



Math 673

Multigrid Methods: A Mostly Matrix-Based Approach

Chapter 05: Multigrid

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Chapter 05, Part 2 of 2

Multigrid



Convergence of the Two-Grid Method Revisited



Theorem (Convergence of the Two-Grid Method)

Suppose that $L = 1$ (two-grid) $m_1 = m \geq 1$ and $m_2 = 0$ (one-sided). Suppose that Assumptions (G0, strong Galerkin condition), (A0, strong approximation property), and (S1, first smoothing property) all hold. Then

$$\left\| \mathbf{u}_1^E - \text{TG} \left(\mathbf{f}_1, \mathbf{u}_1^{(0)} \right) \right\|_{A_1} \leq C_{A0} C_{S1} m^{-1/2} \left\| \mathbf{u}_1^E - \mathbf{u}_1^{(0)} \right\|_{A_1},$$

where

$$A_1 \mathbf{u}_1^E = \mathbf{f}_1.$$

Written another way,

$$\left\| \mathbf{e}_1^{k+1} \right\|_{A_1} \leq C_{A0} C_{S1} m^{-1/2} \left\| \mathbf{e}_1^k \right\|_{A_1},$$

where $\mathbf{e}_1^k = \mathbf{u}_1^E - \mathbf{u}_1^{(0)}$.



Proof.

Recall that, in the present case,

$$E_1 = (I_1 - \tilde{N}_1) K_1^m,$$

and

$$\mathbf{e}_1^{k+1} = E_1 \mathbf{e}_1^k,$$

or, equivalently

$$\mathbf{u}_1^E - \text{TG}(\mathbf{f}_1, \mathbf{u}_1^{(0)}) = E_1 (\mathbf{u}_1^E - \mathbf{u}_1^{(0)}).$$

When we prove (G0) and (A0) imply (A1) in the last slide deck, we also see that

$$\left\| (I_\ell - \tilde{N}_\ell) \mathbf{u}_\ell \right\|_{A_\ell} \leq C_{A0} \rho_\ell^{-1/2} \|A_\ell \mathbf{u}_\ell\|_\ell, \quad (1)$$

for any $\mathbf{u}_\ell \in \mathbb{R}^{n_\ell}$.



Proof (Cont.)

Applying (1) (with $\ell = 1$), and using Assumption (S1), we have

$$\begin{aligned}\left\| \mathbf{e}_1^{k+1} \right\|_{A_1} &= \left\| \left(I_1 - \tilde{\Pi}_1 \right) K_1^m \mathbf{e}_1^k \right\|_{A_1} \\ &\stackrel{(1)}{\leq} C_{A0} \rho_1^{-1/2} \left\| A_1 K_1^m \mathbf{e}_1^k \right\|_1 \\ &\stackrel{(S1)}{\leq} C_{A0} \rho_1^{-1/2} C_{S1} \rho_1^{1/2} m^{-1/2} \left\| \mathbf{e}_1^k \right\|_{A_1} \\ &= C_{A0} C_{S1} m^{-1/2} \left\| \mathbf{e}_1^k \right\|_{A_1}.\end{aligned}$$





Next, we prove the two-grid method converges even with a significantly weakened approximation property. Before we get to that result, we need a technical lemma.

Lemma

For Richardson's smoother we have the following stabilities:

$$\|K_\ell \mathbf{v}_\ell\|_{A_\ell} \leq \|\mathbf{v}_\ell\|_{A_\ell}, \quad (2)$$

$$(K_\ell \mathbf{v}_\ell, \mathbf{v}_\ell)_\ell \leq (\mathbf{v}_\ell, \mathbf{v}_\ell)_\ell, \quad (3)$$

for all $\mathbf{v}_\ell \in \mathbb{R}^{n_\ell}$, $\ell \geq 0$



Proof.

Let $\mathbf{v}_\ell \in \mathbb{R}^{n_\ell}$ be arbitrary. Suppose that $B_\ell := \{\mathbf{w}_\ell^{(1)}, \mathbf{w}_\ell^{(2)}, \dots, \mathbf{w}_\ell^{(n_\ell)}\}$ is an orthonormal basis of eigenvectors of A_ℓ with respect to $(\cdot, \cdot)_\ell$. Then, there exist unique constants $\alpha_1, \alpha_2, \dots, \alpha_{n_\ell} \in \mathbb{R}$, such that

$$\mathbf{v}_\ell = \sum_{k=1}^{n_\ell} \alpha_k \mathbf{w}_\ell^{(k)}.$$

Recall

$$K_\ell = I_\ell - \Lambda_\ell^{-1} A_\ell,$$

with

$$\rho(A_\ell) =: \rho_\ell \leq \Lambda_\ell \leq C_R \rho_\ell,$$

where $C_R \geq 1$ is independent of ℓ . Then

$$K_\ell \mathbf{w}_\ell^{(k)} = \mu_\ell^{(k)} \mathbf{w}_\ell^{(k)},$$

where

$$\mu_\ell^{(k)} := \left(1 - \frac{\lambda_\ell^{(k)}}{\Lambda_\ell} \right).$$



Proof (Cont.)

The $\lambda_\ell^{(k)}$ are the positive eigenvalues for the SPD matrix A_ℓ , and the $\mu_\ell^{(k)}$ are the eigenvalues for K_ℓ , $k = 1, \dots, n_\ell$. Thus

$$\begin{aligned}\|K_\ell \mathbf{v}_\ell\|_{A_\ell}^2 &= (K_\ell \mathbf{v}_\ell, A_\ell K_\ell \mathbf{v}_\ell)_\ell \\ &= \sum_{k=1}^{n_\ell} \left(\mu_\ell^{(k)}\right)^2 \lambda_\ell^{(k)} \alpha_k^2.\end{aligned}$$

Recall for the Richardson's smoother, we have

$$\Lambda_\ell \geq \rho_\ell = \rho(A_\ell), \quad 1 \leq \ell \leq L, \quad (4)$$

and thus,

$$0 \leq \mu_\ell^{(k)} = 1 - \frac{\lambda_\ell^{(k)}}{\Lambda_\ell} \leq 1, \quad (5)$$

and we have

$$\|K_\ell \mathbf{v}_\ell\|_{A_\ell}^2 \leq \|\mathbf{v}_\ell\|_{A_\ell}^2.$$

Hence,

$$\|K_\ell \mathbf{v}_\ell\|_{A_\ell} \leq \|\mathbf{v}_\ell\|_{A_\ell}.$$



Proof (Cont.)

For the second estimate,

$$\begin{aligned}(\mathbf{K}_\ell \mathbf{v}_\ell, \mathbf{v}_\ell)_\ell &= \left(\mathbf{K}_\ell \sum_{k=1}^{n_\ell} \alpha_k \mathbf{w}_\ell^{(k)}, \sum_{k=1}^{n_\ell} \alpha_k \mathbf{w}_\ell^{(k)} \right)_\ell \\&= \left(\sum_{k=1}^{n_\ell} \alpha_k \mathbf{K}_\ell \mathbf{w}_\ell^{(k)}, \sum_{k=1}^{n_\ell} \alpha_k \mathbf{w}_\ell^{(k)} \right)_\ell \\&= \left(\sum_{k=1}^{n_\ell} \alpha_k \mu_\ell^{(k)} \mathbf{w}_\ell^{(k)}, \sum_{k=1}^{n_\ell} \alpha_k \mathbf{w}_\ell^{(k)} \right)_\ell \\&= \sum_{k=1}^{n_\ell} \alpha_k^2 \mu_\ell^{(k)} \\&\stackrel{(5)}{\leq} \sum_{k=1}^{n_\ell} \alpha_k^2 \\&= (\mathbf{v}_\ell, \mathbf{v}_\ell)_\ell.\end{aligned}$$





Theorem (α -Weak Convergence of the Two-Grid Method)

Suppose that $L = 1$ (two-grid) $m_1 = m \geq 1$ and $m_2 = 0$ (one-sided). Assume that Assumptions (G0, strong Galerkin condition) and (A2, α -weak approximation property), hold. Suppose that smoothing is performed with the Richardson method. Then

$$\left\| \mathbf{u}_1^E - \text{TG} \left(\mathbf{f}_1, \mathbf{u}_1^{(0)} \right) \right\|_{A_1} \leq \left(\frac{C_{A2} C_{S1}}{m^{1/2}} \right)^\alpha \left\| \mathbf{u}_1^E - \mathbf{u}_1^{(0)} \right\|_{A_1}, \quad (6)$$

where

$$A_1 \mathbf{u}_1^E = \mathbf{f}_1.$$

Written another way,

$$\left\| \mathbf{e}_1^{k+1} \right\|_{A_1} \leq \left(\frac{C_{A2} C_{S1}}{m^{1/2}} \right)^\alpha \left\| \mathbf{e}_1^k \right\|_{A_1}, \quad (7)$$

where $\mathbf{e}_1^k = \mathbf{u}_1^E - \mathbf{u}_1^{(0)}$ and $\mathbf{e}_1^{k+1} = \mathbf{u}_1^E - \mathbf{u}_1^{(3)}$.



Proof.

Again

$$E_1 = (I_1 - \tilde{N}_1) K_1^m,$$

and

$$\mathbf{e}_1^{k+1} = E_1 \mathbf{e}_1^k,$$

or, equivalently

$$\mathbf{u}_1^E - \text{TG}(\mathbf{f}_1, \mathbf{u}_1^{(0)}) = E_1 (\mathbf{u}_1^E - \mathbf{u}_1^{(0)}).$$

Since the strong Galerkin condition holds,

$$(I_1 - \tilde{N}_1)^2 = I_1 - \tilde{N}_1,$$

and it is always true that

$$(I_1 - \tilde{N}_1)^* = I_1 - \tilde{N}_1.$$



Proof (Cont.)

Thus, the α -weak approximation can be expressed as follows: for all $\mathbf{u}_\ell \in \mathbb{R}^{n_\ell}$ and $1 \leq \ell \leq L$,

$$\begin{aligned} \left\| (I_1 - \tilde{\Pi}_1) \mathbf{u}_1 \right\|_{A_1}^2 &= \left((I_1 - \tilde{\Pi}_1) \mathbf{u}_1, \mathbf{u}_1 \right)_{A_\ell} \\ &\leq \frac{C_{A2}^{2\alpha}}{\rho_1^\alpha} \|A_1 \mathbf{u}_1\|_1^{2\alpha} \|\mathbf{u}_1\|_{A_1}^{2(1-\alpha)}, \end{aligned}$$

for some $C_{A2} > 0$.



Proof (Cont.)

Next, recall that Richardson's method satisfies Assumption (S1). Using this fact, the stability of Richardson's method, and the α -weak approximation property (A2), we have

$$\begin{aligned}
 \left\| \mathbf{e}_1^{k+1} \right\|_{A_1} &= \left\| \left(I_1 - \tilde{\Pi}_1 \right) K_1^m \mathbf{e}_1^k \right\|_{A_1} \\
 &\leq \frac{C_{A2}^\alpha}{\rho_1^{\alpha/2}} \left\| A_1 K_1^m \mathbf{e}_1^k \right\|_1^\alpha \left\| K_1^m \mathbf{e}_1^k \right\|_{A_1}^{(1-\alpha)} \\
 &\stackrel{(Stability)}{\leq} \frac{C_{A2}^\alpha}{\rho_1^{\alpha/2}} \left\| A_1 K_1^m \mathbf{e}_1^k \right\|_1^\alpha \left\| \mathbf{e}_1^k \right\|_{A_1}^{(1-\alpha)} \\
 &\stackrel{(S1)}{\leq} \frac{C_{A2}^\alpha}{\rho_1^{\alpha/2}} C_{S1}^\alpha \rho_1^{\alpha/2} m^{-\alpha/2} \left\| \mathbf{e}_1^k \right\|_{A_1} \\
 &= \left(\frac{C_{A2} C_{S1}}{m^{1/2}} \right)^\alpha \left\| \mathbf{e}_1^k \right\|_{A_1}.
 \end{aligned}$$





Convergence of the W-Cycle Algorithm



In this section, we will prove that the W-cycle converges, provided that we perform enough smoothing iterations per cycle. The proof of the convergence of the W-cycle algorithm uses a technique called a perturbation argument. Basically, we will show that the error is equal to the error in the two-grid method plus a perturbation that we can control.

Theorem (Convergence of the One-Sided W-Cycle)

Suppose that $p \geq 2$, $m_1 = m \geq 1$, and $m_2 = 0$ (one-sided). Suppose, further, that Assumptions (G0, Galerkin condition) and (A0, strong approximation property) hold and the smoothing is done by Richardson's smoother. Then for any $0 < \gamma < 1$, m can be chosen large enough so that

$$\left\| \mathbf{u}_\ell^E - \text{MG} \left(\mathbf{g}_\ell, \ell, \mathbf{u}_\ell^{(0)} \right) \right\|_{A_\ell} \leq \gamma \left\| \mathbf{u}_\ell^E - \mathbf{u}_\ell^{(0)} \right\|_{A_\ell},$$

for any $\ell \geq 0$, where

$$A_\ell \mathbf{u}_\ell^E = \mathbf{g}_\ell.$$



Proof.

The proof is by induction.

(Base cases): The cases $\ell = 0$, and $\ell = 1$ (which is two-grid) are clearly true.

(Induction hypothesis): Assume

$$\|E_{\ell-1} \mathbf{w}_{\ell-1}\|_{A_{\ell-1}} \leq \gamma \|\mathbf{w}_{\ell-1}\|_{A_{\ell-1}}$$

is true for any $\mathbf{w}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}$.



Proof (Cont.)

(General case): Suppose that $\mathbf{q}_{\ell-1}^{(1,E)}, \mathbf{r}_{\ell-1}^{(1)} \in \mathbb{R}^{n_{\ell-1}}$ satisfy

$$A_{\ell-1} \mathbf{q}_{\ell-1}^{(1,E)} = \mathbf{r}_{\ell-1}^{(1)}.$$

Recall, $\mathbf{q}_{\ell-1}^{(1,E)}$ is the *exact* coarse grid correction. Then

$$\begin{aligned} \mathbf{u}_{\ell}^E - \text{MG}(\mathbf{g}_{\ell}, \ell, \mathbf{u}_{\ell}^{(0)}) &= \mathbf{u}_{\ell}^E - \mathbf{u}_{\ell}^{(2)} \\ &= \mathbf{u}_{\ell}^E - \left\{ \mathbf{u}_{\ell}^{(1)} + P_{\ell-1} \mathbf{q}_{\ell-1}^{(1)} \right\} \\ &= \mathbf{u}_{\ell}^E - \left(\mathbf{u}_{\ell}^{(1)} + P_{\ell-1} \mathbf{q}_{\ell-1}^{(1,E)} \right) \\ &\quad + P_{\ell-1} \left(\mathbf{q}_{\ell-1}^{(1,E)} - \mathbf{q}_{\ell-1}^{(1)} \right) \\ &= \mathbf{u}_{\ell}^E - \text{TG}(\mathbf{g}_{\ell}, \mathbf{u}_{\ell}^{(0)}) + P_{\ell-1} \left(\mathbf{q}_{\ell-1}^{(1,E)} - \mathbf{q}_{\ell-1}^{(1)} \right). \end{aligned}$$



Proof (Cont.)

Suppose that $m \in \mathbb{N}$ satisfies

$$0 < \left(\frac{C_{A0} C_{S1}}{\gamma - \gamma^p} \right)^2 \leq m. \quad (8)$$

We have proved Richardson's smoother satisfies Assumption (S1) in the last slide deck. Thus,

$$\begin{aligned} \left\| \mathbf{u}_\ell^E - \text{MG} \left(\mathbf{g}_\ell, \ell, \mathbf{u}_\ell^{(0)} \right) \right\|_{A_\ell} &\leq \left\| \mathbf{u}_\ell^E - \text{TG} \left(\mathbf{g}_\ell, \mathbf{u}_\ell^{(0)} \right) \right\|_{A_\ell} \\ &\quad + \left\| \mathbf{P}_{\ell-1} \left(\mathbf{q}_{\ell-1}^{(1,E)} - \mathbf{q}_{\ell-1}^{(1)} \right) \right\|_{A_\ell} \\ &\stackrel{(\text{TG Conv.})}{\leq} C_{A0} C_{S1} m^{-1/2} \left\| \mathbf{u}_\ell^E - \mathbf{u}_\ell^{(0)} \right\|_{A_\ell} \\ &\quad + \left\| \mathbf{P}_{\ell-1} \left(\mathbf{q}_{\ell-1}^{(1,E)} - \mathbf{q}_{\ell-1}^{(1)} \right) \right\|_{A_\ell}. \end{aligned} \quad (9)$$



Proof (Cont.)

Now, observe that, for any $\mathbf{w}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}$,

$$\begin{aligned}\|\mathbf{P}_{\ell-1}\mathbf{w}_{\ell-1}\|_{\mathbf{A}_{\ell}}^2 &= (\mathbf{P}_{\ell-1}\mathbf{w}_{\ell-1}, \mathbf{P}_{\ell-1}\mathbf{w}_{\ell-1})_{\mathbf{A}_{\ell}} \\ &= (\mathbf{P}_{\ell-1}\mathbf{w}_{\ell-1}, \mathbf{A}_{\ell}\mathbf{P}_{\ell-1}\mathbf{w}_{\ell-1})_{\ell} \\ &= \left(\mathbf{w}_{\ell-1}, \mathbf{P}_{\ell-1}^{\top}\mathbf{A}_{\ell}\mathbf{P}_{\ell-1}\mathbf{w}_{\ell-1}\right)_{\ell-1} \\ &= (\mathbf{w}_{\ell-1}, \mathbf{R}_{\ell-1}\mathbf{A}_{\ell}\mathbf{P}_{\ell-1}\mathbf{w}_{\ell-1})_{\ell-1} \\ &\stackrel{(G0)}{=} (\mathbf{w}_{\ell-1}, \mathbf{A}_{\ell-1}\mathbf{w}_{\ell-1})_{\ell-1} \\ &= (\mathbf{w}_{\ell-1}, \mathbf{w}_{\ell-1})_{\mathbf{A}_{\ell-1}} \\ &= \|\mathbf{w}_{\ell-1}\|_{\mathbf{A}_{\ell-1}}^2.\end{aligned}$$



Proof (Cont.)

In the proof of the Multigrid error relation theorem in the last slide deck, we showed that

$$\begin{aligned}
 \mathbf{q}_{\ell-1}^{(1,E)} - \mathbf{q}_{\ell-1}^{(1)} &= \mathbf{E}_{\ell-1}^p \mathbf{q}_{\ell-1}^{(1,E)} \\
 &\stackrel{(\text{MG Err. Rel.})}{=} \mathbf{E}_{\ell-1}^p \Pi_{\ell-1} \left(\mathbf{u}_{\ell}^E - \mathbf{u}_{\ell}^{(1)} \right) \\
 &= \mathbf{E}_{\ell-1}^p \Pi_{\ell-1} \mathbf{K}_{\ell}^m \left(\mathbf{u}_{\ell}^E - \mathbf{u}_{\ell}^{(0)} \right).
 \end{aligned}$$

Using the induction hypothesis,

$$\begin{aligned}
 \left\| \Pi_{\ell-1} \left(\mathbf{q}_{\ell-1}^{(1,E)} - \mathbf{q}_{\ell-1}^{(1)} \right) \right\|_{A_{\ell}} &= \left\| \mathbf{q}_{\ell-1}^{(1,E)} - \mathbf{q}_{\ell-1}^{(1)} \right\|_{A_{\ell-1}} \\
 &= \left\| \mathbf{E}_{\ell-1}^p \Pi_{\ell-1} \mathbf{K}_{\ell}^m \left(\mathbf{u}_{\ell}^E - \mathbf{u}_{\ell}^{(0)} \right) \right\|_{A_{\ell-1}} \\
 &\stackrel{\text{Ind. Hyp.}}{\leq} \gamma^p \left\| \Pi_{\ell-1} \mathbf{K}_{\ell}^m \left(\mathbf{u}_{\ell}^E - \mathbf{u}_{\ell}^{(0)} \right) \right\|_{A_{\ell-1}}.
 \end{aligned}$$



Proof (Cont.)

Since we are assuming the Galerkin condition (Assumption (G0)) holds, it follows that

$$\|\Pi_{\ell-1} \mathbf{w}_\ell\|_{A_{\ell-1}} = \|\tilde{\Pi}_\ell \mathbf{w}_\ell\|_{A_\ell}.$$

Furthermore,

$$\begin{aligned} \|\tilde{\Pi}_\ell \mathbf{w}_\ell\|_{A_\ell}^2 &= (\tilde{\Pi}_\ell \mathbf{w}_\ell, \tilde{\Pi}_\ell \mathbf{w}_\ell)_{A_\ell} \\ &= (\tilde{\Pi}_\ell^2 \mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} \\ &= (\tilde{\Pi}_\ell \mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} \\ &\stackrel{\text{C.S.}}{\leq} \|\tilde{\Pi}_\ell \mathbf{w}_\ell\|_{A_\ell} \|\mathbf{w}_\ell\|_{A_\ell}. \end{aligned}$$

So, we have the stability

$$\|\tilde{\Pi}_\ell \mathbf{w}_\ell\|_{A_\ell} \leq \|\mathbf{w}_\ell\|_{A_\ell}. \quad (10)$$



Proof (Cont.)

Therefore,

$$\begin{aligned}
 \left\| P_{\ell-1} \left(\mathbf{q}_{\ell-1}^{(1,E)} - \mathbf{q}_{\ell-1}^{(1)} \right) \right\|_{A_\ell} &\leq \gamma^p \left\| \Pi_{\ell-1} \mathbf{K}_\ell^m \left(\mathbf{u}_\ell^E - \mathbf{u}_\ell^{(0)} \right) \right\|_{A_{\ell-1}} \\
 &= \gamma^p \left\| \tilde{\Pi}_\ell \mathbf{K}_\ell^m \left(\mathbf{u}_\ell^E - \mathbf{u}_\ell^{(0)} \right) \right\|_{A_\ell} \\
 &\stackrel{(10)}{\leq} \gamma^p \left\| \mathbf{K}_\ell^m \left(\mathbf{u}_\ell^E - \mathbf{u}_\ell^{(0)} \right) \right\|_{A_\ell} \\
 &\stackrel{(\text{Stability})}{\leq} \gamma^p \left\| \mathbf{u}_\ell^E - \mathbf{u}_\ell^{(0)} \right\|_{A_\ell}.
 \end{aligned}$$



Proof (Cont.)

Combining this with estimate (9), we have

$$\begin{aligned}
 & \left\| \mathbf{u}_\ell^E - \text{MG} \left(\mathbf{g}_\ell, \ell, \mathbf{u}_\ell^{(0)} \right) \right\|_{A_\ell} \\
 & \leq C_{A0} C_{S1} m^{-1/2} \left\| \mathbf{u}_\ell^E - \mathbf{u}_\ell^{(0)} \right\|_{A_\ell} + \left\| \mathbf{P}_{\ell-1} \left(\mathbf{q}_{\ell-1}^{(1,E)} - \mathbf{q}_{\ell-1}^{(1)} \right) \right\|_{A_\ell} \\
 & \leq C_{A0} C_{S1} m^{-1/2} \left\| \mathbf{u}_\ell^E - \mathbf{u}_\ell^{(0)} \right\|_{A_\ell} + \gamma^p \left\| \mathbf{u}_\ell^E - \mathbf{u}_\ell^{(0)} \right\|_{A_\ell} \\
 & \leq \left(C_{A0} C_{S1} m^{-1/2} + \gamma^p \right) \left\| \mathbf{u}_\ell^E - \mathbf{u}_\ell^{(0)} \right\|_{A_\ell} \\
 & \stackrel{(8)}{\leq} \left(C_{A0} C_{S1} \left(\left(\frac{C_{A0} C_{S1}}{\gamma - \gamma^p} \right)^2 \right)^{-1/2} + \gamma^p \right) \left\| \mathbf{u}_\ell^E - \mathbf{u}_\ell^{(0)} \right\|_{A_\ell} \\
 & = \gamma \left\| \mathbf{u}_\ell^E - \mathbf{u}_\ell^{(0)} \right\|_{A_\ell}.
 \end{aligned}$$





Remark

Notice that we need $p > 1$ for this argument to work. Otherwise $\gamma - \gamma^p$ is zero and m would need to be infinitely large to get convergence.



We can weaken our approximation property assumptions and still achieve convergence.

Theorem (α -Weak Convergence of the One-Sided W-Cycle)

Suppose that $p \geq 2$, $m_1 = m \geq 1$, and $m_2 = 0$ (one-sided). Suppose, further, that Assumptions (G0, strong Galerkin condition) and (A2, α -weak approximation property) hold and the smoothing is done by Richardson's smoother. Then for any $0 < \gamma < 1$, m can be chosen large enough so that

$$\left\| \mathbf{u}_\ell^E - \text{MG} \left(\mathbf{g}_\ell, \ell, \mathbf{u}_\ell^{(0)} \right) \right\|_{A_\ell} \leq \gamma \left\| \mathbf{u}_\ell^E - \mathbf{u}_\ell^{(0)} \right\|_{A_\ell},$$

for any $\ell \geq 0$, where

$$A_\ell \mathbf{u}_\ell^E = \mathbf{g}_\ell.$$

In particular, it suffices to choose $m \in \mathbb{N}$ such that

$$\left(\frac{C_{A2} C_{S1}}{(\gamma - \gamma^p)^{1/\alpha}} \right)^2 \leq m$$

to achieve the desired contraction $\gamma \in (0, 1)$.



Proof.

The proof is quite similar to the previous case and is based on a perturbation argument. But, one needs to use the α -weak convergence of the two-grid method. □



Convergence of the Simple Symmetric V-Cycle



In this section, we will prove that the simple symmetric V-cycle algorithm ($p = 1$ and $m_1 = m_2 = 1$) converges. First we need a new, and useful, smoothing assumption.

Definition (Assumption (S2))

We say that the multigrid algorithm satisfies the **second smoothing property**, equivalently, **Assumption (S2)**, iff there is some $C_{S2} > 0$ such that

$$\|\mathbf{v}_\ell\|_\ell^2 \leq \rho_\ell C_{S2}^2 (\bar{K}_\ell \mathbf{v}_\ell, \mathbf{v}_\ell)_\ell, \quad (11)$$

for all $\mathbf{v}_\ell \in \mathbb{R}^{n_\ell}$ and $\ell \geq 1$, where

$$\bar{K}_\ell := (I_\ell - K_\ell^* K_\ell) A_\ell^{-1}.$$



Lemma

Richardson's smoother satisfies Assumption (S2) with $S_\ell = \Lambda_\ell^{-1} \mathbf{I}_\ell = S_\ell^\top$.



Proof.

Recall

$$\rho_\ell \leq \Lambda_\ell \leq C_R \rho_\ell,$$

for some $C_R \geq 1$ that is independent of ℓ . Then

$$\begin{aligned} \bar{K}_\ell &= (I_\ell - K_\ell^* K_\ell) A_\ell^{-1} \\ &= \left\{ I_\ell - \left(I_\ell - \Lambda_\ell^{-1} A_\ell \right) \left(I_\ell - \Lambda_\ell^{-1} A_\ell \right) \right\} A_\ell^{-1} \\ &= \left(I_\ell - \left\{ I_\ell - 2\Lambda_\ell^{-1} A_\ell + \Lambda_\ell^{-2} A_\ell^2 \right\} \right) A_\ell^{-1} \\ &= 2\Lambda_\ell^{-1} I_\ell - \Lambda_\ell^{-2} A_\ell. \end{aligned}$$

Define

$$J_\ell := \rho_\ell C_R \bar{K}_\ell - I_\ell.$$

If we can show that J_ℓ is SPSPD with respect to $(\cdot, \cdot)_\ell$ then we get (S2) with $C_{S2}^2 = C_R$.



Proof (Cont.)

J_ℓ is clearly symmetric with respect to $(\cdot, \cdot)_\ell$. Now let $\{\mathbf{v}_\ell^{(1)}, \mathbf{v}_\ell^{(2)}, \dots, \mathbf{v}_\ell^{(n_\ell)}\}$ be the orthonormal basis of eigenvectors of A_ℓ with respect to $(\cdot, \cdot)_\ell$. Then

$$\begin{aligned} J_\ell \mathbf{v}_\ell^{(k)} &= \rho_\ell C_R \bar{K}_\ell \mathbf{v}_\ell^{(k)} - \mathbf{v}_\ell^{(k)} \\ &= \rho_\ell C_R \left(2\Lambda_\ell^{-1} I_\ell - \Lambda_\ell^{-2} A_\ell \right) \mathbf{v}_\ell^{(k)} - \mathbf{v}_\ell^{(k)} \\ &= 2\rho_\ell C_R \Lambda_\ell^{-1} \mathbf{v}_\ell^{(k)} - \rho C_R \Lambda_\ell^{-2} \lambda_\ell^{(k)} \mathbf{v}_\ell^{(k)} - \mathbf{v}_\ell^{(k)} \\ &= \left(2\rho_\ell C_R \Lambda_\ell^{-1} - \rho C_R \Lambda_\ell^{-2} \lambda_\ell^{(k)} - 1 \right) \mathbf{v}_\ell^{(k)}. \end{aligned}$$

Set

$$\eta_\ell^{(k)} := 2\rho_\ell C_R \Lambda_\ell^{-1} - \rho C_R \Lambda_\ell^{-2} \lambda_\ell^{(k)} - 1.$$



Proof (Cont.)

We want to show that $\eta_\ell^{(k)} \geq 0$ for all $1 \leq k \leq n_\ell$.

$$\begin{aligned}
 \eta_\ell^{(k)} &= 2C_R \frac{\rho_\ell}{\Lambda_\ell} - C_R \frac{\rho_\ell \lambda_\ell^{(k)}}{\Lambda_\ell^2} - 1 \\
 &\geq 2C_R \frac{\rho_\ell}{\Lambda_\ell} - C_R \frac{\rho_\ell}{\Lambda_\ell} - 1 \quad (\text{since } -\lambda_\ell^{(k)} \geq -\Lambda_\ell) \\
 &= C_R \frac{\rho_\ell}{\Lambda_\ell} - 1 \\
 &\geq 1 - 1 \quad (\text{since } C_R \rho_\ell \geq \Lambda_\ell) \\
 &= 0.
 \end{aligned}$$

Thus the eigenvalues of $J_\ell, \eta_\ell^{(k)}$, are all non-negative and J_ℓ is SPSP. This implies

$$0 \leq (J_\ell \mathbf{v}_\ell, \mathbf{v}_\ell)_\ell = \rho_\ell C_R (\bar{K}_\ell \mathbf{v}_\ell, \mathbf{v}_\ell)_\ell - (\mathbf{v}_\ell, \mathbf{v}_\ell)_\ell,$$

and (S2) follows with $C_{S2}^2 = C_R$. □



Next, we need two more technical lemmas.

Lemma

Let $J_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$ and $J_\ell = J_\ell^*$. Then

$$(J_\ell \mathbf{v}_\ell, J_\ell \mathbf{v}_\ell)_{A_\ell} - (J_\ell^2 \mathbf{v}_\ell, J_\ell^2 \mathbf{v}_\ell)_{A_\ell} \leq (\mathbf{v}_\ell, \mathbf{v}_\ell)_{A_\ell} - (J_\ell \mathbf{v}_\ell, J_\ell \mathbf{v}_\ell)_{A_\ell}, \quad (12)$$

for any $\mathbf{v}_\ell \in \mathbb{R}^{n_\ell}$



Proof.

Since A_ℓ is SPD.

$$\begin{aligned}
 0 &\leq \left\| (I_\ell - J_\ell^2) \mathbf{v}_\ell \right\|_{A_\ell}^2 \\
 &= \left((I_\ell - J_\ell^2) \mathbf{v}_\ell, (I_\ell - J_\ell^2) \mathbf{v}_\ell \right)_{A_\ell} \\
 &= \left(I_\ell \mathbf{v}_\ell - J_\ell^2 \mathbf{v}_\ell, I_\ell \mathbf{v}_\ell - J_\ell^2 \mathbf{v}_\ell \right)_{A_\ell} \\
 &= (\mathbf{v}_\ell, \mathbf{v}_\ell)_{A_\ell} - (J_\ell^2 \mathbf{v}_\ell, \mathbf{v}_\ell)_{A_\ell} - (\mathbf{v}_\ell, J_\ell^2 \mathbf{v}_\ell)_{A_\ell} + (J_\ell^2 \mathbf{v}_\ell, J_\ell^2 \mathbf{v}_\ell)_{A_\ell} \\
 &= (\mathbf{v}_\ell, \mathbf{v}_\ell)_{A_\ell} - (J_\ell \mathbf{v}_\ell, J_\ell \mathbf{v}_\ell)_{A_\ell} - (J_\ell \mathbf{v}_\ell, J_\ell \mathbf{v}_\ell)_{A_\ell} + (J_\ell^2 \mathbf{v}_\ell, J_\ell^2 \mathbf{v}_\ell)_{A_\ell} \\
 &= (\mathbf{v}_\ell, \mathbf{v}_\ell)_{A_\ell} - 2(J_\ell \mathbf{v}_\ell, J_\ell \mathbf{v}_\ell)_{A_\ell} + (J_\ell^2 \mathbf{v}_\ell, J_\ell^2 \mathbf{v}_\ell)_{A_\ell}.
 \end{aligned}$$

So

$$(J_\ell \mathbf{v}_\ell, J_\ell \mathbf{v}_\ell)_{A_\ell} - (J_\ell^2 \mathbf{v}_\ell, J_\ell^2 \mathbf{v}_\ell)_{A_\ell} \leq (\mathbf{v}_\ell, \mathbf{v}_\ell)_{A_\ell} - (J_\ell \mathbf{v}_\ell, J_\ell \mathbf{v}_\ell)_{A_\ell}.$$





Lemma

For any $\mathbf{v}_\ell \in \mathbb{R}^{n_\ell}$

$$(\Pi_{\ell-1} \mathbf{v}_\ell, \Pi_{\ell-1} \mathbf{v}_\ell)_{A_{\ell-1}} = (\mathbf{v}_\ell, \mathbf{v}_\ell)_{A_\ell} - \left((I_\ell - \tilde{\Pi}_\ell) \mathbf{v}_\ell, \mathbf{v}_\ell \right)_{A_\ell}. \quad (13)$$



Proof.

Recall that we always have

$$R_{\ell-1}A_\ell = A_{\ell-1}\Pi_{\ell-1},$$

and

$$\tilde{\Pi}_\ell = P_{\ell-1}A_{\ell-1}^{-1}R_{\ell-1}A_\ell = P_{\ell-1}\Pi_{\ell-1}.$$

Then

$$\begin{aligned} (\Pi_{\ell-1}\mathbf{v}_\ell, \Pi_{\ell-1}\mathbf{v}_\ell)_{A_{\ell-1}} &= (\Pi_{\ell-1}\mathbf{v}_\ell, A_{\ell-1}\Pi_{\ell-1}\mathbf{v}_\ell)_{\ell-1} \\ &= (\Pi_{\ell-1}\mathbf{v}_\ell, R_{\ell-1}A_\ell\mathbf{v}_\ell)_{\ell-1} \\ &= (R_{\ell-1}^\top \Pi_{\ell-1}\mathbf{v}_\ell, A_\ell\mathbf{v}_\ell)_\ell \\ &= (P_{\ell-1}\Pi_{\ell-1}\mathbf{v}_\ell, A_\ell\mathbf{v}_\ell)_\ell \\ &= (\tilde{\Pi}_\ell\mathbf{v}_\ell, A_\ell\mathbf{v}_\ell)_\ell \\ &= (\tilde{\Pi}_\ell\mathbf{v}_\ell, \mathbf{v}_\ell)_{A_\ell} \\ &= (\mathbf{v}_\ell, \mathbf{v}_\ell)_{A_\ell} - ((I_\ell - \tilde{\Pi}_\ell)\mathbf{v}_\ell, \mathbf{v}_\ell)_{A_\ell}. \end{aligned}$$



The simple symmetric V-cycle method is gotten by setting $m_1 = m_2 = 1$. It is somewhat surprising that the method converges, because only one pre-smoothing and one post-smoothing iteration is preformed.

Theorem

Suppose that Assumptions (G1, weak Galerkin condition), (A1, weak approximation property), and (S2, second smoothing property) all hold. Suppose that $p = 1$, $m_1 = m_2 = m = 1$, and $S_\ell = S_\ell^\top$. Then

$$0 \leq (E_\ell \mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} \leq \frac{C_{A1}^2 C_{S2}^2}{C_{A1}^2 C_{S2}^2 + 1} (\mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell},$$

for all $\mathbf{u}_\ell \in \mathbb{R}^{n_\ell}$.



Proof.

Recall, since $p = 1$, $m_1 = m_2 = m = 1$, and $S_\ell = S_\ell^\top$,

$$E_\ell = K_\ell \left(I_\ell - \tilde{\Pi}_\ell \right) K_\ell + K_\ell P_{\ell-1} E_{\ell-1} \Pi_{\ell-1} K_\ell.$$

In particular, notice that

$$K_\ell^* = I_\ell - S_\ell^\top A_\ell = I_\ell - S_\ell A_\ell = K_\ell.$$

Now, set

$$T_1 := \left(\left(I_\ell - \tilde{\Pi}_\ell \right) \mathbf{w}_\ell, \mathbf{w}_\ell \right)_{A_\ell},$$

and

$$T_2 := \left(P_{\ell-1} E_{\ell-1} \Pi_{\ell-1} \mathbf{w}_\ell, \mathbf{w}_\ell \right)_{A_\ell},$$

where

$$\mathbf{w}_\ell = K_\ell \mathbf{u}_\ell.$$

Then

$$(E_\ell \mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} = T_1 + T_2.$$



Proof (Cont.)

Let us first consider T_1 :

$$\begin{aligned}
 T_1 &= \left((I_\ell - \tilde{\Pi}_\ell) \mathbf{w}_\ell, \mathbf{w}_\ell \right)_{A_\ell} \\
 &\stackrel{(A1)}{\leq} C_{A1}^2 \rho_\ell^{-1} \|A_\ell \mathbf{w}_\ell\|_\ell^2 \\
 &= C_{A1}^2 \rho_\ell^{-1} \|A_\ell K_\ell \mathbf{u}_\ell\|_\ell^2 \\
 &\stackrel{(S2)}{\leq} C_{A1}^2 \rho_\ell^{-1} C_{S2}^2 \rho_\ell (\bar{K}_\ell A_\ell K_\ell \mathbf{u}_\ell, A_\ell K_\ell \mathbf{u}_\ell)_\ell \\
 &= C_{A1}^2 C_{S2}^2 \left((I_\ell - K_\ell^* K_\ell) A_\ell^{-1} A_\ell K_\ell \mathbf{u}_\ell, A_\ell K_\ell \mathbf{u}_\ell \right)_\ell \\
 &= C_{A1}^2 C_{S2}^2 ((I_\ell - K_\ell^* K_\ell) K_\ell \mathbf{u}_\ell, A_\ell K_\ell \mathbf{u}_\ell)_\ell \\
 &= C_{A1}^2 C_{S2}^2 \{ (K_\ell \mathbf{u}_\ell, A_\ell K_\ell \mathbf{u}_\ell)_\ell - (K_\ell^* K_\ell K_\ell \mathbf{u}_\ell, A_\ell K_\ell \mathbf{u}_\ell)_\ell \} \\
 &= C_{A1}^2 C_{S2}^2 \{ (K_\ell \mathbf{u}_\ell, K_\ell \mathbf{u}_\ell)_{A_\ell} - (K_\ell^* K_\ell K_\ell \mathbf{u}_\ell, K_\ell \mathbf{u}_\ell)_{A_\ell} \} \\
 &= C_{A1}^2 C_{S2}^2 \left\{ (K_\ell \mathbf{u}_\ell, K_\ell \mathbf{u}_\ell)_{A_\ell} - (K_\ell^2 \mathbf{u}_\ell, K_\ell^2 \mathbf{u}_\ell)_{A_\ell} \right\} \\
 &\stackrel{(12)}{\leq} C_{A1}^2 C_{S2}^2 \{ (\mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} - (K_\ell \mathbf{u}_\ell, K_\ell \mathbf{u}_\ell)_{A_\ell} \}. \tag{14}
 \end{aligned}$$



Proof (Cont.)

The proof proceeds by induction. The base case is trivial, and we skip that.

(Induction hypothesis): Assume that, for any $\mathbf{w}_{\ell-1} \in \mathbb{R}^{n_{\ell-1}}$,

$$(\mathbf{E}_{\ell-1} \mathbf{w}_{\ell-1}, \mathbf{w}_{\ell-1})_{A_{\ell-1}} \leq \gamma (\mathbf{w}_{\ell-1}, \mathbf{w}_{\ell-1})_{A_{\ell-1}}, \quad \gamma := \frac{C_{A1}^2 C_{S2}^2}{C_{A1}^2 C_{S2}^2 + 1}.$$



Proof (Cont.)

(General case): Now, we turn to the bound for T_2 . First, note that

$$\begin{aligned} T_2 &= (E_{\ell-1} \Pi_{\ell-1} \mathbf{w}_\ell, R_{\ell-1} A_\ell \mathbf{w}_\ell)_{\ell-1} \\ &= (E_{\ell-1} \Pi_{\ell-1} \mathbf{w}_\ell, A_{\ell-1} \Pi_{\ell-1} \mathbf{w}_\ell)_{\ell-1} \\ &= (E_{\ell-1} \Pi_{\ell-1} \mathbf{w}_\ell, \Pi_{\ell-1} \mathbf{w}_\ell)_{A_{\ell-1}}. \end{aligned}$$

Then

$$\begin{aligned} T_2 &= (E_{\ell-1} \Pi_{\ell-1} \mathbf{w}_\ell, \Pi_{\ell-1} \mathbf{w}_\ell)_{A_{\ell-1}} \\ &\stackrel{\text{ind. hyp.}}{\leq} \gamma (\Pi_{\ell-1} \mathbf{w}_\ell, \Pi_{\ell-1} \mathbf{w}_\ell)_{A_{\ell-1}} \\ &\stackrel{(13)}{=} \gamma \left\{ (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} - \left((I_\ell - \tilde{\Pi}_\ell) \mathbf{w}_\ell, \mathbf{w}_\ell \right)_{A_\ell} \right\} \\ &= \gamma (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} - \gamma T_1. \end{aligned} \tag{15}$$



Proof (Cont.)

To finish up,

$$\begin{aligned}
 (E_\ell \mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} &= T_1 + T_2 \\
 &= (1 - \gamma) T_1 + \gamma T_1 + T_2 \\
 &\stackrel{(15)}{\leq} (1 - \gamma) T_1 + \gamma T_1 + \gamma (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} - \gamma T_1 \\
 &= (1 - \gamma) T_1 + \gamma (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} \\
 &\stackrel{(14)}{\leq} (1 - \gamma) C_{A1}^2 C_{S2}^2 \left\{ (\mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} - (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} \right\} + \gamma (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} \\
 &= \left(1 - \frac{C_{A1}^2 C_{S2}^2}{C_{A1}^2 C_{S2}^2 + 1} \right) C_{A1}^2 C_{S2}^2 \left\{ (\mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} - (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} \right\} \\
 &\quad + \frac{C_{A1}^2 C_{S2}^2}{C_{A1}^2 C_{S2}^2 + 1} (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell}
 \end{aligned}$$



Proof (Cont.)

$$\begin{aligned} &= \frac{C_{A1}^2 C_{S2}^2}{C_{A1}^2 C_{S2}^2 + 1} \left\{ (\mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} - (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} \right\} \\ &\quad + \frac{C_{A1}^2 C_{S2}^2}{C_{A1}^2 C_{S2}^2 + 1} (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} \\ &= \gamma \left\{ (\mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} - (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} \right\} + \gamma (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} \\ &= \gamma (\mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} . \end{aligned}$$





Corollary (Convergence of Simple Symmetric V-Cycle)

Suppose that hypotheses of the last theorem hold and $\mathbf{u}_\ell^{\text{E}}, \mathbf{g}_\ell \in \mathbb{R}^{n_\ell}$ satisfy

$$\mathbf{A}_\ell \mathbf{u}_\ell^{\text{E}} = \mathbf{g}_\ell.$$

Then, given any $\mathbf{u}_\ell^{(0)} \in \mathbb{R}^{n_\ell}$,

$$\begin{aligned} \left\| \mathbf{u}_\ell^{\text{E}} - \mathbf{u}_\ell^{(3)} \right\|_{\mathbf{A}_\ell} &= \left\| \mathbf{u}_\ell^{\text{E}} - \text{MG} \left(\mathbf{g}_\ell, \ell, \mathbf{u}_\ell^{(0)} \right) \right\|_{\mathbf{A}_\ell} \\ &\leq \frac{M}{M+m} \left\| \mathbf{u}_\ell^{\text{E}} - \mathbf{u}_\ell^{(0)} \right\|_{\mathbf{A}_\ell}, \end{aligned}$$

where

$$M = C_{\text{A1}}^2 C_{\text{S2}}^2 \quad \text{and} \quad m = 1.$$



Proof.

We need only to show that

$$\|E_\ell \mathbf{v}_\ell\|_{A_\ell} \leq \frac{M}{M+m} \|\mathbf{v}_\ell\|_{A_\ell},$$

is true for any $\mathbf{v}_\ell \in \mathbb{R}^{n_\ell}$. Since E_ℓ is SPSP w.r.t. $(\cdot, \cdot)_{A_\ell}$, for $\ell \geq 1$, there is a basis of eigenvectors of E_ℓ , $\{\mathbf{w}_\ell^{(1)}, \dots, \mathbf{w}_\ell^{(n_\ell)}\}$, such that

$$E_\ell \mathbf{w}_\ell^{(j)} = \epsilon_\ell^{(j)} \mathbf{w}_\ell^{(j)},$$

$$\left(\mathbf{w}_\ell^{(i)}, \mathbf{w}_\ell^{(j)} \right)_{A_\ell} = \delta_{ij},$$

and

$$0 \leq \epsilon_\ell^{(1)} \leq \epsilon_\ell^{(2)} \leq \dots \leq \epsilon_\ell^{(n_\ell)}.$$



Proof (Cont.)

Suppose

$$\mathbf{v}_\ell = \sum_{k=1}^{n_\ell} c_k \mathbf{w}_\ell^{(k)}.$$

Then

$$(\mathbf{E}_\ell \mathbf{v}_\ell, \mathbf{v}_\ell)_{A_\ell} = \sum_{k=1}^{n_\ell} c_k^2 \epsilon_\ell^{(k)}$$

and

$$(\mathbf{v}_\ell, \mathbf{v}_\ell)_{A_\ell} = \sum_{k=1}^{n_\ell} c_k^2.$$

The last theorem guarantees that

$$\sum_{k=1}^{n_\ell} c_k^2 \epsilon_\ell^{(k)} \leq \frac{M}{M+m} \sum_{k=1}^{n_\ell} c_k^2,$$

for any $c_1, \dots, c_{n_\ell} \in \mathbb{R}$. This implies that

$$0 \leq \epsilon_\ell^{(k)} \leq \frac{M}{M+m}, \quad 1 \leq k \leq n_\ell.$$



Proof (Cont.)

Therefore

$$\begin{aligned}\|\mathbf{E}_\ell \mathbf{v}_\ell\|_{\mathbf{A}_\ell}^2 &= (\mathbf{E}_\ell \mathbf{v}_\ell, \mathbf{E}_\ell \mathbf{v}_\ell)_{\mathbf{A}_\ell} \\ &= \sum_{k=1}^{n_\ell} c_k^2 \left(\epsilon_\ell^{(k)} \right)^2 \\ &\leq \left(\frac{M}{M+m} \right)^2 \sum_{k=1}^{n_\ell} c_k^2 \\ &= \left(\frac{M}{M+m} \right)^2 \|\mathbf{v}_\ell\|_{\mathbf{A}_\ell}^2.\end{aligned}$$





Convergence of the General Symmetric V-Cycle



Now for the general symmetric V-cycle. Here we want to show that we can improve the convergence rate if more smoothing steps are performed. We need a technical lemma first.

Lemma

Suppose that smoothing is done with Richardson's smoother, that is,

$$S_\ell = \Lambda_\ell^{-1} I_\ell,$$

where

$$\rho_\ell \leq \Lambda_\ell \leq C_R \rho_\ell,$$

for some $C_R \geq 1$ that is independent of ℓ . Then,

$$\left((I_\ell - K_\ell) K_\ell^{2m} \mathbf{v}_\ell, \mathbf{v}_\ell \right)_{A_\ell} \leq \frac{1}{2m} \left((I_\ell - K_\ell^{2m}) \mathbf{v}_\ell, \mathbf{v}_\ell \right)_{A_\ell} \quad (16)$$

for any $m \geq 1$ and $\ell \geq 1$.



Proof.

Suppose $i, j \in \mathbb{Z}$ with $0 \leq j \leq i$. Then

$$\begin{aligned}
 \left((I_\ell - K_\ell) K_\ell^i \mathbf{v}_\ell, \mathbf{v}_\ell \right)_{A_\ell} &= \left(A_\ell (I_\ell - K_\ell) K_\ell^i \mathbf{v}_\ell, \mathbf{v}_\ell \right)_\ell \\
 &= \Lambda_\ell^{-1} \left(A_\ell^2 K_\ell^i \mathbf{v}_\ell, \mathbf{v}_\ell \right)_\ell \\
 &= \Lambda_\ell^{-1} \left(K_\ell^i A_\ell \mathbf{v}_\ell, A_\ell \mathbf{v}_\ell \right)_\ell \\
 &\stackrel{(3)}{\leq} \Lambda_\ell^{-1} \left(K_\ell^j A_\ell \mathbf{v}_\ell, A_\ell \mathbf{v}_\ell \right)_\ell \\
 &= \left((I_\ell - K_\ell) K_\ell^j \mathbf{v}_\ell, \mathbf{v}_\ell \right)_{A_\ell}. \tag{17}
 \end{aligned}$$



Proof (Cont.)

Therefore,

$$\begin{aligned}
 & 2m \left((I_\ell - K_\ell) K_\ell^{2m} \mathbf{v}_\ell, \mathbf{v}_\ell \right)_{A_\ell} \\
 &= \underbrace{\left((I_\ell - K_\ell) K_\ell^{2m} \mathbf{v}_\ell, \mathbf{v}_\ell \right)_{A_\ell} + \cdots + \left((I_\ell - K_\ell) K_\ell^{2m} \mathbf{v}_\ell, \mathbf{v}_\ell \right)_{A_\ell}}_{2m} \\
 &\stackrel{(17)}{\leq} \underbrace{\left((I_\ell - K_\ell) K_\ell^0 \mathbf{v}_\ell, \mathbf{v}_\ell \right)_{A_\ell}}_{(j=0)} + \underbrace{\left((I_\ell - K_\ell) K_\ell^1 \mathbf{v}_\ell, \mathbf{v}_\ell \right)_{A_\ell}}_{(j=1)} \\
 &\quad + \cdots + \underbrace{\left((I_\ell - K_\ell) K_\ell^{2m-1} \mathbf{v}_\ell, \mathbf{v}_\ell \right)_{A_\ell}}_{(j=2m-1)} \\
 &= \left((I_\ell - K_\ell^{2m}) \mathbf{v}_\ell, \mathbf{v}_\ell \right)_{A_\ell}.
 \end{aligned}$$

The last equality follows since the sum telescopes. □



Theorem (Braess–Hackbusch Result)

Suppose that Assumptions (G1) and (A1) hold. Suppose $p = 1$, $m_1 = m_2 = m \geq 1$, and smoothing is done with Richardson's smoother. Then

$$0 \leq (\mathbf{E}_\ell \mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} \leq \frac{M}{M + m} (\mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell},$$

for all $\mathbf{u}_\ell \in \mathbb{R}^{n_\ell}$, where

$$M := \frac{C_{A1}^2 C_R}{2}.$$



Proof.

The proof is similar to that of the last theorem. We begin with an expression for the error propagation matrix :

$$E_\ell = K_\ell^m (I_\ell - \tilde{\Pi}_\ell) K_\ell^m + K_\ell P_{\ell-1} E_{\ell-1} P_{\ell-1} \Pi_{\ell-1} K_\ell^m,$$

where

$$K_\ell = I_\ell - \Lambda_\ell^{-1} A_\ell = K_\ell^*,$$

and

$$\rho_\ell \leq \Lambda_\ell \leq C_R \rho_\ell, \quad \exists C_R \geq 1.$$

As before, set

$$T_1 := \left((I_\ell - \tilde{\Pi}_\ell) \mathbf{w}_\ell, \mathbf{w}_\ell \right)_{A_\ell},$$

and

$$T_2 := (P_{\ell-1} E_{\ell-1} P_{\ell-1} \mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell},$$

where

$$\mathbf{w}_\ell = K_\ell^m \mathbf{u}_\ell.$$

Then

$$(E_\ell \mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} = T_1 + T_2.$$



Proof (Cont.)

We first estimate T_1 :

$$\begin{aligned}
 T_1 &= \left((I_\ell - \tilde{\Pi}_\ell) \mathbf{w}_\ell, \mathbf{w}_\ell \right)_{A_\ell} \\
 &\stackrel{(A1)}{\leq} C_{A1}^2 \rho_\ell^{-1} \|A_\ell \mathbf{w}_\ell\|_\ell^2 \\
 &= C_{A1}^2 \rho_\ell^{-1} \|A_\ell K_\ell^m \mathbf{u}_\ell\|_\ell^2 \\
 &= C_{A1}^2 \rho_\ell^{-1} (A_\ell K_\ell^m \mathbf{u}_\ell, A_\ell K_\ell^m \mathbf{u}_\ell)_\ell \\
 &= C_{A1}^2 \rho_\ell^{-1} (A_\ell^2 K_\ell^m \mathbf{u}_\ell, K_\ell^m \mathbf{u}_\ell)_\ell \\
 &= C_{A1}^2 \rho_\ell^{-1} (A_\ell K_\ell^m \mathbf{u}_\ell, K_\ell^m \mathbf{u}_\ell)_{A_\ell} \\
 &= C_{A1}^2 \rho_\ell^{-1} \Lambda_\ell ((I_\ell - K_\ell) K_\ell^m \mathbf{u}_\ell, K_\ell^m \mathbf{u}_\ell)_{A_\ell} \\
 &= C_{A1}^2 \rho_\ell^{-1} \Lambda_\ell \left((I_\ell - K_\ell) K_\ell^{2m} \mathbf{u}_\ell, \mathbf{u}_\ell \right)_{A_\ell} \\
 &\stackrel{(16)}{\leq} \frac{C_{A1}^2 \rho_\ell^{-1} \Lambda_\ell}{2m} \left((I_\ell - K_\ell^{2m}) \mathbf{u}_\ell, \mathbf{u}_\ell \right)_{A_\ell}
 \end{aligned}$$



Proof (Cont.)

$$\begin{aligned}
 &\leq \frac{C_{A1}^2 C_R}{2m} \left((I_\ell - K_\ell^{2m}) \mathbf{u}_\ell, \mathbf{u}_\ell \right)_{A_\ell} \\
 &= \frac{M}{m} \left\{ (\mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} - (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} \right\}
 \end{aligned} \tag{18}$$

Set

$$\gamma := \frac{M}{M + m}.$$

Exactly as in the proof of the last theorem, an induction argument yields

$$T_2 \leq \gamma (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} - \gamma T_1. \tag{19}$$



Proof (Cont.)

Therefore,

$$\begin{aligned} (E_\ell \mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} &= T_1 + T_2 \\ &= (1 - \gamma) T_1 + \gamma T_1 + T_2 \\ &\stackrel{(19)}{\leq} (1 - \gamma) T_1 + \gamma T_1 + \gamma (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} - \gamma T_1 \\ &\stackrel{(18)}{\leq} (1 - \gamma) \frac{M}{m} \left\{ (\mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell} - (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} \right\} + \gamma (\mathbf{w}_\ell, \mathbf{w}_\ell)_{A_\ell} \\ &= \gamma (\mathbf{u}_\ell, \mathbf{u}_\ell)_{A_\ell}. \end{aligned}$$





Corollary (Convergence of General Symmetric V-Cycle)

Suppose that hypotheses of the last theorem hold and $\mathbf{u}_\ell^E, \mathbf{g}_\ell \in \mathbb{R}^{n_\ell}$ satisfy

$$A_\ell \mathbf{u}_\ell^E = \mathbf{g}_\ell.$$

Then, given any $\mathbf{u}_\ell^{(0)} \in \mathbb{R}^{n_\ell}$,

$$\begin{aligned} \left\| \mathbf{u}_\ell^E - \mathbf{u}_\ell^{(3)} \right\|_{A_\ell} &= \left\| \mathbf{u}_\ell^E - \text{MG} \left(\mathbf{g}_\ell, \ell, \mathbf{u}_\ell^{(0)} \right) \right\|_{A_\ell} \\ &\leq \frac{M}{M+m} \left\| \mathbf{u}_\ell^E - \mathbf{u}_\ell^{(0)} \right\|_{A_\ell}, \end{aligned}$$

where

$$M = \frac{C_{A1}^2 C_{S2}^2}{2} \quad \text{and} \quad m \geq 1.$$

Proof.

The proof is exactly that same as that for the last corollary. □