Assignment 4

CIS 621: Algorithms and Complexity

Problem 1 (10 points) For the graph $G = (\mathcal{U}, \mathcal{E})$, where $\mathcal{U}$ is the set of vertices and $\mathcal{E}$ is the set of edges, we define the following nonlinear integer program, where $w_{i,j} \geq 0$, $\forall (i,j) \in \mathcal{E}$ and k is a nonnegative integer:

$$\begin{aligned} \max \quad & \sum_{(i,j)\in\mathcal{E}} w_{i,j}(x_i + x_j - 2x_i x_j) \\ s.t. \quad & \sum_{i\in\mathcal{U}} x_i = k\,, \\ & x_i \in \{0,1\},\ \forall i \in \mathcal{U}\,. \end{aligned}$$

- Show that the following linear program is a relaxation of the above problem:

$$\begin{aligned} \max \quad & \sum_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j} \\ s.t. \quad & z_{i,j} \leq x_i + x_j,\ \forall (i,j) \in \mathcal{E}\,, \\ & z_{i,j} \leq 2 - x_i - x_j,\ \forall (i,j) \in \mathcal{E}\,, \\ & \sum_{i\in\mathcal{U}} x_i = k\,, \\ & 0 \leq x_i \leq 1,\ \forall i \in \mathcal{U}\,, \\ & 0 \leq z_{i,j} \leq 1,\ \forall (i,j) \in \mathcal{E}\,. \end{aligned}$$

- Let $F(x) = \sum_{(i,j)\in\mathcal{E}} w_{i,j}(x_i + x_j - 2x_i x_j)$ be the objective function of the nonlinear integer program. Show that for any (x, z) that is feasible to the linear program, $F(x) \geq \frac{1}{2}\sum_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j}$.

Solution (1) First, any x_i, x_j that are feasible for the integer program are also feasible for the linear program. Second, in the linear program, denoting z^*, x^* as the optimal solution, we have $z^*_{i,j} = \min\{x^*_i + x^*_j, 2 - x^*_i - x^*_j\}$, $\forall i$, j. Let's consider all possible x^*_i, x^*_j in the four cases of (i) $x^*_i = 0$, $x^*_j = 0$, (ii) $x^*_i = 0$, $x^*_j = 1$, (iii) $x^*_i = 1$, $x^*_j = 0$, and (iv) $x^*_i = 1$, $x^*_j = 1$. No matter in which case, the objective function value of the linear program equals that of the integer program. Consequently, the linear program is a relaxation to the integer program.

(2) If $x_i + x_j \leq 2 - x_i - x_j$, i.e., $x_i + x_j \leq 1$, then $\frac{1}{2}\sum_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j} \leq \frac{1}{2}\sum_{(i,j)\in\mathcal{E}} w_{i,j}(x_i + x_j)$. So, we need to show $x_i + x_j \leq 2(x_i + x_j - 2x_i x_j)$. It actually already holds, because $x_i + x_j = 2(x_i + x_j) - (x_i + x_j) \leq 2(x_i + x_j) - (x_i + x_j)^2 \leq 2(x_i + x_j) - 4x_i x_j$.

If $x_i + x_j \geq 2 - x_i - x_j$, i.e., $x_i + x_j \geq 1$, then $\frac{1}{2}\sum_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j} \leq \frac{1}{2}\sum_{(i,j)\in\mathcal{E}} w_{i,j}(2 - x_i - x_j)$. So, we need to show $2 - x_i - x_j \leq 2(x_i + x_j - 2x_i x_j)$. It actually already holds, because $2 - x_i - x_j \leq (x_i + x_j)(2 - (x_i + x_j)) = 2(x_i + x_j) - (x_i + x_j)^2 \leq 2(x_i + x_j) - 4x_i x_j$.

Problem 2 (10 points) For the directed graph $G = (\mathcal{U}, \mathcal{E})$, where $\mathcal{U}$ is the set of vertices and $\mathcal{E}$ is the set of directed edges, we want to partition $\mathcal{U}$ into two sets $\mathcal{V}$ and $\mathcal{W} = \mathcal{U} \setminus \mathcal{V}$ in order to maximize the total weight of the edges going from $\mathcal{V}$ to $\mathcal{W}$ (i.e., the edges (i,j) with $i \in \mathcal{V}$ and $j \in \mathcal{W}$).

- Give a randomized $\frac{1}{4}$-approximation algorithm for this problem.
- Show that the following linear program is a relaxation of this problem:

$$\begin{aligned} \max \quad & \sum_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j} \\ s.t. \quad & z_{i,j} \leq x_i,\ \forall (i,j) \in \mathcal{E}\,, \\ & z_{i,j} \leq 1 - x_j,\ \forall (i,j) \in \mathcal{E}\,, \end{aligned}$$

$$0 \le x_i \le 1,\ \forall i \in \mathcal{U}\,,$$
$$0 \le z_{i,j} \le 1,\ \forall (i,j) \in \mathcal{E}\,.$$

- For the above linear program, give a randomized $\frac{1}{2}$-approximation algorithm based on rounding x_i, $\forall i \in \mathcal{U}$ to 1 with the probability of $\frac{1}{2}x_i + \frac{1}{4}$.

Solution (1) Consider an algorithm that places i in $\mathcal{V}$ with the probability p and in $\mathcal{W}$ with the probability $1-p$. For any edge (i,j), let's use $x_{i,j}$ to denote counting in its edge weight if $x_{i,j}=1$ and excluding its edge weight if $x_{i,j}=0$. Thus, for this algorithm, we have $E(\sum_{(i,j)\in\mathcal{E}} w_{i,j}x_{i,j}) = \sum_{(i,j)\in\mathcal{E}} w_{i,j}E(x_{i,j}) = p(1-p)\sum_{(i,j)\in\mathcal{E}} w_{i,j} \ge p(1-p)OPT$, where OPT is the optimal sum of the weights of the edges going from $\mathcal{V}$ to $\mathcal{W}$. That is, we have a $p(1-p)$-approximation algorithm. Let $p(1-p)=\frac{1}{4}$, and we get $p=\frac{1}{2}$.

(2) Omitted; similar to "(1)" of the previous problem in this assignment.

(3) Denoting $x_i, \forall i$ and $z_{i,j}, \forall i,j$ as the optimal fractional solution, and $\bar{x}_i, \forall i$ and $\bar{z}_{i,j}, \forall i,j$ as the optimal integer solution by rounding x_i and $z_{i,j}$, respectively. We have $E(\sum_{(i,j)\in\mathcal{E}} w_{i,j}\bar{z}_{i,j}) = \sum_{(i,j)\in\mathcal{E}} w_{i,j}E(\bar{z}_{i,j})$. If $\bar{x}_i \le 1-\bar{x}_j$, then $E(\bar{z}_{i,j}) = E(\bar{x}_i) = \frac{1}{2}x_i + \frac{1}{4} \ge \frac{1}{2}x_i = \frac{1}{2}z_{i,j}$; if $\bar{x}_i > 1-\bar{x}_j$, then $E(\bar{z}_{i,j}) = E(1-\bar{x}_j) = 1-(\frac{1}{2}x_j+\frac{1}{4}) \ge \frac{1}{2}(1-x_j) = \frac{1}{2}z_{i,j}$. Therefore, it is a $\frac{1}{2}$-approximation algorithm.