}\right) - \mathcal{I}_{\ell-1} v_\ell\left(\boldsymbol{N}_{\ell-1,j}\right) = 0,$$

for every $1 \le j \le n_{\ell-1}$. This decomposition must be unique. To see why, suppose this is not the case. Then

$$v_\ell = v_{\ell-1}^{(i)} + w_\ell^{(i)}, \quad i = 1, 2,$$

with

$$v_{\ell-1}^{(i)} \in V_{\ell-1} \quad \text{and} \quad w_\ell^{(i)} \in W_\ell.$$

So,

$$\begin{aligned} 0 &= \left(v_{\ell-1}^{(1)} - v_{\ell-1}^{(2)}\right) + \left(w_\ell^{(1)} - w_\ell^{(2)}\right) \\ &=: v_{\ell-1} - w_\ell. \end{aligned}$$

Therefore,

$$V_{\ell-1} \ni v_{\ell-1} = w_\ell \in W_\ell.$$

Clearly, both functions must be identically zero. This proves (10). Identity (11) follows from (10). $\square$

Definition

For $1 \le \ell \le L$, define $\mathcal{B}_\ell^W := \{\phi_{\ell,i}\}_{i=1}^{m_\ell} \subset W_\ell$, via

$$\phi_{\ell,i} \in W_\ell, \quad 1 \le i \le m_\ell,$$

and

$$\phi_{\ell,i}\left(\boldsymbol{N}_{\ell,j}^W\right) = \delta_{i,j}, \quad 1 \le i,j \le m_\ell,$$

where

$$\left\{\boldsymbol{N}_{\ell,j}^W\right\}_{j=1}^{m_\ell} := \{\boldsymbol{N}_{\ell,j}\}_{j=1}^{n_\ell} \setminus \{\boldsymbol{N}_{\ell-1,j}\}_{j=1}^{n_{\ell-1}}.$$

Define $\mathcal{B}_0^W := \mathcal{B}_0^V$.

Nodal Basis Functions for W_2

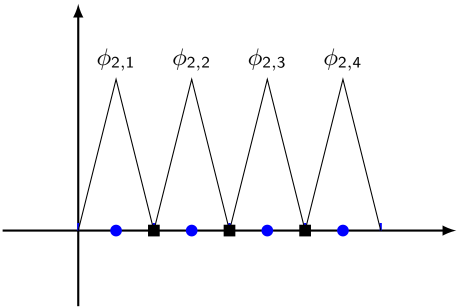

Figure: The nodal basis functions for W_2 in 1D: $\Omega = (0, 1)$, $\ell = 2$, $n_1 = 3$, $m_2 = 4$, $n_2 = 7$.

Lemma

$\mathcal{B}_\ell^W$ is a bona fide basis for W_ℓ, for each $1 \leq \ell \leq L$.

Proof.

Exercise. $\square$

Lemma

Suppose $W_\ell \subseteq V_\ell, 0 \leq \ell \leq L$, *are as defined earlier. Then*

$$\mathcal{H}_\ell := \cup_{j=0}^{\ell} \mathcal{B}_j^W$$

is a bona fide basis for V_ℓ, *for any* $1 \leq \ell \leq L$.

Proof.

The result follows if we can show that

$$\operatorname{span}(\mathcal{H}_\ell) = V_\ell,$$

and $\mathcal{H}_\ell$ is linearly independent.

Suppose $v_\ell \in V_\ell$ is arbitrary. Then, there exist unique $w_j \in W_j$, $0 \le j \le \ell$, such that

$$v_\ell = w_0 + w_1 + \cdots + w_\ell.$$

In fact, we can write down this decomposition explicitly:

$$v_\ell = \mathcal{I}_0 v_\ell + (\mathcal{I}_1 v_\ell - \mathcal{I}_0 v_\ell) + (\mathcal{I}_2 v_\ell - \mathcal{I}_1 v_\ell) + \cdots + (v_\ell - \mathcal{I}_{\ell-1} v_\ell).$$

Setting

$$w_0 := \mathcal{I}_0 v_\ell,$$

$$w_j := \mathcal{I}_j v_\ell - \mathcal{I}_{j-1} v_\ell, \quad 1 \le j \le \ell,$$

and noticing that

$$\mathcal{I}_\ell v_\ell = v_\ell,$$

gives the result.

Proof (Cont.)

Now, since $w_j \in W_j$, $0 \le j \le \ell$, there are unique coefficients $c_{j,1}, \cdots, c_{j,m_j} \in \mathbb{R}$ such that

$$w_j = \sum_{k=1}^{m_j} c_{j,k}\phi_{j,k}.$$

Hence

$$v_\ell = \sum_{j=0}^{\ell} w_j = \sum_{j=0}^{\ell}\sum_{k=1}^{m_j} c_{j,k}\phi_{j,k}$$

and

$$V_\ell \subseteq \mathrm{span}(\mathcal{H}_\ell).$$

On the other hand, it should be clear that

$$\mathrm{span}(\mathcal{H}_\ell) \subseteq V_\ell.$$

Since

$$\#(\mathcal{H}_\ell) = \#(\mathcal{B}_\ell^V) = n_\ell,$$

it follows that $\mathcal{H}_\ell$ is linearly independent, since $\mathcal{B}_\ell^V$ is a basis. $\square$

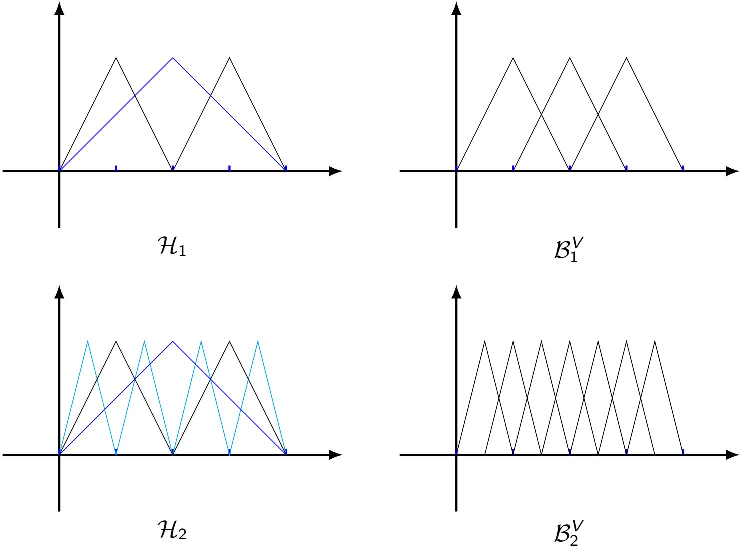

Figure: Hierarchical versus standard Lagrange nodal bases.