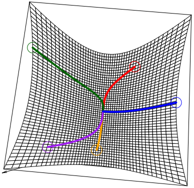
Convex Sets

why convexity?

Why convexity? Simply put: because we can broadly understand and solve convex optimization problems

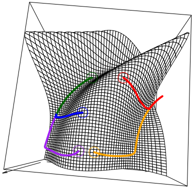
Nonconvex problems are mostly treated on a case by case basis

a convex optimization problem is of the form



$$\begin{aligned} \min_{x \in D} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \le 0, \ i = 1, \dots m \\ & h_j(x) = 0, \ j = 1, \dots r \end{aligned}$$

where f and g_i, $i = 1, \dots m$ are all convex, and h_j, $j = 1, \dots r$ are affine.



Outline

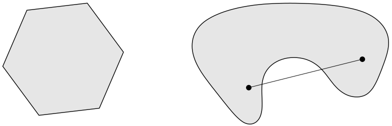
- Convex sets
- Examples
- Key properties
- Operations preserving convexity

Convex sets

Convex set: $C \subseteq \mathbb{R}^n$ such that

$$x, y \in C \implies tx + (1-t)y \in C \text{ for all } 0 \le t \le 1$$

In words, line segment joining any two elements lies entirely in set



Convex combination of $x_1, \dots x_k \in \mathbb{R}^n$: any linear combination

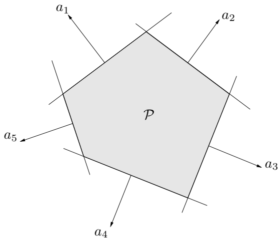
$$\theta_1 x_1 + \dots + \theta_k x_k$$

with $\theta_i \ge 0$, $i = 1, \dots k$, and $\sum_{i=1}^k \theta_i = 1$. Convex hull of a set C, $\mathrm{conv}(C)$, is all convex combinations of elements. Always convex

Examples of convex sets

- Trivial ones: empty set, point, line
- Norm ball: $\{x : \|x\| \leq r\}$, for given norm $\|\cdot\|$, radius r
- Hyperplane: $\{x : a^T x = b\}$, for given a, b
- Halfspace: $\{x : a^T x \leq b\}$
- Affine space: $\{x : Ax = b\}$, for given A, b

- Polyhedron: $\{x : Ax \leq b\}$, where inequality $\leq$ is interpreted componentwise. Note: the set $\{x : Ax \leq b, Cx = d\}$ is also a polyhedron.



Operations preserving convexity

- Intersection: the intersection of convex sets is convex
- Scaling and translation: if C is convex, then

  $$aC + b = \{ax + b : x \in C\}$$

  is convex for any a, b
- Affine images and preimages: if $f(x) = Ax + b$ and C is convex then

  $$f(C) = \{f(x) : x \in C\}$$

  is convex, and if D is convex then

  $$f^{-1}(D) = \{x : f(x) \in D\}$$

  is convex

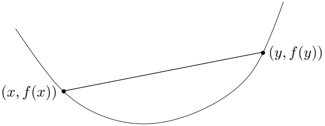
Convex Functions

Convex functions

Convex function: $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathrm{dom}(f) \subseteq \mathbb{R}^n$ convex, and

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \quad \text{for } 0 \le t \le 1$$

and all $x, y \in \mathrm{dom}(f)$



In words, f lies below the line segment joining $f(x), f(y)$

Concave function: opposite inequality above, so that

$$f \text{ concave} \iff -f \text{ convex}$$

Important modifiers:

- Strictly convex: $f\big(tx + (1-t)y\big) < tf(x) + (1-t)f(y)$ for $x \neq y$ and $0 < t < 1$. In words, f is convex and has greater curvature than a linear function

Note: strictly convex $\Rightarrow$ convex

(Analogously for concave functions)

Examples of convex functions

- Univariate functions:
  - Exponential function: e^{ax} is convex for any a over $\mathbb{R}$
  - Power function: x^a is convex for $a \geq 1$ or $a \leq 0$ over $\mathbb{R}_+$ (nonnegative reals)
  - Power function: x^a is concave for $0 \leq a \leq 1$ over $\mathbb{R}_+$
  - Logarithmic function: $\log x$ is concave over $\mathbb{R}_{++}$
- Affine function: $a^T x + b$ is both convex and concave
- Quadratic function: $\frac{1}{2} x^T Q x + b^T x + c$ is convex provided that $Q \succeq 0$ (positive semidefinite)

- Indicator function: if C is convex, then its indicator function

$$I_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$$

  is convex

- Support function: for any set C (convex or not), its support function

$$I_C^*(x) = \max_{y \in C} \; x^T y$$

  is convex

- Max function: $f(x) = \max\{x_1, \ldots x_n\}$ is convex

Key properties of convex functions

- First-order characterization: if f is differentiable, then f is convex if and only if $\mathrm{dom}(f)$ is convex, and

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

  for all $x, y \in \mathrm{dom}(f)$. Therefore for a differentiable convex function $\nabla f(x) = 0 \iff x$ minimizes f

- Second-order characterization: if f is twice differentiable, then f is convex if and only if $\mathrm{dom}(f)$ is convex, and $\nabla^2 f(x) \succeq 0$ for all $x \in \mathrm{dom}(f)$

Operations preserving convexity

- Nonnegative linear combination: $f_1, \dots f_m$ convex implies $a_1 f_1 + \dots + a_m f_m$ convex for any $a_1, \dots a_m \geq 0$
- Pointwise maximization: if f_s is convex for any $s \in S$, then $f(x) = \max_{s \in S} \ f_s(x)$ is convex. Note that the set S here (number of functions f_s) can be infinite
- Partial minimization: if $g(x, y)$ is convex in x, y, and C is convex, then $f(x) = \min_{y \in C} \ g(x, y)$ is convex

More operations preserving convexity

- Affine composition: f convex implies $g(x) = f(Ax + b)$ convex
- General composition: suppose $f = h \circ g$, where $g : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$. Then:
  - f is convex if h is convex and nondecreasing, g is convex
  - f is convex if h is convex and nonincreasing, g is concave
  - f is concave if h is concave and nondecreasing, g concave
  - f is concave if h is concave and nonincreasing, g convex

  How to remember these? Think of the chain rule when $n = 1$:

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$