

Math 674

Multigrid Methods: A Mostly Matrix-Based Approach

Chapter 09: Additive Preconditioners Based on Subspace Decompositions

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Chapter 09, Part 2 of 2 Additive Preconditioners Based on Subspace Decompositions



Hierarchical Basis Preconditioner



Now, we need to connect the spaces W_j to V_ℓ where $0 \le j \le \ell$. In so doing, we will have the tools to build a preconditioner based on the hierarchical basis. Be careful, the number of indices in this section can get a little overwhelming.

Proposition

Let $\mathcal{B}_{j}^{W}=\{\phi_{j,i}\}_{i=1}^{m_{j}}$ and $\mathcal{B}_{\ell}^{V}=\{\psi_{\ell,i}\}_{i=1}^{n_{\ell}}$ be the usual bases for W_{j} and V_{ℓ} , respectively. For each $0\leq j\leq \ell$, there are unique numbers

$$q_{j,k,i}^{\ell} \in \mathbb{R}, \quad 1 \leq k \leq n_{\ell}, \quad 1 \leq i \leq m_{j},$$

such that

$$\phi_{j,i} = \sum_{k=1}^{n_\ell} q_{j,k,i}^\ell \psi_{\ell,k}. \tag{1}$$

Proof.

Exercise.





Definition (Hierarchical Prolongation Matrix)

Define the matrix $\mathsf{Q}_{i}^{\ell} \in \mathbb{R}^{n_{\ell} imes m_{j}}$ via

$$\left[Q_j^\ell\right]_{k,i}:=q_{j,k,i}^\ell,\quad 1\leq k\leq n_\ell,\quad 1\leq i\leq m_j.$$

 Q_j^{ℓ} is called a **hierarchical prolongation matrix**.



Lemma

Suppose that Q_j^ℓ is a hierarchical prolongation matrix and $\mathbf{w}_j \in \mathbb{R}^{m_j}$ is the coordinate vector of the function $\mathbf{w}_j \in W_j$ with respect to the basis \mathcal{B}_j^W . Then,

$$\operatorname{rank}(\mathsf{Q}_j^\ell)=m_j,$$

and the coordinate vector of $w_j \in V_\ell$ in the basis \mathcal{B}_ℓ^V is simply

$$\mathsf{Q}_{j}^{\ell}\mathbf{w}_{j}\in\mathbb{R}^{n_{\ell}}.$$

Proof.

Exercise.



Remark

Note that the family of spaces W_j are hierarchical, but are not nested

$$W_0 \not\subset W_1 \not\subset W_2 \cdots$$
.

Furthermore, it makes no sense to stack the prolongation matrices as we did in the past:

$$\mathsf{Q}_j^\ell \neq \mathsf{Q}_k^\ell \mathsf{Q}_j^k,$$

for $j < k < \ell$. In fact, the product on the right hand side is not usually defined.



Definition

Define the bilinear form $C_j:W_j\times W_j\to\mathbb{R}$ via

$$C_{j}\left(w_{j},v_{j}\right) \coloneqq \sum_{r=1}^{m_{j}} w_{j}\left(\boldsymbol{N}_{j,r}^{W}\right) v_{j}\left(\boldsymbol{N}_{j,r}^{W}\right), \quad \forall \ w_{j},v_{j} \in W_{j}.$$

Let $\mathcal{B}_j^W=\{\phi_{j,i}\}_{i=1}^{m_j}$ be the usual basis for W_j . Define the matrix $\mathsf{C}_j\in\mathbb{R}^{m_j\times m_j}$ via

$$[C_{j}]_{i,k} := C_{j} (\phi_{j,i}, \phi_{j,k})$$

$$= \sum_{r=1}^{m_{j}} \phi_{j,i} (\mathbf{N}_{j,r}^{W}) \phi_{j,k} (\mathbf{N}_{j,r}^{W})$$

$$= \sum_{r=1}^{m_{j}} \delta_{ir} \delta_{kr}$$

$$= \delta_{ik}. \tag{2}$$



Definition (Hierarchical Basis Preconditioner)

Suppose that $\mathcal{B}_{\ell}^V = \{\psi_{\ell,i}\}_{i=1}^{n_\ell}$ is the usual basis for the finite element space V_ℓ . Let $A_L \in \mathbb{R}^{n_L \times n_L}$ be the SPD matrix defined via

$$[\mathsf{A}_L]_{i,j} = \mathsf{a}(\psi_{L,j},\psi_{L,i}), \quad 1 \leq i,j \leq \mathsf{n}_L,$$

where

$$a(u,v) = (\nabla u, \nabla v)_{L^2}, \quad \forall \ u,v \in H_0^1(\Omega).$$

The hierarchical basis preconditioner for A_L is defined as

$$C_{H} = \sum_{\ell=0}^{L} Q_{\ell}^{L} C_{\ell}^{-1} Z_{\ell}^{L} = \sum_{\ell=0}^{L} Q_{\ell}^{L} Z_{\ell}^{L}, \tag{3}$$

where C_ℓ is as in (2), $Q_\ell^L \in \mathbb{R}^{n_L \times m_\ell}$ is the hierarchical prolongation matrix from a previous definition and

$$\mathsf{Z}_\ell^{\mathit{L}} = \left(\mathsf{Q}_\ell^{\mathit{L}}\right)^{ op}$$
 .

Lemma



Assumption (SS1) holds for the hierarchical basis decomposition. In particular, for any object

$$u_L \in \mathbb{R}^{n_L} \overset{\mathcal{B}_L^V}{\leftrightarrow} u_L \in V_L$$

there exist unique objects

$$\mathbf{w}_{\ell} \in \mathbb{R}^{m_{\ell}} \overset{\mathcal{B}_{\ell}^{W}}{\leftrightarrow} \mathbf{w}_{\ell} \in W_{\ell}, \quad 0 \leq \ell \leq L,$$

such that

$$\boldsymbol{u}_{L} = \sum_{\ell=0}^{L} Q_{\ell}^{L} \boldsymbol{w}_{\ell} \in \mathbb{R}^{n_{L}} \overset{\mathcal{B}_{L}^{V}}{\leftrightarrow} u_{L} = \sum_{\ell=0}^{L} w_{\ell} \in V_{L}.$$

Furthermore, the hierarchical basis preconditioner, B_H, defined in (3), is SPD.

Proof.

This follows from the lemmas on the last slide deck. The details are left for an exercise.

Remark

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Our goal is now to show that

$$\lambda_{\min}(\mathsf{C}_H\mathsf{A}_L) \geq C \left(1 + |\mathsf{log}(h_L)|^2\right)^{-1},$$

and

$$\lambda_{\max}(\mathsf{C}_H\mathsf{A}_L) \leq C,$$

where these constants are positive and independent of L. If this is the case,

$$\frac{\lambda_{\mathsf{max}}}{\lambda_{\mathsf{min}}} =: \kappa(\mathsf{C}_H \mathsf{A}_L) \leq C \left(1 + \left| \mathsf{log}(\mathit{h}_L) \right|^2 \right).$$

This estimate is quite useful, since the logarithmic dependence on h_L is quite weak. For example, suppose

$$h_L=\frac{1}{2^L},$$

which is entirely reasonable. Then

$$|\log(h_L)|^2 = L^2 |\log(1/2)|^2$$
.

Our analysis that follows will only work for d = 2.



Now, we need some technical lemmas. For more details, see the book by Brenner and Scott.

Theorem (Mean-Zero Poincaré)

Suppose that Ω is an open polyhedral set in \mathbb{R}^d . Then, for every $u \in H^1(\Omega)$,

$$\|u - \bar{u}\|_{L^2(\Omega)} \le C \|\nabla u\|_{L^2(\Omega)},$$
 (4)

for some constant C>0 that is independent of u by dependent upon Ω , where \bar{u} is the average of u:

$$\bar{u} := \frac{1}{|\Omega|} \int_{\Omega} u(x) dx.$$

As a consequence, for every $u \in H^1(\Omega)$,

$$||u - \bar{u}||_{H^{1}(\Omega)} \le C |u - \bar{u}|_{H^{1}(\Omega)} = C |u|_{H^{1}(\Omega)},$$
 (5)

for some constant C > 0 that is independent of u by dependent upon Ω .



Theorem (Inverse inequality)

Suppose that Ω is an open polygonal domain in \mathbb{R}^d , \mathcal{T}_ℓ , $0 \leq \ell \leq L$ is a nested family of triangulations of Ω , and S_ℓ , $0 \leq \ell \leq L$, are the associated piecewise-linear finite element spaces. Assume that $1 \leq q \leq \infty$. Then, for all $v \in S_\ell$ and all $K \in \mathcal{T}_\ell$,

$$||v||_{H^1(K)} \le Ch_{\ell}^{-1+d/2-d/q} ||v||_{L^q(K)},$$
 (6)

for some constant C>0 that is independent of ℓ but depends on the shape of K.

Proof.

See Section 5.3 in the book by Brenner and Scott.



Theorem

Suppose that Ω is an open polyhedral domain in \mathbb{R}^d , \mathcal{T}_ℓ , $0 \leq \ell \leq L$ is a nested family of triangulations of Ω , and S_ℓ , $0 \leq \ell \leq L$, are the associated piecewise-linear finite element spaces. Then, for all $v_\ell \in S_\ell$, $\ell \geq 1$,

$$\|v_{\ell} - \mathcal{I}_{\ell-1}v_{\ell}\|_{L^{2}(\Omega)} \le Ch_{\ell} |v_{\ell}|_{H^{1}(\Omega)}, \tag{7}$$

for some constant C > 0 that is independent of ℓ .

Proof.

Note that the previous interpolation error estimate does not cover this case, since v_{ℓ} is not in $H^2(\Omega)$. However, the stated result still holds since we are interpolating a very specific class of functions. The proof of the one dimensional case is left as an exercise

In two space dimensions $H^1\hookrightarrow L^p$, for any $1\leq p<\infty$. We cannot quite get control for $p=\infty$. But, if the function space is finite dimensional we can almost get control of the $p=\infty$ case. Here is the result from Section 4.9 in the book by Brenner and Scott.



Theorem

Suppose that Ω is an open polygonal domain in \mathbb{R}^2 and \mathcal{T}_ℓ , $0 \le \ell \le L$ is a nested family of triangulations of Ω . Then, for any $v_\ell \in V_\ell$,

$$\|v_\ell\|_{L^\infty(\Omega)} \leq C\sqrt{1+\left|\log(h_\ell)\right|}\left|v_\ell\right|_{H^1(\Omega)},$$

for some constant C>0 that is independent of ℓ but depends upon the shape of Ω . Further, for all $v_{\ell} \in S_{\ell}$ and any $K \in \mathcal{T}_{\ell}$,

$$\left\| \mathsf{v}_{\ell} - \bar{\mathsf{v}}_{\ell} \right\|_{L^{\infty}(\mathcal{K})} \leq C \sqrt{1 + \left| \mathsf{log}(h_{\ell}) \right|} \left| \mathsf{v}_{\ell} \right|_{H^{1}(\mathcal{K})},$$

for some constant C>0 that is independent of ℓ but depends upon the shape of the triangle $K\in \mathcal{T}_\ell$, where

$$\bar{v}_{\ell} = \frac{1}{|K|} \int_{K} v_{\ell}(x) dx.$$



Lemma

Suppose that $0 \le j < \ell$. For any $v_{\ell} \in S_{\ell}$,

$$\|v_{\ell} - \bar{v}_{j,\ell}\|_{L^{\infty}(K_j)} \leq C\sqrt{1 + \left|\log\left(\frac{h_j}{h_{\ell}}\right)\right|} |v_{\ell}|_{H^1(K_j)}, \tag{8}$$

for some constant C>0 that is independent of j and ℓ but depends upon the shape of the triangle $K_j\in\mathcal{T}_j$, where

$$\bar{v}_{j,\ell}=\frac{1}{|K_j|}\int_{K_j}v_\ell(x)\,dx.$$

Proof.

Exercise.





Lemma

Assume that $\Omega \subset \mathbb{R}^2$ is a polygonal domain. Suppose that $\mathcal{I}_\ell : C(\overline{\Omega}) \to V_\ell$, $0 \le \ell \le L$, is the Lagrange nodal interpolation operator, and $\mathcal{I}_{-1} \equiv 0$. Then, for all $u_L \in V_L$,

$$\|\mathcal{I}_{\ell}u_{L}-\mathcal{I}_{\ell-1}u_{L}\|_{L^{2}(\Omega)}\leq Ch_{\ell}\left(1+\sqrt{L-\ell}\right)|u_{L}|_{H^{1}(\Omega)}.$$
 (9)

for some constant C>0 that is independent of but depends upon the shape of Ω .

Proof.

Define the piecewise constant function \bar{u}_I^ℓ such that

$$ar{u}_L^\ell|_K := rac{1}{|K|} \int_K u_L(x) \, dx, \quad \forall \, K \in \mathcal{T}_\ell.$$

Then,

$$\begin{split} \|\mathcal{I}_{\ell}u_{L} - \mathcal{I}_{\ell-1}u_{L}\|_{L^{2}(\Omega)}^{2} &= \|\mathcal{I}_{\ell}u_{L} - \mathcal{I}_{\ell-1}\left[\mathcal{I}_{\ell}[u_{L}]\right]\|_{L^{2}(\Omega)}^{2} \\ &\stackrel{(7)}{\leq} Ch_{\ell}^{2} \sum_{K \in \mathcal{T}_{\ell}} |\mathcal{I}_{\ell}[u_{L}]|_{H^{1}(K)}^{2} \\ &= Ch_{\ell}^{2} \sum_{K \in \mathcal{T}_{\ell}} \left|\mathcal{I}_{\ell}u_{L} - \bar{u}_{L}^{\ell}\right|_{H^{1}(K)}^{2} \\ &\stackrel{(6)}{\leq} Ch_{\ell}^{2} \sum_{K \in \mathcal{T}_{\ell}} \left\|\mathcal{I}_{\ell}u_{L} - \bar{u}_{L}^{\ell}\right\|_{L^{\infty}(K)}^{2} \\ &\stackrel{(8)}{\leq} Ch_{\ell}^{2} \sum_{K \in \mathcal{T}_{\ell}} \left\|u_{L} - \bar{u}_{L}^{\ell}\right\|_{L^{\infty}(K)}^{2} \\ &\stackrel{(8)}{\leq} Ch_{\ell}^{2} \sum_{K \in \mathcal{T}_{\ell}} \left(1 + \left|\log\left(\frac{h_{\ell}}{h_{L}}\right)\right|\right) |u_{L}|_{H^{1}(K)}^{2} \\ &= Ch_{\ell}^{2} \left(1 + \left|\log\left(\frac{h_{\ell}}{h_{L}}\right)\right|\right) |u_{L}|_{H^{1}(\Omega)}^{2} \,. \end{split}$$



Now, notice that

$$h_\ell = h_0 2^{-\ell} \quad 1 \le \ell \le L.$$

So,

$$\log(h_\ell/h_L) = \log(2^{L-\ell}) = (L-\ell)\log(2).$$

The result follows.



Lemma

There is some constant $C_7 > 0$, independent of L, such that

$$\lambda_{\min}(\mathsf{C}_{\mathsf{H}}\mathsf{A}_{L}) \geq \frac{\mathsf{C}_{7}}{1 + |\mathsf{log}(h_{L})|^{2}}.\tag{10}$$

Proof.

By definition, for any $w_{\ell,1}, w_{\ell,2} \in W_{\ell}$

$$C_{\ell}(w_{\ell,1}, w_{\ell,2}) = \sum_{i=1}^{m_{\ell}} w_{\ell,1}(N_{\ell,i}^{W}) w_{\ell,2}(N_{\ell,i}^{W}).$$

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Proof (Cont.)

Let

$$\mathbf{w}_{\ell,\alpha} \in \mathbb{R}^{m_{\ell}} \overset{\mathcal{B}_{\ell}^{W}}{\leftrightarrow} \mathbf{w}_{\ell,\alpha} \in W_{\ell}, \quad \alpha = 1, 2.$$

Then,

$$egin{aligned} \left(\mathsf{C}_{\ell} oldsymbol{w}_{\ell,1}, oldsymbol{w}_{\ell,2}
ight)_{\ell} &= \sum_{i=1}^{m_{\ell}} \left[oldsymbol{w}_{\ell,1}\right]_{i} \left[oldsymbol{w}_{\ell,2}\right]_{i} \ &= \sum_{i=1}^{m_{\ell}} w_{\ell,1} (oldsymbol{N}_{\ell,i}^{W}) w_{\ell,2} (oldsymbol{N}_{\ell,i}^{W}) \ &= C_{\ell} \left(w_{\ell,1}, w_{\ell,2}\right) \ &= C_{\ell} \left(w_{\ell,2}, w_{\ell,1}\right) \ &=: \left\langle w_{\ell,1}, w_{\ell,2} \right\rangle_{C_{\ell}}. \end{aligned}$$

This last object is like a mass-lumping inner product. All that is missing is a factor of h_{ℓ}^2 .



There are constants $C_3>0$, $C_4>0$ such that, for all $0\leq \ell \leq L$,

$$C_3 h_\ell^2 \langle w_\ell, w_\ell \rangle_{\mathsf{C}_\ell} \le \|w_\ell\|_{L^2(\Omega)}^2 \le C_4 h_\ell^2 \langle w_\ell, w_\ell \rangle_{\mathsf{C}_\ell}, \tag{11}$$

for all $w_{\ell} \in W_{\ell}$. Therefore, for any $w_{\ell} \in W_{\ell} \overset{\mathcal{B}_{\ell}^{W}}{\leftrightarrow} \mathbf{w}_{\ell} \in \mathbb{R}^{m_{\ell}}$,

$$(C_{\ell} \mathbf{w}_{\ell}, \mathbf{w}_{\ell})_{\ell} = h_{\ell}^{-2} h_{\ell}^{2} \langle w_{\ell}, w_{\ell} \rangle_{C_{\ell}}$$

$$\stackrel{(11)}{\leq} C_{3}^{-1} h_{\ell}^{-2} \| w_{\ell} \|_{L^{2}(\Omega)}^{2}$$

$$= C_{3}^{-1} h_{\ell}^{-2} \| w_{\ell} - \mathcal{I}_{\ell-1} w_{\ell} \|_{L^{2}(\Omega)}^{2}$$

$$\stackrel{(7)}{\leq} C_{3}^{-1} C \| w_{\ell} \|_{H^{1}(\Omega)}^{2}$$

$$\stackrel{(6)}{\leq} C_{3}^{-1} C h_{\ell}^{-2} \| w_{\ell} \|_{L^{2}(\Omega)}^{2}$$

$$\stackrel{(11)}{\leq} C_{3}^{-1} C C_{4} (C_{\ell} \mathbf{w}_{\ell}, \mathbf{w}_{\ell})_{\ell}. \tag{12}$$



Therefore, there are constants $C_5 > 0$, $C_6 > 0$, such that we have the equivalence

$$C_5 \sum_{\ell=0}^{L} |w_{\ell}|_{H^1(\Omega)}^2 \leq \sum_{\ell=0}^{L} (C_{\ell} \mathbf{w}_{\ell}, \mathbf{w}_{\ell})_{\ell} \leq C_6 \sum_{\ell=0}^{L} |w_{\ell}|_{H^1(\Omega)}^2,$$
 (13)

for any collection (w_ℓ) , with $w_\ell \in W_\ell$, in general. Now, let $u_L \in V_L$ be given and

$$u_L = \sum_{\ell=0}^L w_\ell, \quad \exists! \ w_\ell \in W_\ell, \quad 0 \le \ell \le L.$$

Recall that

$$w_{\ell} = \mathcal{I}_{\ell} u_{L} - \mathcal{I}_{\ell-1} u_{L}, \quad 1 < \ell < L,$$

and

$$w_0 = \mathcal{I}_0 u_L$$
.



We make the usual identification $w_\ell \in W_\ell \overset{\mathcal{B}_\ell^W}{\leftrightarrow} w_\ell \in \mathbb{R}^{m_\ell}$, and we observe that

$$(\boldsymbol{w}_{\ell})_{\ell=0}^{L} \in \mathsf{Q}[\boldsymbol{u}_{L}],$$

with respect to the hierarchical prolongation matrices from Definition 1. Then, from (12)

$$\begin{split} \sum_{\ell=0}^{L} \left(\mathsf{C}_{\ell} \boldsymbol{w}_{\ell}, \boldsymbol{w}_{\ell} \right)_{\ell} & \leq & C_{3}^{-1} C \sum_{\ell=0}^{L} h_{\ell}^{-2} \left\| \boldsymbol{w}_{\ell} \right\|_{L^{2}(\Omega)}^{2} \\ & \leq & C \sum_{\ell=0}^{L} \left(1 + \sqrt{L - \ell} \right)^{2} \left| \boldsymbol{u}_{L} \right|_{H^{1}(\Omega)}^{2} \\ & \leq & C \sum_{\ell=0}^{L} \left(1 + L - \ell \right) \left| \boldsymbol{u}_{L} \right|_{H^{1}(\Omega)}^{2} \\ & \leq & C \left(1 + L + L^{2} \right) \left| \boldsymbol{u}_{L} \right|_{H^{1}(\Omega)}^{2} \\ & \leq & C L^{2} \left| \boldsymbol{u}_{L} \right|_{H^{1}(\Omega)}^{2}. \end{split}$$

But

$$|u_L|_{H^1(\Omega)}^2 = a(u_L, u_L) = (A_L u_L, u_L)_L,$$

and

$$\begin{aligned} |\log(h_L)|^2 &= \left|\log(h_0 2^{-L})\right|^2 \\ &= |\log(h_0) - L \log(2)|^2 \\ &= \log^2(h_0) - 2 \log(h_0) L \log(2) + L^2 \log^2(2). \end{aligned}$$

So,

$$L^2 \leq C \left(1 + \left|\log(h_L)\right|^2\right), \quad \exists \ C > 0.$$

Thus,

$$\sum_{\ell=0}^{L} \left(\mathsf{C}_{\ell} \boldsymbol{w}_{\ell}, \boldsymbol{w}_{\ell} \right) \leq C \left(1 + \left| \mathsf{log}(\boldsymbol{h}_{L}) \right|^{2} \right) \left(\mathsf{A}_{L} \boldsymbol{u}_{L}, \boldsymbol{u}_{L} \right)_{L},$$

and

$$\lambda_{\min}(\mathsf{C}_{\mathrm{H}}\mathsf{A}_{\mathit{L}}) \geq \mathit{C}_{7}\left(1 + \left|\mathsf{log}(\mathit{h}_{\mathit{L}})\right|^{2}\right)^{-1}.$$





In the last line we use the "big" theorem from the first slide deck:

Theorem (Eigenvalues of CA)

Suppose that Assumption (SS1) holds for the set of prolongation matrices $\{Q_j\}_{j=0}^L$ and C is an additive subspace preconditioner with respect to $\{Q_j\}_{j=0}^L$. The eigenvalues of CA are positive, provided A is SPD with respect to $(\cdot\,,\,\cdot)$. Moreover

$$\lambda_{\max}(\mathsf{CA}) = \max_{\boldsymbol{u} \in \mathbb{R}^n_{\star}} \frac{(\mathsf{A}\boldsymbol{u}, \boldsymbol{u})}{\min\limits_{(\boldsymbol{w}_{\ell}) \in \mathsf{Q}[\boldsymbol{u}]} \sum_{\ell=0}^{L} (\mathsf{C}_{\ell}\boldsymbol{w}_{\ell}, \boldsymbol{w}_{\ell})_{\ell}}, \tag{14}$$

$$\lambda_{\min}(\mathsf{CA}) = \min_{\boldsymbol{u} \in \mathbb{R}_{\star}^{n}} \frac{(\mathsf{A}\boldsymbol{u}, \boldsymbol{u})}{\min_{(\boldsymbol{w}_{\ell}) \in \mathsf{Q}[\boldsymbol{u}]} \sum_{\ell=0}^{L} (\mathsf{C}_{\ell}\boldsymbol{w}_{\ell}, \boldsymbol{w}_{\ell})_{\ell}}.$$
 (15)

Next, we need a little technical lemma, a kind of convolution result.



Lemma

Let $a_j, b_j \ge 0, -\infty < j < \infty$, with

$$s_1 \coloneqq \sum_{j=-\infty}^{\infty} a_j \le \infty,$$

and

$$s_2 := \sum_{j=-\infty}^{\infty} b_j \leq \infty.$$

Then

$$\sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} a_{j-k} b_k \right)^2 \le s_1^2 s_2. \tag{16}$$

Proof.

Exercise.



Lemma

For any $v_{\ell} \in V_{\ell}$ and $v_k \in V_k$, $0 \le \ell \le k \le L$, and d=2, there is a constant C > 0 such that

$$\int_{\Omega} \nabla v_{\ell} \cdot \nabla v_{k} \, dx \leq 2^{(\ell-k)/2} C |v_{\ell}|_{H^{1}(\Omega)} \left(h_{k}^{-1} \|v_{k}\|_{L^{2}(\Omega)} \right). \tag{17}$$



Proof.

For any $K \in \mathcal{T}_{\ell}$, since $\Delta v_{\ell}|_{K} \equiv 0$,

$$\begin{split} \int_{K} \nabla v_{\ell} \cdot \nabla v_{k} \, dx &= \int_{\partial K} \frac{\partial v_{\ell}}{\partial n} v_{k} ds \\ &\leq C h_{\ell}^{-1} \left| v_{\ell} \right|_{H^{1}(K)} \int_{\partial K} \left| v_{k} \right| ds \\ &\leq C h_{\ell}^{-1} \left| v_{\ell} \right|_{H^{1}(K)} \left(h_{k} \sum_{\mathbf{N}_{k} \in \partial K} \left| v_{k}(\mathbf{N}_{k}) \right| \right) \\ &\overset{\text{C.S.}}{\leq} C h_{\ell}^{-1} \left| v_{\ell} \right|_{H^{1}(K)} h_{k} \left(\frac{h_{\ell}}{h_{k}} \right)^{1/2} \left(\sum_{\mathbf{N}_{k} \in \partial K} \left| v_{k}(\mathbf{N}_{k}) \right|^{2} \right)^{1/2} \\ &\leq C \left(\frac{h_{k}}{h_{\ell}} \right)^{1/2} \left| v_{\ell} \right|_{H^{1}(K)} h_{k}^{-1} \left\| v_{k} \right\|_{L^{2}(K)}. \end{split}$$



Thus,

$$\int_{\Omega} \nabla v_{\ell} \cdot \nabla v_{k} \, dx = \sum_{K \in \mathcal{T}_{\ell}} \int_{K} \nabla v_{\ell} \cdot \nabla v_{k} \, dx
\leq C2^{(\ell-k)/2} \sum_{K \in \mathcal{T}_{\ell}} |v_{\ell}|_{H^{1}(K)} \, h_{k}^{-1} \, ||v_{k}||_{L^{2}(K)}
\stackrel{C.S.}{\leq} C2^{(\ell-k)/2} \, |v_{\ell}|_{H^{1}(\Omega)} \, h_{k}^{-1} \, ||v_{k}||_{L^{2}(\Omega)} \, .$$



Lemma (Strengthened Cauchy-Schwarz Inequality)

For any $w_\ell \in W_\ell$ and $w_k \in W_k$, $0 \le \ell \le k \le L$, there is a constant C > 0 such that

$$\int_{\Omega} \nabla w_{\ell} \cdot \nabla w_{k} \, d\mathbf{x} \leq 2^{(\ell-k)/2} C \left| w_{\ell} \right|_{H^{1}(\Omega)} \left| w_{k} \right|_{H^{1}(\Omega)}. \tag{18}$$

Proof.

Observe that

$$w_k = w_k - \mathcal{I}_{k-1} w_k.$$

We use the interpolation error estimate

$$\|w_k - \mathcal{I}_{k-1}w_k\|_{L^2(\Omega)} \leq Ch_k |w_k|_{H^1(\Omega)},$$

to conclude that

$$\|w_k\|_{L^2(\Omega)} \leq Ch_k |w_k|_{H^1(\Omega)}$$
.



Now, we use the last result. Since $w_{\ell} \in V_{\ell}$ and $w_k \in V_k$,

$$\int_{\Omega} \nabla w_{\ell} \cdot \nabla w_{k} \, dx \leq C2^{(\ell-k)/2} |w_{\ell}|_{H^{1}(\Omega)} h_{k}^{-1} ||w_{k}||_{L^{2}(\Omega)}
\leq 2^{(\ell-k)/2} C |w_{\ell}|_{H^{1}(\Omega)} |w_{k}|_{H^{1}(\Omega)}.$$



Lemma

There is a constant $C_8 > 0$, independent of L, such that

$$\lambda_{\mathsf{max}}(\mathsf{C}_{\mathsf{H}}\mathsf{A}_{\mathit{L}}) \leq \mathit{C}_{8}.$$

Proof.

Let $v_L \in V_L$ be arbitrary.

$$v_L \in V_L \stackrel{\mathcal{B}_L}{\leftrightarrow} \mathbf{v}_L \in \mathbb{R}^{n_L}$$
.

There exist unique $w_\ell \in W_\ell \overset{\mathcal{B}_\ell^W}{\leftrightarrow} \mathbf{w}_\ell \in \mathbb{R}^{m_\ell}$, $\ell = 0, \dots, L$, such that

$$v_L = \sum_{\ell=0}^L w_\ell \overset{\mathcal{B}_L}{\leftrightarrow} v_L = \sum_{\ell=0}^L Q_\ell^L w_\ell.$$

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Then,

$$\begin{aligned} (\mathbf{v}_{L}, \mathbf{v}_{L})_{A_{L}} &= (\mathbf{v}_{L}, A_{L} \mathbf{v}_{L})_{L} \\ &= a(\mathbf{v}_{L}, \mathbf{v}_{L}) \\ &= a \left(\sum_{\ell=0}^{L} w_{\ell}, \sum_{k=0}^{L} w_{k} \right) \\ &= \int_{\Omega} \left(\nabla \sum_{\ell=0}^{L} w_{\ell} \right) \cdot \left(\nabla \sum_{k=0}^{L} w_{k} \right) d\mathbf{x} \\ &= \sum_{\ell,k=0}^{L} \int_{\Omega} \nabla w_{\ell} \cdot \nabla w_{k} d\mathbf{x} \\ &\stackrel{(18)}{\leq} C \sum_{\ell,k=0}^{L} 2^{-|\ell-k|/2} |w_{\ell}|_{H^{1}(\Omega)} |w_{k}|_{H^{1}(\Omega)} \\ &\leq C \sum_{\ell=0}^{L} \left(\sum_{k=0}^{L} 2^{-|\ell-k|/2} |w_{k}|_{H^{1}(\Omega)} \right) |w_{\ell}|_{H^{1}(\Omega)} . \end{aligned}$$



Continuing with the estimate,

$$(\mathbf{v}_{L}, \mathbf{v}_{L})_{A_{L}} \overset{\text{C.S.}}{\leq} C \left\{ \sum_{\ell=0}^{L} \left(\sum_{k=0}^{L} 2^{-|\ell-k|/2} |w_{k}|_{H^{1}(\Omega)} \right)^{2} \right\}^{1/2} \left\{ \sum_{\ell=0}^{L} |w_{\ell}|_{H^{1}(\Omega)}^{2} \right\}^{1/2}$$

$$\overset{(16)}{\leq} C \left\{ \sum_{\ell=0}^{L} |w_{\ell}|_{H^{1}(\Omega)}^{2} \right\}^{1/2} \left\{ \sum_{\ell=0}^{L} |w_{\ell}|_{H^{1}(\Omega)}^{2} \right\}^{1/2}$$

$$= C \sum_{\ell=0}^{L} |w_{\ell}|_{H^{1}(\Omega)}^{2}$$

$$\overset{(13)}{\leq} \frac{C}{C_{5}} \sum_{\ell=0}^{L} (\mathbf{w}_{\ell}, \mathbf{w}_{\ell})_{C_{\ell}} .$$



Recall that, since decompositions are unique,

$$\lambda_{\max}(C_{H}A_{L}) \stackrel{\text{(14)}}{=} \max_{\mathbf{v}_{L} \in \mathbb{R}_{\star}^{n_{L}}} \frac{(\mathbf{v}_{L}, \mathbf{v}_{L})_{A_{L}}}{\sum_{\ell=0}^{L} (\mathbf{w}_{\ell}, \mathbf{w}_{\ell})_{C_{\ell}}}$$

$$= \max_{\mathbf{v}_{L} \in \mathbb{R}_{\star}^{n_{L}}} \frac{\frac{C}{C_{5}} \sum_{\ell=0}^{L} (\mathbf{w}_{\ell}, \mathbf{w}_{\ell})_{C_{\ell}}}{\sum_{\ell=0}^{L} (\mathbf{w}_{\ell}, \mathbf{w}_{\ell})_{C_{\ell}}}$$

$$\leq C_{8},$$

using the estimate on the previous slide.



Theorem

$$\kappa(\mathsf{C}_{\mathsf{H}}\mathsf{A}_{\mathsf{L}}) = \frac{\lambda_{\mathsf{max}}(\mathsf{C}_{\mathsf{H}}\mathsf{A}_{\mathsf{L}})}{\lambda_{\mathsf{min}}(\mathsf{C}_{\mathsf{H}}\mathsf{A}_{\mathsf{L}})} \le \frac{C_8}{C_7} \left(1 + \left| \mathsf{log}(h_{\mathsf{L}}) \right|^2 \right). \tag{19}$$

Proof.

The result follows from the last few lemmas.



The BPX Preconditioner

The BPX Preconditioner



The BPX preconditioner has a slightly better performance than the hierarchical basis preconditioner, in the sense that the logarithmic dependence on h_L can be removed.



Definition (BPX Preconditioner)

Define the bilinear form $C_\ell:V_\ell imes V_\ell o\mathbb{R}$ via

$$C_{\ell}\left(w_{\ell},v_{\ell}
ight) = \sum_{i=1}^{n_{\ell}} w_{\ell}(\mathbf{N}_{\ell,i})v_{\ell}(\mathbf{N}_{\ell,i}), \quad \forall w_{\ell},v_{\ell} \in V_{\ell}.$$

The associated matrix $C_{\ell} \in \mathbb{R}^{n_{\ell} \times n_{\ell}}$ is defined as

$$[\mathsf{C}_\ell]_{j,k} = C_\ell (\psi_{\ell,j}, \psi_{\ell,k}) = \delta_{j,k}, \quad 1 \leq j, k \leq n_\ell,$$

where $\mathcal{B}_\ell = \{\psi_{\ell,j}\}_{j=1}^{n_\ell}$ is the standard Lagrange nodal basis for the piecewise linear finite element space V_ℓ , $0 \le \ell \le L$. The **BPX preconditioner** is precisely

$$C_{BPX} := \sum_{\ell=0}^{L} P_{\ell,L} C_{\ell}^{-1} R_{\ell,L} = \sum_{\ell=0}^{L} P_{\ell,L} R_{\ell,L},$$
 (20)

where $\mathsf{P}_{\ell,L} \in \mathbb{R}^{n_L \times n_\ell}$ is the standard multilevel prolongation matrix and $\mathsf{R}_{\ell,L} = \mathsf{P}_{\ell,L}^\top$.



Remark

In the BPX framework, we effectively are taking

$$Q_{\ell}^L = P_{\ell,L}$$
 and $W_{\ell} = V_{\ell}$.



Assumption (SS1) holds for the BPX framework, that is, for every $u_L \in V_L$, there exists $v_\ell \in V_\ell$, $0 \le \ell \le L$, such that

$$u_L = \sum_{\ell=0}^L v_\ell,$$

or, equivalently

$$\mathbf{u}_L = \sum_{\ell=0}^L \mathsf{P}_{\ell,L} \mathbf{v}_{\ell},$$

with

$$V_{\ell} \ni v_{\ell} \stackrel{\mathcal{B}_{\ell}}{\leftrightarrow} \mathbf{v}_{\ell} \in \mathbb{R}^{n_{\ell}},$$

and

$$V_L \ni u_L \stackrel{\mathcal{B}_L}{\leftrightarrow} \boldsymbol{u}_L \in \mathbb{R}^{n_L}.$$

This decomposition is not unique.

Proof.

Exercise.



Let $0 \le j \le \ell$. For any $v_i \in V_i$ and $v_\ell \in V_\ell$,

$$\int_{\Omega} \nabla v_j \cdot \nabla v_\ell \, dx \le C 2^{-|j-\ell|/2} \frac{\|v_j\|_{L^2(\Omega)}}{h_j} \frac{\|v_\ell\|_{L^2(\Omega)}}{h_\ell}, \tag{21}$$

for some C > 0.

Proof.

This is follows from (17) and the inverse inequality

$$|v_j|_{H^1(\Omega)} \leq ch_j^{-1} ||v_j||_{L^2(\Omega)}$$
.



For some $C_9 > 0$ that is independent of L,

$$\lambda_{\max}(C_{\mathrm{BPX}}A_L) \leq C_9.$$

for some $C_9 > 0$ that is independent of L.

Proof.

Let $u_L \in V_L$ be arbitrary. There exist $v_\ell \in V_\ell$, $0 \le \ell \le L$, such that

$$u_L = \sum_{\ell=0}^L v_\ell,$$

or

$$\mathbf{u}_{L} = \sum_{\ell=0}^{L} \mathsf{P}_{\ell,L} \mathbf{v}_{\ell}, \quad V_{\ell} \ni \mathbf{v}_{\ell} \stackrel{\mathcal{B}_{\ell}}{\leftrightarrow} \mathbf{v}_{\ell} \in \mathbb{R}^{n_{\ell}}.$$

As usual, we write

$$(\mathbf{v}_{\ell}) \in \mathsf{Q}[\mathbf{u}_{L}],$$

though the decomposition is not unique.

T

Proof (Cont.)

Then,

$$(\boldsymbol{u}_{L}, \boldsymbol{u}_{L})_{A_{L}} = \sum_{\ell=0}^{L} \sum_{j=0}^{L} a(\boldsymbol{v}_{j}, \boldsymbol{v}_{\ell})$$

$$\stackrel{(21)}{\leq} C \sum_{\ell=0}^{L} \sum_{j=0}^{L} 2^{-|j-\ell|/2} h_{j}^{-1} \|\boldsymbol{v}_{\ell}\|_{L^{2}(\Omega)} h_{\ell} \|\boldsymbol{v}_{k}\|_{L^{2}(\Omega)}$$

$$\stackrel{(16)}{\leq} C \sum_{j=0}^{L} h_{j}^{-2} \|\boldsymbol{v}_{j}\|_{L^{2}(\Omega)}$$

$$\stackrel{(??)}{\leq} C \sum_{j=0}^{L} (\boldsymbol{v}_{j}, \boldsymbol{v}_{j})_{C_{j}}$$

$$= C \sum_{i=0}^{L} (C_{j} \boldsymbol{v}_{j}, \boldsymbol{v}_{j})_{j} .$$



Now, for $(\mathbf{v}_{\ell}) \in \mathbb{Q}[\mathbf{u}_{L}]$, as above,

$$\lambda_{\max}(\mathsf{C}_{\mathrm{BPX}}\mathsf{A}_{L}) \stackrel{\text{(14)}}{=} \max_{\substack{\boldsymbol{u}_{L} \in \mathbb{R}_{\star}^{n_{L}} \\ \boldsymbol{u}_{L} \in \mathbb{R}_{\star}^{n_{L}}}} \frac{(\boldsymbol{u}_{L}, \boldsymbol{u}_{L})_{\mathsf{A}_{L}}}{\min_{\substack{\boldsymbol{w}_{\ell}) \in \mathsf{Q}[\boldsymbol{u}_{L}] \\ \boldsymbol{u}_{L} \in \mathbb{Q}[\boldsymbol{u}_{L}]}} \sum_{\ell=0}^{L} (\boldsymbol{w}_{\ell}, \boldsymbol{w}_{\ell})_{\mathsf{C}_{\ell}}}$$

$$\leq \max_{\boldsymbol{u}_{L} \in \mathbb{R}_{\star}^{n_{L}}} \frac{C \sum_{\ell=0}^{L} (\mathsf{C}_{\ell} \boldsymbol{v}_{\ell}, \boldsymbol{v}_{\ell})_{\ell}}{\min_{\substack{\boldsymbol{w}_{\ell}) \in \mathsf{Q}[\boldsymbol{u}_{L}] \\ \boldsymbol{u}_{\ell} \in \mathbb{Q}[\boldsymbol{u}_{L}]}} \sum_{\ell=0}^{L} (\boldsymbol{w}_{\ell}, \boldsymbol{w}_{\ell})_{\mathsf{C}_{\ell}}}$$

$$\leq C_{9}.$$

Recall that the minimum in the denominator is achievable for some $(w_\ell) \in Q[u_L]$, so we have taken $v_\ell = w_\ell$ in the third step to conclude the upper bound.



There is a constant $C_{10} > 0$ that is independent of L, such that

$$\lambda_{\min}\left(\mathsf{C}_{\mathrm{BPX}}\mathsf{A}_{\mathit{L}}\right) \geq \mathit{C}_{10}.$$

for some C > 10 that is independent of L.

Proof.

Let $u_L \in V_L$ be arbitrary. Set

$$v_{\ell} := \mathcal{R}_{\ell} u_L - \mathcal{R}_{\ell-1} u_L, \quad 0 \le \ell \le L,$$

where $\mathcal{R}_{\ell}: H^1_0(\Omega) \to V_{\ell}$ is the Ritz projection, for $0 \le \ell \le L$, and $R_{-1} \equiv 0$. Since

$$\mathcal{R}_L u_L = u_L$$

it follows that

$$u_L = \sum_{\ell=0}^L v_\ell \in V_L \stackrel{\mathcal{B}_\ell}{\leftrightarrow} u_L = \sum_{\ell=0}^L \mathsf{P}_{\ell,L} v_\ell \in \mathbb{R}^{n_L}, \quad v_\ell \in V_\ell \stackrel{\mathcal{B}_\ell}{\leftrightarrow} v_\ell \in \mathbb{R}^{n_\ell}.$$



Moreover,

$$a(v_j, v_\ell) = 0, \quad 0 \le j \ne \ell \le L. \tag{22}$$

To see this, recall that, in general,

$$a(\mathcal{R}_{\ell}u_{L}, w_{\ell}) = a(u_{L}, w_{\ell}), \quad \forall w_{\ell} \in V_{\ell}.$$

Suppose $j < \ell$, for definiteness. Then, since $V_j \subset V_\ell$,

$$a(\mathcal{R}_{\ell}u_L, w_j) = a(u_L, w_j), \quad \forall w_j \in V_j.$$

In particular, since

$$v_j := \mathcal{R}_j u_L - \mathcal{R}_{j-1} u_L \in V_j \subset V_\ell$$

it follows that

$$a(\mathcal{R}_{\ell}u_{L},v_{j})=a(u_{L},v_{j}).$$

Likewise,

$$a(\mathcal{R}_{\ell-1}u_L,v_j)=a(u_L,v_j).$$

Subtracting, we have

$$a(\mathcal{R}_{\ell}u_{L}-\mathcal{R}_{\ell-1}u_{L},v_{i})=0.$$



To make further progress, let us assume that Ω is convex. Then the standard regularity condition holds. And, for $1 \le \ell \le L$,

$$h_{\ell}^{-2} \| v_{\ell} \|_{L^{2}(\Omega)}^{2} = h_{\ell}^{-2} \| \mathcal{R}_{\ell} u_{L} - \mathcal{R}_{\ell-1} u_{L} \|_{L^{2}(\Omega)}^{2}$$

$$= h_{\ell}^{-2} \| \mathcal{R}_{\ell} u_{L} - \mathcal{R}_{\ell-1} \mathcal{R}_{\ell} u_{L} \|_{L^{2}(\Omega)}^{2}$$
(Nitsche)
$$\leq Ch_{\ell}^{-2} h_{\ell}^{2} \| \mathcal{R}_{\ell} u_{L} - \mathcal{R}_{\ell-1} \mathcal{R}_{\ell} u_{L} \|_{H^{1}(\Omega)}^{2}$$

$$= C \| \mathcal{R}_{\ell} u_{L} - \mathcal{R}_{\ell-1} \mathcal{R}_{\ell} u_{L} \|_{H^{1}(\Omega)}^{2}$$

$$= C \| v_{\ell} \|_{H^{1}(\Omega)}^{2}. \tag{23}$$



To see that $\mathcal{R}_{\ell-1} = \mathcal{R}_{\ell-1}\mathcal{R}_{\ell}$, a fact that we used above, let $u \in H_0^1(\Omega)$ be arbitrary. Then

$$a(\mathcal{R}_{\ell-1}(\mathcal{R}_{\ell}u), w_{\ell-1}) = a(\mathcal{R}_{\ell}u, w_{\ell-1}), \quad \forall w_{\ell-1} \in V_{\ell-1}.$$

But, since $V_{\ell-1} \subset V_{\ell}$, we also have

$$a(\mathcal{R}_{\ell}u, w_{\ell-1}) = a(u, w_{\ell-1}), \quad \forall w_{\ell-1} \in V_{\ell-1}.$$

Also observe that

$$a(\mathcal{R}_{\ell-1}u, w_{\ell-1}) = a(u, w_{\ell-1}), \quad \forall w_{\ell-1} \in V_{\ell-1}.$$

Hence,

$$a(\mathcal{R}_{\ell-1}(\mathcal{R}_{\ell}u), w_{\ell-1}) = a(\mathcal{R}_{\ell-1}u, w_{\ell-1}), \quad \forall w_{\ell-1} \in V_{\ell-1}.$$

And we conclude that $\mathcal{R}_{\ell-1} = \mathcal{R}_{\ell-1} \mathcal{R}_{\ell}$ since

$$\mathcal{R}_{\ell-1}(\mathcal{R}_{\ell}u), \mathcal{R}_{\ell-1}u \in V_{\ell-1}.$$



Estimate (22) holds trivially for $\ell = 0$. Therefore,

$$\sum_{\ell=0}^{L} (C_{\ell} \mathbf{v}_{\ell}, \mathbf{v}_{\ell})_{\ell} \leq C \sum_{\ell=0}^{L} h_{\ell}^{-2} \|\mathbf{v}_{\ell}\|_{L^{2}(\Omega)}^{2} \\
\leq C \sum_{\ell=0}^{L} |\mathbf{v}_{\ell}|_{H^{1}(\Omega)}^{2} \\
\stackrel{(22)}{=} C |\mathbf{u}_{L}|_{H^{1}(\Omega)}^{2}.$$
(24)



So, finally,

$$\lambda_{\min}(\mathsf{C}_{\mathrm{BPX}}\mathsf{A}_{L}) = \min_{\substack{\boldsymbol{u}_{L} \in \mathbb{R}^{n_{L}}_{\star} \\ \boldsymbol{w}_{\ell} \in \mathsf{Q}[\boldsymbol{u}_{L}]}} \frac{(\boldsymbol{u}_{L}, \boldsymbol{u}_{L})_{\mathsf{A}_{L}}}{\min_{\substack{(\boldsymbol{w}_{\ell}) \in \mathsf{Q}[\boldsymbol{u}_{L}] \\ \boldsymbol{w}_{\ell} \in \mathsf{Q}[\boldsymbol{u}_{L}]}} \sum_{\ell=0}^{L} (\boldsymbol{w}_{\ell}, \boldsymbol{w}_{\ell})_{\mathsf{C}_{\ell}}}$$

$$\stackrel{(24)}{\geq} \min_{\substack{\boldsymbol{u}_{L} \in \mathbb{R}^{n_{L}}_{\star} \\ \boldsymbol{U}_{L} \in \mathbb{R}^{n_{L}}_{\star}}} \frac{(\mathsf{A}_{L}\boldsymbol{u}, \boldsymbol{u})_{L}}{C |\boldsymbol{u}_{L}|_{H^{1}(\Omega)}}$$

$$= C_{10}.$$



Theorem

$$\kappa\left(\mathsf{C}_{\mathrm{BPX}}\mathsf{A}_{L}\right) = \frac{\lambda_{\mathsf{max}}\left(\mathsf{C}_{\mathrm{BPX}}\mathsf{A}_{L}\right)}{\lambda_{\mathsf{min}}\left(\mathsf{C}_{\mathrm{BPX}}\mathsf{A}_{L}\right)} \leq \frac{C_{9}}{C_{10}}.\tag{25}$$

Proof.

Follows from the previous lemmas.