RECURRENCE EQUATIONS – ANALYSIS.

MERGE & SORT ALGORITHM. (Ex of Divide & Conquer Algorithm)

EXAMPLE: 2 9 17 1 25 16 18 5

Sort the above in increasing order.

SOLUTION:

2 9 17 1 || 25 16 18 5

2 9 | 17 1 || 25 16 | 18 5

$\underbrace{2 \quad 9}_{\text{sort}}$ $\underbrace{17 \quad 1}_{\text{sort}}$

2 9 || 1 17 || 16 25 | 5 18

Merge. MERGE

1 2 9 17 || 5 16 18 25

$\underbrace{1 \quad 2 \quad 9 \quad 17}_{\text{sorted.}}$ $\underbrace{5 \quad 16 \quad 18 \quad 25}_{\text{sorted.}}$

$\underbrace{1 \quad 2 \quad 5 \quad 9 \quad || \quad 16 \quad 17 \quad 18 \quad 25}_{\text{sorted.}}$

Suppose n = #s do be sorted.

$T(n)$ = # of steps required to sort n #s.

$$T(n) = \underbrace{T\left(\frac{n}{2}\right)}_{\text{sort } 1/2} + \underbrace{T\left(\frac{n}{2}\right)}_{\text{sort } 1/2} + \underbrace{cn}_{\text{merge}} \quad ; \quad c = \text{constant}, \; c > 0.$$

$$T(1) = c_1.$$

SUMMARY:

$$\left.\begin{aligned} T(1) &= c. \\ T(n) &= 2T\left(\frac{n}{2}\right) + cn. \quad n > 1 \end{aligned}\right\} \text{recursive equation.}$$

ANALYSIS: Let $n = 2^m$; $m = 1, 2, 3 \ldots$

$$\boxed{\begin{array}{l} n = 2^m \\ \log_{2} n = m \end{array}}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + cn$$

$$= 2\left(2T\left(\frac{n}{2^2}\right) + c\frac{n}{2}\right) + cn$$

recursive

$$= 2^2 T\left(\frac{n}{2^2}\right) + cn + cn$$

$$= 2^2 T\left(\frac{n}{2^2}\right) + 2cn \longrightarrow$$

$$= 2^2\left[2T\left(\frac{n}{2^3}\right) + c\frac{n}{2^2}\right] + 2cn$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + 3cn \longrightarrow$$

$$= 2^4 T\left(\frac{n}{2^4}\right) + 4cn$$

$$\vdots$$

$$= 2^m T\left(\frac{n}{2^m}\right) + mcn$$

$$= n\,T(1) + mcn$$

$$= nc + mcn = cn(m+1)$$

$$= cn(\log_2 n + 1) = \underset{\omega}{\Omega}(n \log n)$$

$$T(n) = \Theta(n \log n) = O(n \log n)$$

EXAMPLE: $T(n) = 5T\left(\frac{n}{4}\right) + n$

Solution: $T(n) = 5T(n/4) + n$

$$T\left(\frac{n}{4}\right) = 5T\left(\frac{n}{4} \cdot \frac{1}{4}\right) + \frac{n}{4}$$

$$T(n) = 5\left(5T\left(\frac{n}{4^2}\right) + \frac{n}{4}\right) + n.$$

$$= 5^2\, T\left(\frac{n}{4^2}\right) + \frac{5}{4}n. + n.$$

$$T\left(\frac{n}{4^2}\right) = 5T\left(\frac{n}{4^3}\right) + \frac{n}{4^2}$$

$$\Rightarrow T(n) = 5^2\left(5T\left(\frac{n}{4^3}\right) + \frac{n}{4^2}\right) + \frac{5}{4}n + n.$$

$$= 5^3\, T\left(\frac{n}{4^3}\right) + \frac{5^2}{4^2}n. + \frac{5}{4}n + n.$$

$$\vdots$$

$$= 5^r T\left(\frac{n}{4^r}\right) + n\left(\left(\frac{5}{4}\right)^{r-1} + \left(\frac{5}{4}\right)^{r-2} + \dots + 1\right).$$

Let $n = 4^r$, $\log_4 n = r$

$$= 5^r T(1) + n.\frac{\left(\frac{5}{4}\right)^r - 1}{\frac{5}{4} - 1}.$$

$T(1) = c$

$$= 5^r c + n\left(\left(\frac{5}{4}\right)^r - 1\right)4.$$

$$= 5^r c + 4^r\left[\left(\frac{5}{4}\right)^r - 1\right]4.$$

$$= 5^r c + (5^r - 4^r)4. \qquad (\log_4 5 > 1).$$

$5^r = 5^{\log_4 n}$
$= n^{\log_4 5}$

$$= 5^r(c+4) - 4^r.4.$$

const.

$$= n^{\log_4 5}(c+4) - 4n.$$

$$= \Theta\left(n^{\log_4 5}\right).$$

EXAMPLE $T(n) = 7T\left(\frac{n}{7}\right) + n.$

Solution:

$$T\left(\frac{n}{7}\right) = 7T\left(\frac{n}{7^2}\right) + \frac{n}{7}$$

$$T(n) = 7\left(7T\left(\frac{n}{7^2}\right) + \frac{n}{7}\right) + n.$$

$$= 7^2 T\left(\frac{n}{7^2}\right) + n + n.$$

$$= 7^2 T\left(\frac{n}{7^2}\right) + 2n.$$

$$T\left(\frac{n}{7^2}\right) = 7T\left(\frac{n}{7^3}\right) + \frac{n}{7^2}$$

$$T(n) = 7^2\left(7T\left(\frac{n}{7^3}\right) + \frac{n}{7^2}\right) + 2n.$$

$$= 7^3 T\left(\frac{n}{7^3}\right) + 3n.$$

$$= 7^r T\left(\frac{n}{7^r}\right) + rn.$$

$n = 7^r$

$T(1) = c.$

$\log_7 n = r$

$$= n\,T(1) + rn.$$

$$= cn + rn = n(r + c)$$

$$= n.(\log_7 n + c) \quad \text{ignore "c"}$$

$$= \Theta(n \log_7 n) = \Theta(n \log n)$$

ignore 7.

can be big OH

EXAMPLE p71. DAS GUPTA, PAPADIMITROV & VAZIRAH

2.5) a) $T(n) = 2T\left(\frac{n}{3}\right) + 1 \quad ; \quad T(1) = c$

Solution:

$$T(n) = 2T\left(\frac{n}{3}\right) + 1$$

$$= 2\left(2T\left(\frac{n}{3^2}\right) + 1\right) + 1$$

$$= 2^2\, T\left(\frac{n}{3^2}\right) + 2 + 1$$

$$= 2^2\left(2T\left(\frac{n}{3^3}\right) + 1\right) + 2 + 1$$

$$= 2^3\, T\left(\frac{n}{3^3}\right) + 4 + 2 + 1$$

$$= 2^r\, T\left(\frac{n}{3^r}\right) + 2^{r-1} + 2^{r-2} + \dots + 2 + 1$$

Let $n = 3^r$

$$= 2^r T(1) + \frac{2^r - 1}{2 - 1}$$

$$= 2^r c + (2^r - 1)$$

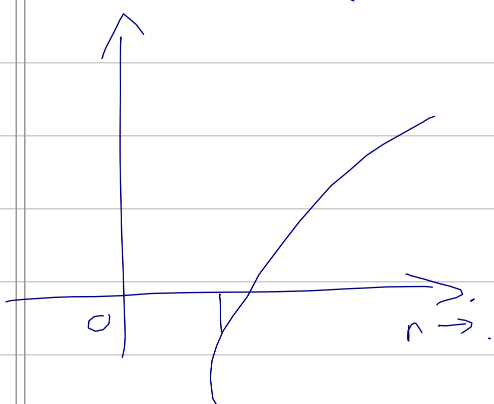
for large n, then r is large.

$$T(n) = 2^r(c+1) = 2^r c_1 \quad ; \quad c_1 = c + 1$$

$$= \Theta(2^r)$$

$$= \Theta\left(n^{\log_3 2}\right)$$

$$\left(\begin{aligned} n &= 3^r \\ \log_3 n &= r \\ 2^r &= 2^{\log_3 n} \\ &= n^{\log_3 2} \end{aligned}\right)$$



EXAMPLE: $T(n) = 9T\left(\frac{n}{3}\right) + n^2$

Solution:

$$T(n) = 9T\left(\frac{n}{3}\right) + n^2$$

$$T\left(\frac{n}{3}\right) = 9T\left(\frac{n}{3^2}\right) + \frac{n^2}{3^2}$$

$$T(n) = 9\left(9T\left(\frac{n}{3^2}\right) + \left(\frac{n}{3}\right)^2\right) + n^2$$

$$= 9^2 T\left(\frac{n}{3^2}\right) + n^2 + n^2$$

$$= 9^2 T\left(\frac{n}{3^2}\right) + 2n^2$$

$$T\left(\frac{n}{3^2}\right) = 9T\left(\frac{n}{3^3}\right) + \left(\frac{n}{3^2}\right)^2$$

$$= 9T\left(\frac{n}{3^3}\right) + \frac{n^2}{3^4}$$

$$T(n) = 9^2\left[9T\left(\frac{n}{3^3}\right) + \frac{n^2}{3^4}\right] + 2n^2$$

$$= 9^3 T\left(\frac{n}{3^3}\right) + 3n^2$$

$$\vdots$$

$$= 9^r T\left(\frac{n}{3^r}\right) + rn^2$$

let $n = 3^r$, $9^r = 3^{2r} = n^2$

$T(1) = c$

$r = 3^r \Rightarrow \log_3 n = r$

$$T(n) = n^2 T(1) + rn^2$$

$$= n^2 c + rn^2 = n^2(r + c)$$

$$= n^2(\log_3 n + c) = \Theta(n^2 \log_3 n)$$

$$= \Theta(n^2 \log n)$$

EXAMPLE

Solution.

$$T(n) = 8T\left(\frac{n}{2}\right) + n^3.$$

$$T(n) = 8T\left(\frac{n}{2}\right) + n^3.$$

$$= 8\left[8T\left(\frac{n}{2^2}\right) + \frac{n^3}{2^3}\right] + n^3.$$

$$= 8^2 T\left(\frac{n}{2^2}\right) + 2n^3$$

$$= 8^2\left[8T\left(\frac{n}{2^3}\right) + \frac{n^3}{2^6}\right] + 2n^3.$$

$$= 8^3 T\left(\frac{n}{2^3}\right) + 3n^3$$

$$\vdots$$

$$= 8^r T\left(\frac{n}{2^r}\right) + rn^3.$$

let $n = 2^r$, $r = \log_2 n$, $8^r = 2^{3r} = n^3$

$$T(n) = n^3 T(1) + rn^3.$$

$$= n^3 c + n^3 r = n^3(r + c).$$

$$= n^3(\log_2 n + c)$$

$$= n^3(\log n)$$

$$= \Theta(n^3 \log n).$$

(ignore c, assume $\log_2 n = \log n$)

Ex: $T(n) = T(n-1) + 2$.

Solution: $T(n) = T(n-1) + 2$.

$= (T(n-2) + 2) + 2$.

$= T(n-2) + 2 + 2$.

$= (T(n-3) + 2) + 2 + 2$.

$= T(n-3) + 3 \times 2$.

$\vdots$

$= T(n-r) + r \times 2$.

let $n - r = 1$, $T(1) = c$

$T(n) = T(1) + 2(n-1)$

$= c + 2(n-1)$

$= 2n + c - 1$

$= \Theta(n)$. ← strictly it should be ω, but can write Θ or O.

EXAMPLE: $T(n) = T(\sqrt{n}) + 1$

Sol: $T(n) = T(\sqrt{n}) + 1$

$= T(n^{1/2}) + 1$.

$= T(n^{1/4}) + 2$

$= T(n^{1/2^2}) + 2$

$= T(n^{\frac{1}{2^3}}) + 3$

$\vdots$

$= \underbrace{T\left(n^{\frac{1}{2^r}}\right)}_{\text{const}} + r = \Theta(r)$.

$n^{\frac{1}{2^r}} = a \Rightarrow n = a^{2^r}$ $\quad = \Theta(\log \log n)$

$2^r = \log_a n$.

$r = \log_2 \log_a n$.

EXAMPLE $T(n) = T(n-1) + n^c \qquad c >> 1.$

Sol: $T(n) = T(n-1) + n^c$

$= T(n-2) + (n-1)^c + n^c$

$= T(n-3) + (n-2)^c + (n-1)^c + n^c$

$\vdots$

$= T(n-r) + (n-r+1)^c + (n-r+2)^c + \dots + n^c$

let $n-r=1$, $T(1)=k$

$T(n) = T(1) + 2^c + 3^c + \dots + n^c$

$= k + 2^c + 3^c + \dots + n^c$

$$\underbrace{1+2+3\dots}\ = \frac{n(n+1)}{2}$$

Sum proportional to n^2.

$\approx k + 1^c + 2^c + \dots + n^c = \Theta(n^{c+1})$

$S_c = 1^c + 2^c + \dots + n^c$; $S_1 = 1+2+\dots+n = \frac{n(n+1)}{2} \propto n^2$

$S_2 = 1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6} \propto n^3$

$S_3 = 1^3+2^3+\dots+n^3 = S_1^2 = \left[\frac{n(n+1)}{2}\right]^2 \propto n^4$

$S_c \propto n^{c+1}$

$1^c + 2^c + \dots + n^c = \sum_{j=1}^{n} j^c \approx \int_1^n x^c\,dx = \frac{x^{c+1}}{c+1}\Big|_1^n = \frac{1}{c+1}\left[n^{c+1} - 1\right]$

$\approx \frac{1}{c+1} n^{c+1} \propto n^{c+1}$

THEOREM. $a = 1, 2, 3 \dots$

$b = 2, 3, 4 \dots$

$n = 1, 2, 3, \dots$, $c > 0$.

Ex 11: $f(n) = \sqrt{n}$; $g(n) = (\log n)^3$.

Sol: $f(n) = n^{1/2}$; $g(n) = (\log n)^3$.

$$L = \lim \frac{f(n)}{g(n)} = \frac{n^{1/2}}{(\log n)^3} \to \infty$$

$\therefore f(n) = \omega(g(n))$.

$\log n \leq n$.
for large n.