CIS 621 Assignment 4

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Problem 1

For the graph $G = (\mathcal{U}, \mathcal{E})$, where $\mathcal{U}$ is the set of vertices and $\mathcal{E}$ is the set of edges, we define the following nonlinear integer program, where $w_{i,j} \geq 0, \forall (i,j) \in \mathcal{E}$ and k is a nonnegative integer:

$$\begin{aligned} &\sup \sum_{(i,j)\in\mathcal{E}} w_{i,j}(x_i + x_j - 2x_i x_j) \\ &s.t. \sum_{i\in\mathcal{U}} x_i = k \\ &\quad x_i \in \{0,1\}, \forall i \in \mathcal{U} \end{aligned}$$

Show that the following linear program is a relaxation of the above problem:

$$\begin{aligned} &\sup \sum_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j} \\ s.t. \quad & z_{i,j} \leq x_i + x_j, \forall (i,j) \in \mathcal{E} \\ & z_{i,j} \leq 2 - x_i - x_j, \forall (i,j) \in \mathcal{E} \\ & \sum_{i\in\mathcal{U}} x_i = k \\ & 0 \leq x_i \leq 1, \forall i \in \mathcal{U} \\ & 0 \leq z_{i,j} \leq 1, \forall (i,j) \in \mathcal{E} \end{aligned}$$

Also, let $F(x) = \sum\limits_{(i,j)\in\mathcal{E}} w_{i,j}(x_i + x_j - 2x_i x_j)$ be the objective function of the nonlinear integer program. Show that for any (x, z) that is feasible to the linear program, $F(x) \geq \frac{1}{2} \sum\limits_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j}$

Part 1)

If we remember the definition of linear relaxation we see that it is

$$x_i \in \{0,1\} \mapsto 0 \leq x_i \leq 1$$

We can see that the conditions on x_i match this. We also need to show that there is an injective map from the linear program to the integer linear program, that is that the integer linear program is a subset of the linear program. There is a 1 to 1 map from ILP to LP, but ILP is not onto LP.

x_i	x_j	ILP	LP
0	0	0	0
1	0	1	1
0	1	1	1
1	1	0	0

Figure 1: Table of injective map from ILP $\mapsto$ LP

Part 2)

To show that $F(x) \geq \frac{1}{2} \sum\limits_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j}$ we can similarly show an injective map. First we will use the condition that $z_{i,j} \leq x_i + x_j$ This last condition fails, so we need to

x_i	x_j	$F(x)$	$\frac{1}{2}\sum w_{i,j}z_{i,j}$
0	0	0	0
1	0	1	1
0	1	1	1
1	1	0	$\frac{1}{2}$

check our other constraint, that $z_{i,j} \leq 2 - x_i - x_j$ Looking at this condition we can see that when $x_i = x_j = 1$

x_i	x_j	$F(x)$	$\frac{1}{2}\sum w_{i,j}z_{i,j}$
1	1	0	0

This shows that any feasible (x, z) to the **integer** linear program, but we need to show for the linear program. We could do a similar table but we'd need an infinite number of values. But we know that the ILP provides bounds to the LP, because of the constraints. We can see by graphing the LP constraints.
From this graph we see the conditions of $z_{i,j}$, where it must be less than this. From this graph we can see that the bounds given in the above table show that all (x, z) that are feasible, under the constraints, satisfy this condition.

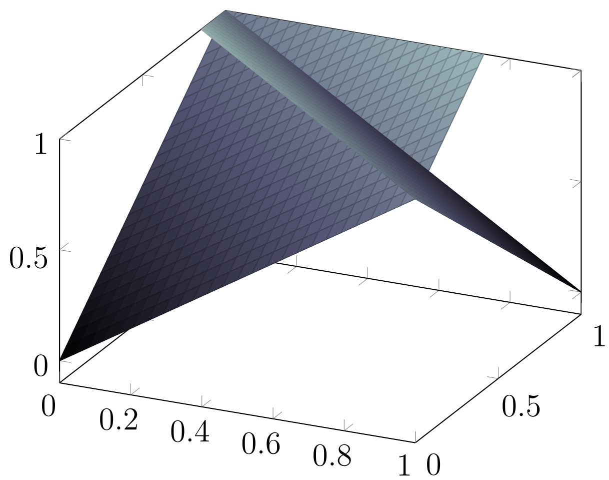

Figure 2: Bounds of $z_{i,j}$

Problem 2

For the directed graph $G = (\mathcal{U}, \mathcal{E})$, where $\mathcal{U}$ is the set of vertices and $\mathcal{E}$ is the set of directed edges, we want to partition $\mathcal{U}$ into two sets, $\mathcal{V}$ and $\mathcal{W} = \mathcal{U}/\mathcal{V}$, in order to maximize the total weight of the edges going from $\mathcal{V}$ to $\mathcal{W}$ (edges (i, j) with $i \in \mathcal{V}$ and $j \in \mathcal{W}$)

- Give a randomized $\frac{1}{4}$-approximation algorithm for this problem.
- Show that the following linear program is a relaxation of this problem.

$$\begin{aligned} max \sum_{(i,j)\in\mathcal{E}} & w_{i,j} z_{i,j} \\ s.t. \quad & z_{i,j} \leq x_i, \forall (i,j) \in \mathcal{E} \\ & z_{i,j} \leq 1 - x_j, \forall (i,j) \in \mathcal{E} \\ & 0 \leq x_i \leq 1, \forall i \in \mathcal{U} \\ & 0 \leq z_{i,j} \leq 1, \forall (i,j) \in \mathcal{E} \end{aligned}$$

- For the above linear program, give a randomized $\frac{1}{2}$-approximation algorithm based on rounding $x_i \forall i \in \mathcal{U}$ to 1, with the probability of $\frac{1}{2}x_i + \frac{1}{4}$

Part 1)

For the $\frac{1}{4}$ approximation we can look at the expectation value of the function, we only need to consider $z_{i,j}$

$$\begin{aligned}\langle z_{i,j}\rangle &= Pr(vertex\ within\ cut \wedge from\ i\ to\ j)\\ & Pr(vertex\ within\ cut)Pr(from\ i\ to\ j)\\ & \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\\ & \frac{1}{4}\end{aligned}$$

Part 2)

For this we need to describe the original problem. We can say

$$\begin{aligned}\max \qquad & \sum_{(i,j)\in\mathcal{E}} w_{i,j}(1-x_i)x_j\\ s.t. \qquad & x \in \{0,1\}\end{aligned}$$

We can similarly use an injective map to show that the LP (given) is a relaxation of the above ILP.

x_i	x_j	ILP	LP
0	0	0	0
1	0	0	0
0	1	1	1
1	1	0	0

It is easy to see that in the ILP that any time $x_j = 0$ or $x_i = 1$ then the objective function is 0. We can do the same thing by looking at the LP constraints, showing that $z_{i,j} \leq x_i$ and $z_{i,j} \leq 1 - x_j$. Similar to problem 1 we need both constraints.

Part 3)

If we look back at our expectation values we can see that the probability that $Pr(x \in \mathcal{V}) = \frac{1}{2}$, and from here it is easy to see that the rounding is $\frac{1}{2}x_i$ (given that it is bound between 0 and 1). We can then add the $\frac{1}{4}$ approximation algorithm to get the result