Augmenting Data Structures

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- So far, we've only looked at one design technique *(What is it?)*

Augmenting Data Structures

- This course is supposed to be about design and analysis of algorithms
- So far, we've only looked at one design technique: *divide and conquer*
- Next up: augmenting data structures
  - Or, "One good thief is worth ten good scholars"

Dynamic Order Statistics

- We've seen algorithms for finding the ith element of an unordered set in $O(n)$ time
- Next, a structure to support finding the ith element of a dynamic set in $O(\lg n)$ time
  - *What operations do dynamic sets usually support?*
  - *What structure works well for these?*
  - *How could we use this structure for order statistics?*
  - *How might we augment it to support efficient extraction of order statistics?*

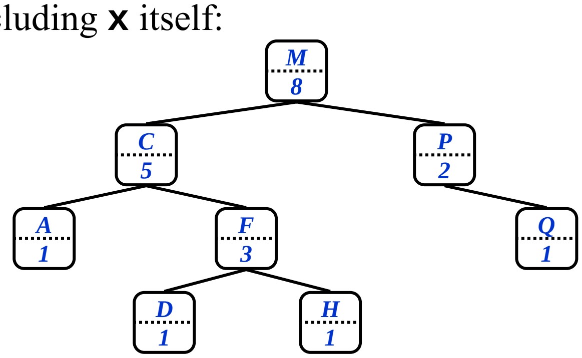
Dynamic Order Statistics – contd.

- We've seen algorithms for finding the *i*th element of an unordered set in O(*n*) time
- *OS-Trees*: a structure to support finding the *i*th element of a dynamic set in O(lg *n*) time
  - Support standard dynamic set operations (**`Insert()`**, **`Delete()`**, **`Min()`**, **`Max()`**, **`Succ()`**, **`Pred()`**)
  - Also support these order statistic operations:

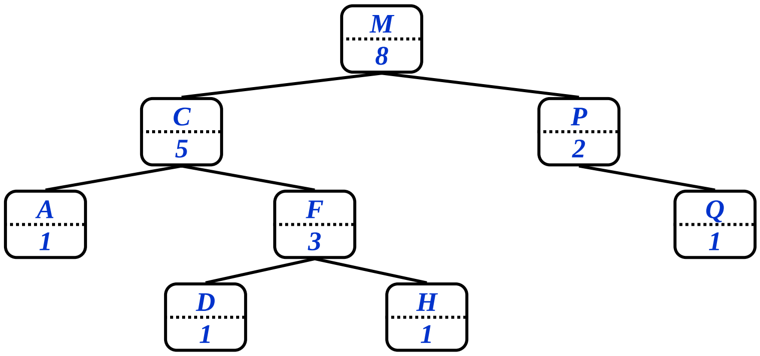
```
void OS-Select(root, i);
int OS-Rank(x);
```

Order Statistic Trees

- OS Trees augment red-black trees:
  - Associate a *size* field with each node in the tree
  - **`x->size`** records the size of subtree rooted at **`x`**, including **`x`** itself:



Selection On OS Trees



How can we use this property
to select the ith element of the set?

OS-Select

```
OS-Select(x, i)
{
    r = x->left->size + 1;
    if (i == r)
        return x;
    else if (i < r)
        return OS-Select(x->left, i);
    else
        return OS-Select(x->right, i-r);
}
```

OS-Select Example

- Example: show OS-Select(*root*, 5):

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OS-Select(x, i)
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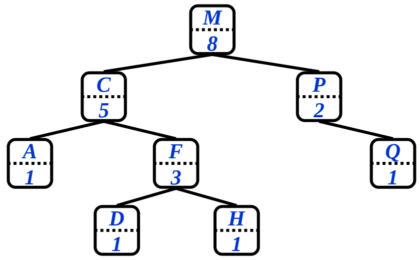

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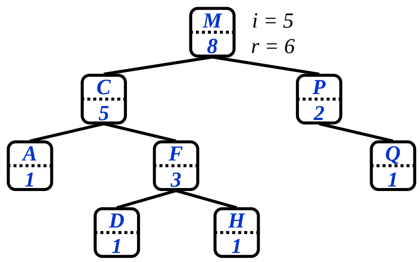

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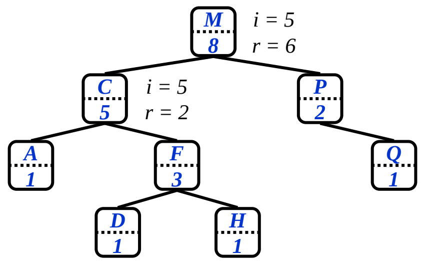

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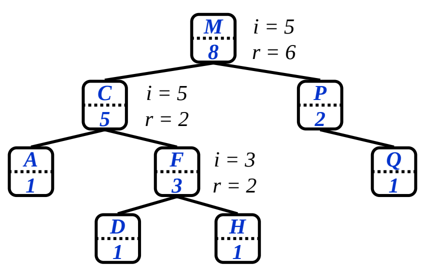

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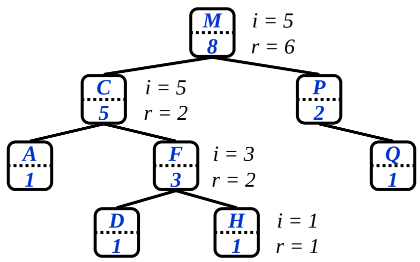

Review: OS-Select

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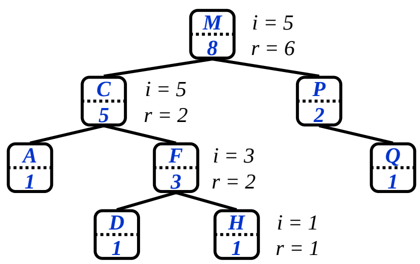

Note: use a sentinel NIL element at the leaves with size = 0 to simplify code, avoid testing for NULL

OS-Select: A Subtlety

```
OS-Select(x, i)
{
    r = x->left->size + 1;
    if (i == r)
        return x;
    else if (i < r)
        return OS-Select(x->left, i);
    else
        return OS-Select(x->right, i-r);
}
```

Oops…

- *What happens at the leaves?*
- *How can we deal elegantly with this?*

OS-Select

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    else
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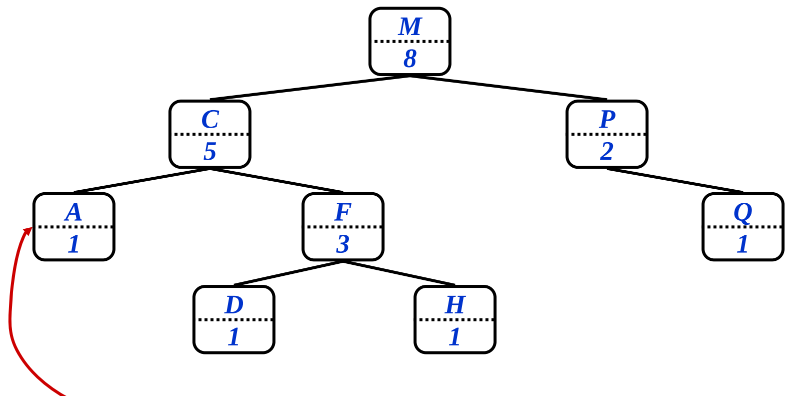
}
```

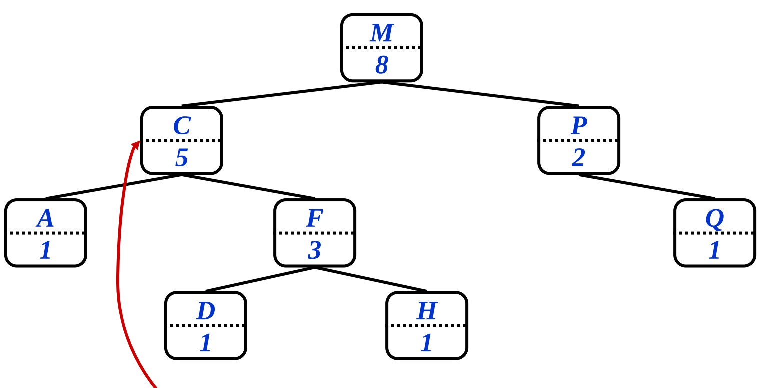
- *What will be the running time?*

Determining The Rank Of An Element



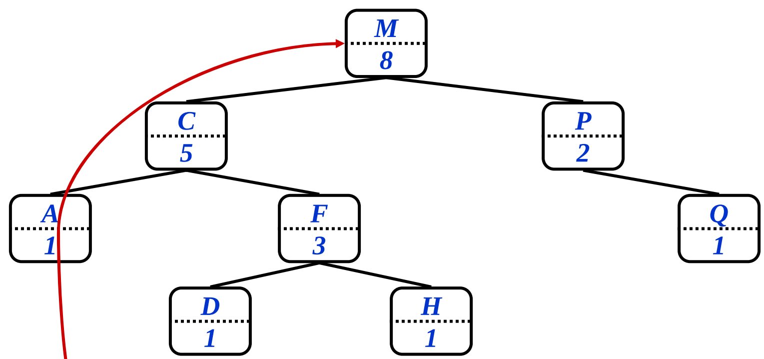
What is the rank of this element?

Determining The Rank Of An Element



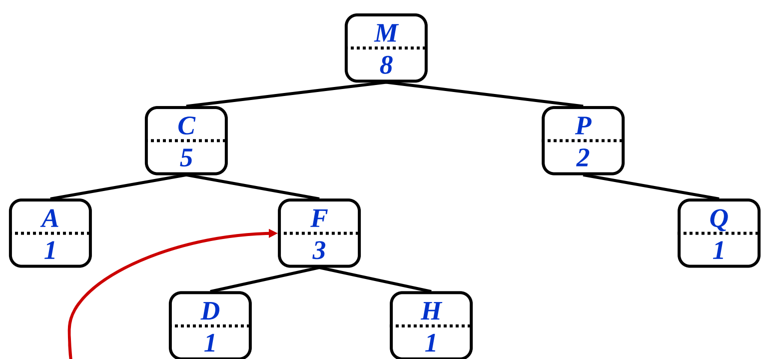
Of this one? Why?

Determining The Rank Of An Element



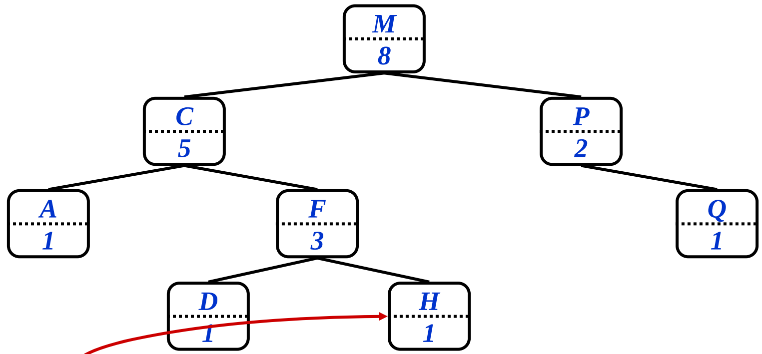
Of the root? What's the pattern here?

Determining The Rank Of An Element



What about the rank of this element?

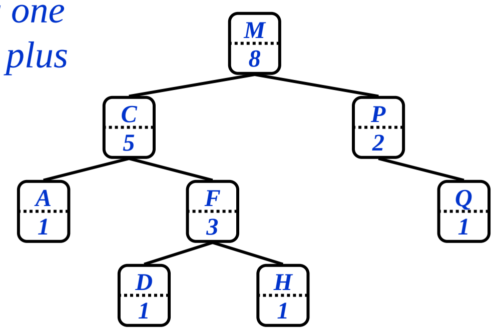
Determining The Rank Of An Element



This one? What's the pattern here?

Review: Determining The Rank Of An Element

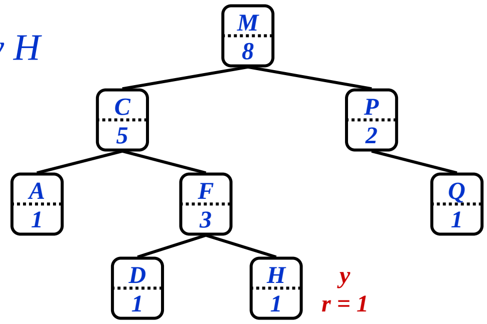
Idea: rank of right child x *is one more than its parent's rank, plus the size of* x*'s left subtree*




```
OS-Rank(T, x)
{
    r = x->left->size + 1;
    y = x;
    while (y != T->root)
        if (y == y->p->right)
            r = r + y->p->left->size + 1;
        y = y->p;
    return r;
}
```

Review: Determining The Rank Of An Element

Example 1:
find rank of element with key H

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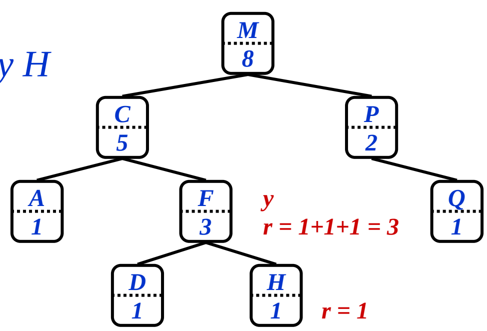


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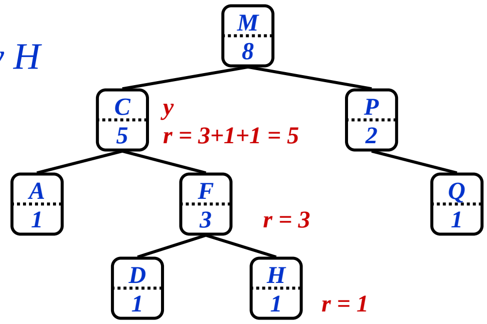


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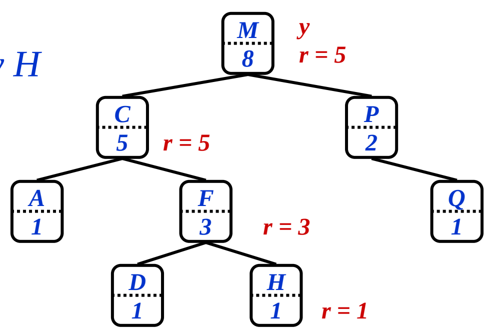


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Review: Determining The Rank Of An Element

Example 2:

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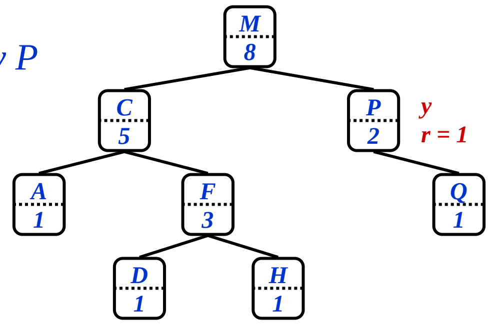


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Review: Determining The Rank Of An Element

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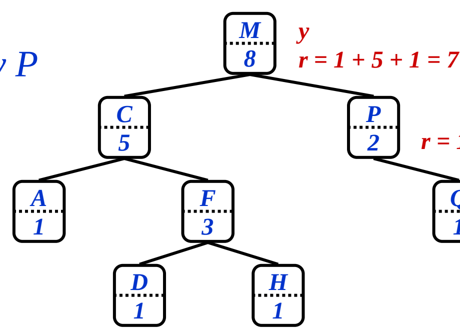


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}
```

- *What will be the running time?*

OS-Trees: Maintaining Sizes

- So we've shown that with subtree sizes, order statistic operations can be done in O(lg n) time
- Next step: maintain sizes during Insert() and Delete() operations
  - *How would we adjust the size fields during insertion on a plain binary search tree?*

OS-Trees: Maintaining Sizes

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OS-Trees: Maintaining Sizes

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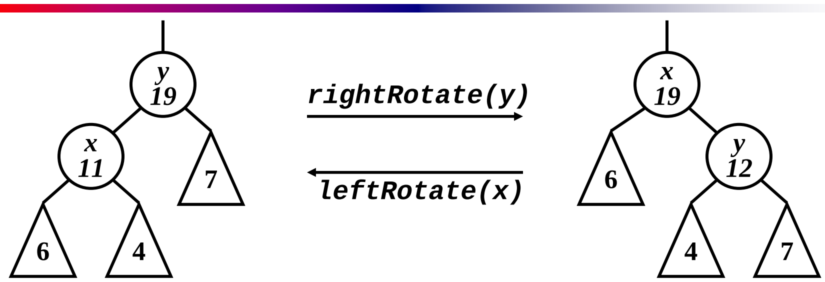
- Next step: maintain sizes during Insert() and Delete() operations
  - *How would we adjust the size fields during insertion on a plain binary search tree?*
  - A: increment sizes of nodes traversed during search

  - *Why won't this work on red-black trees?*

Review: Maintaining Subtree Sizes

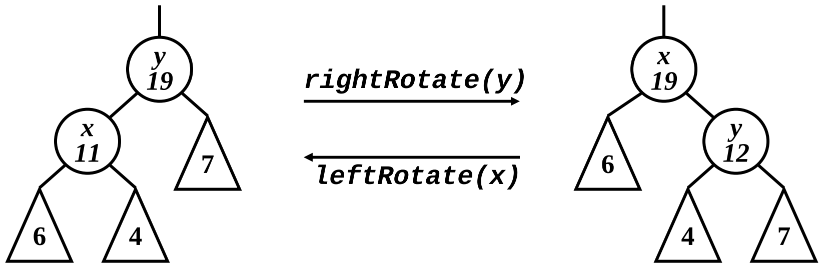
- So by keeping subtree sizes, order statistic operations can be done in O(lg n) time
- Next: maintain sizes during Insert() and Delete() operations
  - Insert(): Increment size fields of nodes traversed during search down the tree
  - Delete(): Decrement sizes along a path from the deleted node to the root
  - Both: Update sizes correctly during rotations

Maintaining Size Through Rotation



- Salient point: rotation invalidates only *x* and *y*
- Can recalculate their sizes in constant time
  - *Why?*

Reivew: Maintaining Subtree Sizes



- Note that rotation invalidates only *x* and *y*
- Can recalculate their sizes in constant time
- Thm 15.1: can compute any property in O(lg n) time that depends only on node, left child, and right child

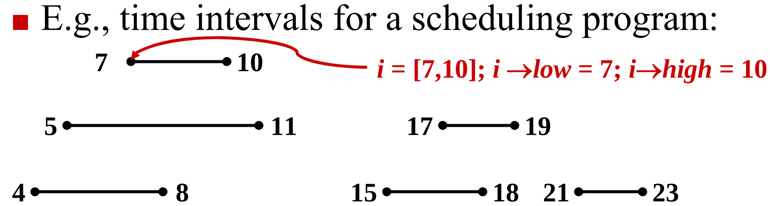
Augmenting Data Structures: Methodology

- Choose underlying data structure
  - E.g., red-black trees
- Determine additional information to maintain
  - E.g., subtree sizes
- Verify that information can be maintained for operations that modify the structure
  - E.g., Insert(), Delete()   (don't forget rotations!)
- Develop new operations
  - E.g., OS-Rank(), OS-Select()

Interval Trees

Interval Trees

- The problem: maintain a set of intervals
  - E.g., time intervals for a scheduling program:



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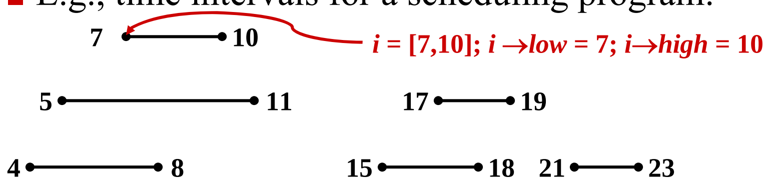

  - Query: find an interval in the set that overlaps a given query interval
    - [14,16] → [15,18]
    - [16,19] → [15,18] or [17,19]
    - [12,14] → NULL

Interval Trees

- Following the methodology:
  - Pick underlying data structure
  - Decide what additional information to store
  - Figure out how to maintain the information
  - Develop the desired new operations

Review: Interval Trees

- Following the methodology:
  - Pick underlying data structure
    - Red-black trees will store intervals, keyed on $i \rightarrow low$
  - Decide what additional information to store
    - Store the maximum endpoint in the subtree rooted at i
  - Figure out how to maintain the information
    - Update max as traverse down during insert
    - Recalculate max after delete with a traversal up the tree
    - Update during rotations
  - Develop the desired new operations

Interval Trees

- Following the methodology:
  - *Pick underlying data structure*
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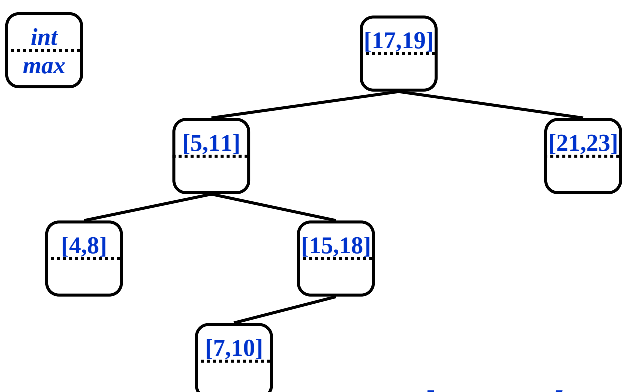
Interval Trees

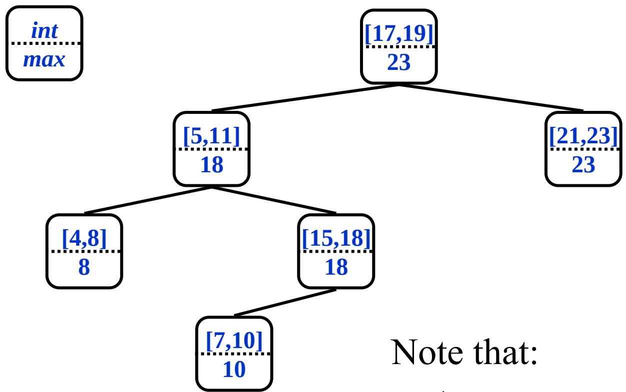
- Following the methodology:
  - Pick underlying data structure
    - Red-black trees will store intervals, keyed on $i \rightarrow low$
  - *Decide what additional information to store*
    - We will store *max*, the maximum endpoint in the subtree rooted at *i*
  - Figure out how to maintain the information
  - Develop the desired new operations

Interval Trees



What are the max fields?

Interval Trees



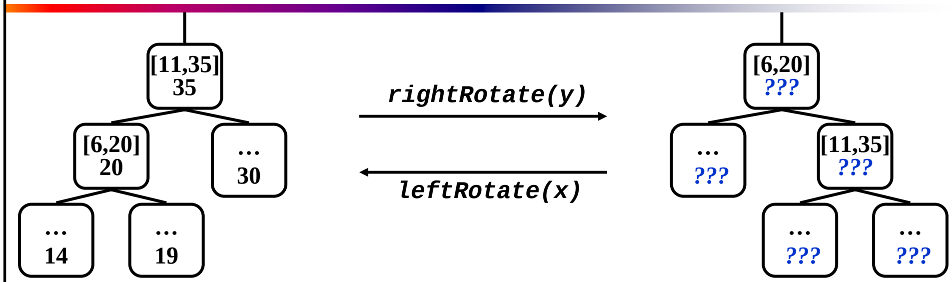
Note that:

$$x \to \max = \max \begin{cases} x \to high \\ x \to left \to \max \\ x \to right \to \max \end{cases}$$

Interval Trees

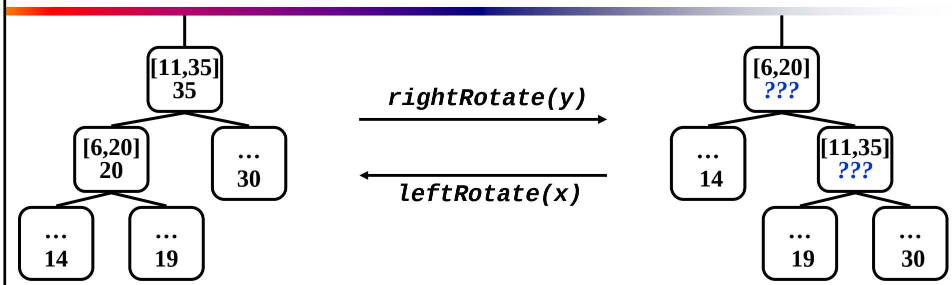
- Following the methodology:
  - Pick underlying data structure
    - Red-black trees will store intervals, keyed on $i \to low$
  - Decide what additional information to store
    - Store the maximum endpoint in the subtree rooted at i
  - *Figure out how to maintain the information*
    - *How would we maintain max field for a BST?*
    - *What's different?*
  - Develop the desired new operations

Interval Trees



- *What are the new max values for the subtrees?*

Interval Trees



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- *What are the new max values for x and y?*

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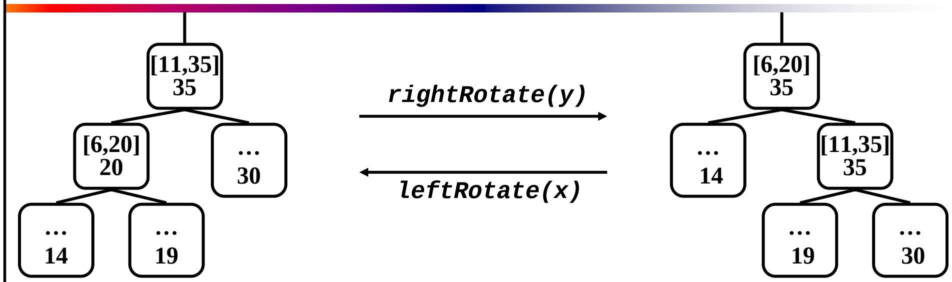


- *What are the new max values for the subtrees?*
- A: Unchanged
- *What are the new max values for x and y?*
- A: root value unchanged, recompute other

Interval Trees

- Following the methodology:
  - Pick underlying data structure
    - Red-black trees will store intervals, keyed on $i \rightarrow low$
  - Decide what additional information to store
    - Store the maximum endpoint in the subtree rooted at i
  - Figure out how to maintain the information
    - Insert: update max on way down, during rotations
    - Delete: similar
  - *Develop the desired new operations*

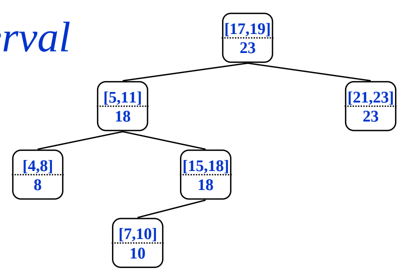
Searching Interval Trees

```
IntervalSearch(T, i)
{
    x = T->root;
    while (x != NULL && !overlap(i, x->interval))
        if (x->left != NULL && x->left->max ≥ i->low)
            x = x->left;
        else
            x = x->right;
    return x
}
```

- *What will be the running time?*

IntervalSearch() Example

- *Example: search for interval overlapping [14,16]*

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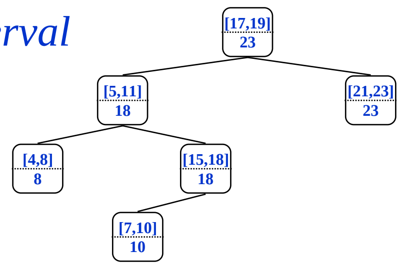
IntervalSearch() Example

- *Example: search for interval overlapping [12,14]*

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        else
            x = x->right;
    return x
}
```

Correctness of IntervalSearch()

- Key idea: need to check only 1 of node's 2 children
  - Case 1: search goes right
    - Show that ∃ overlap in right subtree, or no overlap at all
  - Case 2: search goes left
    - Show that ∃ overlap in left subtree, or no overlap at all

Correctness of IntervalSearch()

- Case 1: if search goes right, ∃ overlap in the right subtree or no overlap in either subtree
  - If ∃ overlap in right subtree, we're done
  - Otherwise:
    - x→left = NULL, or x → left → max < x → low (*Why?*)
    - Thus, no overlap in left subtree!

```
while (x != NULL && !overlap(i, x->interval))
        if (x->left != NULL && x->left->max ≥ i->low)
            x = x->left;
        else
            x = x->right;
    return x;
```

Correctness of IntervalSearch()

- Case 2: if search goes left, ∃ overlap in the left subtree or no overlap in either subtree
  - If ∃ overlap in left subtree, we're done
  - Otherwise:
    - i →low ≤ x →left →max, by branch condition
    - x →left →max = y →high for some y in left subtree
    - Since i and y don't overlap and i →low ≤ y →high, i →high < y →low
    - Since tree is sorted by low's, i →high < any low in right subtree
    - Thus, no overlap in right subtree

```
while (x != NULL && !overlap(i, x->interval))
        if (x->left != NULL && x->left->max ≥ i->low)
            x = x->left;
        else
            x = x->right;
    return x;
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