STAT201A – Sec. 102
Homework #6.

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#1.

Proof. So, given $N_{(0,t)} = n$ and $N_{(0,t)} \sim$ Poisson(λ), and the fact that cutting up $(0,t)$ into $(0,a) \sqcup (a,b) \sqcup (b,t)$ yields 3 independent Poisson processes with rates $a\lambda/t$, $(b-a)\lambda/t$ and $(t-b)\lambda/t$ intuition immediately has that $(N_{(0,a)}, N_{(a,b)}, N_{(b,t)}) \sim$ Multinomial with n trials, and 3 outcomes with probabilities $p_1 = a/t$, $p_2 = (b-a)/t$ and $p_3 = (t-b)/t$. Why?

$$\frac{P\left((N_{(0,a)}, N_{(a,b)}, N_{(b,t)}) = (x_1,x_2,x_3), N_{(0,t)} = n\right)}{P(N_{(0,t)} = n)} = P\left((N_{(0,a)}, N_{(a,b)}, N_{(b,t)}) = (x_1,x_2,x_3) \,\middle|\, N_{(0,t)} = n\right)$$

and $N_{(0,a)} + N_{(a,b)} + N_{(b,t)} = N_{(0,t)}$, hence using this fact and the independence of these subprocesses, we have that

$$\begin{aligned}
P\left((N_{(0,a)}, N_{(a,b)}, N_{(b,t)}) \right. &= \left.(x_1,x_2,x_3), N_{(0,t)} = n\right) \\
&= P\left((N_{(0,a)}, N_{(a,b)}, N_{(b,t)}) = (x_1,x_2,n-(x_1+x_2))\right) \\
&= P(N_{(0,a)} = x_1)P(N_{(a,b)} = x_2)P(N_{(b,t)} = n-(x_1+x_2)) \\
&= \left(\frac{a}{t}\right)^{x_1} \frac{e^{-\frac{a\lambda}{t}}\lambda^{x_1}}{x_1!} \left(\frac{b-a}{t}\right)^{x_2} \frac{e^{-\frac{(b-a)\lambda}{t}}\lambda^{x_2}}{x_2!} \left(\frac{t-b}{t}\right)^{n-x_1-x_2} \frac{e^{-\frac{(t-b)\lambda}{t}}\lambda^{n-x_1-x_2}}{(n-x_1-x_2)!} \\
&= \frac{e^{-\lambda(\frac{a}{t}+\frac{b-a}{t}+\frac{t-b}{t})}\lambda^{x_1+x_2+n-x_1-x_2}}{x_1!x_2!(n-x_1-x_2)!} \left(\frac{a}{t}\right)^{x_1} \left(\frac{b-a}{t}\right)^{x_2} \left(\frac{t-b}{t}\right)^{n-x_1-x_2} \\
&= e^{-\lambda}\frac{\lambda^n}{n!}\binom{n}{x_1,x_2,n-x_1-x_2} \left(\frac{a}{t}\right)^{x_1} \left(\frac{b-a}{t}\right)^{x_2} \left(\frac{t-b}{t}\right)^{n-x_1-x_2}
\end{aligned}$$

Hence,

$$P\left((N_{(0,a)}, N_{(a,b)}, N_{(b,t)}) = (x_1,x_2,x_3) \,\middle|\, N_{(0,t)} = n\right) = \binom{n}{x_1,x_2,x_3} \left(\frac{a}{t}\right)^{x_1} \left(\frac{b-a}{t}\right)^{x_2} \left(\frac{t-b}{t}\right)^{x_3}$$

□

#2.

Proof. Let δt denote a small neighborhood about $t \in (0,1)$ of width dt, and note that

$$P(T_r \in dt, N_{(0,1)} = n) = P(N_{(0,t)} = r-1, N_{\delta t} = 1, N_{(t,1)} = n-r)$$

Since the N_I's are independent, this identity amounts to saying:

$$\begin{aligned}
P(T_r \in dt, N_{(0,1)} = n) &= P(N_{(0,t)} = r-1)P(N_{\delta t} = 1)P(N_{(t,1)} = n-r) \\
&= \left(e^{-t\lambda}\frac{(t\lambda)^{r-1}}{(r-1)!}\right)\left(e^{-dt\lambda}\lambda dt\right)\left(e^{-(1-t)\lambda}\frac{((1-t)\lambda)^{n-r}}{(n-r)!}\right) \\
&= \frac{e^{-(1+dt)\lambda}\lambda^n}{n!}\left(\frac{n!}{(r-1)!(n-r)!}t^{r-1}(1-t)^{n-r}\,dt\right) \\
&\approx P_n\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}t^{r-1}(1-t)^{s-1}\,dt
\end{aligned}$$

where $s = n - r + 1$, and $P_n = P(N_{(0,1)} = n)$. Hence,

$$P(T_r \in dt \,\big|\, N_{(0,1)} = n) = \frac{P(T_r \in dt, N_{(0,1)} = n)}{P_n} = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} t^{r-1}(1-t)^{s-1}$$

which is the density for a βeta random variable with parameters r and s. □

#3.

Proof. Using the same technique as in question #2 (and letting $t < s$):

$$\begin{aligned}
P(T_1 \in dt, T_n \in ds, N_{(0,1)} = n) &= P(N_{\delta t} = 1, N_{(t,s)} = n-2, N_{\delta s} = 1) \\
&= \left(e^{-dt\,\lambda}\lambda\, dt\right)\left(e^{-(s-t)\lambda}\frac{(s-t)^{n-2}\lambda^{n-2}}{(n-2)!}\right)\left(e^{-ds\,\lambda}\lambda\, ds\right) \\
&= e^{-(s-t+dt+ds)\lambda}\frac{\lambda^n}{n!}\binom{n}{1,1,n-2}(s-t)^{n-2}\, dt\, ds \\
&\approx P_n\; n(n-1)(s-t)^{n-2}\, dt\, ds
\end{aligned}$$

Hence,

$$P(T_1 \in dt, T_n \in ds \,\big|\, N_{(0,1)} = n) = \frac{P(T_1 \in dt, T_n \in ds, N_{(0,1)} = n)}{P_n} = n(n-1)(s-t)^{n-2}\, dt\, ds$$

And this happens to be the joint density of $(U_{(1)}, U_{(n)})$ the first and last order statistics for $U_1, \ldots, U_n \overset{iid}{\sim}$ Uniform([0,1]) random variables. □

#4.

Proof. Given that for $r \in \mathbb{N}$ we may express $T_r = \sum_{i=1}^r W_i \sim \Gamma(r, \lambda)$, where $W_i \overset{iid}{\sim}$ Exp(λ), we have that $T_{r+s} = \sum_{i=1}^{r+s} W_i$, and thus

$$\begin{aligned}
\operatorname{cov}(T_r, T_{r+s}) &= \operatorname{cov}\left(\sum_{i=1}^r W_i, \sum_{i=1}^r W_i + \sum_{j=r+1}^s W_j\right) \\
&= \operatorname{cov}\left(\sum_{i=1}^r W_i, \sum_{i=1}^r W_i\right) + \operatorname{cov}\left(\sum_{i=1}^r W_i, \sum_{j=r+1}^s W_j\right) \\
&= \operatorname{var}(T_r) + \sum_{i=1}^r \sum_{j=r+1}^s \operatorname{cov}(W_i, W_j) \\
&= \frac{r}{\lambda^2}
\end{aligned}$$

Since $W_i \perp\!\!\!\perp W_j$ for $i \neq j$, and the variance of a $\Gamma(r, \lambda)$ random variable is r/λ^2. Consequently,

$$\rho(T_r, T_{r+s}) = \frac{\operatorname{cov}(T_r, T_{r+s})}{\sigma_r \sigma_{r+s}} = \frac{r/\lambda^2}{\sqrt{r}/\lambda \cdot \sqrt{r+s}/\lambda} = \sqrt{\frac{r}{r+s}}$$

This indicates that $\rho(T_r, T_{r+s}) \to 0$ as $s \to \infty$, which makes sense as the greater $s - r$ becomes, the less it would makes sense for information about T_r to help us determine anything about T_{r+s}. □