



Math 673/4

# Multigrid Methods: A Mostly Matrix-Based Approach

## Chapter 05: Multigrid: Algorithms and Abstract Convergence Theory

Abner J. Salgado and Steven M. Wise

asalgad1@utk.edu swise1@utk.edu  
University of Tennessee

F24/S25



# Chapter 05, Part 1 of 3

## Multigrid: Algorithms and Abstract Convergence Theory

# Introduction



The idea behind multigrid is to replace the exact solution in the coarse-grid correction by a recursive application of a “two-grid” method. In this chapter, we will learn how to extend the two-grid method to obtain various general multigrid algorithms.

This chapter borrows heavily from the elegant presentations in the books by Brenner and Scott (2008), and Braess (2007). The books by Hackbusch (1985) and Wesseling (2004) are also excellent resources and go into significantly greater depth.



# Basic Building Blocks of Multigrid

# Canonical Level Inner Products



Suppose that

$$1 \leq n_0 < n_1 < \cdots < n_\ell < \cdots < n_L \in \mathbb{N}.$$

We refer to  $\mathbb{R}^{n_\ell}$  as the level  $\ell$  space, or, sometimes more informally, as level  $\ell$ .

Symmetry and transposition are understood with respect to the canonical product on level  $\ell$ ,  $(\cdot, \cdot)_\ell : \mathbb{R}^{n_\ell} \times \mathbb{R}^{n_\ell} \rightarrow \mathbb{R}$ , which is defined as

$$(\mathbf{u}_\ell, \mathbf{v}_\ell)_\ell := \mathbf{v}_\ell^\top \mathbf{u}_\ell = \sum_{j=1}^{n_\ell} u_{\ell,j} v_{\ell,j},$$

for all  $\mathbf{u}_\ell, \mathbf{v}_\ell \in \mathbb{R}^{n_\ell}$ .



We will need the following simple property in this and following chapters.

### Proposition (Coarse-Fine Transposition)

*Suppose that  $1 \leq n_{\ell-1} < n_\ell$  and  $B \in \mathbb{R}^{n_{\ell-1} \times n_\ell}$  is arbitrary. Then, for any  $\mathbf{u}_\ell \in \mathbb{R}^{n_\ell}$  and any  $\mathbf{v}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}$ ,*

$$(B\mathbf{u}_\ell, \mathbf{v}_{\ell-1})_{\ell-1} = \left( \mathbf{u}_\ell, B^\top \mathbf{v}_{\ell-1} \right)_\ell,$$

*where  $B^\top \in \mathbb{R}^{n_\ell \times n_{\ell-1}}$  is the usual matrix transpose of  $B$ .*

Proof.

Exercise. □

## Prolongation and Restriction



We will assume in this chapter, and most everywhere else, unless otherwise noted, that the restriction matrices,

$$R_{\ell-1} \in \mathbb{R}^{n_{\ell-1} \times n_{\ell}}, \quad 1 \leq \ell \leq L,$$

are full rank, i.e.,

$$\text{rank}(R_{\ell-1}) = n_{\ell-1}.$$

We assume that the prolongation matrices satisfy

$$P_{\ell-1} = R_{\ell-1}^T \in \mathbb{R}^{n_{\ell} \times n_{\ell-1}}, \quad 1 \leq \ell \leq L.$$

Thus, the prolongation matrices are also of full rank and satisfy

$$(R_{\ell-1} \mathbf{u}_{\ell}, \mathbf{v}_{\ell-1})_{\ell-1} = (\mathbf{u}_{\ell}, P_{\ell-1} \mathbf{v}_{\ell-1})_{\ell},$$

for any  $\mathbf{u}_{\ell} \in \mathbb{R}^{n_{\ell}}$  and any  $\mathbf{v}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}$ ,

# A Family of SPD Stiffness Matrices



We assume that we have in our possession a family of (square) SPD stiffness matrices

$$\mathbf{A}_\ell \in \mathbb{R}^{n_\ell \times n_\ell}, \quad 0 \leq \ell \leq L.$$

We define the energy inner product on level  $\ell$ ,  $(\cdot, \cdot)_{\mathbf{A}_\ell}$ , via

$$(\mathbf{u}_\ell, \mathbf{v}_\ell)_{\mathbf{A}_\ell} := (\mathbf{A}_\ell \mathbf{u}_\ell, \mathbf{v}_\ell)_\ell,$$

for all  $\mathbf{u}_\ell, \mathbf{v}_\ell \in \mathbb{R}^{n_\ell}$ .



# Basic Building Blocks of Multigrid



Our goal is to build an efficient iterative solver for the equation

$$A_L \mathbf{u}_L^{\mathbb{E}} = \mathbf{f}_L,$$

given  $\mathbf{f}_L \in \mathbb{R}^{n_L}$ . As usual, we define

$$\mathbf{e}_L^{\square} := \mathbf{u}_L^{\mathbb{E}} - \mathbf{u}_L^{\square},$$

where  $\mathbf{u}_L^{\square} \in \mathbb{R}^{n_L}$  is an approximate solution. Similarly,

$$\mathbf{r}_L^{\square} := \mathbf{f}_L - A_L \mathbf{u}_L^{\square} = A_L \mathbf{e}_L^{\square}.$$



## Definition (Multigrid Operator)

Suppose  $\ell \in \{0, 1, \dots, L\}$ ,  $\mathbf{g}_\ell \in \mathbb{R}^{n_\ell}$ , and  $\mathbf{u}_\ell^{(0)} \in \mathbb{R}^{n_\ell}$  are given. The vector  $\mathbf{u}_\ell^{(3)} \in \mathbb{R}^{n_\ell}$  is computed via the recursive **multigrid operator**,

$$\mathbf{u}_\ell^{(3)} := \text{MG} \left( \mathbf{g}_\ell, \ell, \mathbf{u}_\ell^{(0)} \right), \quad (1)$$

as follows:

- If  $\ell = 0$ , then

$$\mathbf{u}_0^{(3)} := \mathbf{u}_0^{(1,E)} := \mathbf{A}_0^{-1} \mathbf{g}_0. \quad (2)$$



## Definition (Multigrid Operator Cont.)

- Otherwise, if  $1 \leq \ell \leq L$ , then

- pre-smoothing:

- $\mathbf{u}_\ell^{(1,0)} := \mathbf{u}_\ell^{(0)}$ ;
- $\mathbf{u}_\ell^{(1,\sigma+1)} := \mathbf{u}_\ell^{(1,\sigma)} + S_\ell \left( \mathbf{g}_\ell - \mathbf{A}_\ell \mathbf{u}_\ell^{(1,\sigma)} \right), \quad 0 \leq \sigma \leq m_1 - 1$ ;
- $\mathbf{u}_\ell^{(1)} := \mathbf{u}_\ell^{(1,m_1)}$ ;

- coarse grid correction:

- $\mathbf{r}_\ell^{(1)} := \mathbf{g}_\ell - \mathbf{A}_\ell \mathbf{u}_\ell^{(1)}$ ;
- $\mathbf{r}_{\ell-1}^{(1)} := \mathbf{R}_{\ell-1} \mathbf{r}_\ell^{(1)}$ ;
- $\mathbf{q}_{\ell-1}^{(1,0)} := \mathbf{0}$ ;
- $\mathbf{q}_{\ell-1}^{(1,\sigma+1)} := \text{MG} \left( \mathbf{r}_{\ell-1}^{(1)}, \ell - 1, \mathbf{q}_{\ell-1}^{(1,\sigma)} \right), \quad 0 \leq \sigma \leq p - 1$ ;
- $\mathbf{q}_{\ell-1}^{(1)} := \mathbf{q}_{\ell-1}^{(1,p)}$ ;
- $\mathbf{q}_\ell^{(1)} := \mathbf{P}_{\ell-1} \mathbf{q}_{\ell-1}^{(1)}$ ;
- $\mathbf{u}_\ell^{(2)} := \mathbf{u}_\ell^{(1)} + \mathbf{q}_\ell^{(1)}$ ;

- post-smoothing:

- $\mathbf{u}_\ell^{(3,0)} := \mathbf{u}_\ell^{(2)}$ ;
- $\mathbf{u}_\ell^{(3,\sigma+1)} := \mathbf{u}_\ell^{(3,\sigma)} + S_\ell^\top \left( \mathbf{g}_\ell - \mathbf{A}_\ell \mathbf{u}_\ell^{(3,\sigma)} \right), \quad 0 \leq \sigma \leq m_2 - 1$ ;
- and, finally,

$$\mathbf{u}_\ell^{(3)} := \mathbf{u}_\ell^{(3,m_2)}. \quad (3)$$



## Definition (Multigrid Operator Cont.)

The vector  $\mathbf{q}_{\ell-1}^{(1)} \in \mathbb{R}^{n_{\ell-1}}$  is called the **multigrid coarse grid correction**. The vector

$$\mathbf{q}_{\ell-1}^{(1,E)} := \mathbf{A}_{\ell-1}^{-1} \mathbf{r}_{\ell-1}^{(1)},$$

the **exact coarse grid correction**.



## Remark

*In the two-grid method, we found that  $\mathbf{q}_{\ell-1}^{(1,E)} = \mathbf{q}_{\ell-1}^{(1)}$ . In the multigrid method, this is no longer the case, and we can only expect that  $\mathbf{q}_{\ell-1}^{(1,E)} \approx \mathbf{q}_{\ell-1}^{(1)}$ . Much of multigrid analysis revolves around estimating the difference*

$$\mathbf{q}_{\ell-1}^{(1,E)} - \mathbf{q}_{\ell-1}^{(1)}.$$



## Definition

Let  $m_1$  and  $m_2$  be nonnegative integers and  $p$  be a positive integer. Suppose that  $\mathbf{u}_L^k \in \mathbb{R}^{n_L}$  is given. Then

$$\mathbf{u}_L^{k+1} = \text{MG} \left( \mathbf{f}_L, L, \mathbf{u}_L^k \right)$$

defines the **generic multigrid algorithm** for solving

$$A_L \mathbf{u}_L^{\text{E}} = \mathbf{f}_L.$$



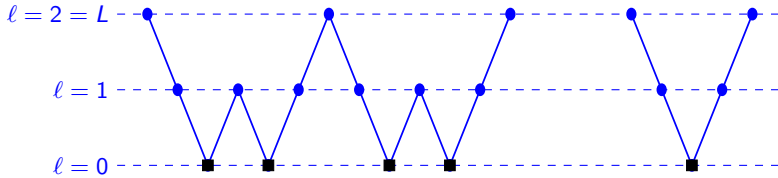
## Definition (Variants of the Multigrid Method)

A multigrid algorithm is called **one-sided** iff  $m_2 = 0$  and  $m_1 \geq 1$ .

The algorithm is called a **W-cycle** iff  $p \geq 2$  and is called a **V-cycle** iff  $p = 1$ .

The algorithm is called **symmetrized** iff  $m_1 = m_2 = m$ . We say that the algorithm is a **simple symmetric V-cycle** iff  $p = 1$  and  $m_1 = m_2 = 1$ .

See the figure on the next page to obtain a more geometric understand of the above definition.



**Figure:** One full  $p = 2$  W-cycle (left) and one full V-cycle (right) for three levels,  $L = 2$ .





## Remark

*In the case  $L = 0$ , we just do a direct solve. When  $L = 1$ , we recover the two-grid (two-level) method.*



We now want to find the error transfer matrix for the general multigrid algorithm.

## Definition

Let  $\ell \geq 0$ . Define, for  $\ell = 0$ ,

$$E_0 := O_{n_0} \in \mathbb{R}^{n_0 \times n_0},$$

where  $O_{n_0} \in \mathbb{R}^{n_0 \times n_0}$  is the zero matrix, and, for  $\ell \geq 1$ ,

$$E_\ell := (K_\ell^*)^{m_2} (I_\ell - P_{\ell-1} (I_{\ell-1} - E_{\ell-1}^P) \Pi_{\ell-1}) K_\ell^{m_1} \in \mathbb{R}^{n_\ell \times n_\ell}, \quad (4)$$

where

$$\left. \begin{aligned} K_\ell &:= I_\ell - S_\ell A_\ell \\ K_\ell^* &= I_\ell - S_\ell^\top A_\ell \end{aligned} \right\} \in \mathbb{R}^{n_\ell \times n_\ell},$$

and

$$\Pi_{\ell-1} := A_{\ell-1}^{-1} R_{\ell-1} A_\ell \in \mathbb{R}^{n_{\ell-1} \times n_\ell}.$$

For further use, let us also define

$$\tilde{\Pi}_\ell := P_{\ell-1} \Pi_{\ell-1} \in \mathbb{R}^{n_\ell \times n_\ell}.$$



## Remark

Observe that, for any  $\mathbf{u}_\ell, \mathbf{v}_\ell \in \mathbb{R}^{n_\ell}$ ,

$$(\mathbf{K}_\ell \mathbf{u}_\ell, \mathbf{v}_\ell)_{A_\ell} = (\mathbf{u}_\ell, \mathbf{K}_\ell^* \mathbf{v}_\ell)_{A_\ell}.$$

More generally, we use  $\mathbf{B}^*$  to denote the adjoint of  $\mathbf{B} \in \mathbb{R}^{n_\ell \times n_\ell}$  with respect to the inner product  $(\cdot, \cdot)_{A_\ell}$ .



## Theorem (Multigrid Error Relation)

Suppose that  $\mathbf{u}_\ell^{\text{E}}, \mathbf{g}_\ell \in \mathbb{R}^{n_\ell}$  satisfy

$$A_\ell \mathbf{u}_\ell^{\text{E}} = \mathbf{g}_\ell.$$

Then, given any  $\mathbf{u}_\ell^{(0)} \in \mathbb{R}^{n_\ell}$ ,

$$\mathbf{u}_\ell^{\text{E}} - \text{MG}(\mathbf{g}_\ell, \ell, \mathbf{u}_\ell^{(0)}) = \mathbf{E}_\ell(\mathbf{u}_\ell^{\text{E}} - \mathbf{u}_\ell^{(0)}),$$

where  $\mathbf{E}_\ell$  is the recursively-defined matrix in equation (4). In particular

$$\mathbf{e}_L^{k+1} = \mathbf{E}_L \mathbf{e}_L^k.$$



## Proof.

The proof is by induction. We will use the notation from the definition of the multigrid operator throughout.

(Base cases): Cases  $\ell = 0$  and  $\ell = 1$  (two-grid method) are clear.

(Induction hypothesis): Assume that the result is true for level  $\ell - 1$ . In other words, suppose that  $\mathbf{u}_{\ell-1}^E, \mathbf{g}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}$  satisfy

$$A_{\ell-1} \mathbf{u}_{\ell-1}^E = \mathbf{g}_{\ell-1}.$$

Then, given any initial vector  $\mathbf{u}_{\ell-1}^{(0)} \in \mathbb{R}^{n_{\ell-1}}$ ,

$$\mathbf{u}_{\ell-1}^E - \text{MG}(\mathbf{g}_{\ell-1}, \ell - 1, \mathbf{u}_{\ell-1}^{(0)}) = \mathbf{E}_{\ell-1}(\mathbf{u}_{\ell-1}^E - \mathbf{u}_{\ell-1}^{(0)}).$$



## Proof (Cont.)

(Generic case): Suppose that  $\mathbf{q}_{\ell-1}^{(1,E)}, \mathbf{r}_{\ell-1}^{(1)} \in \mathbb{R}^{n_{\ell-1}}$  satisfy

$$A_{\ell-1} \mathbf{q}_{\ell-1}^{(1,E)} = \mathbf{r}_{\ell-1}^{(1)}.$$

Then, using the induction hypothesis,

$$\mathbf{q}_{\ell-1}^{(1,E)} - \text{MG} \left( \mathbf{r}_{\ell-1}^{(1)}, \ell - 1, \mathbf{0} \right) = E_{\ell-1} \left( \mathbf{q}_{\ell-1}^{(1,E)} - \mathbf{0} \right) = E_{\ell-1} \mathbf{q}_{\ell-1}^{(1,E)}.$$

Written in the notation defining the multigrid operator,

$$\mathbf{q}_{\ell-1}^{(1,E)} - \mathbf{q}_{\ell-1}^{(1,1)} = E_{\ell-1} \mathbf{q}_{\ell-1}^{(1,E)}.$$

Applying the induction hypothesis again,

$$\begin{aligned} \mathbf{q}_{\ell-1}^{(1,E)} - \mathbf{q}_{\ell-1}^{(1,2)} &= \mathbf{q}_{\ell-1}^{(1,E)} - \text{MG} \left( \mathbf{r}_{\ell-1}^{(1)}, \ell - 1, \mathbf{q}_{\ell-1}^{(1,1)} \right) \\ &= E_{\ell-1} \left( \mathbf{q}_{\ell-1}^{(1,E)} - \mathbf{q}_{\ell-1}^{(1,1)} \right) \\ &= E_{\ell-1}^2 \mathbf{q}_{\ell-1}^{(1,E)}. \end{aligned}$$



## Proof (Cont.)

Continuing in this fashion, for any  $p \in \mathbb{N}$ ,

$$\mathbf{q}_{\ell-1}^{(1,E)} - \mathbf{q}_{\ell-1}^{(1,p)} = \mathbb{E}_{\ell-1}^p \mathbf{q}_{\ell-1}^{(1,E)}.$$

Consequently,

$$\mathbf{q}_{\ell-1}^{(1,E)} - \mathbf{q}_{\ell-1}^{(1)} = \mathbb{E}_{\ell-1}^p \mathbf{q}_{\ell-1}^{(1,E)}.$$

Solving for  $\mathbf{q}_{\ell-1}^{(1)}$ , we have

$$\mathbf{q}_{\ell-1}^{(1)} = (\mathbf{I}_{\ell-1} - \mathbb{E}_{\ell-1}^p) \mathbf{q}_{\ell-1}^{(1,E)}.$$

Now,

$$\begin{aligned} \mathbf{q}_{\ell-1}^{(1,E)} &:= \mathbf{A}_{\ell-1}^{-1} \mathbf{r}_{\ell-1}^{(1)} \\ &= \mathbf{A}_{\ell-1}^{-1} \mathbf{R}_{\ell-1} \left( \mathbf{g}_{\ell} - \mathbf{A}_{\ell} \mathbf{u}_{\ell}^{(1)} \right) \\ &= \mathbf{A}_{\ell-1}^{-1} \mathbf{R}_{\ell-1} \mathbf{A}_{\ell} \left( \mathbf{u}_{\ell}^E - \mathbf{u}_{\ell}^{(1)} \right) \\ &= \Pi_{\ell-1} \left( \mathbf{u}_{\ell}^E - \mathbf{u}_{\ell}^{(1)} \right). \end{aligned} \tag{5}$$



## Proof (Cont.)

Hence

$$\mathbf{q}_{\ell-1}^{(1)} = (\mathbf{I}_{\ell-1} - \mathbf{E}_{\ell-1}^P) \Pi_{\ell-1} \left( \mathbf{u}_{\ell}^E - \mathbf{u}_{\ell}^{(1)} \right).$$

For the error after post-smoothing,

$$\mathbf{u}_{\ell}^E - \mathbf{u}_{\ell}^{(3)} = (\mathbf{K}_{\ell}^*)^{m_2} \left( \mathbf{u}_{\ell}^E - \mathbf{u}_{\ell}^{(2)} \right);$$

for the error after coarse grid correction,

$$\begin{aligned} \mathbf{u}_{\ell}^E - \mathbf{u}_{\ell}^{(2)} &= \mathbf{u}_{\ell}^E - \left( \mathbf{u}_{\ell}^{(1)} + \mathbf{P}_{\ell-1} \mathbf{q}_{\ell-1}^{(1)} \right) \\ &= \mathbf{u}_{\ell}^E - \mathbf{u}_{\ell}^{(1)} - \mathbf{P}_{\ell-1} (\mathbf{I}_{\ell-1} - \mathbf{E}_{\ell-1}^P) \Pi_{\ell-1} \left( \mathbf{u}_{\ell}^E - \mathbf{u}_{\ell}^{(1)} \right) \\ &= (\mathbf{I}_{\ell} - \mathbf{P}_{\ell-1} (\mathbf{I}_{\ell-1} - \mathbf{E}_{\ell-1}^P) \Pi_{\ell-1}) \left( \mathbf{u}_{\ell}^E - \mathbf{u}_{\ell}^{(1)} \right); \end{aligned}$$

and, for the error after pre-smoothing,

$$\mathbf{u}_{\ell}^E - \mathbf{u}_{\ell}^{(1)} = \mathbf{K}_{\ell}^{m_1} \left( \mathbf{u}_{\ell}^E - \mathbf{u}_{\ell}^{(0)} \right).$$





## Proof (Cont.)

Putting it all together, we have

$$\mathbf{u}_\ell^{\text{E}} - \mathbf{u}_\ell^{(3)} \stackrel{(4)}{=} \mathbf{E}_\ell \left( \mathbf{u}_\ell^{\text{E}} - \mathbf{u}_\ell^{(0)} \right).$$





## Corollary

For every  $0 \leq \ell \leq L$ , there is a (recursively defined) matrix  $B_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$ , such that,

$$E_\ell = I_\ell - B_\ell A_\ell$$

and

$$\text{MG}(\mathbf{g}_\ell, \ell, \mathbf{z}_\ell) = \mathbf{z}_\ell + B_\ell (\mathbf{g}_\ell - A_\ell \mathbf{z}_\ell).$$

In other words, the multigrid method is a GLIS.

## Proof.

Exercise.





# Galerkin Conditions



## Definition (Assumptions (G0) and (G1))

We say that the stiffness matrices satisfy the **strong Galerkin condition**, equivalently, **Assumption (G0)** holds, iff

$$A_{\ell-1} = R_{\ell-1} A_{\ell} P_{\ell-1}, \quad 1 \leq \ell \leq L. \quad (6)$$

We say that the **weak Galerkin condition** holds, equivalently, **Assumption (G1)** holds, iff

$$(\mathbf{v}_{\ell}, \mathbf{v}_{\ell})_{A_{\ell}} \geq (\Pi_{\ell-1} \mathbf{v}_{\ell}, \Pi_{\ell-1} \mathbf{v}_{\ell})_{A_{\ell-1}}, \quad (7)$$

for all  $\mathbf{v}_{\ell} \in \mathbb{R}^{n_{\ell}}$  and all  $1 \leq \ell \leq L$ .



We will show that  $(G0) \implies (G1)$ . To do this, we first need the following result.

### Lemma

*If Assumption (G0, strong Galerkin condition) holds then*

$$\tilde{\Pi}_\ell = \tilde{\Pi}_\ell^2 \tag{8}$$

*But  $\tilde{\Pi}_\ell^* = \tilde{\Pi}_\ell$  holds even without Assumption (G0).*



## Proof.

If the strong Galerkin condition holds then

$$A_{\ell-1} = R_{\ell-1} A_{\ell} P_{\ell-1}, \quad 1 \leq \ell \leq L.$$

Recall that

$$\tilde{\Pi}_{\ell} = P_{\ell-1} A_{\ell-1}^{-1} R_{\ell-1} A_{\ell}, \quad 1 \leq \ell \leq L.$$

Consequently,

$$\begin{aligned} \tilde{\Pi}_{\ell}^2 &= P_{\ell-1} A_{\ell-1}^{-1} R_{\ell-1} A_{\ell} P_{\ell-1} A_{\ell-1}^{-1} R_{\ell-1} A_{\ell} \\ &= P_{\ell-1} A_{\ell-1}^{-1} (R_{\ell-1} A_{\ell} P_{\ell-1}) A_{\ell-1}^{-1} R_{\ell-1} A_{\ell} \\ &= P_{\ell-1} A_{\ell-1}^{-1} A_{\ell-1} A_{\ell-1}^{-1} R_{\ell-1} A_{\ell} \\ &= P_{\ell-1} A_{\ell-1}^{-1} R_{\ell-1} A_{\ell} \\ &= \tilde{\Pi}_{\ell}. \end{aligned}$$



## Proof (Cont.)

Next, let  $\mathbf{u}_\ell, \mathbf{v}_\ell \in \mathbb{R}^{n_\ell}$  be arbitrary. Then using the definitions of  $\Pi_{\ell-1}$  and  $\tilde{\Pi}_\ell$ ,

$$\begin{aligned} \left( \tilde{\Pi}_\ell \mathbf{u}_\ell, \mathbf{v}_\ell \right)_{A_\ell} &= \left( A_\ell \tilde{\Pi}_\ell \mathbf{u}_\ell, \mathbf{v}_\ell \right)_\ell \\ &= \left( A_\ell P_{\ell-1} A_{\ell-1}^{-1} R_{\ell-1} A_\ell \mathbf{u}_\ell, \mathbf{v}_\ell \right)_\ell \\ &= \left( \left( A_\ell P_{\ell-1} A_{\ell-1}^{-1} R_{\ell-1} \right) A_\ell \mathbf{u}_\ell, \mathbf{v}_\ell \right)_\ell \\ &= \left( A_\ell \mathbf{u}_\ell, \left( A_\ell P_{\ell-1} A_{\ell-1}^{-1} R_{\ell-1} \right)^\top \mathbf{v}_\ell \right)_\ell \\ &= \left( A_\ell \mathbf{u}_\ell, \tilde{\Pi}_\ell \mathbf{v}_\ell \right)_\ell \\ &= \left( \mathbf{u}_\ell, \tilde{\Pi}_\ell \mathbf{v}_\ell \right)_{A_\ell}. \end{aligned}$$

So,

$$\tilde{\Pi}_\ell^* = \tilde{\Pi}_\ell.$$





## Corollary

*It always holds that*

$$\left(I_\ell - \tilde{\Pi}_\ell\right)^* = I_\ell - \tilde{\Pi}_\ell. \quad (9)$$

*If Assumption (G0) holds, then*

$$\left(I_\ell - \tilde{\Pi}_\ell\right)^2 = I_\ell - \tilde{\Pi}_\ell. \quad (10)$$





## Lemma $((G0) \implies (G1))$

*Assumption (G0) implies Assumption (G1).*

### Proof.

First, for any  $\mathbf{v}_\ell \in \mathbb{R}^{n_\ell}$ , consider

$$\begin{aligned}
(\Pi_{\ell-1} \mathbf{v}_\ell, \Pi_{\ell-1} \mathbf{v}_\ell)_{A_{\ell-1}} &= (\Pi_{\ell-1} \mathbf{v}_\ell, A_{\ell-1} \Pi_{\ell-1} \mathbf{v}_\ell)_{\ell-1} \\
&= \left( \Pi_{\ell-1} \mathbf{v}_\ell, A_{\ell-1} A_{\ell-1}^{-1} R_{\ell-1} A_\ell \mathbf{v}_\ell \right)_{\ell-1} \\
&= (\Pi_{\ell-1} \mathbf{v}_\ell, R_{\ell-1} A_\ell \mathbf{v}_\ell)_{\ell-1} \\
&= \left( R_{\ell-1}^\top \Pi_{\ell-1} \mathbf{v}_\ell, A_\ell \mathbf{v}_\ell \right)_\ell \\
&= (P_{\ell-1} \Pi_{\ell-1} \mathbf{v}_\ell, A_\ell \mathbf{v}_\ell)_\ell \\
&= (P_{\ell-1} \Pi_{\ell-1} \mathbf{v}_\ell, \mathbf{v}_\ell)_{A_\ell} \\
&= \left( \tilde{\Pi}_\ell \mathbf{v}_\ell, \mathbf{v}_\ell \right)_{A_\ell}.
\end{aligned}$$



## Proof (Cont.)

Using the last calculation

$$\begin{aligned}(\mathbf{v}_\ell, \mathbf{v}_\ell)_{A_\ell} - (\Pi_{\ell-1} \mathbf{v}_\ell, \Pi_{\ell-1} \mathbf{v}_\ell)_{A_{\ell-1}} &= (\mathbf{v}_\ell, \mathbf{v}_\ell)_{A_\ell} - (\tilde{\Pi}_\ell \mathbf{v}_\ell, \mathbf{v}_\ell)_{A_\ell} \\&= \left( (I_\ell - \tilde{\Pi}_\ell) \mathbf{v}_\ell, \mathbf{v}_\ell \right)_{A_\ell} \\&\stackrel{(10)}{=} \left( (I_\ell - \tilde{\Pi}_\ell)^2 \mathbf{v}_\ell, \mathbf{v}_\ell \right)_{A_\ell} \\&\stackrel{(9)}{=} \left( (I_\ell - \tilde{\Pi}_\ell) \mathbf{v}_\ell, (I_\ell - \tilde{\Pi}_\ell) \mathbf{v}_\ell \right)_{A_\ell} \\&= \left\| (I_\ell - \tilde{\Pi}_\ell) \mathbf{v}_\ell \right\|_{A_\ell}^2 \\&\geq 0.\end{aligned}$$

Thus (G0) implies (G1). □



### Definition (Assumption (G2))

We say that **Assumption (G2)** holds iff, for all  $\mathbf{u}_\ell \in \mathbb{R}^{n_\ell}$  and all  $1 \leq \ell \leq L$ ,

$$\left( \left( \mathbf{I}_\ell - \tilde{\mathbf{N}}_\ell \right) \mathbf{u}_\ell, \mathbf{u}_\ell \right)_{\mathbf{A}_\ell} \geq 0. \quad (11)$$



## Corollary

*Assumption (G1) is equivalent to Assumption (G2).*

## Proof.

We showed in the proof of the last lemma, using only the definitions of  $\Pi_{\ell-1}$  and  $\tilde{\Pi}_\ell$ , that

$$(\mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} - (\Pi_{\ell-1} \mathbf{u}_\ell, \Pi_{\ell-1} \mathbf{u}_\ell)_{A_{\ell-1}} = \left( (I_\ell - \tilde{\Pi}_\ell) \mathbf{u}_\ell, \mathbf{u}_\ell \right)_{A_\ell}.$$

So (G1) holds iff (G2) holds. □



## Theorem

*If  $m_1 = m_2 = m$ , then, for all  $0 \leq \ell \leq L$*

$$E_\ell = E_\ell^*.$$

*If, in addition, Assumption (G1), holds, or, equivalently, Assumption (G2) holds, then*

$$(E_\ell \mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} \geq 0, \quad \forall \mathbf{u}_\ell \in \mathbb{R}^{n_\ell}.$$



## Proof.

The proof is by induction.

(Base cases): The case  $\ell = 0$  is trivial. For  $\ell = 1$  (the two-grid algorithm) the proof follows as in a previous chapter.

(Induction hypothesis): Assume that

$$(E_{\ell-1} \mathbf{u}_{\ell-1}, \mathbf{v}_{\ell-1})_{A_{\ell-1}} = (\mathbf{u}_{\ell-1}, E_{\ell-1} \mathbf{v}_{\ell-1})_{A_{\ell-1}},$$

and

$$(E_{\ell-1} \mathbf{u}_{\ell-1}, \mathbf{u}_{\ell-1})_{A_{\ell-1}} \geq 0$$

for all  $\mathbf{u}_{\ell-1}, \mathbf{v}_{\ell-1} \in \mathbb{R}^{n_\ell}$ .



## Proof (Cont.)

(General case): We will make use of the definitions of  $\Pi_{\ell-1}$  and  $\tilde{\Pi}_\ell$ , a number of times:

$$\Pi_{\ell-1} = A_{\ell-1}^{-1} R_{\ell-1} A_\ell.$$

So

$$A_{\ell-1} \Pi_{\ell-1} = R_{\ell-1} A_\ell.$$

And

$$\tilde{\Pi}_\ell = P_{\ell-1} \Pi_{\ell-1} = P_{\ell-1} A_{\ell-1}^{-1} R_{\ell-1} A_\ell.$$



## Proof (Cont.)

Thus, we have

$$\begin{aligned}
 & (E_\ell \mathbf{u}_\ell, \mathbf{v}_\ell)_{A_\ell} \\
 &= ((K_\ell^*)^m (I_\ell - P_{\ell-1} (I_{\ell-1} - E_{\ell-1}^P) \Pi_{\ell-1}) K_\ell^m \mathbf{u}_\ell, \mathbf{v}_\ell)_{A_\ell} \\
 &= ((I_\ell - P_{\ell-1} (I_{\ell-1} - E_{\ell-1}^P) \Pi_{\ell-1}) K_\ell^m \mathbf{u}_\ell, K_\ell^m \mathbf{v}_\ell)_{A_\ell} \\
 &= ((I_\ell - P_{\ell-1} \Pi_{\ell-1} + P_{\ell-1} E_{\ell-1}^P \Pi_{\ell-1}) K_\ell^m \mathbf{u}_\ell, K_\ell^m \mathbf{v}_\ell)_{A_\ell} \\
 &= ((I_\ell - P_{\ell-1} \Pi_{\ell-1}) K_\ell^m \mathbf{u}_\ell, K_\ell^m \mathbf{v}_\ell)_{A_\ell} + (P_{\ell-1} E_{\ell-1}^P \Pi_{\ell-1} K_\ell^m \mathbf{u}_\ell, K_\ell^m \mathbf{v}_\ell)_{A_\ell} \\
 &= \left( (I_\ell - \tilde{\Pi}_\ell) K_\ell^m \mathbf{u}_\ell, K_\ell^m \mathbf{v}_\ell \right)_{A_\ell} + (P_{\ell-1} E_{\ell-1}^P \Pi_{\ell-1} K_\ell^m \mathbf{u}_\ell, K_\ell^m \mathbf{v}_\ell)_{A_\ell} \\
 &= \left( K_\ell^m \mathbf{u}_\ell, (I_\ell - \tilde{\Pi}_\ell) K_\ell^m \mathbf{v}_\ell \right)_{A_\ell} + (P_{\ell-1} E_{\ell-1}^P \Pi_{\ell-1} K_\ell^m \mathbf{u}_\ell, A_\ell K_\ell^m \mathbf{v}_\ell)_\ell \\
 &= \left( K_\ell^m \mathbf{u}_\ell, (I_\ell - \tilde{\Pi}_\ell) K_\ell^m \mathbf{v}_\ell \right)_{A_\ell} + (E_{\ell-1}^P \Pi_{\ell-1} K_\ell^m \mathbf{u}_\ell, R_{\ell-1} A_\ell K_\ell^m \mathbf{v}_\ell)_{\ell-1} \\
 &= \left( K_\ell^m \mathbf{u}_\ell, (I_\ell - \tilde{\Pi}_\ell) K_\ell^m \mathbf{v}_\ell \right)_{A_\ell} + (E_{\ell-1}^P \Pi_{\ell-1} K_\ell^m \mathbf{u}_\ell, A_{\ell-1} \Pi_{\ell-1} K_\ell^m \mathbf{v}_\ell)_{\ell-1}
 \end{aligned}$$





## Proof (Cont.)

$$\begin{aligned}
 &= \left( \mathbf{K}_\ell^m \mathbf{u}_\ell, \left( \mathbf{I}_\ell - \tilde{\Pi}_\ell \right) \mathbf{K}_\ell^m \mathbf{v}_\ell \right)_{\mathbf{A}_\ell} + \left( \mathbf{E}_{\ell-1}^p \Pi_{\ell-1} \mathbf{K}_\ell^m \mathbf{u}_\ell, \Pi_{\ell-1} \mathbf{K}_\ell^m \mathbf{v}_\ell \right)_{\mathbf{A}_{\ell-1}} \\
 &= \left( \mathbf{K}_\ell^m \mathbf{u}_\ell, \left( \mathbf{I}_\ell - \tilde{\Pi}_\ell \right) \mathbf{K}_\ell^m \mathbf{v}_\ell \right)_{\mathbf{A}_\ell} + \left( \Pi_{\ell-1} \mathbf{K}_\ell^m \mathbf{u}_\ell, \mathbf{E}_{\ell-1}^p \Pi_{\ell-1} \mathbf{K}_\ell^m \mathbf{v}_\ell \right)_{\mathbf{A}_{\ell-1}} \\
 &= \left( \mathbf{K}_\ell^m \mathbf{u}_\ell, \left( \mathbf{I}_\ell - \tilde{\Pi}_\ell \right) \mathbf{K}_\ell^m \mathbf{v}_\ell \right)_{\mathbf{A}_\ell} + \left( \mathbf{A}_{\ell-1} \Pi_{\ell-1} \mathbf{K}_\ell^m \mathbf{u}_\ell, \mathbf{E}_{\ell-1}^p \Pi_{\ell-1} \mathbf{K}_\ell^m \mathbf{v}_\ell \right)_{\ell-1} \\
 &= \left( \mathbf{K}_\ell^m \mathbf{u}_\ell, \left( \mathbf{I}_\ell - \tilde{\Pi}_\ell \right) \mathbf{K}_\ell^m \mathbf{v}_\ell \right)_{\mathbf{A}_\ell} + \left( \mathbf{R}_{\ell-1} \mathbf{A}_\ell \mathbf{K}_\ell^m \mathbf{u}_\ell, \mathbf{E}_{\ell-1}^p \Pi_{\ell-1} \mathbf{K}_\ell^m \mathbf{v}_\ell \right)_{\ell-1} \\
 &= \left( \mathbf{K}_\ell^m \mathbf{u}_\ell, \left( \mathbf{I}_\ell - \tilde{\Pi}_\ell \right) \mathbf{K}_\ell^m \mathbf{v}_\ell \right)_{\mathbf{A}_\ell} + \left( \mathbf{A}_\ell \mathbf{K}_\ell^m \mathbf{u}_\ell, \mathbf{P}_{\ell-1} \mathbf{E}_{\ell-1}^p \Pi_{\ell-1} \mathbf{K}_\ell^m \mathbf{v}_\ell \right)_\ell \\
 &= \left( \mathbf{K}_\ell^m \mathbf{u}_\ell, \left( \mathbf{I}_\ell - \tilde{\Pi}_\ell \right) \mathbf{K}_\ell^m \mathbf{v}_\ell \right)_{\mathbf{A}_\ell} + \left( \mathbf{K}_\ell^m \mathbf{u}_\ell, \mathbf{P}_{\ell-1} \mathbf{E}_{\ell-1}^p \Pi_{\ell-1} \mathbf{K}_\ell^m \mathbf{v}_\ell \right)_{\mathbf{A}_\ell} \\
 &= \left( \mathbf{u}_\ell, \mathbf{E}_\ell \mathbf{v}_\ell \right)_{\mathbf{A}_\ell} .
 \end{aligned}$$

Hence, symmetry is proven.



## Proof (Cont.)

Notice we have not yet used Assumption (G2), only the definitions of  $\Pi_{\ell-1}$  and  $\tilde{\Pi}_\ell$ , which are assumed to always hold.

Now, we setting  $\mathbf{v}_\ell = \mathbf{u}_\ell$  in the last calculation, we have

$$\begin{aligned}
 (\mathbf{E}_\ell \mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} &= \underbrace{\left( (\mathbf{I}_\ell - \tilde{\Pi}_\ell) \mathbf{K}_\ell^m \mathbf{u}_\ell, \mathbf{K}_\ell^m \mathbf{u}_\ell \right)_{A_\ell}}_{\geq 0, \text{ Assumption (G2)}} \\
 &\quad + \underbrace{\left( \mathbf{E}_{\ell-1}^p \Pi_{\ell-1} \mathbf{K}_\ell^m \mathbf{u}_\ell, \Pi_{\ell-1} \mathbf{K}_\ell^m \mathbf{u}_\ell \right)_{A_{\ell-1}}}_{\geq 0, \text{ induction hypothesis}} \\
 &\geq 0,
 \end{aligned}$$

for any  $\mathbf{u}_\ell \in \mathbb{R}^{n_\ell}$ . The result is proven. □



## Remark

*In the proof above we used the fact that any positive integer power of a self-adjoint positive semi-definite matrix is also positive semi-definite. This is easy to prove and is left as an exercise.*



## Corollary

*If  $m_1 = m_2 = m$ , then, for all  $0 \leq \ell \leq L$ ,*

$$\mathbf{B}_\ell = \mathbf{B}_\ell^\top,$$

*where  $\mathbf{B}_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$  is the multigrid iterator matrix.*

Proof.

Exercise. □



# Strong and Weak Approximation Properties



## Definition (Assumptions (A0), (A1), and (A2))

We say that the multigrid algorithm satisfies the **strong approximation property**, equivalently, **Assumption (A0)**, iff, for all  $\mathbf{u}_\ell \in \mathbb{R}^{n_\ell}$  and  $1 \leq \ell \leq L$ ,

$$\left\| \left( \mathbf{I}_\ell - \tilde{\mathbf{P}}_\ell \right) \mathbf{u}_\ell \right\|_\ell^2 \leq \frac{C_{A0}^2}{\rho_\ell} \left\| \left( \mathbf{I}_\ell - \tilde{\mathbf{P}}_\ell \right) \mathbf{u}_\ell \right\|_{A_\ell}^2, \quad (12)$$

for some  $C_{A0} > 0$  that is independent of  $\ell$ , where  $\rho_\ell = \rho(A_\ell)$ . The multigrid algorithm satisfies the **weak approximation property**, equivalently,

**Assumption (A1)**, iff, for all  $\mathbf{u}_\ell \in \mathbb{R}^{n_\ell}$  and  $1 \leq \ell \leq L$ ,

$$\left( \left( \mathbf{I}_\ell - \tilde{\mathbf{P}}_\ell \right) \mathbf{u}_\ell, \mathbf{u}_\ell \right)_{A_\ell} \leq \frac{C_{A1}^2}{\rho_\ell} \|A_\ell \mathbf{u}_\ell\|_\ell^2, \quad (13)$$

for some  $C_{A1} > 0$  that is independent of  $\ell$ . The multigrid algorithm satisfies the  $\alpha$ -**weak approximation property**, equivalently, **Assumption (A2)**, iff there is a constant  $\alpha \in (0, 1]$ , such that, for all  $\mathbf{u}_\ell \in \mathbb{R}^{n_\ell}$  and  $1 \leq \ell \leq L$ ,

$$\left( \left( \mathbf{I}_\ell - \tilde{\mathbf{P}}_\ell \right) \mathbf{u}_\ell, \mathbf{u}_\ell \right)_{A_\ell} \leq \frac{C_{A2}^{2\alpha}}{\rho_\ell^\alpha} \|A_\ell \mathbf{u}_\ell\|_\ell^{2\alpha} \|\mathbf{u}_\ell\|_{A_\ell}^{2(1-\alpha)}, \quad (14)$$

for some  $C_{A2} > 0$  that is independent of  $\ell$ .



## Theorem

*If the strong Galerkin condition, Assumption (G0), holds, then (A0) implies (A1).*



## Proof.

Since Assumption (G0) holds

$$\left( I_\ell - \tilde{\Pi}_\ell \right)^2 \stackrel{(10)}{=} I_\ell - \tilde{\Pi}_\ell.$$

Therefore,

$$\begin{aligned}
\left\| \left( I_\ell - \tilde{\Pi}_\ell \right) \mathbf{u}_\ell \right\|_{A_\ell}^2 &= \left( \left( I_\ell - \tilde{\Pi}_\ell \right) \mathbf{u}_\ell, \left( I_\ell - \tilde{\Pi}_\ell \right) \mathbf{u}_\ell \right)_{A_\ell} \\
&= \left( \left( I_\ell - \tilde{\Pi}_\ell \right)^2 \mathbf{u}_\ell, \mathbf{u}_\ell \right)_{A_\ell} \\
&= \left( \left( I_\ell - \tilde{\Pi}_\ell \right) \mathbf{u}_\ell, \mathbf{u}_\ell \right)_{A_\ell} \\
&= \left( \left( I_\ell - \tilde{\Pi}_\ell \right) \mathbf{u}_\ell, A_\ell \mathbf{u}_\ell \right)_\ell \\
&\stackrel{\text{C.S.}}{\leq} \left\| \left( I_\ell - \tilde{\Pi}_\ell \right) \mathbf{u}_\ell \right\|_\ell \|A_\ell \mathbf{u}_\ell\|_\ell \\
&\stackrel{(A0)}{\leq} C_{A0} \rho_\ell^{-1/2} \left\| \left( I_\ell - \tilde{\Pi}_\ell \right) \mathbf{u}_\ell \right\|_{A_\ell} \|A_\ell \mathbf{u}_\ell\|_\ell.
\end{aligned}$$





## Proof (Cont.)

Thus

$$\left\| \left( \mathbf{I}_\ell - \tilde{\mathbf{P}}_\ell \right) \mathbf{u}_\ell \right\|_{\mathbf{A}_\ell} \leq C_{A0} \rho_\ell^{-1/2} \|\mathbf{A}_\ell \mathbf{u}_\ell\|_\ell. \quad (15)$$

Squaring, we have

$$\left( \left( \mathbf{I}_\ell - \tilde{\mathbf{P}}_\ell \right) \mathbf{u}_\ell, \mathbf{u}_\ell \right)_{\mathbf{A}_\ell} \leq C_{A0}^2 \rho_\ell^{-1} \|\mathbf{A}_\ell \mathbf{u}_\ell\|_\ell^2,$$

which is the desired result with  $C_{A1} = C_{A0}$ . □



# The First Smoothing Property



## Definition (Richardson's Smoother)

Suppose that  $\mathbf{u}_\ell^{\text{E}}, \mathbf{g}_\ell \in \mathbb{R}^{n_\ell}$  satisfy

$$\mathbf{A}_\ell \mathbf{u}_\ell^{\text{E}} = \mathbf{g}_\ell.$$

**Richardson's method** for approximating  $\mathbf{u}_\ell^{\text{E}}$  is the GLIS defined via

$$\mathbf{u}_\ell^{(\sigma+1)} = \mathbf{u}_\ell^{(\sigma)} + \omega \left( \mathbf{g}_\ell - \mathbf{A}_\ell \mathbf{u}_\ell^{(\sigma)} \right),$$

where  $\omega > 0$  is a parameter. Its error transfer matrix is

$$\mathbf{K}_\ell = \mathbf{I}_\ell - \omega \mathbf{A}_\ell = \mathbf{K}_\ell^*.$$



## Definition (Richardson's Smoother Cont.)

Suppose that  $\Lambda_\ell > 0$  is a number satisfying

$$C_R \rho_\ell \geq \Lambda_\ell \geq \rho_\ell = \rho(A_\ell), \quad 1 \leq \ell \leq L, \quad (16)$$

for some  $C_R \geq 1$  that is independent of  $\ell$ , where  $\rho(A_\ell)$  is the spectral radius of  $A_\ell$ . Choosing

$$\omega := \frac{1}{\Lambda_\ell},$$

we obtain **Richardson's smoother**, and in this case the error transfer matrix is

$$K_\ell = I_\ell - \Lambda_\ell^{-1} A_\ell = K_\ell^*.$$



## Definition (Assumption (S1))

We say that the multigrid algorithm satisfies the **first smoothing property**, equivalently, **Assumption (S1)**, iff

$$\|K_\ell^m \mathbf{u}_\ell\|_{A_\ell^2} \leq C_{S1} \rho_\ell^{1/2} m^{-1/2} \|\mathbf{u}_\ell\|_{A_\ell}, \quad (17)$$

for all  $\mathbf{u}_\ell \in \mathbb{R}^{n_\ell}$  and  $1 \leq \ell \leq L$ , for some  $C_{S1} > 0$  that is independent of  $\ell$ .



## Theorem

*Richardson's smoother satisfies the first smoothing property, Assumption (S1).*

## Proof.

The proof is similar to the proof of the two-grid version.

