9/13/2011

for $O = (W, A, Y) \sim P_0 \in \mathcal{M}_{NP}$

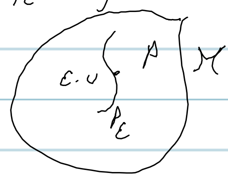
$\Psi : \mathcal{M} \to \mathbb{R}$

$\Psi(P) = E_P[E_P(Y \mid A=1, W)] = \sum_w \sum_y y P(y \mid A=1, w) P_W(w)$

$P_{W,\varepsilon}(w) = (1 + \varepsilon S_W(w)) P_W(w), \quad E_P(S_W(w)) = 0$

$P_{A,\varepsilon}(A=1 \mid W=w) = (1 + \varepsilon S_A(a \mid w)) p(a \mid w), \quad E_P(S(A \mid w) \mid w) = 0$

$P_{Y\varepsilon}(Y=y \mid A=1, W=w) = (1 + \varepsilon S_Y(y \mid a, w)) p(y \mid a, w), \quad E_P(S_Y(y \mid A, w) \mid A, w) = 0$



$\frac{\partial}{\partial \varepsilon} \log p_\varepsilon(w, A, Y) = \frac{\partial}{\partial \varepsilon} \log(P_{W,\varepsilon}(w) \cdot P_{A,\varepsilon}(A \mid w) \cdot P_{Y\varepsilon}(Y \mid A$

$= S_W(w) + S_A(A \mid w) + S_Y(Y \mid A, w)$

Tanget space $\equiv T(P) = \{S(w); E(S) = 0\} + \{S(A \mid w); E(S \mid w) = 0\}$
$+ \{S(O) \cdot E(S \mid A, W) = 0\}$

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{how rich ?}}$

$= L_0^2(P)$

<u>Orthogonal decomposition of $L_0^2(P)$</u>

$S(O) = (S(O) - E(S \mid A, W)) + (E(S \mid A, W) - E(S \mid W)) + (E(S \mid W) - E(S))$

for $S \in L_0^2(P)$ cancels cancels 0

projection of O on

Note: $E[S_W(W) \cdot S_A(A \mid W)] = 0$

$E(S_Y(Y \mid A, W) \cdot S_W(W)) = 0$

*all these inner products are 0

so $H_W \oplus H_A \oplus H_Y$ for these orthogonal Hilbert spaces

$S(O) = \underbrace{(S(O) - E(S \mid A, W))}_{\pi[S \mid H_Y]} + \underbrace{(E(S \mid A, W) - E(S \mid W))}_{\pi[S \mid H_A]} + \underbrace{(E(S \mid W) - E(S))}_{\pi[S \mid H_W]}$

check that this is orthogonal

* if take $S - \pi(S \mid H_Y)$, it is not $\perp$ to H_Y — ie $S - \pi[S \mid H_A] \perp$

$E[S \mid A, W] \perp H_Y$ is true (it holds) — cont.

↳ true : $E[E[S \mid A, W] \cdot S(Y \mid A, W)] = 0$

cont. $E[(\underbrace{S - E[S|A,W]}_{=0} + E(S|W)) \cdot S(A|W)]$

$= E[\underbrace{E(S - E(S|A,W)|A,W)}_{=0} + E(S|W) S(A|W)]$

$= E[\underbrace{E(S|W)}_{=S} \cdot \underbrace{S(A|W)}_{=0}]$ Note: take conditional expectation

$= E[E(S|W) \underbrace{E(S(A|W)|W)}_{=0}] = 0$

* Can think of H_A, H_W, H_Y as tangent spaces
↳ orthogonal sum is whole tangent space.
$L_0^2(P)$

Now, take $\frac{\partial}{\partial \varepsilon} \Psi(P_\varepsilon)\Big|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} \sum_w \sum_y y(1 + \varepsilon S_Y(y|1,w)) p(y|1,w)(1 + \varepsilon S_W(w)) p(w)$

$= \sum_w \sum_y y S_Y(y|1,w) p(y|1,w) p(w) + \sum_w \underbrace{\sum_y y\, p(y|1,w)}_{\bar{Q}(1,w)} S_W(w) p(w)$

have to write as expectation

$= \sum_w \sum_y \sum_a y I(a=1) S_Y(y|a,w) p(y|a,w) p(a,w) p(w) \cdot \left(\frac{1}{p(a=1|w)}\right)$
$+ \sum_w \bar{Q}(1,w) \cdot S_W(w) p(w)$ → function of w w/ mean 0

$= E\left[Y \frac{I(A=1)}{P(A=1|W)} \cdot S_Y[Y|A,W]\right] + E(\bar{Q}(1,W) S_W(W))$

we want inner product $\langle D, S_W + S_A + S_Y \rangle$

has mean 0

$= E\left[\left(Y - E(Y|A,W)\right) \cdot \underbrace{\frac{I(A=1)}{P(A=1|W)}}_{\text{fixed}} \underbrace{S_Y(Y|A,W)}_{0}\right] + E[(\bar{Q}(1,W) - \Psi(P)) S_W(W)]$

constant ; $= 0$ mean

$= E\left(\underbrace{(Y - E(Y|A,W))}_{\text{becomes } 0} \cdot \underbrace{\frac{I(A=1)}{P(A=1|W)}}_{\text{fixed}} \underbrace{[S_Y(Y|A,W) + S_W(W) + S_A(A|W)]}_{\text{mean } 0}\right)$

$+ E\left((\bar{Q}(1,W) - \Psi(P)) \underbrace{(S_W(W) + S_A(A|W) + S_Y(Y|A,W))}_{\text{mean } 0}\right)$

∴ We have shown that $\frac{\partial}{\partial \varepsilon} \Psi(P_\varepsilon)\Big|_{\varepsilon=0} = \underbrace{E_P[D^*(P)(O) \cdot [S_W + S_A + S_Y](O)]}_{\langle D^*, S \rangle_{L_0^2(P)}}$,

where $D^*(P)(O) = \frac{I(A=1)}{P(A=1|W)}(Y - E(Y|A,W)) + E(Y|A=1,W) - \Psi(P)$

canonical gradient, aka efficient influence curve.

If we have model type $P_{\psi,\eta}$
then score $S_\psi = \frac{\partial}{\partial \psi} \log P_{\psi,\eta}$

$$\frac{\partial}{\partial \varepsilon} \log P_{\psi,\eta_\varepsilon} \Big|_{\varepsilon=0}$$

T_η = nuisance tangent space

S^*_ψ = eff score = $S_\psi - \pi[S_\psi / T_\eta]$

$\Downarrow$

canonical gradient = $\frac{\partial}{\partial \psi} E[S^*_\psi]^{-1} S^*_\psi$

*If we want to do it, read appendix A15 (p. 558)

$\hookrightarrow P_{\beta,\theta}(Y=1/A,W) = \exp(A(B_0 + B_1 W)) \cdot \theta_0(W)$ — can fluctuate this

$P_{Y/A,W}$ $\qquad$ $\theta_0(W)$ " $P(Y=1/A=0,W)$

$$\underset{\substack{\uparrow\\ \beta,\eta}}{P}(O) \cdot P_W(W)\, P_A(A/W) \cdot P_{B,\theta}[Y/A,W]$$

$\hat{\eta} = (P_W, P_{A/W}, \theta)$ of param interest ; nuisance param

$\pi(\text{score} / T_W \oplus T_{A/W} \oplus T_\theta)$
$= \pi(\text{score}/T_W) + \pi(\text{score}/T_{A/W}) + \pi(\text{score}/T_\theta)$

Model $\mathcal{M}(g_0)$: $g_0(a/w) = P_0(A=a/W=w)$ is known

$\Psi: \mathcal{M}(g_0) \to \mathbb{R}$, $\Psi(P) = E_P(E_P(Y/A=1,W))$

$\frac{\partial}{\partial \varepsilon} \Psi(P_\varepsilon)\big|_{\varepsilon=0} = E[D(P)(O)(S_W + S_Y)]$, where

$$D(P)(O) = \left[\frac{I(A=1)}{g_0(1/W)} Y - \Psi\right] + f(A/W) \quad \forall f \ni E_{g_0} f(A/W) = 0$$

If $f(A/W) = f^*(A/W) = \frac{-I(A=1)}{g_0(1,W)} \bar{Q}(A,W) + \bar{Q}(1,W)$
$= \bar{Q}(1,W)\left[1 - \frac{I(A=1)}{g_0(1,W)}\right]$

*will give us efficient influence curve.

Theorem: For $P = Q, g$
$P \in \mathcal{M}$
$\Psi(Q)$

The canonical gradient of $\Psi: \mathcal{M} \to \mathbb{R}$
equals " " of $\Psi: \mathcal{M}(g_0) \to \mathbb{R}$

*Look into efficiency bounds.