Fast Fourier Transform

Fourier, Jean Baptiste Joseph. Fourier was born on 21 March, 1768 in Auxerre, Bourgogne, France. He was orphaned at the age of nine. Besides contemplating a career in priesthood at a very young age, Fourier demonstrated an aptitude for mathematics beginning at the age of thirteen. Young Fourier was nominated to study at the École Normale in Paris in the year 1794. In the year 1795, he was appointed to a position in École Polytechnique.

Fourier discovered that any periodic motion can be expressed as a superposition of sinusoidal and cosinusoidal waves. Based upon partial differential equations, Fourier developed a mathematical theory of heat and expounded it in *Théorie Analytique de la Chaleur*, (1822). It is in this treatise that he developed the concept of Fourier series. He also initiated the study of Fourier integrals.

Fourier series has found multiple applications in mathematics, mathematical physics, engineering, and several other scientific disciplines. William Thompson Kelvin, a British mathematician called Fourier's work a *mathematical poem.* Fourier died on 16 May, 1830 in Paris, France.

1. **Introduction**

A transform is a mapping of a function from one space to another. Specially crafted transforms help us see patterns. A problem or a physical scenario can sometimes be addressed more easily in the transform domain than in the time domain. The purpose of transform analysis is to represent a function as a linear combination of some basis functions. This is in a sense a decomposition of a function into some "elementary" functions. These elementary functions are the building blocks of the transform. Transforms can either be continuous or be discrete.

2. **Discrete Fourier Transform**

The discrete Fourier transform (DFT) is an important tool in the study of signals. It essentially approximates a periodic sequence of discrete set of points by finite sums of trigonometric (sine and cosine) functions. Let $\mathbb{Z}_N = \{0, 1, 2, \ldots, N-1\}$.

Definition. *Let* $N \in \mathbb{P}$ *and* $\omega_N = e^{2\pi i/N}, i = \sqrt{-1}, \pi = 3.1415926535897\ldots$. *The discrete Fourier transform of the sequence of complex numbers*

$$\{y(0), y(1), \ldots, y(N-1)\}$$

is a sequence of complex numbers

$$\{Y(0), Y(1), \ldots, Y(N-1)\}$$

where

$$Y(m) = \frac{1}{\sqrt{N}} \sum_{n=0}^{(N-1)} y(n) \, \omega_N^{mn}, \qquad m \in \mathbb{Z}_N$$

$\square$

In the above definition, the arguments of $y(\cdot)$ and $Y(\cdot)$ are computed modulo N.

Observation. The inverse of DFT is

$$y(n) = \frac{1}{\sqrt{N}} \sum_{m=0}^{(N-1)} Y(m) \, \omega_N^{-mn}, \qquad n \in \mathbb{Z}_N$$

$\square$

The DFT and its inverse are also sometimes denoted by $\Im_N [y(n)] \triangleq Y(m)$, and $\Im_N^{-1} [Y(m)] \triangleq y(n)$ respectively. Some elementary properties of DFT are summarized below.

Properties of DFT

Let $N \in \mathbb{P}, \alpha_1, \alpha_2 \in \mathbb{C}$, and

$$\Im_N [y(n)] = Y(m), \quad \Im_N [y_1(n)] = Y_1(m), \quad \Im_N [y_2(n)] = Y_2(m)$$

1. Periodicity: $Y(m) = Y(m+N)$
2. Linearity: $\Im_N [\alpha_1 y_1(n) + \alpha_2 y_2(n)] = \alpha_1 Y_1(m) + \alpha_2 Y_2(m)$
3. Time reversal: $\Im_N [y(-n)] = Y(-m)$
4. Conjugate function: $\Im_N \left[\overline{y(n)}\right] = \overline{Y(-m)}$
5. Symmetry or duality: $\Im_N [Y(n)] = y(-m)$
6. Time shift: $\Im_N [y(n - n_0)] = \omega_N^{mn_0} Y(m)$, $n_0 \in \mathbb{Z}$
7. Frequency shift: $\Im_N \left[\omega_N^{nk} y(n)\right] = Y(m+k)$, $k \in \mathbb{Z}$

8. Circular convolution: Let

$$\{x(0), x(1), \ldots, x(N-1)\} \quad \text{and} \quad \{y(0), y(1), \ldots, y(N-1)\}$$

be two periodic complex sequences of period N each. The circular convolution of these two sequences is a periodic sequence of period N. Let this convolved sequence be $\{w(0), w(1), \ldots, w(N-1)\}$, where

$$w(n) = \sum_{k=0}^{(N-1)} x(k)\, y(n-k), \quad n \in \mathbb{Z}_N$$

In the above equation, $(n-k)$ is computed modulo N. Therefore this convolution is circular. It can be shown that if $\Im_N[x(n)] = X(m), \Im_N[y(n)] = Y(m)$, and $\Im_N[w(n)] = W(m)$ then

$$W(m) = \sqrt{N} X(m)\, Y(m), \qquad m \in \mathbb{Z}_N$$

Similarly discrete Fourier transform of the sequence $x(n)\, y(n), n \in \mathbb{Z}_N$ is the sequence

$$\frac{1}{\sqrt{N}} \sum_{k=0}^{(N-1)} X(k)\, Y(m-k), \quad m \in \mathbb{Z}_N$$

9. Parseval's relationships:

$$\begin{aligned} \sum_{k=0}^{(N-1)} x(k)\, \overline{y(k)} &= \sum_{k=0}^{(N-1)} \overline{y(k) \frac{1}{\sqrt{N}} \sum_{j=0}^{(N-1)} X(j)\, \omega_N^{-jk}} \\ &= \frac{1}{\sqrt{N}} \sum_{j=0}^{(N-1)} X(j) \sum_{k=0}^{(N-1)} \overline{y(k) \omega_N^{-jk}} \end{aligned}$$

Thus

$$\sum_{k=0}^{(N-1)} x(k)\, \overline{y(k)} = \sum_{j=0}^{(N-1)} X(j)\, \overline{Y(j)}$$

Therefore

$$\sum_{k=0}^{(N-1)} |x(k)|^2 = \sum_{j=0}^{(N-1)} |X(j)|^2$$

□

Computation of DFT

A direct computation of the DFT of the complex sequence

$$\{y(0), y(1), \ldots, y(N-1)\} \triangleq \{y(n) \mid y(n) \in \mathbb{C},\ n \in \mathbb{Z}_N\}$$

requires up to N^2 complex multiplication and addition operations. Therefore the computational complexity of a direct computation of DFT of size N is $O(N^2)$ operations.

3. Fast Fourier Transform

Computationally efficient algorithms exist to compute DFT. Such algorithms are called fast Fourier transforms (FFT). It is assumed in these algorithms, that it is more expensive to do multiplication, than either an addition or subtraction operation. We discuss a computationally efficient algorithms to compute DFT.

A FFT algorithm originally due to the celebrated German mathematician J. C. F. Gauss, and later rediscovered independently by James W. Cooley (1926-) and John W. Tukey (1915-2000). Cooley and Tukey devised a computerized algorithm to implement the discrete Fourier transform.

The Cooley-Tukey FFT algorithm achieves reduction in the number of operations by using the principle of divide and conquer. The genesis of the FFT algorithm is first given. Let $N = 2S$, and split the sequence

$$\{y(n) \mid y(n) \in \mathbb{C}, n = 0, 1, \ldots, (N-1)\}$$

into two sequences:

$$\{p(n) \mid p(n) = y(2n), n = 0, 1, \ldots, (S-1)\}$$
$$\{q(n) \mid q(n) = y(2n+1), n = 0, 1, \ldots, (S-1)\}$$

These are the sequences with even and odd indices respectively. Let $\Im_S[p(n)] = P(m)$ and $\Im_S[q(n)] = Q(m)$. Since

$$\omega_N^{2km} = \omega_S^{km}, \quad \text{and} \quad \omega_N^{(2k+1)m} = \omega_S^{km}\omega_N^m$$

It follows that

$$Y(m) = P(m) + \omega_N^m Q(m), \qquad 0 \le m \le (N-1)$$

In the computation of $Y(m)$'s, it should be noted that $P(m)$ and $Q(m)$ are each periodic in m with period S. Also observe that,

$$P(m+S) = P(m), \quad Q(m+S) = Q(m), \ \omega_N^{S+m} = -\omega_N^m, \quad \text{for } 0 \le m \le (S-1)$$

The transform coefficients $Y(m)$ for $0 \le m \le N$ can be computed as follows.

$$\begin{aligned} Y(m) &= P(m) + \omega_N^m Q(m), & 0 \le m \le (S-1) \\ Y(m+S) &= P(m) - \omega_N^m Q(m), & 0 \le m \le (S-1) \end{aligned}$$

Computation of $P(m)$ and $Q(m)$ for $0 \le m \le (S-1)$, requires $(S-1)^2$ multiplications each. Therefore computation of $Y(m)$'s after this splitting requires $2(S-1)^2 + (S-1)$ multiplication operations, while a direct computation requires $(2S-1)^2$ such operations. Therefore there is a reduction in the multiplicative complexity, approximately by a factor of two. Let the complexity of computing DFT of size N be $\mathcal{C}(N)$. Therefore, if such splitting operations are used, then $\mathcal{C}(N) \simeq 2\mathcal{C}(N/2) + N/2$, where $\mathcal{C}(2) = 1$.

Let $N = 2^K$, and successively use the spliting operation to compute $P(m)$'s and $Q(m)$'s, and so on. Then it can be shown that $\mathcal{C}(N) \simeq NK/2$. Thus the computational complexity of the Cooley-Tukey FFT algorithm is $O(N \log N)$.