Homework 1: Solution

Deadline: May 03rd, 2019

P2. Problem Solving [50 points]

Q1. [21 pts] Consider a security game with four targets. The payoffs are given in the following table. In each cell, the first number is the defender payoff and the second is the attacker payoff.

	t_1	t_2	t_3	t_4
Covered	(1,0)	(3,0)	(8,0)	(8,-1)
Uncovered	(-1,1)	(0,2)	(0,4)	(-4,4)

Find the SSE of the game using the ORIGAMI algorithm when the number of defender resources is (i) one; (ii) two; and (iii) three. You must provide the computation step by step.

Solution. Note that the attack set consists of all targets with the highest expected utility for the attacker. The general idea of ORIGAMI is to incrementally assign coverage probabilities to targets to expand the attack set until all security resources are used.

Number of security resources is 1*:* initially, the coverage probabilities $c_1 = c_2 = c_3 = c_4 = 0$. Targets t_3 and t_4 has the highest expected utility for the attacker, which is 4. ORIGAMI increases coverage probabilities at these targets such that the attacker's expected utility at these two targets will be equal to the reward of the attacker at target t_2. That is:

$$c_4 \times (-1) + (1 - c_4) \times 4 = 2 \implies c_4 = \frac{2}{5}$$
$$c_3 \times 0 + (1 - c_3) \times 4 = 2 \implies c_3 = \frac{1}{2}$$

Since the current $c_3 + c_4 = 0.9 < 1$, we can assign more coverage probabilities to the targets. Note that at the current stage, the attack set has three targets (t_2, t_3, t_4). The next step is to try to increase coverage probabilities at these targets such that the attacker's expected utility at these targets in the attack set are all equal to the attacker's reward at target t_1. That is:

$$c_4 \times (-1) + (1 - c_4) \times 4 = 1 \implies c_4 = \frac{3}{5}$$
$$c_3 \times 0 + (1 - c_3) \times 4 = 1 \implies c_3 = \frac{3}{4}$$
$$c_2 \times 0 + (1 - c_2) \times 2 = 1 \implies \frac{1}{2}$$

However, the current total coverage probabilities $c_2+c_3+c_4 = \frac{3}{5}+\frac{3}{4}+\frac{1}{2} > 1$ which means that such probability assignment is not feasible. Therefore, as the final step, ORIGAMI assign all security resources to the three targets in the attack set (t_2, t_3, t_4). That is:

$$
\begin{aligned}
& c_4 \times (-1) + (1-c_4) \times 4 = c_3 \times 0 + (1-c_3) \times 4 = 1 \implies c_3 = \frac{5 \times c_4}{4} \\
& c_4 \times (-1) + (1-c_4) \times 4 = c_2 \times 0 + (1-c_2) \times 2 \implies c_2 = \frac{5 \times c_4 - 2}{2} \\
& c_4 + c_3 + c_2 = 1
\end{aligned}
$$

By replacing $c_3 = \frac{5 \times c_4}{4}$ and $c_2 = \frac{5 \times c_4 - 2}{2}$ in the third equation, we obtain:

$$
\begin{aligned}
c_4 + \frac{5 \times c_4}{4} + \frac{5 \times c_4 - 2}{2} &= 1 \implies c_4 = \frac{8}{19} \approx 0.42 \\
c_3 = \frac{5 \times c_4}{4} &= \frac{10}{19} \approx 0.53 \\
c_2 = \frac{5 \times c_4 - 2}{2} &= \frac{1}{19} \approx 0.05.
\end{aligned}
$$

As a result, the defender's strategy is $(c_4, c_3, c_2, c_1) = (0.42, 0.53, 0.05, 0)$. Solving the game with two and three resources is similar.

Q2. [29 pts] Roger has invited Caleb to his party. Roger must choose whether or not to hire a clown. Simultaneously, Caleb must decide whether or not to go the party. Caleb likes Roger but he hates clowns. Caleb's payoff from going to the party is 4 if there is no clown, but 0 if there is a clown there. Caleb's payoff from not going to the party is 3 if there is no clown at the party, but 1 if there is a clown at the party. Roger likes clowns (he especially likes Caleb's reaction to them but does not like paying for them). Roger's payoff if Caleb comes to the party is 4 if there is no clown, but $8 - x$ if there is a clown (x is the cost of a clown). Roger's payoff if Caleb does not come to the party is 2 if there is no clown, but $3 - x$ if there is a clown there.

1. (5 pts) Write down the payoff matrix of this game.

   Solution. The payoff matrix is determined as follows:

	Party	No Party
Clown	$8-x, 0$	$3-x, 1$
No Clown	$4, 4$	$2, 3$

   Table 1: Payoff matrix

   in which the row player is Roger and the column player is Caleb.

2. (24 pts) Find any dominated strategies and the Nash equilibrium of the game (with explanation) when (i) $x = 0$; (ii) $x = 2$; (iii) $x = 3$; and (iv) $x = 5$.

   Solution.

$x = 0$**.** The No Clown strategy is the dominated strategy for Roger. A pure Nash equilibrium is (Clown, No Party)

$x = 2$**.** There is no dominated strategy or pure Nash equilibrium. Denote by $(p, 1-p)$ Roger's strategy and $(q, 1-q)$ Caleb's strategy in the equilibrium. Then we have:

Roger is indifferent between Clown and No Clown, which means:

$$q \times 6 + (1-q) \times 1 = q \times 4 + (1-q) \times 2 \implies q = \frac{1}{3}$$

Caleb is indifferent between Party and No Party, which means:

$$p \times 0 + (1-p) \times 4 = p \times 1 + (1-p) \times 3 \implies p = \frac{1}{2}$$

The Nash equilibrium of the game is (($\frac{1}{2}$ Clown, $\frac{1}{2}$ No Clown), ($\frac{1}{3}$ Party, $\frac{2}{3}$ No Party))

$x = 3$ **and** $x = 5$ Nash equilibrium/dominated strategy can be found in a similar approach.