SAT-based Problems

- SATISFIABILITY (SAT, CNF-SAT)
  - Input
    - Set of variables $X = \{x_i\}$
    - Set of clauses $C = \{c_j\}$ where each clause is a conjunction of literals
  - Yes/No Question
    - Is there a truth assignment to the variables that satisfies all the clauses?
  - Example Input
    - $X = \{x_1, x_2, x_3, x_4\}$
    - $C = \{(x_1 \vee x_2 \vee x_3 \vee x_4), (x_1 \vee \neg x_2), (\neg x_1 \vee \neg x_3 \vee x_4)\}$
- KSAT (K-CNF SAT): SAT with the additional restriction that each clause has at most K literals
  - The above example would satisfy 4SAT but not 3SAT because of the first clause.
- 3-OCC-3SAT: Each variable appears at most 3 times, each literal at most twice
  - Example Input
    - $X = \{x_1, x_2, x_3, x_4\}$
    - $C = \{(x_1 \vee x_2 \vee x_3), (x_1 \vee \neg x_2), (\neg x_1 \vee \neg x_3 \vee x_4)\}$
    - Note that x_1 appears two times and $\neg x_1$ appears once so we satisfy the constraints for x_1. The same holds for all other literals.
- MAX-2SAT
  - Input
    - Set of variables $X = \{x_i\}$
    - Set of clauses $C = \{c_j\}$ where each clause is a conjunction of at most two literals
    - Integer K
  - Yes/No Question
    - Can K clauses in C be satisfied by some truth assignment to the variables?
  - Example Input Instance
    - $X = \{x_1, x_2, x_3\}$
    - $C = \{(x_1 \vee x_2), (x_1 \vee \neg x_2), (\neg x_1 \vee \neg x_3)\}$
    - $K = 2$
- Circuit SAT
  - Input: Boolean circuit with variable input gates
  - Y/N Question: Is there an assignment of truth values to input gates that leads to the circuit evaluating to true?

Graph Theory Problems

- HAMILTIONIAN PATH
  - Input: Graph $G = (V, E)$
  - Yes/No Question: Does there exist a Hamiltonian Path (a path that visits each node exactly once) in G?
- HAMILTIONIAN CYCLE
  - Input: Graph $G = (V, E)$
  - Yes/No Question: Does there exist a Hamiltonian Cycle (a cycle that visits each node exactly once) in G?
- INDEPENDENT SET
  - Input: Graph $G = (V, E)$
  - Yes/No Question: Does there exist an independent set (a set of nodes $C \subseteq V$ such that no two nodes in C are connected by an edge in E) in G of size K?
- CLIQUE
  - Input: Graph $G = (V, E)$
  - Yes/No Question: Does there exist a clique (a set of nodes $C \subseteq V$ such that every pair of nodes in C are connected by an edge in E) in G of size K?
- VERTEX COVER
  - Input: Graph $G = (V, E)$
  - Yes/No Question: Does there exist a vertex cover (a set of nodes C such that for every edge $(u, v) \in E$, at least one of u and v is in C) in G of size at most K?
- Traveling Salesperson Problem (TSP)
  - Input
    * List of n cities
    * List of all integer distances $d(i, j)$ between cities i and j
    * Integer bound B
  - Yes/No Question
    * Is there a tour of the n cities (visit each city once returning to origin city) with total distance traveled at most B?
- Common Variations of TSP
  - Triangle Inequality TSP: The main difference is that for any three cities i, j, and k, it must be the case that $d(i, j) \leq d(i, k) + d(k, j)$.
  - Bottleneck TSP: The goal is to minimize the MAXIMUM intercity distance traversed rather than the total distance traveled.

Some Other Problems

- Integer Programming
  - Input
    * A set v of integer variables
    * A set of linear inequalities over these variables
    * A linear objective function $f(v)$ to maximize (or minimize)
    * An integer B
  - Yes/No Question
    * Is there an assignment of integers to v such that all inequalities are true and $f(v) \geq B$?
  - Example Input Instance
    * $V = \{v_1, v_2\}$
    * $v_1 \geq 1, v_2 \geq 0, v_1 + v_2 \leq 3$
    * $f(v) = 2v_2$
    * $B = 3$
- Subset Sum
  - Input
    * A set of positive integers S
    * An integer B
  - Yes/No Question
    * Does there exist an $S' \subseteq S$ whose elements sum to B?
  - Example Input Instance
    * $S = \{1, 3, 10, 25, 37\}$
    * $t = 40$
- Partition Problem (Set Partition Problem)
  - Input
    * A set of positive integers S
  - Yes/No Question
    * Does there exist an $S' \subset S$ such that the sum of elements in S' is the same as the sum of elements in $S - S'$?
  - Example Input Instance
    * $S = \{1, 3, 10, 25, 37\}$

Example Reduction: HP $\leq_p$ SAT

- Input to the reduction (input to HAM PATH)
  - Graph $G = (V, E)$
- Output of the reduction (input to SAT)
  - Boolean expression ϕ in CNF with variables X and clauses C
- Description of ϕ in terms of G
  - If $|V| = \{1, 2, \cdots, n\}$, then $X = \{x_{iv} \mid 1 \leq i, v \leq n\}$
    * Meaning of x_{iv}: node v is the i^{th} node of the Hamiltonian path
  - The set of clauses C will enforce this meaning:

    * For nodes $1 \leq v \leq n$, node v must appear in the path
      $(x_{1v} \vee x_{2v} \vee \cdots \vee x_{nv})$
      n clauses of length n

    * For nodes $1 \leq v \leq n$, node v must appear only once in the path
      $(\neg x_{iv} \vee \neg x_{jv})$ for $1 \leq i < j \leq n$
      $O(n^3)$ clauses of length 2

    * For $1 \leq i \leq n$, some node v must be the i^{th} node
      $(x_{i1} \vee x_{i2} \vee \cdots \vee x_{in})$
      n clauses of length n

    * For $1 \leq i \leq n$, two nodes v and w cannot both be the i^{th}
      $(\neg x_{iv} \vee \neg x_{iw})$ for $1 \leq v < w \leq n$
      $O(n^3)$ clauses of length 2

    * For each non-edge $(v, w) \notin E$, w cannot follow v in the Hamiltonian Path
      $(\neg x_{iv} \vee \neg x_{i+1,w})$ for $1 \leq i \leq n - 1$
      $O(n^3)$ clauses of length 2

- Example inputs
  - Graph $G_1 = (V_1, E_1)$
    * $V_1 = \{v_1, v_2, v_3\}$ and $E_1 = \{(v_1, v_2), (v_2, v_3)\}$
  - Graph $G_2 = (V_2, E_2)$
    * $V_2 = \{v_1, v_2, v_3\}$ and $E_2 = \{(v_1, v_2)\}$