Assignment 4

CIS 621: Algorithms and Complexity

Problem 1 (10 points) For the graph $G = (\mathcal{U}, \mathcal{E})$, where $\mathcal{U}$ is the set of vertices and $\mathcal{E}$ is the set of edges, we define the following nonlinear integer program, where $w_{i,j} \geq 0$, $\forall (i,j) \in \mathcal{E}$ and k is a nonnegative integer:

$$\max \quad \sum_{(i,j) \in \mathcal{E}} w_{i,j}(x_i + x_j - 2x_i x_j)$$

$$s.t. \quad \sum_{i \in \mathcal{U}} x_i = k\,,$$

$$x_i \in \{0, 1\}, \ \forall i \in \mathcal{U}\,.$$

- Show that the following linear program is a relaxation of the above problem:

$$\max \quad \sum_{(i,j) \in \mathcal{E}} w_{i,j} z_{i,j}$$

$$s.t. \quad z_{i,j} \leq x_i + x_j, \ \forall (i,j) \in \mathcal{E}\,,$$

$$z_{i,j} \leq 2 - x_i - x_j, \ \forall (i,j) \in \mathcal{E}\,,$$

$$\sum_{i \in \mathcal{U}} x_i = k\,,$$

$$0 \leq x_i \leq 1, \ \forall i \in \mathcal{U}\,,$$

$$0 \leq z_{i,j} \leq 1, \ \forall (i,j) \in \mathcal{E}\,.$$

- Let $F(x) = \sum_{(i,j) \in \mathcal{E}} w_{i,j}(x_i + x_j - 2x_i x_j)$ be the objective function of the nonlinear integer program. Show that for any (x, z) that is feasible to the linear program, $F(x) \geq \frac{1}{2} \sum_{(i,j) \in \mathcal{E}} w_{i,j} z_{i,j}$.

Problem 2 (10 points) For the directed graph $G = (\mathcal{U}, \mathcal{E})$, where $\mathcal{U}$ is the set of vertices and $\mathcal{E}$ is the set of directed edges, we want to partition $\mathcal{U}$ into two sets $\mathcal{V}$ and $\mathcal{W} = \mathcal{U} \setminus \mathcal{V}$ in order to maximize the total weight of the edges going from $\mathcal{V}$ to $\mathcal{W}$ (i.e., the edges (i,j) with $i \in \mathcal{V}$ and $j \in \mathcal{W}$).

- Give a randomized $\frac{1}{4}$-approximation algorithm for this problem.

- Show that the following linear program is a relaxation of this problem:

$$\max \quad \sum_{(i,j) \in \mathcal{E}} w_{i,j} z_{i,j}$$

$$s.t. \quad z_{i,j} \leq x_i, \ \forall (i,j) \in \mathcal{E}\,,$$

$$z_{i,j} \leq 1 - x_j, \ \forall (i,j) \in \mathcal{E}\,,$$

$$0 \leq x_i \leq 1, \ \forall i \in \mathcal{U}\,,$$

$$0 \leq z_{i,j} \leq 1, \ \forall (i,j) \in \mathcal{E}\,.$$

- For the above linear program, give a randomized $\frac{1}{2}$-approximation algorithm based on rounding x_i, $\forall i \in \mathcal{U}$ to 1 with the probability of $\frac{1}{2}x_i + \frac{1}{4}$.