Linear Algebra: Linear Dependence and Independence

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Introduction to linear independence

If two vectors are *linearly dependent*, any linear combination of the vectors will produce a scalar combination of one vector. For example,

$$\begin{aligned} \hat{\mathbf{a}} &= (2,3) \\ \hat{\mathbf{b}} &= (4,6) \\ c_1\hat{\mathbf{a}} + c_2\hat{\mathbf{b}} &= c_1\hat{\mathbf{a}} + 2c_2\hat{\mathbf{b}} \\ &= (c_1 + 2c_2)\hat{\mathbf{a}} = c_3\hat{\mathbf{a}} \end{aligned}$$

Hence $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are linearly dependent.

Two vectors are linearly dependent if one vector in the set can be represented by a linear combination of another vector in that set.

Even a set of three vectors in $\mathbb{R}^2$ would be linearly dependent, because the third vector could be represented by a linear combination of the other two vectors *if they span the vector space*. The third vector would have to be outside of the $\mathbb{R}^2$ plane to be linearly independent.

A good *basis* for $\mathbb{R}^2$ would be two, linearly independent vectors. Hence, unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ would be a good basis for $\mathbb{R}^2$.

More on linear independence

A set $s = \{v_1, v_2 \dots v_n\}$ is *linearly dependent* $\iff c_1v_1 + c_2v_2 + \dots c_nv_n = \mathbf{0}$, for some c_i where at least one is $\neq 0$. This means that one vector can be represented by the sum of the other vectors:

$$\begin{aligned} v_1 &= a_2v_2 + a_3v_3 + \dots a_nv_n \\ 0 &= -1v_1 + a_2v_2 + a_3v_3 + \dots a_nv_n \end{aligned}$$

Assuming $c_1 \neq 0$, you can divide both sides by c_1, yielding:

$$\begin{aligned} v_1 + \frac{c_2}{c_1}v_2 + \dots \frac{c_n}{c_1} &= \mathbf{0} \\ \frac{c_2}{c_1}v_2 + \dots \frac{c_n}{c_1}v_n &= -v_1 \end{aligned}$$

Hence, if at least one of these constants is nonzero, you can represent v_1 as a linear combination of the other vectors.

Testing for linear independence

If you have a linear combination of two vectors $c_1v_1 + c_2v_2 = \mathbf{0}$:

- If c_1 or c_2 are nonzero $\rightarrow$ they are dependent.
- Else if c_1 *and* c_2 are both zero $\rightarrow$ they are independent.

For example:

$$v_1 = (2, 1), v_2 = (3, 2)$$
$$2c_1 + 3c_2 = 0$$
$$c_1 + 2c_2 = 0$$
$$c_1 + \frac{3}{2}c_2 = 0$$
$$\frac{1}{2}c_2 = 0$$
$$c_2 = 0 = c_1$$

Hence, these vectors are linearly independent.

For three vectors in $\mathbb{R}^2$:

$$v_1 = (2, 1), v_2 = (3, 2), v_3 = (1, 2)$$
$$2c_1 + 3c_2 + c_3 = 0$$
$$c_1 + 2c_2 + 2c_3 = 0$$

Picking a $c_3 = -1$:

$$2c_1 + 3c_2 - 1 = 0$$
$$2c_1 + 4c_2 - 4 = 0$$
$$-c_2 + 3 = 0$$
$$c_2 = 3$$
$$c_1 = -4, c_2 = 3, c_3 = -1$$

If you had three vectors in two-dimensional space, one of them has to be linearly dependent.

If you can show that you can satisfy $c_1v_1 + c_2v_2 = \mathbf{0}$ with at least one nonzero constant, the vectors are linearly dependent.

Span and linear independence example

1. Does the following set of vectors span $\mathbb{R}^3$?
2. Are the vectors linearly independent?

Testing for span

$$s = \{v_1 = (1, -1, 2), v_2 = (2, 1, 3), v_3 = (-1, 0, 2)\}$$

Can $c_1v_1 + c_2v_2 + c_3v_3 = (a, b, c)$ for any a, b, c?

$$\begin{gathered} c_1 + 2c_2 - c_3 = a \\ -c_1 + c_2 = b \\ 2c_1 + 3c_2 + 2c_3 = c \end{gathered}$$

$$b + a = 3c_2 - c_3$$

Multiplying a by -2:

$$c - 2a = -c_2 + 4c_3$$

To eliminate the $-c_2$ term in the last equation, we multiply it by 3 and add it to the $b + a$ term.

$$\begin{gathered} 11c_3 = 3c - 5a + b \\ c_3 = \frac{1}{11}(3c - 5a + b) \\ c_2 = \frac{1}{3}(b + a + c_3) \\ c_1 = a - 2c_2 + c_3 \end{gathered}$$

Hence, with any a, b, c, you can find c_1, c_2, c_3. They are then said to span $\mathbb{R}^3$.

Testing for linear independence

$$c_1v_1 + c_2v_2 + c_3v_3 = \mathbf{0}$$

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It's the same procedure as the last problem, but using $\mathbf{0}$ instead of a, b, c.

$$c_3 = \frac{1}{11}(3(0) - 5(0) + 0) = 0$$
$$c_2 = \frac{1}{3}(0 + 0 + 0) = 0$$
$$c_1 = (0 - 2(0) + 0) = 0$$

Since all of these constants are equal to zero, these vectors are linearly independent.