Appendix

Recycling Rates

The recycling rate is defined as the total catch divided by the number of individual fish that comprise the catch. Each individual fish is caught between 0 and infinity times, so the following relationship holds:

$$\sum_{0}^{\infty} f_i(t) = 1, \tag{1}$$

where $f_i(t)$ is the proportion of the population caught i times (See Table A1 for complete list of symbols).

The number of fish that have been caught at least once at time t is $N \sum_{i=1}^{\infty} f_i(t)$, where N is the population size which is assumed constant throughout time t. Denoting catch at time t as $C(t)$, the recycling rate, $R(t)$, can be written as

$$R(t) = \frac{C(t)}{N \sum_{i=1}^{\infty} f_i(t)} = \frac{C(t)}{N(1 - f_0(t))}. \tag{2}$$

Typically, we are interested in the recycling rate at the end of a fishing season, which we denote as simply $R = R(1)$, in which case the recycling rate can be simplified:

$$R = \frac{C(1)}{N(1 - f_0(1))}. \tag{3}$$

Notice that $C(1)$ is the catch per one season of angling effort. Therefore, $C(1)/N$ is the proportion of fish caught in one season, which we term seasonal catchability coefficient, Q, and so

$$R = \frac{Q}{1 - f_0(1)}. \tag{4}$$

Since Q is derived from direct estimates of catch and population size, the only remaining unknown is $f_0(1)$.

The General Solution

In order to find $f_0(1)$, we will need to derive expressions for the fractional catch history, $f_i(t)$, as well as the fractional reporting history, $g_i(t)$, which will be needed to fit tag return data to our analytic model.

Suppose that the individual catchability of a fish can vary from 0 to some maximum catchability $\varphi_{\max}$. Then an individual fish has individual catchability $s\varphi_{\max}$, where $s \in [0, 1]$. The exact distribution of fish selectivity is determined by a probability density function (pdf) $h(s)$. This pdf satisfies the following normalization condition:

$$\int_0^1 h(s)ds = 1. \tag{5}$$

The catchability coefficient is the average of the individual catchabilities. Thus

$$Q = \varphi_{\max} \int_0^1 sh(s)ds \tag{6}$$

can be used to determine $\varphi_{\max}$ if $h(s)$ and Q are known.

Our next step is to find an expression for our fractional catch history, in which $g_i(t)$ represents the fraction of the tagged fish reported i times at time t. If $g_i(s, t)$ is the probability density function for fish that have catchability $s\varphi_{\max}$ and have been reported i times at time t, then

$$g_i(t) = \int_0^1 g_i(t, s)ds. \tag{7}$$

At the beginning of the season, no fish have been reported, and the distribution of selectivities is deteremined by $h(s)$, so when $t = 0$, the following equations hold:

$$g_0(0, s) = h(s) \tag{8}$$

and for $i \neq 0$,

$$g_i(0, s) = 0. \tag{9}$$

The selectivity of a fish is fixed and we are assuming a fixed population, so we can actually consider each (perhaps infinitely small) subpopulation with selectivity s separately. To determine how the density function varies with time, we fix s and solve $g_i(t, s)$ for each i.

Once the season begins, fish will begin being reported. Fish with selectivity s are caught at a rate of $s\varphi_{\max}$ (times per season), and reported at a rate of $\lambda s\varphi_{\max}$ (times per season). Thus, for each s, we get the following differential equations:

$$\frac{dg_0(t, s)}{dt} = -\lambda s\varphi_{\max} g_0(t, s) \tag{10}$$

$$\frac{dg_i(t, s)}{dt} = \lambda s\varphi_{\max} g_{i-1}(t, s) - \lambda s\varphi_{\max} g_i(t, s). \tag{11}$$

Equations (8) and (9) give us the initial conditions for how the population begins at $t = 0$, and equations (10) and (11) describe how $g_i(t, s)$ evolves with time. These differential equations form a complete initial value problem and have the neat closed form solution

$$g_i(t, s) = \frac{h(s)}{i!} e^{-\lambda s\varphi_{\max} t} (\lambda s\varphi_{\max} t)^i. \tag{12}$$

This is a Poisson distribution with parameter $\lambda s \varphi_{\max} t$ multiplied by the density of selectivity, $h(s)$.

Now, to determine the fraction of fish that have been caught i times, we use equation (7) to obtain

$$g_i(t) = \int_0^1 \frac{h(s)}{i!} e^{-\lambda s \varphi_{\max} t} (\lambda s \varphi_{\max} t)^i ds = \frac{(\lambda \varphi_{\max} t)^i}{i!} \int_0^1 h(s) e^{-\lambda s \varphi_{\max} t} s^i ds. \tag{13}$$

Ultimately, we are interested in the fraction of fish that have been caught, rather than reported, i times. If the reporting rate is 100 percent, then every caught fish is reported, and $f_i(t) = g_i(t)$. By substituting $\lambda = 1$ into equation (13),

$$f_i(t) = \frac{(\varphi_{\max} t)^i}{i!} \int_0^1 h(s) e^{-s \varphi_{\max} t} s^i ds. \tag{14}$$

We cannot continue past this point without knowing more about the shape of $h(s)$. For non-trivial selectivity distributions, there may not be a closed form solution for the integral. However, any selectivity curve can be inserted into equation (14) and solved numerically by a computer to obtain $f_0(t)$ and therefore compute the recycling rate with equation (4).

Simplifying Assumptions

By making certain assumptions about $h(s)$ we can generate some simple solutions that are easy to apply. The first assumption is that some proportion of the population, p, is vulnerable to angling while another proportion, $(1 - p)$, is not vulnerable. When $p < 1$, the selectivity frequency distribution will become zero-inflated leading to a higher recycling rate to compensate for the uncatchable fraction. We can make additional assumptions about the shape of $h(s)$ for the remaining catchable fraction.

Equal Selectivity

Suppose we have a simple selectivity distribution, with fraction p having selectivity $s = 1$ and fraction $(1-p)$ having selectivity $s = 0$. (Setting $p = 1$ means there is no zero-inflation and all fish are equally catchable.) With this distribution, $h(s)$ is the sum of two Dirac delta functions $\delta(x)$ (Weisstein 2021a):

$$h(s) = p\delta(s-1) + (1-p)\delta(s). \tag{15}$$

We can substitute this into (13) to get an expression for $g_0(t)$:

$$g_0(t) = \int_0^1 \Big(p\delta(s-1) + (1-p)\delta(s)\Big) e^{-\lambda s \varphi_{\max} t} ds = pe^{-\lambda \varphi_{\max} t} + (1-p). \tag{16}$$

For $i \neq 0$, there is no contribution from the uncatchable fraction, and

$$g_i(t) = \frac{(\lambda \varphi_{\max} t)^i}{i!} \int_0^1 \Big(p\delta(s-1) + (1-p)\delta(s)\Big) e^{-\lambda s \varphi_{\max} t} s^i ds = p\frac{(\lambda \varphi_{\max} t)^i}{i!} e^{-\lambda \varphi_{\max} t}. \tag{17}$$

Tag return data can be fit to equations (16) and (17) to estimate p after using equation (6) to solve for $\varphi_{\max}$,

$$\varphi_{\max} = \frac{Q}{\int_0^1 s\Big(p\delta(s-1) + (1-p)\delta(s)\Big) ds} = \frac{Q}{p}. \tag{18}$$

Using the estimated p and setting $\lambda = 1$ in (16) yields

$$f_0(t) = pe^{-\varphi_{\max} t} + (1-p). \tag{19}$$

We set t to one season and combine (3), (19), (18) to get our final result:

$$R = \frac{Q/p}{1 - e^{-Q/p}}. \tag{20}$$

If it is reasonable to assume that $p = 1$, then no data fitting is required and the recycling rate can be estimated directly as

$$R = \frac{Q}{1 - e^{-Q}}. \tag{21}$$

Uniformly Distributed Selectivity

We pursue a similar strategy with a slightly different selectivity distribution. Now we assume that there is a catchable fraction p whose selectivity is uniformly distributed between 0 and 1. Once again, there is an uncatchable fraction $1 - p$. In this case, $h(s)$ has only one delta function:

$$h(s) = p + (1 - p)\delta(s). \tag{22}$$

This is no longer a discrete distribution, so the derivation is more complex than the equal selectivity case. Now, when $i = 0$,

$$g_0(t) = (1 - p) + p \int_0^1 e^{-\lambda s \varphi_{\max} t} ds = (1 - p) + p \frac{1 - e^{-\lambda \varphi_{\max} t}}{\lambda \varphi_{\max} t}. \tag{23}$$

When $i \neq 0$, our solution includes the lower incomplete gamma function, $\gamma(a, x)$ (Weisstein 2021b).

$$g_i(t) = \frac{(\lambda \varphi_{\max} t)^i}{i!} p \int_0^1 e^{-\lambda s \varphi_{\max} t} s^i ds = \frac{(\lambda \varphi_{\max} t)^{-1}}{i!} \gamma(i + 1, \lambda \varphi_{\max} t). \tag{24}$$

As in the previous case, tag return data can be fitted to equations (23) and (24) after substituting for $\varphi_{\max}$

$$\varphi_{\max} = \frac{Q}{\int_0^1 s\Big(p + (1-p)\delta(s)\Big)ds} = \frac{Q}{p\int_0^1 s ds} = \frac{2Q}{p}. \tag{25}$$

Once again, we get $f_0(t)$ by setting $\lambda = 1$ in (23):

$$f_0(t) = (1-p) + p\frac{1 - e^{-\varphi_{\max} t}}{\varphi_{\max} t}, \tag{26}$$

and the recycling rate is found by combining equations (3), (26), and (25):

$$R = \frac{2\left(\frac{Q}{p}\right)^2}{2\frac{Q}{p} - 1 + e^{-2\frac{Q}{p}}}. \tag{27}$$

If we can assume $p = 1$, then no tag return data is required and the recycling rate simplifies to

$$R = \frac{2(Q)^2}{2Q - 1 + e^{-2Q}}. \tag{28}$$

Citations

Weisstein, E. W. 2021a. ”Delta Function.” From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/DeltaFunction.html.

Weisstein, E. W. 2021b. ”Incomplete Gamma Function.” From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/IncompleteGammaFunction.html

Table A1. Description of symbols used in the appendix.

Symbol	Description
$R(t)$	Recycling rate; the ratio of total catch to individuals caught at least once
t	Time. A fishing season begins at $t = 0$ and ends at $t = 1$.
N	The number of fish in the population
$C(t)$	The number of fish caught through time t
Q	Seasonal Catchability coefficient; the proportion of the population caught in one season of angling
λ	Tag reporting rate by anglers
s	Selectivity of a fish, between 0 and 1
$h(s)$	Selectivity distribution function, or selectivity probability density function
$\varphi_{\max}$	Maximum individual catchability, or the number of times the most catchable fish is expected to be caught in a season
φ	Individual catchability, or the number of times a specific fish is expected to be caught in a season, $\varphi = s\varphi_{\max}$
p	The fraction of the population that is available to angling
$f_i(t)$	The fraction of fish that have been caught exactly i times at time t
$g_i(t)$	The fraction of fish that have been *reported* exactly i times at time t
$f_i(t, s)$	The probability density function for fish with selectivity s that have been caught exactly i times at time t
$g_i(t, s)$	The probability density function for fish with selectivity s that have been *reported* exactly i times at time t