FFT or
Fast Fourier Transform

- **FFT**

$$A(x) = a_0 + a_1 x + \ldots.. + a_{n-1} x^{n-1}$$
$$B(x) = b_0 + b_1 x + \ldots.. + b_{n-1} x^{n-1}$$
$$C(x) = A(x)B(x)$$
$$= c_0 + c_1 x + c_2 x^2 + \ldots.. + c_{2n-3} x^{2n-3} + c_{2n-2} x^{2n-2}$$
$$c_j = \sum_{k=0}^{j} a_k b_{j-k}$$

•Coefficient representation:

How to evaluate $A(x_0)$?

p2.

- Horner's rule:

$$A(x_0) = a_0 + x_0(a_1 + x_0(a_2 + \ldots.. + x_0(a_{n-2} + x_0 a_{n-1})\ldots))$$

$\theta(n)$

- Point-value representation:

A point-value representation of a polynomial A(x) of degree-bound n is a set of n point-value pairs $\{(x_0, y_0), (x_1, y_1), \ldots..(x_{n-1}, y_{n-1})\}$, where $y_k = A(x_k)$

$$A: \{(x_0, y_0), (x_1, y_1), \ldots..(x_{2n-1}, y_{2n-1})\}$$
$$B: \{(x_0, y_0'), (x_1, y_1'), \ldots..(x_{2n-1}, y_{2n-1}')\}$$
$$C: \{(x_0, y_0 y_0'), (x_1, y_1 y_1'), \ldots..(x_{2n-1}, y_{2n-1} y_{2n-1}')\}$$

- **Thm1**

For any set $\{(x_0, y_0), (x_1, y_1), \ldots..(x_{n-1}, y_{n-1})\}$ of n point-value pairs, there is a unique poly A(x) of degree $\leq$ n-1, such that $y_k = A(x_k)$ for k=0, 1,...., n-1

Pf:

$$\text{if } X = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{pmatrix} \text{ then}$$

$$X \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

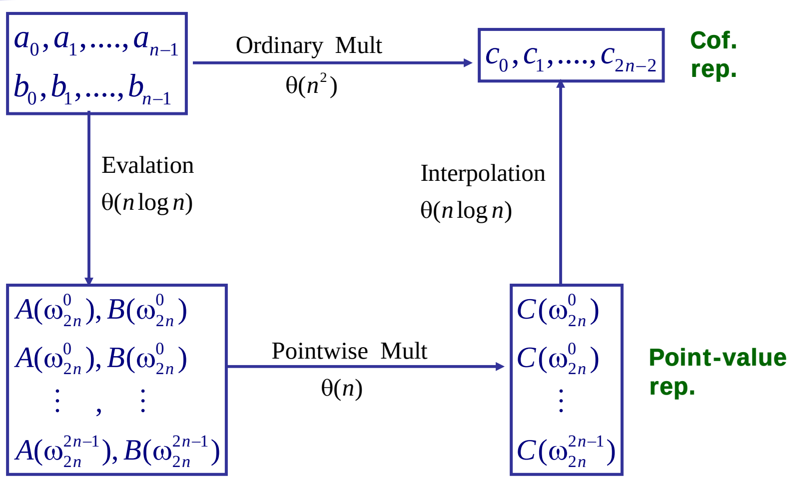
$$\det(X)=\prod_{j<k}(x_k-x_j)\neq 0 \text{ if } x_k\text{'s are distinct}$$

Thus, a_j's can be solved uniquely.

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = X^{-1} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

- n-point interpolation:
  Lagrange's formula:

$$A(x)=\sum_{k=0}^{n-1} y_k \frac{\prod_{j\neq k}(x-x_j)}{\prod_{j\neq k}(x_k-x_j)}$$



- **Thm2**
  The product of 2 polynomials of deg-bound n can be computed in time θ(nlog n), with both the input and output representation in coefficient form

  Complex roots of unity:

  $\omega^n = 1$, $e^{2\pi ik/n}$ for k=0, 1, ..., n-1 $e^{iu} = \cos u + i \sin u$

  $\omega_n = e^{2\pi i/n}$, the principal n-th root of unity

  $\omega_n^0, \omega_n^1, \omega_n^2, \ldots, \omega_n^{n-1}$

  $\omega_n^n = ?$, $\omega_n^0 = 1$

  $\omega_n^j \omega_n^k = \omega_n^{(j+k) \bmod n}$

  $\omega_n^{-1} = \omega_n^{n-1}$

p7.

- **Lemma 3 (Cancellation Lemma)**
  n, k, d: non-negative integers, $\omega_{dn}^{dk} = \omega_n^k$

Pf:

$$\omega_{dn}^{dk} = (e^{2\pi i/dn})^{dk} = (e^{2\pi i/n})^k = \omega_n^k$$

- **Cor. 4 n: even positive integer**

$$\omega_n^{n/2} = \omega_2 = -1$$

p8.

- **Lemma 5 (Halving lemma)**

  n: even positive integer

  The squares of the n complex n-th roots of unity are n/2 complex (n/2)th roots of unity.

  Pf:

  $(\omega_n^k)^2 = \omega_n^{2k} = \omega_{n/2,}^k$ where $k \in Z^+ \cup \{0\}$

  $(\omega_n^{k+n/2})^2 = \omega_n^{2k+n} = \omega_n^{2k} = \omega_{n/2}^k$

  $\Rightarrow \omega_n^k$ and $\omega_n^{k+n/2}$ have the same squre

- **Lemma 6 (Summation lemma)**

  $$n \in Z^+,\ k \in Z^+ \cup \{0\},\ n \nmid k,\ \sum_{j=0}^{n-1} (\omega_n^k)^j = 0$$

  Pf:

  $$\sum_{j=0}^{n-1} (\omega_n^k)^j = \frac{(\omega_n^k)^n - 1}{\omega_n^k - 1} = \frac{(\omega_n^n)^k - 1}{\omega_n^k - 1} = 0$$

p9.

- **DFT**

  Evaluate $A(x) = \sum_{j=0}^{n-1} a_j x^j$ *at* $\omega_n^0, \omega_n^1, \ldots\ldots, \omega_n^{n-1}$,

  Assume n is a power of 2

  Let $a = \langle a_0, a_1, \ldots., a_{n-1} \rangle$, and $y_k = A(\omega_n^k) = \sum_{j=0}^{n-1} a_j \omega_n^{kj}$

  $y = \langle y_0, y_1, \ldots., y_{n-1} \rangle$ is the DFT of the coefficient vector $\mathrm{a} = \langle \mathrm{a}_0, \mathrm{a}_1, \ldots., \mathrm{a}_{n-1} \rangle$,

  $y = \mathrm{DFT}_n(a)$

p10.

- Interpolation at the complex roots of unity:

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix}$$

$$y = V_n a, \quad (V_n)_{k,j} = \omega_n^{kj}$$

$$a = V_n^{-1} y$$

p11.

- **Thm 7**

$$j,k = 0,1,\ldots\ldots,n-1, \quad (V_n^{-1})_{j,k} = \omega_n^{-kj} / n$$

Pf:

$$V_n^{-1}V_n = I_n, \quad (V_n^{-1}V_n)_{j,j'} = \sum_{k=0}^{n-1} (V_n^{-1})_{j,k} (V_n)_{k,j'}$$

$$= \sum_{k=0}^{n-1} \frac{1}{n} \omega_n^{-kj} \omega_n^{kj'}$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \omega_n^{k(j'-j)} \quad \text{-------}(*)$$

if j=j', then (*)=1, if $j \neq j'$, then by lemma 6, (*)=0

$\because$ -(n-1) < j'-j <n-1, and $n \nmid (j'-j)$

p12.

- **FFT**

$$A^{[0]}(x) = a_0 + a_2 x + a_4 x^2 + \ldots.. + a_{n-2} x^{n/2-1}$$

$$A^{[1]}(x) = a_1 + a_3 x + a_5 x^2 + \ldots.. + a_{n-1} x^{n/2-1}$$

$$(*) \quad A(x) = A^{[0]}(x^2) + x A^{[1]}(x^2)$$

Thus evaluating A(x) at $\omega_n^0, \omega_n^1, \ldots.., \omega_n^{n-1}$ reduce to

1. evaluating $A^{[0]}(x)$ and $A^{[1]}(x)$ at

$$(\omega_n^0)^2, (\omega_n^1)^2, \ldots.., (\omega_n^{n-1})^2$$

2. combining the results according to (*)

p13.

Let $\begin{cases} y_k^{[0]} = A^{[0]}(\omega_{n/2}^k) \\ y_k^{[1]} = A^{[1]}(\omega_{n/2}^k) \end{cases}$

$$\begin{aligned} y_k = A(\omega_n^k) &= A^{[0]}(\omega_n^{2k}) + \omega_n^{2k} A^{[1]}(\omega_n^{2k+n}) \\ &= A^{[0]}(\omega_{n/2}^k) - \omega_n^k A^{[1]}(\omega_{n/2}^k) \\ &= y_k^{[0]} + \omega_n^k y_k^{[1]} \end{aligned}$$

$$\begin{aligned} y_{k+n/2} = A(\omega_n^{k+n/2}) &= A^{[0]}(\omega_n^{2k+n}) + \omega_n^{k+n/2} A^{[1]}(\omega_n^{2k}) \\ &= A^{[0]}(\omega_{n/2}^k) + \omega_n^{k+n/2} A^{[1]}(\omega_{n/2}^k) \\ &= y_k^{[0]} + \omega_n^{k+n/2} y_k^{[1]} = y_k^{[0]} - \omega_n^k y_k^{[1]} \end{aligned}$$

p14.

```
Recursive-FFT(a)
{  n=length[a];    /* n: power of 2  */
   if n=1 the return a;
   ω_n = e^{2πi/n};
   ω=1
   a^[0] = (a_0, a_2, ....., a_{n-2});
   a^[1] = (a_1, a_3, ....., a_{n-1});
   y^[0] = Recursive-FFT(a^[0]);
   y^[1] = Recursive-FFT(a^[1]);
   for  k=0  to  (n/2 - 1)  do
   {
        y_k = y_k^[0] + ω y_k^[1];
        y_{k+n/2} = y_k^[0] - ω y_k^[1];
        ω=ωω_n;                    }
}
```

$T(n) = 2T(n/2) + \theta(n)$

$=\theta(n\log n)$

- **Thm 8 (Convolution thm)**

  n: power of 2

  a,b: vectors of length n

  $a \otimes b = \mathrm{DFT}^{-1}(\mathrm{DFT}_{2n}(a) \bullet \mathrm{DFT}_{2n}(b))$

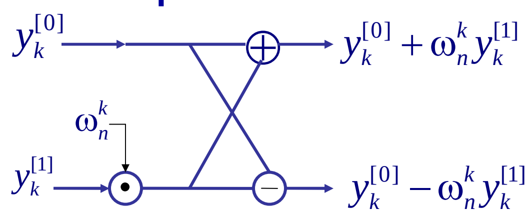
  Componentwise product

  $a \otimes b =< a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_0, ... >$

  j-th elt: $\sum_{k=0}^{j} a_k b_{j-k}$

p16.

- **Efficient FFT implement**



$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)$

(a_0, a_2, a_4, a_6) $\quad$ (a_1, a_3, a_5, a_7)

(a_0, a_4) $\quad$ (a_2, a_6) $\quad$ (a_1, a_5) $\quad$ (a_3, a_7)

a_0 $\quad$ a_4 $\quad$ a_2 $\quad$ a_6 $\quad$ a_1 $\quad$ a_5 $\quad$ a_3 $\quad$ a_7

000 100 010 110 001 101 011 111

Idea :

for s=1 to (lg n) do
  for k=0 to n-1 by 2^s
    do combine the two 2^{s-1} – element DFT's in $y^{[1]}$
    $y^{[0]}$ A[k..k+2^{s-1} –1] and A[k+2^{s-1}..k+2^{s-1} –1]
      into one 2^s – element DFT in A[k..k+2^s –1]

```
FFT-Base(a)
{  n = length[a];
   for s=1 to (lg n) do
   {  m=2^s;
      ω_m = e^(2πi/m);
      for k=0 to n-1 by m do
      {  ω=1;
         for j=0 to (m/2 - 1) do
         {  t=ωA[k + j + m/2];
            u=A[k+j];
            A[k+j]=u+t;
            A[k+j+m/2]=u-t;
            ω = ωω_m;            }
      }   }   }
```

Iterative-FFT(a)

{ Bit-Reverse-Copy(a, A);

n=length[a];

for s=1 to (lg n) do

{ m=2^s; $\omega_m = e^{2\pi i/m}$; $\omega = 1$;

for j=0 to (m/2 -1) do

{ for k=j to n-1 by m do

{ t = $\omega A[k+m/2]$;

u = A[k];

A[k]=u+t;

A[k+m/2]=u-t; }

$\omega = \omega\omega_m$

}

}

}

Bit-Reverse-Copy(a,A)

{ n=length[a];

for k=0 to n-1 do

A[rev(k)] = a_k

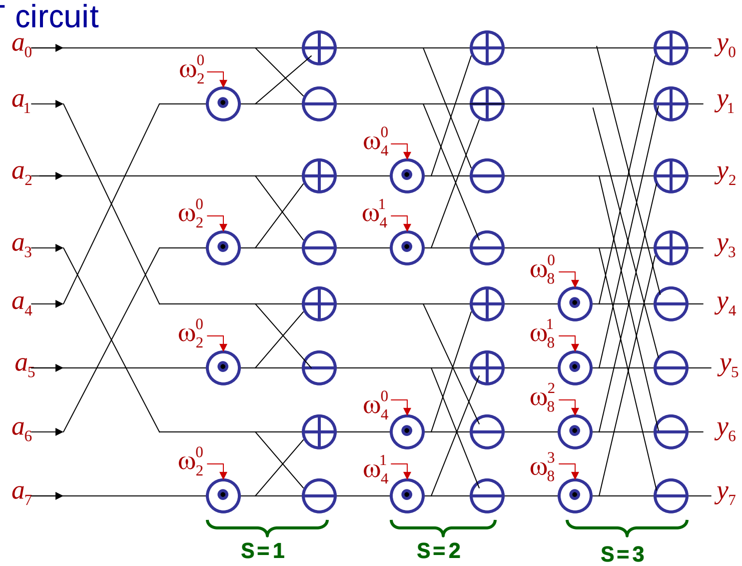
}

eg

rev(011)=110, rev(001)=100

p19.

FFT circuit



$n = 8$

In general, depth: θ(lg n)

size : θ(nlg n)