Homework 2: Solution

Deadline: May 17th, 2019

Problem Solving [50 points]

Q1. Perfect information EFG [20 points]. Consider the "cross-out game." In this game, one writes down the numbers 1, 2, 3. Person 1 starts by crossing out any one number or any two adjacent numbers: for example, person 1 might cross out 1, might cross out 1 and 2, or might cross out 2 and 3. Then person 2 also crosses out either one number or two adjacent numbers. For example, starting from 1, 2, 3, say person 1 crosses out 1. Then person 2 can either cross out 2, cross out 3, or cross out both 2 and 3. Play continues like this. Once a number is crossed out, it cannot be crossed out again. Also, if for example person 1 crosses out 2 in her first move, person 2 cannot then cross out both 1 and 3, because 1 and 3 are not adjacent (even though 2 is crossed out). The winner is the person who crosses out the last number.

1. Model this as an extensive form game. Show a subgame perfect Nash equilibrium of this game by drawing appropriate arrows in the game tree.

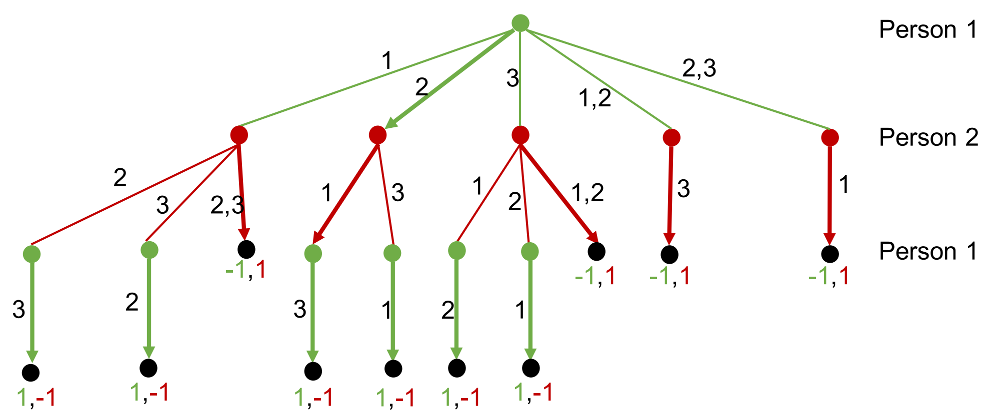

Figure 1: Extensive Form Games

2. Now instead of just three numbers, say that you start with m numbers. In other words, you have the numbers 1, 2, 3,..., m. Can person 1 always win this game? (Hint: look at m = 4, m = 5,etc. first to get some ideas.)

Solution. Yes, person 1 can always win. If m is even, person 1 first crosses out two numbers $(\frac{m}{2}, \frac{m}{2}+1)$. If m is odd, person 1 first crosses out the number in the middle $(\frac{m+1}{2})$. This choice divides the original array into 2 sub-arrays with equal sizes. Then whenever person 2 chooses an action of crossing out in one sub-array, person 1 will mimic the action of person 2 in the other sub-array until the game ends.

Q2. Imperfect information EFG. Consider the following imperfect information extensive form game with two players: player 1 and player 2.

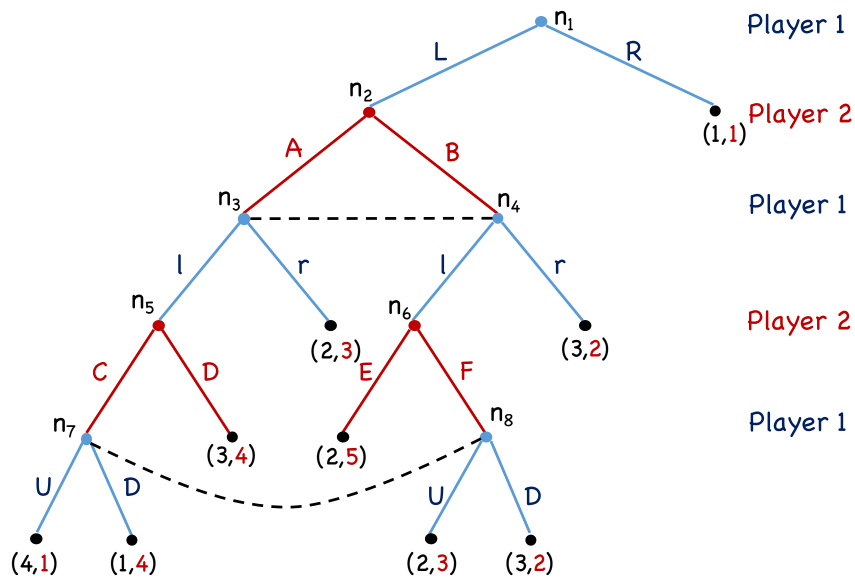

Figure 2: Caption

1. (10 points) Provide an example of a realization plan for each player.

Solution. A realization plan for player 1:

$$
\begin{aligned}
&r_1(\emptyset) = 1.0\\
&r_1(L) = 0.5, r_1(R) = 0.5\\
&r_1(Ll) = 0.2, r_1(Lr) = 0.3\\
&r_1(LlU) = 0.1, r_1(LlD) = 0.1.
\end{aligned}
$$

A realization plan for player 2:

$$
\begin{aligned}
&r_2(\emptyset) = 1.0\\
&r_2(A) = 0.5, r_2(B) = 0.5\\
&r_2(AC) = 0.2, r_2(AD) = 0.3\\
&r_2(BE) = 0.3, r_2(BF) = 0.2.
\end{aligned}
$$

2. (10 points) Find the behavioral strategy of each player corresponding to the provided realization plan.

Solution. The corresponding behavioral strategy of player 1:

$$\beta_1(L) = 0.5, \beta_1(R) = 0.5$$
$$\beta_1(l) = \frac{r_1(Ll)}{r_1(L)} = 0.4, \beta_1(r) = 0.6$$
$$\beta_1(U) = \frac{r_1(LlU)}{r_1(Ll)} = 0.5, \beta_1(D) = 0.5$$

The corresponding behavioral strategy for player 2:

$$\beta_2(A) = \beta_2(B) = 0.5$$
$$\beta_2(C) = \frac{r_2(AC)}{r_2(A)} = 0.4, \beta_2(D) = 0.6$$
$$\beta_2(E) = \frac{r_2(BE)}{r_2(B)} = 0.6, \beta_2(F) = 0.4$$

3. (Grads only) (10 points) Given the provided realization plan of player 1, write down the linear program to compute an optimal realization plan of player 2 (Reference: textbook).

Solution. The payoff function (with the corresponding realization plan of player 1):

	∅	A	B	AC	AD	BE	BF
∅ (1.0)	0,0	0,0	0,0	0,0	0,0	0,0	0,0
L (0.5)	0,0	0,0	0,0	0,0	0,0	0,0	0,0
R (0.5)	1,1	1,1	1,1	1,1	1,1	1,1	1,1
Ll (0.2)	0,0	0,0	0,0	0,0	3,4	2,5	0,0
Lr (0.3)	0,0	2,3	3,2	2,3	2,3	3,2	3,2
LlU (0.1)	0,0	0,0	0,0	4,1	3,4	2,5	2,3
LlD (0.1)	0,0	0,0	0,0	1,4	3,4	2,5	3,2

Table 1: Payoff function of the sequence form

We can remove the first three rows and the first column to obtain the reduced form:

	A	B	AC	AD	BE	BF
Ll (0.2)	0,0	0,0	0,0	3,4	2,5	0,0
Lr (0.3)	2,3	3,2	2,3	2,3	3,2	3,2
LlU (0.1)	0,0	0,0	4,1	3,4	2,5	2,3
LlD (0.1)	0,0	0,0	1,4	3,4	2,5	3,2

Table 2: Payoff function of the sequence form

The coefficient of each sequence of player 2 in the objective of the optimization program:

$$\begin{aligned}
r_2(A) &: 0.3 \times 3 = 0.9 \\
r_2(B) &: 0.3 \times 2 = 0.6 \\
r_2(AC) &: 0.3 \times 3 + 0.1 \times 1 + 0.1 \times 4 = 1.4 \\
r_2(AD) &: 0.2 \times 4 + 0.3 \times 3 + 0.1 \times 4 + 0.1 \times 4 = 2.5 \\
r_2(BE) &: 0.2 \times 5 + 0.3 \times 2 + 0.1 \times 5 + 0.1 \times 5 = 2.6 \\
r_2(BF) &: 0.3 \times 2 + 0.1 \times 3 + 0.1 \times 2 = 1.1
\end{aligned}$$

The optimization program:

$$\begin{aligned}
\max\ & 0.9r_2(A) + 0.6r_2(B) + 1.4r_2(AC) + 2.5r_2(AD) + 2.6r_2(BE) + 1.1r_2(BF) \\
\text{s.t. } & r_2(A) + r_2(B) = 1.0 \\
& r_2(AC) + r_2(AD) = r_2(A) \\
& r_2(BE) + r_2(BF) = r_2(B) \\
& r_2(A), r_2(B), r_2(AC), r_2(AD), r_2(BE), r_2(BF) \geq 0.
\end{aligned}$$

Q3. Coalition structure generation (undergrads only) (10 points). Consider the following coalitional game with 5 players. Compute an optimal coalition structure.
$v(\{1\}) = 30$, $v(\{2\}) = 40$, $v(\{3\}) = 25$, $v(\{4\}) = 45$, $v(\{5\}) = 35$
$v(\{1,2\}) = 50$,$v(\{1,3\}) = 60$, $v(\{1,4\}) = 80$, $v(\{1,5\}) = 70$, $v(\{2,3\}) = 55$, $v(\{2,4\}) = 70$, $v(\{2,5\}) = 50$, $v(\{3,4\}) = 80$, $v(\{3,5\}) = 65$, $v(\{4,5\}) = 85$
$v(\{1,2,3\}) = 90$, $v(\{1,2,4\}) = 120$, $v(\{1,2,5\}) = 115$, $v(\{1,3,4\}) = 100$, $v(\{1,3,5\}) = 90$, $v(\{1,4,5\}) = 125$, $v(\{2,3,4\}) = 115$, $v(\{2,3,5\}) = 85$, $v(\{2,4,5\}) = 130$, $v(\{3,4,5\}) = 100$
$v(\{1,2,3,4\}) = 140$, $v(\{1,2,3,5\}) = 165$,$v(\{1,2,4,5\}) = 130$, $v(\{1,3,4,5\}) = 175$, $v(\{2,3,4,5\}) = 160$
$v(\{1,2,3,4,5\}) = 200$

Solution. By following the dynamic programming approach as described in the lecture slides, the optimal coalition structure is $\{\{2\},\{1,3,4,5\}\}$ with the corresponding value is 215.