Steven T. Fisher
CS 791
MW 2:30 pm - 3:45 pm
HW4

Problem (4.1)**.** *Consider the optimization problem*

$$\begin{array}{ll} \textit{minimize} & x^2+1 \\ \textit{subject to} & (x-2)(x-4) \leq 0 \end{array}$$

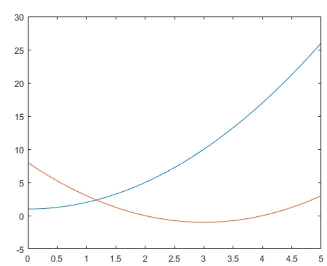
with variable $x \in \mathbf{R}$

(a) *Analysis of primal problem.* Give the feasible set, the optimal value, and the optimal solution.

Solution: Using the following CVX code:

```
cx_begin
    variable x(1)
    minimize( x^2 + 1 )
    subject to
        (x - 2)*(x - 4) <= 0
cvx_end
```

We find the the optimal value is $x^* = 2$ and the optimal solution is $p^* = 5$. The graph of the feasible set is below:

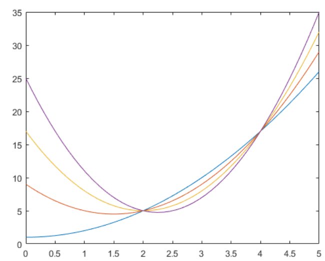


(b) *Lagrangian and dual function.* Plot the objective x^2+1 versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x,\lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^* \geq \inf_x L(x,\lambda)$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g.

Solution: The Lagragian for our problem is given by:

$$L(x, \lambda) = (1 + \lambda)x^2 - 6\lambda x + (1 + 8\lambda)$$

The graph of the feasible set is below:



The graph to the right shows the plot of $g(\lambda)$ with is unbounded below. Also, note that g is concave and is equal to $p^* = 5$, when $\lambda = 2$.

(c) *Lagrange dual problem.* State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ^*. Does strong duality hold?

Solution: The dual problem is given by:

$$\begin{array}{ll} \text{maximize} & -9\lambda^2/(1+\lambda) + 8\lambda + 1 \\ \text{subject to} & \lambda \geq 0 \end{array}$$

From (b) we noted that the max occurs when $\lambda = 2$, with a solution $d^* = 5$. Therefore, strong duality holds for our problem.

(d) *Sensitivity analysis.* Let $p^*(u)$ denote the optimal value of the problem

$$\begin{array}{ll} \text{minimize} & x^2 + 1 \\ \text{subject to} & (x-2)(x-4) \leq u \end{array}$$

as a function of the parameter u. Plot $p^*(u)$. Verify that $\dfrac{dp^*(0)}{du} = -\lambda^*$.

Solution: Similar to our dual problem, our problem is infeasible if $u < -1$, This is due to the fact the the $\inf_x x^2 - 6x + 8) = -1$. Solving, our equation for when $u \geq -1$, we will find the roots to our equation and determine the interval for where we have a feasible set. This will give the interval:

$$\left[3 - \sqrt{1+u}, 3 + \sqrt{1+u}\right]$$

Now, our optimal point will occur in the feasible reagion and should occur at $x^* = 3 - \sqrt{1+u}$. So, for p^* we have:

$$p^* = \begin{cases} \infty & u < -1 \\ 11 + u - 6\sqrt{1+u} & -1 \leq u < 8 \\ 1 & u \geq 8 \end{cases}$$

Now, if we differentiate p^* we get:

$$\frac{dp^*(0)}{du} = 2 = -\lambda$$

Problem (4.2)**.** *Consider the quadratic program*

$$\begin{array}{ll} \textit{minimize} & x_1^2 + 2x_2^2 - x_1x_2 - x_1 \\ & x_1 + 2x_2 \leq u_1 \\ \textit{subject to} & x_1 - 4x_2 \leq u_2 \\ & 5x_1 + 76x_2 \leq 1 \end{array}$$

with variables x_1, x_2, and parameters u_1, u_2. Solve this QP, for parameter values $u_1 = -2$, $u_2 = -3$, to find optimal primal values x_1^* and x_2^*, and optimal dual variable values λ_1^*, λ_2^*, and λ_3^*. Let p^* denote the optimal objective value. Verify that the KKT conditions hold for the optimal primal and dual variables you found(within reasonable numerical accuracy).

Matlab hint. See the CVX user's guide to find out how to retrieve optimal dual variables. To specify the quadratic objective use ***quad_form()****.*

Solution: The Matlab source code to solving this problem is:

```
Q = [1 -1/2; -1/2 2];
f = [-1 0]';
A = [1 2; 1 -4; 5 76];
b = [-2 -3 1]';
cvx_begin
variable x(2)
dual variable lambda
minimize(quad_form(x,Q)+f'*x)
subject to
lambda: A*x <= b
cvx_end
p_star = cvx_optval
x
lambda
A*x-b
2*Q*x+f+A'*lambda
```

This returns the following for the solutions:

$$\lambda_1^*, \lambda_2^*, \lambda_3^* \geq 0$$

$$p^* = 8.2222, x_1^* = -2.3333, x_2^* = 0.1667, \lambda_1^* = 1.8994, \lambda_2^* = 3.4684, \lambda_3^* = 0.0931$$

Now, we need to verify the KTT conditions:

$$x_1^* + 2x_2^* \leq u_1 \Rightarrow -2.3333 + 2(0.1667) = -1.9999 \approx -2 \leq -2$$
$$x_1^* - 4x_2^* \leq u_2 \Rightarrow -2.3333 - 4(0.1667) \approx -3 \leq -3$$
$$5x_1^* + 76x_2^* \leq 1 \Rightarrow -11.6665 + 12.6692 = 1.0027 \approx 1 \leq 1$$

And, using MatLab we verified that $Ax - b = 0$

Problem (4.3)**.** *Find the dual function of the LP*

$$\begin{array}{ll} \textit{minimize} & c^T x \\ \textit{subject to} & Gx \preccurlyeq h, Ax = b \end{array}$$

Give the dual problem, and make the implicit equality constraints explicit.

Solution: In order to obtain the dual problem, we first must find the Lagrangian. Using the definition we have

$$L(x, \lambda, \mu) \quad = c^T x + \lambda^T (Gx - h) + \mu^T (Ax - b) \quad = (c^T + \lambda^T G + \mu^T A)x - h\lambda^T - \mu^T b$$

Now, that we know what the Lagrangian is, we need the dual function

$$g(\lambda, \mu) = \inf_x L(x, \lambda, \mu) = \begin{cases} -\lambda^T h - \mu^T b & c + G^T \lambda + A^T \mu = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Now, that we have the dual function then our dual problem is

$$\begin{array}{ll} \text{maximize} & g(\lambda, \mu) \\ \text{subject to} & \lambda \succcurlyeq 0 \end{array}$$

The last thing that we need, is to make the implicit constraints explicit. So, our dual problem is:

$$\begin{array}{ll} \text{maximize} & -\lambda^T h - \mu^T b \\ \text{subject to} & c + G^T \lambda + A^T \mu = 0 \\ & \lambda \succcurlyeq 0 \end{array}$$

Problem (4.4)**.** *The relative entropy between two vectors $x, y \in \mathbf{R}^n_{++}$ is defined as $\sum_{k=1}^n x_k \log(x_k/y_k)$. This is a convex function, jointly in x and y. In the following problem we calculate the vector x that minimizes the relative entropy with a given vector y, subject to equality contraints on x:*

$$\begin{array}{ll} \textit{minimize} & \sum_{k=1}^n x_k \log(x_k/y_k) \\ \textit{subject to} & Ax = b, 1^T x = 1 \end{array}$$

The optimization variable is $x \in \mathbf{R}^n$. The domain of the objective function is $\mathbf{R}^n_{++}$. The parameters $y \in \mathbf{R}^n_{++}$, $A \in \mathbf{R}^{m \times n}$, and $b \in \mathbf{R}^m$ are given. Derive the Lagrange dual of this problem and simplity it to get

$$\textit{maximize } b^T z - \log \sum_{k=1}^n y_k e^{a_k^T z}$$

(a_k is the kth column of A).

Solution: We will begin by deriving the Lagrange dual function, which is given by:

$$L(x, z, \mu) = \sum_k^n x_k \log(\frac{x_k}{y_k}) + b^T z - z^T Ax + \mu - \mu 1^T x$$

Now, if we minimized with respect to x_k we get:

$$1 + \log(\frac{x_k}{y_k}) - a_k^T z - \mu \qquad k = 1, \ldots, n$$

Solving the above for x_k we get:

$$\begin{aligned} 1 + \log(\frac{x_k}{y_k}) - a_k^T z - \mu &= 0 \\ \Rightarrow \log(\frac{x_k}{y_k}) &= a_k^T + \mu - 1 \\ \Rightarrow \frac{x_k}{y_k} &= e^{a_k^T + \mu - 1} \\ \Rightarrow x_k &= y_k e^{a_k^T + \mu - 1} \end{aligned}$$

Now that we have the value for x_k, we will substitute it back into L:

$$\begin{aligned} \sum_k^n x_k \log(\frac{x_k}{y_k}) + b^T z - z^T Ax + \mu - \mu 1^T x &= \sum_k^n y_k e^{a_k^T + \mu - 1} \log(\frac{y_k e^{a_k^T + \mu - 1}}{y_k}) + b^T z - z^T Ax + \mu - \mu 1^T x \\ &= \sum_k^n y_k e^{a_k^T + \mu - 1} \log(e^{a_k^T + \mu - 1}) + b^T z - z^T Ax + \mu - \mu 1^T x \\ &= \sum_k^n y_k e^{a_k^T + \mu - 1} a_k^T + \mu - 1 + b^T z - z^T Ax + \mu - \mu 1^T x \\ &= \sum_k^n y_k e^{a_k^T + \mu - 1} a_k^T + \mu - 1 + b^T z - z^T Ax + \mu - \mu 1^T x \end{aligned}$$

Simplifying we get:

$$g(z, \mu) = b^T z + \mu - \sum_{k}^{n} y_k^{a_k^t + \mu - 1}$$

Now, simplifying with respect to μ, we get:

$$g(z, \mu) = b^T z - \log \sum_{k=1}^{n} y_k e^{a_k^T z}$$