CIS 621 Assignment 4

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Problem 1

For the graph $G = (\mathcal{U}, \mathcal{E})$, where $\mathcal{U}$ is the set of vertices and $\mathcal{E}$ is the set of edges, we define the following nonlinear integer program, where $w_{i,j} \geq 0, \forall (i,j) \in \mathcal{E}$ and k is a nonnegative integer:

$$\begin{aligned}
&\sup \sum_{(i,j)\in\mathcal{E}} w_{i,j}(x_i + x_j - 2x_i x_j) \\
&s.t. \sum_{i\in\mathcal{U}} x_i = k \\
&\quad\quad x_i \in \{0,1\}, \forall i \in \mathcal{U}
\end{aligned}$$

Show that the following linear program is a relaxation of the above problem:

$$\begin{aligned}
&\sup \sum_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j} \\
s.t. \quad & z_{i,j} \leq x_i + x_j, \forall (i,j) \in \mathcal{E} \\
& z_{i,j} \leq 2 - x_i - x_j, \forall (i,j) \in \mathcal{E} \\
& \sum_{i\in\mathcal{U}} x_i = k \\
& 0 \leq x_i \leq 1, \forall i \in \mathcal{U} \\
& 0 \leq z_{i,j} \leq 1, \forall (i,j) \in \mathcal{E}
\end{aligned}$$

Also, let $F(x) = \sum\limits_{(i,j)\in\mathcal{E}} w_{i,j}(x_i + x_j - 2x_i x_j)$ be the objective function of the nonlinear integer program. Show that for any (x, z) that is feasible to the linear program, $F(x) \geq \frac{1}{2} \sum\limits_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j}$

Part 1)

If we remember the definition of linear relaxation we see that it is

$$x_i \in \{0,1\} \mapsto 0 \leq x_i \leq 1$$

We can see from the given problem that

$$z_{i,j} = x_i + x_j - 2x_ix_j$$

We may also notice that $z_{i,j} = 0 \forall x_i, x_j \in \{0,1\}$. Clearly this needs a relaxation so we can approximate a solution. From here we can easily check the given linear program is a relaxation of the initial integer program.
It is trivial to see that $z_{i,j} \leq x_i + x_j \because$ the negative term ensures that $z_{i,j}$ is smaller. Similarly we can see that if we remove the x_ix_j term from the 2 that we obtain a maximal value and thus our $z_{i,j}$ has to be smaller than that. Finally, we see that we have the relaxation terms where $x_{i,j}$ and $z_{i,j}$ are not limited to integer values. Therefore we can conclude that this is a relaxation of the initial problem.