Math 673/4

Multigrid Methods: A Mostly Matrix-Based Approach

Chapter 07: Cell-Centered Finite Difference Methods and Multigrid

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Chapter 07, Part 1 of 3

Cell-Centered Finite Difference Methods and Multigrid

Objectives

In this chapter we will develop the cell-centered finite difference method to approximate the solution to the standard two point boundary value problem,

$$\begin{cases} -u'' &= f, & \text{in} & \Omega = (0,1), \\ u &= 0, & \text{on} & \partial\Omega = \{0,1\} \end{cases} \tag{1}$$

and its two dimensional analogue.

We will describe two-grid and multigrid methods to find approximations in an efficient way. The reader will observe that the techniques for the analysis of the resulting iterative methods will be similar to those for the finite element approximations. The details, however, are different and require some careful attention.

Discretization in One Dimension

Grid Structure

We begin by creating multilevel grids that cover the interval $\Omega = (0, 1)$. The construction, however, is different from that of earlier chapters. For $\ell = 0, 1, 2, \dots, L$, we define the number of grid points and the uniform grid spacing by

$$n_\ell := q \cdot 2^\ell, \qquad h_\ell := \frac{1}{n_\ell},$$

where $q \in \mathbb{N}$. The first notable difference from before is that, for $1 \leq \ell \leq L$, the grids we construct here will have an even number of degrees of freedom n_ℓ. The level-0 grid has size $n_0 = q$, which may be even or odd, as desired by the analyst.

Definition (Cell-Centered Grid)

Let $0 \leq \ell \leq L$. The **cell-centered grid points** are defined via

$$x_{\ell,i} := \left(i - \frac{1}{2}\right) h_\ell, \quad i = 0, 1, 2, \ldots, n_\ell, n_\ell + 1. \tag{2}$$

The **set of cell-centered points** is denoted by C_ℓ and defined as

$$C_\ell := \{x_{\ell,i} \mid 0 \leq i \leq n_\ell + 1\}.$$

The grid points $x_{\ell,0} < 0$ and $x_{\ell,n_\ell+1} > 1$ are called **ghost points**. The **set of interior cell-centered points** is denoted by C_ℓ° and defined as

$$C_\ell^\circ := \{x_{\ell,i} \mid 1 \leq i \leq n_\ell\}.$$

The **vertex points** are defined via

$$x_{\ell,i+1/2} := ih_\ell, \quad i = 0, 1, \ldots, n_\ell. \tag{3}$$

The disjoint **grid cells** are

$$K_{\ell,i} := \left(x_{\ell,i-1/2}, x_{\ell,i+1/2}\right], \quad i = 1, 2, \ldots, n_\ell. \tag{4}$$

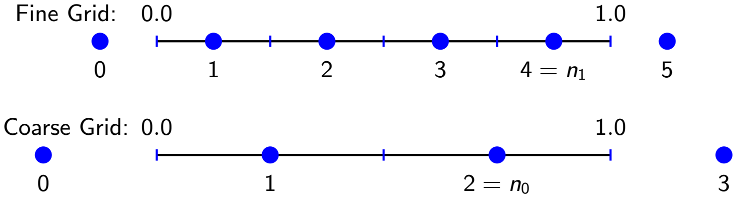

Figure: A uniform and nested two-level cell-centered grid with $q = 2$. Notice that, for the cell-centered finite difference case, the coarse and fine grid points (blue dots) are not collocated.

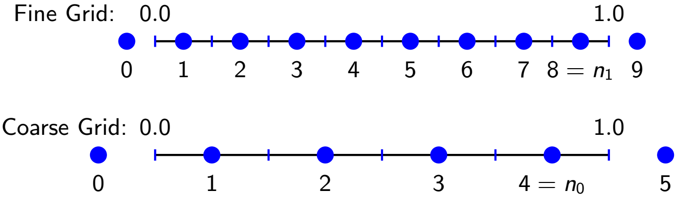

Figure: A uniform and nested two-level cell-centered grid with $q = 4$. Notice that, for the cell-centered finite difference case, the coarse and fine grid points (blue dots) are not collocated.

Remark (Nested Grid)

We refer to the cell centered multilevel grid structure as being nested and uniform, even though, as we will see, the cell-centered grid points of level-$(\ell-1)$ *never appear in level-*ℓ *grid. See figures on the previous slides.*

There are two equivalent ways of viewing functions built upon cell-centered grids that will be useful for us.

Definition (Grid Functions)

For every $0 \leq \ell \leq L$, we define the space of **piecewise constant functions** as

$$V_\ell := \left\{ v : (0,1] \to \mathbb{R} \;\middle|\; v|_{K_{\ell,i}} \in \mathbb{P}_0,\ 1 \leq i \leq n_\ell \right\}.$$

The spaces of **cell-centered grid functions** and **interior cell-centered grid functions** are defined, respectively, as

$$\mathcal{C}_\ell := \{ v_\ell : C_\ell \to \mathbb{R} \} \quad \text{and} \quad \mathcal{C}_\ell^\circ := \{ v_\ell : C_\ell^\circ \to \mathbb{R} \}.$$

The subspace of **homogeneous Dirichlet cell-centered grid functions** is denoted

$$\mathcal{C}_{\ell,0} := \{ v_\ell \in \mathcal{C}_\ell \mid v_\ell(x_{\ell,0}) = -v_\ell(x_{\ell,1}),\ v_\ell(x_{\ell,n_\ell}) = -v_\ell(x_{\ell,n_\ell+1}) \}.$$

Remark

In an obvious way, we can identify the two spaces $\mathcal{C}_{\ell,0}$ and $\mathcal{C}_{\ell}^{\circ}$. Given any function $f \in \mathcal{C}_{\ell,0}$, we can find its unique counterpart in $\mathcal{C}_{\ell}^{\circ}$, and vice versa. Nevertheless, it will be useful to distinguish the two and retain different notation for them.

Notation

For grid functions, we will use the convenient and standard notation

$$v_{\ell,i} := v_\ell(x_{\ell,i}), \quad 0 \leq i \leq n_\ell + 1.$$

In other words, we prefer to use subscripts when referring to the values, or entries, of grid functions.

Next, we recall that, for a set $A \subset \mathbb{R}^n$, it is standard to denote by χ_A its characteristic function, i.e.,

$$\chi_A(\boldsymbol{x}) := \begin{cases} 1, & \boldsymbol{x} \in A, \\ 0, & \boldsymbol{x} \in \mathbb{R}^n \setminus A. \end{cases}$$

Definition (Basis Functions)

For every $0 \leq \ell \leq L$, define the **basis of piecewise constant functions** via

$$\mathcal{B}_\ell := \{\psi_{\ell,j}\}_{j=1}^{n_\ell} \subset V_\ell,$$

where

$$\psi_{\ell,j} := \chi_{K_{\ell,j}}, \quad 1 \leq j \leq n_\ell.$$

Define the **canonical grid function basis**

$$\mathcal{G}_\ell := \{g_{\ell,j}\}_{j=1}^{n_\ell} \subset \mathcal{C}_{\ell,0},$$

where $g_{\ell,j} \in \mathcal{C}_{\ell,0}$ satisfies

$$g_{\ell,j}(x_{\ell,i}) = \delta_{i,j}, \quad 1 \leq i,j \leq n_\ell.$$

Proposition (Bases)

For every $0 \le \ell \le L$, $\mathcal{B}_\ell$ *is indeed a basis of* V_ℓ, *and* $\mathcal{G}_\ell$ *is a basis for* $\mathcal{C}_{\ell,0}$. *Thus,*

$$\dim(V_\ell) = n_\ell = \dim(\mathcal{C}_{\ell,0}).$$

Proof.

Exercise. □

Identifications

In light of the previous result, there is a one-to-one correspondence between vectors in $\mathbb{R}^{n_\ell}$ and functions in V_ℓ. This will be denoted, as usual, by

$$v_\ell \in V_\ell \stackrel{\mathcal{B}_\ell}{\longleftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}.$$

This identification connects the coordinate vector $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$ with its unique piecewise constant function $v_\ell \in V_\ell$ through the basis $\mathcal{B}_\ell$ so that

$$v_\ell = \sum_{j=1}^{n_\ell} v_{\ell,j}\psi_{\ell,j}.$$

Likewise, there is a one-to-one correspondence between vectors in $\mathbb{R}^{n_\ell}$ and grid functions in $\mathcal{C}_{\ell,0}$, which also has dimension equal to n_ℓ. This correspondence is denoted by

$$v_\ell \in \mathcal{C}_{\ell,0} \stackrel{\mathcal{G}_\ell}{\longleftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}.$$

Once again, this identification connects the coordinate vector $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$ with its unique grid function $v_\ell \in \mathcal{C}_{\ell,0}$ via

$$v_\ell = \sum_{j=1}^{n_\ell} v_{\ell,j} g_{\ell,j}.$$

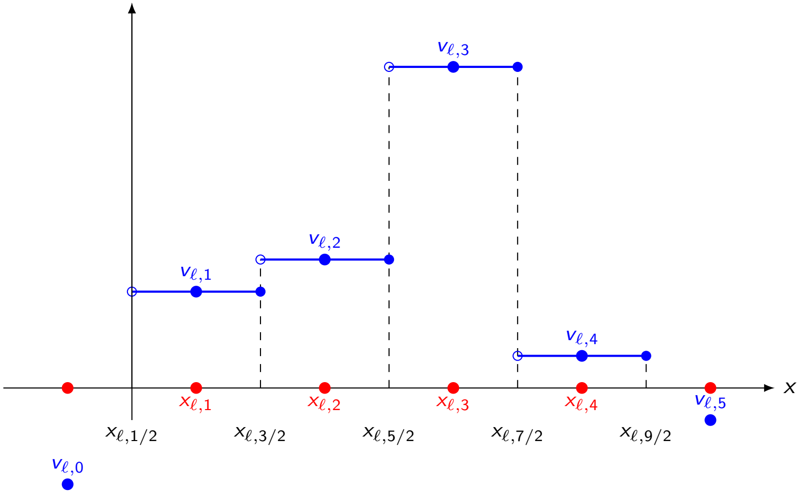

Figure: A piecewise constant function $v_\ell \in V_\ell$. We use the same symbol for the corresponding grid function $v_\ell \in \mathcal{C}_{\ell,0}$, which has values only at the cell-centered points and has homogeneous Dirichlet boundary values.

Finite Difference Approximation

The finite difference approximation of the one-dimensional Poisson problem (1) can be expressed as follows: given $f_{\ell,i} := f(x_{\ell,i})$, $1 \leq i \leq n_\ell$, find the grid function

$$u_\ell = (u_{\ell,0}, u_{\ell,1}, u_{\ell,2}, \ldots, u_{\ell,n_\ell}, u_{\ell,n_\ell+1}) \in \mathcal{C}_\ell,$$

such that

$$\begin{cases} \dfrac{u_{\ell,0} + u_{\ell,1}}{2} = 0, \\ \dfrac{-u_{\ell,i-1} + 2u_{\ell,i} - u_{\ell,i+1}}{h_\ell^2} = f_{\ell,i}, \quad 1 \leq i \leq n_\ell, \\ \dfrac{u_{\ell,n_\ell} + u_{n_\ell+1}}{2} = 0. \end{cases} \tag{5}$$

Finite Difference Approximation

Equivalently, we can express the problem as follows: find the grid function

$$u_\ell = (u_{\ell,0}, u_{\ell,1}, u_{\ell,2}, \dots, u_{\ell,n_\ell}, u_{\ell,n_\ell+1}) \in \mathcal{C}_{\ell,0},$$

such that

$$\frac{-u_{\ell,i-1} + 2u_{\ell,i} - u_{\ell,i+1}}{h_\ell^2} = f_{\ell,i}, \quad 1 \le i \le n_\ell, \tag{6}$$

where the boundary conditions are built into the space of solutions.

Convergence

In the cell-centered finite difference method

$$u_{\ell,i} \approx u(x_{\ell,i}) = u\left(\left(i - \frac{1}{2}\right) h_\ell\right),$$

where the function u is the solution to the boundary value problem. In fact, one can show that, under certain reasonable assumptions,

$$\sqrt{h_\ell \sum_{i=1}^{n_\ell} \left| u(x_{\ell,i}) - u_{\ell,i} \right|^2} \leq C h_\ell^2,$$

where $C > 0$ is a constant that is independent of h_ℓ. See Salgado and Wise (2023).

Matrix Form

Let us now set

$$\boldsymbol{u}_\ell = \begin{bmatrix} u_{\ell,1} \\ u_{\ell,2} \\ \vdots \\ u_{\ell,n_\ell-1} \\ u_{\ell,n_\ell} \end{bmatrix} \in \mathbb{R}^{n_\ell}, \quad \boldsymbol{f}_\ell = \begin{bmatrix} f_{\ell,1} \\ f_{\ell,2} \\ \vdots \\ f_{\ell,n_\ell-1} \\ f_{\ell,n_\ell} \end{bmatrix} \in \mathbb{R}^{n_\ell},$$

and

$$\mathsf{A}_\ell := \frac{1}{h_\ell^2} \begin{bmatrix} 3 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 3 \end{bmatrix} \in \mathbb{R}^{n_\ell \times n_\ell}.$$

$\mathsf{A}_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$ is called the cell-centered stiffness matrix. Note how it differs from the stiffness matrices we encountered in previous chapters. In matrix form, the finite difference approximation is as follows: find $\boldsymbol{u}_\ell \in \mathbb{R}^{n_\ell}$, such that

$$\mathsf{A}_\ell \boldsymbol{u}_\ell = \boldsymbol{f}_\ell. \tag{7}$$

Definition (Level-ℓ Stiffness Matrix)

For $0 \leq \ell \leq L$, the **level-ℓ stiffness matrix** is given by

$$\mathsf{A}_\ell := \frac{1}{h_\ell^2} \begin{bmatrix} 3 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 3 \end{bmatrix} \in \mathbb{R}^{n_\ell \times n_\ell}. \tag{8}$$

Theorem (Stiffness Matrix is SPD)

The level-ℓ stiffness matrix, $\mathsf{A}_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$, is SPD. Its eigenvalues are, for $k = 1, 2, \ldots, n_\ell$,

$$\lambda_\ell^{(k)} = \frac{4}{h_\ell^2} \sin^2\left(\frac{k\pi h_\ell}{2}\right) = \frac{2}{h_\ell^2}\left(1 - \cos(k\pi h_\ell)\right), \tag{9}$$

and the corresponding eigenvectors are

$$\left[\boldsymbol{v}_\ell^{(k)}\right]_i = v_{\ell,i}^{(k)} = \sin\left(k\pi x_{\ell,i}\right), \quad 1 \le i \le n_\ell. \tag{10}$$

Proof.

The matrix A_ℓ is clearly symmetric. We will show that it is positive definite by showing that all of its eigenvalues are positive. To see this, we first of all observe that, for all $k = 1, \ldots, n_\ell$,

$$\sin(k\pi x_{\ell,0}) = -\sin(k\pi x_{\ell,1}),$$
$$\sin(k\pi x_{\ell,n_\ell+1}) = -\sin(k\pi x_{\ell,n_\ell}).$$

Proof (Cont.)

Thus, for $1 \leq i \leq n_\ell$,

$$h_\ell^2 \left[\mathsf{A}_\ell \boldsymbol{v}_\ell^{(k)} \right]_i = -\sin(k\pi x_{\ell,i-1}) + 2\sin(k\pi x_{\ell,i}) - \sin(k\pi x_{\ell,i+1})$$
$$= 2\left[1 - \cos(k\pi h_\ell)\right]\sin(k\pi x_{\ell,i}).$$

This shows that, for $1 \leq i \leq n_\ell$,

$$\left[\mathsf{A}_\ell \boldsymbol{v}_\ell^{(k)} \right]_i = \lambda_\ell^{(k)} v_{\ell,i}^{(k)}.$$

Since the eigenvalues are positive, A_ℓ is SPD. $\square$

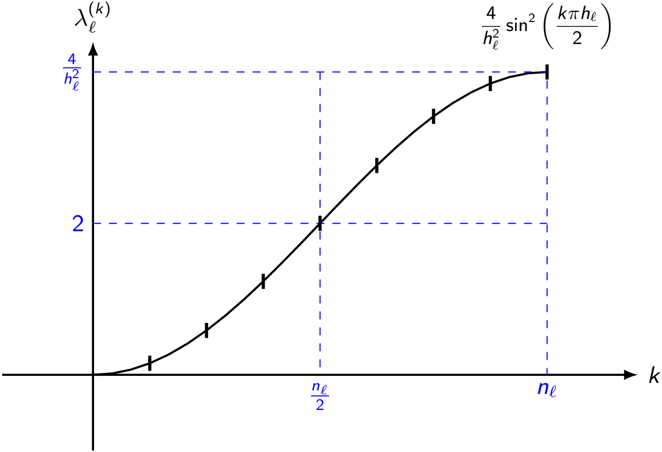

Figure: Eigenvalues of the level-ℓ stiffness matrix A_ℓ. The values of $\lambda_\ell^{(k)}$ are plotted for $n_\ell = 8$.

Remark (Scaling)

Observe that, for the one-dimensional cell-centered finite difference stiffness matrix, the largest eigenvalue always satisfies the equality

$$h_\ell^2 \lambda_\ell^{(n_\ell)} = 4\sin^2\left(\frac{n_\ell \pi}{2n_\ell}\right) = 4.$$

For the finite element discretization described in an earlier chapter, we observed that the largest eigenvalue satisfied the strict inequality

$$h_\ell \lambda_\ell^{(n_\ell)} = 4\sin^2\left(\frac{n_\ell \pi}{2(n_\ell+1)}\right) < 4,$$

with

$$\lim_{n_\ell \uparrow \infty} h_\ell \lambda_\ell^{(n_\ell)} = 4.$$

Recall, however, that the scalings we used for the respective stiffness matrices, and therefore, their respective eigenvalues, were different.

Proposition (Orthogonality)

Assume that $\ell \geq 1$, so that n_ℓ is even. The eigenvectors of the level-ℓ stiffness matrix are pair-wise orthogonal and satisfy the normalization conditions

$$\left(\boldsymbol{v}_\ell^{(k)}, \boldsymbol{v}_\ell^{(k)}\right)_\ell = \left(\boldsymbol{v}_\ell^{(k)}\right)^\top \boldsymbol{v}_\ell^{(k)} = \beta_{\ell,k} := \begin{cases} \frac{n_\ell}{2}, & 1 \leq k \leq n_\ell - 1, \\ n_\ell, & k = n_\ell. \end{cases}$$

Proof.

Exercise. □

Theorem (Condition Number Estimate)

There are positive constants $C_1 \leq C_2$ such that, for every ℓ, the spectral condition number of the level-ℓ stiffness matrix, A_ℓ, that is,

$$\kappa_2(\mathsf{A}_\ell) := \|\mathsf{A}_\ell\| \left\|\mathsf{A}_\ell^{-1}\right\| = \frac{\lambda_\ell^{(n_\ell)}}{\lambda_\ell^{(1)}},$$

satisfies the estimates

$$C_1 h_\ell^{-2} \leq \kappa_2(\mathsf{A}_\ell) \leq C_2 h_\ell^{-2}.$$

Proof.

The proof is similar to one from an earlier chapter. □

The Damped Jacobi and Richardson Smoothers

Damped Jacobi

After discretization, at every level $0 \le \ell \le L$, we have a family of linear systems to consider:

$$\mathsf{A}_\ell \boldsymbol{u}_\ell = \boldsymbol{f}_\ell \in \mathbb{R}^{n_\ell}. \tag{11}$$

To approximate the solution of (11) at a given level ℓ, one could consider using a GLIS. First, let us recall the damped Jacobi method. This method requires the canonical splitting of A_ℓ, namely,

$$\mathsf{A}_\ell = \mathsf{D}_\ell - \mathsf{U}_\ell - \mathsf{L}_\ell.$$

where

$$h_\ell^2 \mathsf{D}_\ell = \begin{bmatrix} 3 & & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & 2 & \\ & & & & 3 \end{bmatrix}, \quad h_\ell^2 \mathsf{U}_\ell = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} \in \mathbb{R}^{n_\ell \times n_\ell},$$

and, of course, $\mathsf{L}_\ell = \mathsf{U}_\ell^\top$.

Damped Jacobi

The damped Jacobi method reads

$$\boldsymbol{z}_\ell^{\sigma+1} := \mathsf{D}_\ell^{-1}\left(\mathsf{U}_\ell + \mathsf{U}_\ell^\top\right)\boldsymbol{u}_\ell^\sigma + \mathsf{D}_\ell^{-1}\boldsymbol{f}_\ell,$$

$$\boldsymbol{u}_\ell^{\sigma+1} := \omega\boldsymbol{z}_\ell^{\sigma+1} + (1-\omega)\boldsymbol{u}_\ell^\sigma,$$

where $0 < \omega \leq 1$. Eliminating $\boldsymbol{z}_\ell^{\sigma+1}$, we have the equivalent version

$$\boldsymbol{u}_\ell^{\sigma+1} = \boldsymbol{u}_\ell^\sigma + \omega\mathsf{D}_\ell^{-1}\left(\boldsymbol{f}_\ell - \mathsf{A}_\ell\boldsymbol{u}_\ell^\sigma\right).$$

In our multigrid terminology, the Damped Jacobi Smoother is selected by using the iterator matrix

$$\mathsf{S}_\ell = \omega\mathsf{D}_\ell^{-1},$$

and the associated error transfer is

$$\mathsf{K}_\ell = \mathsf{I}_\ell - \mathsf{S}_\ell\mathsf{A}_\ell = \mathsf{I}_\ell - \omega\mathsf{D}_\ell^{-1}\mathsf{A}_\ell.$$

The difficulty for our analysis with the damped Jacobi method is that the diagonal elements are no longer constant. This means that the eigenvectors of K_ℓ no longer coincide with those of A_ℓ. **Consequently, we will abandon it!**

Richardson's Smoother

Instead, we will only consider Richardson's method as a smoother, which, as we will see, has more desirable properties. Unlike what we encountered in an earlier chapter, this change is not merely cosmetic. Richardson's smoother can be expressed generically as

$$\boldsymbol{u}_\ell^{\sigma+1} = \boldsymbol{u}_\ell^{\sigma} + \kappa \left(\boldsymbol{f}_\ell - \mathsf{A}_\ell \boldsymbol{u}_\ell^{\sigma} \right),$$

where $\kappa > 0$. We will typically choose

$$\kappa^{-1} = \Lambda_\ell := \lambda_\ell^{(n_\ell)} = \frac{4}{h_\ell^2},$$

where Λ_ℓ is precisely the spectral radius of A_ℓ. We refer to this choice as the Λ–Richardson smoother.

Richardson's Smoother

For now, we want to let $\kappa > 0$ remain variable, or generic, in a sense. To do so, let us choose

$$\kappa := \omega \frac{h_\ell^2}{2}, \quad \omega > 0.$$

Thus, the generic form of Richardson's smoother that we will consider is

$$\boldsymbol{u}_\ell^{\sigma+1} = \boldsymbol{u}_\ell^{\sigma} + \omega \frac{h_\ell^2}{2} \left(\boldsymbol{f}_\ell - \mathsf{A}_\ell \boldsymbol{u}_\ell^{\sigma} \right), \quad \omega > 0,$$

and the error transfer matrix for the generic Richardson smoother is

$$\mathsf{K}_\ell = \mathsf{K}_\ell(\omega) = \mathsf{I}_\ell - \frac{\omega h_\ell^2}{2} \mathsf{A}_\ell. \tag{12}$$

If we set $\omega = \frac{1}{2}$, we obtain the Λ–Richardson smoother, which is the one typically used in multigrid applications.

Theorem (Eigen-Pairs of Generic Richardson)

Assume that the error transfer matrix for the generic Richardson smoother K_ℓ*, when applied to the model problem* (11)*, is given by* (12)*. The eigenvectors of* K_ℓ *are the same as those for the level-*ℓ *stiffness matrix,* A_ℓ*, that is,*

$$\left[\boldsymbol{v}_\ell^{(k)}\right]_i = v_{\ell,i}^{(k)} = \sin(k\pi x_{\ell,i}), \quad 1 \leq i \leq n_\ell,$$

for $k = 1, \ldots, n_\ell$*. The eigenvalues of* K_ℓ *are*

$$\mu_\ell^{(k)}(\omega) = 1 - 2\omega \sin^2\left(\frac{k\pi h_\ell}{2}\right), \quad 1 \leq k \leq n_\ell. \tag{13}$$

Proof.

Exercise. □

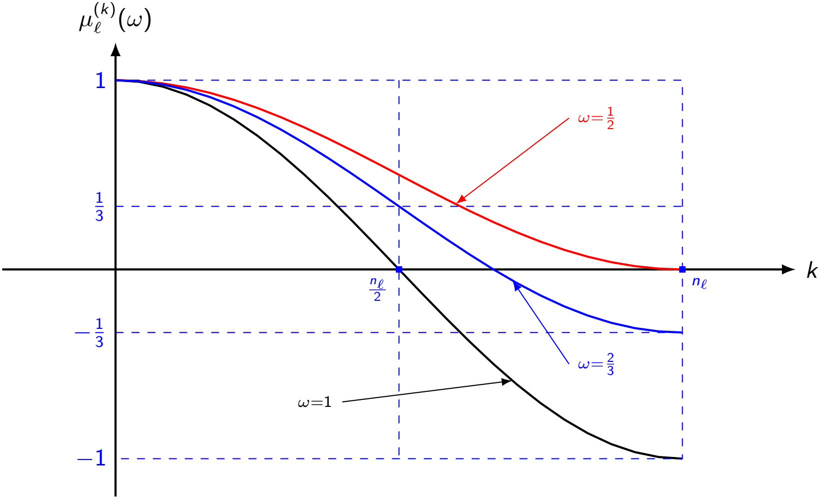

Figure: Plots of the eigenvalues of K_ℓ as functions of k, for various values of $\omega \in (0, 1]$.

Remark (Exactness)

Observe that for the cell-centered finite difference case we have the equality

$$\mu_\ell^{(n_\ell)}(\omega) = 1 - 2\omega \sin^2\left(\frac{n_\ell \pi}{2n_\ell}\right) = 1 - 2\omega.$$

On the other hand, for the one-dimensional finite element case, we have the inequality

$$\mu_\ell^{(n_\ell)}(\omega) = 1 - 2\omega \sin^2\left(\frac{n_\ell \pi}{2(n_\ell + 1)}\right) > 1 - 2\omega.$$

The next result shows why the performance of Richardson's method, as a standalone solver, is degraded as $h_\ell \to 0$.

Theorem (Spectral Radius of Generic Richardson)

Let $1 \leq \ell \leq L$ *and assume that the error transfer matrix for the generic Richardson smoother* K_ℓ*, when applied to the model problem* (11)*, is given by* (12)*. Then, as* $h_\ell \to 0$,

$$\rho(\mathsf{K}_\ell) = \mu_1^{(1)}(\omega) = 1 - \omega\Theta(h_\ell^2),$$

for all $0 < \omega \leq 1$*, that is, there exists a constant* $C > 0$*, independent of* h_ℓ *and* ω*, such that*

$$0 < 1 - C\omega h_\ell^2 \rho(\mathsf{K}_\ell) < 1.$$

Proof.

Exercise. □

The Λ–Richardson Smoother

Clearly the Λ–Richardson smoother has the following eigenvalues:

$$\mu_\ell^{(k)}\left(\frac{1}{2}\right) = \frac{1}{2}\cos(k\pi h_\ell) + 1 - \frac{1}{2}$$
$$= 1 - \frac{\lambda_\ell^{(k)}}{\Lambda_\ell}.$$

By looking at the figure on the next slide, it clearly follows that, for $\ell \geq 1$,

$$\left|\mu_\ell^{(k)}\left(\frac{1}{2}\right)\right| = \mu_\ell^{(k)}\left(\frac{1}{2}\right) \leq \mu_\ell^{(n_{\ell-1})}\left(\frac{1}{2}\right) = \frac{1}{2}, \quad n_{\ell-1} = \frac{n_\ell}{2} \leq k \leq n_\ell.$$

In other words the Λ–Richardson smoother reduces all high frequency modes of the error by at least half.

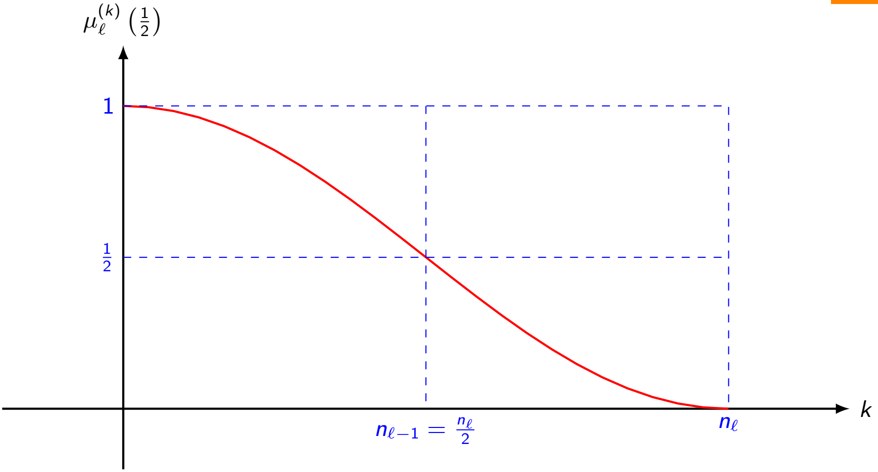

Figure: Eigenvalues of the error transfer matrix for the Λ–Richardson smoother.

The First Smoothing Property

Orthogonal Diagonalization of A_ℓ

The one-dimensional cell-centered finite difference (level-ℓ) stiffness matrix, introduced earlier, is SPD. Therefore, orthogonally diagonalizable, i.e.,

$$A_\ell = V_\ell D_\ell V_\ell^\top,$$

where V_ℓ is the orthogonal matrix containing the normalized eigenvectors of A_ℓ,

$$V_\ell = \begin{bmatrix} | & | & & | \\ \tilde{\boldsymbol{v}}_\ell^{(1)} & \tilde{\boldsymbol{v}}_\ell^{(2)} & \cdots & \tilde{\boldsymbol{v}}_\ell^{(n_\ell)} \\ | & | & & | \end{bmatrix} \in \mathbb{R}^{n_\ell \times n_\ell},$$

and D_ℓ is the diagonal matrix containing the positive eigenvalues of A_ℓ,

$$D_\ell = \mathrm{diag}\left[\lambda_\ell^{(1)}, \lambda_\ell^{(2)}, \ldots, \lambda_\ell^{(n_\ell)}\right] \in \mathbb{R}^{n_\ell \times n_\ell}.$$

The eigenvalues, $\lambda_\ell^{(k)}$, of A_ℓ are defined in (9).

Matrix Square Roots

Since A_ℓ is SPD, we can define its square root as

$$\mathsf{A}_\ell^{1/2} = \mathsf{V}_\ell \mathsf{D}_\ell^{1/2} \mathsf{V}_\ell^\top,$$

where

$$\mathsf{D}_\ell^{1/2} = \operatorname{diag}\left[\sqrt{\lambda_\ell^{(1)}}, \sqrt{\lambda_\ell^{(2)}}, \ldots, \sqrt{\lambda_\ell^{(n_\ell)}}\right] \in \mathbb{R}^{n_\ell \times n_\ell}.$$

Of course, $\mathsf{A}_\ell^{1/2}$ is SPD, and $\mathsf{A}_\ell^{1/2}\mathsf{A}_\ell^{1/2} = \mathsf{A}_\ell$.

Norms Defined by A_ℓ

Observe now that, for $\boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}$,

$$\begin{aligned}
\|\boldsymbol{v}_\ell\|_{A_\ell} &= \sqrt{(\boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_{A_\ell}} \\
&= \sqrt{(A_\ell \boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_\ell} \\
&= \sqrt{\left(A_\ell^{1/2} \boldsymbol{v}_\ell, A_\ell^{1/2} \boldsymbol{v}_\ell\right)_\ell} \\
&= \left\| A_\ell^{1/2} \boldsymbol{v}_\ell \right\|_\ell .
\end{aligned}$$

In a similar way,

$$\begin{aligned}
\|\boldsymbol{v}_\ell\|_{A_\ell^2} &:= \sqrt{(A_\ell^2 \boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_\ell} \\
&= \sqrt{(A_\ell \boldsymbol{v}_\ell, A_\ell \boldsymbol{v}_\ell)_\ell} \\
&= \|A_\ell \boldsymbol{v}_\ell\|_\ell .
\end{aligned}$$

Let us now establish a smoothing property for the cell-centered finite difference case in one space dimension.

Theorem (Smoothing Property)

Let $1 \le \ell \le L$. Assume that smoothing is carried out by the Λ–Richardson smoother, that is, $\omega = \frac{1}{2}$. There is some constant $C > 0$, such that, for any $m_1 \in \mathbb{N}$ and all $\boldsymbol{w}_\ell \in \mathbb{R}^{n_\ell}$,

$$\left\| \mathsf{K}_\ell^{m_1} \boldsymbol{w}_\ell \right\|_{\mathsf{A}_\ell^2} \le C \sqrt{\frac{\Lambda_\ell}{m_1}} \left\| \boldsymbol{w}_\ell \right\|_{\mathsf{A}_\ell}. \tag{14}$$

In particular, we can take

$$C = 2^{-1/2}.$$

In other words, the Λ-Richardson smoother satisfies the first smoothing property (S1).

Proof.

First, observe that

$$\|\mathsf{K}_\ell^{m_1}\boldsymbol{w}_\ell\|_{\mathsf{A}_\ell^2}^2 = \|\mathsf{A}_\ell\mathsf{K}_\ell^{m_1}\boldsymbol{w}_\ell\|_\ell^2$$
$$= \left(\mathsf{A}_\ell\mathsf{K}_\ell^{m_1}\boldsymbol{w}_\ell, \mathsf{A}_\ell\mathsf{K}_\ell^{m_1}\boldsymbol{w}_\ell\right)_\ell .$$

Using the eigenvector basis, let us write

$$\boldsymbol{w}_\ell = \sum_{k=1}^{n_\ell} w_{\ell,k}\boldsymbol{v}_\ell^{(k)}.$$

Proof (Cont.)

Then, using the identity

$$\left(\boldsymbol{v}_\ell^{(k)}, \boldsymbol{v}_\ell^{(k)}\right)_\ell = \left(\boldsymbol{v}_\ell^{(k)}\right)^\top \boldsymbol{v}_\ell^{(k)} = \beta_{\ell,k} := \begin{cases} \dfrac{n_\ell}{2}, & 1 \leq k \leq n_\ell - 1, \\ n_\ell, & k = n_\ell, \end{cases}$$

we have

$$\begin{aligned} \|\mathsf{K}_\ell^{m_1} \boldsymbol{w}_\ell\|_{\mathsf{A}_\ell^2}^2 &= \sum_{k=1}^{n_\ell} \beta_{\ell,k} \left(\lambda_\ell^{(k)} w_{\ell,k}\right)^2 \left(\mu_\ell^{(k)}\left(\frac{1}{2}\right)\right)^{2m_1} \\ &= \Lambda_\ell \sum_{k=1}^{n_\ell} \beta_{\ell,k} \left(\frac{\lambda_\ell^{(k)}}{\Lambda_\ell}\right)\left(1 - \frac{\lambda_\ell^{(k)}}{\Lambda_\ell}\right)^{2m_1} \lambda_\ell^{(k)} w_{\ell,k}^2 \\ &\leq \Lambda_\ell G(m_1) \sum_{k=1}^{n_\ell} \beta_{\ell,k} \lambda_\ell^{(k)} w_{\ell,k}^2 \\ &= \Lambda_\ell G(m_1) \|\boldsymbol{w}_\ell\|_{\mathsf{A}_\ell}^2 . \end{aligned}$$

Proof (Cont.)

Here we have defined

$$G(m) := \max_{1\leq k\leq n_\ell} \left(\frac{\lambda_\ell^{(k)}}{\Lambda_\ell}\right)\left(1-\frac{\lambda_\ell^{(k)}}{\Lambda_\ell}\right)^{2m}.$$

Upon rescaling and using a lemma from an earlier chapter, we have

$$\begin{aligned} G(m) &\leq \max_{0\leq x\leq 1} x(1-x)^{2m} \\ &\leq \frac{1}{2m}. \end{aligned}$$

Therefore,

$$\|\mathsf{K}_\ell^{m_1}\boldsymbol{w}_\ell\|_{\mathsf{A}_\ell^2} \leq \sqrt{\frac{1}{2}}\sqrt{\frac{\Lambda_\ell}{m_1}}\,\|\boldsymbol{w}_\ell\|_{\mathsf{A}_\ell},$$

and the result follows with $C = 2^{-1/2}$. □