Stat 201A, Fall 2012
HOMEWORK 1 (due Tuesday 8/28)

1. Useful bounds. Let $A_1, A_2, \ldots, A_n$ be events.

a) Prove **Boole's inequality** for two events: $P(A_1 \cup A_2) \le P(A_1) + P(A_2)$

b) Use **a** and induction to prove Boole's inequality for n events:

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) \quad \le \quad \sum_{i=1}^{n} P(A_i)$$

This is also known as the first Bonferroni inequality. Bonferroni gave a further sequence of bounds, using the inclusion-exclusion formulas. The first one of those is Lemma 1.6.

2. Here are a couple of standard uses of what you proved above.

a) A professor will be at a conference next Wednesday, Thursday, and Friday. The weather report for the city where the conference is being held says that the chance of rain is

Wednesday	Thursday	Friday
10%	20%	25%

Fill in the blanks with the sharpest bounds you can find:
The chance that it rains sometime during those three days is at least ___________ and at most ___________.

b) Suppose that I place n bets, and that I have chance p_i of winning Bet i, for $1 \le i \le n$. Rather greedily, I want to know the chance that I win all the bets. The trouble is that the bets are all dependent on each other in various complicated ways, so there's no straightforward calculation that I win them all. So fill in the blank with a lower bound: The chance that I win all the bets is at least ___________.
[Notice that the bound is only interesting when the p_i's are large. It is frequently used in statistics, for example in situations where several different quantities are predicted using the same data and you need some idea about the chance that all the predictions are good. This is called *Bonferroni's Method*.]

3. Text problem 1.10.7. That's Problem 7 in Section 1.10. You can of course follow the hint given, or use what you proved above along with Theorem 1.8.

4. Text problem 1.10.4. I'm not sure I like the hint. If you use induction, you won't get very far with the general index set. Try the usual set theoretic method of taking a point in the set on the left hand side and showing that it must also be in the set on the right, and vice versa.

5. Text 1.10.8. Use a few of the results you've proved above.

6. That's enough set theory for one homework. Switch gears to something very concrete: randomly shuffling a standard deck of 52 cards. That means all shuffles are equally likely. Formally, you're working with the uniform distribution on all permutations of the 52 cards.

Just so that we're all speaking the same language, I'll describe the entire deck. A standard deck contains four suits, called hearts, diamonds, spades, and clubs. The first two suits are red and the others are black. Each suit consists of 13 cards, labeled Ace, 2, 3, 4, ..., 10, Jack, Queen, King. These labels are called ranks.

a) How many shuffles are there? In how many shuffles is the 17th card red? What is the chance that the 17th card is red? Does the number 17 appear in your answer?

b) I deal a poker hand. That's 5 cards dealt at random without replacement from the deck. What is the chance that the last two cards that I deal are aces? How would your answer have been affected had I dealt a bridge hand (13 cards) instead of a poker hand?

c) I shuffle a deck and deal all the cards one by one. What is the chance that the ace of spades appears before all the red cards?

7. Look, no algebra! The *combinatorial method* uses counting arguments to prove algebraic facts. It can be fast, elegant, and revealing. Let n be a positive integer.

a) Show, simply by counting the same thing in two different ways, **without using any factorial formulas**, that

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

b) Use a combinatorial argument to show that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$

8. A coin that lands heads with probability p is tossed repeatedly. Let k be a positive integer, and let T be the number of tosses till there are at least k heads and at least k tails. Find the distribution of T.

[Hint: As always when finding a distribution, start by working out the possible values of T. Then think about what must happen on the last toss.]