Simulating Stock Prices

- **The geometric Brownian motion stock price model**
  - Recall that a rv Y is said to be *lognormal* if $X = \ln(Y)$ is a normal random variable.
  - Alternatively, Y is a lognormal rv if $Y = e^X$, where X is a normal rv.
  - If v and σ, are the mean and standard deviation of X, the mean and variance of Y are given by

$$E[Y] = e^{v+\sigma^2/2}, \quad \text{var}[Y] = e^{2v+\sigma^2}(e^{\sigma^2} - 1).$$

  - Note that $E[Y] \neq e^{E[X]} = e^v$ although $Y = e^X$.
  - A popular stock price model based on the lognormal distribution is the *geometric Brownian motion* model, which relates the stock prices at time 0, S_0, and time $t > 0$, S_t by the following relation:

$$\ln(S_t) = \ln(S_0) + (\mu - \sigma^2/2)t + \sigma z(t) \quad ,$$

where, μ and $\sigma > 0$ are constants and $z(t)$ is a normal rv with mean 0 and variance t.[1]

  - It follows that $\ln(S_t / S_0)$ is a normal random variable with mean $(\mu - \sigma^2/2)t$ and variance $\sigma^2 t$.

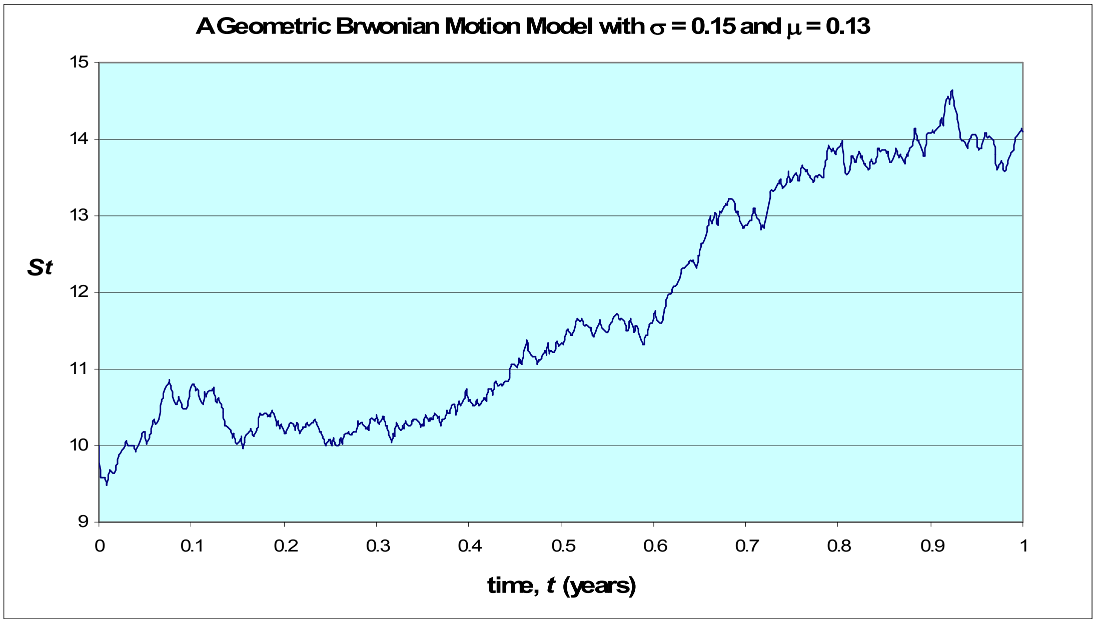
[1] $z(t)$ is called a Brownian motion.

- That is, S_t / S_0 is a lognormal rv with mean and variance

  $E[S_t / S_0] = e^{\mu t - \sigma^2 t/2 + \sigma^2 t/2} = e^{\mu t}$,

  $\mathrm{var}[S_t / S_0] = e^{2(\mu t - \sigma^2 t/2) + \sigma^2 t/2}(e^{\sigma^2 t} - 1) = e^{\mu t}(e^{\sigma^2 t} - 1)$.

- Note that μ can be seen as the stock rate of return assuming continuous compounding. So μ is called the *expected return* of the stock.
- In addition, σ measures the variability of the stock price. So σ is called the *volatility* of the stock price.
- Typical values for these parameters are μ = 13% and σ = 15% when time *t* is measured in years.
- The main idea behind the geometric Brownian motion model is that the probability of a certain percentage change in the stock price within a time *t* is the same at all times.
- This is a memoryless or Markovian behavior indicating that past stock values won't help in predicting future values.
- In addition, the *expected* value and variance of the stock price typically follow an increasing trend.



- **Simulating geometric Brownian motion stock prices**
  - The key idea for simulating a stock price is that $\ln(S_t / S_0)$ is normally distributed with mean $(\mu - \sigma^2/2)t$ and variance $\sigma^2 t$.
  - An algorithm for simulating the stock price at a time $t > 0$, given that current stock price (at $t = 0$) is S_0 is as follows.

    1. Generate $Z \sim N(0,1)$
    2. Set $\mu_t = (\mu - \sigma^2/2)t$ and $\sigma_t = \sigma t^{0.5}$.
    3. Set $S_t = S_0 \times e^{\mu t + \sigma t Z}$
  - In practice the expected return, μ, is too difficult to estimate accurately, while the volatility σ can be estimated reasonably well from historical data.

- For estimating μ one is better off making a subjective estimate or a probability distribution.
- Estimating σ can be made based on historical data. However, an implied volatility approach is often used.
- The idea of implied volatility is to find σ based on the market prices of certain financial instruments.
- Among the widely used instruments for this purpose are *European stock options*.

- **European options and the Black – Scholes model**
  - A *European call option* is a financial instrument that gives its holder the right, but not the obligation, to *buy* one (or more) share(s) of stock price for a *strike price K* per share at a *maturity time T* in the future.
  - The buyer of the option pays a price or a *premium*, C, in exchange for it.
  - Obviously, a European call option is beneficial only when the stock price at time T exceeds $K+C$.
  - A *European put option* is a financial instrument that gives its holder the right, but not the obligation, to *sell* one (or more) share(s) of stock price for a *strike price K* per share at a *maturity time T* in the future,
  - The buyer of the option pays a premium, P, in exchange for it.

- Assuming a geometric Brownian motion stock model, Black and Scholes (1973) derived a key result given C (or P) as a function of T, K, μ, σ, and the interest rate r assuming continuous compounding.
- The Black-Scholes formula for call and put options is given by the following theorem.

Theorem *The price at time* 0 *of a European call* and put *options with strike price K and maturity T on an underlying stock with volatility* σ *are*

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2) \ ,$$

$$P = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \, ,$$

where S_0 *is the stock price at time* 0*,*

$$d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}, \ d_2 = d_1 - \sigma\sqrt{T} \ \ and \ \ N(x) = \int_{-\infty}^{x} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \ is$$

the standard Normal cdf.

- The implied volatility, $\hat{\sigma}$, of a stock is the value of σ which make the option price specified by the Black-Scholes formula equal to market prices of the options listed in financial newspapers and websites.
- Finding $\hat{\sigma}$ is done numerically.

- **Example 1.**
  - On June 30, 1998 Dell stock sold for \$94. A European put with a strike price of \$80 expiring on November 22, 1998 was selling for \$5.25. The current 90 day T-Bill (bond) rate is 5.5%. What is the implied volatility for Dell?
  - Solution: See Excel file Dell_stock.xls.

- **Example 2 (model with random μ).**
  - Simulate the daily Dell stock in Example 1 between July 1, 1998 till the end of 1998. The expected stock return is believed to equally likely take on values 10%, 20%, 30%, and 40%.
  - Solution: See Excel file Dell_stock.xls.

- **Example 3 (model with μ based on analyst forecast).**
  - Simulate the daily Dell stock in Example 1 between July 1, 1998 till the end of 1998. Analysts consensus view is that Dell stock will be selling for \$120 on 1/1/1999.
  - Here we need to solve for μ that makes the expected stock price equal to \$120 on 1/1/1999. Recall that the expected stock price at time t is $E[S_t] = S_0 e^{\mu t}$.

- Setting $t = 0$, at 06/30/1998, then $t = 185$ on 1/1/1999. Then, we can solve for μ as follows.

$$110 = 94e^{\mu(185/365)} \Rightarrow \ln(110) = \ln(94) + 0.5068\mu \Rightarrow \mu = 0.31.$$

- See Excel file Dell_stock.xls for the complete simulation.

- **Example 4 (Strong Buy/Strong Sell Consensus).**
  - On July 10, 2000, Business Week reported the results of a study that estimated how well analysts' consensus of Strong Buy (1) to Strong Sell (5) forecast annual return on a stock. They found the following predictions for annual returns (relative to the market, assessed usually via an index fund).

Rating	**Average Excess Return over Market**
Strong Buy = 1	+4.5%
Buy = 2	+3.5%
Hold = 3	+0.5%
Sell = 4	−1.0%
Strong Sell = 5	−8.5%

  - Suppose that Dell stock in Example 1 got a rating of 1.6, and the market return is equally likely to be −10%, −5%, 0%, 5%, 10%, 15%. Simulate the daily Dell stock between July 1, 1998 and the end of 1998.

- First, we find the average excess return of Dell over market by interpolating in the Business Week table as $0.4\times4.5+0.6\times3.5 = +3.9\%$.
- Then, μ is set as 3.9% plus the simulated market return.
- See Excel file Dell_stock.xls for the complete simulation.

- **Generating Stock Prices by Bootstrapping.**
  - As an alternative approach to geometric Brownian motion, we use bootstrapping to simulate future stock prices based on a sample of stock history.
  - The idea behind bootstrapping is assuming that every past value is equally likely to occur in the future.

- **Example 5 (Bootstrapping).**
  - Simulate the first 3 months of 1999 of the Dell stock price in Example 1 based on data from the last three months of 1998 by bootstrapping.
  - See Excel file Dell_stock.xls for the complete simulation.