Math 673/4

Multigrid Methods: A Mostly Matrix-Based Approach

Chapter 07: Cell-Centered Finite Difference Methods and Multigrid

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Chapter 07, Part 2 of 3

Cell-Centered Finite Difference Methods and Multigrid

Prolongation, Restriction, and Galerkin Failure

Prolongation and Restriction

As in the finite element setting, prolongation and restriction matrices communicate information between a coarse-level, say, the level-$(\ell - 1)$ grid, and a fine-level, the level-ℓ grid.

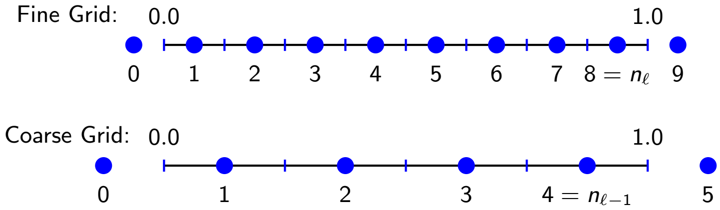

Figure: A uniform and nested two-level cell-centered grid. The prolongation operator moves information from the coarse level (level-$(\ell - 1)$) grid to the fine level (level-ℓ) grid.

Example (Non-nested Grid Points)

Let $n_\ell = 8$ and $n_{\ell-1} = 4$. A uniform and nested two-level grid is shown in the figure on the previous slide. The prolongation operator moves information from the coarse level (level-$(\ell - 1)$) grid to the fine level (level-ℓ) grid. The restriction operator moves information in the opposite direction. Since the values at the ghost cells will be set using boundary conditions, only the interior points are involved in the prolongation and restriction processes. For example, to compute the prolongation of the 4 interior degrees of freedom in the coarse grid in this example, we need a matrix having 8 rows and 4 columns.

Definition (Prolongation and Restriction)

Suppose the multi-level cell-centered grid is uniform and nested, and $1 \le \ell \le L$. Recall that, by construction $n_\ell = 2n_{\ell-1}$, for $1 \le \ell \le L$. The **cell-centered prolongation matrix**, $\mathsf{P}_{\ell-1} \in \mathbb{R}^{n_\ell \times n_{\ell-1}}$, has the form

$$\mathsf{P}_{\ell-1} = \begin{bmatrix} 1 & 0 & & 0 & 0 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & & 0 & 1 \\ 0 & 0 & & 0 & 1 \end{bmatrix}.$$

Definition (Prolongation and Restriction (Cont.))

Equivalently, the action of $\mathsf{P}_{\ell-1}$ on an arbitrary vector $\boldsymbol{v}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}$ is

$$\begin{aligned} [\mathsf{P}_{\ell-1}\boldsymbol{v}_{\ell-1}]_{2i-1} &= v_{\ell-1,i}, \quad 1 \le i \le n_{\ell-1}, \\ [\mathsf{P}_{\ell-1}\boldsymbol{v}_{\ell-1}]_{2i} &= v_{\ell-1,i}, \quad 1 \le i \le n_{\ell-1}. \end{aligned}$$

The **cell-centered restriction matrix**, $\mathsf{R}_{\ell-1} \in \mathbb{R}^{n_{\ell-1} \times n_\ell}$, is defined via

$$\mathsf{R}_{\ell-1} = \frac{1}{2}\mathsf{P}_{\ell-1}^\top. \tag{1}$$

Proposition

Suppose the multi-level cell-centered grid is uniform and nested, and $1 \leq \ell \leq L$. *Then*

$$\mathsf{R}_{\ell-1}\mathsf{P}_{\ell-1} = \mathsf{I}_{n_{\ell-1}}.$$

Proof.

Exercise. □

Proposition (Imbalanced Galerkin Condition)

Suppose the multi-level cell-centered grid is uniform and nested. The strong Galerkin condition (G0) fails to hold in this setting. However, for $1 \le \ell \le L$,

$$\mathsf{R}_{\ell-1}\mathsf{A}_{\ell}\mathsf{P}_{\ell-1} = 2\mathsf{A}_{\ell-1}, \tag{2}$$

so that an imbalanced Galerkin condition, that is, Assumption (G3), holds with $r = \frac{1}{2}$.

Proof.

Exercise. □

The fact that the standard Galerkin conditions fail has important implications for the analysis of the cell centered multigrid method. For instance, the coarse grid Ritz projection matrix is no longer a projection, as the following result affirms.

Proposition (Not a Projection)

Let $1 \leq \ell \leq L$. Recall that the matrix $\tilde{\Pi}_\ell$ is generally defined via

$$\tilde{\Pi}_\ell = \mathsf{P}_{\ell-1}\mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell.$$

In the present setting, $\tilde{\Pi}_\ell$ is not a projection matrix. In particular,

$$\tilde{\Pi}_\ell^2 = 2\tilde{\Pi}_\ell.$$

Furthermore,

$$(\mathsf{I}_\ell - \tilde{\Pi}_\ell)^2 = \mathsf{I}_\ell.$$

On the other hand, we always have

$$\tilde{\Pi}_\ell^* = \tilde{\Pi}_\ell \quad \textit{and} \quad (\mathsf{I}_\ell - \tilde{\Pi}_\ell)^* = \mathsf{I}_\ell - \tilde{\Pi}_\ell.$$

Proof.

Using the definitions and the imbalanced Galerkin condition, we have

$$\begin{aligned}
\tilde{\Pi}_\ell^2 &= \mathsf{P}_{\ell-1}\mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell\mathsf{P}_{\ell-1}\mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell \\
&\stackrel{(2)}{=} \mathsf{P}_{\ell-1}\mathsf{A}_{\ell-1}^{-1}(2\mathsf{A}_{\ell-1})\mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell \\
&= 2\mathsf{P}_{\ell-1}\mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell \\
&= 2\tilde{\Pi}_\ell.
\end{aligned}$$

The other calculations are left for an exercise. $\square$

Corollary (Coarse-Grid Ritz Error)

Let $1 \leq \ell \leq L$. Since, in the present setting, the coarse-level stiffness matrix satisfies

$$\mathsf{A}_{\ell-1} = \frac{1}{2}\mathsf{R}_{\ell-1}\mathsf{A}_\ell\mathsf{P}_{\ell-1},$$

it follows that

$$\left\|(\mathsf{I}_\ell - \tilde{\Pi}_\ell)\boldsymbol{v}_\ell\right\|_{\mathsf{A}_\ell} = \left\|\boldsymbol{v}_\ell\right\|_{\mathsf{A}_\ell}, \quad \forall\, \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}.$$

Consequently,

$$\left\|\mathsf{I}_\ell - \tilde{\Pi}_\ell\right\|_{\mathsf{A}_\ell} = 1.$$

Proof.

Exercise. □

Some Additional Multilevel Tools

Additional Machinery

We have all of the pieces in place to define the multigrid method. In this section, we add some additional machinery that will aid in the analysis of the cell-centered multigrid method in one space dimension. In particular, we need some tools for establishing a suitable approximation property, which is the subject of the next section.

Recall, the definition of the level-ℓ stiffness matrix:

$$\mathsf{A}_\ell := \frac{1}{h_\ell^2}\begin{bmatrix} 3 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 3 \end{bmatrix} \in \mathbb{R}^{n_\ell \times n_\ell}. \tag{3}$$

Definition (Level-ℓ Discrete Operators)

Let $v_\ell \in V_\ell$ be arbitrary and suppose $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$ is its coordinate vector with respect to the basis $\mathcal{B}_\ell$, that is,

$$v_\ell \in V_\ell \stackrel{\mathcal{B}_\ell}{\longleftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}.$$

Suppose that A_ℓ is cell-centered stiffness matrix defined in (3). The **level-ℓ discrete minus Laplacian operator** on V_ℓ, is the operator $-\Delta_\ell^\circ : V_\ell \to V_\ell$ defined by

$$-\Delta_\ell^\circ v_\ell := \sum_{i=1}^{n_\ell} \left[\mathsf{A}_\ell \boldsymbol{v}_\ell\right]_i \psi_{\ell,i}.$$

The **energy form** $\mathcal{A}_\ell(\cdot,\cdot) : V_\ell \times V_\ell \to \mathbb{R}$ is defined via

$$\mathcal{A}_\ell(u_\ell, v_\ell) := \left(-\Delta_\ell^\circ u_\ell, v_\ell\right)_{L^2(0,1)} = -\left(\Delta_\ell^\circ u_\ell, v_\ell\right)_{L^2(0,1)}, \tag{4}$$

for all $u_\ell, v_\ell \in V_\ell$.

Proposition (Symmetric and Bilinear)

For each $0 \leq \ell \leq L$*, the form* $\mathcal{A}_\ell\left(\cdot,\cdot\right)$ *is symmetric, that is,*

$$\mathcal{A}_\ell\left(u_\ell, v_\ell\right) = \mathcal{A}_\ell\left(v_\ell, u_\ell\right), \qquad \forall\, u_\ell, v_\ell \in V_\ell.$$

It is also bilinear, that is, linear in each argument. Furthermore, for all $1 \leq i, j \leq n_\ell$*,*

$$\mathcal{A}_\ell\left(\psi_{\ell,i}, \psi_{\ell,j}\right) = h_\ell\left[\mathsf{A}_\ell\right]_{i,j} = h_\ell\left[\mathsf{A}_\ell\right]_{j,i}. \tag{5}$$

Proof.

Exercise. □

Definition (L^2-Projection onto V_ℓ)

For each $0 \leq \ell \leq L$, we denote the L^2-**projection onto** V_ℓ, by $Q_\ell : L^2(0,1) \to V_\ell$. This operator is defined as follows: for each $v \in L^2(0,1)$, $Q_\ell v \in V_\ell$ is the unique function satisfying

$$(Q_\ell v, w_\ell)_{L^2(0,1)} = (v, w_\ell)_{L^2(0,1)}, \qquad \forall\, w_\ell \in V_\ell.$$

Proposition (Coordinate Vectors)

Let $0 \leq \ell \leq L$ and $v \in L^2(0,1)$. If $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$ is the coordinate vector of $v_\ell = Q_\ell v \in V_\ell$, with respect to the basis $\mathcal{B}_\ell$, then

$$v_{\ell,i} = [\boldsymbol{v}_\ell]_i = \frac{1}{h_\ell} \int_{K_{\ell,i}} v(x)\,\mathrm{d}x, \qquad i = 1, \ldots, n_\ell. \tag{6}$$

Proof.

By definition,

$$v_\ell = Q_\ell v = \sum_{i=1}^{n_\ell} v_{\ell,i} \psi_{\ell,i},$$

and

$$\left(\sum_{i=1}^{n_\ell} v_{\ell,i} \psi_{\ell,i}, \psi_{\ell,j} \right)_{L^2(0,1)} = (v, \psi_{\ell,j})_{L^2(0,1)},$$

for each $j = 1, \ldots, n_\ell$.

Proof (Cont.)

Using linearity and the fact that

$$(\psi_{\ell,i}, \psi_{\ell,j})_{L^2(0,1)} = h_\ell \delta_{i,j},$$

we have

$$\begin{aligned}
(v, \psi_{\ell,j})_{L^2(0,1)} &= \sum_{i=1}^{n_\ell} v_{\ell,i} (\psi_{\ell,i}, \psi_{\ell,j})_{L^2(0,1)} \\
&= \sum_{i=1}^{n_\ell} v_{\ell,i} h_\ell \delta_{i,j} \\
&= h_\ell v_{\ell,j}.
\end{aligned}$$

Hence, we have

$$v_{\ell,j} = \frac{1}{h_\ell} (v, \psi_{\ell,j})_{L^2(0,1)} = \frac{1}{h_\ell} \int_{K_{\ell,j}} v(x)\,\mathrm{d}x. \qquad \square$$

Proposition (Representation)

Assume that $1 \leq \ell \leq L$*, and suppose that* $v_\ell \in V_\ell$ *is arbitrary. Set* $v_{\ell-1} = Q_{\ell-1} v_\ell \in V_{\ell-1}$ *and assume that* $\boldsymbol{v}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}$ *is the coordinate vector of* $v_{\ell-1}$ *with respect to the basis* $\mathcal{B}_{\ell-1}$*, that is,*

$$v_{\ell-1} \in V_{\ell-1} \overset{\mathcal{B}_{\ell-1}}{\longleftrightarrow} \boldsymbol{v}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}.$$

Then, for each $i \leq i \leq n_{\ell-1}$*, we have*

$$v_{\ell-1,i} = \frac{1}{2}\left(v_{\ell,2i-1} + v_{\ell,2i}\right),$$

where

$$v_\ell \in V_\ell \overset{\mathcal{B}_\ell}{\longleftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}.$$

In other words,

$$\boldsymbol{v}_{\ell-1} = \mathrm{R}_{\ell-1} \boldsymbol{v}_\ell.$$

Proof.

Suppose

$$v_\ell = \sum_{i=1}^{n_\ell} v_{\ell,i}\psi_{\ell,i} \in V_\ell \overset{\mathcal{B}_\ell}{\longleftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}.$$

Then, since

$$K_{\ell-1,i} = K_{\ell,2i-1} \cup K_{\ell,2i},$$

it follows that, for $1 \le i \le n_{\ell-1}$,

$$\begin{aligned} v_{\ell-1,i} &\overset{(6)}{=} \frac{1}{h_{\ell-1}} \int_{K_{\ell-1,i}} v_\ell(x)\,\mathrm{d}x \\ &= \frac{1}{h_{\ell-1}} \int_{K_{\ell,2i-1}} v_\ell(x)\,\mathrm{d}x + \frac{1}{h_{\ell-1}} \int_{K_{\ell,2i}} v_\ell(x)\,\mathrm{d}x \\ &= \frac{h_\ell}{h_{\ell-1}} v_{\ell,2i-1} + \frac{h_\ell}{h_{\ell-1}} v_{\ell,2i} \\ &= \frac{1}{2}(v_{\ell,2i-1} + v_{\ell,2i}). \end{aligned}$$

□

Proposition (Prolongation Action)

Let $1 \leq \ell \leq L$. Assume that $v_{\ell-1} \in V_{\ell-1}$ is arbitrary, and

$$v_{\ell-1} \in V_{\ell-1} \overset{\mathcal{B}_{\ell-1}}{\longleftrightarrow} \boldsymbol{v}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}.$$

Then, $v_{\ell-1} \in V_\ell$, and if $\boldsymbol{v}_\ell$ is its coordinate vector with respect to the basis $\mathcal{B}_\ell$, that is,

$$v_{\ell-1} \in V_\ell \overset{\mathcal{B}_\ell}{\longleftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell},$$

then

$$\boldsymbol{v}_\ell = \mathsf{P}_{\ell-1}\boldsymbol{v}_{\ell-1}.$$

Proof.

Exercise. □

Lemma (H^1-Stability)

For every $0 \leq \ell \leq L$ the operator Q_ℓ is H^1-stable in the sense that there is a constant $C_2 > 0$, independent of ℓ, such that, whenever $v \in H_0^1(0,1)$

$$\mathcal{A}_\ell\left(Q_\ell v, Q_\ell v\right) \leq C_2^2 \int_0^1 \left|v'(x)\right|^2 \mathrm{d}x.$$

Proof.

Exercise. □

Proposition (Projection Error Estimate)

There exist a constant $C > 0$, independent of ℓ, such that, for every $0 \leq \ell \leq L$ and all $v \in H^1(0,1)$, we have

$$\|v - Q_\ell v\|_{L^2(0,1)} \leq C h_\ell \|v'\|_{L^2(0,1)}. \tag{7}$$

Proof.

Exercise. □

The α-Weak Approximation Property

The Plan

Recall that we have assumed that smoothing will be carried out via the Λ-Richardson smoother, and the smoothing property has already been established. All of the elements of multigrid are now completely defined. We now have only to prove the convergence of the various algorithms by establishing an appropriate approximation property.

In this section, we demonstrate that the cell-centered multigrid method satisfies the α-weak approximation property with, precisely, $\alpha = \frac{1}{2}$. Our present analysis will be for the one-dimensional case with constant diffusivity, but it can straightforwardly be extended to higher dimensions and non-constant diffusivity. More on that later.

An Almost Equivalent Reformulation

Let $f \in L^2(0,1)$ be given. The cell-centered finite difference problem can be formulated as follows: find $u_\ell \in V_\ell$ such that

$$\mathcal{A}_\ell(u_\ell, v_\ell) = (f, v_\ell)_{L^2(0,1)}, \quad \forall\, v_\ell \in V_\ell. \tag{8}$$

Set

$$f_\ell := Q_\ell f \in V_\ell \xleftrightarrow{\mathcal{B}_\ell} \boldsymbol{f}_\ell \in \mathbb{R}^{n_\ell}$$

and

$$u_\ell \in V_\ell \xleftrightarrow{\mathcal{B}_\ell} \boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}.$$

It is easy to see that

$$\mathsf{A}_\ell \boldsymbol{u}_\ell = \boldsymbol{f}_\ell. \tag{9}$$

Of course, we will be interested primarily in the approximation on some fine level $\ell = L$.

Remark

This matrix problem $\mathrm{A}_\ell \boldsymbol{u}_\ell = \boldsymbol{f}_\ell$ *in* (9) *is analogous, but not precisely equivalent, to that expressed earlier in the cell-centered finite difference context. They appear to be the same, and, to be very clear, the stiffness matrices are precisely the same, down to the very last entry. This is true by design.*

So why, one may ask, are the problems not equivalent? The answer is simple, but subtle. The way $\boldsymbol{f}_\ell$ *is defined on the last slide is different from the way it is defined in the cell-centered finite difference context.*

Theorem (Error Estimate)

Let $f \in L^2(0,1)$ be given. Assume that $u \in H_0^1(0,1) \cap H^2(0,1)$ satisfies

$$(u', v')_{L^2(0,1)} = (f, v)_{L^2(0,1)}, \quad \forall\, v \in H_0^1(0,1).$$

Assume that $u_\ell \in V_\ell$ is the unique solution to (8). Then, there is a constant $C > 0$, independent of ℓ, for which

$$\sqrt{\mathcal{A}_\ell\,(u_\ell - Q_\ell u, u_\ell - Q_\ell u)} \le C h_\ell \|f\|_{L^2(0,1)}. \tag{10}$$

Proof.

See Ewing, Lazarov, and Vassilevski (1991). □

Definition (Discrete Cell-Centered Ritz Projection)

For each $1 \leq \ell \leq L$, the **discrete cell-centered Ritz projection**, $\mathcal{R}_{\ell-1} : V_\ell \to V_{\ell-1}$, is defined as follows: given $v_\ell \in V_\ell$, then $\mathcal{R}_{\ell-1} v_\ell \in V_{\ell-1}$ satisfies

$$\mathcal{A}_{\ell-1} \left(\mathcal{R}_{\ell-1} v_\ell, w_{\ell-1} \right) = \mathcal{A}_\ell \left(v_\ell, w_{\ell-1} \right), \qquad \forall\, w_{\ell-1} \in V_{\ell-1}.$$

Proposition (Ritz Projection Representation)

Let $1 \leq \ell \leq L$. Assume that $v_\ell \in V_\ell$, and set $v_{\ell-1} = \mathcal{R}_{\ell-1} v_\ell \in V_{\ell-1}$. Under the usual coordinate identification, that is,

$$v_{\ell-1} = \mathcal{R}_{\ell-1} v_\ell \in V_{\ell-1} \overset{\mathcal{B}_{\ell-1}}{\longleftrightarrow} \boldsymbol{v}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}},$$
$$v_\ell \in V_\ell \overset{\mathcal{B}_\ell}{\longleftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell},$$

we have

$$\boldsymbol{v}_{\ell-1} = \Pi_{\ell-1} \boldsymbol{v}_\ell,$$

where $\Pi_{\ell-1}$ is defined, as before, by

$$\Pi_{\ell-1} = \mathsf{A}_{\ell-1}^{-1} \mathsf{R}_{\ell-1} \mathsf{A}_\ell \in \mathbb{R}^{n_{\ell-1} \times n_\ell}.$$

Proof.

Clearly, for all $j = 1, \dots, n_{\ell-1}$,

$$\mathcal{A}_{\ell-1}\left(\sum_{i=1}^{n_{\ell-1}} v_{\ell-1,i}\psi_{\ell-1,i}, \psi_{\ell-1,j}\right) = \mathcal{A}_{\ell}\left(\sum_{k=1}^{n_{\ell}} v_{\ell,k}\psi_{\ell,k}, \psi_{\ell-1,j}\right). \tag{11}$$

In addition, the left side of (11) is

$$\begin{aligned}\mathcal{A}_{\ell-1}\left(\sum_{i=1}^{n_{\ell-1}} v_{\ell-1,i}\psi_{\ell-1,i}, \psi_{\ell-1,j}\right) &= \sum_{i=1}^{n_{\ell-1}} v_{\ell-1,i}\mathcal{A}_{\ell-1}(\psi_{\ell-1,i}, \psi_{\ell-1,j}) \\ &\overset{(5)}{=} h_{\ell-1}\sum_{i=1}^{n_{\ell-1}} [\mathsf{A}_{\ell-1}]_{i,j} v_{\ell-1,i} \\ &= h_{\ell-1}[\mathsf{A}_{\ell-1}\boldsymbol{v}_{\ell-1}]_j.\end{aligned}$$

Proof (Cont.)

Next, observe that

$$\psi_{\ell-1,j} = \sum_{s=1}^{n_\ell} p_{\ell-1,s,j} \psi_{\ell,s},$$

where

$$[\mathsf{P}_{\ell-1}]_{s,j} = p_{\ell-1,s,j}, \quad 1 \le s \le n_\ell, \quad 1 \le j \le n_{\ell-1}.$$

Thus, the right-hand side of (11) satisfies,

$$\begin{aligned} \mathcal{A}_\ell \left(\sum_{k=1}^{n_\ell} v_{\ell,k} \psi_{\ell,k}, \sum_{s=1}^{n_\ell} p_{\ell-1,s,j} \psi_{\ell,s} \right) &= \sum_{k,s=1}^{n_\ell} v_{\ell,k} p_{\ell-1,s,j} \mathcal{A}_\ell \left(\psi_{\ell,k}, \psi_{\ell,s} \right) \\ &\stackrel{(5)}{=} h_\ell \sum_{k,s=1}^{n_\ell} v_{\ell,k} p_{\ell-1,s,j} [\mathsf{A}_\ell]_{k,s} \\ &= h_\ell [\mathsf{P}_{\ell-1}^\top \mathsf{A}_\ell \boldsymbol{v}_\ell]_j. \end{aligned}$$

Proof (Cont.)

Thus, for every $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$, (11) is equivalent to

$$\begin{aligned} \mathsf{A}_{\ell-1}\boldsymbol{v}_{\ell-1} &= \frac{h_\ell}{h_{\ell-1}}\mathsf{P}_{\ell-1}^\top \mathsf{A}_\ell \boldsymbol{v}_\ell \\ &= \frac{1}{2}\mathsf{P}_{\ell-1}^\top \mathsf{A}_\ell \boldsymbol{v}_\ell \\ &\overset{(1)}{=} \mathsf{R}_{\ell-1}\mathsf{A}_\ell \boldsymbol{v}_\ell. \end{aligned}$$

This can be rewritten as

$$\boldsymbol{v}_{\ell-1} = \mathsf{A}_{\ell-1}^{-1}\mathsf{R}_{\ell-1}\mathsf{A}_\ell \boldsymbol{v}_\ell = \Pi_{\ell-1}\boldsymbol{v}_\ell. \qquad \square$$

Definition (Lifting Operator)

Let $0 \leq \ell \leq L$. The **lifting operator**, $\mathcal{L} : V_\ell \to H_0^1(0,1)$ is defined as follows. Let $v_\ell \in V_\ell$, then $\mathcal{L}v_\ell \in H_0^1(0,1)$ is the solution of the variational problem

$$\left((\mathcal{L}v_\ell)', w'\right)_{L^2(0,1)} = \left(-\Delta_\ell^\circ v_\ell, w\right)_{L^2(0,1)}, \qquad \forall\, w \in H_0^1(0,1). \tag{12}$$

Proposition (Reverse Stability)

There exist a constant $C > 0$ such that, for every $0 \leq \ell \leq L$ and all $v_\ell \in V_\ell$

$$\|\mathcal{L} v_\ell\|^2_{H^1(0,1)} \leq C \mathcal{A}_\ell (v_\ell, v_\ell)\,. \tag{13}$$

Proof.

By the Poincaré inequality, there is a constant $C_1 > 0$ such that

$$\|\mathcal{L} v_\ell\|^2_{H^1(0,1)} \leq C_1 \left((\mathcal{L} v_\ell)', (\mathcal{L} v_\ell)' \right)_{L^2(0,1)}\,.$$

Now, using the definitions of the lifting operator (12) and the L^2-projection operator,

$$\begin{aligned} \left((\mathcal{L} v_\ell)', (\mathcal{L} v_\ell)' \right)_{L^2(0,1)} &= - \left(\Delta_\ell^\circ v_\ell, \mathcal{L} v_\ell \right)_{L^2(0,1)} \\ &= - \left(\Delta_\ell^\circ v_\ell, Q_\ell \mathcal{L} v_\ell \right)_{L^2(0,1)} \\ &\stackrel{(4)}{=} \mathcal{A}_\ell \left(v_\ell, Q_\ell \mathcal{L} v_\ell \right) \\ &\leq \sqrt{\mathcal{A}_\ell \left(v_\ell, v_\ell \right)} \sqrt{\mathcal{A}_\ell \left(Q_\ell \mathcal{L} v_\ell, Q_\ell \mathcal{L} v_\ell \right)}, \end{aligned}$$

where the last step follows from the Cauchy-Schwartz inequality.

Proof (Cont.)

We may now combine this estimate with the stability of the L^2-projection, to obtain

$$\begin{aligned}
\|\mathcal{L}v_\ell\|^2_{H^1(0,1)} &\leq C_1 \sqrt{\mathcal{A}_\ell\left(v_\ell, v_\ell\right)} \sqrt{\mathcal{A}_\ell\left(Q_\ell \mathcal{L}v_\ell, Q_\ell \mathcal{L}v_\ell\right)} \\
&\leq C_1 C_2 \sqrt{\mathcal{A}_\ell\left(v_\ell, v_\ell\right)} \sqrt{\left((\mathcal{L}v_\ell)', (\mathcal{L}v_\ell)'\right)_{L^2(0,1)}} \\
&\leq C_1 C_2 \sqrt{\mathcal{A}_\ell\left(v_\ell, v_\ell\right)} \|\mathcal{L}v_\ell\|_{H^1(0,1)}.
\end{aligned}$$

Setting $C = (C_1 C_2)^2$, the proof is complete. □

The following result can be understood as a version of the α-weak approximation property, but stated for piecewise-constant functions instead of vectors.

Theorem (Approximation)

For every $1 \leq \ell \leq L$, *and all* $v_\ell \in V_\ell$, *there is a constant* $C > 0$, *independent of* ℓ, *such that*

$$|\mathcal{A}_\ell\left(v_\ell - \mathcal{R}_{\ell-1} v_\ell, v_\ell\right)| \leq C h_\ell \left\|\Delta_\ell^\circ v_\ell\right\|_{L^2(0,1)} \sqrt{\mathcal{A}_\ell\left(v_\ell, v_\ell\right)},$$

Proof.

Let $v_\ell \in V_\ell$ be arbitrary. Then, by triangle inequality,

$$\begin{aligned} |\mathcal{A}_\ell(v_\ell - \mathcal{R}_{\ell-1}v_\ell, v_\ell)| &\le |\mathcal{A}_\ell(v_\ell - Q_\ell \mathcal{L} v_\ell, v_\ell)| \\ &\quad + |\mathcal{A}_\ell(Q_\ell \mathcal{L} v_\ell - Q_{\ell-1}\mathcal{L} v_\ell, v_\ell)| \\ &\quad + |\mathcal{A}_\ell(Q_{\ell-1}\mathcal{L} v_\ell - \mathcal{R}_{\ell-1}v_\ell, v_\ell)| \\ &= T_1 + T_2 + T_3. \end{aligned} \tag{14}$$

We now examine each term separately.

Proof (Cont.)

We now deal with the term T_1. Since $v_\ell \in V_\ell$ is the cell-centered finite difference approximation of $\mathcal{L}v_\ell$, the error estimate (10) guarantees that there exist $C > 0$ such that

$$\begin{aligned} T_1 &\leq \sqrt{\mathcal{A}_\ell \left(v_\ell - Q_\ell \mathcal{L} v_\ell, v_\ell - Q_\ell \mathcal{L} v_\ell\right)} \sqrt{\mathcal{A}_\ell \left(v_\ell, v_\ell\right)} \\ &\overset{(10)}{\leq} C h_\ell \left\| \Delta_\ell^\circ v_\ell \right\|_{L^2(0,1)} \sqrt{\mathcal{A}_\ell \left(v_\ell, v_\ell\right)}. \end{aligned}$$

Proof (Cont.)

To handle the last term, T_3, we begin with a preliminary observation. Let $v_{\ell-1} \in V_{\ell-1}$ be the cell-centered finite difference approximation of $\mathcal{L}v_\ell$ in the space $V_{\ell-1}$. In other words, $v_{\ell-1}$ is the unique function that satisfies

$$\mathcal{A}_{\ell-1}\left(v_{\ell-1}, w_{\ell-1}\right) = -\left(\Delta_\ell^\circ v_\ell, w_{\ell-1}\right)_{L^2(0,1)}, \quad \forall\, w_{\ell-1} \in V_{\ell-1}.$$

On the other hand, since $V_{\ell-1} \subset V_\ell$, the definition (4) of the energy form shows that

$$-\left(\Delta_\ell^\circ v_\ell, w_{\ell-1}\right)_{L^2(0,1)} = \mathcal{A}_\ell\left(v_\ell, w_{\ell-1}\right), \quad \forall\, w_{\ell-1} \in V_{\ell-1}.$$

Finally, recall that, by the definition of the discrete Ritz projection,

$$\mathcal{A}_\ell\left(v_\ell, w_{\ell-1}\right) = \mathcal{A}_{\ell-1}\left(\mathcal{R}_{\ell-1} v_\ell, w_{\ell-1}\right), \quad \forall\, w_{\ell-1} \in V_{\ell-1}.$$

Proof (Cont.)

Let us combine these three identities to obtain that

$$\mathcal{A}_{\ell-1}\left(v_{\ell-1}, w_{\ell-1}\right) = \mathcal{A}_{\ell-1}\left(\mathcal{R}_{\ell-1} v_\ell, w_{\ell-1}\right), \quad \forall\, w_{\ell-1} \in V_{\ell-1}.$$

In other words, $v_{\ell-1} = \mathcal{R}_{\ell-1} v_\ell$. In summary, $\mathcal{R}_{\ell-1} v_\ell \in V_{\ell-1}$ is the cell-centered finite difference approximation of the function $\mathcal{L} v_\ell \in H_0^1(0,1)$. The error estimate (10) then guarantees

$$\sqrt{\mathcal{A}_{\ell-1}\left(\mathcal{R}_{\ell-1} v_\ell - Q_{\ell-1}\mathcal{L} v_\ell, \mathcal{R}_{\ell-1} v_\ell - Q_{\ell-1}\mathcal{L} v_\ell\right)} \le C h_{\ell-1} \left\| \Delta_\ell^\circ v_\ell \right\|_{L^2(0,1)}.$$

Next we observe that, using the imbalanced Galerkin condition, for all $w_{\ell-1} \in V_{\ell-1}$,

$$\mathcal{A}_\ell\left(w_{\ell-1}, w_{\ell-1}\right) = 2\mathcal{A}_{\ell-1}\left(w_{\ell-1}, w_{\ell-1}\right).$$

These last two facts together yield

$$\begin{aligned} T_3 &\le \sqrt{\mathcal{A}_\ell\left(Q_{\ell-1}\mathcal{L} v_\ell - \mathcal{R}_{\ell-1} v_\ell, Q_{\ell-1}\mathcal{L} v_\ell - \mathcal{R}_{\ell-1} v_\ell\right)} \sqrt{\mathcal{A}_\ell\left(v_\ell, v_\ell\right)} \\ &\le \sqrt{2} C h_{\ell-1} \left\| \Delta_\ell^\circ v_\ell \right\|_{L^2(0,1)} \sqrt{\mathcal{A}_\ell\left(v_\ell, v_\ell\right)} \\ &= 2\sqrt{2} C h_\ell \left\| \Delta_\ell^\circ v_\ell \right\|_{L^2(0,1)} \sqrt{\mathcal{A}_\ell\left(v_\ell, v_\ell\right)}. \end{aligned}$$

Proof (Cont.)

Finally we consider the middle term, T_2, of (14). Note that

$$
\begin{aligned}
T_2 &= \left| \left(Q_\ell \mathcal{L} v_\ell - Q_{\ell-1} \mathcal{L} v_\ell, \mathring{\Delta}_\ell v_\ell \right)_{L^2(0,1)} \right| \\
&\leq \| Q_\ell \mathcal{L} v_\ell - Q_{\ell-1} \mathcal{L} v_\ell \|_{L^2(0,1)} \| \mathring{\Delta}_\ell v_\ell \|_{L^2(0,1)} \\
&\leq \left(\| Q_\ell \mathcal{L} v_\ell - \mathcal{L} v_\ell \|_{L^2(0,1)} + \| \mathcal{L} v_\ell - Q_{\ell-1} \mathcal{L} v_\ell \|_{L^2(0,1)} \right) \| \mathring{\Delta}_\ell v_\ell \|_{L^2(0,1)} \\
&\overset{(7)}{\leq} C(h_\ell + h_{\ell-1}) \| \mathcal{L} v_\ell \|_{H^1(0,1)} \| \mathring{\Delta}_\ell v_\ell \|_{L^2(0,1)} \\
&\overset{(13)}{\leq} C h_\ell \| \mathring{\Delta}_\ell v_\ell \|_{L^2(0,1)} \sqrt{\mathcal{A}_\ell (v_\ell, v_\ell)}.
\end{aligned}
$$

It then remains to combine the previously obtained estimates of each one of the terms T_i, $i = 1, 2, 3$, that comprise (14). $\square$

We have reached the point where we can show that the α-weak approximation property (A2) holds with

$$\alpha = \frac{1}{2}.$$

Theorem (α-weak approximation property)

There is a constant $C_{\mathrm{A2}} > 0$ such that, for all $1 \leq \ell \leq L$, we have

$$\left| \left(\left(I - \tilde{\Pi}_\ell \right) \boldsymbol{v}_\ell, \boldsymbol{v}_\ell \right)_{\mathsf{A}_\ell} \right| \leq C_{\mathrm{A2}} \frac{\left\| \mathsf{A}_\ell \boldsymbol{v}_\ell \right\|_\ell}{\sqrt{\rho\left(\mathsf{A}_\ell\right)}} \left\| \boldsymbol{v}_\ell \right\|_{\mathsf{A}_\ell}, \qquad \forall\, \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}.$$

Proof.

Exercise. □

Remark (Convergence)

We have shown that the smoothing property holds for Richardson's smoother. The imbalanced Galerkin condition holds with $r = \frac{1}{2}$. Finally, we have just verified assumption (A2) (the α-Weak Approximation) with $\alpha = \frac{1}{2}$. Thus, we can apply the abstract theory of Section 5.9 to conclude that the two-grid method, the symmetric W-cycle with a large enough number of smoothing steps, and even the generic symmetric W-cycle methods converge. There is nothing more to do! The proofs are merely an application of the aforementioned results.