Steven T. Fisher
CS 791
MW 2:30 pm - 3:45 pm
HW2

Problem (2.1)**.** *Show that $f(x) = e^{\alpha x^T A x}$ is convex, where A is a positive semidefinite symmetric $n \times n$ matrix and α is a positive scalar.*

Proof. Let $f(x) = e^{\alpha x^T A x}$, where A is a positive semidefinite symmetric $n \times n$ matrix and α a positive scalar. We want to show that f is convex. First, we will note that, since $g(x) = \alpha x^T A x$ is a quadratic, with a positive scalar and positive semidefinite symetric matrix, then it is convex. Now, since g is a convex function, then it suffice to show that $e^{g(x)}$ is convex. We will use the Hessian:

$$\begin{aligned}
\nabla f(x) &= e^{g(x)} \nabla g(x) \\
\nabla^2 f(x) &= e^{g(x)} \nabla^2 g(x) + e^{g(x)} \nabla g(x) \nabla g(x)^T \\
&= e^{g(x)} (\nabla^2 g(x) + \nabla g(x) \nabla g(x)^T) \\
&\succeq 0
\end{aligned}$$

Hence, $f(x) = e^{\alpha x^T A x}$ is convex. ∎

Problem (2.2)**.** *Show that* $f(x,t) = -\log(t^2 - x^T x)$ *with* $\mathbf{dom} f = \{(x,t) \in \mathbf{R^n} \times \mathbf{R} | t > \|x\|_2\}$ *is convex. Hint: you can use composition rules here and use convexity of the quadractic over linear function.*

Proof. In order to show that $f(x,t) = -\log(t^2 - x^T x)$ is convex, we will utilize composition rules and the convexity of the quadractic over linear function. First, we need to note that given the following $(\frac{1}{t})y^T y$ is the quadratic over linear function, which is convex on the **dom** f. Therefore, we have $t - (\frac{1}{t})y^T y$ is concave, because we have a convex function being subtracted from a linear function. Now, utilizing the composition rules, we know that since $h = t - (\frac{1}{t})y^T y$ is concave then since $g(y) = -\log y$ is convex and decreasing, we have that $g \circ h$ is convex. Thus, in our case we have $-\log(t - (\frac{1}{t})y^T y)$ is convex on our **dom** f. Thus, we can now $f(x,t) = -\log(t - (\frac{1}{t})x^T x) - \log t = -\log(t^2 - x^T x)$. Hence, our function is convex, since it is the sum of two convex functions. ■

Problem (2.3)**.** *Show that* $f(x) = \dfrac{x^T x}{(\prod_{i=1}^n x_i)^{\frac{1}{n}}}$ *is convex* $\textbf{dom } f = \mathbf{R^n_{++}}$

Hint: Perspective Composition Rule. *Suppose that* $f : \mathbf{R^n} \to \mathbf{R}$ *is a closed proper convex function satisfying* $f(0) \le 0$ *and* $g : \mathbf{R^m} \to \mathbf{R}$ *be a closed proper concave function which is nonnegative on its effective domain, the function* $h(x) = g(x) f(x/g(x))$ *is convex with* $\textbf{dom } h = \{x \in \textbf{dom } g | x/g(x) \in \textbf{dom } f\}$*.*

Proof. Let, $f(x) = x^T x$ and $g(x) = (\prod_{i=1}^n x_i)^{1/n}$. We will first need to show that g is concave. Now,

$$\triangledown^2 g(x) = -\frac{\prod_{i=1}^n x_i^{1/n}}{n^2}(n \text{ diag}^2(z) - zz^T) \quad \text{where } (z_i = \frac{1}{x_i})$$

now multiplying by a vector v we get:

$$\begin{aligned} v^T \triangledown^2 g(x) v &= -\frac{\prod_{i=1}^n x_i^{1/n}}{n^2}\left(\sum_{i=1}^n 1 \sum_{i=1}^n v_i^2 q_i^2 - \left(\sum_{i=1}^n q_i v_i\right)^2\right) \\ &= -\frac{\prod_{i=1}^n x_i^{1/n}}{n^2}(\|a\|_2^2 \|b\|_2^2 - \langle a, b\rangle^2) \le 0 \end{aligned}$$

where we have $a_i = 1$ and $b_i = q_i v_i \ \forall v$. Thus, $\triangledown^2 g(x) \le 0$.

Now, since f is a closed proper convex function with $f(0) \le 0$ and g is a concave funtion. Then by the definition of the Perspective Composition Rule, we have that:

$$\begin{aligned} h(x) &= g(x)(f(x/g(x)) \\ &= (\prod_{i=1}^n x_i)^{1/n} \left(\frac{x}{g(x)}\right)^T \frac{x}{g(x)} \\ &= \frac{x^T x}{(\prod_{i=1}^n x_i)^{1/n}} \end{aligned}$$

Thus, h is convex.

∎

Problem (2.4)**.** *Show the following:*

(a) If f and g are convex, both nondecreasing (or nonincreasing), and positive functions on an interval, then fg convex.

Proof. Let f and g be convex, both nondecreasing (or nonincreasing), and positive functions on an interval. We want to show that fg is convex. Now, since f and g are both positive and convex, then for $0 \leq \theta \leq 1$, we have:

$$\begin{aligned} f(\theta x + (1-\theta)y)g(\theta x + (1-\theta)y) &\leq (\theta f(x) + (1-\theta)f(y))(\theta g(x) + (1-\theta)g(y)) \\ &= \theta f(x)g(x) + (1-\theta)f(y)(g(y) \\ &\quad +\theta(1\theta)(f(y) - f(x))(g(x) - g(y)) \end{aligned}$$

Now, if both f and g are both nondecreasing or nonincreasing then $\theta(1\theta)(f(y) - f(x))(g(x) - g(y)) \leq 0$. Thus, we have that

$$f(\theta x + (1-\theta)y)g(\theta x + (1-\theta)y) \leq \theta f(x)g(x) + (1-\theta)f(y)g(y)$$

Hence, fg is convex. ■

(b) Suppose that $f : \mathbf{R^n} \to \mathbf{R}$ is nonnegative and convex, and $g : \mathbf{R^n} \to \mathbf{R}$ is positive and concave. Show that the function $\frac{f^2}{g}$, with domain **dom** $f \cap$ **dom** g is convex.

Proof. Given f, a nonegative and convex function, and g, a positive and concave function. We want to show that $\dfrac{f^2}{g}$ is convex. With out loss of generality, let us assume that $n = 1$. Let $x, y \in$ **dom** $f \cap$ **dom** g and $\theta \in [0, 1]$. Now since f is convex, then we have

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

Now, since f is nonnegative. Then, so is $\theta f(x)+(1-\theta)f(y)$ is nonnegative. Therefore, we have:

$$f(\theta x + (1-\theta)y)^2 \leq (\theta f(x) + (1-\theta)f(y))^2$$

Similarly, since g is concave then we have:

$$g(\theta x + (1-\theta)y) \geq \theta g(x) + (1-\theta)g(y)$$

Therefore, we have:

$$\frac{f(\theta x + (1-\theta)y)^2}{g(\theta x + (1-\theta)y)} \leq \frac{(\theta f(x) + (1-\theta)f(y))^2}{\theta g(x) + (1-\theta)g(y)}$$

Since, $g(\theta x + (1-\theta)y) \geq 0$, then we have:

$$\frac{f(\theta x + (1-\theta)y)^2}{g(\theta x + (1-\theta)y)} \leq \theta\frac{f(x)^2}{g(x)} + (1-\theta)\frac{f(y)^2}{g(y)}$$

Thus, $\dfrac{f^2}{g}$ is convex. ■