Math 673

Multigrid Methods: A Mostly Matrix-Based Approach

Chapter 09: Additive Preconditioners Based on Subspace Decompositions

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Chapter 09, Part 2 of 2

Additive Preconditioners Based on Subspace Decompositions

Hierarchical Basis Preconditioner

Now, we need to connect the spaces W_j to V_ℓ where $0 \leq j \leq \ell$. In so doing, we will have the tools to build a preconditioner based on the hierarchical basis. Be careful, the number of indices in this section can get a little overwhelming.

Proposition

Let $\mathcal{B}_j^W = \{\phi_{j,i}\}_{i=1}^{m_j}$ and $\mathcal{B}_\ell^V = \{\psi_{\ell,i}\}_{i=1}^{n_\ell}$ be the usual bases for W_j and V_ℓ, respectively. For each $0 \leq j \leq \ell$, there are unique numbers

$$q_{j,k,i}^{\ell} \in \mathbb{R}, \quad 1 \leq k \leq n_\ell, \quad 1 \leq i \leq m_j,$$

such that

$$\phi_{j,i} = \sum_{k=1}^{n_\ell} q_{j,k,i}^{\ell} \psi_{\ell,k}. \tag{1}$$

Proof.

Exercise. □

Definition (Hierarchical Prolongation Matrix)

Define the matrix $\mathsf{Q}_j^\ell \in \mathbb{R}^{n_\ell \times m_j}$ via

$$\left[\mathsf{Q}_j^\ell\right]_{i,k} := q_{j,k,i}^\ell, \quad 1 \le k \le n_\ell, \quad 1 \le i \le m_j.$$

Q_j^ℓ is called a **hierarchical prolongation matrix**.

Lemma

Suppose that Q_j^ℓ *is a hierarchical prolongation matrix and* $\boldsymbol{w}_j \in \mathbb{R}^{m_j}$ *is the coordinate vector of the function* $w_j \in W_j$ *with respect to the basis* $\mathcal{B}_j^W$*. Then,*

$$\operatorname{rank}(\mathsf{Q}_j^\ell) = m_j,$$

and the coordinate vector of $\boldsymbol{w}_j \in V_\ell$ *in the basis* $\mathcal{B}_\ell^V$ *is simply*

$$\mathsf{Q}_j^\ell \boldsymbol{w}_j \in \mathbb{R}^{n_\ell}.$$

Proof.

Exercise. □

Remark

Note that the family of spaces W_j are hierarchical, but are not nested

$$W_0 \not\subset W_1 \not\subset W_2 \cdots .$$

Furthermore, it makes no sense to stack the prolongation matrices as we did in the past:

$$\mathsf{Q}_j^\ell \neq \mathsf{Q}_k^\ell \mathsf{Q}_j^k,$$

for $j < k < \ell$. In fact, the product on the right hand side is not usually defined.

Definition

Define the bilinear form $C_j : W_j \times W_j \to \mathbb{R}$ via

$$C_j\left(w_j, v_j\right) := \sum_{r=1}^{m_j} w_j\left(\boldsymbol{N}_{j,r}^W\right) v_j\left(\boldsymbol{N}_{j,r}^W\right), \quad \forall\ w_j, v_j \in W_j.$$

Let $\mathcal{B}_j^W = \{\phi_{j,i}\}_{i=1}^{m_j}$ be the usual basis for W_j. Define the matrix $\mathsf{C}_j \in \mathbb{R}^{m_j \times m_j}$ via

$$\begin{aligned} \left[\mathsf{C}_j\right]_{i,k} &:= C_j\left(\phi_{j,i}, \phi_{j,k}\right) \\ &= \sum_{r=1}^{m_j} \phi_{j,i}\left(\boldsymbol{N}_{j,r}^W\right) \phi_{j,k}\left(\boldsymbol{N}_{j,r}^W\right) \\ &= \sum_{r=1}^{m_j} \delta_{ir}\delta_{kr} \\ &= \delta_{ik}. \end{aligned} \tag{2}$$

Definition (Hierarchical Basis Preconditioner)

Suppose that $\mathcal{B}_\ell^V = \{\psi_{\ell,i}\}_{i=1}^{n_\ell}$ is the usual basis for the finite element space V_ℓ. Let $\mathsf{A}_L \in \mathbb{R}^{n_L \times n_L}$ be the SPD matrix defined via

$$[\mathsf{A}_L]_{i,j} = a(\psi_{L,j}, \psi_{L,i}), \quad 1 \leq i, j \leq n_L,$$

where

$$a(u, v) = (\nabla u, \nabla v)_{L^2}, \quad \forall\, u, v \in H_0^1(\Omega).$$

The **hierarchical basis preconditioner for** A_L is defined as

$$\mathsf{C}_H = \sum_{\ell=0}^{L} \mathsf{Q}_\ell^L \mathsf{C}_\ell^{-1} \mathsf{Z}_\ell^L = \sum_{\ell=0}^{L} \mathsf{Q}_\ell^L \mathsf{Z}_\ell^L, \tag{3}$$

where C_ℓ is as in (2), $\mathsf{Q}_\ell^L \in \mathbb{R}^{n_L \times m_\ell}$ is the hierarchical prolongation matrix from a previous definition and

$$\mathsf{Z}_\ell^L = \left(\mathsf{Q}_\ell^L\right)^\top.$$

Lemma

Assumption (SS1) holds for the hierarchical basis decomposition. In particular, for any object

$$\boldsymbol{u}_L \in \mathbb{R}^{n_L} \overset{\mathcal{B}_L^V}{\leftrightarrow} u_L \in V_L$$

there exist unique objects

$$\boldsymbol{w}_\ell \in \mathbb{R}^{m_\ell} \overset{\mathcal{B}_\ell^W}{\leftrightarrow} w_\ell \in W_\ell, \quad 0 \leq \ell \leq L,$$

such that

$$\boldsymbol{u} = \sum_{\ell=0}^{L} \mathsf{Q}_\ell^L \boldsymbol{w}_\ell \in \mathbb{R}^{n_L} \overset{\mathcal{B}_L^V}{\leftrightarrow} u_L = \sum_{\ell=0}^{L} w_\ell \in V_L.$$

Furthermore, the hierarchical basis preconditioner, B_H, *defined in* (3), *is SPD.*

Proof.

This follows from the lemmas on the last slide deck. The details are left for an exercise. □

Remark

Our goal is now to show that

$$\lambda_{\min}(\mathsf{C}_H \mathsf{A}_L) \geq C_1 \left(1 + |\log(h_L)|^2\right)^{-1},$$

and

$$\lambda_{\max}(\mathsf{C}_H \mathsf{A}_L) \leq C_2,$$

where $\mathsf{C}_1, C_2 > 0$ *are independent of* L*. If this is the case*

$$\frac{\lambda_{\max}}{\lambda_{\min}} =: \kappa(\mathsf{C}_H \mathsf{A}_L) \leq \frac{C_2}{C_1}\left(1 + |\log(h_L)|^2\right).$$

This estimate is quite useful, since the logarithmic dependence on h_L *is quite weak. For example, suppose*

$$h_L = \frac{1}{2^L},$$

which is entirely reasonable. Then

$$|\log(h_L)|^2 = L^2 \, |\log(1/2)|^2 .$$

Our analysis that follows will only work for $d = 2$.

Now, we need some technical lemmas. For more details, see the book by Brenner and Scott.

Theorem (Mean-Zero Poincaré)

Suppose that Ω is an open polyhedral set in $\mathbb{R}^d$. Then, for every $u \in H^1(\Omega)$,

$$\|u - \bar{u}\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}, \tag{4}$$

for some constant $C > 0$ that is independent of u by dependent upon Ω, where $\bar{u}$ is the average of u:

$$\bar{u} := \frac{1}{|\Omega|} \int_\Omega u(\boldsymbol{x})\, d\boldsymbol{x}.$$

As a consequence, for every $u \in H^1(\Omega)$,

$$\|u - \bar{u}\|_{H^1(\Omega)} \leq C |u - \bar{u}|_{H^1(\Omega)} = C |u|_{H^1(\Omega)}, \tag{5}$$

for some constant $C > 0$ that is independent of u by dependent upon Ω.

Theorem (Inverse inequality)

Suppose that Ω is an open polygonal domain in $\mathbb{R}^d$, $\mathcal{T}_\ell$, $0 \leq \ell \leq L$ is a nested family of triangulations of Ω, and S_ℓ, $0 \leq \ell \leq L$, are the associated piecewise-linear finite element spaces. Assume that $1 \leq q \leq \infty$. Then, for all $v \in S_\ell$ and all $K \in \mathcal{T}_\ell$,

$$\|v\|_{H^1(K)} \leq C h_\ell^{-1+d/2-d/q} \|v\|_{L^q(K)}, \tag{6}$$

for some constant $C > 0$ that is independent of ℓ but depends on the shape of K.

Proof.

See Section 5.3 in the book by Brenner and Scott. □

In two space dimensions $H^1 \hookrightarrow L^p$, for any $1 \le p < \infty$. We cannot quite get control for $p = \infty$. But, if the function space is finite dimensional we can almost get control of the $p = \infty$ case. Here is the result from Section 4.9 in the book by Brenner and Scott.

Theorem

Suppose that Ω is an open polygonal domain in $\mathbb{R}^2$ and $\mathcal{T}_\ell$, $0 \le \ell \le L$ is a nested family of triangulations of Ω. Then, for any $v_\ell \in V_\ell$,

$$\|v_\ell\|_{L^\infty(\Omega)} \le C\sqrt{1 + |\log(h_\ell)|}\, |v_\ell|_{H^1(\Omega)}\,,$$

for some constant $C > 0$ that is independent of ℓ but depends upon the shape of Ω. Further, for all $v_\ell \in S_\ell$ and any $K \in \mathcal{T}_\ell$,

$$\|v_\ell - \bar{v}_\ell\|_{L^\infty(K)} \le C\sqrt{1 + |\log(h_\ell)|}\, |v_\ell|_{H^1(K)}\,,$$

for some constant $C > 0$ that is independent of ℓ but depends upon the shape of the triangle $K \in \mathcal{T}_\ell$, where

$$\bar{v}_\ell = \frac{1}{|K|}\int_K v_\ell(x)\, dx.$$

Lemma

Suppose that $0 \le j < \ell$. *For any* $v_\ell \in S_\ell$,

$$\|v_\ell - \bar{v}_{j,\ell}\|_{L^\infty(K_j)} \le C\sqrt{1 + \left|\log\left(\frac{h_j}{h_\ell}\right)\right|}\,|v_\ell|_{H^1(K_j)}\,, \tag{7}$$

for some constant $C > 0$ *that is independent of* j *and* ℓ *but depends upon the shape of the triangle* $K_j \in \mathcal{T}_j$, *where*

$$\bar{v}_{j,\ell} = \frac{1}{|K_j|}\int_{K_j} v_\ell(\boldsymbol{x})\,d\boldsymbol{x}.$$

Proof.

Exercise. □

Lemma

Assume that $\Omega \subset \mathbb{R}^2$ is a polygonal domain. Suppose that $\mathcal{I}_\ell : C(\overline{\Omega}) \to V_\ell$, $0 \leq \ell \leq L$, is the Lagrange nodal interpolation operator, and $\mathcal{I}_{-1} \equiv 0$. Then, for all $u_L \in V_L$,

$$\left\| \mathcal{I}_\ell u_L - \mathcal{I}_{\ell-1} u_L \right\|_{L^2(\Omega)} \leq C h_\ell \left(1 + \sqrt{L - \ell} \right) |u_L|_{H^1(\Omega)} \,. \tag{8}$$

for some constant $C > 0$ that is independent of but depends upon the shape of Ω.

Proof.

Define the piecewise constant function $\bar{u}_L^\ell$ such that

$$\bar{u}_L^\ell|_K := \frac{1}{|K|} \int_K u_L(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall\, K \in \mathcal{T}_\ell.$$

Proof (Cont.)

Then,

$$
\begin{aligned}
\left\|\mathcal{I}_\ell u_L - \mathcal{I}_{\ell-1} u_L\right\|_{L^2(\Omega)}^2 &= \left\|\mathcal{I}_\ell u_L - \mathcal{I}_{\ell-1}\left[\mathcal{I}_\ell[u_L]\right]\right\|_{L^2(\Omega)}^2 \\
&\leq Ch_\ell^2 \sum_{K\in\mathcal{T}_\ell} \left|\mathcal{I}_\ell[u_L]\right|_{H^1(K)}^2 \\
&= Ch_\ell^2 \sum_{K\in\mathcal{T}_\ell} \left|\mathcal{I}_\ell u_L - \bar{u}_L^\ell\right|_{H^1(K)}^2 \\
&\overset{(6)}{\leq} Ch_\ell^2 \sum_{K\in\mathcal{T}_\ell} \left\|\mathcal{I}_\ell u_L - \bar{u}_L^\ell\right\|_{L^\infty(K)}^2 \\
&\leq Ch_\ell^2 \sum_{K\in\mathcal{T}_\ell} \left\|u_L - \bar{u}_L^\ell\right\|_{L^\infty(K)}^2 \\
&\overset{(7)}{\leq} Ch_\ell^2 \sum_{K\in\mathcal{T}_\ell} \left(1 + \left|\log\left(\frac{h_\ell}{h_L}\right)\right|\right) |u_L|_{H^1(K)}^2 \\
&= Ch_\ell^2 \left(1 + \left|\log\left(\frac{h_\ell}{h_L}\right)\right|\right) |u_L|_{H^1(\Omega)}^2 .
\end{aligned}
$$

Proof (Cont.)

Now, notice that

$$h_\ell = h_0 2^{-\ell} \quad 1 \leq \ell \leq L.$$

So,

$$\log(h_\ell / h_L) = \log(2^{L-\ell}) = (L - \ell)\log(2).$$

The result follows. □

Lemma

There is some constant $C_1 > 0$ such that

$$\lambda_{\min}(\mathsf{B}_H \mathsf{A}_L) \geq \frac{C_1}{1 + |\log(h_L)|^2}. \tag{9}$$

Proof.

By definition, for any $w_{\ell,1}, w_{\ell,2} \in W_\ell$

$$C_\ell\left(w_{\ell,1}, w_{\ell,2}\right) = \sum_{i=1}^{m_\ell} w_{\ell,1}(\boldsymbol{N}_{\ell,i}^W) w_{\ell,2}(\boldsymbol{N}_{\ell,i}^W).$$

Let

$$\boldsymbol{w}_{\ell,\alpha} \in \mathbb{R}^{m_\ell} \overset{\mathcal{B}_\ell^W}{\leftrightarrow} w_{\ell,\alpha}, \quad \alpha = 1, 2.$$

Proof (Cont.)

Then,

$$
\begin{aligned}
(\mathsf{C}_\ell \boldsymbol{w}_{\ell,1}, \boldsymbol{w}_{\ell,2})_\ell &= \sum_{i=1}^{m_\ell} [\boldsymbol{w}_{\ell,1}]_i \, [\boldsymbol{w}_{\ell,2}]_i \\
&= \sum_{i=1}^{m_\ell} w_{\ell,1}(\boldsymbol{N}_{\ell,i}^W) w_{\ell,2}(\boldsymbol{N}_{\ell,i}^W) \\
&= C_\ell \left(w_{\ell,1}, w_{\ell,2} \right) \\
&= C_\ell \left(w_{\ell,2}, w_{\ell,1} \right) \\
&=: \langle w_{\ell,1}, w_{\ell,2} \rangle_{\mathsf{C}_\ell} .
\end{aligned}
$$

This last object is like a mass-lumping inner product. All that is missing is a factor of ${h_\ell}^2$.

There are constants $C_3 > 0$, $C_4 > 0$ such that, for all $0 \leq \ell \leq L$,

$$
C_3 {h_\ell}^2 \langle w_\ell, w_\ell \rangle_{\mathsf{C}_\ell} \leq \| w_\ell \|_{L^2(\Omega)}^2 \leq C_4 {h_\ell}^2 \langle w_\ell, w_\ell \rangle_{\mathsf{C}_\ell} , \qquad (10)
$$

for all $w_\ell \in W_\ell$.

Proof (Cont.)

Therefore, for any $w_\ell \in W_\ell \overset{\mathcal{B}_\ell^W}{\leftrightarrow} \boldsymbol{w}_\ell \in \mathbb{R}^{m_\ell}$,

$$
\begin{aligned}
\left(\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell\right)_\ell &= h_\ell{}^{-2} h_\ell{}^{2} \left\langle w_\ell, w_\ell \right\rangle_{\mathsf{C}_\ell} \\
&\overset{(10)}{\leq} C_3^{-1} h_\ell{}^{-2} \left\| w_\ell \right\|_{L^2(\Omega)}^2 \\
&= C_3^{-1} h_\ell{}^{-2} \left\| w_\ell - \mathcal{I}_{\ell-1} w_\ell \right\|_{L^2(\Omega)}^2 \\
&\overset{\text{(interp. err.)}}{\leq} C_3^{-1} C \left| w_\ell \right|_{H^1(\Omega)}^2 \\
&\overset{(6)}{\leq} C_3^{-1} C h_\ell{}^{-2} \left\| w_\ell \right\|_{L^2(\Omega)}^2 \\
&\overset{(10)}{\leq} C_3^{-1} C C_4 \left(\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell\right)_\ell . \qquad (11)
\end{aligned}
$$

Proof (Cont.)

Therefore, there are constants $C_5 > 0$, $C_6 > 0$, such that we have the equivalence

$$C_5 \sum_{\ell=0}^{L} |w_\ell|^2_{H^1(\Omega)} \leq \sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell \leq C_6 \sum_{\ell=0}^{L} |w_\ell|^2_{H^1(\Omega)}, \tag{12}$$

for any collection (w_ℓ), with $w_\ell \in W_\ell$, in general. Now, let $u_L \in V_L$ be given and

$$u_L = \sum_{\ell=0}^{L} w_\ell, \quad \exists!\, w_\ell \in W_\ell, \quad 0 \leq \ell \leq L.$$

Recall that

$$w_\ell = \mathcal{I}_\ell u_L - \mathcal{I}_{\ell-1} u_L, \quad 1 \leq \ell \leq L,$$

and

$$w_0 = \mathcal{I}_0 u_L.$$

We make the usual identification $w_\ell \in W_\ell \overset{\mathcal{B}_\ell^W}{\leftrightarrow} \boldsymbol{w}_\ell \in \mathbb{R}^{m_\ell}$, and we observe that

$$(\boldsymbol{w}_\ell)_{\ell=0}^{L} \in \mathsf{Q}[\boldsymbol{u}_L].$$

Proof (Cont.)

Then, from (11)

$$
\begin{aligned}
\sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell &\leq C_3^{-1} C \sum_{\ell=0}^{L} h_\ell{}^{-2} \|w_\ell\|^2_{L^2(\Omega)} \\
&\stackrel{(8)}{\leq} C \sum_{\ell=0}^{L} \left(1 + \sqrt{L-\ell}\right)^2 |u_L|^2_{H^1(\Omega)} \\
&\leq C \sum_{\ell=0}^{L} (1 + L - \ell) |u_L|^2_{H^1(\Omega)} \\
&\leq C \left(1 + L + L^2\right) |u_L|^2_{H^1(\Omega)} \\
&\stackrel{L\geq 1}{\leq} C L^2 |u_L|^2_{H^1(\Omega)} .
\end{aligned}
$$

Proof (Cont.)

But

$$|u_L|^2_{H^1(\Omega)} = a(u_L, u_L) = (\mathsf{A}_L \boldsymbol{u}_L, \boldsymbol{u}_L)_L,$$

and

$$\begin{aligned} |\log(h_L)|^2 &= \left|\log(h_0 2^{-L})\right|^2 \\ &= |\log(h_0) - L\log(2)|^2 \\ &= \log^2(h_0) - 2\log(h_0)L\log(2) + L^2\log^2(2). \end{aligned}$$

So,

$$L^2 \le C\left(1 + |\log(h_L)|^2\right), \quad \exists\, C > 0.$$

Proof (Cont.)

Thus,

$$\sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell) \leq C \left(1 + |\log(h_L)|^2\right) (\mathsf{A}\boldsymbol{u}_L, \boldsymbol{u}_L)_L,$$

and it follows from the big theorem of the last slide deck that

$$\lambda_{\min}(\mathsf{C}_H \mathsf{A}_L) \geq C_1 \left(1 + |\log(h_L)|^2\right)^{-1}.$$

$\square$

For reference, here is that "big" theorem.

Theorem (Eigenvalues of CA)

Suppose that Assumption (SS1) holds for the set of prolongation matrices $\{\mathsf{Q}_j\}_{j=0}^{L}$ *and* C *is an additive subspace preconditioner with respect to* $\{\mathsf{Q}_j\}_{j=0}^{L}$*. The eigenvalues of* CA *are positive, provided* A *is SPD with respect to* $(\cdot\,,\cdot)$*. Moreover*

$$\lambda_{\max}(\mathsf{CA}) = \max_{\boldsymbol{u}\in\mathbb{R}^n_\star} \frac{(\mathsf{A}\boldsymbol{u},\boldsymbol{u})}{\displaystyle\min_{(\boldsymbol{w}_\ell)\in\mathsf{Q}[\boldsymbol{u}]} \sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell}, \tag{13}$$

$$\lambda_{\min}(\mathsf{CA}) = \min_{\boldsymbol{u}\in\mathbb{R}^n_\star} \frac{(\mathsf{A}\boldsymbol{u},\boldsymbol{u})}{\displaystyle\min_{(\boldsymbol{w}_\ell)\in\mathsf{Q}[\boldsymbol{u}]} \sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell}. \tag{14}$$

Next, we need a little technical lemma, a kind of convolution result.

Lemma

Let $a_j, b_j \geq 0, -\infty < j < \infty$, *with*

$$s_1 := \sum_{j=-\infty}^{\infty} a_j \leq \infty,$$

and

$$s_2 := \sum_{j=-\infty}^{\infty} b_j \leq \infty.$$

Then

$$\sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} a_{j-k} b_k \right)^2 \leq s_1^2 s_2. \tag{15}$$

Proof.

Exercise. □

Lemma

For any $v_\ell \in V_\ell$ and $v_k \in V_k$, $0 \le \ell \le k \le L$, and d=2, there is a constant $C > 0$ such that

$$\int_\Omega \nabla v_\ell \cdot \nabla v_k d\boldsymbol{x} \le 2^{(\ell-k)/2} C \left| v_\ell \right|_{H^1(\Omega)} \left(h_k^{-1} \left\| v_k \right\|_{L^2(\Omega)} \right). \qquad (16)$$

Proof.

For any $K \in \mathcal{T}_\ell$, since $\Delta v_\ell|_K \equiv 0$,

$$
\begin{aligned}
\int_K \nabla v_\ell \cdot \nabla v_k d\boldsymbol{x} &= \int_{\partial K} \frac{\partial v_\ell}{\partial n} v_k ds \\
&\leq C h_\ell^{-1} |v_\ell|_{H^1(K)} \int_{\partial K} |v_k| ds \\
&\leq C h_\ell^{-1} |v_\ell|_{H^1(K)} \left(h_k \sum_{\boldsymbol{N}_k \in \partial K} \left| v_k(\boldsymbol{N}_k) \right| \right) \\
&\overset{\text{C.S.}}{\leq} C h_\ell^{-1} |v_\ell|_{H^1(K)} \left(h_k \left(\frac{h_\ell}{h_k} \right)^{1/2} \left(\sum_{\boldsymbol{N}_k \in \partial K} \left| v_k(\boldsymbol{N}_k) \right|^2 \right)^{1/2} \right) \\
&\leq C \left(\frac{h_k}{h_\ell} \right)^{1/2} |v_\ell|_{H^1(K)} h_k^{-1} \| v_k \|_{L^2(K)} .
\end{aligned}
$$

Proof (Cont.)

Thus,

$$\begin{aligned}
\int_\Omega \nabla v_\ell \cdot \nabla v_k \, d\boldsymbol{x} &= \sum_{K \in \mathcal{T}_\ell} \int_K \nabla v_\ell \cdot \nabla v_k \, d\boldsymbol{x} \\
&\leq C 2^{(\ell-k)/2} \sum_{K \in \mathcal{T}_\ell} |v_\ell|_{H^1(K)} h_k^{-1} \|v_k\|_{L^2(K)} \\
&\stackrel{\text{C.S.}}{\leq} C 2^{(\ell-k)/2} |v_\ell|_{H^1(\Omega)} h_k^{-1} \|v_k\|_{L^2(\Omega)} .
\end{aligned}$$

□

Lemma (Strengthened Cauchy-Schwarz Inequality)

For any $w_\ell \in W_\ell$ and $w_k \in W_k$, $0 \leq \ell \leq k \leq L$, there is a constant $C > 0$ such that

$$\int_\Omega \nabla w_\ell \cdot \nabla w_k d\boldsymbol{x} \leq 2^{(\ell-k)/2} C \left| w_\ell \right|_{H^1(\Omega)} \left| w_k \right|_{H^1(\Omega)} . \tag{17}$$

Proof.

Observe that

$$w_k = w_k - \mathcal{I}_{\ell-1}(w_k).$$

We use the interpolation error estimate

$$\|w_k - \mathcal{I}_{k-1}(w_k)\|_{L^2(\Omega)} \leq Ch_k |w_k|_{H^1(\Omega)},$$

to conclude that

$$\|w_k\|_{L^2(\Omega)} \leq Ch_k |w_k|_{H^1(\Omega)}.$$

Now, we use the last result. Since $w_\ell \in V_\ell$ and $w_k \in V_k$,

$$\begin{aligned} \int_\Omega \nabla w_\ell \cdot \nabla w_k d\mathbf{x} &\leq C 2^{(\ell-k)/2} |w_\ell|_{H^1(\Omega)} h_k^{-1} \|w_k\|_{L^2(\Omega)} \\ &\leq 2^{(\ell-k)/2} C |w_\ell|_{H^1(\Omega)} |w_k|_{H^1(\Omega)} \end{aligned}$$

□

Lemma

There is a constant $C_2 > 0$ *such that*

$$\lambda_{\max}(\mathsf{B}_H \mathsf{A}_L) \leq C_2,$$

independent of L.

Proof.

Let $v_L \in V_L$ be arbitrary.

$$v_L \in V_L \overset{\mathcal{B}_L}{\leftrightarrow} \boldsymbol{v} \in \mathbb{R}^n.$$

There exist unique $w_\ell \in W_\ell \overset{\mathcal{B}_\ell^W}{\leftrightarrow} \boldsymbol{w}_\ell \in \mathbb{R}^{m_\ell}$ such that

$$v_L = \sum_{\ell=0}^{L} w_\ell \leftrightarrow \boldsymbol{v} = \sum_{\ell=0}^{L} Q_\ell^L \boldsymbol{w}_\ell.$$

Then

$$\begin{aligned}
(\boldsymbol{v}, \boldsymbol{v})_{\mathrm{A}_L} &= (\boldsymbol{v}, A_L \boldsymbol{v}) \\
&= a(\boldsymbol{v}, \boldsymbol{v}) \\
&= a\left(\sum_{\ell=0}^{L} w_\ell, \sum_{k=0}^{L} w_k\right) \\
&= \int_\Omega \left(\nabla \sum_{\ell=0}^{L} w_\ell\right) \cdot \left(\nabla \sum_{k=0}^{L} w_k\right) d\boldsymbol{x}
\end{aligned}$$

Proof (Cont.)

$$
\begin{aligned}
&= \sum_{\ell,k=0}^{L} \int_\Omega \nabla w_\ell \cdot \nabla w_k \, dx \\
&\overset{(17)}{\leq} C \sum_{\ell,k=0}^{L} 2^{-|\ell-k|/2} \left| w_\ell \right|_{H^1(\Omega)} \left| w_k \right|_{H^1(\Omega)} \\
&\leq C \sum_{\ell=0}^{L} \left(\sum_{k=0}^{L} 2^{-|\ell-k|/2} \left| w_k \right|_{H^1(\Omega)} \right) \left| w_\ell \right|_{H^1(\Omega)} \qquad (18) \\
&\overset{\text{C.S.}}{\leq} C \left\{ \sum_{\ell=0}^{L} \left(\sum_{k=0}^{L} 2^{-|\ell-k|/2} \left| w_k \right|_{H^1(\Omega)} \right)^2 \right\}^{1/2} \left\{ \sum_{\ell=0}^{L} \left| w_\ell \right|_{H^1(\Omega)}^2 \right\}^{1/2} \\
&\overset{(15)}{\leq} C \left\{ \sum_{\ell=0}^{L} \left| w_\ell \right|_{H^1(\Omega)}^2 \right\}^{1/2} \left\{ \sum_{\ell=0}^{L} \left| w_\ell \right|_{H^1(\Omega)}^2 \right\}^{1/2} \\
&= C \sum_{\ell=0}^{L} \left| w_\ell \right|_{H^1(\Omega)}^2 \overset{(12)}{\leq} C_2 \sum_{\ell=0}^{L} (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{C_\ell} .
\end{aligned}
$$

Proof (Cont.)

Recall that, since decomposition are unique

$$
\begin{aligned}
\lambda_{\max}(\mathsf{B}_H \mathsf{A}_L) &\overset{(13)}{=} \max_{\boldsymbol{u}\in\mathbb{R}^n_*} \frac{(\boldsymbol{u},\boldsymbol{u})_{\mathsf{A}_L}}{\sum_{\ell=0}^{L}(\boldsymbol{w}_\ell,\boldsymbol{w}_\ell)_{\mathsf{C}_\ell}} \\
&\overset{(18)}{=} \max_{\boldsymbol{u}\in\mathbb{R}^n_*} \frac{\mathbb{C}_2 \sum_{\ell=0}^{L}(\boldsymbol{w}_\ell,\boldsymbol{w}_\ell)_{\mathsf{C}_\ell}}{\sum_{\ell=0}^{L}(\boldsymbol{w}_\ell,\boldsymbol{w}_\ell)_{\mathsf{C}_\ell}} \\
&\leq C_2.
\end{aligned}
$$

□

Theorem

There is a constant $C > 0$ independent of L, such that

$$\kappa(\mathsf{B}_H \mathsf{A}_L) = \frac{\lambda_{\max}(\mathsf{B}_H \mathsf{A}_L)}{\lambda_{\min}(\mathsf{B}_H \mathsf{A}_L)} \leq C \left(1 + |\log(h_L)|^2\right). \tag{19}$$

independent of L.

Proof.

Exercise. $\square$

The BPX Preconditioner

The BPX Preconditioner

The BPX preconditioner has a slightly better perforance than the hierarchical basis preconditioner, in the sense that the logarithmic dependence on h_L can be removed. For this method we choose

$$W_\ell := V_\ell, \quad 0 \leq \ell \leq L.$$

Thus $W_L = V_L$ and

$$m_\ell = n_\ell, \quad 0 \leq \ell \leq L.$$

Definition

Define the operator $C_\ell : V_\ell \to V_\ell'$ via

$$C_\ell \left[v_{\ell,1} \right] \left(v_{\ell,2} \right) = \sum_{i=1}^{n_\ell} v_{\ell,1}(\boldsymbol{N}_{\ell,i}^W) v_{\ell,2}(\boldsymbol{N}_{\ell,i}^W).$$

The matrix $\mathsf{C}_\ell \in \mathbb{R}^{m_\ell \times m_\ell}$ is defined as

$$\left[\mathsf{C}_\ell \right]_{j,k} = C_\ell \left[\phi_{\ell,j} \right] \left(\phi_{\ell,k} \right) = \delta_{j,k}, \quad 1 \le j, k \le n_\ell,$$

where $\mathcal{B}_\ell = \{ \phi_{\ell,j} \}_{j=1}^{n_\ell}$ is the Lagrange nodal basis for the piecewise linear FE space $V_\ell, 0 \le \ell \le L$. The BPX preconditioner is

$$C_{BPX} := \sum_{\ell=0}^{L} \mathsf{P}_\ell^L \mathsf{C}_\ell^{-1} \mathcal{R}_\ell^L = \sum_{\ell=0}^{L} \mathsf{P}_\ell^L \mathcal{R}_\ell^L, \tag{20}$$

where $\mathsf{P}_\ell^L \in \mathbb{R}^{n \times n_\ell}$ is the standard prolongation matrix from Chapter 6 and $\mathcal{R}_\ell^L = \left(\mathsf{P}_\ell^L \right)^T$.

Lemma

Assumption (SS1) holds for the BPX framework, i.e., for every $u_L \in V_L$, there exists $v_\ell \in V_\ell, 0 \leq \ell \leq L$, such that

$$u_L = \sum_{\ell=0}^{L} v_\ell,$$

or, equivalently

$$\boldsymbol{u} = \sum_{\ell=0}^{L} \mathrm{P}_\ell^I \boldsymbol{v}_\ell,$$

with

$$V_\ell \ni v_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^n,$$

and

$$V_L \ni u_L \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{u} \in \mathbb{R}^n.$$

Proof.

This is trivial because of the nestedness of the the spaces

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{L-1} \subset V_l.$$

□

Remark

Note that the decomposition is no longer unique.

Lemma

For any $v_j \in V_j, v_\ell \in V_\ell$*,*

$$\int_\Omega \nabla v_j \cdot \nabla v_\ell dx \leq C 2^{-|j-\ell|/2} \left(h_j^{-1} \|v_j\|_{L^2(\Omega)} \right) \left(h_\ell^{-1} \|v_\ell\|_{L^2(\Omega)} \right), \tag{21}$$

for some $C > 0$*.*

Proof.

This is follows from (16) and the inverse inequality

$$|v_j|_{H^1(\Omega)} \leq c h_j^{-1} \|v_j\|_{L^2(\Omega)} .$$

□

Lemma

For some $C_2 > 0$ that is independent of L,

$$\lambda_{\max}\left(B_{BPX}A_L\right) \leq C_2.$$

for some $C > 0$.

Proof.

Let $u_L \in V_L$ be arbitrary. There exists $v_\ell \in V_\ell, 0 \leq \ell \leq L$, such that

$$u_L = \sum_{\ell=0}^{L} v_\ell,$$

or

$$\boldsymbol{u} = \sum_{\ell=0}^{L} \mathrm{P}_\ell^l \boldsymbol{v}_\ell.$$

Proof (Cont.)

The decomposition is not unique, however. Then

$$
\begin{aligned}
(\boldsymbol{u}, \boldsymbol{u})_{A_L} &= (\boldsymbol{u}, A_L \boldsymbol{u}) \\
&= a(\boldsymbol{u}, \boldsymbol{u}) \\
&= a\left(\sum_{j=0}^{L} v_j, \sum_{\ell=0}^{L} v_\ell \right) \\
&= \sum_{\ell, j=0}^{L} a(v_j, v_\ell) \\
&\overset{(21)}{\leq} C \sum_{\ell, j=0}^{L} 2^{-|j-\ell|/2} h_j^{-1} \|v_\ell\|_{L^2(\Omega)} h_\ell \|v_k\|_{L^2(\Omega)} \\
&\overset{(15)}{\leq} C \sum_{j=0}^{L} h_j^{-2} \|v_j\|_{L^2(\Omega)} \\
&\overset{\text{MG Norm Equiv.}}{\leq} C_2 \sum_{j=0}^{L} (\boldsymbol{v}_j, \boldsymbol{v}_j)_{C_j} = C_2 \sum_{j=0}^{L} (C_j \boldsymbol{v}_j, \boldsymbol{v}_j)_j
\end{aligned}
$$

Proof (Cont.)

Now,

$$\begin{aligned}
\lambda_{\max}(C_{BPX}A_L) &\overset{\text{Eigenvalues of CA}}{=} \max_{\boldsymbol{u}\in\mathbb{R}^n_*} \frac{(\boldsymbol{u},\boldsymbol{u})_{\mathsf{A}_L}}{\displaystyle\min_{\boldsymbol{u}=\sum_{\ell=0}^{L}\mathsf{P}_\ell^L\boldsymbol{v}'_\ell}\sum_{\ell=0}^{L}\left(\boldsymbol{u}'_\ell,\boldsymbol{u}'_\ell\right)_{\mathsf{C}_\ell}} \\
&\leq \max_{\boldsymbol{u}\in\mathbb{R}^n_*} \frac{C_2\sum_{\ell=0}^{L}(\mathsf{C}_\ell\boldsymbol{w}_\ell,\boldsymbol{w}_\ell)_\ell}{\displaystyle\min_{\boldsymbol{v}'_\ell}\sum_{\ell=0}^{L}(\mathsf{C}_\ell\boldsymbol{w}_\ell,\boldsymbol{w}_\ell)} \\
&\leq C_2.
\end{aligned}$$

Recall that the minimum was achievable, so we could take $\boldsymbol{v}_\ell=\boldsymbol{v}'_\ell$. $\square$

Lemma

There is a constant $C_1 > 0$ that is independent of L, such that

$$\lambda_{\min}\left(B_{BPX}A_L\right) \geq C_1.$$

for some $C > 0$.

Proof.

Let $u_L \in V_L$ be arbitrary. Set

$$v_\ell =: \mathcal{R}_\ell u_L - R_{\ell-1} u_L, \quad 0 \leq \ell \leq L,$$

where $\mathcal{R}_\ell : H_0^1(\Omega) \to V_\ell$ is the Ritz projection for $0 \leq \ell \leq L$ and $R_{-1} \equiv 0$. Since

$$\mathcal{R}_\ell u_L = u_L,$$

it follows that

$$u_L = \sum_{\ell=0}^{L} v_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{u}_\ell = \sum_{\ell=0}^{L} \mathsf{P}_\ell^L v_\ell.$$

Moreover,

$$a(v_j, v_\ell) = 0, \quad 0 \leq j \neq \ell \leq L. \tag{22}$$

To see this, recall that, in general,

$$a(R_j u_L, v_j') = a(u_L, v_j'), \quad \forall v_j' \in V_j.$$

Proof (Cont.)

Suppose $j < \ell$, for definiteness. Then

$$a(R_j u_L, v'_\ell) = a(u_L, v'_\ell), \quad \forall\, v'_\ell \in V_\ell.$$

In particular, since

$$v_j := R_j u_L - R_{j-1} u_L \in V_j \subset V_\ell,$$

and

$$a(\mathcal{R}_\ell u_L, v_j) = a(u_L, v_j),$$

likewise

$$a(R_{\ell-1} u_L, v_j) = a(u_L, v_j),$$

Subtracting, we have

$$a(\mathcal{R}_\ell u_L - R_{\ell-1} u_L, v_j) = 0$$

Proof (Cont.)

To make further progress, let us assume that Ω is convex. Then the standard regularity condition holds. And, for $1 \leq \ell \leq L$,

$$\begin{aligned} {h_\ell}^{-2} \|v_\ell\|^2_{L^2(\Omega)} &= {h_\ell}^{-2} \|\mathcal{R}_\ell u_L - R_{\ell-1} u_L\|^2_{L^2(\Omega)} \\ &= {h_\ell}^{-2} \|\mathcal{R}_\ell u_L - R_{\ell-1} \mathcal{R}_\ell u_L\|^2_{L^2(\Omega)} \\ &\overset{(??)}{\leq} C{h_\ell}^{-2}{h_\ell}^{2} |\mathcal{R}_\ell u_L - R_{\ell-1} \mathcal{R}_\ell u_L|^2_{H^1(\Omega)} \\ &= C |\mathcal{R}_\ell u_L - R_{\ell-1} \mathcal{R}_\ell u_L|^2_{H^1(\Omega)} \\ &= C |v_\ell|^2_{H^1(\Omega)} . \end{aligned} \tag{23}$$

To see that $R_{\ell-1} = R_{\ell-1}\mathcal{R}_\ell$, let $u \in H^1_0(\Omega)$ be arbitrary. Then

$$a(R_{\ell-1}(\mathcal{R}_\ell u), v'_{\ell-1}) = a(\mathcal{R}_\ell u, v'_{\ell-1}), \quad \forall v'_{\ell-1} \in V_{\ell-1}.$$

But,

$$a(\mathcal{R}_\ell u, v'_{\ell-1}) = a(u, v'_{\ell-1}), \quad \forall v'_{\ell-1} \in V_{\ell-1}.$$

Proof (Cont.)

Since

$$a(\mathcal{R}_\ell u, v'_\ell) = a(u, v'_\ell), \quad \forall v'_\ell \in V_\ell,$$

and

$$V_{\ell-1} \subset V_\ell.$$

But

$$a(R_{\ell-1} u, v'_{\ell-1}) = a(u, v'_{\ell-1}), \quad \forall v'_{\ell-1} \in V_{\ell-1}.$$

Hence

$$a(R_{\ell-1}(\mathcal{R}_\ell u), v'_{\ell-1}) = a(\mathrm{R}_{\ell-1} u, v'_{\ell-1}), \quad \forall v'_{\ell-1} \in V_{\ell-1}.$$

And we conclude that $R_{\ell-1} = R_{\ell-1}\mathcal{R}_\ell$ since

$$R_{\ell-1}(\mathcal{R}_\ell u), R_{\ell-1} u \in V_{\ell-1}.$$

Estimate (22) holds trivially for $\ell = 0$.

Proof (Cont.)

Finally,

$$\begin{aligned}
\sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_\ell &\overset{\text{MG Norm Equiv.}}{\leq} C \sum_{\ell=0}^{L} h_\ell^{-2} \|v_\ell\|_{L^2(\Omega)}^2 \\
&\overset{(23)}{\leq} C_1^{-1} \sum_{\ell=0}^{L} |v_\ell|_{H^1(\Omega)}^2 \\
&\overset{(22)}{=} C_1^{-1} |u_L|_{H^1(\Omega)}^2 .
\end{aligned} \tag{24}$$

Proof (Cont.)

Also,

$$
\begin{aligned}
\lambda_{\min}(C_{BPX}A_L) &= \min_{\boldsymbol{u}\in\mathbb{R}^n_*} \frac{(\boldsymbol{u},\boldsymbol{u})_{\mathsf{A}_L}}{\displaystyle\min_{\boldsymbol{u}=\sum_{\ell=0}^{L}\mathsf{P}_\ell^L\boldsymbol{v}'_\ell}\sum_{\ell=0}^{L}\left(\boldsymbol{u}'_\ell,\boldsymbol{u}'_\ell\right)_{\mathsf{C}_\ell}} \\
&\geq \min_{\boldsymbol{u}\in\mathbb{R}^n_*} \frac{(\mathsf{A}_L\boldsymbol{u},\boldsymbol{u})_L}{\displaystyle\min_{\boldsymbol{v}'_\ell}\sum_{\ell=0}^{L}(\mathsf{C}_\ell\boldsymbol{v}_\ell,\boldsymbol{v}_\ell)} \\
&\geq \min_{\boldsymbol{u}\in\mathbb{R}^n_*} \frac{(\mathsf{A}_L\boldsymbol{u},\boldsymbol{u})_L}{C_1^{-1}\left|u_L\right|_{H^1(\Omega)}} \\
&= C_1.
\end{aligned}
$$

□

Theorem

$$\kappa\left(B_{BPX}A_L\right) = \frac{\lambda_{\max}\left(B_{BPX}A_L\right)}{\lambda_{\min}\left(B_{BPX}A_L\right)} \leq \frac{C_2}{C_1}.$$

Proof.

Follows from previous Lemmas. The details are left for an exercise. □