Fibonacci Heaps

Priority Queues

		Heaps				
Operation	**Linked List**	**Binary**	**Binomial**	**Fibonacci †**	**Relaxed**
make-heap	1	1	1	1	1
insert	1	log N	log N	1	1
find-min	N	1	log N	1	1
delete-min	N	log N	log N	log N	log N
union	1	N	log N	1	1
decrease-key	1	log N	log N	1	1
delete	N	log N	log N	log N	log N
is-empty	1	1	1	1	1

† amortized

this time

Fibonacci Heaps

Fibonacci heap history. **Fredman and Tarjan (1986)**

- Ingenious data structure and analysis.
- Original motivation: O(m + n log n) shortest path algorithm.
  - also led to faster algorithms for MST, weighted bipartite matching
- Still ahead of its time.

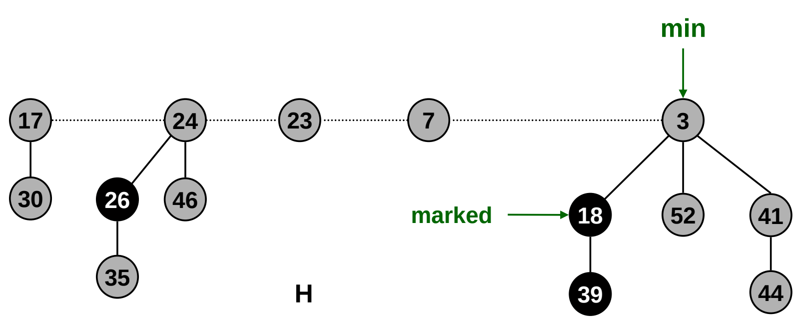
Fibonacci heap intuition.

- Similar to binomial heaps, but less structured.
- Decrease-key and union run in O(1) time.
- "Lazy" unions.

Fibonacci Heaps: Structure

Fibonacci heap.

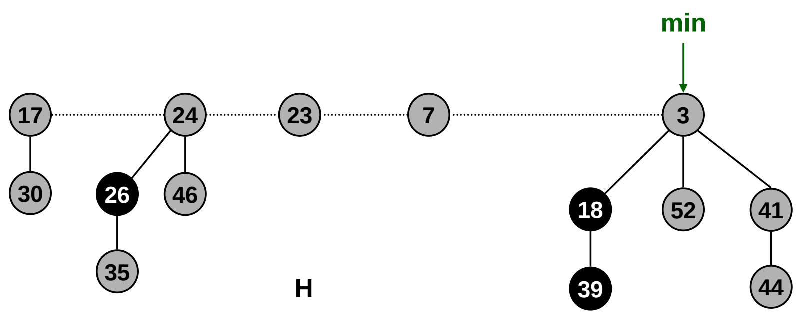
- Set of min-heap ordered trees.



Fibonacci Heaps: Implementation

Implementation.

- Represent trees using left-child, right sibling pointers and circular, doubly linked list.
  - can quickly splice off subtrees
- Roots of trees connected with circular doubly linked list.
  - fast union
- Pointer to root of tree with min element.
  - fast find-min



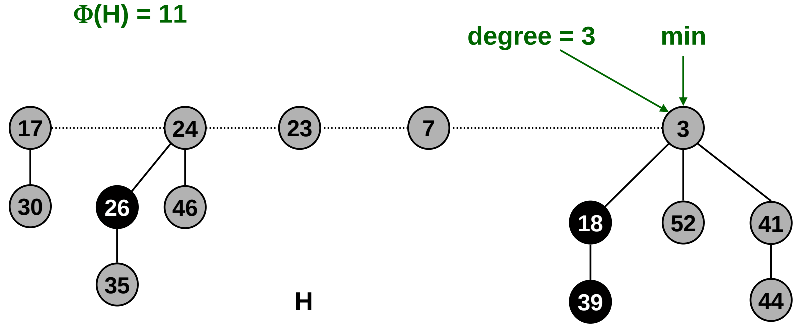
Fibonacci Heaps: Potential Function

Key quantities.

- Degree[x] = degree of node x.
- Mark[x] = mark of node x (black or gray).
- t(H) = # trees.
- m(H) = # marked nodes.
- Φ(H) = t(H) + 2m(H) = potential function.

t(H) = 5, m(H) = 3

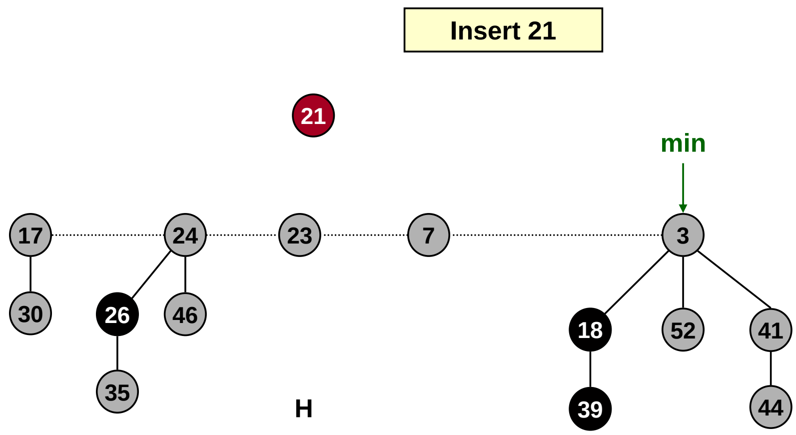
Φ(H) = 11



Fibonacci Heaps: Insert

Insert.

- Create a new singleton tree.
- Add to left of min pointer.
- Update min pointer.



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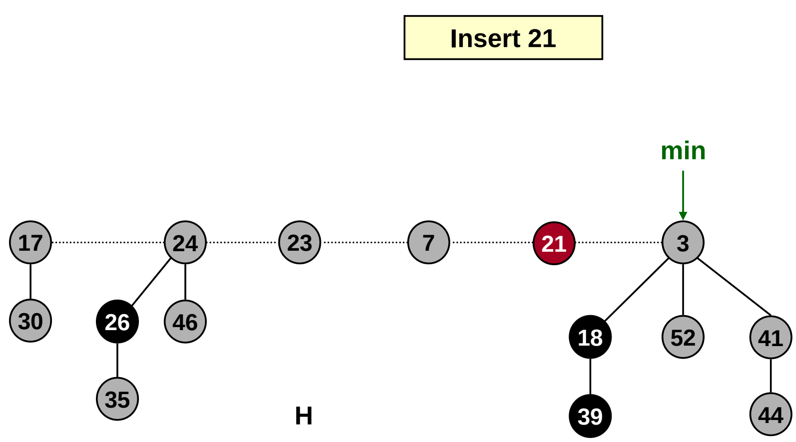


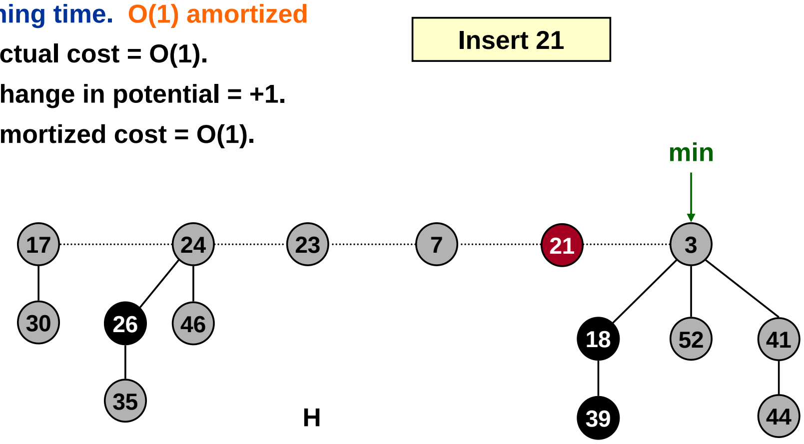
Fibonacci Heaps: Insert

Insert.

- Create a new singleton tree.
- Add to left of min pointer.
- Update min pointer.

Running time. O(1) amortized

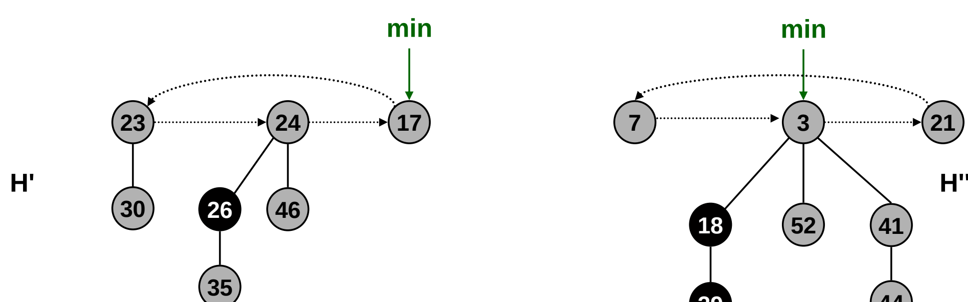
- Actual cost = O(1).
- Change in potential = +1.
- Amortized cost = O(1).



Fibonacci Heaps: Union

Union.

- Concatenate two Fibonacci heaps.
- Root lists are circular, doubly linked lists.



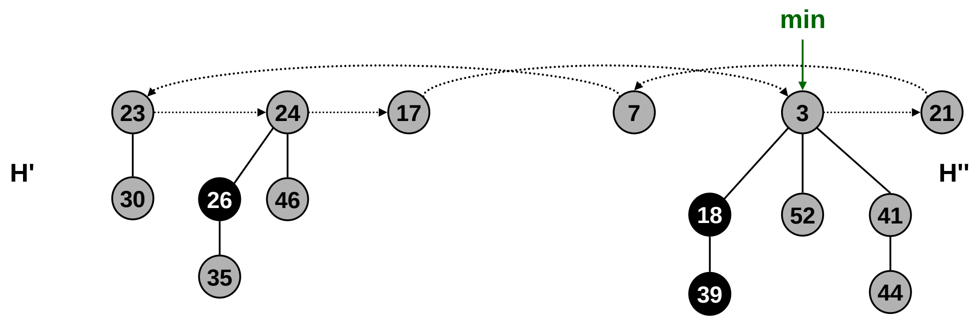
Fibonacci Heaps: Union

Union.

- Concatenate two Fibonacci heaps.
- Root lists are circular, doubly linked lists.

Running time. **O(1) amortized**

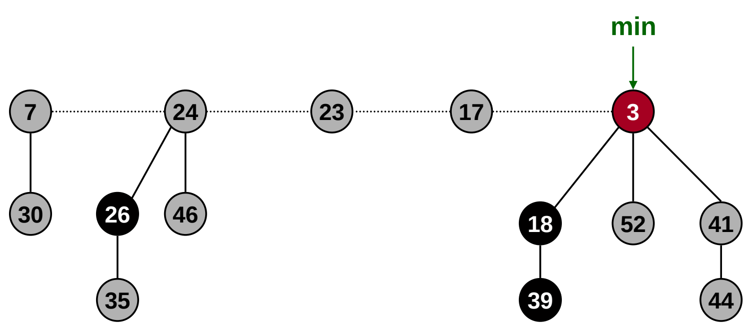
- Actual cost = O(1).
- Change in potential = 0.
- Amortized cost = O(1).



Fibonacci Heaps: Delete Min

Delete min.

- Delete min and concatenate its children into root list.
- Consolidate trees so that no two roots have same degree.



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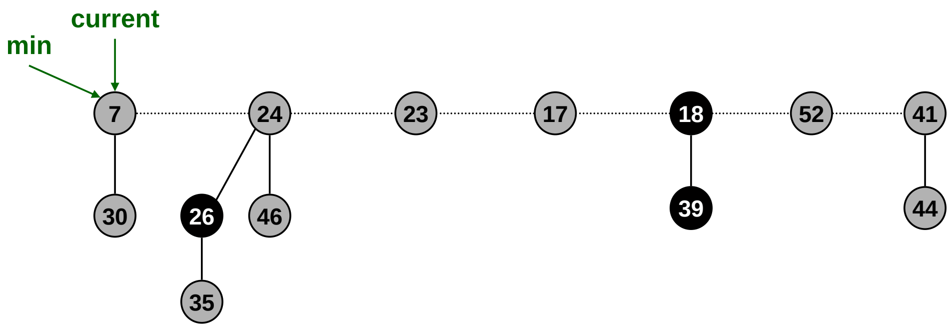

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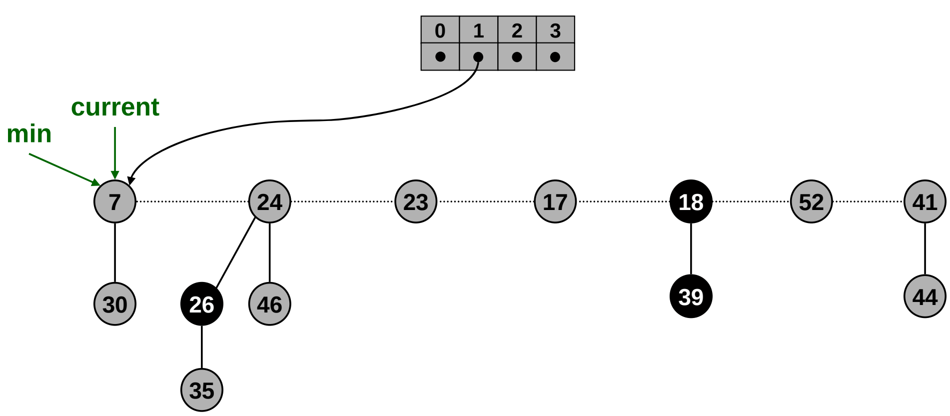

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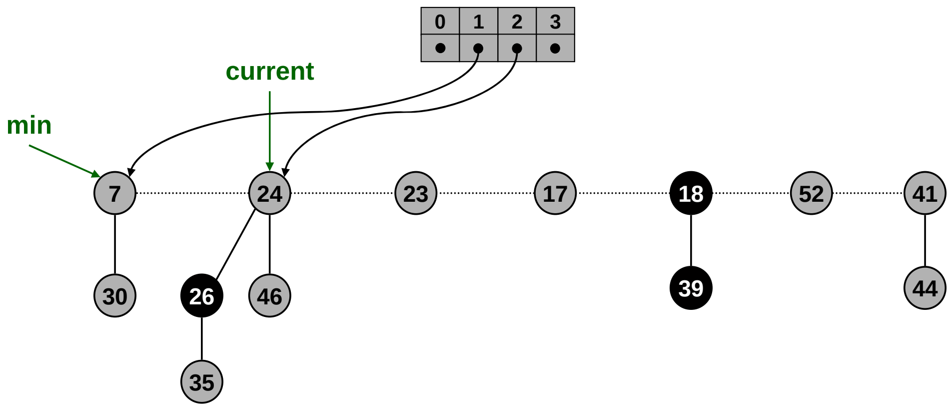

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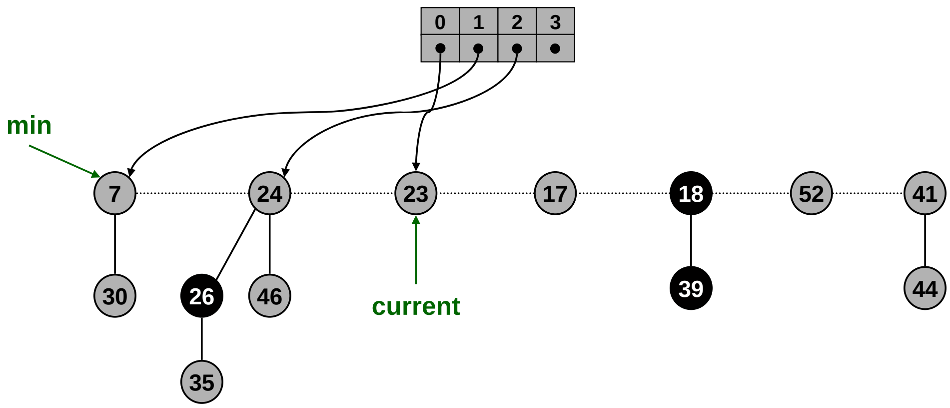

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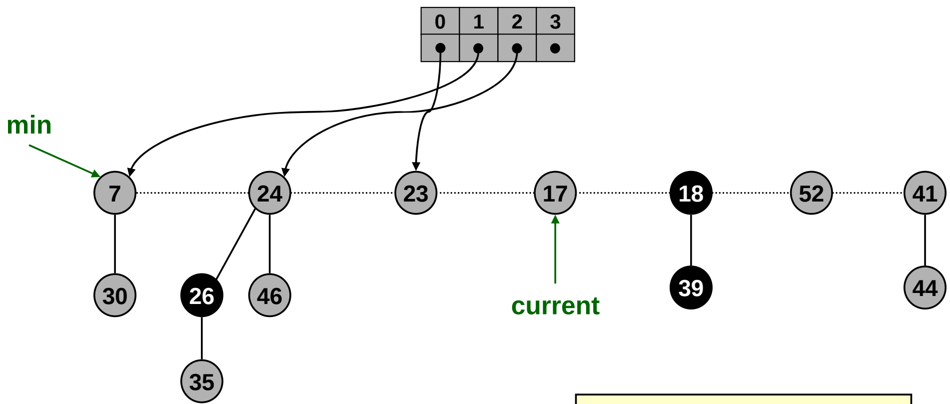

Merge 17 and 23 trees.

Fibonacci Heaps: Delete Min

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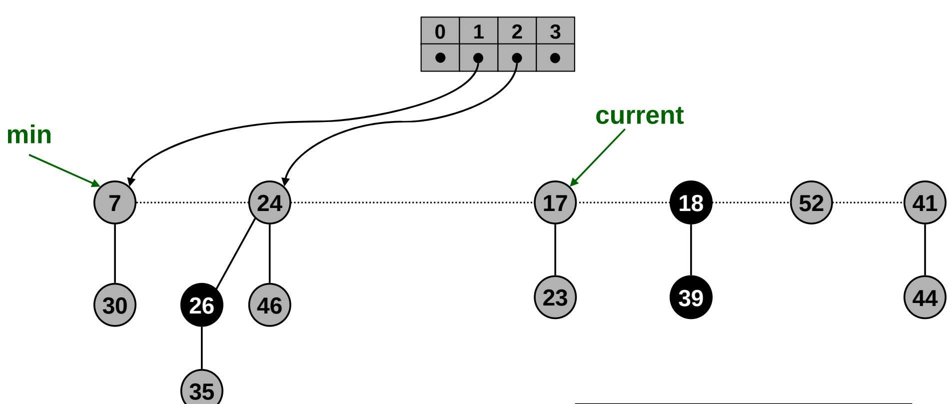

Merge 7 and 17 trees.

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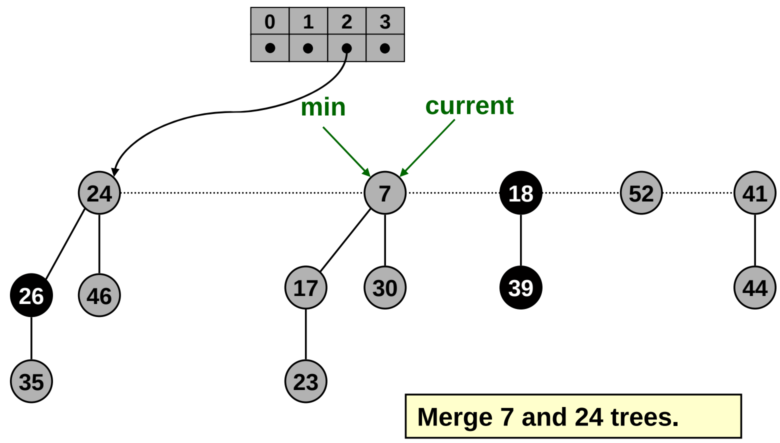

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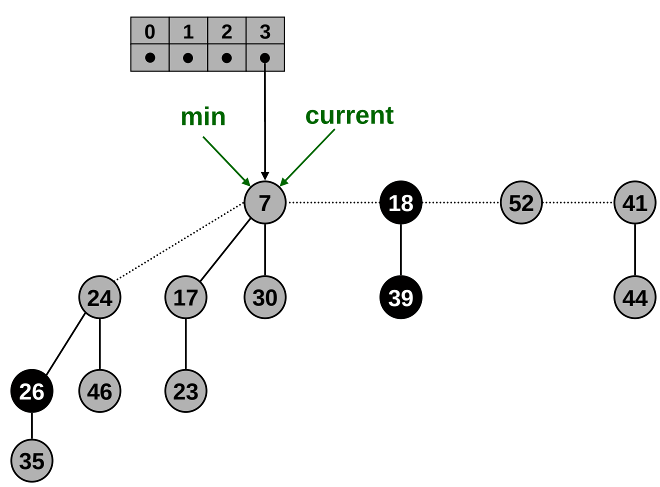

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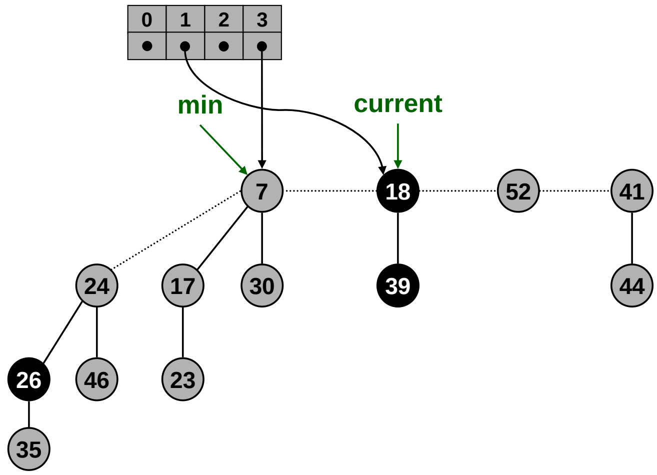

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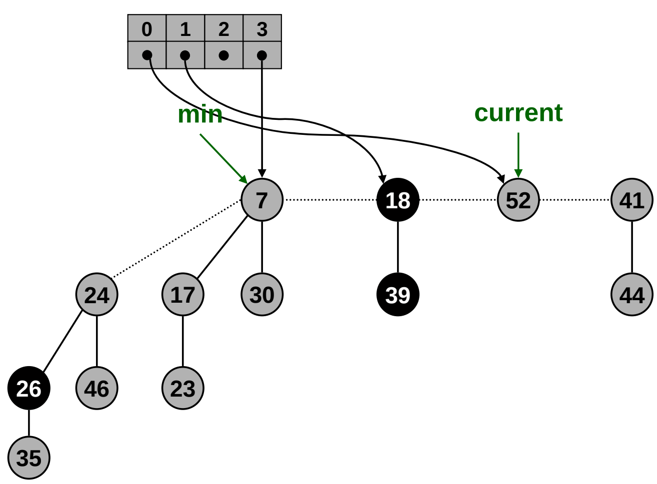

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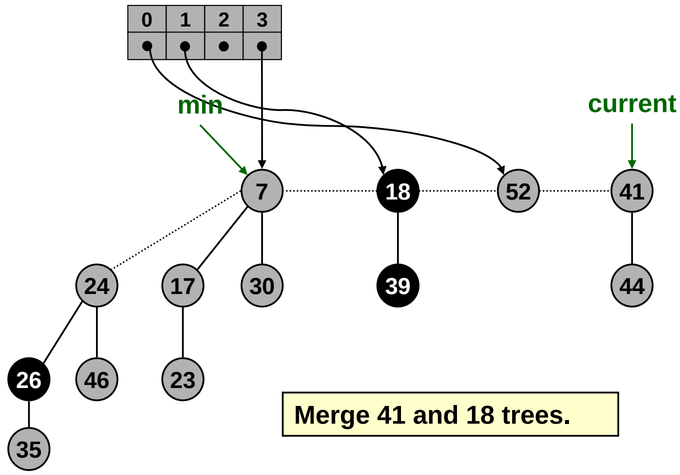

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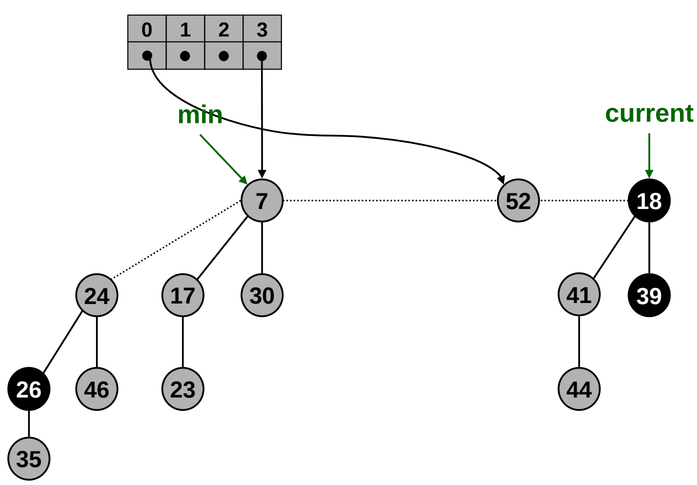

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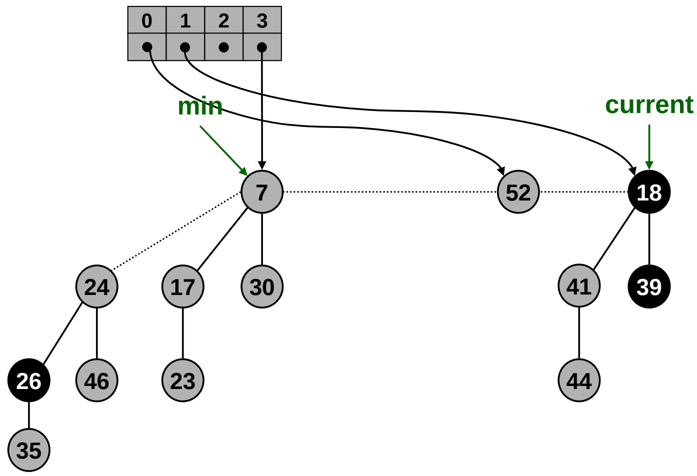

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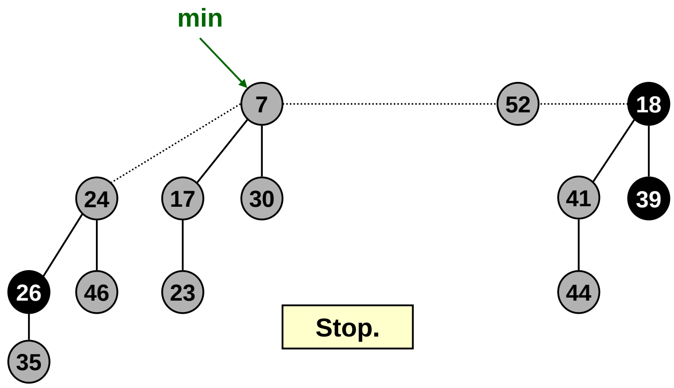


Fibonacci Heaps: Delete Min Analysis

Notation.
- D(n) = max degree of any node in Fibonacci heap with n nodes.
- t(H) = # trees in heap H.
- $\Phi(H)$ = t(H) + 2m(H).

Actual cost. O(D(n) + t(H))
- O(D(n)) work adding min's children into root list and updating min.
  - at most D(n) children of min node
- O(D(n) + t(H)) work consolidating trees.
  - work is proportional to size of root list since number of roots decreases by one after each merging
  - $\leq$ D(n) + t(H) - 1 root nodes at beginning of consolidation

Amortized cost. O(D(n))
- t(H') $\leq$ D(n) + 1 since no two trees have same degree.
- $\Delta\Phi(H) \leq$ D(n) + 1 - t(H).

Fibonacci Heaps: Delete Min Analysis

Is amortized cost of O(D(n)) good?

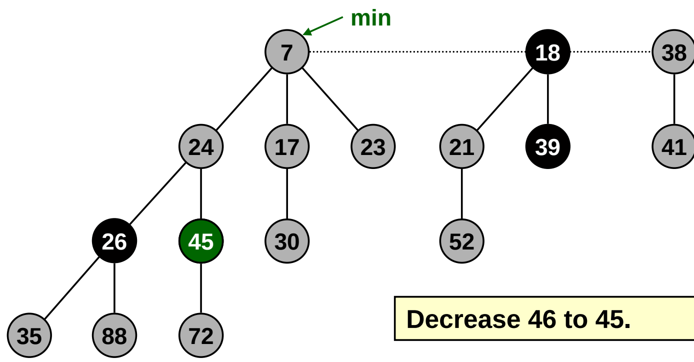
- Yes, if only Insert, Delete-min, and Union operations supported.
  - in this case, Fibonacci heap contains only binomial trees since we only merge trees of equal root degree
  - this implies D(n) $\leq \lfloor \log_2 N \rfloor$

- Yes, if we support Decrease-key in clever way.
  - we'll show that D(n) $\leq \lfloor \log_\phi N \rfloor$, where ϕ is golden ratio
  - $\phi^2 = 1 + \phi$
  - $\phi = (1 + \sqrt{5}) / 2 = 1.618\ldots$
  - limiting ratio between successive Fibonacci numbers!

Fibonacci Heaps: Decrease Key

Decrease key of element x to k.

- Case 0: min-heap property not violated.
  - decrease key of x to k
  - change heap min pointer if necessary

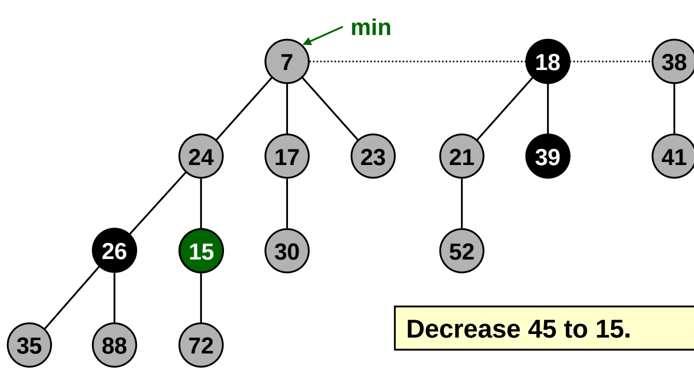


Decrease 46 to 45.

Fibonacci Heaps: Decrease Key

Decrease key of element x to k.

- Case 1: parent of x is unmarked.
  - decrease key of x to k
  - cut off link between x and its parent
  - mark parent
  - add tree rooted at x to root list, updating heap min pointer



Decrease 45 to 15.

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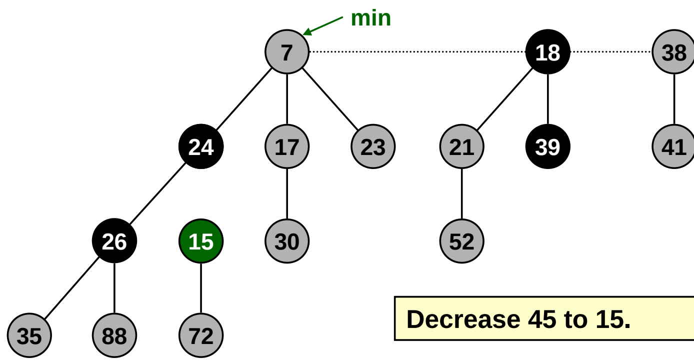

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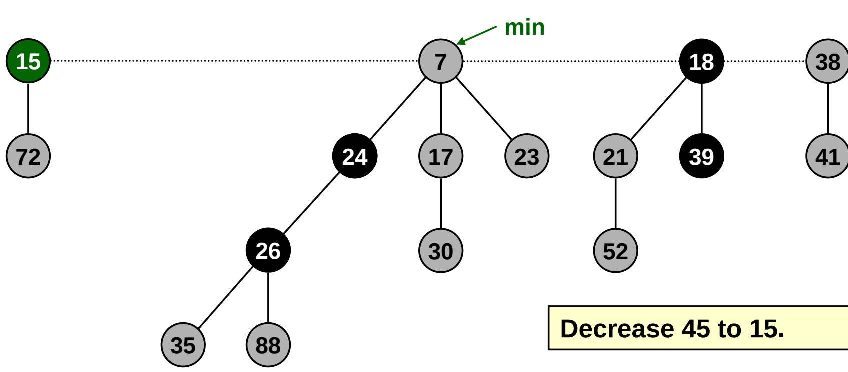
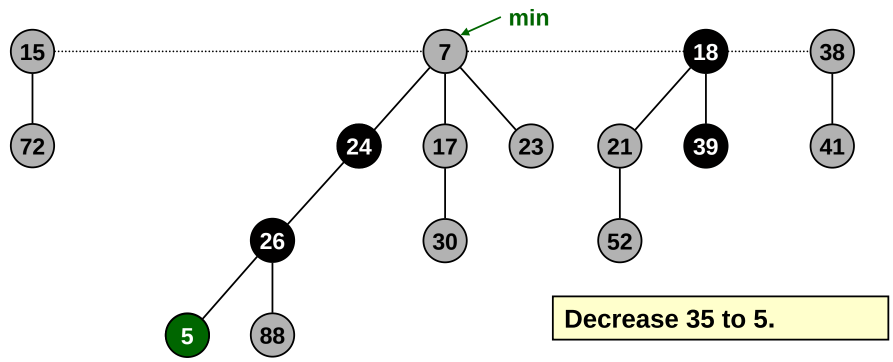


Decrease 45 to 15.

Fibonacci Heaps: Decrease Key

Decrease key of element x to k.

- Case 2: parent of x is marked.
  - decrease key of x to k
  - cut off link between x and its parent p[x], and add x to root list
  - cut off link between p[x] and p[p[x]], add p[x] to root list
    - If p[p[x]] unmarked, then mark it.
    - If p[p[x]] marked, cut off p[p[x]], unmark, and repeat.



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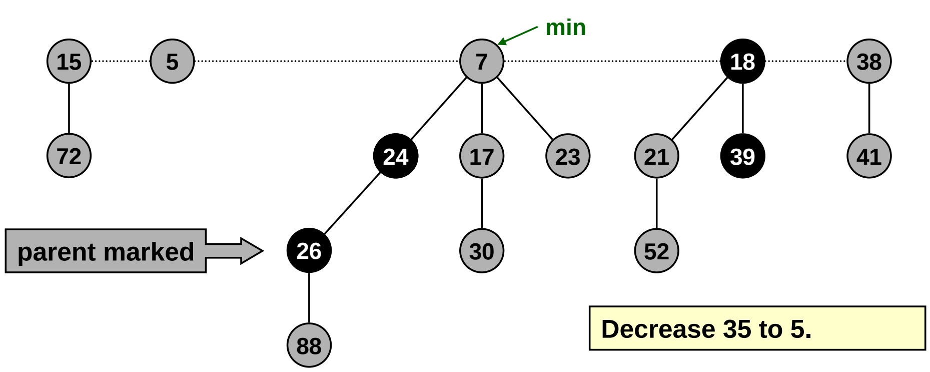

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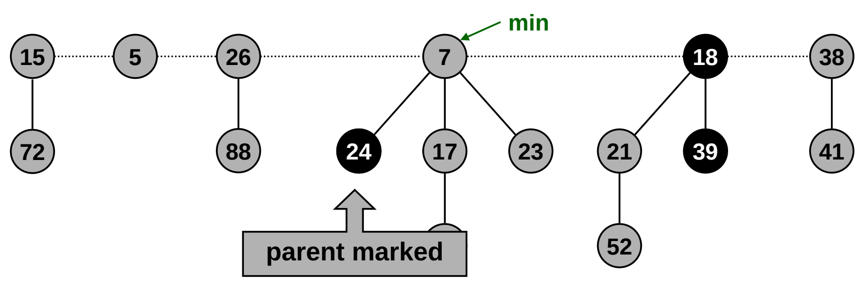

Decrease 35 to 5.

Fibonacci Heaps: Decrease Key

Decrease key of element x to k.

- Case 2: parent of x is marked.
  - decrease key of x to k
  - cut off link between x and its parent p[x], and add x to root list
  - cut off link between p[x] and p[p[x]], add p[x] to root list
    - If p[p[x]] unmarked, then mark it.
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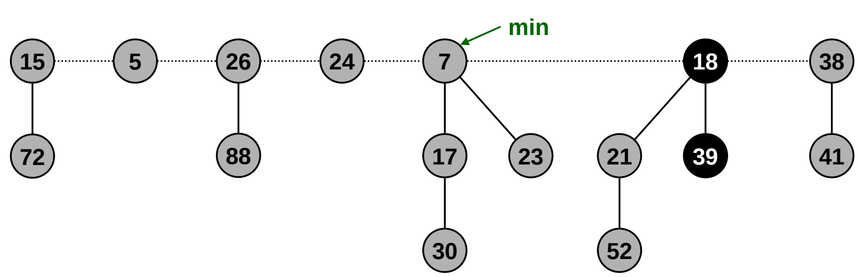


Decrease 35 to 5.

Fibonacci Heaps: Decrease Key Analysis

Notation.

- $t(H)$ = # trees in heap H.
- $m(H)$ = # marked nodes in heap H.
- $\Phi(H) = t(H) + 2m(H)$.

Actual cost. $O(c)$

- $O(1)$ time for decrease key.
- $O(1)$ time for each of c cascading cuts, plus reinserting in root list.

Amortized cost. $O(1)$

- $t(H') = t(H) + c$
- $m(H') \leq m(H) - c + 2$
  - each cascading cut unmarks a node
  - last cascading cut could potentially mark a node
- $\Delta\Phi \leq c + 2(-c + 2) = 4 - c$.

Fibonacci Heaps: Delete

Delete node x.

- Decrease key of x to $-\infty$.
- Delete min element in heap.

Amortized cost. $O(D(n))$

- $O(1)$ for decrease-key.
- $O(D(n))$ for delete-min.
- $D(n)$ = max degree of any node in Fibonacci heap.

Fibonacci Heaps: Bounding Max Degree

Definition. D(N) = max degree in Fibonacci heap with N nodes.
Key lemma. $D(N) \le \log_\phi N$, where $\phi = (1 + \sqrt{5}) / 2$.
Corollary. Delete and Delete-min take O(log N) amortized time.

Lemma. Let x be a node with degree k, and let $y_1, \ldots, y_k$ denote the children of x in the order in which they were linked to x. Then:

$$\text{degree}(y_i) \ge \begin{cases} 0 & \text{if } i = 1 \\ i-2 & \text{if } i \ge 1 \end{cases}$$

Proof.

- When y_i is linked to x, $y_1, \ldots, y_{i-1}$ already linked to x,
  $\Rightarrow$ degree(x) = i - 1
  $\Rightarrow$ degree(y_i) = i - 1 since we only link nodes of equal degree
- Since then, y_i has lost at most one child
  - otherwise it would have been cut from x
- Thus, degree(y_i) = i - 1 or i - 2

Fibonacci Heaps: Bounding Max Degree

Key lemma. In a Fibonacci heap with N nodes, the maximum degree of any node is at most $\log_\phi N$, where $\phi = (1 + \sqrt{5}) / 2$.

Proof of key lemma.

- For any node x, we show that size(x) $\ge \phi^{\text{degree(x)}}$.
  - size(x) = # node in subtree rooted at x
  - taking base ϕ logs, degree(x) $\le \log_\phi$ (size(x)) $\le \log_\phi$ N.
- Let s_k be min size of tree rooted at any degree k node.
  - trivial to see that $s_0 = 1$, $s_1 = 2$
  - s_k monotonically increases with k
- Let x* be a degree k node of size s_k, and let $y_1, \ldots, y_k$ be children in order that they were linked to x*.

Assume k ≥ 2

$$\begin{aligned} s_k &= \text{size}(x^*) \\ &= 2 + \sum_{i=2}^{k} size(y_i) \\ &\ge 2 + \sum_{i=2}^{k} s_{\deg[y_i]} \\ &\ge 2 + \sum_{i=2}^{k} s_{i-2} \\ &= 2 + \sum_{i=0}^{k-2} s_i \end{aligned}$$

Fibonacci Facts

Definition. The Fibonacci sequence is:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

- 1, 2, 3, 5, 8, 13, 21, . . .
- Slightly nonstandard definition.

Fact F1. $F_k \geq \phi^k$, where $\phi = (1 + \sqrt{5}) / 2 = 1.618\ldots$

Fact F2. For $k \geq 2$, $F_k = 2 + \sum_{i=0}^{k-2} F_i$

$$\begin{aligned} s_k &= \text{size}\,(x^*) \\ &= 2 + \sum_{i=2}^{k} size(y_i) \\ &\geq 2 + \sum_{i=2}^{k} s_{\deg[y_i]} \\ &\geq 2 + \sum_{i=2}^{k} s_{i-2} \\ &= 2 + \sum_{i=0}^{k-2} s_i \end{aligned}$$

Consequence. $s_k \geq F_k \geq \phi^k$.

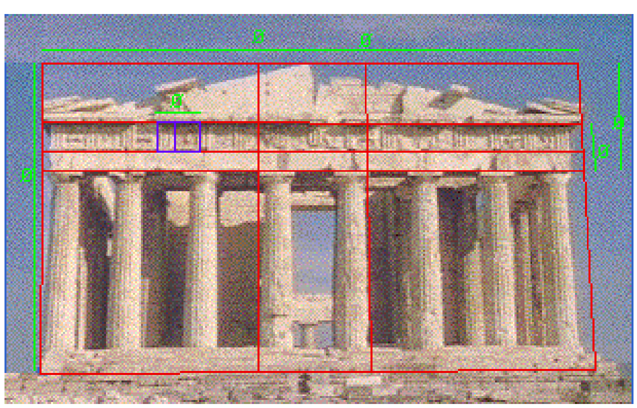
- This implies that $\text{size}(x) \geq \phi^{\text{degree}(x)}$ for all nodes x.

Golden Ratio

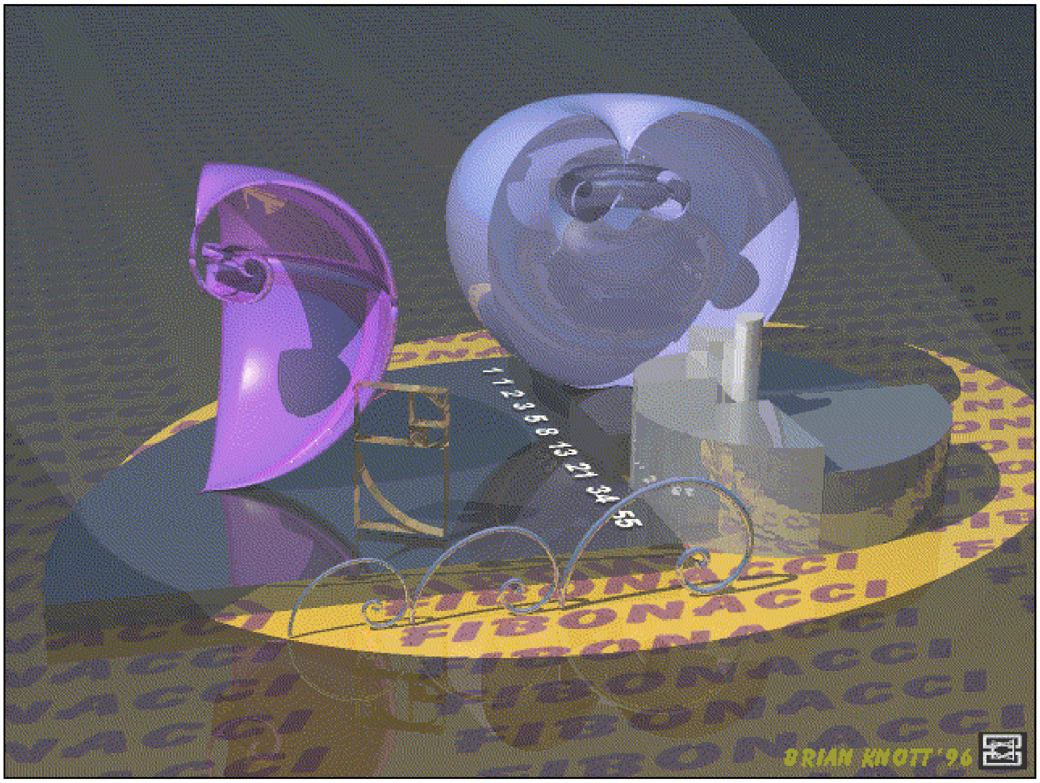
Definition. The Fibonacci sequence is: 1, 2, 3, 5, 8, 13, 21, . . .

Definition. The golden ratio $\phi = (1 + \sqrt{5}) / 2 = 1.618\ldots$

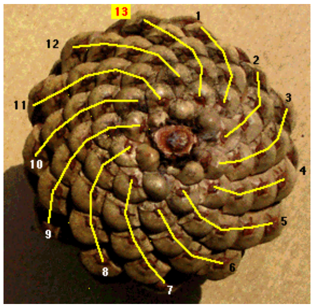
- Divide a rectangle into a square and smaller rectangle such that the smaller rectangle has the same ratio as original one.



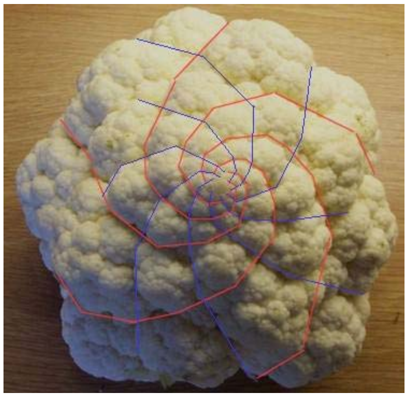
Parthenon, Athens Greece



Fibonacci Numbers and Nature



Pinecone



Cauliflower

Fibonacci Proofs

Fact F1. $F_k \geq \phi^k$.

Proof. (by induction on k)

- **Base cases:**
  - $F_0 = 1$, $F_1 = 2 \geq \phi$.
- **Inductive hypotheses:**
  - $F_k \geq \phi^k$ and $F_{k+1} \geq \phi^{k+1}$

$$\begin{aligned} F_{k+2} &= F_k + F_{k+1} \\ &\geq \varphi^k + \varphi^{k+1} \\ &= \varphi^k(1+\varphi) \\ &= \varphi^k(\varphi^2) \\ &= \varphi^{k+2} \end{aligned}$$

$\phi^2 = \phi + 1$

Fact F2. For $k \geq 2$, $F_k = 2 + \sum_{i=0}^{k-2} F_i$

Proof. (by induction on k)

- **Base cases:**
  - $F_2 = 3$, $F_3 = 5$
- **Inductive hypotheses:**

$$F_k = 2 + \sum_{i=0}^{k-2} F_i$$

$$\begin{aligned} F_{k+2} &= F_k + F_{k+1} \\ &= 2 + \sum_{i=0}^{k-2} F_i + F_{k+1} \\ &= 2 + \sum_{i=0}^{k} F_k \end{aligned}$$