Design and Analysis of Algorithms
Lecture-4: Heapsort

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Overview

- Sorting overview
- Heaps
- Heapify
- BuildHeap
- Heapsort
- Priority Queues

- Part of the slides are based on material from Prof. Jianhua Ruan, The University of Texas at San Antonio

Why Sorting?

- It comes up all the time
- It can bottlenose an app in terms of either time or space
- There’s a lot of sorting algorithms – rich & interesting as a problem
- We can prove a lower bounds!

Analyzing sorts

- Running time
- Space – is it in-place?
- Stable: do equal elements get shifted?

Insertion Sort

- Main idea: Insert each element into its proper place in sorted order.
  - A[0] is sorted by itself
  - Then consider A[1]. Swap with first element if necessary.
  - Then consider A[2]. Put into place with first 2.
  - Etc…
  - After ith pass, first i elements are sorted. They are the same first i elements from the unsorted set.

Insertion Sort

- $O(n^2)$ sort
- Stable
- Is especially bad if elements are in reverse sorted order
- Already sorted order?

Selection Sort

- Main idea: find the smallest, then the next smallest, etc.
- When you find smallest, swap it with A[o]

- After ith pass, A[0] – A[i] has the correct sorted elements.

Selection Sort

- $O(n^2)$
- Stable
- Does it perform better/worse for already sorted input? Reverse-sorted input?

Bubble Sort

- Main idea: Go through each element, and if it's out of order with its neighbor, swap them.
- If you can go through entire array with no swaps, you're done
- After ith pass, at least last i positions are sorted and in final correct order.

Bubble Sort

- $O(n^2)$
- Worst case time has a big constant
- What if already sorted?
- "In short, the bubble sort seems to have nothing to recommend it, except a catchy name and the fact that it leads to some interesting theoretical problems."

  Don Knuth*, The Art of Computer Programming: Vol. 3, Sorting and Searching*

Merge Sort

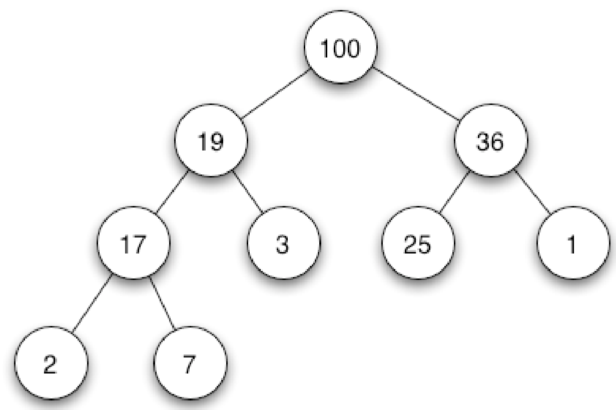
- Divide-and-conquer
- If there is only 0 or 1 element, it’s done
- Otherwise, recursively sort ½ the set
- Then merge them together (O(n))

Merge Sort

- O(nlgn)
- No real best or worst case
- Can be made to be in-place

Heap Sort

- Heap: A full binary tree maintained so that the biggest element is always the root.
- What is the length of a path from root to leaf?



Heap Sort

- Main idea: Create a heap out of the elements.
- Inserting each element takes time lgn
- Delete biggest and re-heapify (also lgn)
- Delete biggest and re-heapify for all n nodes = O(nlgn)

Heap Sort

- O(nlgn)
- In-place
- Quite a bit of shuffling memory

Quicksort

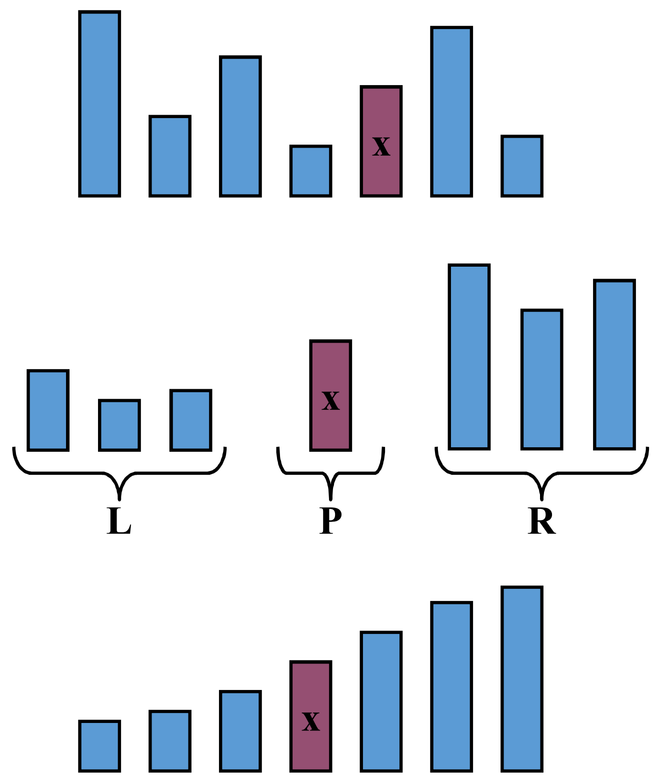
- Main idea:
  - Find a Pivot element
  - Split array into elements less than pivot, equal to pivot, and greater than pivot, called partitioning
  - Recursively sort the pieces

Divide and Conquer

1. Pick a pivot element

2. Put everything <pivot on the left and everything > pivot on right.

3. Sort the left and right

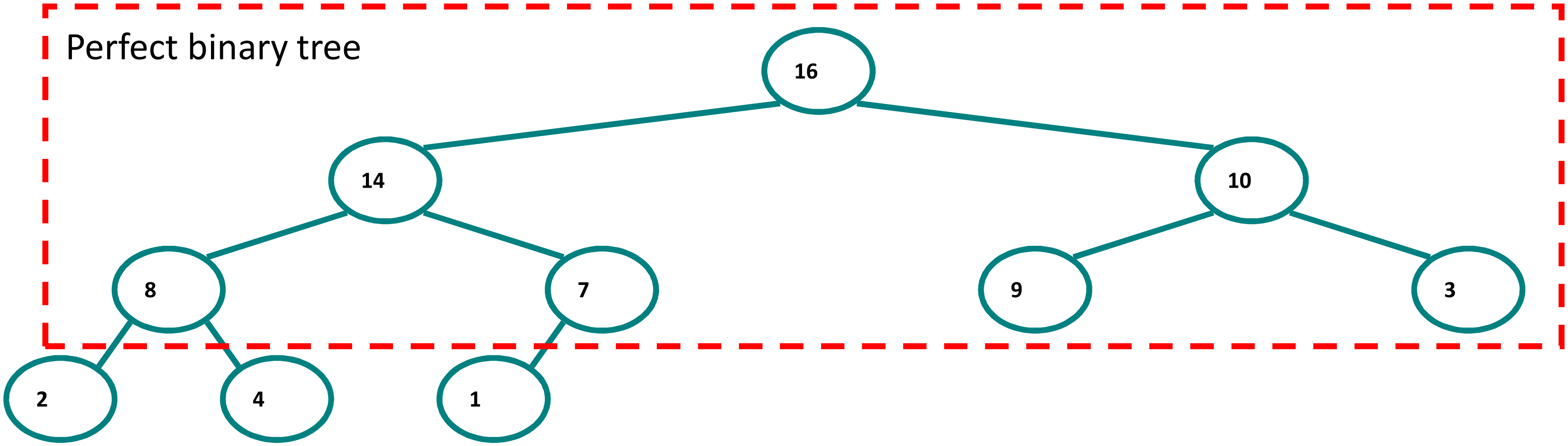


Heap

- A heap is a data structure that allows you to quickly retrieve the largest (or smallest) element
- It takes time $\Theta(n)$ to build the heap
- If you need to retrieve largest element, second largest, third largest…, in long run the time taken for building heaps will be rewarded

Heaps

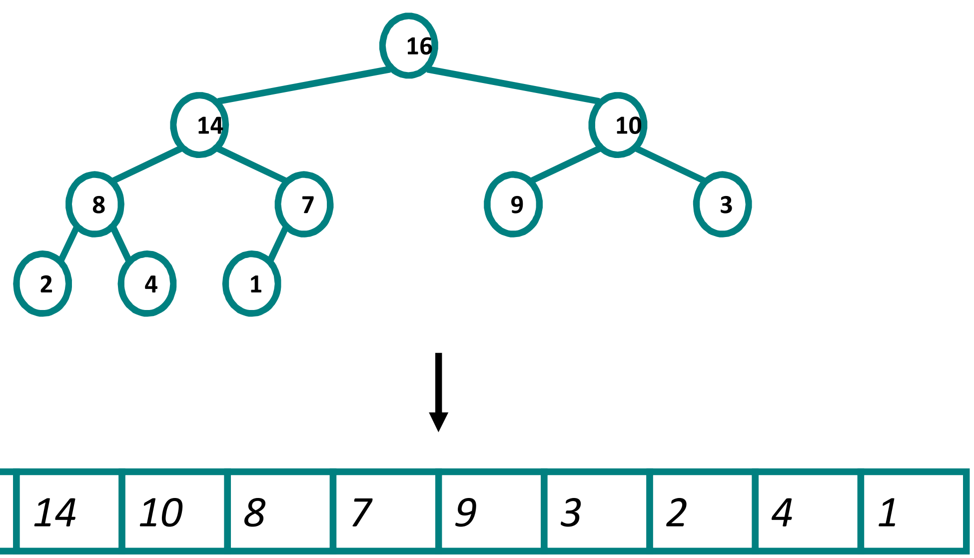
- A *heap* can be seen as a complete binary tree:



*A **complete binary tree** is a binary tree in which every level, except possibly the last, is completely filled, and all nodes are as far left as possible*

Heaps

- In practice, heaps are usually implemented as arrays:



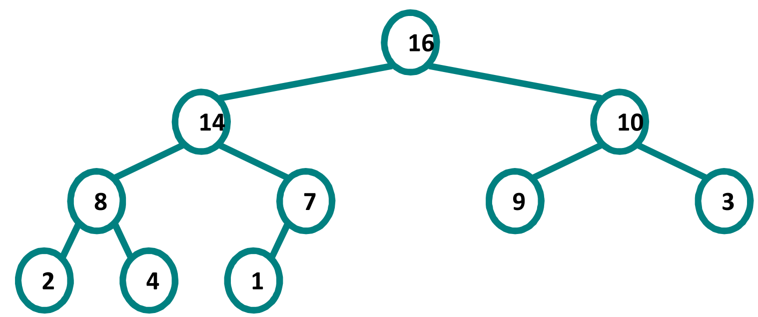
Heaps

- To represent a complete binary tree as an array:
  - The root node is A[1]
  - Node *i* is A[*i*]
  - The parent of node *i* is A[*i*/2] (note: integer divide)
  - The left child of node *i* is A[2*i*]
  - The right child of node *i* is A[2*i* + 1]

A =

16	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*

=



Referencing Heap Elements

- So…

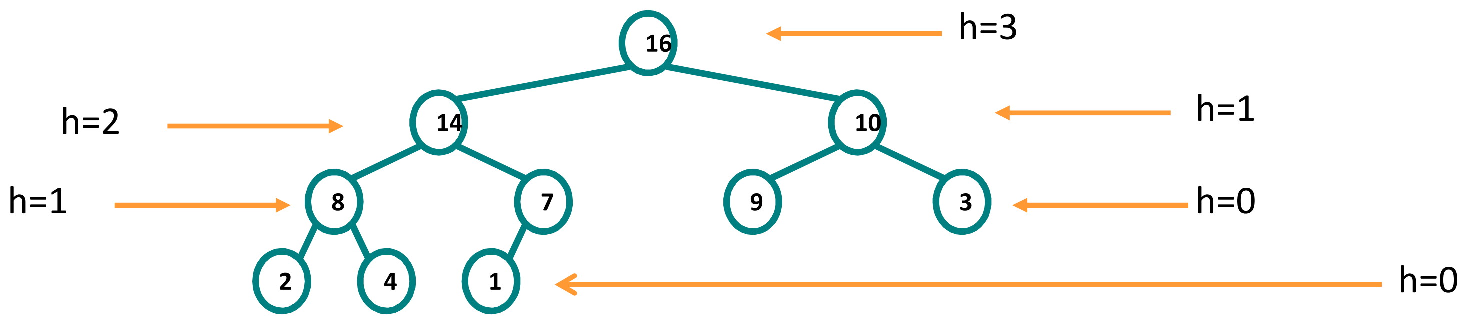
```
Parent(i)
 {return ⌊i/2⌋;}

Left(i)
 {return 2*i;}

right(i)
 {return 2*i + 1;}
```

Heap Height

- Definitions:
  - The *height of a node* in the tree = the number of edges on the longest downward path to a leaf
  - The *height of a tree* = the height of its root



- *What is the height of an n-element heap? Why?*
- $\lfloor \log_2(n) \rfloor$. Basic heap operations take at most time proportional to the height of the heap

The Heap Property

- Heaps also satisfy the *heap property*:

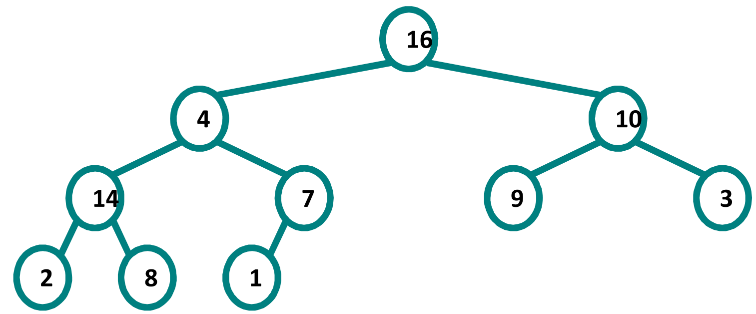
  $A[Parent(i)] \geq A[i]$ for all nodes $i > 1$

  - In other words, the value of a node is at most the value of its parent
  - The value of a node should be greater than or equal to both its left and right children
    - And all of its descendents
  - *Where is the largest element in a heap stored?*

Are they heaps?

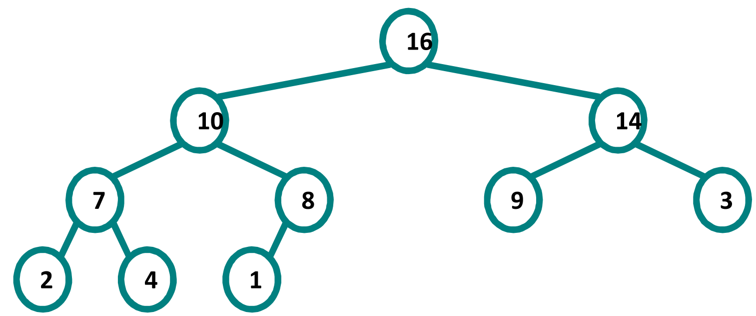
16	*4*	*10*	*14*	*7*	*9*	*3*	*2*	*8*	*1*

=



16	*10*	*14*	*7*	*8*	*9*	*3*	*2*	*4*	*1*

=



Violation to heap property: a node has value less than one of its children
How to find that?
How to resolve that?

Exercise

- What are the max and min number of elements in a heap of height *h*?
- Is {23, 17, 14, 6, 13, 10, 1, 5, 7, 12} a max-heap?
- Show that in the array representation of the heap, the leaves are the nodes indexed by $\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n$

Heap Operations: Heapify()

- **`Heapify()`**: maintain the heap property
  - Given: a node *i* in the heap with children *l* and *r*
  - Given: two subtrees rooted at *l* and *r*, assumed to be heaps
  - Problem: The subtree rooted at *i* may violate the heap property
  - Action: let the value of the parent node "sift down" so subtree at *i* satisfies the heap property
    - Fix up the relationship between i, l, and r recursively

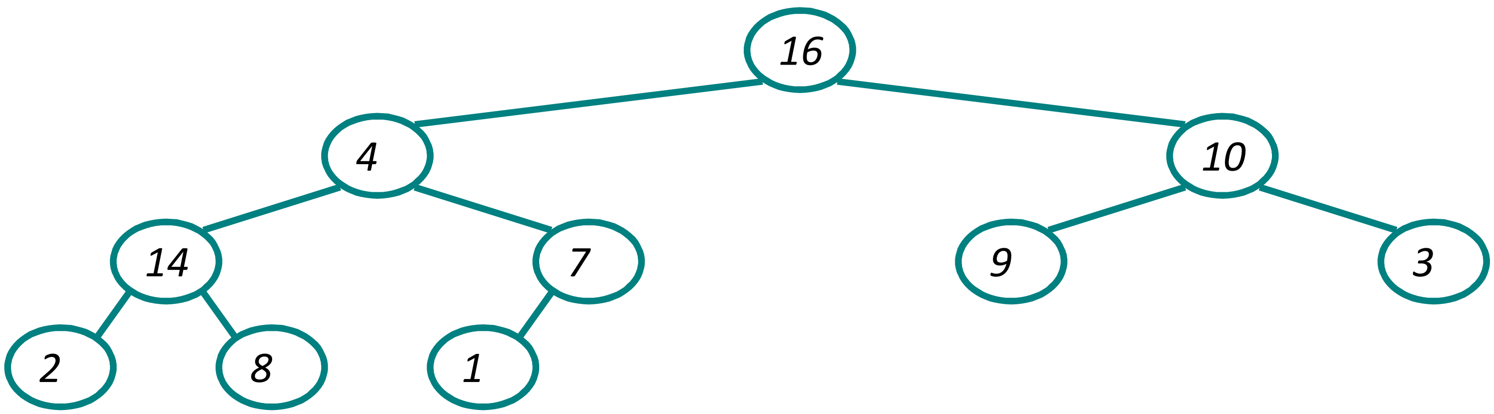
Heap Operations: Heapify()

```
Heapify(A, i)
{ // precondition: subtrees rooted at l and r are heaps
  l = Left(i); r = Right(i);
  if (l <= heap_size(A) && A[l] > A[i])
        largest = l;
  else
        largest = i;
  if (r <= heap_size(A) && A[r] > A[largest])
        largest = r;
  if (largest != i) {
        Swap(A, i, largest);
        Heapify(A, largest);
  }
} // postcondition: subtree rooted at i is a heap
```

Among A[l], A[i], A[r], which one is largest?

If violation, fix it.

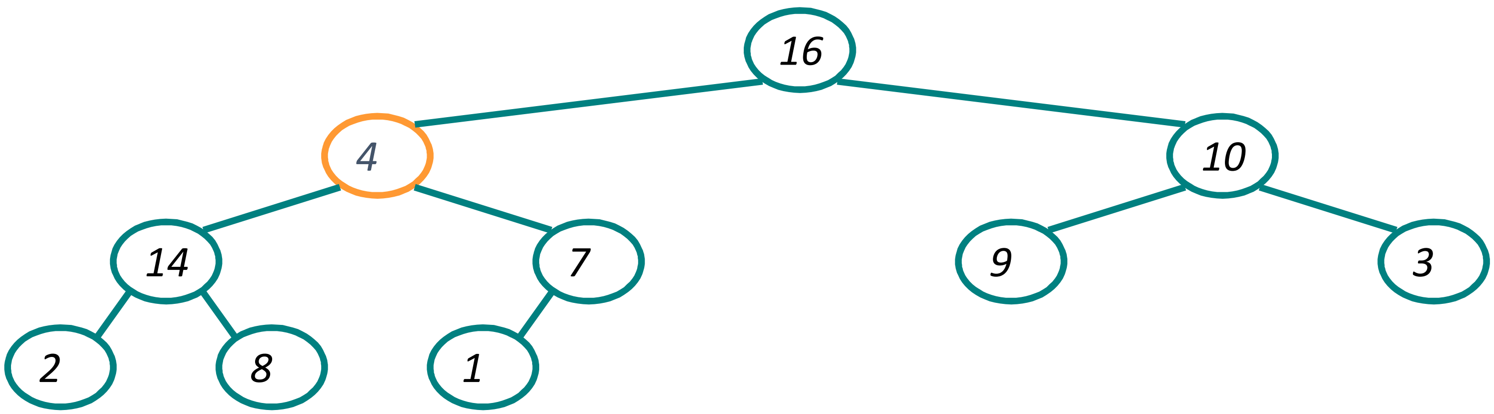
Heapify() Example



A =

16	4	10	14	7	9	3	2	8	1

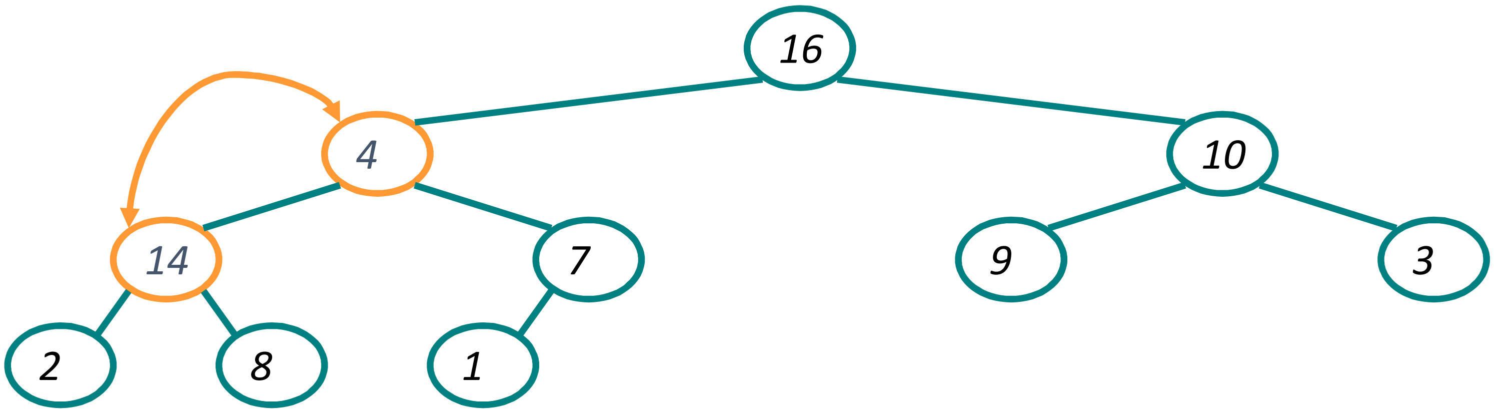
Heapify() Example



A =

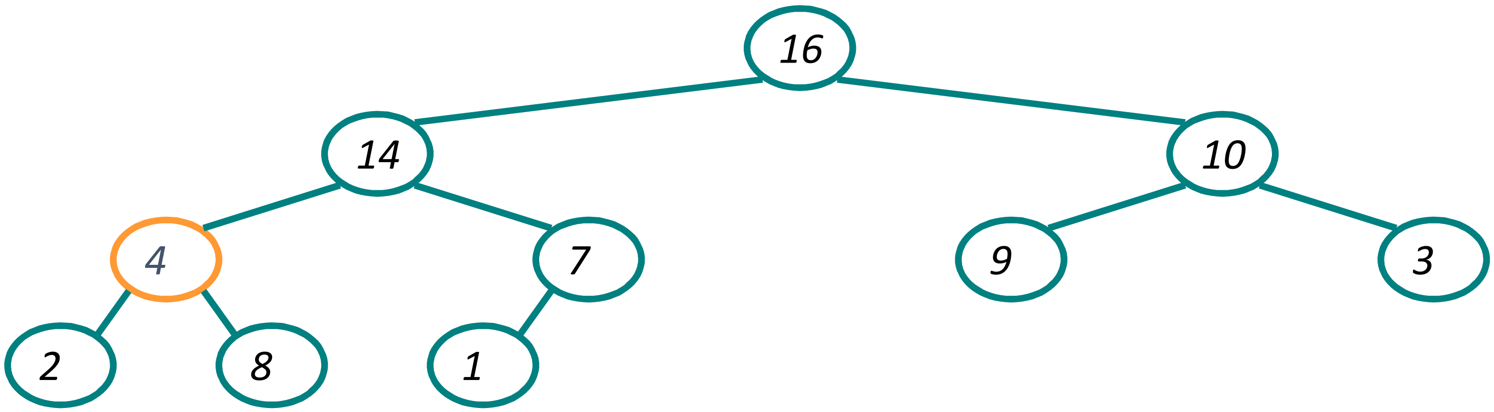
16	4	10	14	7	9	3	2	8	1

Heapify() Example



A =									
16	4	10	14	7	9	3	2	8	1

Heapify() Example



A =

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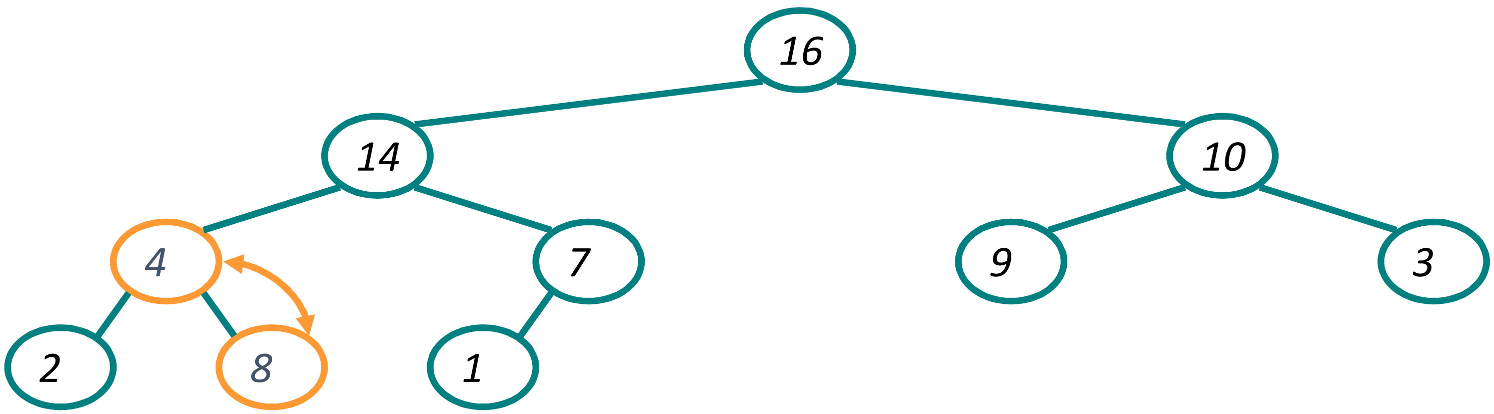
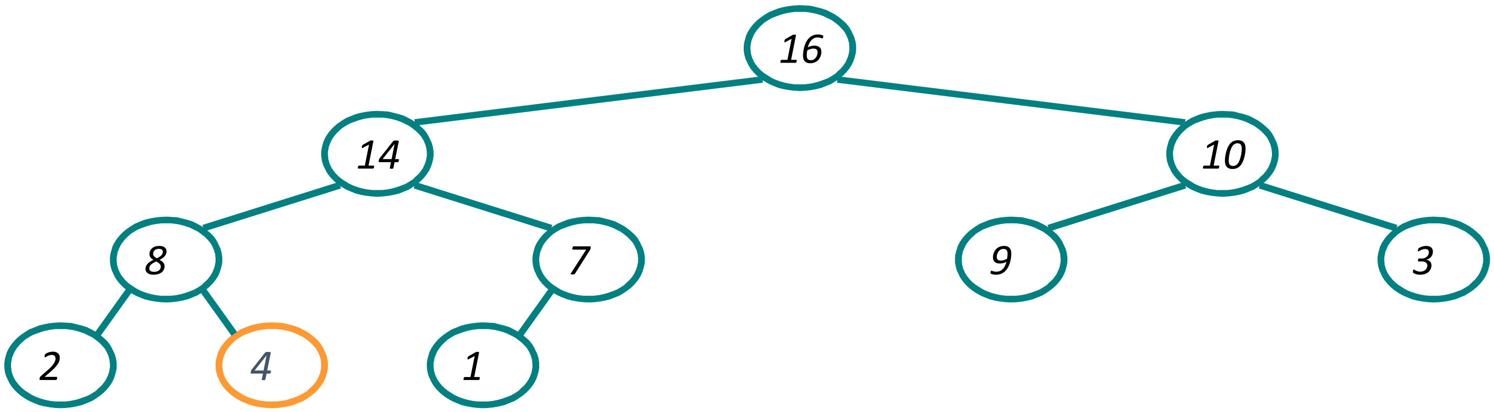
Heapify() Example



A =	16	14	10	4	7	9	3	2	8	1

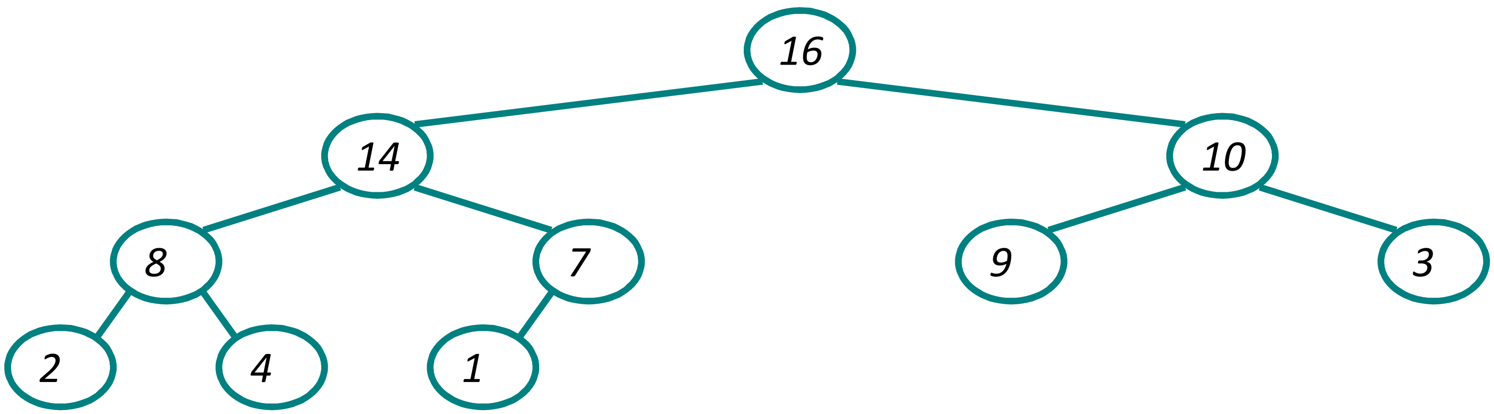
Heapify() Example



A =

16	14	10	8	7	9	3	2	4	1

Heapify() Example



A =									
16	14	10	8	7	9	3	2	4	1

Analyzing Heapify(): Informal

- *Aside from the recursive call, what is the running time of* **`Heapify()`***?*
- *How many times can* **`Heapify()`** *recursively call itself?*
- *What is the worst-case running time of* **`Heapify()`** *on a heap of size n?*

Analyzing Heapify(): Formal

- Fixing up relationships between *i*, *l*, and *r* takes $\Theta(1)$ time
- *If the heap at i has n elements, how many elements can the subtrees at l or r have?*
  - Draw it
- Answer: $2n/3$ (worst case: bottom row 1/2 full)
- So time taken by **`Heapify()`** is given by

  $T(n) \leq T(2n/3) + \Theta(1)$

Analyzing Heapify(): Formal

- So in the worst case we have

  $T(n) = T(2n/3) + \Theta(1)$

- By case 2 of the Master Theorem, ($\Theta\ (1) = \Theta(n^{\log_{1.5} a})$)

  $T(n) = O(\lg n)$

- Thus, **`Heapify()`** takes logarithmic time

Heap Operations: BuildHeap()

- We can build a heap in a bottom-up manner by running **`Heapify()`** on successive subarrays
  - Fact: for array of length *n*, all elements in range A[$\lfloor n/2 \rfloor$ + 1 .. n] are heaps (*Why?*)
  - So:
    - Walk backwards through the array from n/2 to 1, calling **`Heapify()`** on each node.
    - Order of processing guarantees that the children of node *i* are heaps when *i* is processed

- Fact: for array of length *n*, all elements in range A[$\lfloor n/2 \rfloor$ + 1 .. n] are heaps (*Why?*) ➔ *They are all leaves, which are single-node heap*

Heap size	# leaves	# internal nodes
1	1	0
2	1	1
3	2	1
4	2	2
5	3	2

0 <= # leaves - # internal nodes <= 1

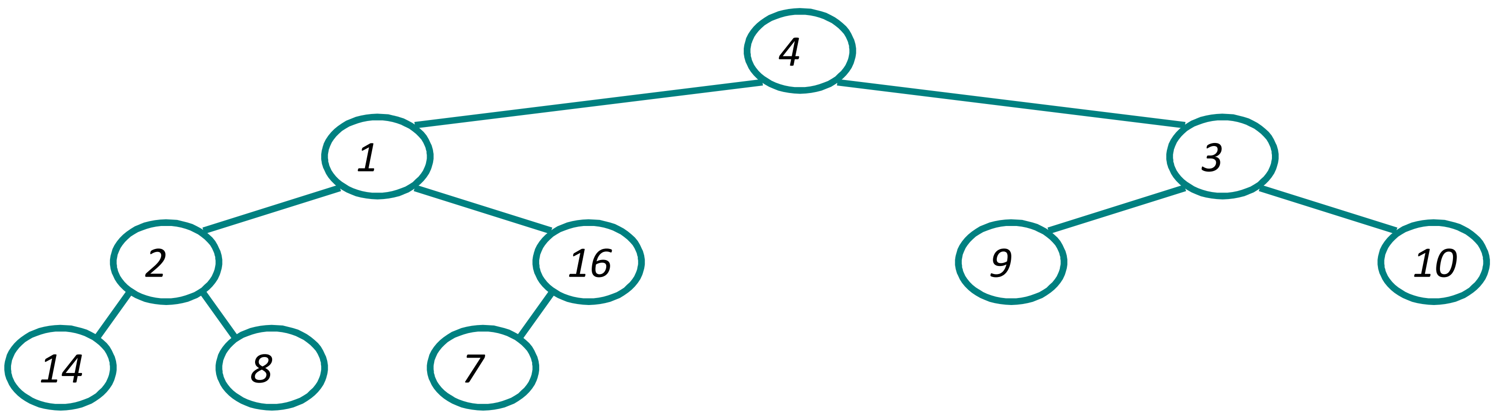
# of internal nodes = $\lfloor n/2 \rfloor$

BuildHeap()

```
// given an unsorted array A, make A a heap
BuildHeap(A)
{
 heap_size(A) = length(A);
 for (i = ⌊length[A]/2⌋ downto 1)
     Heapify(A, i);
}
```

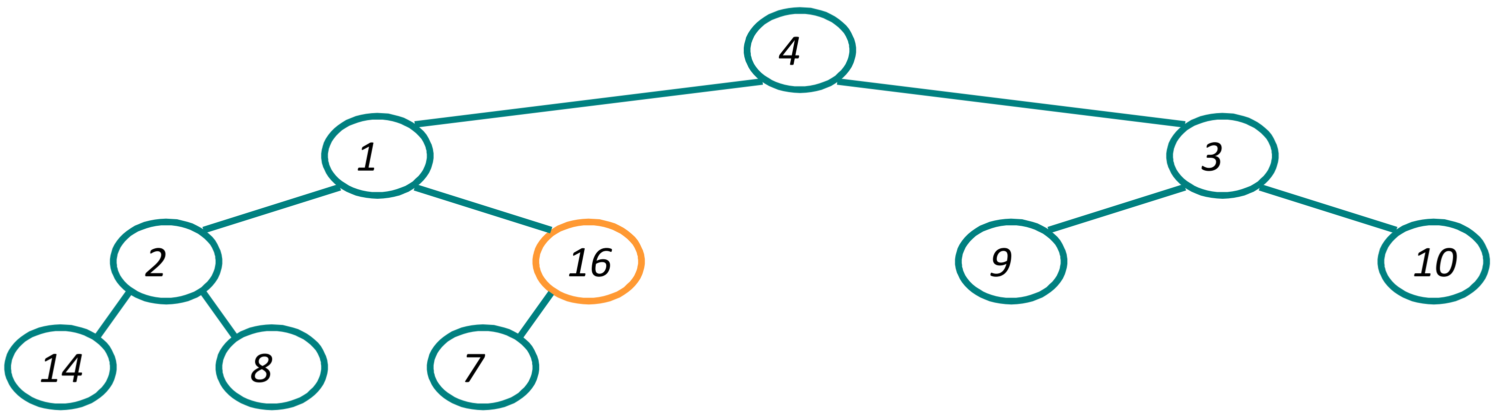
BuildHeap() Example

- Work through example
  A = {4, 1, 3, 2, 16, 9, 10, 14, 8, 7}

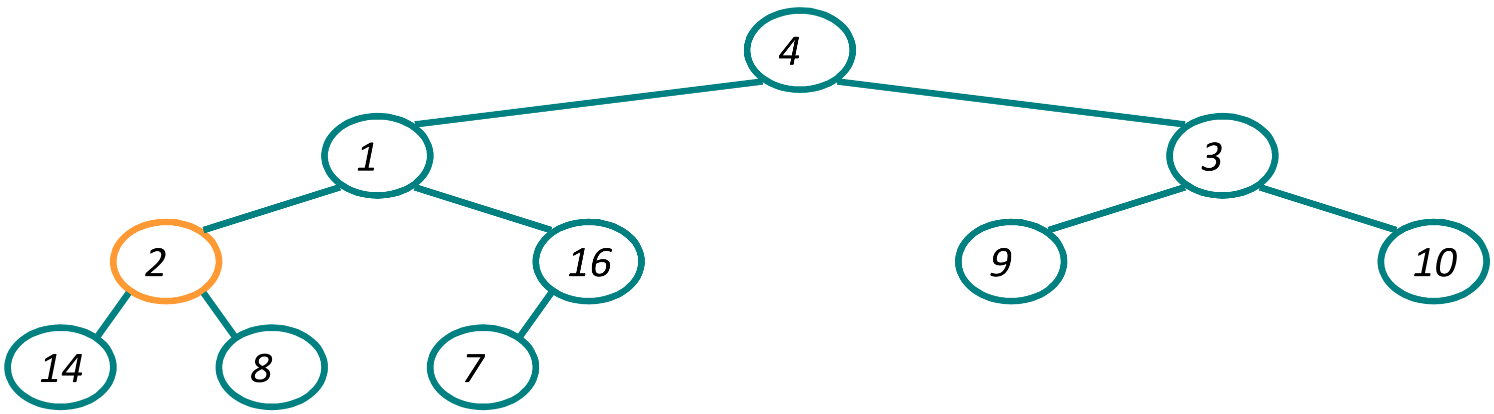


A =

4	*1*	*3*	*2*	*16*	*9*	*10*	*14*	*8*	*7*

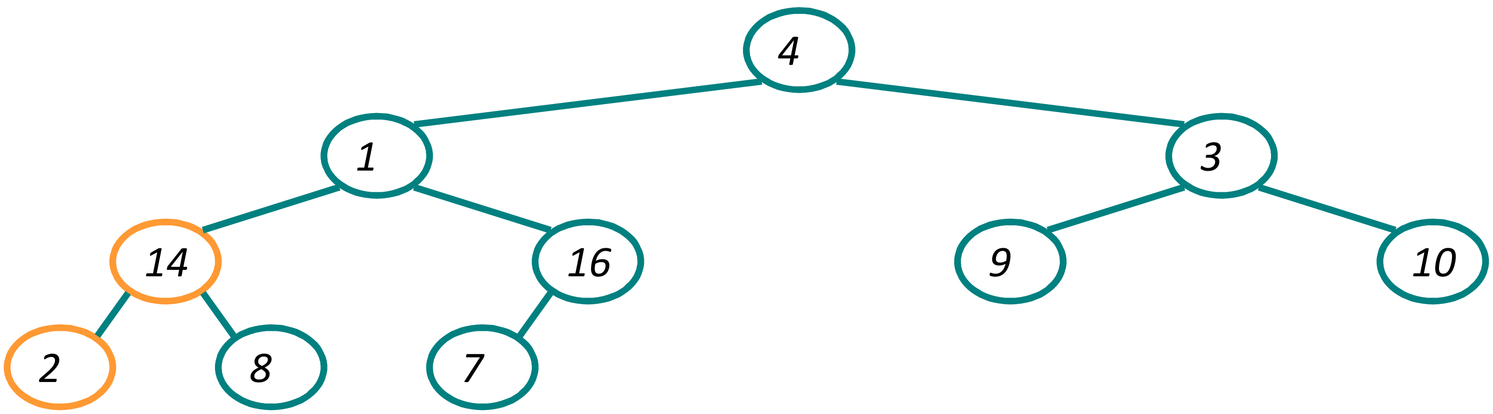


A =	*4*	*1*	*3*	*2*	*16*	*9*	*10*	*14*	*8*	*7*



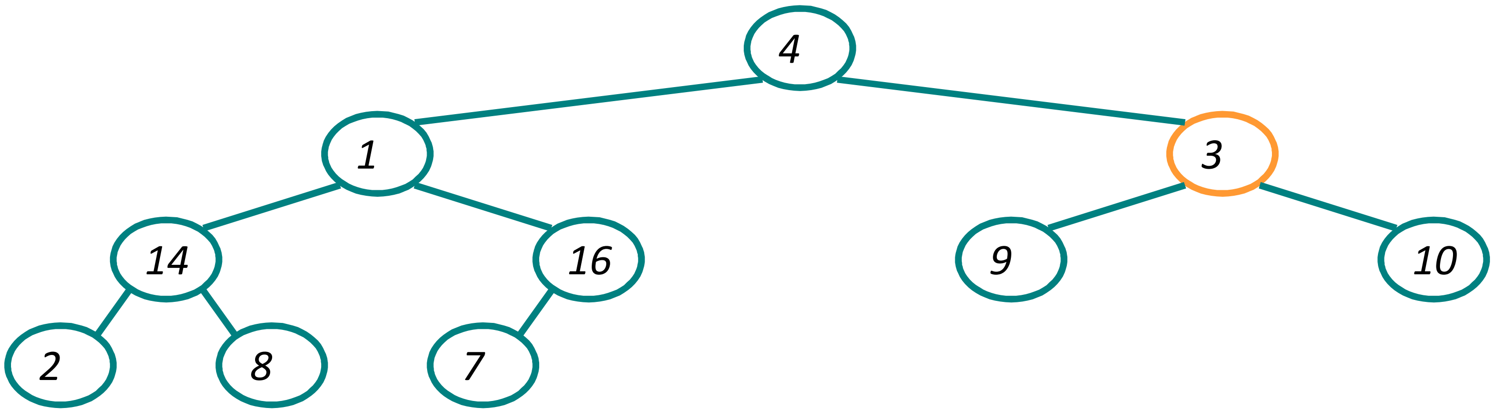
A =

4	1	3	2	16	9	10	14	8	7



A =

4	*1*	*3*	*14*	*16*	*9*	*10*	*2*	*8*	*7*



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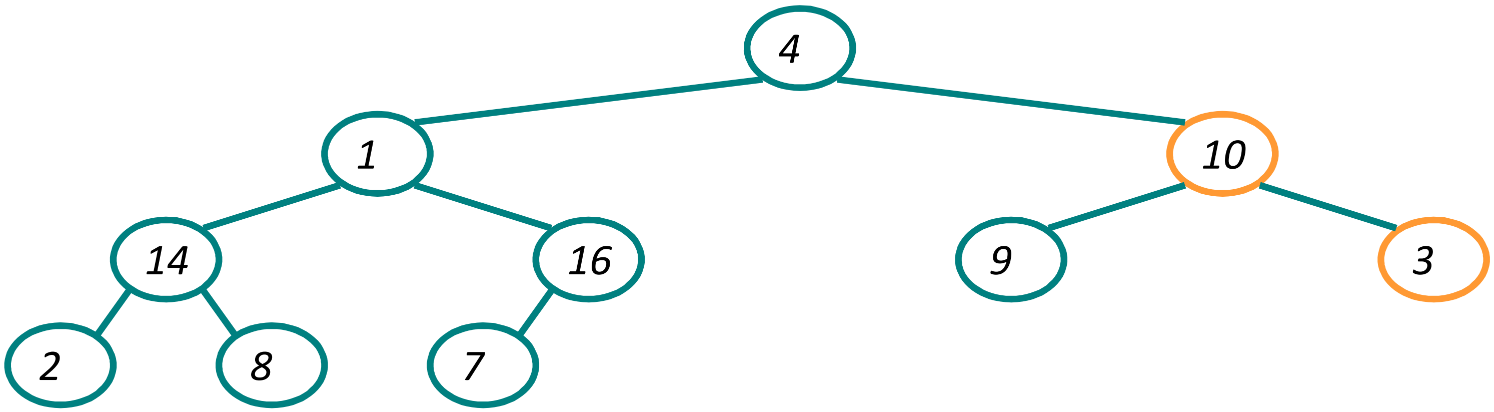

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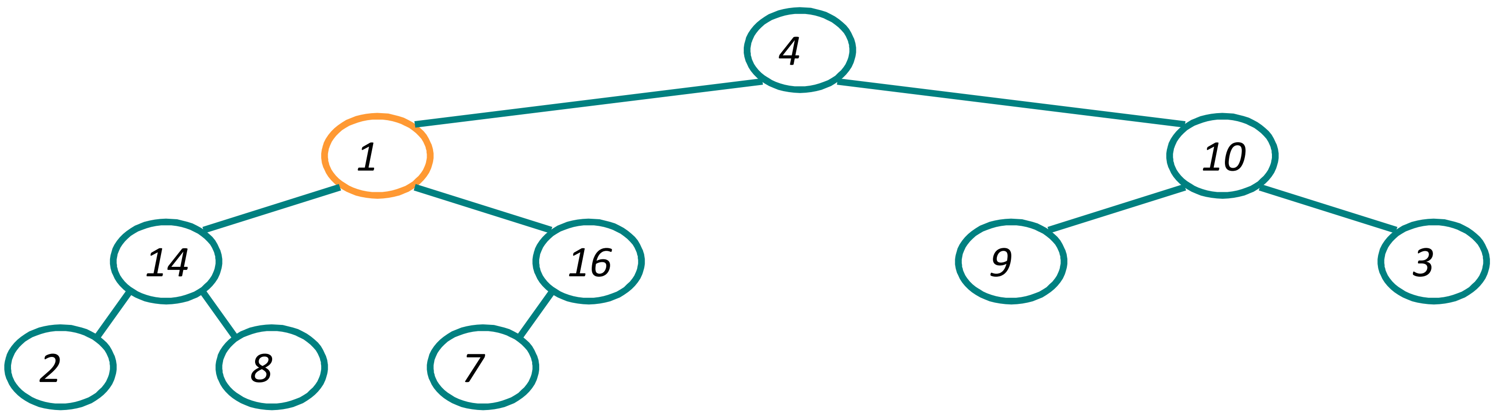
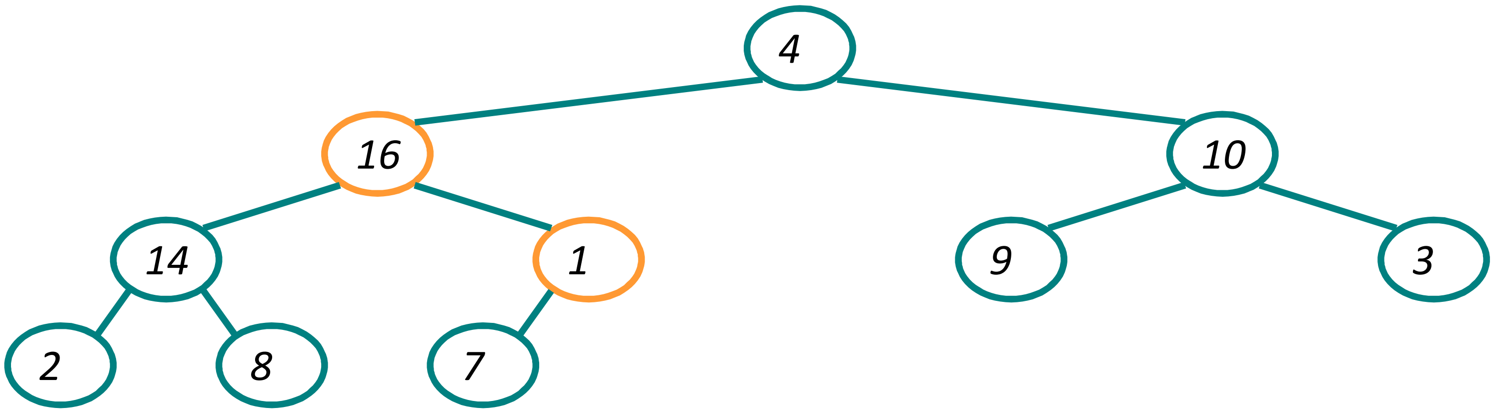
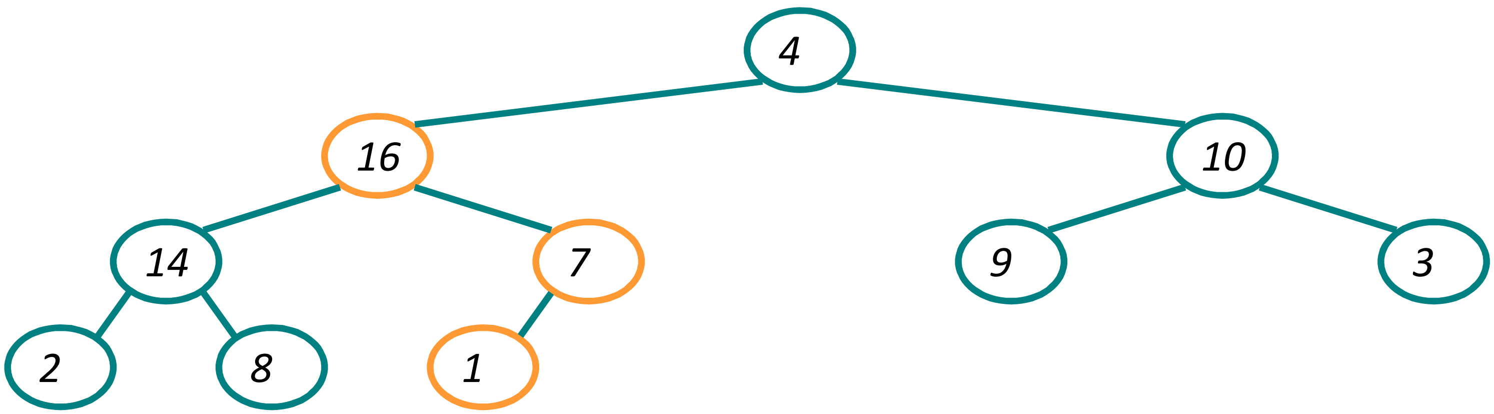


A =	4	1	10	14	16	9	3	2	8	7



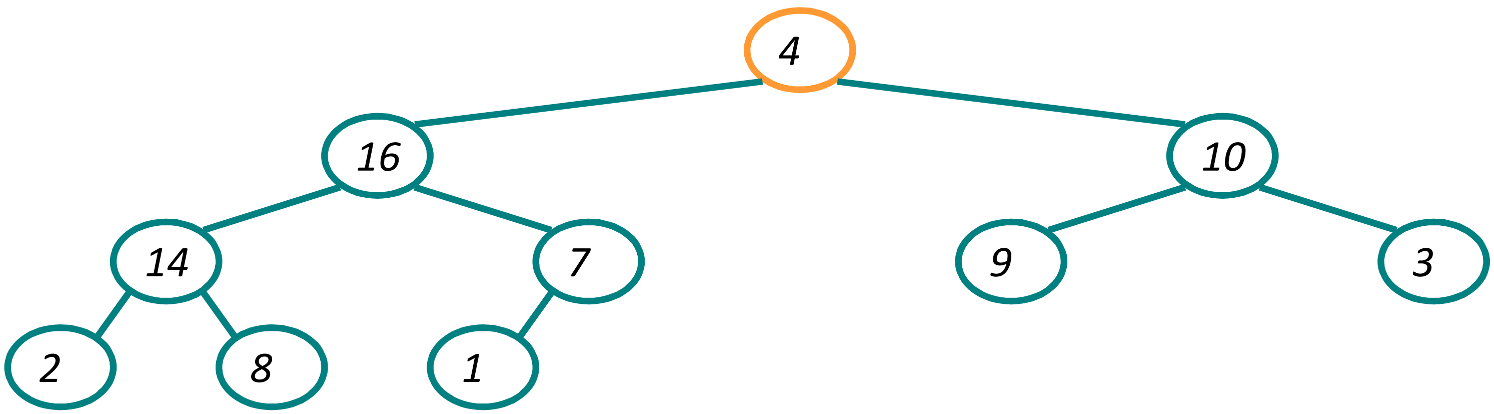
A =

4	16	10	14	1	9	3	2	8	7



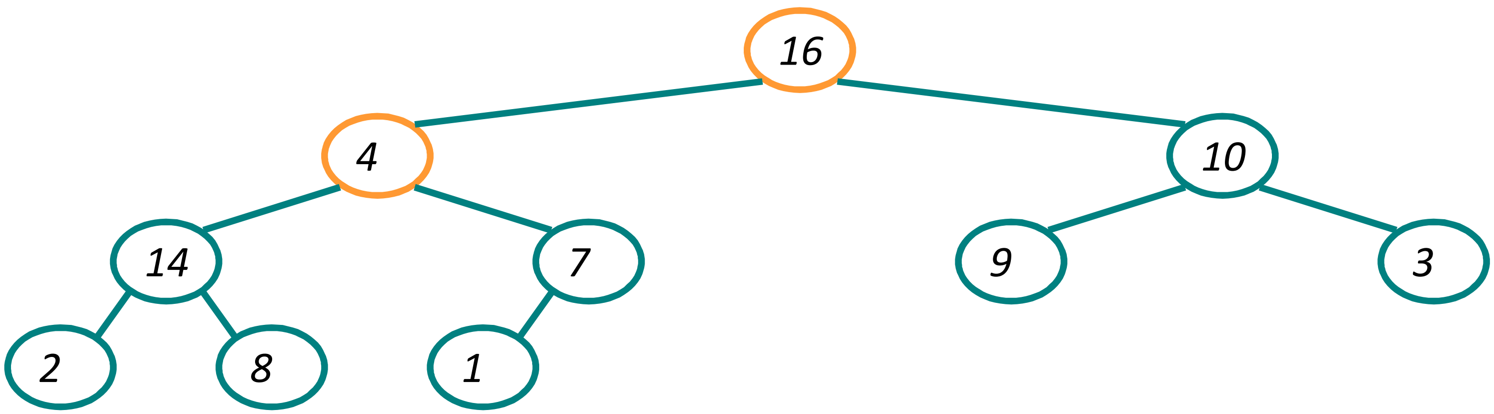
A =

4	16	10	14	7	9	3	2	8	1

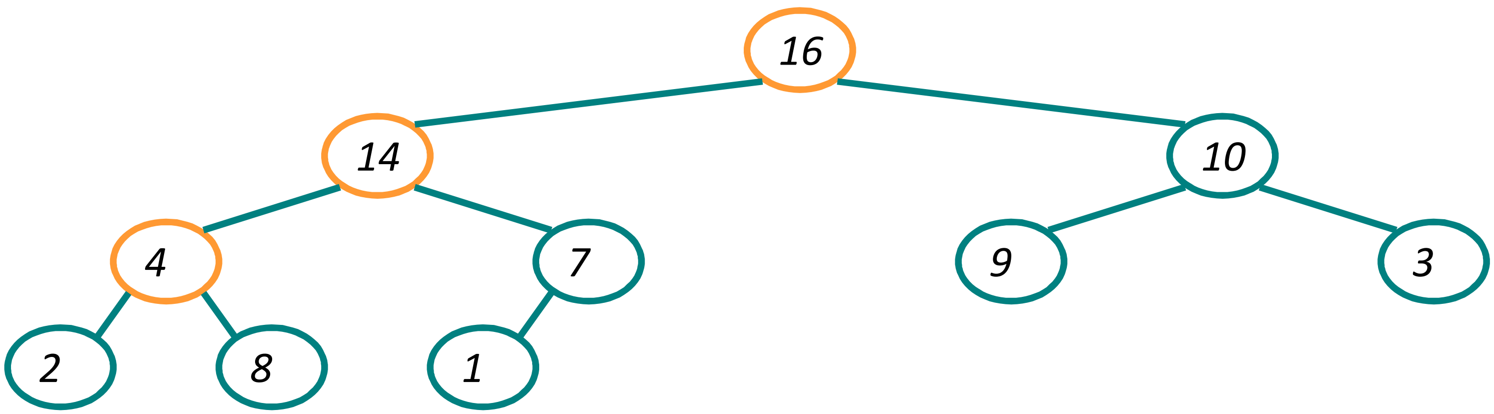


A =

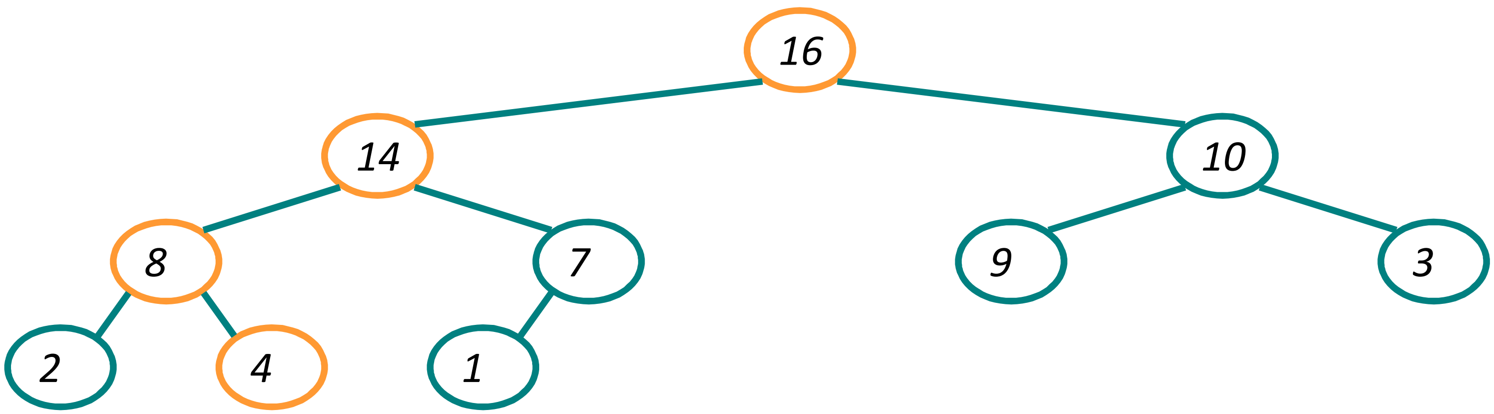
4	16	10	14	7	9	3	2	8	1



A =	*16*	*4*	*10*	*14*	*7*	*9*	*3*	*2*	*8*	*1*



A =	16	14	10	4	7	9	3	2	8	1



A =

16	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*

Analyzing BuildHeap()

- Each call to **`Heapify()`** takes O(lg *n*) time
- There are O(*n*) such calls (specifically, $\lfloor n/2 \rfloor$)
- Thus the running time is O(*n* lg *n*)
  - *Is this a correct asymptotic upper bound?*
  - *Is this an asymptotically tight bound?*
- A tighter bound is O(*n*)
  - *How can this be? Is there a flaw in the above reasoning?*

Analyzing BuildHeap(): Tight

- To **`Heapify()`** a subtree takes O(h) time where h is the height of the subtree
  - h = O(lg m), m = # nodes in subtree
  - The height of most subtrees is small
- Fact: an n-element heap has at most $\lceil n/2^{h+1} \rceil$ nodes of height h *(why?)*

$$T(n) \le \sum_{h=1}^{\lg n} \left\lceil \frac{n}{2^{h+1}} \right\rceil h \le \sum_{h=1}^{\lg n} \frac{nh}{2^h} = n \sum_{h=1}^{\lg n} \frac{h}{2^h} \le 2n$$

- Therefore T(n) = O(n)

- Fact: an n-element heap has at most $\lceil n/2^{h+1} \rceil$ nodes of height h *(why?)*
- $\lceil n/2 \rceil$ leaf nodes (h = 0): f(0) = $\lceil n/2 \rceil$
- f(1) $\leq$ ($\lceil n/2 \rceil$+1)/2 = $\lceil n/4 \rceil$
- The above fact can be proved using induction
- Assume f(h) $\leq \lceil n/2^{h+1} \rceil$
- f(h+1) $\leq$ (f(h)+1)/2 $\leq \lceil n/2^{h+2} \rceil$

$$T(n) \le \sum_{h=1}^{\lg n} \left\lceil \frac{n}{2^{h+1}} \right\rceil h \le \sum_{h=1}^{\lg n} \frac{nh}{2^h} = n \sum_{h=1}^{\lg n} \frac{h}{2^h} \le 2n$$

$$\sum_{h=1}^{\lg n} \frac{h}{2^h} \le \sum_{h=1}^{\infty} \frac{h}{2^h} = 2$$

Appendix A.8

$$\sum_{k=0}^{\infty} k x^k = \frac{x}{(1-x)^2}$$

for $|x| < 1$.

$$T(n) \le 2n$$

Therefore, building a heap takes Θ(n) time!!

Idea of heap sort

HeapSort(A[1..n])

Build a heap from A

For $i = n$ down to 1

Retrieve largest element from heap

Put element at end of A

Reduce heap size by one

end

Key:

1. Build a heap in linear time
2. Retrieve largest element (and make it ready for next retrieval) in O(log n) time

Heapsort

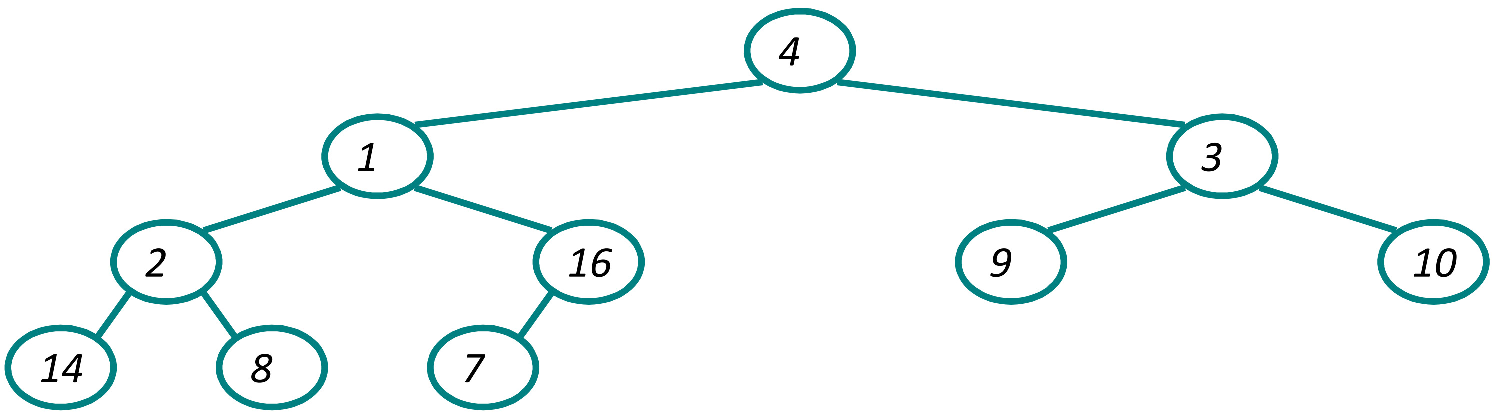
- Given **`BuildHeap()`**, an **in-place** sorting algorithm is easily constructed:
  - Maximum element is at A[1]
  - Discard by swapping with element at A[n]
    - Decrement heap_size[A]
    - A[n] now contains correct value
  - Restore heap property at A[1] by calling **`Heapify()`**
  - Repeat, always swapping A[1] for A[heap_size(A)]

Heapsort

```
Heapsort(A)
{
    BuildHeap(A);
    for (i = length(A) downto 2)
    {
        Swap(A[1], A[i]);
        heap_size(A) -= 1;
        Heapify(A, 1);
    }
}
```

Heapsort Example

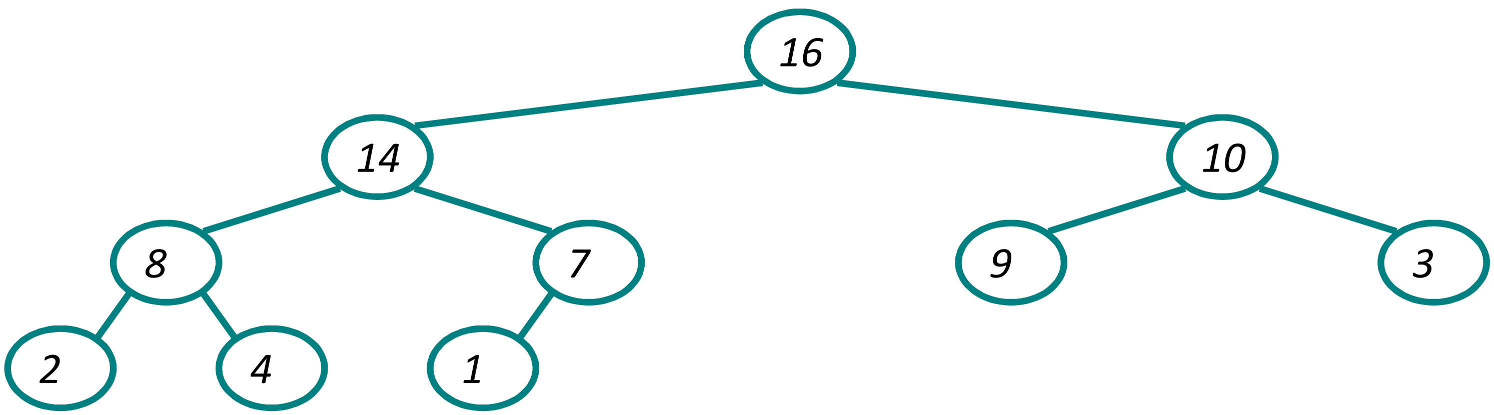
- Work through example
  A = {4, 1, 3, 2, 16, 9, 10, 14, 8, 7}



A =	*4*	*1*	*3*	*2*	*16*	*9*	*10*	*14*	*8*	*7*

Heapsort Example

- First: build a heap

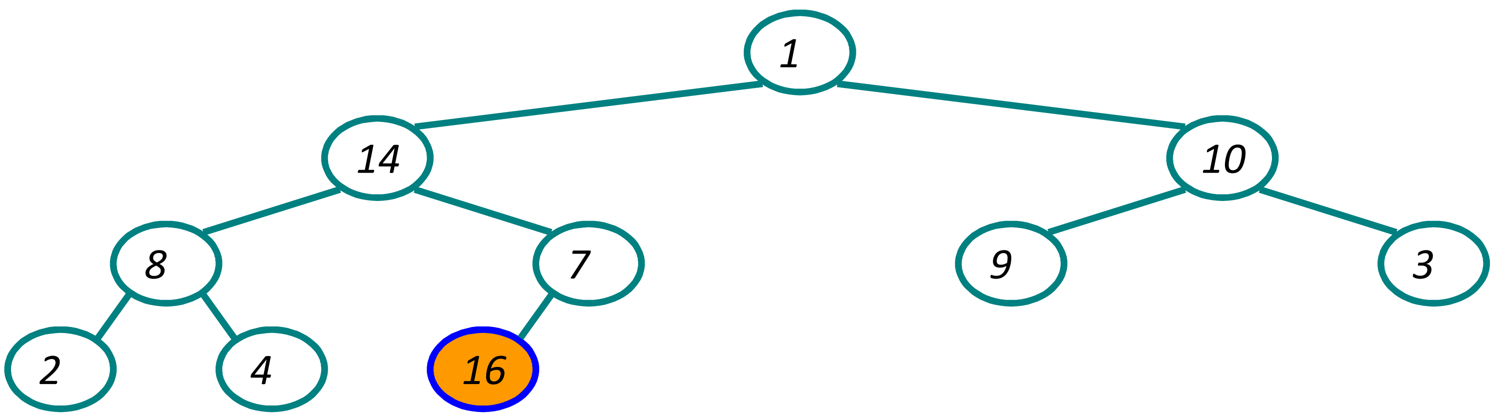


A =

16	14	10	8	7	9	3	2	4	1

Heapsort Example

- Swap last and first

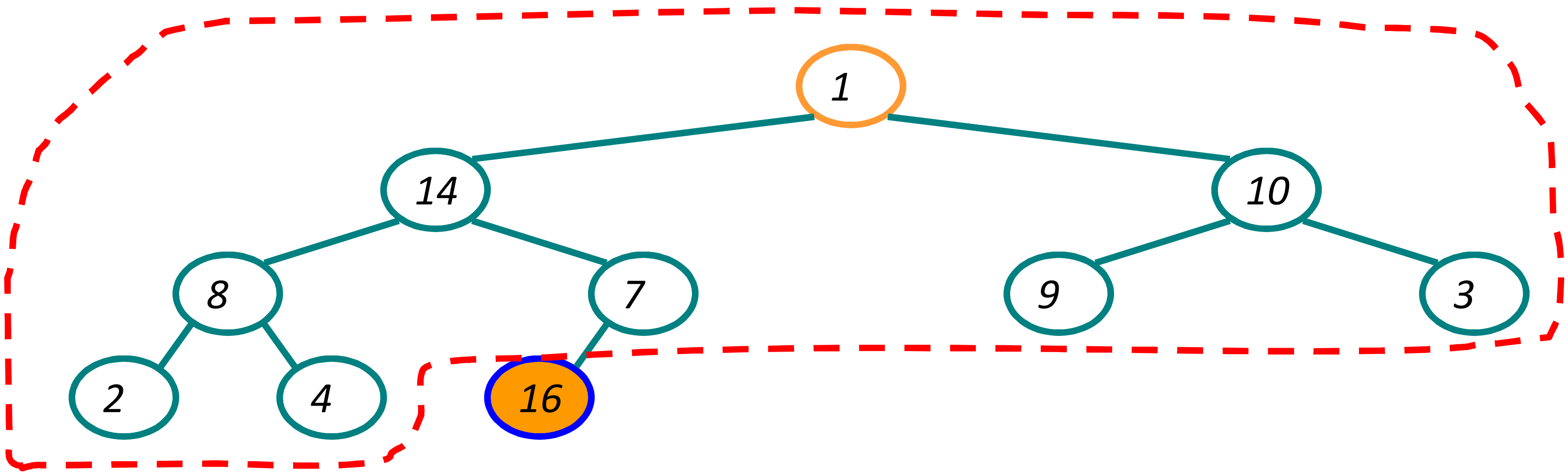


A =

1	14	10	8	7	9	3	2	4	16

Heapsort Example

- Last element sorted

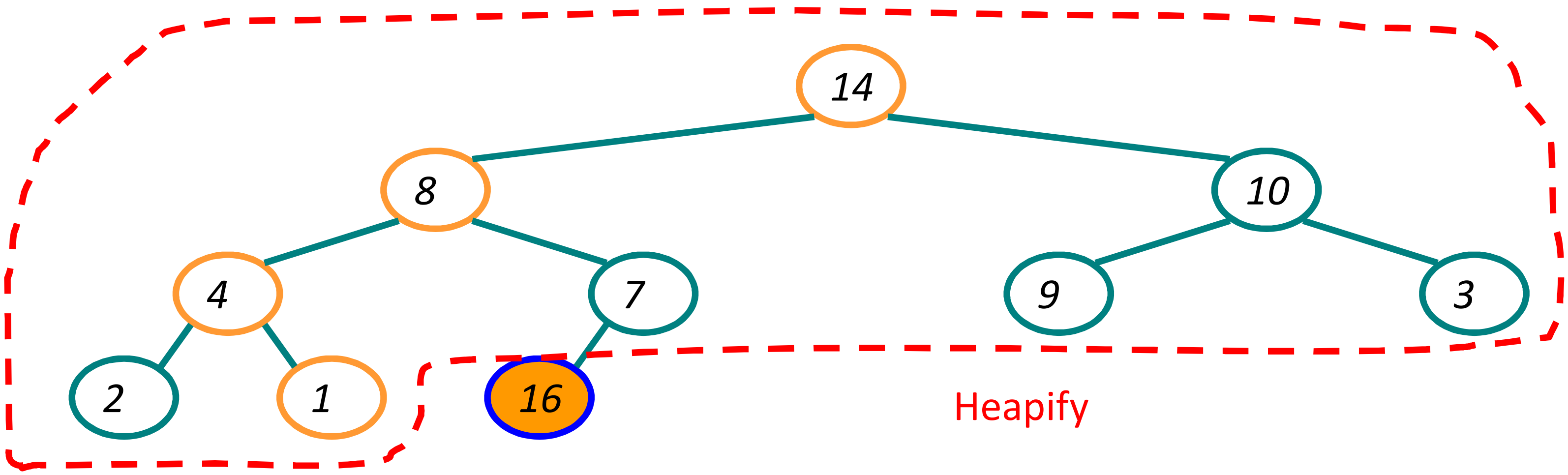


A =

1	14	10	8	7	9	3	2	4	16

Heapsort Example

- Restore heap on remaining unsorted elements

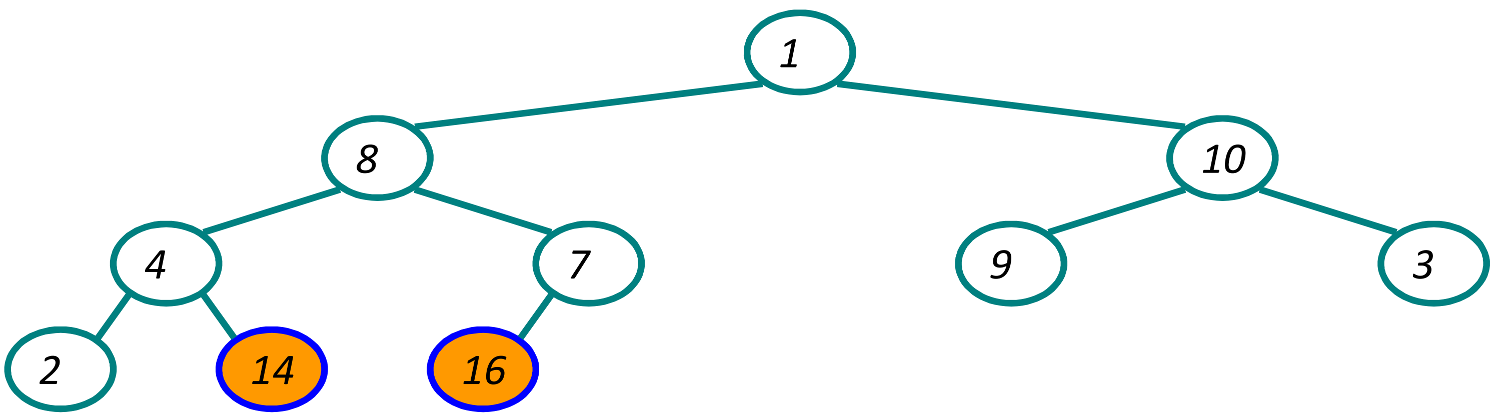


A =

14	8	10	4	7	9	3	2	1	16

Heapsort Example

- Repeat: swap new last and first

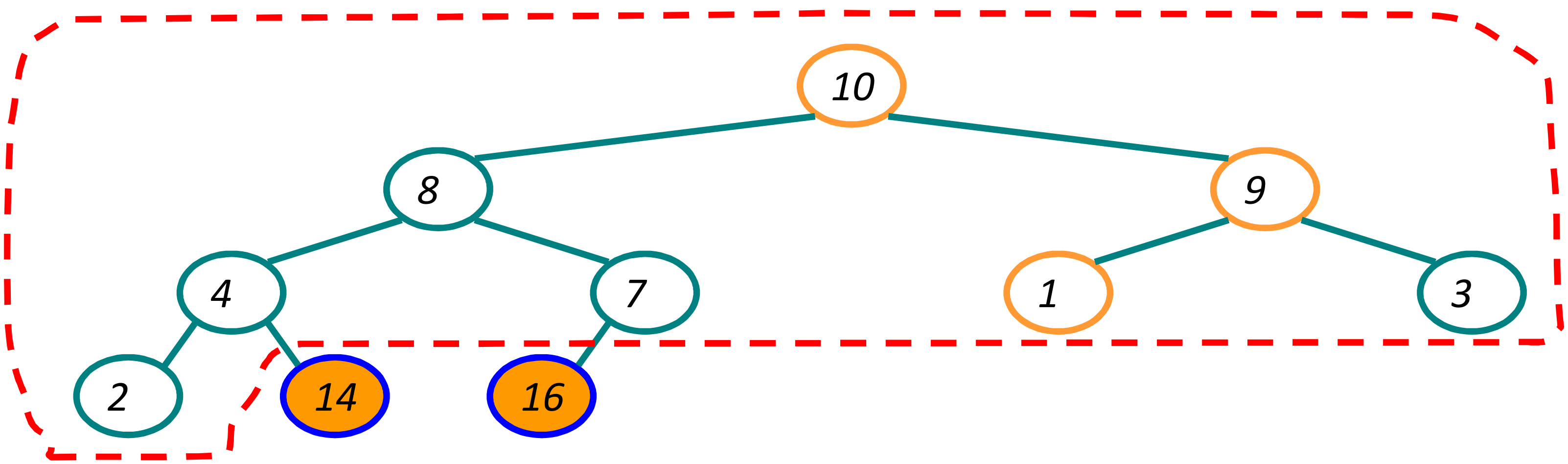


A =

1	*8*	*10*	*4*	*7*	*9*	*3*	*2*	*14*	*16*

Heapsort Example

- Restore heap

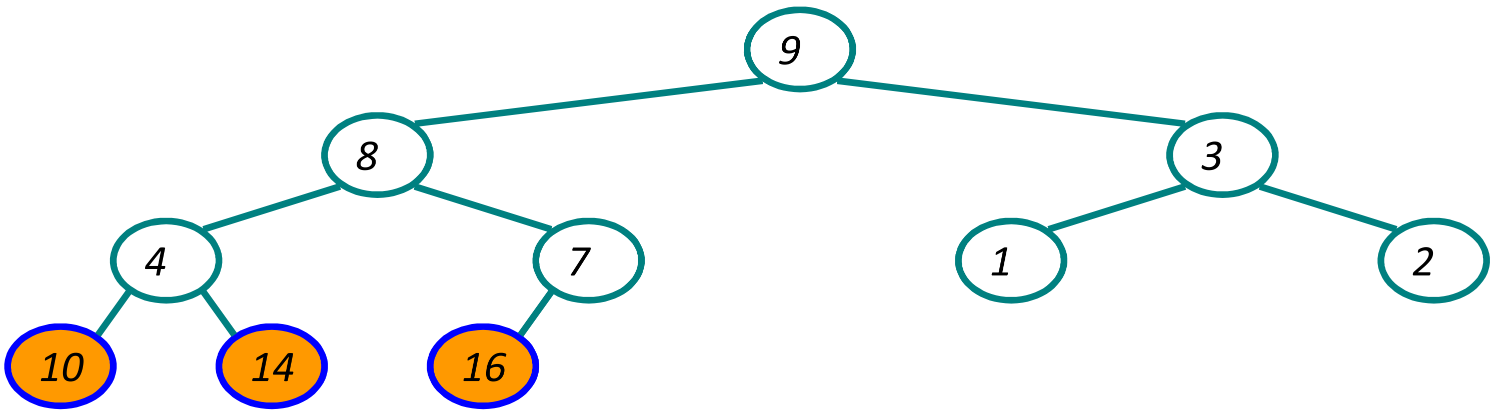


A =

10	8	9	4	7	1	3	2	14	16

Heapsort Example

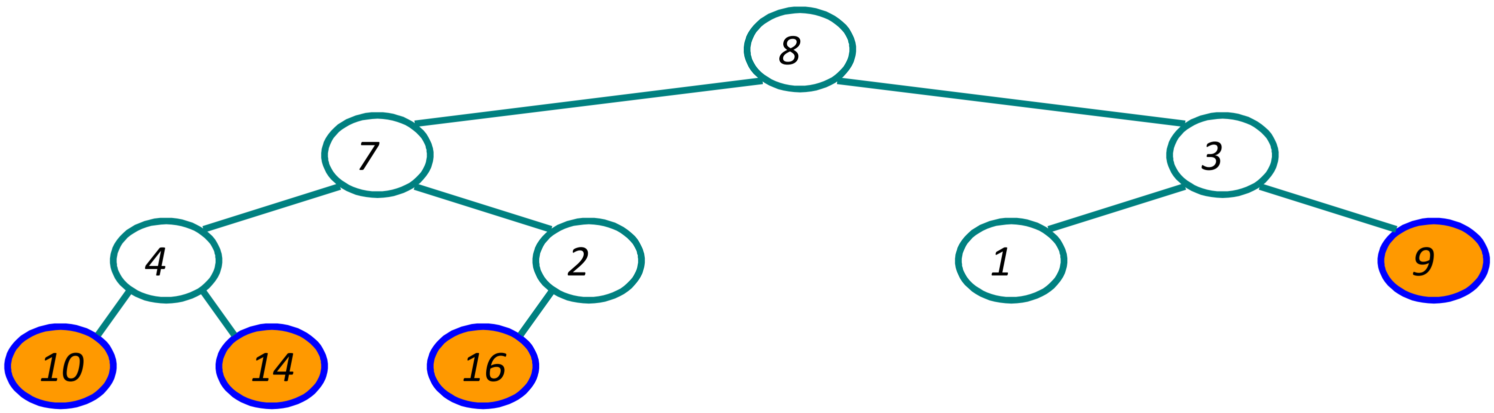
- Repeat



A =									
9	8	3	4	7	1	2	10	14	16

Heapsort Example

- Repeat

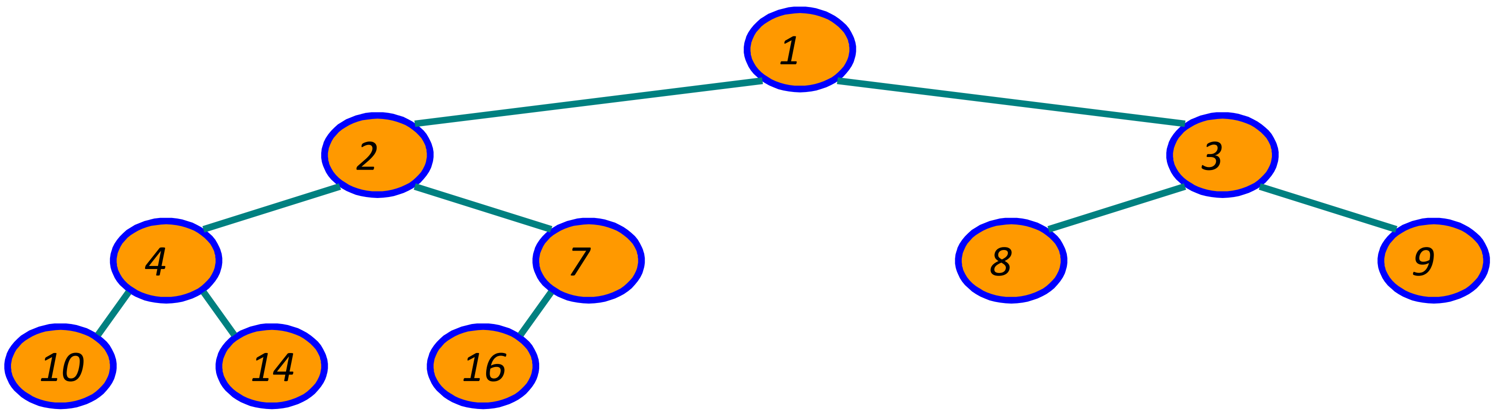


A =

8	7	3	4	2	1	9	10	14	16

Heapsort Example

- Repeat



A =

1	*2*	*3*	*4*	*7*	*8*	*9*	*10*	*14*	*16*

Analyzing Heapsort

- The call to **`BuildHeap()`** takes $O(n)$ time
- Each of the n - 1 calls to **`Heapify()`** takes $O(\lg n)$ time
- Thus the total time taken by **`HeapSort()`**
  $= O(n) + (n - 1) O(\lg n)$
  $= O(n) + O(n \lg n)$
  $= O(n \lg n)$

Comparison

	Time complexity	Stable?	In-place?
Merge sort			
Quick sort			
Heap sort			

Comparison

	Time complexity	Stable?	In-place?
Merge sort	Θ (n log n)	Yes	No
Quick sort	Θ(n log n) expected. Θ(n^2) worst case	No	Yes
Heap sort	Θ (n log n)	No	Yes

Priority Queues

- Heapsort is a nice algorithm, but in practice Quicksort usually wins
- The heap data structure is incredibly useful for implementing priority queues
  - A data structure for maintaining a set S of elements, each with an associated value or key
  - Supports the operations Insert(), Maximum(), ExtractMax(), changeKey()
- What might a priority queue be useful for?

Your personal travel destination list

- You have a list of places that you want to visit, each with a preference score
- Always visit the place with highest score
- Remove a place after visiting it
- You frequently add more destinations
- You may change score for a place when you have more information
- What's the best data structure?

Priority Queue Operations

- Insert(S, x) inserts the element x into set S
- Maximum(S) returns the element of S with the maximum key
- ExtractMax(S) removes and returns the element of S with the maximum key
- ChangeKey(S, i, key) changes the key for element i to something else
- How could we implement these operations using a heap?

Implementing Priority Queues

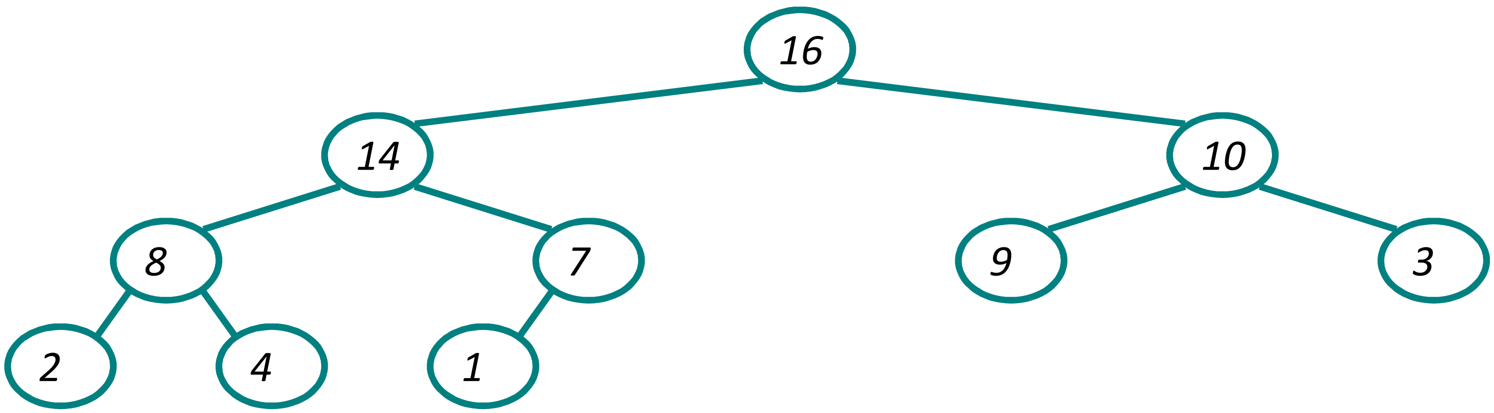
```
HeapMaximum(A)
{

    return A[1];
}
```

Implementing Priority Queues

```
HeapExtractMax(A)
{
    if (heap_size[A] < 1) { error; }
    max = A[1];
    A[1] = A[heap_size[A]]
    heap_size[A] --;
    Heapify(A, 1);
    return max;
}
```

HeapExtractMax Example

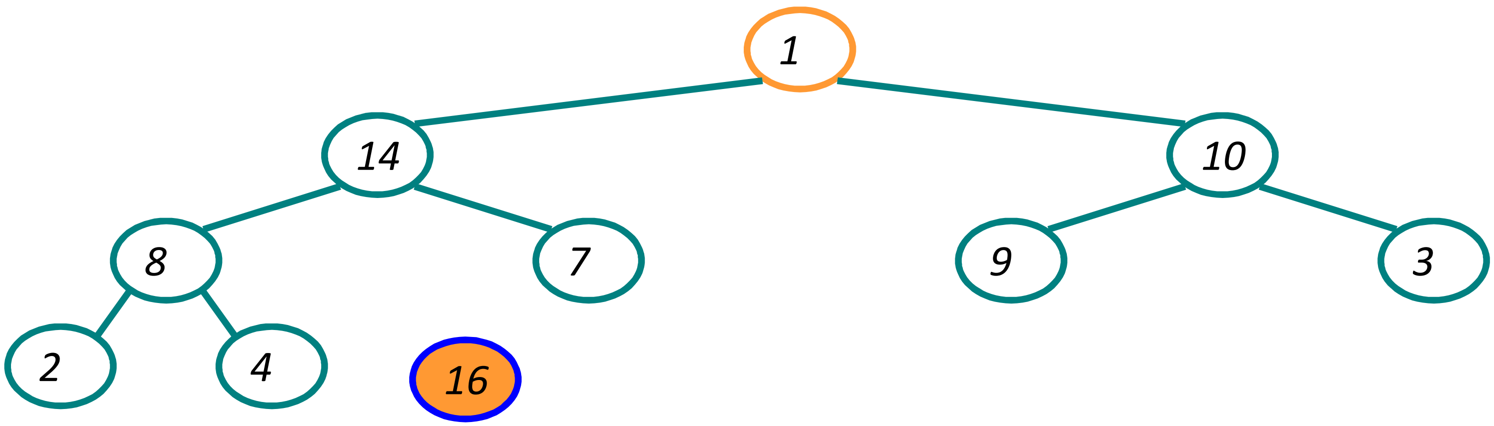


A =

16	14	10	8	7	9	3	2	4	1

HeapExtractMax Example

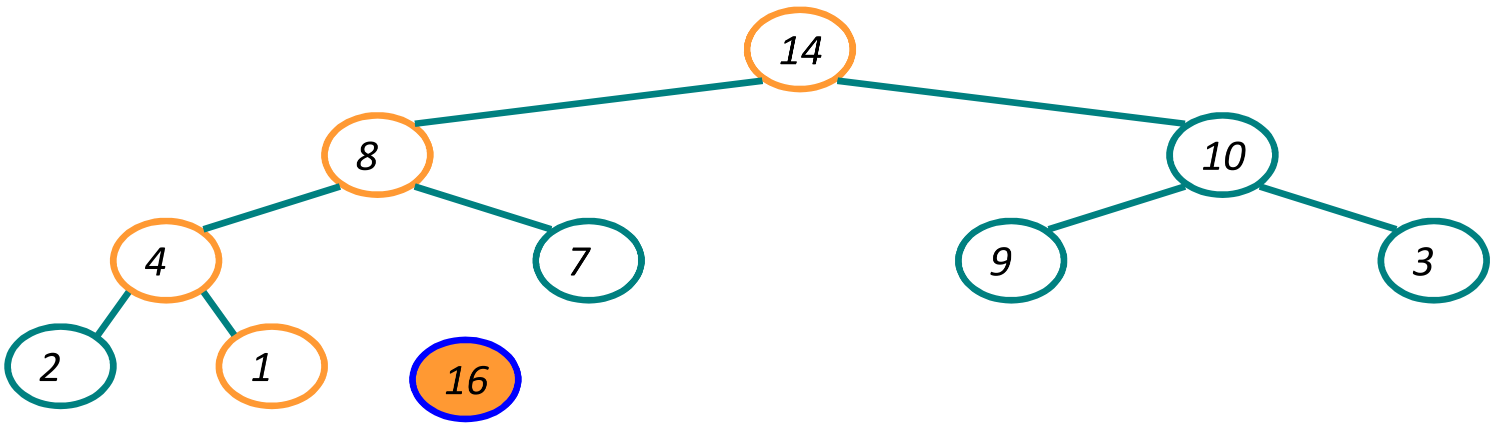
Swap first and last, then remove last



A =	*1*	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*16*

HeapExtractMax Example

Heapify



A =	14	8	10	4	7	9	3	2	1		16

Implementing Priority Queues

```
HeapChangeKey(A, i, key){
    if (key <= A[i]){ // decrease key
        A[i] = key;
        heapify(A, i);
    } else { // increase key
        A[i] = key;
        while (i>1 & A[parent(i)]<A[i])
            swap(A[i], A[parent(i)];
    }
}
```

Sift down

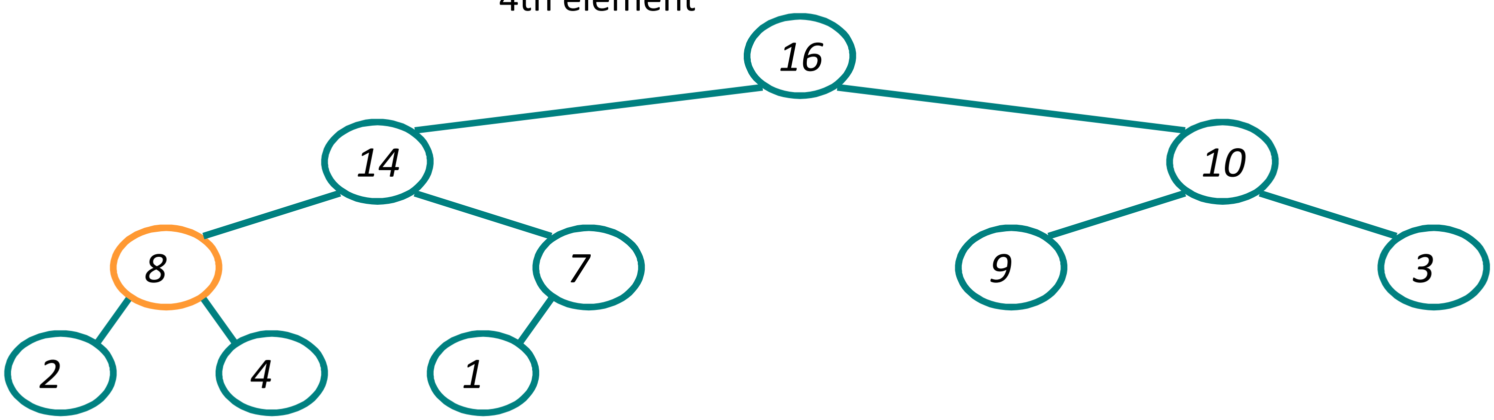
Bubble up

HeapChangeKey Example

HeapChangeKey(A, 4, 15)

Change key value to 15

4th element

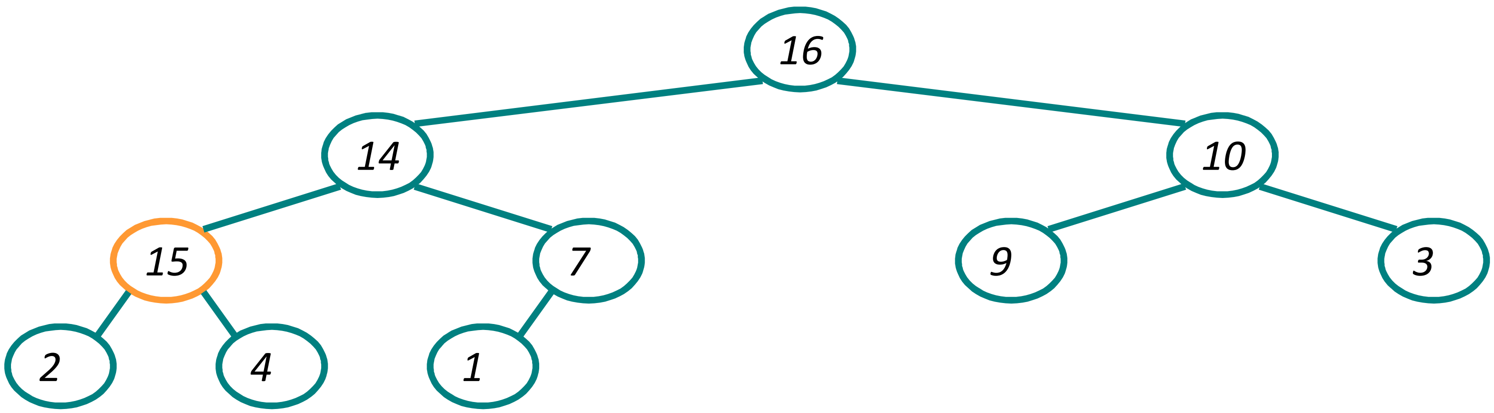


A =

16	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*

HeapChangeKey Example

HeapChangeKey(A, 4, 15)

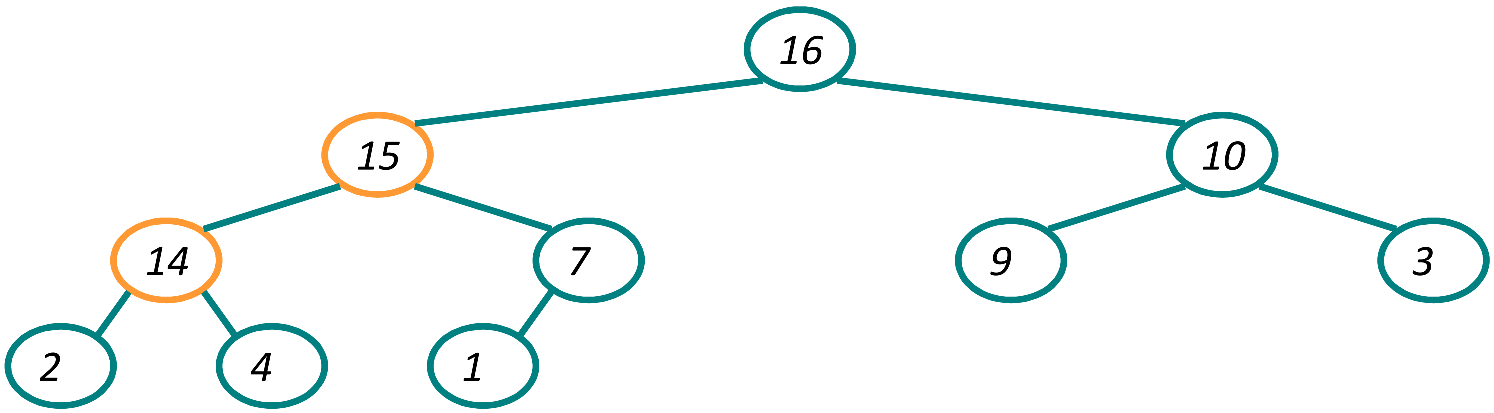


A =

16	14	10	15	7	9	3	2	4	1

HeapChangeKey Example

HeapChangeKey(A, 4, 15)



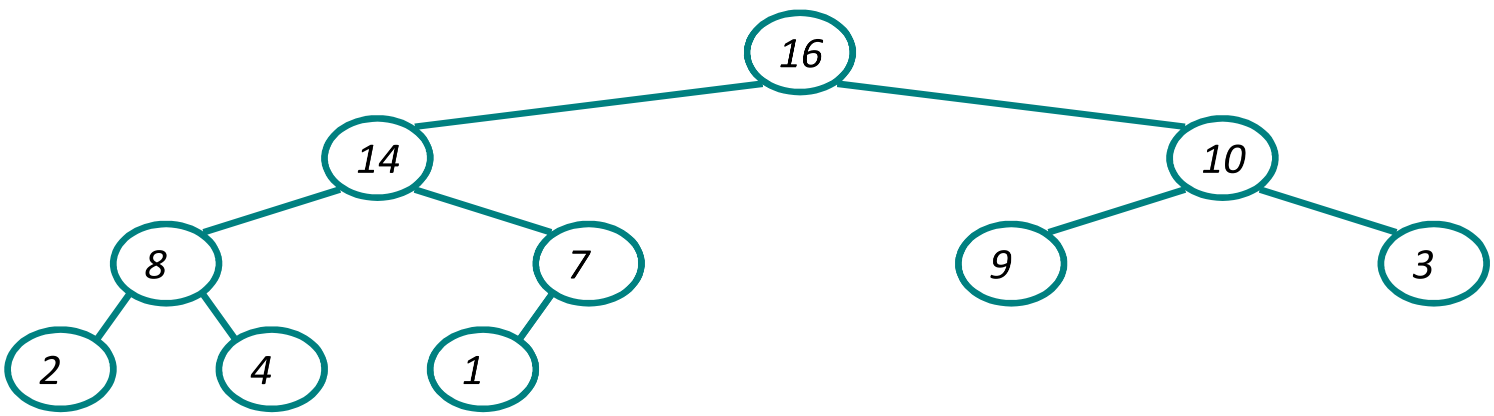
A =	*16*	*15*	*10*	*14*	*7*	*9*	*3*	*2*	*4*	*1*

Implementing Priority Queues

```
HeapInsert(A, key) {
    heap_size[A] ++;
    i = heap_size[A];
    A[i] = -∞;
    HeapChangeKey(A, i, key);
}
```

HeapInsert Example

HeapInsert(A, 17)

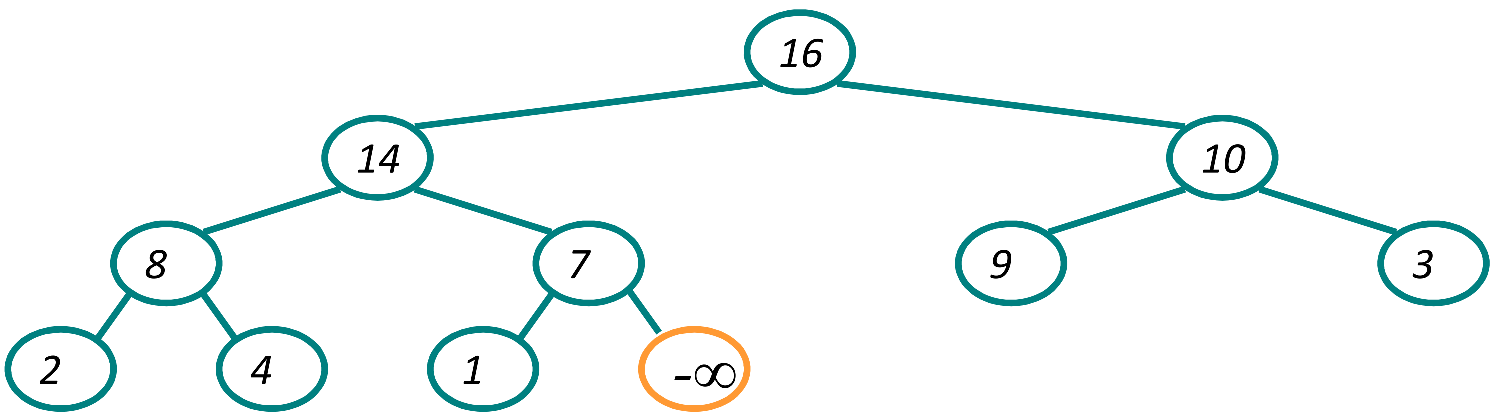


A =

16	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*

HeapInsert Example

HeapInsert(A, 17)



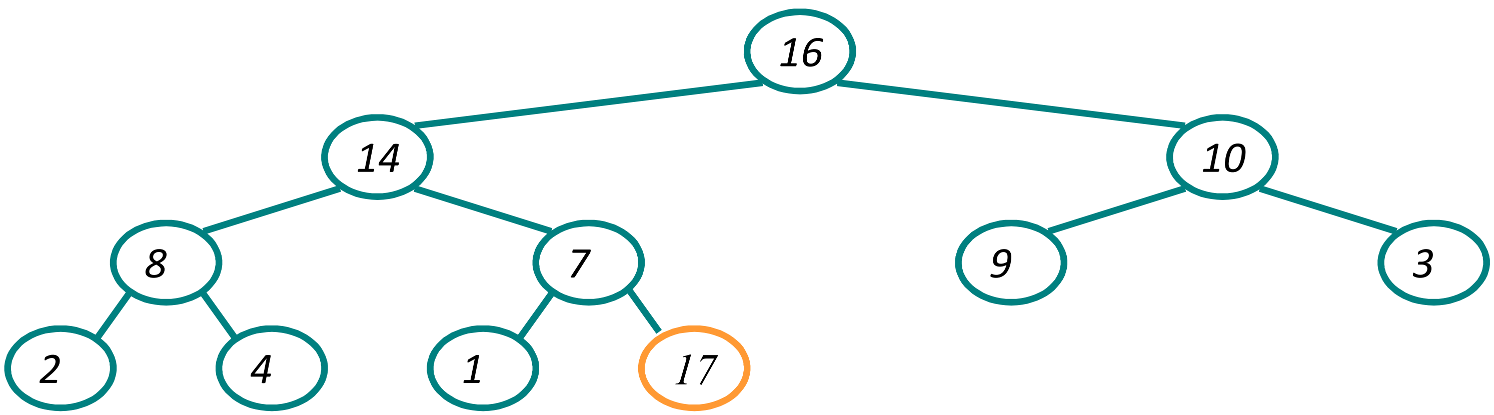
-∞ makes it a valid heap

A =

16	14	10	8	7	9	3	2	4	1	-∞

HeapInsert Example

HeapInsert(A, 17)

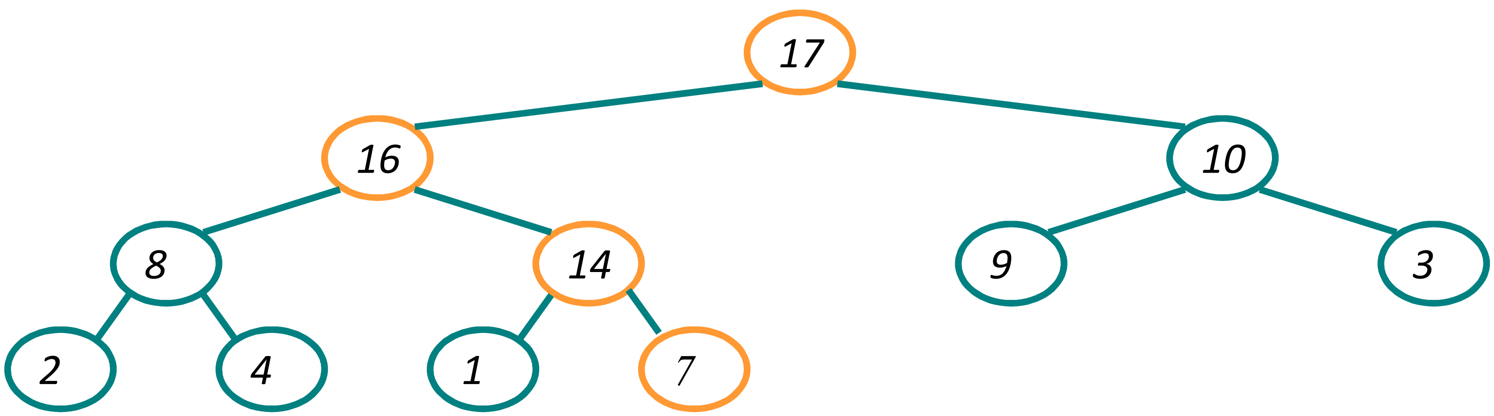


Now call HeapChangeKey

A =	*16*	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*	*17*

HeapInsert Example

HeapInsert(A, 17)



A =

17	*16*	*10*	*8*	*14*	*9*	*3*	*2*	*4*	*1*	*7*

T(n)

- Heapify: $\Theta(\log n)$
- BuildHeap: $\Theta(n)$
- HeapSort: $\Theta(n\log n)$

- HeapMaximum: $\Theta(1)$
- HeapExtractMax: $\Theta(\log n)$
- HeapChangeKey: $\Theta(\log n)$
- HeapInsert: $\Theta(\log n)$

If we use a sorted array / linked list

- Sort: $\Theta(n \log n)$
- Afterwards:

- arrayMaximum: $\Theta(1)$
- arrayExtractMax: $\Theta(n)$ or $\Theta(1)$
- arrayChangeKey: $\Theta(n)$
- arrayInsert: $\Theta(n)$