Linear Algebra: Subspaces and the Basis for a Subspace

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Linear Subspaces

If a set V is a *subspace* of $\mathbb{R}^n$:

1. V contains $\mathbf{0}$.
2. **Closure under scalar multiplication:** If $\hat{\mathbf{x}}$ in V, then $c\hat{\mathbf{x}}$ is in V.
3. **Closure under addition:** If $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ in V, then $\hat{\mathbf{a}} + \hat{\mathbf{b}}$ is in V.

For example, take the simple example $V = \{\mathbf{0}\}$.

1. Does V contain $\mathbf{0}$? Yes.
2. Is there closure under scalar multiplication? $c * \mathbf{0} = \mathbf{0}$, so yes.
3. Is there closure under addition? $\mathbf{0} + \mathbf{0} = \mathbf{0}$.

Hence, this trivial example *is* a subspace of $\mathbb{R}^m$.

For a less trivial example:

$$S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0\}$$

Is S a subspace of $\mathbb{R}^2$?

1. Does V contain $\mathbf{0}$? Yes.
2. Is there closure under scalar multiplication? **No**, because scalar multiplication by a negative number would not be a member of S!
3. Is there closure under addition? Yes, because all vectors have a positive x_1 value so the resultant vector would have a $x_1 \geq 0$.

Hence, since all three conditions are not true, this is not a subspace of $\mathbb{R}^2$.

If we define $U = \operatorname{span}(v_1, v_2, v_3)$, is it a valid subspace of $\mathbb{R}^n$?

1. Does U contain $\mathbf{0}$? Yes.

$$0v_1 + 0v_2 + 0v_3 = \mathbf{0}$$

2. Is there closure under scalar multiplication? Yes.

$$a * \hat{\mathbf{x}} = ac_1v_1 = ac_2v_2 + ac_3v_3 = c_4v_1 + c_5v_2 + c_6v_3$$

3. Is there closure under addition? Yes.

$$\hat{\mathbf{x}} + \hat{\mathbf{y}} = (c_1 + d_1)v_1 + (c_2 + d_2)v_2 + (c_3 + d_3)v_3$$

Hence, U is a valid subspace of $\mathbb{R}^n$

Is the set $U = \text{span}((1,1))$ a valid subspace of $\mathbb{R}^2$?

1. Does U contain $\mathbf{0}$? Yes.
$$0(1,1) = (0,0)$$
2. Is there closure under scalar multiplication? Yes.
$$\hat{\mathbf{a}} = c(1,1)$$
3. Is there closure under addition? Yes.
$$c_1(1,1) + c_2(1,1) = (c_1 + c_2)(1,1)$$

Hence, U is a valid subspace of $\mathbb{R}$.

Basis of a subspace

A *basis* of a subspace is a collection of linearly independent vectors, such that when you take the span of all those vectors, you can construct any of the vectors in the subspace.

$$S = \{v_1, v_2, \dots v_n\}$$
$$V = \text{span}(S) | S \text{ is linearly independent.}$$

In this case, S is a basis for V.

A basis can be thought of a "minimum" set of vectors that spans the subspace.