STAT201A – Sec. 102
Homework #7.

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#1.

Proof. a) Given $X_i \sim \text{Poisson}(\lambda_i)$,

$$\begin{aligned}\psi_{X_i}(t) &= E(e^{tX_i}) \\ &= \sum_{x=0}^{\infty} e^{tx} e^{-\lambda_i} \frac{\lambda_i^x}{x!} \\ &= e^{-\lambda_i} \sum_{x=0}^{\infty} \frac{(e^t\lambda_i)^x}{x!} \\ &= e^{-\lambda_i} e^{e^t\lambda_i} \\ &= \exp\left\{\lambda_i(e^t-1)\right\}\end{aligned}$$

b) Now, supposing $X_1 \perp\!\!\!\perp X_2$ and using the fact that $\psi_{X+Y} = \psi_X\psi_Y$ for $X \perp\!\!\!\perp Y$, we have

$$\begin{aligned}\psi_{X_1+X_2}(t) &= \psi_{X_1}(t)\psi_{X_2}(t) \\ &= \exp\left\{\lambda_1(e^t-1)\right\}\exp\left\{\lambda_2(e^t-1)\right\} \\ &= \exp\left\{(\lambda_1+\lambda_2)(e^t-1)\right\} \\ &= \psi_Y(t)\end{aligned}$$

where $Y \sim \text{Poisson}(\lambda_1+\lambda_2)$

□

#2.

Proof. Let X have density $f_X(x) = \frac{1}{2}e^{-|x|}I_{\mathbb{R}}(x)$. Then,

$$\begin{aligned}\psi_X(t) &= E(e^{tX}) \\ &= \frac{1}{2}\int_{\mathbb{R}} e^{tx}e^{-|x|}\,dx \\ &= \frac{1}{2}\left(\int_{-\infty}^{0} e^{tx}e^{x}\,dx + \int_0^{\infty} e^{tx}e^{-x}\,dx\right) \\ &= \frac{1}{2}\left(\int_{-\infty}^{0} e^{(t+1)x}\,dx + \int_0^{\infty} e^{(t-1)x}\,dx\right)\end{aligned}$$

Clearly, the first integral will not converge unless $t+1>0$ and the second will not converge unless $t-1<0$. That is, both will not converge unless $-1<t<1$. In which case,

$$\psi_X(t) = \frac{1}{2}\left(\frac{1}{1+t}+\frac{1}{1-t}\right) = \frac{1}{1-t^2}$$

Finally, since $-1<t<1$ (and therefore $0<t^2<1$, we have that

$$\frac{1}{1-t^2} = \sum_{k=0}^{\infty}(t^2)^k$$

Hence,

$$\frac{d^n}{dt^n}\psi_X(t) = \sum_{k=0}^{\infty}[2k]_{n-1}t^{2k-n}$$

Thus,

$$\psi_X^{(n)}(0) = \begin{cases} 0 & \text{if } n = 2k+1 \text{ for some } k \in \mathbb{N} \\ (2k)! & \text{if } n = 2k \text{ for some } k \in \mathbb{N} \end{cases}$$

□

#3.

a) Recall that for $X > 0$ and $c > 0$, Markov's inequality says that

$$P(X \geq c) \leq \frac{E(X)}{c}$$

Moreover, we have that for any random variable X, the transformation e^{tX} is non-negative, and thus applying Markov's inequality to e^{tX}, with $c = e^{tx}$ for $t > 0$ and $x \in \mathbb{R}$ yields

$$P\left(e^{tX} \geq e^{tx}\right) \leq \frac{E(e^{tX})}{e^{tx}}$$

However, $\exp$ being a monotonically increasing function allows us to write $P(X \geq x) = P(e^{tX} \geq e^{tx})$, and thus

$$P(X \geq x) \leq e^{-tx}\psi_X(t)$$

b) Using the result in class that gives

$$\psi_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$$

for $X \sim \Gamma\text{amma}(r, \lambda)$, we have that

$$P\left(X \geq \frac{2r}{\lambda}\right) \leq e^{-2tr/\lambda}\left(\frac{\lambda}{\lambda - t}\right)^r = \left(e^{2t/\lambda}(1 - t/\lambda)\right)^{-r}$$

but since this holds for all $t < \lambda$, setting $t = \lambda/2$ yields the bound

$$P\left(X \geq \frac{2r}{\lambda}\right) \leq (e(1 - 1/2))^{-r} = (2/e)^r$$

The bound from Markov is

$$P\left(X \geq \frac{2r}{\lambda}\right) \leq \frac{E(X)}{2r/\lambda} = \frac{r/\lambda}{2r/\lambda} = \frac{1}{2}$$

#4.

Proof. a) i. Since R uses

$$P(X=k) = \binom{n-1+k}{n-1} q^k p^n$$

as the PDF for $X \sim$ NBin(n,p), we have that $P(X \leq 310)$, for $X \sim$ NBin(100,1/4) gives the probability that we need to wait at most 410 trials for 100 success. The following code finds the complement of that probability to be 0.3693

```
p <- 1/4
num_of_heads <- 100
prob_of_410_or_less <- pnbinom(q=310,size=num_of_heads,prob=p)
prob_of_411_or_more <- 1 - prob_of_410_or_less
```

ii. Since we know that geometric random variables are special cases of negative binomials, and in particular, $X \sim$ NBin(n,p) implies that $X = \sum_{i=1}^n G_i$ where $G_i \overset{iid}{\sim}$ Geo(p), then via the central limit theorem

$$\begin{aligned}
P(X \geq 411) &= P\left(\frac{1}{100}\sum_{i=1}^{100} G_i \geq 4.11\right) \\
&= P\left(\frac{\bar{G}_{(100)} - 1/p}{\sigma/\sqrt{n}} \geq \frac{4.11-4}{\sqrt{(3/4)(1/4)^{-2}/10}}\right) \\
&\approx P(Z \geq 0.3175) \\
&= 1 - \Phi(0.3175) \\
&= 0.3754
\end{aligned}$$

Where $G_i \overset{iid}{\sim}$ Geo$(1/4)$ and hence $\sigma = \sqrt{12}$ and $Z \sim$ N(0,1). This is without any correction. To use a correction, we'll consider shifting 411 down to 410.5, and looking at

$$P(X \geq 411) \approx P\left(\frac{1}{100}\sum_{i=1}^{100} G_i \geq 4.105\right) \approx P(Z \geq 0.3031) = 0.3809$$

b) i. The chance that the 100th heads is on the 410th toss is found via

```
dnbinom(x=310,size=100,prob=1/4)

[1] 0.01073
```

ii. As in part ii. of part a), we'll beg to the CLT:

```
n <- 100
p <- 1/4
q <- 1 - p
sigma <- sqrt(q/p^2)
z1 <- (409.5/n - 1/p)/(sigma/sqrt(n))
z2 <- (410.5/n - 1/p)/(sigma/sqrt(n))
pnorm(z2) - pnorm(z1)

[1] 0.01105
```

Hence,

$$P(X = 410) = P(409.5 < X < 410.5) = \Phi(0.3031) - \Phi(0.2742) = 0.011$$

□

#5.

Proof. a) So, n large puts us in position to use a version of the central limit theorem.

$$\begin{aligned} P(\overline{X}_{(n)} - c_n \leq \mu \leq \overline{X}_{(n)} + c_n) &= 1 - \left(P(\overline{X}_{(n)} < \mu - c_n) + P(X_n > \mu + c_n)\right) \\ &= 1 - \left(P\left(\frac{\overline{X}_{(n)} - \mu}{\sigma/\sqrt{n}} < -\frac{c_n\sqrt{n}}{\sigma}\right) + 1 - P\left(\frac{\overline{X}_{(n)} - \mu}{\sigma/\sqrt{n}} \leq \frac{c_n\sqrt{n}}{\sigma}\right)\right) \\ &\approx 1 - \left(\Phi\left(-\frac{c_n\sqrt{n}}{\sigma}\right) + 1 - \Phi\left(\frac{c_n\sqrt{n}}{\sigma}\right)\right) \\ &= 1 - 2\Phi\left(-\frac{c_n\sqrt{n}}{\sigma}\right) = 1 - 2\Phi(-z) \end{aligned}$$

Thus, solving for z such that $1 - 2\Phi(-z) \approx 95\% \iff \Phi(-z) \approx 2.5\%$ and letting $c_n = z\sigma/\sqrt{n}$ will give us our desired constant. Using a normal table, we have that $1 - \Phi(1.96) = 0.025$, so setting $c_n = 1.96\sigma/\sqrt{n}$ does the trick.

b) Using the result that $c_n = 1.96\sigma/\sqrt{n}$ from above, we know that to obtain a confidence interval of width no larger than .01, $2c_n \leq 0.01$. That is,

$$2c_n \leq 0.01 \iff \frac{2 \cdot 1.96 \cdot \sigma}{0.01} \leq \sqrt{n} \iff 392^2\sigma^2 \leq n$$

Since this holds for arbitrary σ^2, it must be that $n \geq \sup_{\sigma^2} 392^2\sigma^2$ and we saw earlier that for a bernoulli trial with success rate p, σ^2 is maximal at $p = 1/2$. Thus, $n \geq 392^2/4 = 38,416$.

□

#6.

Proof. Given $X_1, X_2, \ldots$ iid with mean μ (and assuming ψ_{X_i} exists), we have

$$\psi_{\overline{X}_{(n)}}(t) = \prod_{i=1}^{n} \psi_{\frac{1}{n}X_i}(t) = \prod_{i=1}^{n} \psi_{X_i}(t/n) = \psi_{X_1}(t/n)^n$$

Expanding $\psi_X(t)$ in a Maclaurin series, we have that

$$\psi_X(t) = 1 + \mu t + o(t)$$

Hence,

$$\begin{aligned}\psi_{X_1}(t/n)^n &= \left(1 + \frac{\mu t}{n} + o\left(\frac{\mu t}{n}\right)\right)^n \\ &= \left(1 + \frac{\mu t + no\left(\frac{\mu t}{n}\right)}{n}\right)^n\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{n\to\infty} \psi_{X_1}(t/n)^n &= \lim_{n\to\infty} \left(1 + \frac{\mu t + a_n}{n}\right)^n \\ &= e^{\mu t + a}\end{aligned}$$

where $a_n \to a$. However,

$$\begin{aligned}a &= \lim_{n\to\infty} no\left(\frac{\mu t}{n}\right) \\ &= \lim_{n\to\infty} \frac{o\left(\frac{\mu t}{n}\right)}{1/n} \\ &= \lim_{x\to 0} \frac{o(x)}{x} = 0\end{aligned}$$

Hence, $\psi_{\overline{X}_{(n)}}(t) = \psi_{X_1}(t/n)^n \to e^{\mu t}$ which is the mgf of the constant μ. $\square$