CS 791: Special Topics in Convex Optimization

Instructor: Lei Yang

Department of Computer Science and Engineering, UNR – Spring 2018

Monday, Wednesday 2:30PM - 3:45PM, PE102

Instructor

Dr. Lei Yang

Department of Computer Science and Engineering

University of Nevada, Reno, 89557-0171

Office: SEM 212

Telephone: (775) 682-6872

Email: leiy@unr.edu

Office Hours: Monday, Wednesday 10:00AM-11:30AM

Please make an appointment before you come (email)

Textbook

- **Primary Textbook**
  - S. Boyd and L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.

    http://web.stanford.edu/~boyd/cvxbook/

- **Other References**
  - Ben-Tal and A. Nemirovski, Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering, SIAM, 2001
  - D. Bertsekas and J. N. Tsitsiklis, Parallel and Distributed Computation: Numerical Methods. 1st Edition, Prentice Hall, 1989
  - Matlab and the CVX optimization environment will be used for analysis and programming synthesis. CVX can be downloaded from Stanford University

Course Topics and Goals

- Topics
  - Convex sets, functions, optimization problems
  - Examples and applications
  - Algorithms

- Goals
  - Recognize/formulate problems as convex optimization problems
  - Develop code for convex optimization problems

"No pain, no gain", otherwise, we are not special.
We will also learn the principle of "No Free Lunch".

Course Format

- Your feedback is keenly sought – please let me know asap what you like and dislike, don't wait.
- Read the textbook. It is an entry point for you to access related subjects.
- Everyone is expected to participate in class discussion.
- Project report
- Presentations

Point distribution (tentative)

- Attendance/Participation 5%
- Quizzes 10%
- Homework 10%
- Semester (Midterm) Exams 40%
- Project 35%

Due Dates and Times

- All work needs to be turned in on time, in class.
- Late assignments will incur strict penalties. Assignments turned in after the due date and time will be graded as late. The penalty for late assignments will be 20%. No assignments will be accepted 24 hours after the assigned deadline.
- Discussion of assigned problems is not only allowed but also encouraged. However, each student is expected to turn in his/her own write-up.
- All exams (quizzes, midterm and the final exam) are to be treated as individual and not collective efforts, unless explicitly indicated otherwise.
- There will be no make-up exams. There will be NO exceptions, unless there is a real emergency (e.g., car accident), an excused (e.g., medical) absence, or an exemption has been granted by the instructor in advance. For more details, see the Class Absence Policy (UAM 3,020).

Homework Assignments

- All HWs will be individual assignments
  - These assignments must be done individually
  - Cheating will not be tolerated
  - You are encouraged to discuss homework with others but what you turn in must be your own work (copying is cheating and will be dealt with appropriately)
- Academic Dishonesty (https://www.unr.edu/student-conduct/policies/university-policies-and-guidelines/academic-standards/policy)

Project

- Individual project
  - Mandatory tasks
  - 1 research-related task
- Final report
  - Class presentation and/or demo
- **One key goal** of this course is to take advantage of your intelligence and (limited) experience (so you’re audacious and creative) to expand your knowledge in creating something useful and interesting

Prerequisites

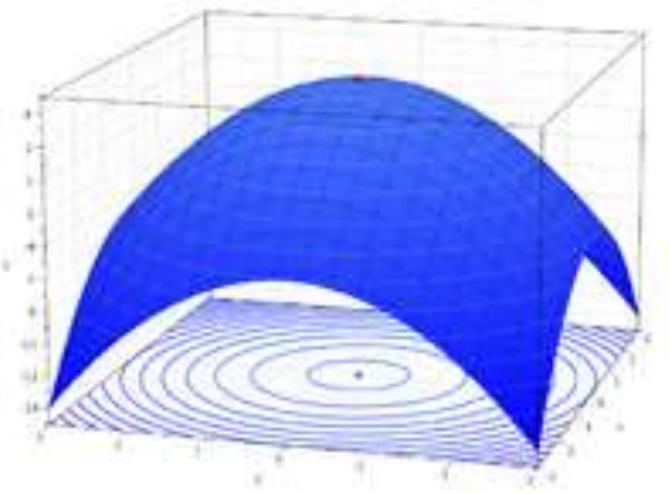
- Linear algebra
- Advanced calculus
- Basic knowledge of Matlab

Course Topics

- 1. History, context, and applications
- 2. Vector spaces, matrix algebra and decompositions
- 3. Least squares
- 4. Linear programming
- 5. Convex sets, functions and optimization
- 6. Duality, Lagrange multipliers, KKT conditions
- 7. Approximation and fitting problems
- 8. Newton's method
- 9. Interior point methods
- 10. Applications to signal processing, communications, and control problems.

Mathematical Optimization

- In mathematics, computer science and operations research, **mathematical optimization** (alternatively, optimization or mathematical programming) is the selection of a best element (with regard to some criteria) from some set of available alternatives.



$f(x,y) = -(x^2 + y^2) + 4$

The global maximum at (0, 0, 4) is indicated by a red dot.

Mathematical Optimization

- (Mathematical) optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \ldots, m \end{array}$$

  - $x = (x_1, \ldots, x_n)$: optimization variables
  - $f_0 : \mathbf{R}^n \to \mathbf{R}$: objective function
  - $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$: constraint functions

- Optimal solution x^* has smallest value of f_0 among all vectors that satisfy the constraints

Examples

- **Portfolio optimization**
  - variables: amounts invested in different assets
  - constraints: budget, max./min. investment per asset, minimum return
  - objective: overall risk or return variance

- **Device sizing in electronic circuits**
  - variables: device widths and lengths
  - constraints: manufacturing limits, timing requirements, maximum area
  - objective: power consumption

- **Data fitting**
  - variables: model parameters
  - constraints: prior information, parameter limits
  - objective: measure of misfit or prediction error

Solving Optimization Problems

- **General nonlinear optimization problem**
  - often difficult to solve, e.g., if nonconvex
  - methods involve some compromise, e.g., very long computational time
- **Local optimization**
  - find a suboptimal solution
  - computationally fast but initial point dependent
- **Global optimization**
  - find a global optimal solution
  - computationally slow

In convex optimization, the art and challenge is in problem formulation. In nonconvex optimization, the art and challenge is in problem structure. Convex optimization plays an important role in exploiting this problem structure

Perspective

Widely known: linear programming is powerful and easy to solve

Modified view: ... the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity.

– SIAM Review 1993, R. Rockafellar

- Local optimality is also global optimality
- Lagrange duality theory well developed
- Know a lot about the problem and solution structures
- Efficiently compute the solutions numerically

So we'll start with convex optimization introduction, then specialize into different applications involving convex problems (and nonconvex problems)

Least-squares

$$\text{minimize} \quad \|Ax - b\|_2^2$$

- **solving least-squares problems**
  - analytical solution: $x^* = \left(A^T A\right)^{-1} A^T b$
  - reliable and efficient algorithms and software
  - computation time proportional to $n^2 k$ ($A \in \mathbf{R}^{k \times n}$); less if structured
  - a mature technology

- **using least-squares**
  - least-squares problems are easy to recognize
  - a few standard techniques increase flexibility (e.g., including weights, adding regularization terms)

Linear Programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

- **solving linear programs**
  - no analytical formula for solution
  - reliable and efficient algorithms and software
  - computation time proportional to n^2m if $m \geq n$; less with structure
  - a mature technology
- **using linear programming**
  - not as easy to recognize as least-squares problems
  - a few standard tricks used to convert problems into linear programs (e.g., problems involving ℓ1- or ℓ1-norms, piecewise-linear functions)

Convex Optimization Problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \ldots, m \end{array}$$

- objective and constraint functions are convex:

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$$

if $\alpha + \beta = 1,\ \alpha \geq 0,\ \beta \geq 0$

- includes least-squares problems and linear programs as special cases

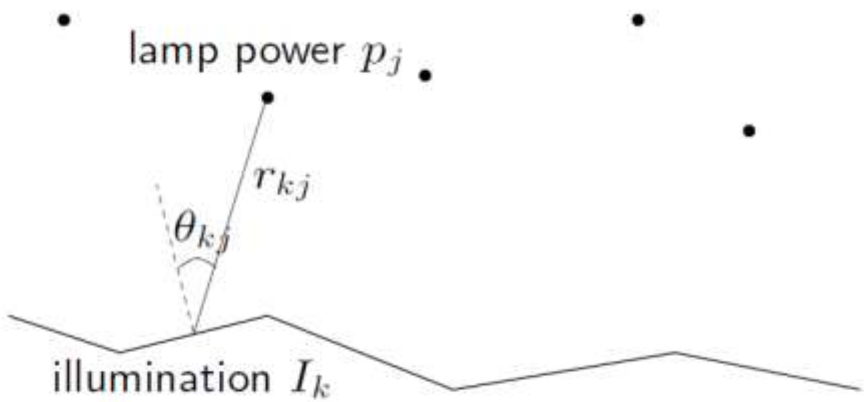
Convex Optimization Problem

- **solving convex optimization problems**
  - no analytical solution
  - reliable and efficient algorithms
  - computation time (roughly) proportional to $max\{n^3, n^2m, F\}$, where F is cost of evaluating f_i's and their first and second derivatives
  - almost a technology

- **using convex optimization**
  - often difficult to recognize
  - many tricks for transforming problems into convex form
  - surprisingly many problems can be solved via convex optimization

Example

- m lamps illuminating n (small, flat) patches



- intensity I_k at patch k depends linearly on lamp powers p_j:

$$I_k = \sum_{j=1}^{m} a_{kj} p_j, \qquad a_{kj} = r_{kj}^{-2} \max\{\cos\theta_{kj}, 0\}$$

- **problem**: achieve desired illumination I_{des} with bounded lamp powers

$$\begin{array}{ll} \text{minimize} & \max_{k=1,\ldots,n} |\log I_k - \log I_{\text{des}}| \\ \text{subject to} & 0 \le p_j \le p_{\max}, \quad j = 1, \ldots, m \end{array}$$

How to solve?

- Use uniform power: $p_j = p$, vary p
- Use least-squares:

$$\text{minimize} \quad \sum_{k=1}^{n} (I_k - I_{\text{des}})^2$$

round p_j if $p_j > p_{\max}$ or $p_j < 0$

- Use weighted least-squares:

$$\text{minimize} \quad \sum_{k=1}^{n} (I_k - I_{\text{des}})^2 + \sum_{j=1}^{m} w_j (p_j - p_{\text{max}}/2)^2$$

iteratively adjust weights w_j until $0 \leq p_j \leq p_{\max}$

- Use linear programming:

$$\begin{array}{ll} \text{minimize} & \max_{k=1,\ldots,n} |I_k - I_{\text{des}}| \\ \text{subject to} & 0 \leq p_j \leq p_{\text{max}}, \quad j = 1, \ldots, m \end{array}$$

which can be solved via linear programming

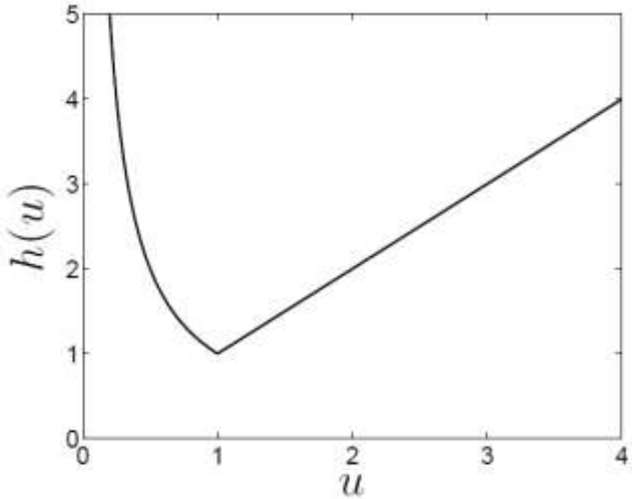
- Of course, these are approximate (suboptimal) ‘solutions’

How to solve?

- Use convex optimization: problem is equivalent to

$$\begin{array}{ll}\text{minimize} & f_0(p) = \max_{k=1,\ldots,n} h(I_k/I_{\text{des}}) \\ \text{subject to} & 0 \le p_j \le p_{\max}, \quad j = 1, \ldots, m\end{array}$$

with $h(u) = \max\{u, 1/u\}$



f_0 is convex because maximum of convex functions is convex

- **exact** solution obtained with effort

Additional Constraints

- Does adding 1 or 2 below complicate the problem?

 1. no more than half of total power is in any 10 lamps

 2. no more than half of the lamps are on ($p_j > 0$)

answer: with (1), still easy to solve; with (2), extremely difficult

Summary

- All the problems can be formulated as mathematical optimization problems

- General optimization problems are difficult to solve

- Certain problem classes (convex optimization) can be solved efficiently and reliably
  - Recognize/formulate problems (such as the illumination problem) as convex optimization problems

Thank you!