Due Date: **<u>April 18, 2018, Beginning of the class</u>**
How to submit: **<u>Type your HW and submit hard copy in class</u>**

4.1 Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & x^2 + 1 \\ \text{subject to} & (x-2)(x-4) \leq 0 \end{array}$$

with variable $x \in \mathbf{R}$

(a) *Analysis of primal problem.* Give the feasible set, the optimal value, and the optimal solution.
(b) *Lagrangian and dual function.* Plot the objective $x^2 + 1$ versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^* \geq \inf_x L(x, \lambda)$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g.
(c) *Lagrange dual problem.* State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ^*. Does strong duality hold?
(d) *Sensitivity analysis.* Let $p^*(u)$ denote the optimal value of the problem

$$\begin{array}{ll} \text{minimize} & x^2 + 1 \\ \text{subject to} & (x-2)(x-4) \leq u \end{array}$$

as a function of the parameter u. Plot $p^*(u)$. Verify that $\frac{dp^*(0)}{du} = -\lambda^*$.

4.2 Consider the quadratic program

$$\begin{array}{ll} \text{minimize} & x_1^2 + 2x_2^2 - x_1 x_2 - x_1 \\ & x_1 + 2x_2 \leq u_1 \\ \text{subject to} & x_1 - 4x_2 \leq u_2 \\ & 5x_1 + 76x_2 \leq 1 \end{array}$$

with variables x_1, x_2, and parameters u_1, u_2.

Solve this QP, for parameter values $u_1 = -2$, $u_2 = -3$, to find optimal primal variable values x_1^* and x_2^*, and optimal dual variable values λ_1^*, λ_2^* and λ_3^*. Let p^* denote the optimal objective value. Verify that the KKT conditions hold for the optimal primal and dual variables you found (within reasonable numerical accuracy).

Matlab hint: See the CVX users' guide to find out how to retrieve optimal dual variables. To specify the quadratic objective, use **quad_form()**.

4.3 Find the dual function of the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preccurlyeq h, Ax = b \end{array}$$

Give the dual problem, and make the implicit equality constraints explicit.

4.4 The relative entropy between two vectors x, $y \in \mathbf{R}^n_{++}$ is defined as $\sum_{k=1}^n x_k \log(x_k/y_k)$. This is a convex function, jointly in x and y. In the following problem we calculate the vector x that minimizes the relative entropy with a given vector y, subject to equality constraints on x:

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^n x_k \log(x_k/y_k) \\ \text{subject to} & Ax = b, 1^T x = 1 \end{array}$$

The optimization variable is $x \in \mathbf{R}^n$. The domain of the objective function is $\mathbf{R}^n_{++}$. The parameters $y \in \mathbf{R}^n_{++}$, $A \in \mathbf{R}^{m \times n}$, and $b \in \mathbf{R}^m$ are given. Derive the Lagrange dual of this problem and simplify it to get

$$\text{maximize } b^T z - \log \sum_{k=1}^n y_k e^{a_k^T z}$$

(a_k is the kth column of A).