Math 673

Multigrid Methods: A Mostly Matrix-Based Approach

Chapter06: Multigrid and the Conforming Finite Element Method

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Chapter 06, Part 1 of 2

Multigrid and the Conforming Finite Element Method

Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ or 3, be an open polyhedral domain. Often, we will also assume that Ω is convex. The weak form of the model problem may be expressed as follows: given $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v)_{L^2(\Omega)}, \quad \forall\, v \in H_0^1(\Omega), \tag{1}$$

where

$$a(u, v) := (\nabla u, \nabla v)_{L^2(\Omega)}. \tag{2}$$

The finite element approximation is based on this weak formulation. In this chapter, we will use the theory of the last chapter to prove that basic multigrid algorithms will converge when applied to the finite element approximation of the model problem. The path forward is simple. We need only check that the basic assumptions hold.

Nested Families of Finite Element Spaces

Let $\mathcal{T}_0$ be a conforming triangulation of Ω. This means that there are no "hanging nodes". We define $\mathcal{T}_1$ to be the triangulation of Ω that results from bisecting ($d = 1$) or quadrisecting ($d = 2$) simplices of the triangulation $\mathcal{T}_0$. See the two following two figures. The refinement of the tetrahedra in $d = 3$ is more complicated, and we skip that case for the sake of brevity.

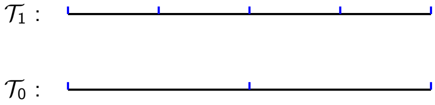


Figure: Bisecting the triangulation $\mathcal{T}_0$ in 1D.

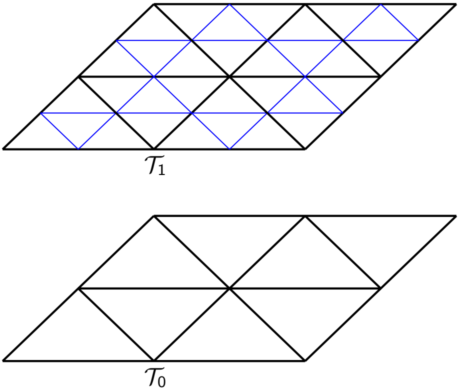


Figure: Quadrisecting the triangulation $\mathcal{T}_0$ in 2D.

For $d = 2$, we connect the edge midpoints. Observe that the daughter triangles are similar to the mother. Continuing, we can recursively define a nested family of the conforming triangulations, indexed by ℓ,

$$\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_\ell, \cdots, \mathcal{T}_L.$$

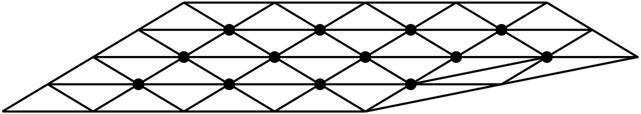
Definition (Finite Element Space)

Let $\{\mathcal{T}_\ell\}_{\ell=0}^{L}$ be a nested family of triangulations of Ω. For $0 \leq \ell \leq L$, define the **grid spacing**, h_ℓ, via

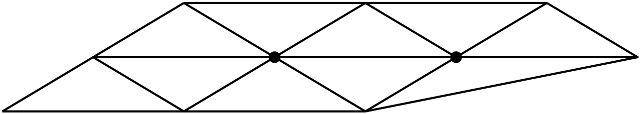
$$h_\ell := \max_{K \in \mathcal{T}_\ell} \mathrm{diam}(K).$$

Subordinate to $\mathcal{T}_\ell$, define the **finite element space**, V_ℓ, via

$$V_\ell := \left\{ v_\ell \in C^0(\overline{\Omega}) \;\middle|\; v_\ell|_K \in \mathbb{P}_1(K),\ \forall\, K \in \mathcal{T}_\ell,\ v_\ell|_{\partial\Omega} \equiv 0 \right\}.$$



$\mathcal{T}_1 : n_1 = 13.$



$\mathcal{T}_0 : n_0 = 2$

Figure: Quadrisecting a complicated triangulation $\mathcal{T}_0$ in 2D. Interior vertices are marked with a filled dot.

Definition (Lagrange Nodal Basis)

Let $0 \leq \ell \leq L$, and suppose V_ℓ and $\mathcal{T}_\ell$ are defined as above. Suppose that $\{\boldsymbol{N}_{\ell,j}\}_{j=1}^{n_\ell}$ is the set of interior vertices of the triangulation $\mathcal{T}_\ell$. By $\mathcal{B}_\ell$ we denote the **Lagrange nodal basis** for V_ℓ, that is,

$$\mathcal{B}_\ell := \{\psi_{\ell,i}\}_{i=1}^{n_\ell}\,, \tag{3}$$

where $\psi_{\ell,i} \in V_\ell$ is the unique function with the property that

$$\psi_{\ell,i}(\boldsymbol{N}_{\ell,j}) = \delta_{i,j}.$$

The functions $\psi_{\ell,i}$ are called **hat functions**.

Remark

We will always assume that $n_0 > 0$. In other words, there is always at least interior vertex.

See the next two figures for plots of hat functions in two dimensions.

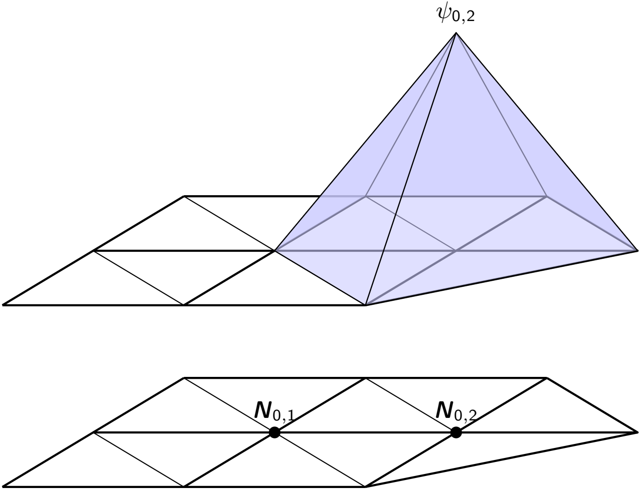


Figure: Interior vertices $\boldsymbol{N}_{0,1}$ and $\boldsymbol{N}_{0,2}$ in the triangulation $\mathcal{T}_0$, and the basis function, $\psi_{0,2}$.

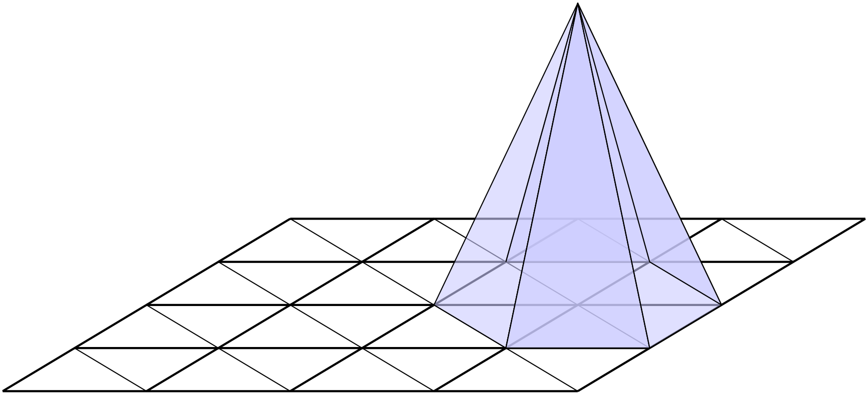


Figure: A uniform mesh in 2D and a Langrange nodal basis (hat) function.

The following is easily established.

Proposition

Let $0 \leq \ell \leq L$, and suppose V_ℓ and $\mathcal{T}_\ell$ are defined as above. Suppose that $\mathcal{B}_\ell := \{\psi_{\ell,i}\}_{i=1}^{n_\ell}$ is as defined in (3). Then, $\mathcal{B}_\ell$ is a bona fide basis of V_ℓ.

Proof.

Exercise. □

Lemma

For $0 \leq \ell \leq L$, with V_ℓ and $\mathcal{T}_\ell$ defined as above, we have

$$V_0 \subset V_1 \subset V_2 \subset \cdots V_\ell \subset V_L \subset H_0^1(\Omega),$$

and each V_ℓ is a finite dimensional vector space with

$$\dim(V_\ell) = n_\ell.$$

Furthermore,

$$0 < n_0 < n_1 < n_2 < \cdots < n_\ell < \cdots < n_L < \infty,$$

where n_ℓ is precisely the number of interior vertices of the triangulation $\mathcal{T}_\ell$. Finally,

$$h_{\ell-1} = 2h_\ell, \quad 1 \leq \ell \leq L.$$

Proof.

Exercise. $\square$

Remark

Suppose $u_\ell \in V_\ell$ is the unique function whose coordinate vector is $\boldsymbol{u}_\ell$ with basis $\mathcal{B}_\ell$. To express this connection we write

$$u_\ell \in V_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}.$$

Lemma

Suppose that $\mathcal{B}_\ell$ is the Lagrange nodal basis for V_ℓ, $0 \leq \ell \leq L$. Then, for every ℓ, with $1 \leq \ell \leq L$, there exist unique numbers

$$p_{\ell-1,j,i} \in \mathbb{R}, \quad 1 \leq j \leq n_\ell, \quad 1 \leq i \leq n_{\ell-1},$$

with the property that

$$\psi_{\ell-1,i} = \sum_{j=1}^{n_\ell} p_{\ell-1,j,i} \psi_{\ell,j}, \tag{4}$$

for each $1 \leq i \leq n_{\ell-1}$.

Proof.

Since $V_{\ell-1}$ is a linear subspace of V_ℓ and $\mathcal{B}_\ell$ is a basis for the latter, for every $\psi_{\ell-1,i} \in \mathcal{B}_{\ell-1} \subset V_{\ell-1} \subset V_\ell$, there exists unique coefficients

$$p_{\ell-1,j,i} \in \mathbb{R}, \quad 1 \le j \le n_\ell,$$

such that

$$\psi_{\ell-1,i} = \sum_{j=1}^{n_\ell} p_{\ell-1,j,i} \psi_{\ell,j}.$$

Recall that basis representation are unique. □

Definition (Prolongation Matrix)

For $1 \leq \ell \leq L$, define the **prolongation matrix**,

$$\mathsf{P}_{\ell-1} \in \mathbb{R}^{n_\ell \times n_{\ell-1}},$$

via

$$[\mathsf{P}_{\ell-1}]_{i,j} := p_{\ell-1,i,j}. \tag{5}$$

Lemma

Let $1 \leq \ell \leq L$, and suppose $v_{\ell-1} \in V_{\ell-1}$ is arbitrary. Suppose $\boldsymbol{v}_{\ell-1}$ is the coordinate vector of $v_{\ell-1}$ in the Lagrange nodal basis $\mathcal{B}_{\ell-1}$ and $\boldsymbol{v}_\ell$ is the coordinate vector of $v_{\ell-1}$ in the basis $\mathcal{B}_\ell$. Then

$$\boldsymbol{v}_\ell = \mathsf{P}_{\ell-1}\boldsymbol{v}_{\ell-1}.$$

Proof.

We write, as usual,

$$[\boldsymbol{v}_{\ell-1}]_i = v_{\ell-1,i}, \quad 1 \leq i \leq n_{\ell-1},$$

and

$$[\boldsymbol{v}_\ell]_i = v_{\ell,i}, \quad 1 \leq i \leq n_\ell.$$

Thus

$$\begin{aligned} v_{\ell-1} &= \sum_{i=1}^{n_{\ell-1}} v_{\ell-1,i}\psi_{\ell-1,i} \\ &\overset{(4)}{=} \sum_{i=1}^{n_{\ell-1}} v_{\ell-1,i} \sum_{j=1}^{n_\ell} p_{\ell-1,j,i}\psi_{\ell,j} \\ &= \sum_{j=1}^{n_\ell} \left\{ \sum_{i=1}^{n_{\ell-1}} p_{\ell-1,j,i} v_{\ell-1,i} \right\} \psi_{\ell,j} \\ &= \sum_{j=1}^{n_\ell} [\mathsf{P}_{\ell-1}\boldsymbol{v}_{\ell-1}]_j \, \psi_{\ell,j}. \end{aligned}$$

Proof (Cont.)

But

$$v_{\ell-1} = \sum_{j=1}^{n_\ell} [\boldsymbol{v}_\ell]_j \, \psi_{\ell,j}.$$

Since basis representations are unique,

$$\boldsymbol{v}_\ell = \mathsf{P}_{\ell-1} \boldsymbol{v}_{\ell-1}.$$

$\square$

Definition

For $1 \leq \ell \leq L$, define the **restriction matrix** via

$$\mathsf{R}_{\ell-1} := \mathsf{P}_{\ell-1}^{T} \in \mathbb{R}^{n_{\ell-1} \times n_{\ell}}.$$

Lemma

For $1 \leq \ell \leq L$ *and suppose* $\mathsf{R}_{\ell-1}$ *and* $\mathsf{P}_{\ell-1}$ *are defined as above. Then*

$$\operatorname{rank}(\mathsf{P}_{\ell-1}) = \operatorname{rank}(\mathsf{R}_{\ell-1}) = n_{\ell-1}.$$

Proof.

Suppose

$$\mathsf{P}_{\ell-1}\boldsymbol{v}_{\ell-1} = \mathbf{0} \in \mathbb{R}^{n_\ell}.$$

This represents a linear combination of the $n_{\ell-1}$ columns of $\mathsf{P}_{\ell-1}$. Using the notation from the last lemma and its proof, we have

$$\boldsymbol{v}_\ell = \mathbf{0} \in \mathbb{R}^{n_\ell},$$

where $\boldsymbol{v}_{\ell-1}$ and $\boldsymbol{v}_\ell$ are coordinate vectors of some function $v_{\ell-1} \in V_{\ell-1}$ in the bases $\mathcal{B}_{\ell-1}$ and $\mathcal{B}_\ell$, respectively.

The only way that $\boldsymbol{v}_\ell = \mathbf{0}$ is if $v_{\ell-1} \equiv 0$ in $V_{\ell-1}$. But then $\boldsymbol{v}_{\ell-1} = \mathbf{0}$. Thus, the columns of $\mathsf{P}_{\ell-1}$ are linearly independent and

$$\operatorname{rank}(\mathsf{P}_{\ell-1}) = n_{\ell-1}.$$

□

The Stiffness Matrices

Definition (Stiffness Matrix)

Suppose $0 \leq \ell \leq L$. Assume that V_ℓ and $\mathcal{T}_\ell$ are defined as above. The **level-ℓ stiffness matrix** $\mathsf{A}_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$ is defined via

$$[\mathsf{A}_\ell]_{i,j} := a\left(\psi_{\ell,j}, \psi_{\ell,i}\right),$$

for all $1 \leq i, j \leq n_\ell$, where $a(\,\cdot\,,\,\cdot\,)$ is the energy inner product defined in (2).

We need the following well-known result to show that the stiffness matrix is positive definite.

Theorem (Poincaré Inequality)

Suppose $d = 1, 2$ *or* 3 *and* Ω *is an open polyhedral domain. There is a constant* $C_{\mathrm{P}} > 0$*, depending only on the domain* Ω*, such that*

$$C_{\mathrm{P}} \left\| v \right\|_{L^2(\Omega)}^2 \leq a\left(v, v\right), \tag{6}$$

for all $v \in H_0^1(\Omega)$.

Lemma

The stiffness matrices $\mathsf{A}_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$ *are all SPD. Moreover, they satisfy the Galerkin condition (Assumption (G0)), that is, for* $1 \leq \ell \leq L$,

$$\mathsf{A}_{\ell-1} = \mathsf{R}_{\ell-1}\mathsf{A}_\ell\mathsf{P}_{\ell-1}. \tag{7}$$

Proof.

(Symmetry):

$$\begin{aligned}[\mathsf{A}_\ell]_{i,j} &= a\left(\psi_{\ell,j}, \psi_{\ell,i}\right) \\ &= a\left(\psi_{\ell,i}, \psi_{\ell,j}\right) \\ &= [\mathsf{A}_\ell]_{j,i}\,.\end{aligned}$$

(Positivity): Let $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$ be arbitrary and suppose $v_\ell \in V_\ell$ is the unique function with coordinations $\boldsymbol{v}_\ell$. Then, using the Poincaré inequality,

$$\begin{aligned}0 \leq C_{\mathrm{P}} \left\|v_\ell\right\|^2_{L^2(\Omega)} &\overset{(6)}{\leq} a\left(v_\ell, v_\ell\right) \\ &= a\left(\sum_{j=1}^{n_\ell} v_{\ell,j}\psi_{\ell,j}, \sum_{j=1}^{n_\ell} v_{\ell,i}\psi_{\ell,i}\right) \\ &= \sum_{i=1}^{n_\ell}\sum_{j=1}^{n_\ell} v_{\ell,i}\, a\left(\psi_{\ell,j}, \psi_{\ell,i}\right) v_{\ell,j} \\ &= \boldsymbol{v}_\ell^T \mathsf{A}_\ell \boldsymbol{v}_\ell.\end{aligned}$$

But $v_\ell \equiv 0$ iff $\boldsymbol{v}_\ell = \mathbf{0}$. Hence A_ℓ is SPD.

Proof (Cont.)

(Galerkin condition): By defintion

$$
\begin{aligned}
[\mathsf{A}_{\ell-1}]_{i,j} &= a\left(\psi_{\ell-1,j}, \psi_{\ell-1,i}\right) \\
&\overset{(4)}{=} a\left(\sum_{s=1}^{n_\ell} p_{\ell-1,s,j}\psi_{\ell,s}, \sum_{t=1}^{n_\ell} p_{\ell-1,t,i}\psi_{\ell,t}\right) \\
&= \sum_{s=1}^{n_\ell}\sum_{t=1}^{n_\ell} p_{\ell-1,s,j}\, a\left(\psi_{\ell,s}, \psi_{\ell,t}\right) p_{\ell-1,t,i} \\
&= \sum_{t=1}^{n_\ell}\sum_{s=1}^{n_\ell} p_{\ell-1,t,i}\, a\left(\psi_{\ell,s}, \psi_{\ell,t}\right) p_{\ell-1,s,j} \\
&\overset{(5)}{=} \sum_{t=1}^{n_\ell}\sum_{s=1}^{n_\ell} [\mathsf{R}_{\ell-1}]_{i,t} [\mathsf{A}_\ell]_{t,s} [\mathsf{P}_{\ell-1}]_{s,j} .
\end{aligned}
$$

Thus,

$$
\mathsf{A}_{\ell-1} = \mathsf{R}_{\ell-1}\mathsf{A}_\ell\mathsf{P}_{\ell-1}.
$$

$\square$

Remark

Observe that this last result also confirms the fact that $\mathsf{P}_{\ell-1}$ *and* $\mathsf{R}_{\ell-1}$ *are of full rank. Otherwise,* $\mathsf{A}_{\ell-1}$ *could not be positive definite.*

Inner Products and Norms

Now, for our level-ℓ Euclidean inner products and norms, we use the now-familiar notation:

$$(\boldsymbol{u}_\ell, \boldsymbol{v}_\ell)_\ell := \boldsymbol{v}_\ell^T \boldsymbol{u}_\ell, \quad \forall\, \boldsymbol{u}_\ell, \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell},$$

and

$$\|\boldsymbol{u}_\ell\|_\ell := \sqrt{(\boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_\ell} \quad \forall\, \boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}.$$

The stiffness-matrix induced inner products and norms are, recall,

$$(\boldsymbol{u}_\ell, \boldsymbol{v}_\ell)_{\mathsf{A}_\ell} := (\mathsf{A}_\ell \boldsymbol{u}_\ell, \boldsymbol{v}_\ell)_\ell, \quad \forall\, \boldsymbol{u}_\ell, \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell},$$

and

$$\|\boldsymbol{u}_\ell\|_{\mathsf{A}_\ell} := \sqrt{(\boldsymbol{u}_\ell, \boldsymbol{u}_\ell)_{\mathsf{A}_\ell}} \quad \forall\, \boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}.$$

Next we estimate the size of the condition number of A_ℓ. To do this we need a couple of results whose proofs can be found in most good finite element books.

Lemma (Norm Equivalence)

Suppose $d = 1, 2$ or 3 and $0 \leq \ell \leq L$. Let $v_\ell \in V_\ell$ be arbitrary and assume that $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$ is its unique coordinate vector in the basis $\mathcal{B}_\ell$. Then, there are constants $C_2 \geq C_1 > 0$, both independent of ℓ and v_ℓ, such that

$$C_1 h_\ell^d \|\boldsymbol{v}_\ell\|_\ell^2 \leq \|v_\ell\|_{L^2(\Omega)}^2 \leq C_2 h_\ell^d \|\boldsymbol{v}_\ell\|_\ell^2 . \tag{8}$$

Lemma (Inverse Inequality)

Suppose $d = 1, 2$ or 3 and $0 \le \ell \le L$. There is a constant $C_3 > 0$ independent of $\ell \ge 0$ such that

$$a(v_\ell, v_\ell) \le C_3 h_\ell^{-2} \|v_\ell\|_{L^2(\Omega)}^2, \tag{9}$$

for all $v_\ell \in V_\ell$. As a consequence of (8),

$$a(v_\ell, v_\ell) \le C_2 C_3 h_\ell^{d-2} \|\boldsymbol{v}_\ell\|_\ell^2, \tag{10}$$

for all $v_\ell \in V_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$.

Remark

These results require some conditions on the underlying family of conforming meshes. such as global quasi-uniformity and shape regularity, which hold thanks to our construction of the family $\mathcal{T}_\ell$, $\ell \geq 0$.

Remark

Estimate (8) *is an example of finite-dimensional norm equivalence, and estimate* (9) *is called an* **inverse inequality**.

Proofs of the following facts can be found in the book by Braess.

Lemma (Asymptotic Sharpness)

Let V_ℓ *and* $\mathcal{T}_\ell$ *be defined as usual. There exists a family of non-trivial functions* $\tilde{v}_\ell \in V_\ell$ *and a constant* $C_4 > 0$*, independent of* ℓ *and* $\tilde{v}_\ell$*, such that, for every* $0 \leq \ell \leq L$

$$C_4 h_\ell^{-2} \left\| \tilde{v}_\ell \right\|_{L^2(\Omega)}^2 \leq a\left(\tilde{v}_\ell, \tilde{v}_\ell\right). \tag{11}$$

There exists a family of non-trivial functions $\hat{v}_\ell \in V_\ell$ *and a constant* $C_5 > 0$*, independent of* ℓ *and* $\hat{v}_\ell$*, such that, for every* $0 \leq \ell \leq L$

$$a\left(\hat{v}_\ell, \hat{v}_\ell\right) \leq C_5 \left\| \hat{v}_\ell \right\|_{L^2(\Omega)}^2. \tag{12}$$

Theorem (Condition Number Estimate)

Let $d = 1, 2$, or 3 and V_ℓ and $\mathcal{T}_\ell$ be defined as usual. There exist constants $C_7 \geq C_6 > 0$, independent of $\ell \geq 0$, such that

$$C_6 h_\ell^{-2} \leq \kappa_2(\mathsf{A}_\ell) = \frac{\lambda_\ell^{(n_\ell)}}{\lambda_\ell^{(1)}} \leq C_7 h_\ell^{-2}. \tag{13}$$

Consequently,

$$\kappa_2(\mathsf{A}_\ell) = \Theta(h_\ell^{-2}), \tag{14}$$

as $\ell \to \infty$. In particular, there are constant $C_7^{(i)}, C_6^{(i)} > 0$ for $i = 1, n_\ell$, such that

$$C_6^{(n_\ell)} h_\ell^{d-2} \leq \lambda_\ell^{(n_\ell)} \leq C_7^{(n_\ell)} h_\ell^{d-2}$$

and

$$C_7^{(1)} h_\ell^{d} \leq \lambda_\ell^{(1)} \leq C_6^{(1)} h_\ell^{d}.$$

Proof.

First we recall some basis facts for the Rayleigh quotient for A_ℓ:

$$R(\boldsymbol{v}_\ell) := \frac{\boldsymbol{v}_\ell^T \mathsf{A}_\ell \boldsymbol{v}_\ell}{\boldsymbol{v}_\ell^T \boldsymbol{v}_\ell}.$$

The smallest and largest eigenvalues satisfy

$$\lambda_\ell^{(1)} = R\left(\boldsymbol{v}_\ell^{(1)}\right) = \min_{\boldsymbol{v}_\ell} R(\boldsymbol{v}_\ell) > 0$$

and

$$\lambda_\ell^{(n_\ell)} = R\left(\boldsymbol{v}_\ell^{(n_\ell)}\right) = \max_{\boldsymbol{v}_\ell} R(\boldsymbol{v}_\ell),$$

where

$$\mathsf{A}_\ell \boldsymbol{v}_\ell^{(k)} = \lambda_\ell^{(k)} \boldsymbol{v}_\ell^{(k)}, \quad 1 \leq k \leq n_\ell.$$

Proof (Cont.)

(Upper bound in (13)): As usual we use the correspondence

$$v_\ell \in V_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}.$$

Then, for arbitrary $v_\ell \in V_\ell$,

$$\begin{aligned} R(\boldsymbol{v}_\ell) &:= \frac{\boldsymbol{v}_\ell^T \mathsf{A}_\ell \boldsymbol{v}_\ell}{\boldsymbol{v}_\ell^T \boldsymbol{v}_\ell} \\ &= \frac{a\left(v_\ell, v_\ell\right)}{\|\boldsymbol{v}_\ell\|_\ell^2} \\ &\overset{(10)}{\leq} \frac{C_2 C_3 h_\ell^{d-2} \|\boldsymbol{v}_\ell\|_\ell^2}{\|\boldsymbol{v}_\ell\|_\ell^2} \\ &=: C_7^{(n_\ell)} h_\ell^{d-2}. \end{aligned}$$

This implies

$$\lambda_\ell^{(n_\ell)} \leq C_7^{(n_\ell)} h_\ell^{d-2}.$$

Proof (Cont.)

Similarly,

$$
\begin{aligned}
R(\boldsymbol{v}_\ell) &:= \frac{a\left(v_\ell, v_\ell\right)}{\left\|\boldsymbol{v}_\ell\right\|_\ell^2} \\
&\overset{(6)}{\geq} \frac{C_{\mathrm{P}}\left\|v_\ell\right\|_{L^2(\Omega)}^2}{\left\|\boldsymbol{v}_\ell\right\|_\ell^2} \\
&\overset{(8)}{\geq} \frac{C_{\mathrm{P}} C_1 h_\ell^d \left\|\boldsymbol{v}_\ell\right\|_\ell^2}{\left\|\boldsymbol{v}_\ell\right\|_\ell^2} \\
&=: C_7^{(1)} h_\ell^d .
\end{aligned}
$$

Therefore,

$$
\lambda_\ell^{(1)} \geq C_7^{(1)} h_\ell^d .
$$

We conclude that

$$
\kappa_2(\mathsf{A}_\ell) = \frac{\lambda_\ell^{(n_\ell)}}{\lambda_\ell^{(1)}} \leq \frac{C_7^{(n_\ell)} h_\ell^{d-2}}{C_7^{(1)} h_\ell^d} =: C_7 h_\ell^{-2}
$$

Proof (Cont.)

(Lower bound in (13)): Next, it follows that

$$
\begin{aligned}
\lambda_\ell^{(n_\ell)} &= R\left(\boldsymbol{v}_\ell^{(n_\ell)}\right) \\
&\geq R(\tilde{\boldsymbol{v}}_\ell) \\
&= \frac{a\left(\tilde{v}_\ell, \tilde{v}_\ell\right)}{\left\|\tilde{\boldsymbol{v}}_\ell\right\|_\ell^2} \\
&\overset{(11)}{\geq} \frac{C_4 h_\ell^{-2} \left\|\tilde{v}_\ell\right\|_{L^2(\Omega)}^2}{\left\|\tilde{\boldsymbol{v}}_\ell\right\|_\ell^2} \\
&\overset{(8)}{\geq} \frac{C_1 C_4 h_\ell^{d-2} \left\|\tilde{\boldsymbol{v}}_\ell\right\|_\ell^2}{\left\|\tilde{\boldsymbol{v}}_\ell\right\|_\ell^2} \\
&=: C_6^{(n_\ell)} h_\ell^{d-2},
\end{aligned}
$$

where we use the correspondence

$$
\tilde{v}_\ell \in V_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \tilde{\boldsymbol{v}}_\ell \in \mathbb{R}^{n_\ell}.
$$

Proof (Cont.)

Finally,

$$\begin{aligned}
\lambda_\ell^{(1)} &= R\left(\boldsymbol{v}_\ell^{(1)}\right) \\
&\leq R(\hat{\boldsymbol{v}}_\ell) \\
&= \frac{a\left(\hat{v}_\ell, \hat{v}_\ell\right)}{\|\hat{\boldsymbol{v}}_\ell\|_\ell^2} \\
&\overset{(12)}{\leq} \frac{C_5 \left\|\hat{v}_\ell\right\|_{L^2(\Omega)}^2}{\|\hat{\boldsymbol{v}}_\ell\|_\ell^2} \\
&\overset{(8)}{\leq} \frac{C_2 C_5 h_\ell^d \left\|\hat{\boldsymbol{v}}_\ell\right\|_\ell^2}{\|\hat{\boldsymbol{v}}_\ell\|_\ell^2} \\
&=: C_6^{(1)} h_\ell^d,
\end{aligned}$$

where we used the correspondence

$$\hat{v}_\ell \in V_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \hat{\boldsymbol{v}}_\ell \in \mathbb{R}^{n_\ell}.$$

Proof (Cont.)

We conclude that

$$\kappa_2(\mathsf{A}_\ell) \geq \frac{C_6^{(n_\ell)} h_\ell^{d-2}}{C_6^{(1)} h_\ell^d} =: C_6 h_\ell^{-2}.$$

$\square$