Due Date: <u>Feb 14, 2018, Beginning of the class</u>
How to submit: <u>Hard copy in the Class</u>

1.1 Let $C \in \mathbf{R}^n$ be a convex set, with $x_1, \dots, x_k \in C$, and let $\theta_1, \dots, \theta_k \in \mathbf{R}$ satisfy $\theta_i \geq 0$, $\theta_1 + \cdots + \theta_k = 1$. Show that $\theta_1 x_1 + \cdots + \theta_k x_k \in C$. (The definition of convexity is that this holds for k = 2; you must show it for arbitrary k.) Hint. Use induction on k.

1.2 Show that the convex hull of a set S is the intersection of all convex sets that contain S.

1.3 Let a and b be distinct points in $\mathbf{R}^n$. Show that the set of all points that are closer (in Euclidean norm) to a than b, i.e., $\{x \mid \|x - a\|_2 \leq \|x - b\|_2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \leq d$. Draw a picture.

1.4 Show that:

(a) The intersection $\bigcap_{i \in I} C_i$ of a collection $\{C_i | i \in I\}$ of cones is a cone.

(b) The Cartesian product $C_1 \times C_2$ of two cones C_1 and C_2 is a cone.

(c) The vector sum $C_1 + C_2$ of two cones C_1 and C_2 is a cone.

(d) The image and the inverse image of a cone under a linear transformation is a cone.

(e) A subset C is a convex cone if and only if it is closed under addition and positive scalar multiplication, i.e., $C + C \subset C$, and $\gamma C \subset C$ C for all $\gamma > 0$.

1.5 Is the set $\{a \in \mathbf{R}^n | p(0) = 1, |p(t)| \leq 1 \; for \; \alpha \leq t \leq \beta\}$, where $p(t) = a_1 + a_2 t + \cdots + a_k t^{k-1}$, convex? Give the details of your conclusion.

1.6 Let $C \in \mathbf{R}^n$ be the solution set of a quadratic inequality, $C = \{x \in \mathbf{R}^n | x^T A x + b^T x + c \leq 0\}$ with $A \in \mathbf{S}^n$, b$\in \mathbf{R}^n$, and $c \in \mathbf{R}$. Show that C is convex if $A \succcurlyeq 0$.

1.7 Which of the following sets are convex?

(a) A slab, i.e., a set of the form $\{x \in \mathbf{R}^n | \alpha \leq a^T x \leq \beta\}$

(b) A rectangle, i.e., a set of the form $\{x \in \mathbf{R}^n | \alpha_i \leq x_i \leq \beta_i, i = 1, \dots, n\}$

(c) A wedge, i.e., $\{x \in \mathbf{R}^n | {a_1}^T x \leq b_1, {a_2}^T x \leq b_2\}$

(d) The set of points closer to a given point than a given set, i.e., $\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2, \text{for all } y \in S\}$, where $S \subseteq \mathbf{R}^n$.

(e) The set of points closer to one set than another, i.e., $\{x | \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\}$, where $S, T \subseteq \mathbf{R}^n$ and $\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 | z \in S\}$.

(f) The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbf{R}^n$, with S_1 convex

(g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, i.e., the set $\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.