Math 674

Multigrid Methods: A Mostly Matrix-Based Approach

Chapter 09: Additive Preconditioners Based on Subspace Decompositions

Abner J. Salgado and Steven M. Wise

asalgad1@utk.edu swise1@.utk.edu
University of Tennessee

Spring 2025

Chapter 09, Part 2 of 2

Additive Preconditioners Based on Subspace Decompositions

Hierarchical Basis Preconditioner

Now, we need to connect the spaces W_j to V_ℓ where $0 \le j \le \ell$. In so doing, we will have the tools to build a preconditioner based on the hierarchical basis. Be careful, the number of indices in this section can get a little overwhelming.

Proposition

Let $\mathcal{B}_j^W = \{\phi_{j,i}\}_{i=1}^{m_j}$ and $\mathcal{B}_\ell^V = \{\psi_{\ell,i}\}_{i=1}^{n_\ell}$ be the usual bases for W_j and V_ℓ, respectively. For each $0 \le j \le \ell$, there are unique numbers

$$q_{j,k,i}^{\ell} \in \mathbb{R}, \quad 1 \le k \le n_\ell, \quad 1 \le i \le m_j,$$

such that

$$\phi_{j,i} = \sum_{k=1}^{n_\ell} q_{j,k,i}^{\ell} \psi_{\ell,k}. \tag{1}$$

Proof.

Exercise. □

Definition (Hierarchical Prolongation Matrix)

Define the matrix $\mathsf{Q}_j^\ell \in \mathbb{R}^{n_\ell \times m_j}$ via

$$\left[\mathsf{Q}_j^\ell\right]_{k,i} := q_{j,k,i}^\ell, \quad 1 \le k \le n_\ell, \quad 1 \le i \le m_j.$$

Q_j^ℓ is called a **hierarchical prolongation matrix**.

Lemma

Suppose that Q_j^ℓ is a hierarchical prolongation matrix and $\boldsymbol{w}_j \in \mathbb{R}^{m_j}$ is the coordinate vector of the function $w_j \in W_j$ with respect to the basis $\mathcal{B}_j^W$. Then,

$$\operatorname{rank}(\mathsf{Q}_j^\ell) = m_j,$$

and the coordinate vector of $w_j \in V_\ell$ in the basis $\mathcal{B}_\ell^V$ is simply

$$\mathsf{Q}_j^\ell \boldsymbol{w}_j \in \mathbb{R}^{n_\ell}.$$

Proof.

Exercise. □

Remark

Note that the family of spaces W_j are hierarchical, but are not nested

$$W_0 \not\subset W_1 \not\subset W_2 \cdots .$$

Furthermore, it makes no sense to stack the prolongation matrices as we did in the past:

$$\mathsf{Q}_j^\ell \neq \mathsf{Q}_k^\ell \mathsf{Q}_j^k,$$

for $j < k < \ell$. In fact, the product on the right hand side is not usually defined.

Definition

Define the bilinear form $C_j : W_j \times W_j \to \mathbb{R}$ via

$$C_j\left(w_j, v_j\right) := \sum_{r=1}^{m_j} w_j\left(\boldsymbol{N}_{j,r}^W\right) v_j\left(\boldsymbol{N}_{j,r}^W\right), \quad \forall\, w_j, v_j \in W_j.$$

Let $\mathcal{B}_j^W = \{\phi_{j,i}\}_{i=1}^{m_j}$ be the usual basis for W_j. Define the matrix $\mathsf{C}_j \in \mathbb{R}^{m_j \times m_j}$ via

$$\begin{aligned} \left[\mathsf{C}_j\right]_{i,k} &:= C_j\left(\phi_{j,i}, \phi_{j,k}\right) \\ &= \sum_{r=1}^{m_j} \phi_{j,i}\left(\boldsymbol{N}_{j,r}^W\right) \phi_{j,k}\left(\boldsymbol{N}_{j,r}^W\right) \\ &= \sum_{r=1}^{m_j} \delta_{ir}\delta_{kr} \\ &= \delta_{ik}. \end{aligned} \tag{2}$$

Definition (Hierarchical Basis Preconditioner)

Suppose that $\mathcal{B}_\ell^V = \{\psi_{\ell,i}\}_{i=1}^{n_\ell}$ is the usual basis for the finite element space V_ℓ. Let $\mathsf{A}_L \in \mathbb{R}^{n_L \times n_L}$ be the SPD matrix defined via

$$[\mathsf{A}_L]_{i,j} = a(\psi_{L,j}, \psi_{L,i}), \quad 1 \leq i, j \leq n_L,$$

where

$$a(u, v) = (\nabla u, \nabla v)_{L^2}, \quad \forall\ u, v \in H_0^1(\Omega).$$

The **hierarchical basis preconditioner for** A_L is defined as

$$\mathsf{C}_{\mathrm{H}} = \sum_{\ell=0}^{L} \mathsf{Q}_\ell^L \mathsf{C}_\ell^{-1} \mathsf{Z}_\ell^L = \sum_{\ell=0}^{L} \mathsf{Q}_\ell^L \mathsf{Z}_\ell^L, \tag{3}$$

where C_ℓ is as in (2), $\mathsf{Q}_\ell^L \in \mathbb{R}^{n_L \times m_\ell}$ is the hierarchical prolongation matrix from a previous definition and

$$\mathsf{Z}_\ell^L = \left(\mathsf{Q}_\ell^L\right)^\top.$$

Lemma

Assumption (SS1) holds for the hierarchical basis decomposition. In particular, for any object

$$\boldsymbol{u}_L \in \mathbb{R}^{n_L} \overset{\mathcal{B}_L^V}{\leftrightarrow} u_L \in V_L$$

there exist unique objects

$$\boldsymbol{w}_\ell \in \mathbb{R}^{m_\ell} \overset{\mathcal{B}_\ell^W}{\leftrightarrow} w_\ell \in W_\ell, \quad 0 \leq \ell \leq L,$$

such that

$$\boldsymbol{u}_L = \sum_{\ell=0}^{L} \mathsf{Q}_\ell^L \boldsymbol{w}_\ell \in \mathbb{R}^{n_L} \overset{\mathcal{B}_L^V}{\leftrightarrow} u_L = \sum_{\ell=0}^{L} w_\ell \in V_L.$$

Furthermore, the hierarchical basis preconditioner, C_{H}*, defined in* (3)*, is SPD.*

Proof.

This follows from the lemmas on the last slide deck. The details are left for an exercise. □

Remark

Our goal is now to show that

$$\lambda_{\min}(\mathsf{C}_{\mathrm{H}}\mathsf{A}_L) \geq C\left(1 + |\log(h_L)|^2\right)^{-1},$$

and

$$\lambda_{\max}(\mathsf{C}_{\mathrm{H}}\mathsf{A}_L) \leq C,$$

where these constants are positive and independent of L. If this is the case,

$$\frac{\lambda_{\max}}{\lambda_{\min}} =: \kappa(\mathsf{C}_{\mathrm{H}}\mathsf{A}_L) \leq C\left(1 + |\log(h_L)|^2\right).$$

This estimate is quite useful, since the logarithmic dependence on h_L is quite weak. For example, suppose

$$h_L = \frac{1}{2^L},$$

which is entirely reasonable. Then

$$|\log(h_L)|^2 = L^2\,|\log(1/2)|^2.$$

Our analysis that follows will only work for $d = 2$.

Now, we need some technical lemmas. For more details, see the book by Brenner and Scott.

Theorem (Mean-Zero Poincaré)

Suppose that Ω is an open polyhedral set in $\mathbb{R}^d$. Then, for every $u \in H^1(\Omega)$,

$$\|u - \bar{u}\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}, \tag{4}$$

for some constant $C > 0$ that is independent of u by dependent upon Ω, where $\bar{u}$ is the average of u:

$$\bar{u} := \frac{1}{|\Omega|} \int_\Omega u(\boldsymbol{x})\, d\boldsymbol{x}.$$

As a consequence, for every $u \in H^1(\Omega)$,

$$\|u - \bar{u}\|_{H^1(\Omega)} \leq C |u - \bar{u}|_{H^1(\Omega)} = C |u|_{H^1(\Omega)}, \tag{5}$$

for some constant $C > 0$ that is independent of u by dependent upon Ω.

Theorem (Inverse inequality)

Suppose that Ω is an open polygonal domain in $\mathbb{R}^d$, $\mathcal{T}_\ell$, $0 \leq \ell \leq L$ is a nested family of triangulations of Ω, and S_ℓ, $0 \leq \ell \leq L$, are the associated piecewise-linear finite element spaces. Assume that $1 \leq q \leq \infty$. Then, for all $v \in S_\ell$ and all $K \in \mathcal{T}_\ell$,

$$\|v\|_{H^1(K)} \leq C h_\ell^{-1+d/2-d/q} \|v\|_{L^q(K)}, \tag{6}$$

for some constant $C > 0$ that is independent of ℓ but depends on the shape of K.

Proof.

See Section 5.3 in the book by Brenner and Scott. □

Theorem

Suppose that Ω is an open polyhedral domain in $\mathbb{R}^d$, $\mathcal{T}_\ell$, $0 \le \ell \le L$ is a nested family of triangulations of Ω, and S_ℓ, $0 \le \ell \le L$, are the associated piecewise-linear finite element spaces. Then, for all $v_\ell \in S_\ell$, $\ell \ge 1$,

$$\left\| v_\ell - \mathcal{I}_{\ell-1} v_\ell \right\|_{L^2(\Omega)} \le C h_\ell \left| v_\ell \right|_{H^1(\Omega)}, \tag{7}$$

for some constant $C > 0$ that is independent of ℓ.

Proof.

Note that the previous interpolation error estimate from Chapter 6 does not cover this case, since v_ℓ is not in $H^2(\Omega)$. However, the stated result still holds since we are interpolating a very specific class of functions. The proof of the one dimensional case is left as an exercise. □

In two space dimensions $H^1 \hookrightarrow L^p$, for any $1 \leq p < \infty$. We cannot quite get control for $p = \infty$. But, if the function space is finite dimensional we can get control of the $p = \infty$ case, at the cost of an h-dependence. Here is the result from Section 4.9 in the book by Brenner and Scott.

Theorem

Suppose that Ω is an open polygonal domain in $\mathbb{R}^2$ and $\mathcal{T}_\ell$, $0 \leq \ell \leq L$ is a nested family of triangulations of Ω. Then, for any $v_\ell \in V_\ell$,

$$\|v_\ell\|_{L^\infty(\Omega)} \leq C\sqrt{1 + |\log(h_\ell)|}\, |v_\ell|_{H^1(\Omega)} ,$$

for some constant $C > 0$ that is independent of ℓ but depends upon the shape of Ω. Further, for all $v_\ell \in S_\ell$ and any $K \in \mathcal{T}_\ell$,

$$\|v_\ell - \bar{v}_\ell\|_{L^\infty(K)} \leq C\sqrt{1 + |\log(h_\ell)|}\, |v_\ell|_{H^1(K)} ,$$

for some constant $C > 0$ that is independent of ℓ but depends upon the shape of the triangle $K \in \mathcal{T}_\ell$, where

$$\bar{v}_\ell = \frac{1}{|K|}\int_K v_\ell(x)\, dx.$$

Lemma

Suppose that $0 \leq j < \ell$. *For any* $v_\ell \in S_\ell$,

$$\|v_\ell - \bar{v}_{j,\ell}\|_{L^\infty(K_j)} \leq C\sqrt{1 + \left|\log\left(\frac{h_j}{h_\ell}\right)\right|}\,|v_\ell|_{H^1(K_j)}\,, \tag{8}$$

for some constant $C > 0$ *that is independent of* j *and* ℓ *but depends upon the shape of the triangle* $K_j \in \mathcal{T}_j$, *where*

$$\bar{v}_{j,\ell} = \frac{1}{|K_j|}\int_{K_j} v_\ell(\boldsymbol{x})\,d\boldsymbol{x}.$$

Proof.

Exercise. □

Lemma

Assume that $\Omega \subset \mathbb{R}^2$ is a polygonal domain. Suppose that $\mathcal{I}_\ell : C(\overline{\Omega}) \to V_\ell$, $0 \le \ell \le L$, is the Lagrange nodal interpolation operator, and $\mathcal{I}_{-1} \equiv 0$. Then, for all $u_L \in V_L$,

$$\left\| \mathcal{I}_\ell u_L - \mathcal{I}_{\ell-1} u_L \right\|_{L^2(\Omega)} \le C h_\ell \left(1 + \sqrt{L - \ell} \right) |u_L|_{H^1(\Omega)} . \tag{9}$$

for some constant $C > 0$ that is independent of but depends upon the shape of Ω.

Proof.

Define the piecewise constant function $\bar{u}_L^\ell$ such that

$$\bar{u}_L^\ell |_K := \frac{1}{|K|} \int_K u_L(\boldsymbol{x}) \, d\boldsymbol{x}, \quad \forall \, K \in \mathcal{T}_\ell .$$

Proof (Cont.)

Then,

$$
\begin{aligned}
\left\|\mathcal{I}_\ell u_L - \mathcal{I}_{\ell-1} u_L\right\|_{L^2(\Omega)}^2 &= \left\|\mathcal{I}_\ell u_L - \mathcal{I}_{\ell-1}\left[\mathcal{I}_\ell[u_L]\right]\right\|_{L^2(\Omega)}^2 \\
&\overset{(7)}{\leq} C h_\ell^2 \sum_{K\in\mathcal{T}_\ell} \left|\mathcal{I}_\ell[u_L]\right|_{H^1(K)}^2 \\
&= C h_\ell^2 \sum_{K\in\mathcal{T}_\ell} \left|\mathcal{I}_\ell u_L - \bar{u}_L^\ell\right|_{H^1(K)}^2 \\
&\overset{(6)}{\leq} C h_\ell^2 \sum_{K\in\mathcal{T}_\ell} \left\|\mathcal{I}_\ell u_L - \bar{u}_L^\ell\right\|_{L^\infty(K)}^2 \\
&\leq C h_\ell^2 \sum_{K\in\mathcal{T}_\ell} \left\|u_L - \bar{u}_L^\ell\right\|_{L^\infty(K)}^2 \\
&\overset{(8)}{\leq} C h_\ell^2 \sum_{K\in\mathcal{T}_\ell} \left(1 + \left|\log\left(\frac{h_\ell}{h_L}\right)\right|\right) \left|u_L\right|_{H^1(K)}^2 \\
&= C h_\ell^2 \left(1 + \left|\log\left(\frac{h_\ell}{h_L}\right)\right|\right) \left|u_L\right|_{H^1(\Omega)}^2 .
\end{aligned}
$$

Proof (Cont.)

Now, notice that

$$h_\ell = h_0 2^{-\ell} \quad 1 \leq \ell \leq L.$$

So,

$$\log(h_\ell / h_L) = \log(2^{L-\ell}) = (L - \ell)\log(2).$$

The result follows. $\square$

Lemma

There is some constant $C_7 > 0$, independent of L, such that

$$\lambda_{\min}(\mathsf{C}_{\mathrm{H}}\mathsf{A}_L) \geq \frac{C_7}{1 + |\log(h_L)|^2}. \tag{10}$$

Proof.

By definition, for any $w_{\ell,1}, w_{\ell,2} \in W_\ell$

$$C_\ell\left(w_{\ell,1}, w_{\ell,2}\right) = \sum_{i=1}^{m_\ell} w_{\ell,1}(\boldsymbol{N}_{\ell,i}^W) w_{\ell,2}(\boldsymbol{N}_{\ell,i}^W).$$

Proof (Cont.)

Let

$$\boldsymbol{w}_{\ell,\alpha} \in \mathbb{R}^{m_\ell} \overset{\mathcal{B}_\ell^W}{\leftrightarrow} w_{\ell,\alpha} \in W_\ell, \quad \alpha = 1, 2.$$

Then,

$$\begin{aligned}
(\mathsf{C}_\ell \boldsymbol{w}_{\ell,1}, \boldsymbol{w}_{\ell,2})_\ell &= \sum_{i=1}^{m_\ell} [\boldsymbol{w}_{\ell,1}]_i \, [\boldsymbol{w}_{\ell,2}]_i \\
&= \sum_{i=1}^{m_\ell} w_{\ell,1}(\boldsymbol{N}_{\ell,i}^W) w_{\ell,2}(\boldsymbol{N}_{\ell,i}^W) \\
&= C_\ell \left(w_{\ell,1}, w_{\ell,2} \right) \\
&= C_\ell \left(w_{\ell,2}, w_{\ell,1} \right) \\
&=: \left\langle w_{\ell,1}, w_{\ell,2} \right\rangle_{\mathsf{C}_\ell} .
\end{aligned}$$

This last object is like a mass-lumping inner product. All that is missing is a factor of h_ℓ^2.

Proof (Cont.)

There are constants $C_3 > 0$, $C_4 > 0$ such that, for all $0 \leq \ell \leq L$,

$$C_3 h_\ell^2 \langle w_\ell, w_\ell \rangle_{\mathsf{C}_\ell} \leq \|w_\ell\|_{L^2(\Omega)}^2 \leq C_4 h_\ell^2 \langle w_\ell, w_\ell \rangle_{\mathsf{C}_\ell}, \tag{11}$$

for all $w_\ell \in W_\ell$. Therefore, for any $w_\ell \in W_\ell \overset{\mathcal{B}_\ell^W}{\leftrightarrow} \boldsymbol{w}_\ell \in \mathbb{R}^{m_\ell}$,

$$\begin{aligned}
(\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell &= h_\ell^{-2} h_\ell^2 \langle w_\ell, w_\ell \rangle_{\mathsf{C}_\ell} \\
&\overset{(11)}{\leq} C_3^{-1} h_\ell^{-2} \|w_\ell\|_{L^2(\Omega)}^2 \\
&= C_3^{-1} h_\ell^{-2} \|w_\ell - \mathcal{I}_{\ell-1} w_\ell\|_{L^2(\Omega)}^2 \\
&\overset{(7)}{\leq} C_3^{-1} C \, |w_\ell|_{H^1(\Omega)}^2 \\
&\overset{(6)}{\leq} C_3^{-1} C h_\ell^{-2} \|w_\ell\|_{L^2(\Omega)}^2 \\
&\overset{(11)}{\leq} C_3^{-1} C C_4 (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell .
\end{aligned} \tag{12}$$

Proof (Cont.)

Therefore, there are constants $C_5 > 0$, $C_6 > 0$, such that we have the equivalence

$$C_5 \sum_{\ell=0}^{L} |w_\ell|^2_{H^1(\Omega)} \le \sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell \le C_6 \sum_{\ell=0}^{L} |w_\ell|^2_{H^1(\Omega)}, \tag{13}$$

for any collection (w_ℓ), with $w_\ell \in W_\ell$, in general. Now, let $u_L \in V_L$ be given and

$$u_L = \sum_{\ell=0}^{L} w_\ell, \quad \exists!\, w_\ell \in W_\ell, \quad 0 \le \ell \le L.$$

Recall that

$$w_\ell = \mathcal{I}_\ell u_L - \mathcal{I}_{\ell-1} u_L, \quad 1 \le \ell \le L,$$

and

$$w_0 = \mathcal{I}_0 u_L.$$

Proof (Cont.)

We make the usual identification $w_\ell \in W_\ell \overset{\mathcal{B}_\ell^W}{\leftrightarrow} \boldsymbol{w}_\ell \in \mathbb{R}^{m_\ell}$, and we observe that

$$(\boldsymbol{w}_\ell)_{\ell=0}^L \in \mathsf{Q}[\boldsymbol{u}_L],$$

with respect to the hierarchical prolongation matrices from Definition 1. Then, from (12)

$$\begin{aligned}
\sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_\ell &\leq C_3^{-1} C \sum_{\ell=0}^{L} h_\ell^{-2} \|w_\ell\|_{L^2(\Omega)}^2 \\
&\overset{(9)}{\leq} C \sum_{\ell=0}^{L} \left(1 + \sqrt{L-\ell}\right)^2 |u_L|_{H^1(\Omega)}^2 \\
&\leq C \sum_{\ell=0}^{L} (1 + L - \ell) |u_L|_{H^1(\Omega)}^2 \\
&\leq C \left(1 + L + L^2\right) |u_L|_{H^1(\Omega)}^2 \\
&\overset{L \geq 1}{\leq} C L^2 |u_L|_{H^1(\Omega)}^2 .
\end{aligned}$$

Proof (Cont.)

But

$$|u_L|^2_{H^1(\Omega)} = a(u_L, u_L) = (\mathrm{A}_L \boldsymbol{u}_L, \boldsymbol{u}_L)_L,$$

and

$$\begin{aligned} |\log(h_L)|^2 &= \left|\log(h_0 2^{-L})\right|^2 \\ &= |\log(h_0) - L\log(2)|^2 \\ &= \log^2(h_0) - 2\log(h_0)L\log(2) + L^2\log^2(2). \end{aligned}$$

So,

$$L^2 \le C\left(1 + |\log(h_L)|^2\right), \quad \exists\, C > 0.$$

Thus,

$$\sum_{\ell=0}^{L} (\mathrm{C}_\ell \boldsymbol{w}_\ell, \boldsymbol{w}_\ell) \le C\left(1 + |\log(h_L)|^2\right)(\mathrm{A}_L \boldsymbol{u}_L, \boldsymbol{u}_L)_L,$$

and

$$\lambda_{\min}(\mathrm{C}_{\mathrm{H}}\mathrm{A}_L) \ge C_7\left(1 + |\log(h_L)|^2\right)^{-1}.$$

□

In the last line we use the "big" theorem from the first slide deck:

Theorem (Eigenvalues of CA)

Suppose that Assumption (SS1) holds for the set of prolongation matrices $\{\mathsf{Q}_j\}_{j=0}^{L}$ and C is an additive subspace preconditioner with respect to $\{\mathsf{Q}_j\}_{j=0}^{L}$. The eigenvalues of CA are positive, provided A is SPD with respect to $(\cdot,\cdot)$. Moreover

$$\lambda_{\max}(\mathsf{CA}) = \max_{\boldsymbol{u}\in\mathbb{R}^n_\star} \frac{(\mathsf{A}\boldsymbol{u},\boldsymbol{u})}{\displaystyle\min_{(\boldsymbol{w}_\ell)\in\mathsf{Q}[\boldsymbol{u}]}\sum_{\ell=0}^{L}(\mathsf{C}_\ell\boldsymbol{w}_\ell,\boldsymbol{w}_\ell)_\ell}, \tag{14}$$

$$\lambda_{\min}(\mathsf{CA}) = \min_{\boldsymbol{u}\in\mathbb{R}^n_\star} \frac{(\mathsf{A}\boldsymbol{u},\boldsymbol{u})}{\displaystyle\min_{(\boldsymbol{w}_\ell)\in\mathsf{Q}[\boldsymbol{u}]}\sum_{\ell=0}^{L}(\mathsf{C}_\ell\boldsymbol{w}_\ell,\boldsymbol{w}_\ell)_\ell}. \tag{15}$$

Next, we need a little technical lemma, a kind of convolution result.

Lemma

Let $a_j, b_j \geq 0, -\infty < j < \infty$, *with*

$$s_1 := \sum_{j=-\infty}^{\infty} a_j \leq \infty,$$

and

$$s_2 := \sum_{j=-\infty}^{\infty} b_j \leq \infty.$$

Then

$$\sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} a_{j-k} b_k \right)^2 \leq s_1^2 s_2. \tag{16}$$

Proof.

Exercise. □

Lemma

For any $v_\ell \in V_\ell$ and $v_k \in V_k$, $0 \leq \ell \leq k \leq L$, and d=2, there is a constant $C > 0$ such that

$$\int_\Omega \nabla v_\ell \cdot \nabla v_k \, d\boldsymbol{x} \leq 2^{(\ell-k)/2} C \left| v_\ell \right|_{H^1(\Omega)} \left(h_k^{-1} \left\| v_k \right\|_{L^2(\Omega)} \right). \tag{17}$$

Proof.

For any $K \in \mathcal{T}_\ell$, since $\Delta v_\ell|_K \equiv 0$,

$$
\begin{aligned}
\int_K \nabla v_\ell \cdot \nabla v_k \, d\boldsymbol{x} &= \int_{\partial K} \frac{\partial v_\ell}{\partial n} v_k ds \\
&\leq C h_\ell^{-1} |v_\ell|_{H^1(K)} \int_{\partial K} |v_k| ds \\
&\leq C h_\ell^{-1} |v_\ell|_{H^1(K)} \left(h_k \sum_{\boldsymbol{N}_k \in \partial K} \left| v_k(\boldsymbol{N}_k) \right| \right) \\
&\overset{\text{C.S.}}{\leq} C h_\ell^{-1} |v_\ell|_{H^1(K)} h_k \left(\frac{h_\ell}{h_k} \right)^{1/2} \left(\sum_{\boldsymbol{N}_k \in \partial K} \left| v_k(\boldsymbol{N}_k) \right|^2 \right)^{1/2} \\
&\leq C \left(\frac{h_k}{h_\ell} \right)^{1/2} |v_\ell|_{H^1(K)} h_k^{-1} \|v_k\|_{L^2(K)} .
\end{aligned}
$$

Proof (Cont.)

Thus,

$$\begin{aligned}
\int_\Omega \nabla v_\ell \cdot \nabla v_k \, d\boldsymbol{x} &= \sum_{K \in \mathcal{T}_\ell} \int_K \nabla v_\ell \cdot \nabla v_k \, d\boldsymbol{x} \\
&\leq C 2^{(\ell-k)/2} \sum_{K \in \mathcal{T}_\ell} |v_\ell|_{H^1(K)} \, h_k^{-1} \, \|v_k\|_{L^2(K)} \\
&\overset{\text{C.S.}}{\leq} C 2^{(\ell-k)/2} |v_\ell|_{H^1(\Omega)} \, h_k^{-1} \, \|v_k\|_{L^2(\Omega)} .
\end{aligned}$$

□

Lemma (Strengthened Cauchy-Schwarz Inequality)

For any $w_\ell \in W_\ell$ and $w_k \in W_k$, $0 \le \ell \le k \le L$, there is a constant $C > 0$ such that

$$\int_\Omega \nabla w_\ell \cdot \nabla w_k \, d\mathbf{x} \le 2^{(\ell-k)/2} C \left| w_\ell \right|_{H^1(\Omega)} \left| w_k \right|_{H^1(\Omega)} . \tag{18}$$

Proof.

Observe that

$$w_k = w_k - \mathcal{I}_{k-1} w_k .$$

We use the interpolation error estimate

$$\left\| w_k - \mathcal{I}_{k-1} w_k \right\|_{L^2(\Omega)} \le C h_k \left| w_k \right|_{H^1(\Omega)} ,$$

to conclude that

$$\left\| w_k \right\|_{L^2(\Omega)} \le C h_k \left| w_k \right|_{H^1(\Omega)} .$$

Proof (Cont.)

Now, we use the last result. Since $w_\ell \in V_\ell$ and $w_k \in V_k$,

$$\begin{aligned}\int_\Omega \nabla w_\ell \cdot \nabla w_k \, dx &\leq C 2^{(\ell-k)/2} \left| w_\ell \right|_{H^1(\Omega)} h_k^{-1} \left\| w_k \right\|_{L^2(\Omega)} \\ &\leq 2^{(\ell-k)/2} C \left| w_\ell \right|_{H^1(\Omega)} \left| w_k \right|_{H^1(\Omega)} .\end{aligned}$$

$\square$

Lemma

There is a constant $C_8 > 0$, independent of L, such that

$$\lambda_{\max}(\mathsf{C}_{\mathrm{H}}\mathsf{A}_L) \leq C_8.$$

Proof.

Let $v_L \in V_L$ be arbitrary.

$$v_L \in V_L \overset{\mathcal{B}_L}{\leftrightarrow} \boldsymbol{v}_L \in \mathbb{R}^{n_L}.$$

There exist unique $w_\ell \in W_\ell \overset{\mathcal{B}_\ell^W}{\leftrightarrow} \boldsymbol{w}_\ell \in \mathbb{R}^{m_\ell}$, $\ell = 0, \ldots, , L$, such that

$$v_L = \sum_{\ell=0}^{L} w_\ell \overset{\mathcal{B}_L}{\leftrightarrow} \boldsymbol{v}_L = \sum_{\ell=0}^{L} \mathsf{Q}_\ell^L \boldsymbol{w}_\ell.$$

Proof (Cont.)

Then,

$$\begin{aligned}
(\boldsymbol{v}_L, \boldsymbol{v}_L)_{\mathsf{A}_L} &= (\boldsymbol{v}_L, \mathsf{A}_L \boldsymbol{v}_L)_L \\
&= a(v_L, v_L) \\
&= a\left(\sum_{\ell=0}^{L} w_\ell, \sum_{k=0}^{L} w_k \right) \\
&= \int_\Omega \left(\nabla \sum_{\ell=0}^{L} w_\ell \right) \cdot \left(\nabla \sum_{k=0}^{L} w_k \right) \, d\boldsymbol{x} \\
&= \sum_{\ell,k=0}^{L} \int_\Omega \nabla w_\ell \cdot \nabla w_k \, d\boldsymbol{x} \\
&\overset{(18)}{\leq} C \sum_{\ell,k=0}^{L} 2^{-|\ell-k|/2} \left| w_\ell \right|_{H^1(\Omega)} \left| w_k \right|_{H^1(\Omega)} \\
&\leq C \sum_{\ell=0}^{L} \left(\sum_{k=0}^{L} 2^{-|\ell-k|/2} \left| w_k \right|_{H^1(\Omega)} \right) \left| w_\ell \right|_{H^1(\Omega)} .
\end{aligned}$$

Proof (Cont.)

Continuing with the estimate,

$$
\begin{aligned}
(\boldsymbol{v}_L, \boldsymbol{v}_L)_{\mathsf{A}_L} &\overset{\text{C.S.}}{\leq} C \left\{ \sum_{\ell=0}^{L} \left(\sum_{k=0}^{L} 2^{-|\ell-k|/2} \left| w_k \right|_{H^1(\Omega)} \right)^2 \right\}^{1/2} \left\{ \sum_{\ell=0}^{L} \left| w_\ell \right|_{H^1(\Omega)}^2 \right\}^{1/2} \\
&\overset{(16)}{\leq} C \left\{ \sum_{\ell=0}^{L} \left| w_\ell \right|_{H^1(\Omega)}^2 \right\}^{1/2} \left\{ \sum_{\ell=0}^{L} \left| w_\ell \right|_{H^1(\Omega)}^2 \right\}^{1/2} \\
&= C \sum_{\ell=0}^{L} \left| w_\ell \right|_{H^1(\Omega)}^2 \\
&\overset{(13)}{\leq} \frac{C}{C_5} \sum_{\ell=0}^{L} (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{C}_\ell} .
\end{aligned}
$$

Proof (Cont.)

Recall that, since decompositions are unique,

$$
\begin{aligned}
\lambda_{\max}(\mathsf{C}_{\mathrm{H}}\mathsf{A}_L) &\overset{(14)}{=} \max_{\boldsymbol{v}_L\in\mathbb{R}^{n_L}_\star} \frac{(\boldsymbol{v}_L,\boldsymbol{v}_L)_{\mathsf{A}_L}}{\sum_{\ell=0}^{L}(\boldsymbol{w}_\ell,\boldsymbol{w}_\ell)_{\mathsf{C}_\ell}} \\
&= \max_{\boldsymbol{v}_L\in\mathbb{R}^{n_L}_\star} \frac{\frac{C}{C_5}\sum_{\ell=0}^{L}(\boldsymbol{w}_\ell,\boldsymbol{w}_\ell)_{\mathsf{C}_\ell}}{\sum_{\ell=0}^{L}(\boldsymbol{w}_\ell,\boldsymbol{w}_\ell)_{\mathsf{C}_\ell}} \\
&\le C_8,
\end{aligned}
$$

using the estimate on the previous slide. □

Theorem

$$\kappa(\mathsf{C}_{\mathrm{H}}\mathsf{A}_L) = \frac{\lambda_{\max}(\mathsf{C}_{\mathrm{H}}\mathsf{A}_L)}{\lambda_{\min}(\mathsf{C}_{\mathrm{H}}\mathsf{A}_L)} \leq \frac{C_8}{C_7}\left(1 + |\log(h_L)|^2\right). \tag{19}$$

Proof.

The result follows from the last few lemmas. □

The BPX Preconditioner

The BPX Preconditioner

The BPX preconditioner has a slightly better performance than the hierarchical basis preconditioner, in the sense that the logarithmic dependence on h_L can be removed.

Definition (BPX Preconditioner)

Define the bilinear form $C_\ell : V_\ell \times V_\ell \to \mathbb{R}$ via

$$C_\ell\left(w_\ell, v_\ell\right) = \sum_{i=1}^{n_\ell} w_\ell(\boldsymbol{N}_{\ell,i}) v_\ell(\boldsymbol{N}_{\ell,i}), \quad \forall\, w_\ell, v_\ell \in V_\ell.$$

The associated matrix $\mathsf{C}_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$ is defined as

$$\left[\mathsf{C}_\ell\right]_{j,k} = C_\ell\left(\psi_{\ell,j}, \psi_{\ell,k}\right) = \delta_{j,k}, \quad 1 \le j, k \le n_\ell,$$

where $\mathcal{B}_\ell = \{\psi_{\ell,j}\}_{j=1}^{n_\ell}$ is the standard Lagrange nodal basis for the piecewise linear finite element space V_ℓ, $0 \le \ell \le L$. The **BPX preconditioner** is precisely

$$\mathsf{C}_{\mathrm{BPX}} := \sum_{\ell=0}^{L} \mathsf{P}_{\ell,L} \mathsf{C}_\ell^{-1} \mathsf{R}_{\ell,L} = \sum_{\ell=0}^{L} \mathsf{P}_{\ell,L} \mathsf{R}_{\ell,L}, \tag{20}$$

where $\mathsf{P}_{\ell,L} \in \mathbb{R}^{n_L \times n_\ell}$ is the standard multilevel prolongation matrix and $\mathsf{R}_{\ell,L} = \mathsf{P}_{\ell,L}^\top$.

Remark

Note that we have dropped, and will continue to drop, the superscripted V and just write $\mathcal{B}_\ell = \{\psi_{\ell,j}\}_{j=1}^{n_\ell}$ for the standard Lagrange nodal basis for the piecewise linear finite element space V_ℓ, $0 \leq \ell \leq L$.

Remark

In the BPX framework, we effectively are taking

$$\mathsf{Q}_\ell^L = \mathsf{P}_{\ell,L}.$$

Lemma

Assumption (SS1) holds for the BPX framework, that is, for every $u_L \in V_L$, there exists $v_\ell \in V_\ell$, $0 \leq \ell \leq L$, such that

$$u_L = \sum_{\ell=0}^{L} v_\ell,$$

or, equivalently

$$\boldsymbol{u}_L = \sum_{\ell=0}^{L} \mathsf{P}_{\ell,L} \boldsymbol{v}_\ell,$$

with

$$V_\ell \ni v_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell},$$

and

$$V_L \ni u_L \overset{\mathcal{B}_L}{\leftrightarrow} \boldsymbol{u}_L \in \mathbb{R}^{n_L}.$$

This decomposition is not unique.

Proof.

Exercise. □

Lemma

Let $0 \leq j \leq \ell$. For any $v_j \in V_j$ and $v_\ell \in V_\ell$,

$$\int_\Omega \nabla v_j \cdot \nabla v_\ell \, d\mathbf{x} \leq C 2^{-|j-\ell|/2} \frac{\|v_j\|_{L^2(\Omega)}}{h_j} \frac{\|v_\ell\|_{L^2(\Omega)}}{h_\ell}, \tag{21}$$

for some $C > 0$.

Proof.

This is follows from (17) and the inverse inequality

$$|v_j|_{H^1(\Omega)} \leq c h_j^{-1} \|v_j\|_{L^2(\Omega)} .$$

□

Lemma

For some $C_9 > 0$ that is independent of L,

$$\lambda_{\max}\left(\mathsf{C}_{\mathrm{BPX}}\mathsf{A}_L\right) \leq C_9.$$

for some $C_9 > 0$ that is independent of L.

Proof.

Let $u_L \in V_L$ be arbitrary. There exist $v_\ell \in V_\ell$, $0 \leq \ell \leq L$, such that

$$u_L = \sum_{\ell=0}^{L} v_\ell,$$

or

$$\boldsymbol{u}_L = \sum_{\ell=0}^{L} \mathsf{P}_{\ell,L}\boldsymbol{v}_\ell, \quad V_\ell \ni v_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}.$$

As usual, we write

$$(\boldsymbol{v}_\ell) \in \mathsf{Q}[\boldsymbol{u}_L],$$

though the decomposition is not unique.

Proof (Cont.)

Then,

$$
\begin{aligned}
(\boldsymbol{u}_L, \boldsymbol{u}_L)_{\mathsf{A}_L} &= \sum_{\ell=0}^{L} \sum_{j=0}^{L} a(v_j, v_\ell) \\
&\overset{(21)}{\leq} C \sum_{\ell=0}^{L} \sum_{j=0}^{L} 2^{-|j-\ell|/2} h_j^{-1} \|v_\ell\|_{L^2(\Omega)} h_\ell \|v_k\|_{L^2(\Omega)} \\
&\overset{(16)}{\leq} C \sum_{j=0}^{L} h_j^{-2} \|v_j\|_{L^2(\Omega)} \\
&\leq C \sum_{j=0}^{L} (\boldsymbol{v}_j, \boldsymbol{v}_j)_{\mathsf{C}_j} \\
&= C \sum_{j=0}^{L} (\mathsf{C}_j \boldsymbol{v}_j, \boldsymbol{v}_j)_j .
\end{aligned}
$$

Proof (Cont.)

Now, for $(\boldsymbol{v}_\ell) \in \mathrm{Q}[\boldsymbol{u}_L]$, as above,

$$\begin{aligned}
\lambda_{\max}(\mathsf{C}_{\mathrm{BPX}}\mathsf{A}_L) &\overset{(14)}{=} \max_{\boldsymbol{u}_L \in \mathbb{R}^{n_L}_\star} \frac{(\boldsymbol{u}_L, \boldsymbol{u}_L)_{\mathsf{A}_L}}{\displaystyle\min_{(\boldsymbol{w}_\ell)\in \mathrm{Q}[\boldsymbol{u}_L]} \sum_{\ell=0}^{L} (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{C}_\ell}} \\
&\leq \max_{\boldsymbol{u}_L \in \mathbb{R}^{n_L}_\star} \frac{C \sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_\ell}{\displaystyle\min_{(\boldsymbol{w}_\ell)\in \mathrm{Q}[\boldsymbol{u}_L]} \sum_{\ell=0}^{L} (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{C}_\ell}} \\
&\leq C_9.
\end{aligned}$$

Recall that the minimum in the denominator is achievable for some $(\boldsymbol{w}_\ell) \in \mathrm{Q}[\boldsymbol{u}_L]$, so we have taken $\boldsymbol{v}_\ell = \boldsymbol{w}_\ell$ in the third step to conclude the upper bound. □

Before we establish a lower for $\lambda_{\min}\left(\mathsf{C}_{\mathrm{BPX}}\mathsf{A}_L\right)$ we need another technical lemma.

Lemma

Let $u \in H_0^1(\Omega)$ be arbitrary. Then, for any $1 \le \ell \le L$,

$$\mathcal{R}_{\ell-1} u = \mathcal{R}_{\ell-1}(\mathcal{R}_\ell u), \tag{22}$$

where $\mathcal{R}_\ell : H_0^1(\Omega) \to V_\ell$ is the Ritz projection, for $0 \le \ell \le L$. In other words, $\mathcal{R}_{\ell-1} = \mathcal{R}_{\ell-1}\mathcal{R}_\ell$.

Proof.

By definition,

$$a(\mathcal{R}_{\ell-1}(\mathcal{R}_\ell u), w_{\ell-1}) = a(\mathcal{R}_\ell u, w_{\ell-1}), \quad \forall\, w_{\ell-1} \in V_{\ell-1}.$$

But, since $V_{\ell-1} \subset V_\ell$, we also have

$$a(\mathcal{R}_\ell u, w_{\ell-1}) = a(u, w_{\ell-1}), \quad \forall\, w_{\ell-1} \in V_{\ell-1}.$$

Proof (Cont.)

Also observe that

$$a(\mathcal{R}_{\ell-1} u, w_{\ell-1}) = a(u, w_{\ell-1}), \quad \forall\, w_{\ell-1} \in V_{\ell-1}.$$

Hence,

$$a(\mathcal{R}_{\ell-1}(\mathcal{R}_\ell u), w_{\ell-1}) = a(\mathcal{R}_{\ell-1} u, w_{\ell-1}), \quad \forall\, w_{\ell-1} \in V_{\ell-1}.$$

And we conclude that $\mathcal{R}_{\ell-1} = \mathcal{R}_{\ell-1}\mathcal{R}_\ell$ since

$$\mathcal{R}_{\ell-1}(\mathcal{R}_\ell u), \mathcal{R}_{\ell-1} u \in V_{\ell-1}.$$

$\square$

Lemma

There is a constant $C_{10} > 0$ that is independent of L, such that

$$\lambda_{\min}\left(\mathsf{C}_{\mathrm{BPX}}\mathsf{A}_L\right) \geq C_{10}.$$

for some $C > 10$ that is independent of L.

Proof.

Let $u_L \in V_L$ be arbitrary. Set

$$v_\ell := \mathcal{R}_\ell u_L - \mathcal{R}_{\ell-1} u_L, \quad 0 \leq \ell \leq L,$$

where $\mathcal{R}_\ell : H_0^1(\Omega) \to V_\ell$ is the Ritz projection, for $0 \leq \ell \leq L$, and $R_{-1} \equiv 0$. Since

$$\mathcal{R}_L u_L = u_L,$$

it follows that

$$u_L = \sum_{\ell=0}^{L} v_\ell \in V_L \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{u}_L = \sum_{\ell=0}^{L} \mathsf{P}_{\ell,L} \boldsymbol{v}_\ell \in \mathbb{R}^{n_L}, \quad v_\ell \in V_\ell \overset{\mathcal{B}_\ell}{\leftrightarrow} \boldsymbol{v}_\ell \in \mathbb{R}^{n_\ell}.$$

Proof (Cont.)

Moreover,

$$a(v_j, v_\ell) = 0, \quad 0 \le j \neq \ell \le L. \tag{23}$$

To see this, recall that, in general,

$$a(\mathcal{R}_\ell u_L, w_\ell) = a(u_L, w_\ell), \quad \forall\, w_\ell \in V_\ell.$$

Suppose $j < \ell$, for definiteness. Then, since $V_j \subset V_\ell$,

$$a(\mathcal{R}_\ell u_L, w_j) = a(u_L, w_j), \quad \forall\, w_j \in V_j.$$

In particular, since

$$v_j := \mathcal{R}_j u_L - \mathcal{R}_{j-1} u_L \in V_j \subset V_\ell,$$

it follows that

$$a(\mathcal{R}_\ell u_L, v_j) = a(u_L, v_j).$$

Likewise,

$$a(\mathcal{R}_{\ell-1} u_L, v_j) = a(u_L, v_j).$$

Subtracting, we have

$$a(\mathcal{R}_\ell u_L - \mathcal{R}_{\ell-1} u_L, v_j) = 0.$$

Proof (Cont.)

To make further progress, let us assume that Ω is convex. Then the standard regularity condition holds. And, for $1 \leq \ell \leq L$,

$$
\begin{aligned}
h_\ell^{-2} \|v_\ell\|_{L^2(\Omega)}^2 &= h_\ell^{-2} \|\mathcal{R}_\ell u_L - \mathcal{R}_{\ell-1} u_L\|_{L^2(\Omega)}^2 \\
&\overset{(22)}{=} h_\ell^{-2} \|\mathcal{R}_\ell u_L - \mathcal{R}_{\ell-1}(\mathcal{R}_\ell u_L)\|_{L^2(\Omega)}^2 \\
&\overset{\text{(Nitsche)}}{\leq} C h_\ell^{-2} h_\ell^2 |\mathcal{R}_\ell u_L - \mathcal{R}_{\ell-1}\mathcal{R}_\ell u_L|_{H^1(\Omega)}^2 \\
&= C |\mathcal{R}_\ell u_L - \mathcal{R}_{\ell-1}\mathcal{R}_\ell u_L|_{H^1(\Omega)}^2 \\
&= C |v_\ell|_{H^1(\Omega)}^2 . \qquad (24)
\end{aligned}
$$

Estimate (24) holds trivially for $\ell = 0$.

Proof (Cont.)

Therefore,

$$
\begin{aligned}
\sum_{\ell=0}^{L} (\mathsf{C}_\ell \boldsymbol{v}_\ell, \boldsymbol{v}_\ell)_\ell &\leq C \sum_{\ell=0}^{L} h_\ell^{-2} \|v_\ell\|_{L^2(\Omega)}^2 \\
&\overset{(24)}{\leq} C \sum_{\ell=0}^{L} |v_\ell|_{H^1(\Omega)}^2 \\
&\overset{(23)}{=} C\, |u_L|_{H^1(\Omega)}^2 \,. \qquad (25)
\end{aligned}
$$

Proof (Cont.)

So, finally,

$$\begin{aligned} \lambda_{\min}(\mathsf{C}_{\mathrm{BPX}}\mathsf{A}_L) &= \min_{\boldsymbol{u}_L \in \mathbb{R}^{n_L}_\star} \frac{(\boldsymbol{u}_L, \boldsymbol{u}_L)_{\mathsf{A}_L}}{\displaystyle\min_{(\boldsymbol{w}_\ell)\in \mathsf{Q}[\boldsymbol{u}_L]} \sum_{\ell=0}^{L} (\boldsymbol{w}_\ell, \boldsymbol{w}_\ell)_{\mathsf{C}_\ell}} \\ &\overset{(25)}{\geq} \min_{\boldsymbol{u}_L \in \mathbb{R}^{n_L}_\star} \frac{(\mathsf{A}_L \boldsymbol{u}_L, \boldsymbol{u}_L)_L}{C\,|u_L|_{H^1(\Omega)}} \\ &= C_{10}. \end{aligned}$$

□

Theorem

$$\kappa\left(\mathsf{C}_{\mathrm{BPX}}\mathsf{A}_L\right) = \frac{\lambda_{\max}\left(\mathsf{C}_{\mathrm{BPX}}\mathsf{A}_L\right)}{\lambda_{\min}\left(\mathsf{C}_{\mathrm{BPX}}\mathsf{A}_L\right)} \leq \frac{C_9}{C_{10}}. \tag{26}$$

Proof.

Follows from the previous lemmas. □