Design and Analysis of Algorithms
Lecture-5:
Quicksort & Linear Time Sorting

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Overview

- Quicksort
  - Concept
  - Time Complexity Analysis
- More about sorting
  - Theoretical lower-bound
  - Linear-time sorting algorithms
  - Stability of sorting

- Part of the slides are based on material from Prof. Jianhua Ruan, The University of Texas at San Antonio

Quick sort

- Another divide and conquer sorting algorithm – like merge sort
- Anyone remember the basic idea?
- The worst-case and average-case running time?
- Learn some new algorithm analysis tricks

Quicksort

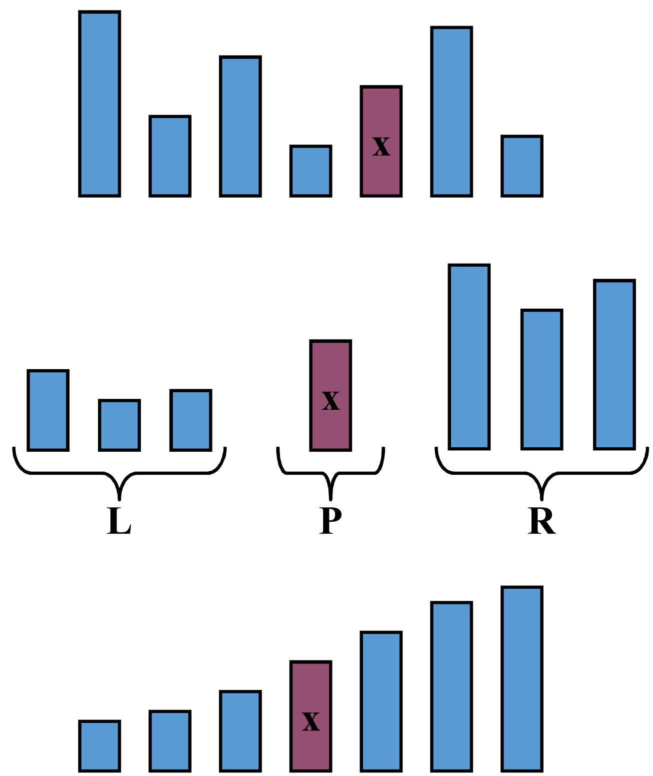
- Main idea:
  - Find a Pivot element
  - Split array into elements less than pivot, equal to pivot, and greater than pivot, called partitioning
  - Recursively sort the pieces

Divide and Conquer

1. Pick a pivot element

2. Put everything <pivot on the left and everything > pivot on right.

3. Recursively Sort the left and right



Quick sort

Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.

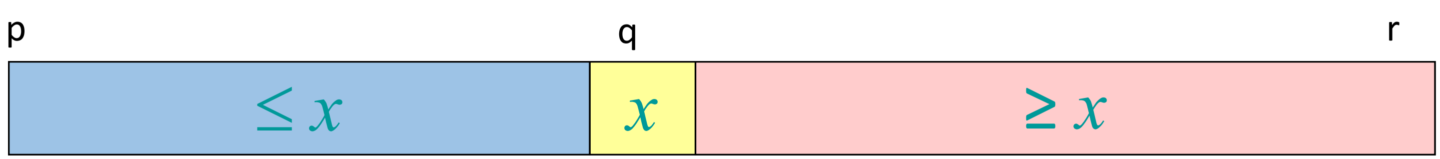
$\leq x$	x	$\geq x$

2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

Key: *Linear-time partitioning subroutine.*

Partition

- All the action takes place in the **`partition()`** function
  - Rearranges the subarray in place
  - End result: two subarrays
    - All values in first subarray $\leq$ all values in second
  - Returns the index of the "pivot" element separating the two subarrays



Pseudocode for quicksort

QUICKSORT(A, p, r)
  if $p < r$
    then $q \leftarrow$ PARTITION(A, p, r)
      QUICKSORT$(A, p, q-1)$
      QUICKSORT$(A, q+1, r)$

Initial call: QUICKSORT$(A, 1, n)$

Idea of partition

- If we are allowed to use a second array, it would be easy

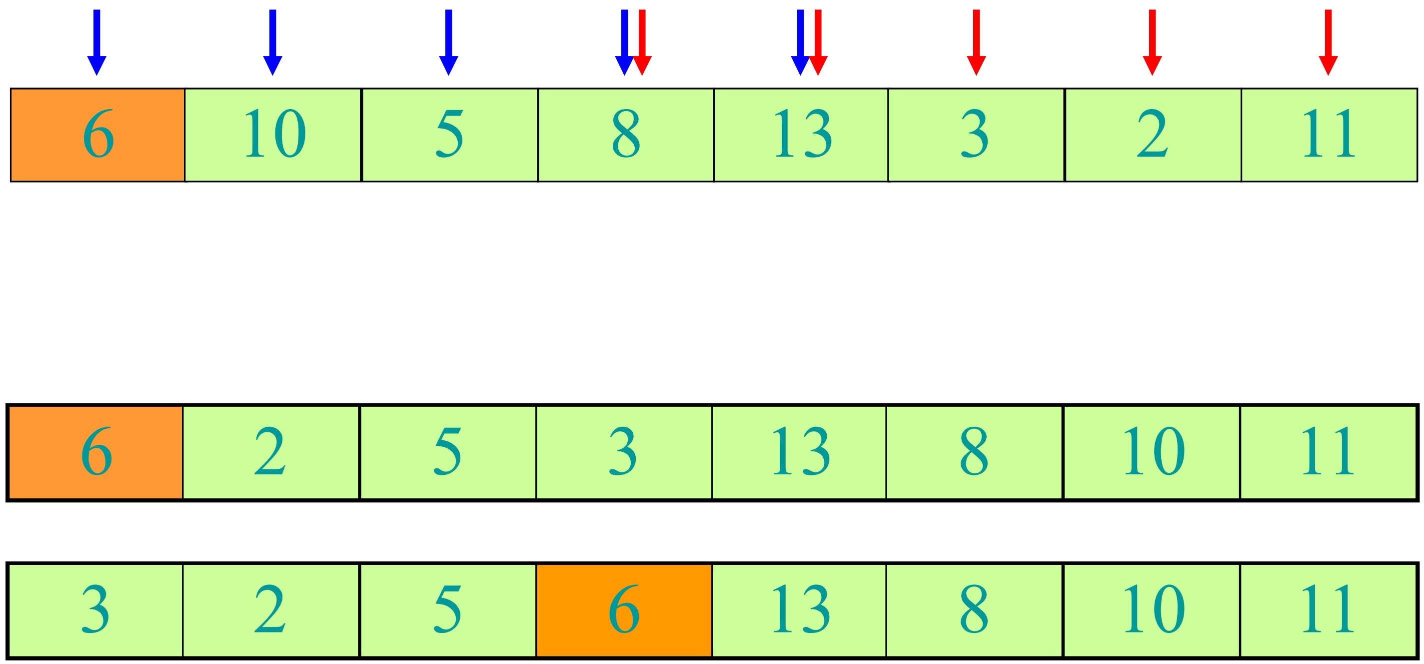
6	10	5	8	13	3	2	11

6	5	3	2	11	13	8	10

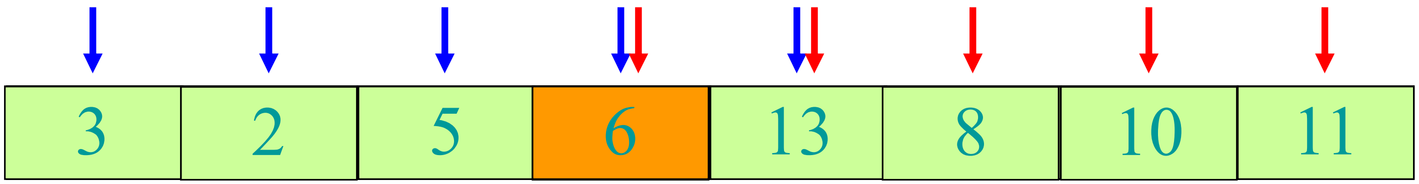
2	5	3	6	11	13	8	10

Another idea

- Keep two iterators: one from head, one from tail



In-place Partition



Partition In Words

- Partition(A, p, r):
  - Select an element to act as the "pivot" (*which?*)
  - Grow two regions, A[p..i] and A[j..r]
    - All elements in A[p..i] <= pivot
    - All elements in A[j..r] >= pivot
  - Increment i until A[i] > pivot
  - Decrement j until A[j] < pivot
  - Swap A[i] and A[j]
  - Repeat until i >= j
  - Swap A[j] and A[p]
  - Return j

Note: different from book's **`partition()`**, **`which`** uses two iterators that both move forward.

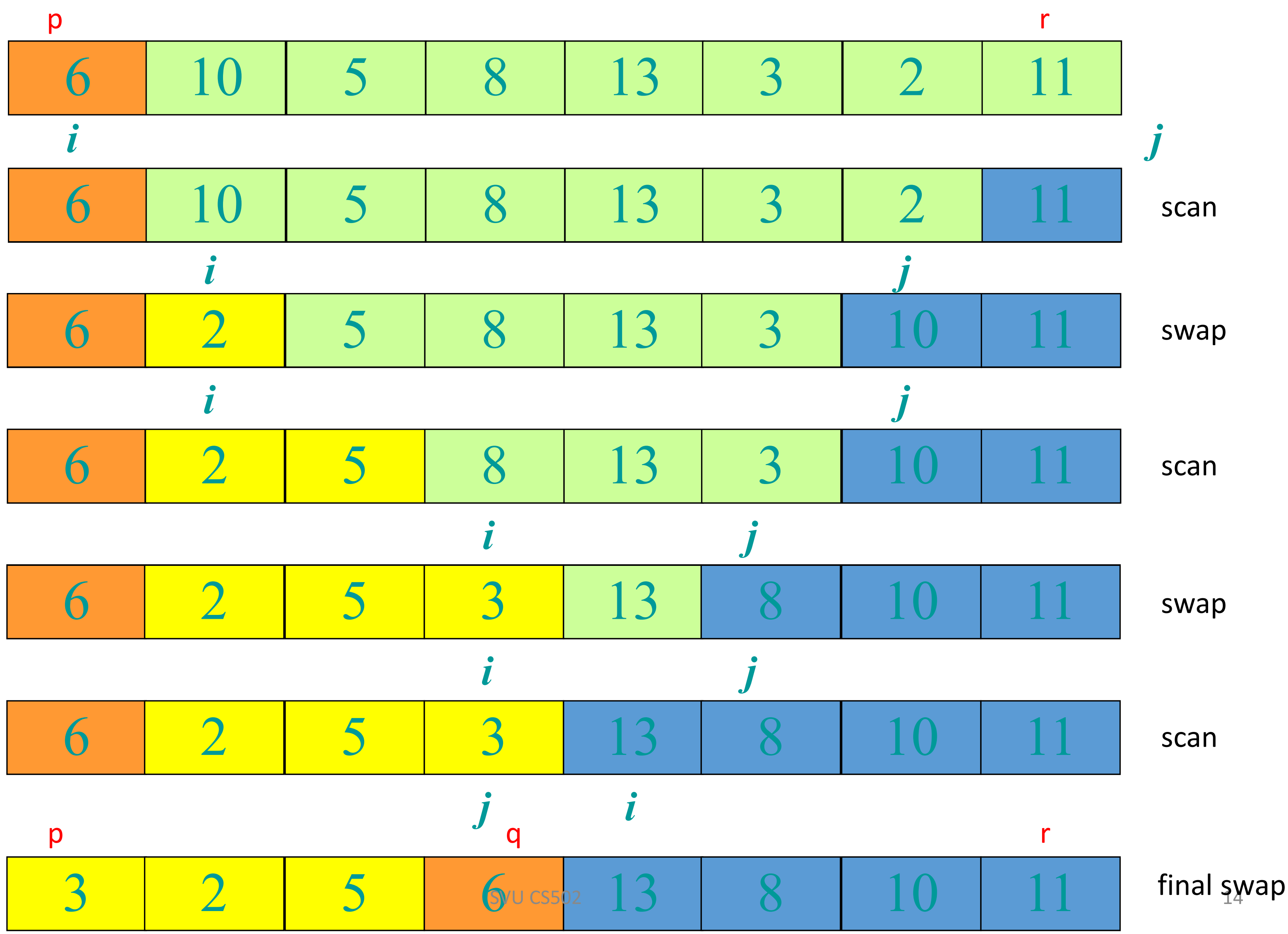
Partition Code

```
Partition(A, p, r)
    x = A[p];        // pivot is the first element
    i = p;
    j = r + 1;
    while (TRUE) {
        repeat
            i++;
        until A[i] > x or i >= j;
        repeat
            j--;
        until A[j] < x or j < i;
        if (i < j)
            Swap (A[i], A[j]);
        else
            break;
    }
    swap (A[p], A[j]);
    return j;
```

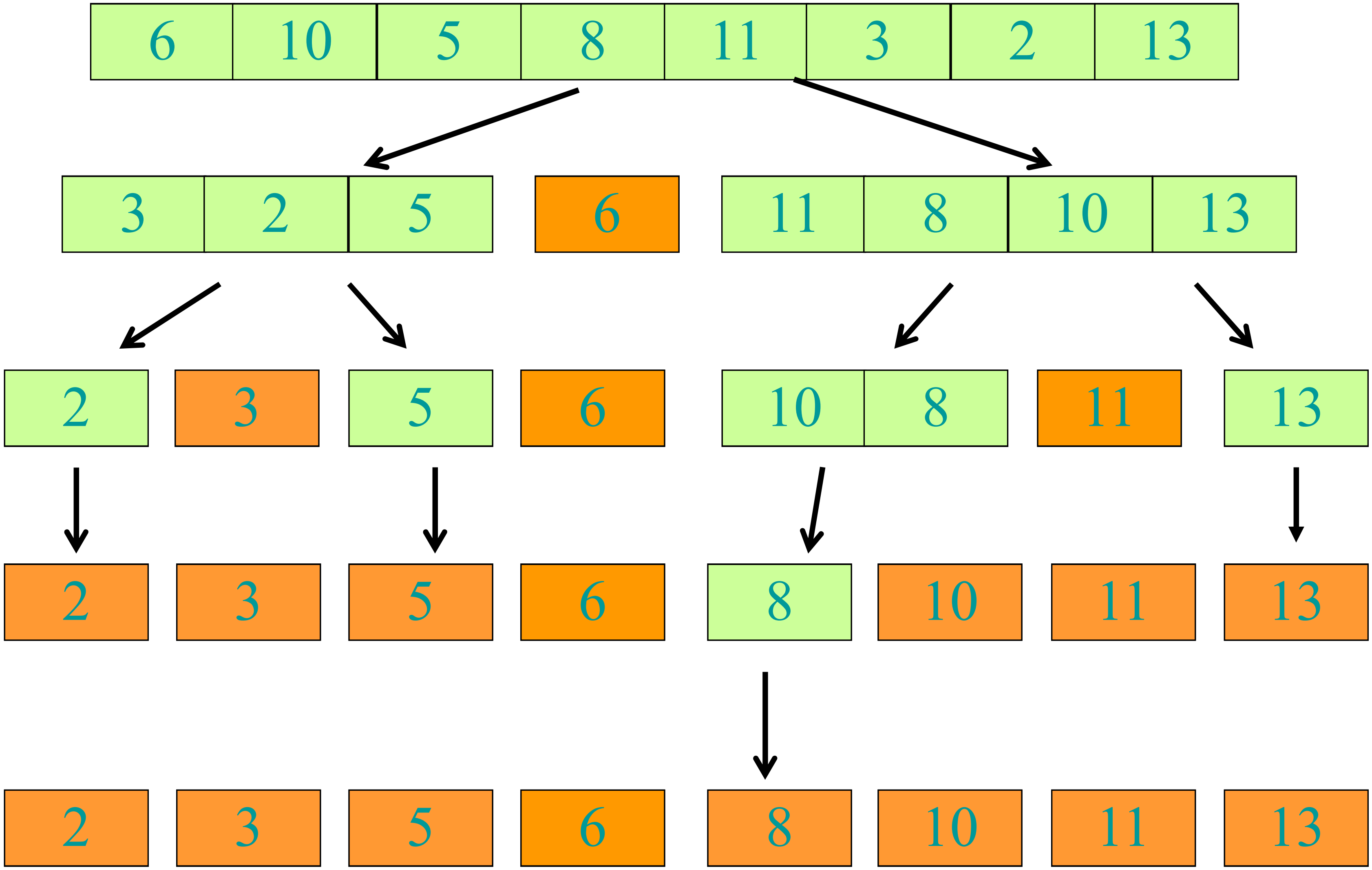
What is the running time of **`partition()`**?

`partition()` *runs in* $\Theta(n)$ *time* x = 6

Partition example



Quick sort
example



Analysis of quicksort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n) =$ worst-case running time on an array of n elements.

Worst-case of quicksort

```
QUICKSORT(A, p, r)
  if p < r
    then q ← PARTITION(A, p, r)
         QUICKSORT(A, p, q–1)
         QUICKSORT(A, q+1, r)
```

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$\begin{aligned} T(n) &= T(0) + T(n-1) + \Theta(n) \\ &= \Theta(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n^2) \qquad \textbf{\textit{(arithmetic series)}} \end{aligned}$$

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + n$$

Worst-case recursion tree

$$T(n) = T(0) + T(n–1) + n$$

$$T(n)$$

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + n$$

$$n$$

$$T(0) \qquad T(n-1)$$

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + n$$

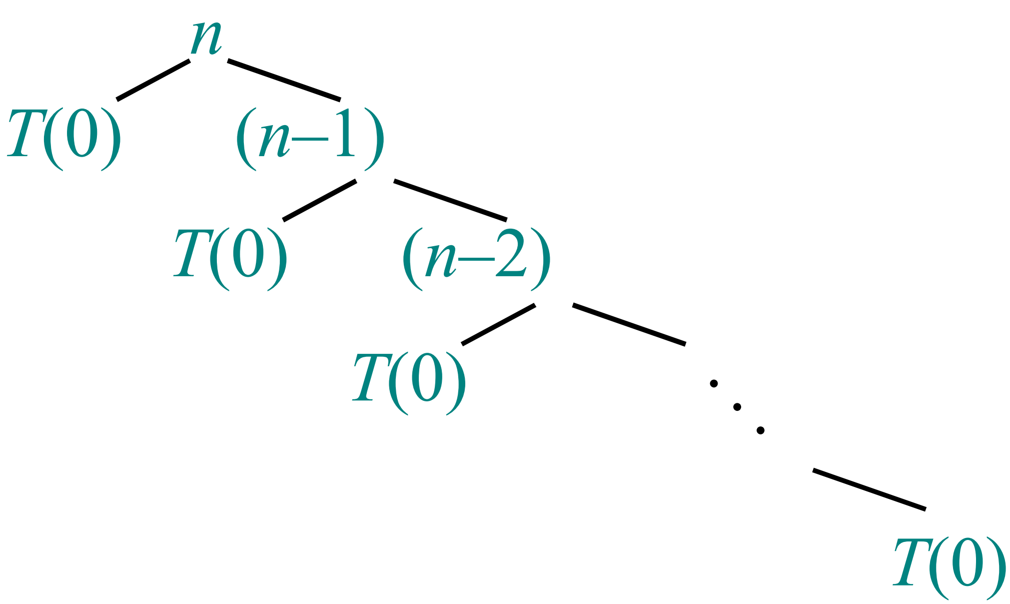
$$n$$

$$T(0) \qquad (n-1)$$

$$T(0) \qquad T(n-2)$$

Worst-case recursion tree

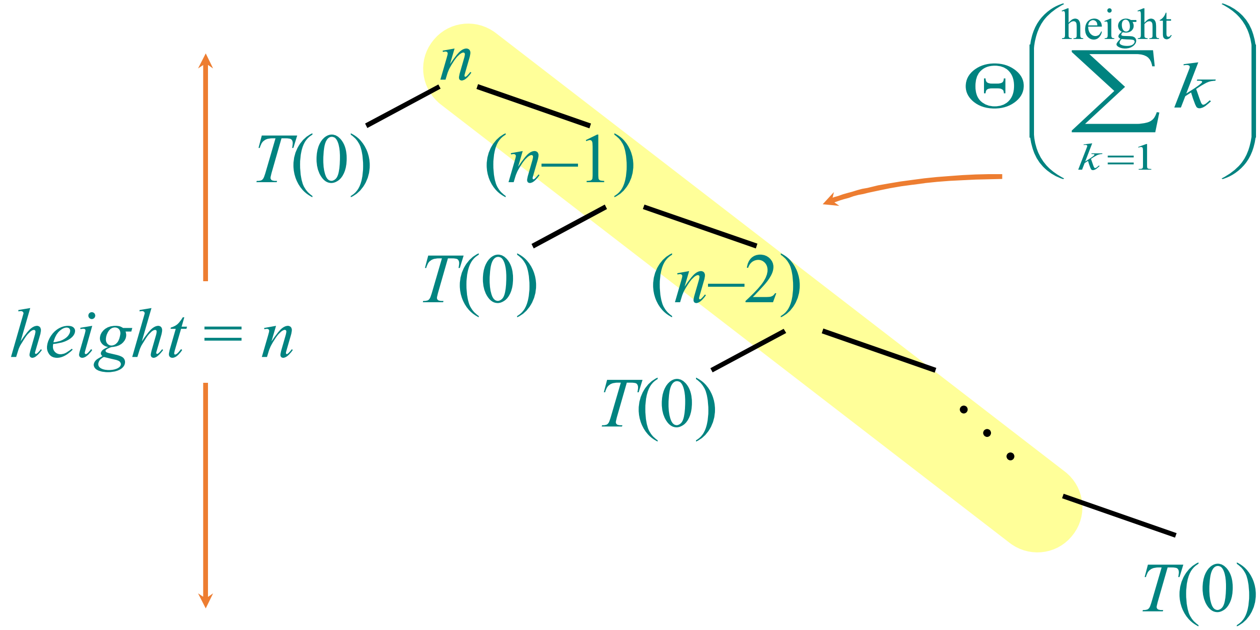
$$T(n) = T(0) + T(n-1) + n$$



Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + n$$

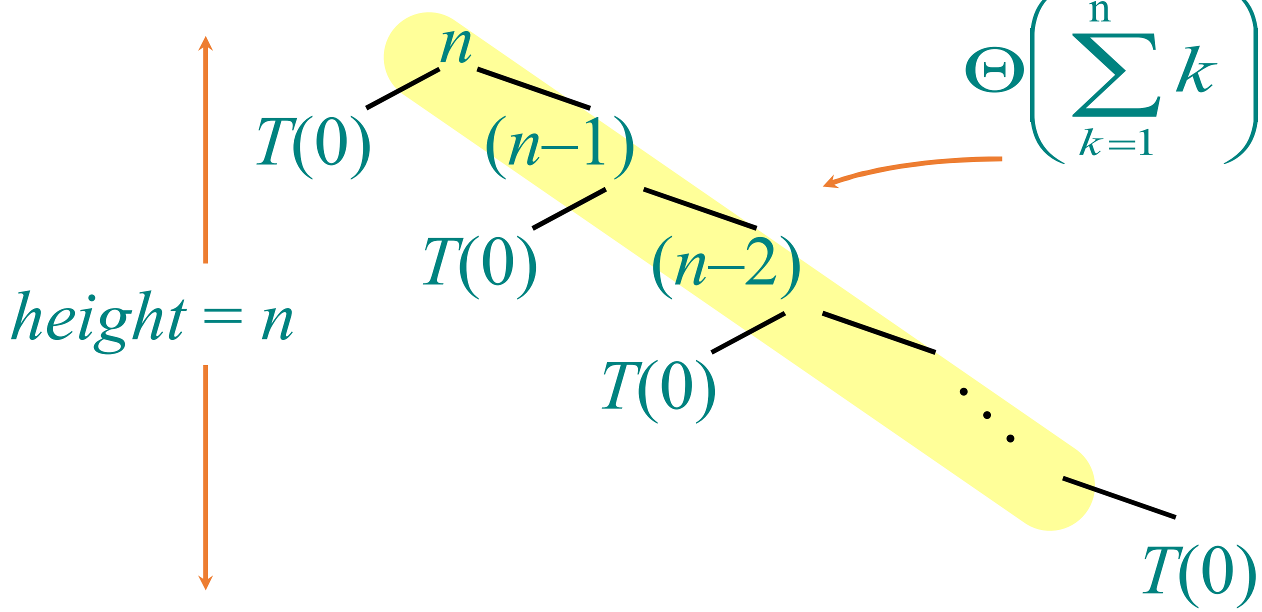
$$\Theta\left(\sum_{k=1}^{\text{height}} k\right) = \Theta\left(n^2\right)$$



Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + n$$

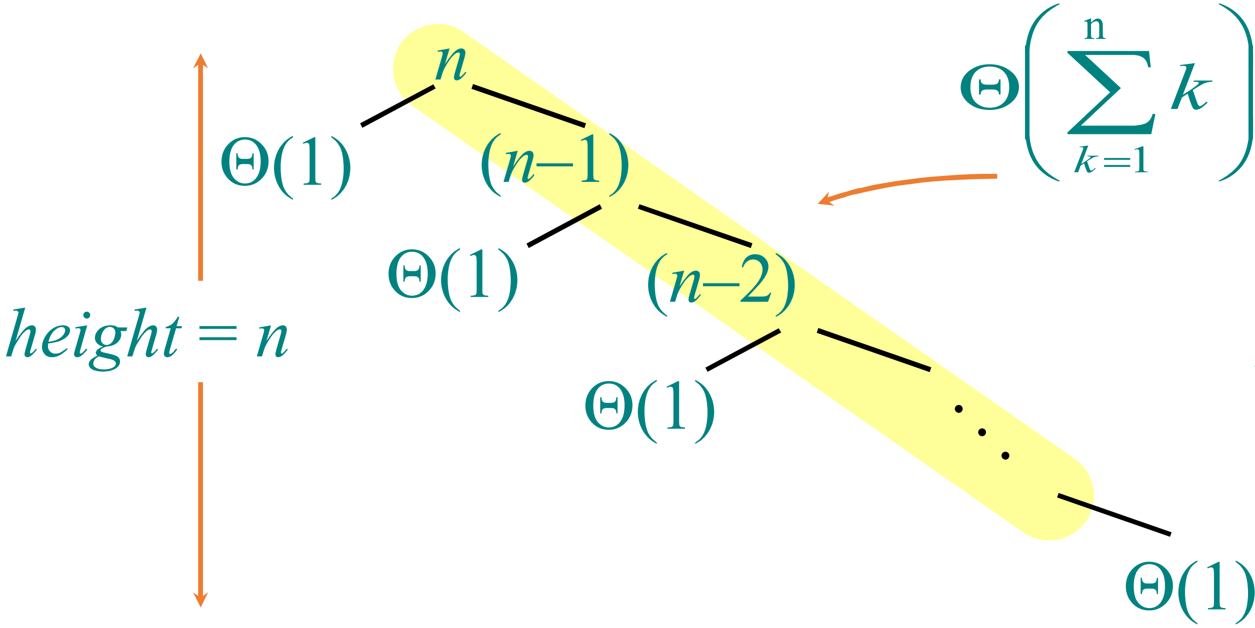
$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta\left(n^2\right)$$



Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + n$$

$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta\left(n^2\right)$$



$$\begin{aligned} T(n) &= \Theta(n) + \Theta(n^2) \\ &= \Theta(n^2) \end{aligned}$$

Best-case analysis

(For intuition only!)

If we're lucky, PARTITION splits the array evenly:

$$\begin{aligned} T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(n \log n) \end{aligned}$$

(same as merge sort)

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

$$T(n) = T\left(\tfrac{1}{10}n\right) + T\left(\tfrac{9}{10}n\right) + \Theta(n)$$

What is the solution to this recurrence?

Analysis of “almost-best” case

$$T(n)$$

Analysis of “almost-best” case

$$n$$

$$T\left(\tfrac{1}{10}n\right) \qquad\qquad T\left(\tfrac{9}{10}n\right)$$

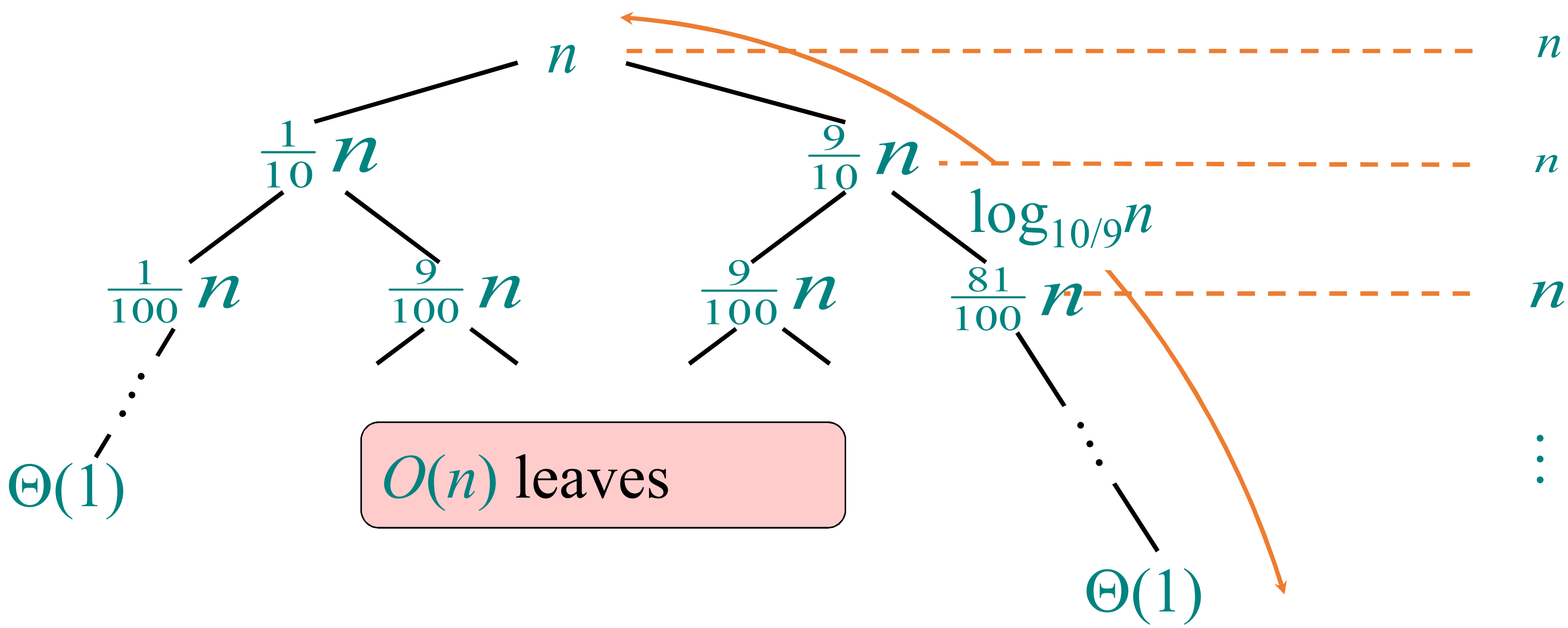
Analysis of “almost-best” case

$$n$$

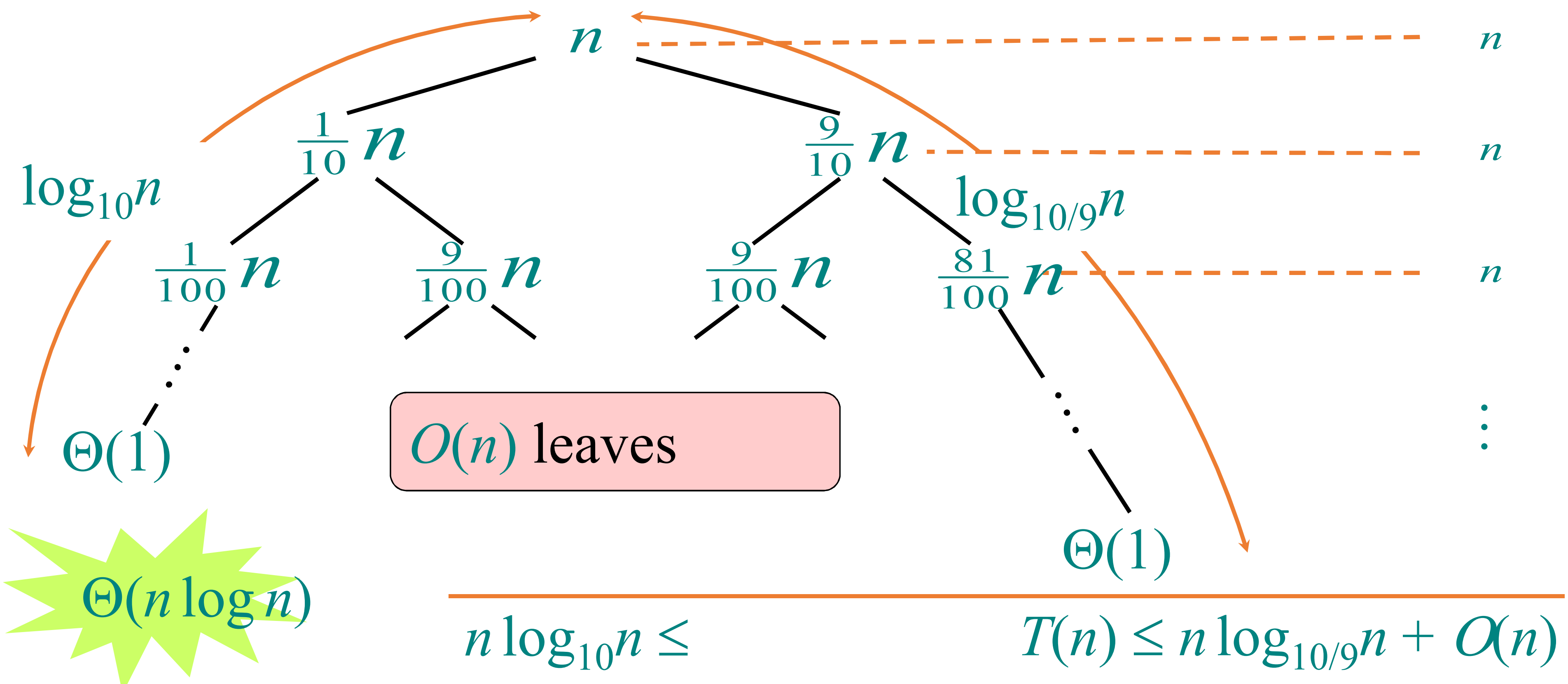
$$\frac{1}{10}n \qquad\qquad \frac{9}{10}n$$

$$T\left(\frac{1}{100}n\right)\; T\left(\frac{9}{100}n\right) \qquad T\left(\frac{9}{100}n\right)\; T\left(\frac{81}{100}n\right)$$

Analysis of “almost-best” case



Analysis of “almost-best” case



Quicksort Runtimes

- Best-case runtime $T_{best}(n) \in \Theta(n \log n)$
- Worst-case runtime $T_{worst}(n) \in \Theta(n^2)$

- Worse than mergesort? Why is it called quicksort then?
- Its average runtime $T_{avg}(n) \in \Theta(n \log n)$
- Better even, the expected runtime of **randomized quicksort** is $\Theta(n \log n)$

Randomized quicksort

- Randomly choose an element as pivot
  - Every time need to do a partition, throw a die to decide which element to use as the pivot
  - Each element has 1/n probability to be selected

```
Rand-Partition(A, p, r)
    d = random();    // a random number between 0 and 1
    index = p + floor((r-p+1) * d);  // p<=index<=r
    swap(A[p], A[index]);
    Partition(A, p, r);  // now do partition using A[p] as pivot
```

Running time of randomized quicksort

$$T(n) = \begin{cases} T(0) + T(n-1) + dn & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + dn & \text{if } 1 : n-2 \text{ split,} \\ \quad\vdots & \\ T(n-1) + T(0) + dn & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

- The expected running time is an average of all cases

Expectation

$$\overline{T}(n) = \frac{1}{n}\sum_{k=0}^{n-1}\left(\overline{T}(k) + \overline{T}(n-k-1)\right) + n$$

$$\overline{T}(n) = \frac{1}{n}\sum_{k=0}^{n-1}\left(\overline{T}(k) + \overline{T}(n-k-1)\right) + n$$

$$= \frac{2}{n}\sum_{k=0}^{n-1}\overline{T}(k) + n$$

Solving recurrence

1. Recursion tree (iteration) method
   - Good for guessing an answer
2. Substitution method
   - Generic method, rigid, but may be hard
3. Master method
   - Easy to learn, useful in limited cases only
   - Some tricks may help in other cases

Substitution method

The most general method to solve a recurrence (prove O and Ω separately):

1. ***Guess*** the form of the solution:
   (e.g. using recursion trees, or expansion)
2. ***Verify*** by induction (inductive step).

Expected running time of Quicksort

$$\overline{T}(n) = \frac{2}{n}\sum_{k=0}^{n-1}\overline{T}(k) + n$$

- Guess $\overline{T}(n) = O(n \log n)$
- We need to show that $\overline{T}(n) \le cn \log n$ for some c and sufficiently large n
- Use $T(n)$ instead of $\overline{T}(n)$ for convenience

- Fact: $$T(n) = \frac{2}{n}\sum_{k=0}^{n-1} T(k) + n$$
- Need to show: *T(n) ≤ c n log (n)*
- Assume: *T(k) ≤ ck log (k) for 0 ≤ k ≤ n-1*
- Proof:

$$\begin{aligned}
T(n) &= \frac{2}{n}\sum_{k=0}^{n-1} T(k) + n \\
&\le \frac{2c}{n}\sum_{k=0}^{n-1} k \log k + n \\
&\le \frac{2c}{n}\left(\frac{n^2}{2}\log n - \frac{n^2}{8}\right) + n
\end{aligned}$$

using the fact that $\sum_{k=0}^{n-1} k \log k \le \frac{n^2}{2}\log n - \frac{n^2}{8}$

$$\begin{aligned}
&\le cn \log n - \frac{cn}{4} + n \\
&\le cn \log n
\end{aligned}$$

if c ≥ 4. Therefore, by defintion, T(n) = Θ (nlogn)

Tightly Bounding
The Key Summation

$$\sum_{k=0}^{n-1} k \lg k = \sum_{k=1}^{n-1} k \lg k$$

$$\lim_{x \to 0} x \lg x = 0$$

$$= \sum_{k=1}^{\lceil n/2 \rceil - 1} k \lg k + \sum_{k=\lceil n/2 \rceil}^{n-1} k \lg k$$

Split the summation for a tighter bound

$$\leq \sum_{k=1}^{\lceil n/2 \rceil - 1} k \lg k + \sum_{k=\lceil n/2 \rceil}^{n-1} k \lg n$$

***The* $\lg k$ *in the second term is bounded by* $\lg n$**

$$= \sum_{k=1}^{\lceil n/2 \rceil - 1} k \lg k + \lg n \sum_{k=\lceil n/2 \rceil}^{n-1} k$$

Move the* $\lg n$ *outside the summation

Tightly Bounding
The Key Summation

$$\sum_{k=1}^{n-1} k \lg k \le \sum_{k=1}^{\lceil n/2 \rceil - 1} k \lg k + \lg n \sum_{k=\lceil n/2 \rceil}^{n-1} k$$

The summation bound so far

$$\le \sum_{k=1}^{\lceil n/2 \rceil - 1} k \lg (n/2) + \lg n \sum_{k=\lceil n/2 \rceil}^{n-1} k$$

The* lg *k in the first term is bounded by* lg *n/2

$$= \sum_{k=1}^{\lceil n/2 \rceil - 1} k (\lg n - 1) + \lg n \sum_{k=\lceil n/2 \rceil}^{n-1} k$$

lg *n/2* = lg *n* - 1

$$= (\lg n - 1) \sum_{k=1}^{\lceil n/2 \rceil - 1} k + \lg n \sum_{k=\lceil n/2 \rceil}^{n-1} k$$

Move (*lg *n* - 1*) outside the summation

Tightly Bounding
The Key Summation

$$\sum_{k=1}^{n-1} k \lg k \le (\lg n - 1) \sum_{k=1}^{\lceil n/2 \rceil - 1} k + \lg n \sum_{k=\lceil n/2 \rceil}^{n-1} k$$

The summation bound so far

$$= \lg n \sum_{k=1}^{\lceil n/2 \rceil - 1} k - \sum_{k=1}^{\lceil n/2 \rceil - 1} k + \lg n \sum_{k=\lceil n/2 \rceil}^{n-1} k$$

Distribute the (lg n - 1)

$$= \lg n \sum_{k=1}^{n-1} k - \sum_{k=1}^{\lceil n/2 \rceil - 1} k$$

The summations overlap in range; combine them

$$= \lg n \left(\frac{(n-1)(n)}{2} \right) - \sum_{k=1}^{\lceil n/2 \rceil - 1} k$$

The Guassian series

Tightly Bounding
The Key Summation

$$\sum_{k=1}^{n-1} k \lg k \le \left(\frac{(n-1)(n)}{2} \right) \lg n - \sum_{k=1}^{\lceil n/2 \rceil - 1} k$$

The summation bound so far

$$\le \frac{1}{2}\left[n(n-1) \right] \lg n - \sum_{k=1}^{n/2-1} k$$

Rearrange first term, place upper bound on second

$$\le \frac{1}{2}\left[n(n-1) \right] \lg n - \frac{1}{2}\left(\frac{n}{2} \right)\left(\frac{n}{2} - 1 \right)$$

Guassian series

$$\le \frac{1}{2}\left(n^2 \lg n - n \lg n \right) - \frac{1}{8} n^2 + \frac{n}{4}$$

Multiply it all out

Tightly Bounding
The Key Summation

$$\sum_{k=1}^{n-1} k \lg k \le \frac{1}{2}\left(n^2 \lg n - n \lg n\right) - \frac{1}{8}n^2 + \frac{n}{4}$$

$$= \frac{1}{2}n^2 \lg n - \frac{1}{8}n^2 + \left(\frac{n}{4} - \frac{n}{2}\lg n\right)$$

$$\le \frac{1}{2}n^2 \lg n - \frac{1}{8}n^2 \text{ when } \quad n \ge 2$$

Done! !!

Comparison

	Time complexity	Stable?	In-place?
Merge sort			
Quick sort			
Heap sort			

Comparison

	Time complexity	Stable?	In-place?
Merge sort	Θ (n log n)	Yes	No
Quick sort	Θ(n log n) expected. Θ(n^2) worst case	No	Yes
Heap sort	Θ (n log n)	No	Yes

More about sorting

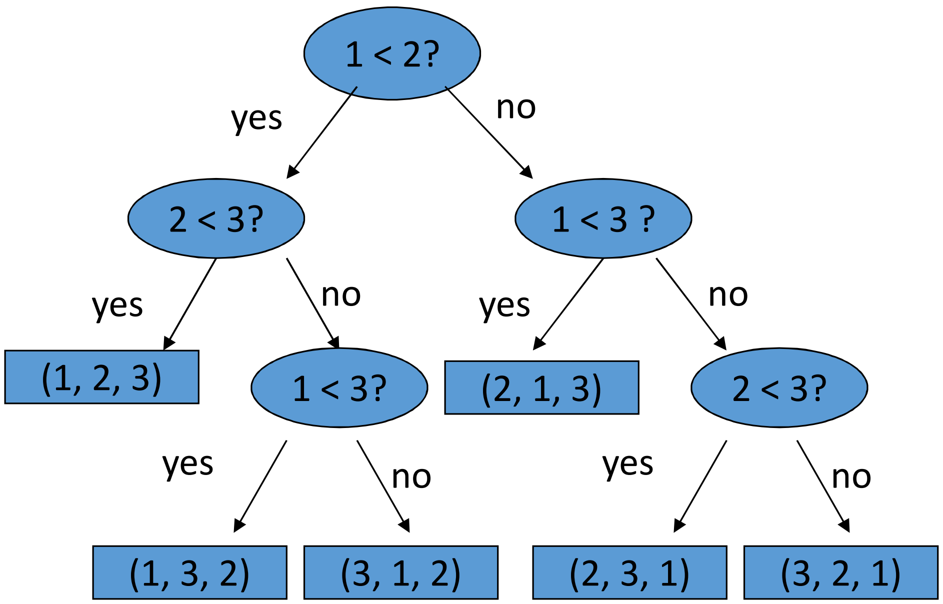
- How many sorting algorithms do you know?
- What are their time complexity?
- What’s common about them?
- Can we do better than $\Theta(n \log n)$?
- Yes and no

Theoretical lower-bound

- Comparison sort: determines the relative order of two elements only by comparison
  - What else can you do …
  - Text book Ch8.1 shows that the theoretical lower-bound for any comparison-based sorting algorithm is $\Theta(n \log n)$

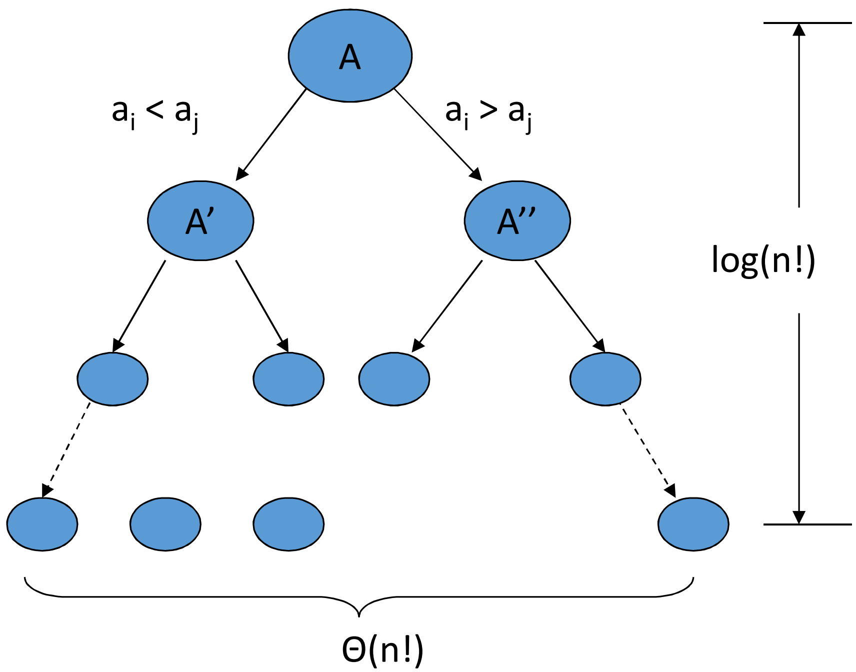
Lower-bound of comparison-based sort

- Assume an array A with 3 distinct elements, a_1, a_2, and a_3
- Use insertion sort
- # comparisons?
- A = [9 6 5]
- A = [5 6 9]
- A = …



Lower-bound of comparison-based sort

- Assume all elements are distinct, each comparison has two possible outcomes: $a_i < a_j$ or $a_i > a_j$
- Based on the outcome, change the relative order of some elements
- Output is a permutation of the input
- A correct sorting algorithm can handle any arbitrary input
- n! possible permutations
- Therefore, at least log(n!) = Θ(nlogn) comparisons in the worst case



Sorting in linear time

- Is there a problem with the theory?
- No. We are going to sort without doing comparison
- How is that possible?
- Key: knowledge about the data
  - Example: Almost sorted? All distinct? Many identical ones? Uniformly distributed?
  - The more you know about your data, the more likely you can have a better algorithm

Counting sort

- Knowledge: the numbers fall in a small range
- Example 1: sort the final exam score of a large class
  - 1000 students
  - Maximum score: 100
  - Minimum score: 0
  - Scores are integers
- Example 2: sort students according to the first letter of their last name
  - Number of students: many
  - Number of letters: 26

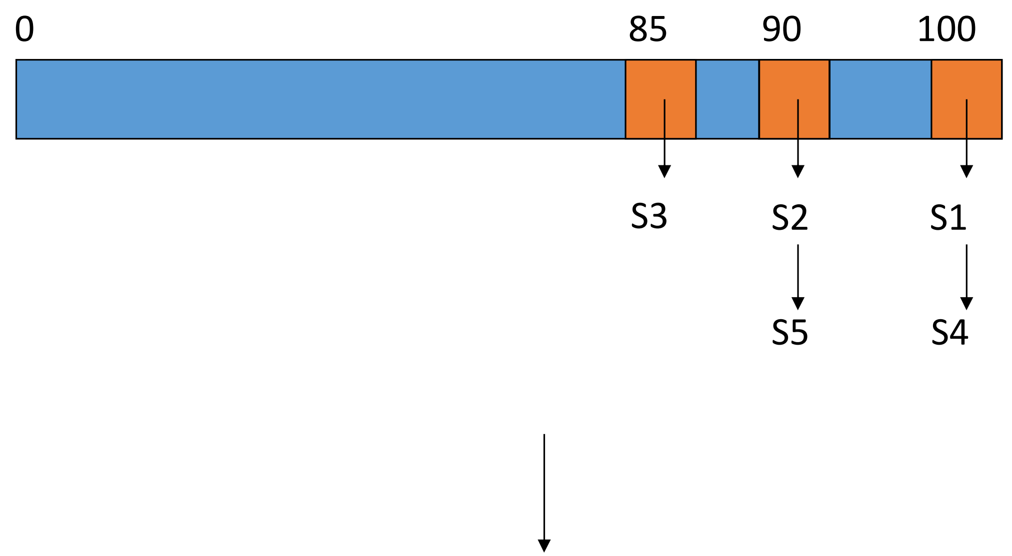
Counting sort

- ***Input***: $A[1 \,.\,.\, n]$, where $A[j] \in \{1, 2, \ldots, k\}$.
- ***Output***: $B[1 \,.\,.\, n]$, sorted.
- ***Auxiliary storage***: $C[1 \,.\,.\, k]$.

- Not an in-place sorting algorithm
- Requires Θ (n+k) additional storage besides the original array

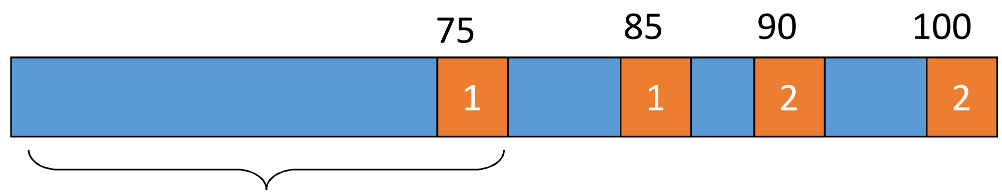
Intuition

- S1: 100
- S2: 90
- S3: 85
- S4: 100
- S5: 90
- …



… S3 … S2, S5, …, S1, S4

Intuition



50 students with score ≤ 75

What is the rank (lowest to highest) for a student with score = 75?

50

200 students with score ≤ 90

What is the rank for a student with score = 90?

200 or 199

Counting sort

1. **for** $i \leftarrow 1$ **to** k
   do $C[i] \leftarrow 0$

Initialize

2. **for** $j \leftarrow 1$ **to** n
   do $C[A[j]] \leftarrow C[A[j]] + 1$ $\triangleright$ $C[i] = |\{\text{key} = i\}|$

Count

3. **for** $i \leftarrow 2$ **to** k
   do $C[i] \leftarrow C[i] + C[i-1]$ $\triangleright$ $C[i] = |\{\text{key} \leq i\}|$

Compute running sum

4. **for** $j \leftarrow n$ **downto** 1
   do $B[C[A[j]]] \leftarrow A[j]$
   $C[A[j]] \leftarrow C[A[j]] - 1$

Re-arrange

Counting-sort example

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:				

B:					

Loop 1: initialization

	1	2	3	4	5
A:	4	1	3	4	3
B:					

	1	2	3	4
C:	0	0	0	0

1. **for** $i \leftarrow 1$ **to** k
   do $C[i] \leftarrow 0$

Loop 2: count

A:

1	2	3	4	5
4	1	3	4	3

C:

1	2	3	4
0	0	0	1

B:

2. **for** $j \leftarrow 1$ **to** n
   do $C[A[j]] \leftarrow C[A[j]] + 1 \quad \triangleright\ C[i] = |\{\text{key} = i\}|$

Loop 2: count

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	1	0	0	1

B:					

2. **for** $j \leftarrow 1$ **to** n

do $C[A[j]] \leftarrow C[A[j]] + 1$ $\quad \triangleright\ C[i] = |\{\text{key} = i\}|$

Loop 2: count

A:

1	2	3	4	5
4	1	3	4	3

C:

1	2	3	4
1	0	1	1

B:

2. **for** $j \leftarrow 1$ **to** n
    do $C[A[j]] \leftarrow C[A[j]] + 1$ $\triangleright$ $C[i] = |\{\text{key} = i\}|$

Loop 2: count

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	1	0	1	2

B:					

2. **for** $j \leftarrow 1$ **to** n

 do $C[A[j]] \leftarrow C[A[j]] + 1$ $\quad \rhd\ C[i] = |\{\text{key} = i\}|$

Loop 2: count

	1	2	3	4	5
A:	4	1	3	4	3

	1	2	3	4
C:	1	0	2	2

B:					

2. **for** $j \leftarrow 1$ **to** n
   do $C[A[j]] \leftarrow C[A[j]] + 1$ $\quad \rhd$ $C[i] = |\{\text{key} = i\}|$

Loop 3: compute running sum

A:

1	2	3	4	5
4	1	3	4	3

C:

1	2	3	4
1	0	2	2

B:

C':

1	1	2	2

3. **for** $i \leftarrow 2$ **to** k
   do $C[i] \leftarrow C[i] + C[i-1]$ $\quad \triangleright\ C[i] = |\{\text{key} \leq i\}|$

Loop 3: compute running sum

	1	2	3	4	5
A:	4	1	3	4	3
B:					

	1	2	3	4
C:	1	0	2	2
C':	1	1	3	2

3. **for** $i \leftarrow 2$ **to** k

   do $C[i] \leftarrow C[i] + C[i-1]$ $\quad \rhd\ C[i] = |\{\text{key} \le i\}|$

Loop 3: compute running sum

	1	2	3	4	5
A:	4	1	3	4	3
B:					

	1	2	3	4
C:	1	0	2	2
C':	1	1	3	5

3. for $i \leftarrow 2$ **to** k

do $C[i] \leftarrow C[i] + C[i–1]$ ⊳ $C[i] = |\{\text{key} \le i\}|$

Loop 4: re-arrange

	1	2	3	4	5
A:	4	1	3	4	3
B:			3		

	1	2	3	4
C:	1	1	3	5
C':	1	1	3	5

4. **for** $j \leftarrow n$ **downto** 1
   do $B[C[A[j]]] \leftarrow A[j]$
   $C[A[j]] \leftarrow C[A[j]] - 1$

Loop 4: re-arrange

	1	2	3	4	5
A:	4	1	3	4	3
B:			3		

	1	2	3	4
C:	1	1	3	5
C':	1	1	2	5

4. **for** $j \leftarrow n$ **downto** 1
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Loop 4: re-arrange

	1	2	3	4	5
A:	4	1	3	4	3
B:			3		4

	1	2	3	4
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Loop 4: re-arrange

	1	2	3	4	5
A:	4	1	3	4	3
B:			3		4

	1	2	3	4
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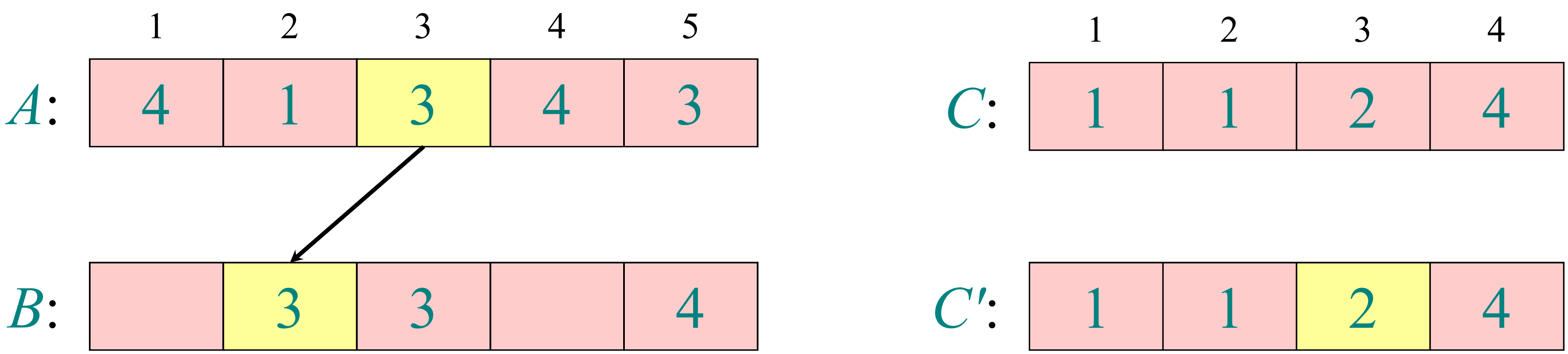

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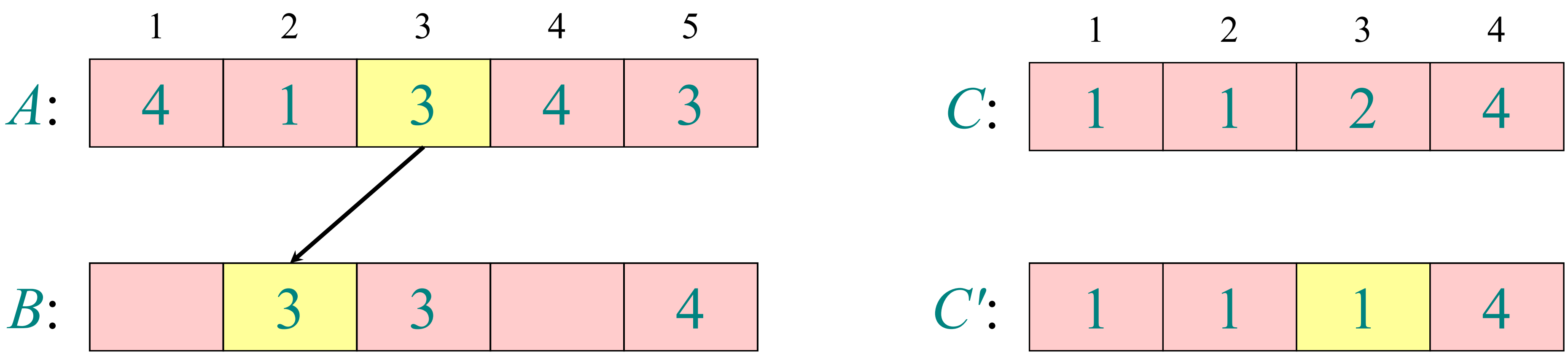

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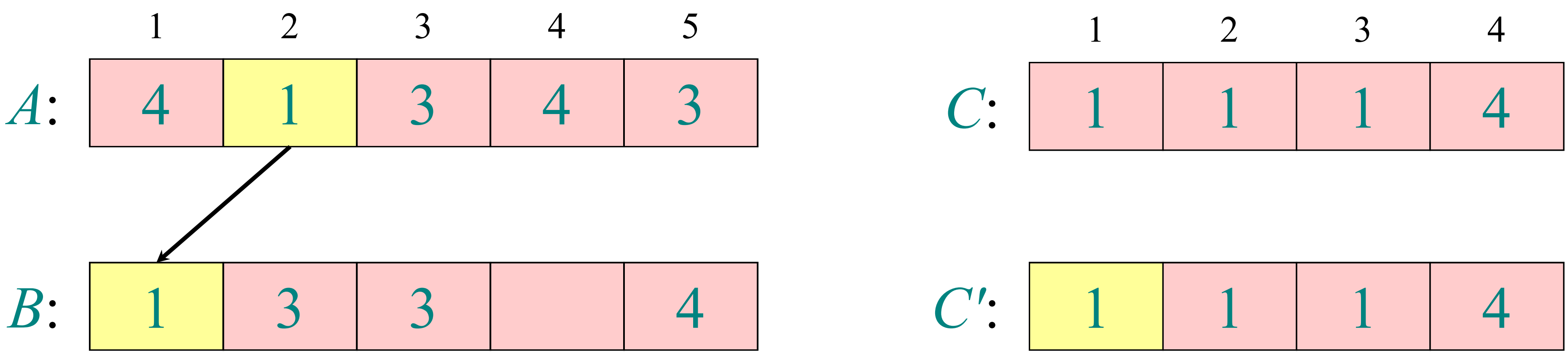
Loop 4: re-arrange



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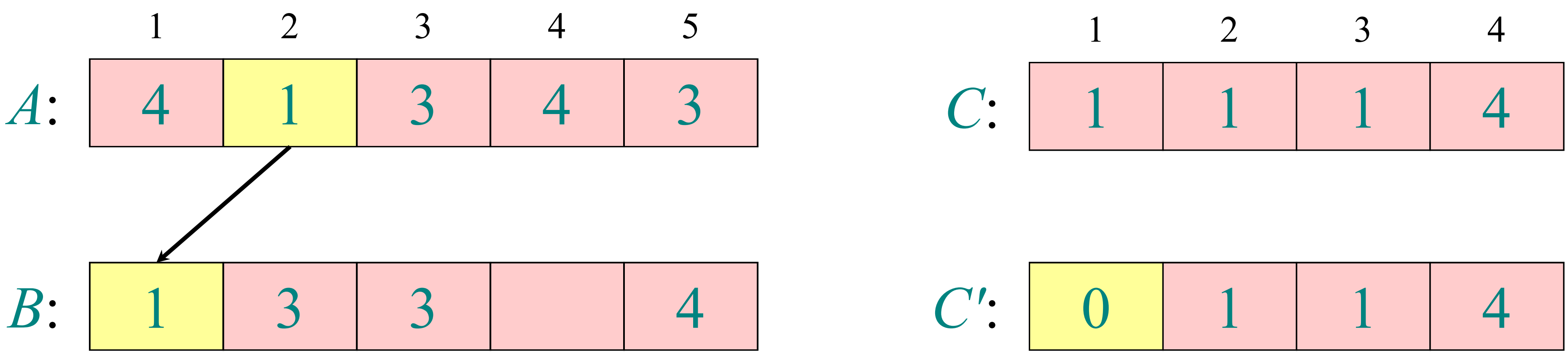
Loop 4: re-arrange



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Loop 4: re-arrange

	1	2	3	4	5
A:	4	1	3	4	3
B:	1	3	3	4	4

	1	2	3	4
C:	0	1	1	4
C':	0	1	1	4

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Analysis

$\Theta(k)$	**1. for** $i \leftarrow 1$ **to** k **do** $C[i] \leftarrow 0$
$\Theta(n)$	**2. for** $j \leftarrow 1$ **to** n **do** $C[A[j]] \leftarrow C[A[j]] + 1$
$\Theta(k)$	**3. for** $i \leftarrow 2$ **to** k **do** $C[i] \leftarrow C[i] + C[i-1]$
$\Theta(n)$	**4. for** $j \leftarrow n$ **downto** 1 **do** $B[C[A[j]]] \leftarrow \mathrm{A}[j]$ $C[A[j]] \leftarrow C[A[j]] - 1$
$\Theta(n + k)$	

Running time

If $k = O(n)$, then counting sort takes $\Theta(n)$ time.

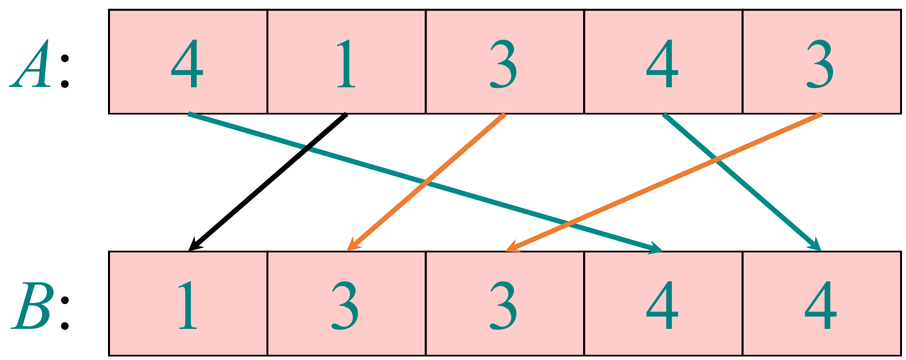
- But, theoretical lower-bound sorting takes $\Omega(n \log n)$ time!
- Problem with the theory?

Answer:

- ***Comparison sorting*** takes $\Omega(n \log n)$ time.
- Counting sort is not a ***comparison sort***.
- In fact, not a single comparison between elements occurs!

Stable sorting

Counting sort is a ***stable*** sort: it preserves the input order among equal elements.



Why this is important?
What other algorithms have this property?

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Stable sort

- Most Θ(n^2) sorting algorithms are stable
  - Standard selection sort is not, but can be made so
- Most Θ(n log n) sorting algorithms are not stable
  - Except merge sort
- Generic way to make any sorting algorithm stable
  - Use two keys, the second key is the original index of the element
  - When two elements are equal, compare their second keys

  5, 6, 5, 1, 2, 3, 2, 6

  (5, 1), (6, 2), (5, 3), (1, 4), (2, 5), (3, 6), (2, 7), (6, 8)

  (5, 1) < (5, 3)

  (2, 5) < (2, 7)

How to sort very large numbers?

19809910912351818 3599
34019954038012811 5295

Those numbers are too large for the int type.
They are represented as strings.

One method: Use comparison-based sorting, but compare strings character by character.

Change if (A[i] < A[j]) to if (compare(A[i], A[j]) < 0)

```
Compare(s, t)
  for i = 1 to length(s)
    if (s[i] < t[i]) return -1;
    else if (s[i] > t[i]) return 1;
  return 0;
```

What's the cost to compare two strings, each with d characters?

$\Theta(d)$

Total cost: $\Theta(d\ n \log n)$

Radix sort

- Similar to sorting address books
- Treat each digit as a key
- Start from the least significant bit

Most significant $\downarrow$ Least significant $\downarrow$

198099109123518183599
340199540380128115295
384700101594539614696
382408360201039258538
614386507628681328936

Radix sort illustration

- Use simpler examples:

7 4 2
7 4 8
0 5 4
6 8 8
4 1 2
2 3 0
9 3 5
1 1 6
1 6 1
4 3 4
3 8 5
6 6 6
0 3 1
0 1 3
3 6 5
1 7 3
0 1 6

Radix sort illustration

- Sort the last digit:

2 3 0
1 6 1
0 3 1
7 4 2
4 1 2
0 1 3
1 7 3
0 5 4
4 3 4
9 3 5
3 8 5
3 6 5
1 1 6
6 6 6
0 1 6
7 4 8
6 8 8

Radix sort illustration

- Sort the second digit:

$\downarrow$

<u>4 1 2</u>
<u>0 1 3</u>
<u>1 1 6</u>
<u>0 1 6</u>
<u>2 3 0</u>
<u>0 3 1</u>
<u>4 3 4</u>
<u>9 3 5</u>
<u>7 4 2</u>
<u>7 4 8</u>
<u>0 5 4</u>
<u>1 6 1</u>
<u>3 6 5</u>
<u>6 6 6</u>
<u>1 7 3</u>
<u>3 8 5</u>
<u>6 8 8</u>

Radix sort illustration

- Sort the first digit:

0 1 3
0 1 6
0 3 1
0 5 4
1 1 6
1 6 1
1 7 3
2 3 0
3 6 5
3 8 5
4 1 2
4 3 4
6 6 6
6 8 8
7 4 2
7 4 8
9 3 5

Time complexity

- Sort each of the d digits by counting sort
- Total cost: $d\ (n + k)$
  - k = 10
  - Total cost: $\Theta(dn)$
- Partition the d digits into groups of 3
  - Total cost: $(n+10^3)d/3$
- We work with binaries rather than decimals
  - Partition d bits into groups of r bits
  - Total cost: $(n+2^r)d/r$
  - Choose $r = log\ n$
  - Total cost: $dn\ /\ log\ n$
  - Compare with $dn\ log\ n$
- Catch: radix sort has a larger hidden constant factor

Space complexity

- Calls counting sort
- Therefore additional storage is needed
- $\Theta(n)$