Linear Algebra: Vector Dot and Cross Products

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Vector dot product and vector length

The *dot product*, $\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$, is defined as the sum of the product of their corresponding components:

$$\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots a_n b_n$$

The vector *magnitude* is defined to be the root mean of squares of all components:

$$\|a\| = \sqrt{a_1^2 + a_2^2 \ldots a_n^2}$$

This means that the magnitude can also be written in terms of the dot product:

$$\|a\| = \sqrt{\hat{\mathbf{a}} \cdot \hat{\mathbf{a}}}$$

Proving vector dot product properties

The dot product is *commutative*: $\hat{v} \cdot \hat{w} = \hat{w} \cdot \hat{v}$:

$$\hat{v} \cdot \hat{w} = v_1 w_1 + v_2 w_2 + \ldots v_n w_n$$
$$\hat{w} \cdot \hat{v} = w_1 v_1 + w_2 v_2 + \ldots w_n v_n$$
$$v_i w_i = w_i v_i \text{ (commuative property)}$$

Is the dot product *distributive*? Does $(\hat{v} \cdot \hat{w}) \cdot \hat{x} = (\hat{v} \cdot \hat{x} + \hat{w} \cdot \hat{x})$

$$(\hat{v} \cdot \hat{w}) \cdot \hat{x} = (v_1 w_1) x_1 + (v_2 w_2) x_2 + \ldots (v_n w_n) x_n$$
$$(\hat{v} \cdot \hat{x} + \hat{w} \cdot \hat{x}) = v_1 x_1 + w_1 x_1 + v_2 x_2 + w_2 x_2 + \ldots v_n x_n + w_n x_n$$

You can factor the x term out, yielding:

$$(\hat{v} \cdot \hat{x} + \hat{w} \cdot \hat{x}) = (v_1 w_1) x_1 + (v_2 w_2) x_2 + \ldots (v_n w_n) x_n$$

Hence, dot products are both communative and distributative. Are they also *associative*? Does $(c\hat{v}) \cdot \hat{w} = c(\hat{v} \cdot \hat{w})$?

$$c(\hat{v}) \cdot \hat{w} = c v_1 w_1 + c v_2 w_2 + \cdots c v_n w_n$$
$$c(\hat{v} \cdot \hat{w}) = c v_1 w_1 + c v_2 w_2 + \cdots c v_n w_n$$

Proof of the Cauchy-Schwarz inequality

The Cauchy-Schwarz inequality $|\hat{x} \cdot \hat{y}| \leq \|x\|\|y\|$, and $|\hat{x} \cdot \hat{y}| = \|x\|\|y\| \iff \hat{x} = c\hat{y}$.

For example, say we have a function

$$\begin{gathered}
p(t) = \|t\hat{y} - \hat{x}\|^2 \geq 0 \\
= (t\hat{y} - \hat{x}) \cdot (t\hat{y} - \hat{x}) \\
= t\hat{y} \cdot t\hat{y} - 2\hat{x} \cdot t\hat{y} + \hat{x} \cdot \hat{x} \\
= (\hat{y} \cdot \hat{y})t^2 - 2(\hat{x} \cdot \hat{y})t + \hat{x} \cdot \hat{x} \geq 0
\end{gathered}$$

Let's define $a = (\hat{y} \cdot \hat{y}), b = 2(\hat{x} \cdot \hat{y}), c = \hat{x} \cdot \hat{x}$.

$$\begin{gathered}
p(t) = at^2 - bt + c \geq 0 \text{ for any t} \\
p(\frac{b}{2a}) = a\frac{b^2}{4a^2} - b\frac{b}{2a} + c \geq 0 \\
= \frac{b^2}{4a} - \frac{b^2}{2a} + c = \frac{b^2 - 2b^2}{4a} + c = \frac{-b^2}{4a} + c \geq 0 \\
c \geq \frac{b^2}{4a} \\
4ac \geq b^2
\end{gathered}$$

Back-substituting a, b, c:

$$\begin{gathered}
4(\hat{y} \cdot \hat{y})(\hat{x} \cdot \hat{x}) \leq (2(\hat{x} \cdot \hat{y}))^2 \\
4(\|\hat{y}\|^2\|\hat{x}\|^2) \leq 2(\hat{x} \cdot \hat{y}) \leq 4(\hat{x} \cdot \hat{y})^2 \\
\|\hat{y}\|^2\|\hat{x}\|^2 \geq (\hat{x} \cdot \hat{y})^2 \\
\|\hat{y}\|\|\hat{x}\| \geq |\hat{x} \cdot \hat{y}|
\end{gathered}$$

What if in the case of $\hat{x} = c\hat{y}$?

$$\begin{gathered}
|\hat{x} \cdot \hat{y}| = |c\hat{y} \cdot \hat{y}| = |c||\hat{y} \cdot \hat{y}| = |c|\|\hat{y}\|^2 \\
= |c|\|\hat{y}\|\|\hat{y}\| = \|c\hat{y}\|\|\hat{y}\| \\
= \|\hat{x}\|\|\hat{y}\|
\end{gathered}$$

Vector triangle inequality

The *triangle inequality* states that the length of the sum of two vectors is less than or equal to the sum of their magnitudes.

If you have $\hat{x}, \hat{y} \in \mathbb{R}^n, x, y \neq 0$, then $|\hat{x} \cdot \hat{y}| \leq \|\hat{x}\|\|\hat{y}\|$, and if $\hat{x} = c\hat{y} \iff |\hat{x} \cdot \hat{y}| = \|\hat{x}\|\|\hat{y}\|$.

You could also add $\hat{x} \cdot \hat{y} \leq |\hat{x} \cdot \hat{y}| \leq \|\hat{x}\|\|\hat{y}\|$.

$$\begin{aligned}
\|\hat{x} + \hat{y}\|^2 &= (\hat{x} + \hat{y}) \cdot (\hat{x} + \hat{y}) \\
&= \|\hat{x}\|^2 + 2(\hat{x} \cdot \hat{y}) + \|\hat{y}\|^2
\end{aligned}$$

$$\begin{aligned}
\|\hat{x} + \hat{y}\|^2 &\leq \|\hat{x}\|^2 + 2\|\hat{x}\|\|\hat{y}\| + \|\hat{y}\|^2 \\
\|\hat{x} + \hat{y}\|^2 &\leq (\|\hat{x}\| + \|\hat{y}\|)^2 \\
\|\hat{x} + \hat{y}\| &\leq \|\hat{x}\| + \|\hat{y}\|
\end{aligned}$$

In the case that $\hat{x} = c\hat{y}, c > 0$, then you could say that $\|\hat{x} + \hat{y}\| = \|\hat{x}\| + \|\hat{y}\|$.

Defining the angle between vectors

If you have vectors $\hat{a}, \hat{b} \in \mathbb{R}^n, \neq 0$, then you could construct a triangle with sides $\|\hat{a}\|, \|\hat{b}\|$, and $\|\hat{a} - \hat{b}\|$. This can work because of the triangle inequality theorem:

$$\|\hat{x} + \hat{y}\| \leq \|\hat{x}\| + \|\hat{y}\|$$

For each of the sides, you can show that you can construct a triangle from nonzero vectors in $\mathbb{R}^n$.

This triangle can allow you to find the *angle* between vectors $\hat{a}$ and $\hat{b}$ using the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos(\theta)$:

$$\begin{aligned}
\|\hat{a} - \hat{b}\|^2 &= \|b\|^2 + \|a\|^2 - 2\|a\|\|b\|\cos\theta \\
(\hat{a} - \hat{b}) \cdot (\hat{a} - \hat{b}) &= \hat{a} \cdot \hat{a} - 2(\hat{a} \cdot \hat{b}) + \hat{b} \cdot \hat{b} \\
(\hat{a} \cdot \hat{b}) &= \|\hat{a}\|\|\hat{b}\|\cos\theta
\end{aligned}$$

This requires two more definitions:

If $\hat{a} = c\hat{b}, c > 0 \implies \theta = 0$. If $\hat{a} = c\hat{b}, c < 0 \implies \theta = 180$.

Using this, you can test if two vectors are perpendicular. This is not defined for zero vectors, because you get $\cos\theta = \frac{0}{0}$.

If $\theta = 90°$, then $\hat{a} \cdot \hat{b} = 0$. These vectors are *orthogonal*, but not necessarily *perpendicular*.

- Two vectors are **orthogonal** if their dot product is equal to 0. This includes the zero vector, so the zero vector is orthogonal to every vector (even itself!).
- Two vectors are **perpendicular** iff they are orthogonal and *both* nonzero.

Defining a plane in $\mathbb{R}^3$ with a point and normal vector

The traditional form to write a plane in $\mathbb{R}^3$ would be to write a linear equation $Ax + By + Cz = D$. Another way to write the equation of the plane is by picking a point on the plane, and a vector which is *normal* to all the vectors on the plane. The vector for the point $\hat{x_0}$ itself does not lie on the plane, but it can be used to construct the plane itself.

If you choose another point $\hat{x} = (x, y, z)$, then you know that $(\hat{x} - \hat{x_0})$ has to be on the plane. Hence,

$$\hat{n} \cdot (\hat{x} - \hat{x_0}) = 0$$

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

This is in the form of $Ax + By + Cz = D$! If you take any point on the plane, you can construct a linear equation for the plane. This is a useful generalization of the plane that can be extended to higher dimensions.

Cross product introduction

The cross product is useful, but much more limited. The cross product is *only* defined in $\mathbb{R}^3$, and produces a vector from two vectors.

If you have $\hat{a} = (a_1, a_2, a_3), \hat{b} = (b_1, b_2, b_3)$, then $\hat{a} \times \hat{b} = ((a_2b_3 - a_3b_2), (a_3b_1 - a_1b_3), (a_1b_2 - a_2b_1))$.

For example, if you have $\hat{a} = (1, -7, 1), \hat{b} = (5, 2, 4)$, then $\hat{a} \times \hat{b} = (-7 * 4 - 2, 5 - 4, 2 - (-7 * 5)) = (-30, 1, 37)$. This vector is orthogonal to $\hat{a}$ and $\hat{b}$!

To figure out the direction, use the *right-hand rule.* Point your pointer in the direction of the first vector, your middle finger in the direction of the other, and your thumb points in the direction of the resultant vector.

Proof: Relationship between cross product and sin of angle

In this section, the proof of $\|\hat{a} \times \hat{b}\| = \|\hat{a}\|\|\hat{b}\| \sin\theta$ is demonstrated.

$$\|\hat{a} \times \hat{b}\|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$
$$= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3b^2b_2^2 + a_3^2b_1^1 - 2a_1a_3b_1b_3 + a_1^2b_3^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2$$

All the middle terms (with -2 as the coefficient) can be written as $-2(a_2a_3b_2b_3 + a_1a_3b_1b_3 + a_1a_2b_1b_2)$.

We know that $\|\hat{a}\|\|\hat{b}\| \cos\theta = \hat{a} \cdot \hat{b} = a_1b_1 + a_2b_2 + a_3b_3$. Squaring both sides:

$$a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + 2(a_1a_2b_1b_1 + a_1a_3b_1b_3 + a_2a_3b_2b_3)$$

This is equal to the middle term of the $\|\hat{a} \times \hat{b}\|^2$! Hence,

$$\|\hat{a} \times \hat{b}\|^2 + \|\hat{a}\|^2\|\hat{b}\|^2 \cos^2\theta = a_1^2(b_1^2 + b_2^2 + b_3^2) + a_2^2(b_1^2 + b_2^2 + b_3^2) + a_3^2(b_1^2 + b_2^2 + b_3^2)$$
$$= (b_1^2 + b_2^2 + b_3^2)(a_1^2 + a_2^2 + a_3^2) = \|\hat{b}\|^2\|\hat{a}\|^2$$
$$\|\hat{a} \times \hat{b}\|^2 = \|\hat{b}\|^2\|\hat{a}\|^2 - \|\hat{b}\|^2\|\hat{a}\|^2 \cos^2\theta$$
$$= \|\hat{a}\|^2\|\hat{b}\|^2(1 - \cos^2\theta) = \|\hat{a}\|^2\|\hat{b}\|^2 \sin^2\theta$$

Hence,

$$\|\hat{a} \times \hat{b}\| = \|\hat{a}\|\|\hat{b}\| \sin\theta$$

Dot and cross product comparison/intuition

First, the dot product is defined to be $\hat{a} \cdot \hat{b} = \|\hat{a}\|\|\hat{b}\| \cos\theta$, and the cross product is defined to be $\|\hat{a} \times \hat{b}\| = \|\hat{a}\|\|\hat{b}\| \sin\theta$.

You can imagine the scalar product as being the length of the projection of $\hat{a}$ multiplied by the length of $\hat{b}$. This is why two orthogonal vectors have a dot product of zero, because vector $\hat{a}$ has no projection on $\hat{b}$.

Similar yet opposite, when you compute the cross product of $\hat{a}$ and $\hat{b}$, then $\|\hat{a} \times \hat{b}\|$ can be thought of as the length of $\hat{b}$ multiplied by the component of $\hat{a}$ perpendicular to $\hat{b}$.

The cross product of two vectors is also equal to the *area of the parallelogram* formed by the two vectors!

Normal vector from plane equation

Earlier in this chapter the method to derive a linear equation for a plane was shown, but how do you find the normal vector from the plane equation $Ax + By + Cz = D$?

Finding the normal vector $\hat{n}$ is incredibly easy - it's $\hat{n} = A\hat{\imath} + B\hat{\jmath} + C\hat{k}$. The constant D makes no difference, because it would just be shifting the normal vector to a different point on the plane, and not actually affecting the direction of the vector itself.

D is also equal to $ax_p + by_p + cz_p$.

Point distance to plane

How do you find the minimum distance between a point not on the plane?

If we take the vector between the origin point and the point off the plane, the vector would be:

$$\hat{f} = (x_0 - x_p)\hat{\imath} + (y_0 - y_p)\hat{\jmath} + (z_0 - z_p)\hat{k}$$

We can use trigonometry to find the minimum distance to the plane by using the *normal vector*. Since the minimum distance from a point p to the plane would be orthogonal to the plane, we can use the vector f and the normal vector to find an angle we can use. After that, you can use trigonometry to find the minimum distance to the plane. The dot product is:

$$\hat{n} \cdot \hat{f} = \frac{Ax_0 - Ax_p + By_0 - By_p + Cz_0 - Cz_p}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{Ax_0 + By_0 + Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}} = \text{ the minimum distance to the plane.}$$

This is the general formula for finding the distance of any point to a plane.