Assignment 1

CIS 621: Algorithms and Complexity

Problem 1 (2 points) Prove the NP-completeness of the following problem by reduction: given n positive integers x_1, x_2, ..., x_n, and another positive integer w, is there a subset of the n integers that add up to exactly w?

Solution 1 This is the subset sum problem. To prove its NP-completeness, we just need to demonstrate that an existing NP-complete problem can reduce to the subset sum problem. Here, we choose the knapsack problem which is a known NP-complete problem. The knapsack problem says the following: given n objects with their values v_1, v_2, ..., v_n and corresponding weights w_1, w_2, ..., w_n (where all values and weights are positive integers), is there a subset of the objects such that the total value of the objects in this subset is no less than V and the total weight of the objects in this subset is no greater than W (where V and W are positive integers)?

To exhibit that the subset sum problem contains the knapsack problem as a special case, we set $x_i = v_i = w_i, \forall i = 1, 2, ..., n$ and set $w = V = W$. That is, for any subset sum problem, we can always construct a corresponding knapsack problem; if one can find a solution to the original subset sum problem, then one also finds a solution to the constructed knapsack problem. Thus, solving subset sum is (at least) as hard as solving knapsack.

Problem 2 Given $a, x \in \mathbb{R}^2$, where $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $b_1, b_2 \in \mathbb{R}$,

- **2.1 (1 point)** prove $a^T x = \|a\|\|x\| \cos\theta$ using the law of cosine, where θ is the angle between a and x;
- **2.2 (1 point)** calculate the distance between the two parallel hyperplanes $\{x|a^T x = b_1\}$ and $\{x|a^T x = b_2\}$.

Solution 2.1 Note $a^T x = a_1x_1 + a_2x_2$, $\|a\|\|x\| \cos\theta = \sqrt{a_1^2 + a_2^2}\sqrt{x_1^2 + x_1^2}\frac{(a_1^2+a_2^2)+(x_1^2+x_1^2)-((a_1-x_1)^2+(a_2-x_2)^2)}{2\sqrt{a_1^2+a_2^2}\sqrt{x_1^2+x_1^2}} = a_1x_1 + a_2x_2$. Therefore we complete the proof. We use the law of cosine to calculate $\cos\theta$.

Solution 2.2 The distance from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to any hyperplane $\{x|a^T x = b\}$ is $\frac{b}{\|a\|}$. Therefore, the distance between the two hyperplanes $\{x|a^T x = b_1\}$ and $\{x|a^T x = b_2\}$ is $\frac{|b_1-b_2|}{||a||}$.

Problem 3 (2 points) Given $x_0, x_1, ..., x_k \in \mathbb{R}^n$. Consider the set of points that are closer to x_0 than any other x_i, i.e., $S = \{x \in \mathbb{R}^n | \|x - x_0\| \le \|x - x_i\|, i = 1, 2, ..., k\}$. Is S a polyhedron? If so, express it in the form of $S = \{x|Ax \preceq b\}$. If not, explain why.

Solution The set $S_i = \{x \in \mathbb{R}^n | \|x - x_0\| \le \|x - x_i\|\}$ is a halfspace, and can be written as $(x_i - x_0)^T (x - x_0) \le \frac{||x_i-x_0||^2}{2}$, $\forall i$. Thus, $A = \begin{bmatrix} (x_1 - x_0)^T \\ (x_2 - x_0)^T \\ \vdots \\ (x_k - x_0)^T \end{bmatrix}$, $b = \begin{bmatrix} \frac{||x_1-x_0||^2}{2} + (x_1 - x_0)^T x_0 \\ \frac{||x_2-x_0||^2}{2} + (x_2 - x_0)^T x_0 \\ \vdots \\ \frac{||x_k-x_0||^2}{2} + (x_k - x_0)^T x_0 \end{bmatrix}$.

Problem 4 (2 points) Let f be a twice differentiable function, with $dom(f)$ convex. Prove f is convex if and only if $(\nabla f(x) - \nabla f(y))^T (x - y) \ge 0$.

Solution Necessary condition: If $x \le y$, by the definition of convex functions, we have $\nabla f(x)^T (y - x) \le f(y) - f(x)$. If $y \le x$, by the definition of convex functions, we have $\nabla f(y)^T (x - y) \le f(x) - f(y)$. Summing up these two inequalities, we complete the proof. (Note this step does not actually require f to be *twice* differentiable.)

Sufficient condition: $(\nabla f(x) - \nabla f(y))^T (x - y) \ge 0$ means $\nabla^2 f(x) = \lim_{x \to y} \frac{\nabla f(x) - \nabla f(y)}{x - y} \ge 0$. Thus f is convex.

Problem 5 Prove the following functions are convex:

- **5.1 (1 point)** $f(x) = \max\{f_1(x), f_2(x), ..., f_m(x)\}$, where $f_i(x), i = 1, 2, ..., m$ are convex;

- **5.2 (1 point)** $f(x) = \min_{y \in C} g(x,y)$, where $g(x,y)$ is convex in both x and y, and C is a convex set;
- **5.3 (2 points)** $f(x,y) = \frac{|x_1|^p + |x_2|^p + \ldots + |x_n|^p}{y^{p-1}}$, where $p > 1, x \in \mathbb{R}^n, y \in \mathbb{R}_{++}$;
- **5.4 (2 points)** $f(x_{11}, ..., x_{1n}, x_{21}, ..., x_{2n}, ..., x_{m1}, ..., x_{mn}) = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} ((x_{ij}+1)\ln(x_{ij}+1) - x_{ij})$, where $x_{ij} \in \mathbb{R}_{++}$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$.

Solution 5.1 Consider $x, y, 0 \le t \le 1$, we need to show $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$. Assume for some n, where $1 \le n \le m$, we have $\max\{f_1(tx+(1-t)y), f_2(tx+(1-t)y), ..., f_m(tx+(1-t)y)\} = f_n(tx+(1-t)y)$. Then, we can derive the following:

$$\begin{aligned}
& f(tx + (1-t)y) \\
&= \max\{f_1(tx + (1-t)y), f_2(tx + (1-t)y), ..., f_m(tx + (1-t)y)\} \\
&= f_n(tx + (1-t)y) \\
&\le tf_n(x) + (1-t)f_n(y) \\
&\le t\max\{f_1(x), f_2(x), ..., f_m(x)\} + (1-t)\max\{f_1(y), f_2(y), ..., f_m(y)\} \\
&= tf(x) + (1-t)f(y)
\end{aligned}$$

Solution 5.2 For any (x_1, y_1), (x_2, y_2), and $0 \le t \le 1$, we have

$$g(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \le tg(x_1, y_1) + (1-t)g(x_2, y_2).$$

That is, for any y_1, y_2, if we let $x_1 \triangleq \arg\min_x g(x, y_1)$ and let $x_2 \triangleq \arg\min_x g(x, y_2)$, we have

$$\begin{aligned}
& \min_x g(x, ty_1 + (1-t)y_2) \\
&\le g(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \\
&\le tg(x_1, y_1) + (1-t)g(x_2, y_2) \\
&= t\min_x g(x, y_1) + (1-t)\min_x g(x, y_2).
\end{aligned}$$

Solution 5.3 Note $f(x,y) = \frac{\|x\|_p^p}{y^{p-1}}$. Consider $x_1, y_1, x_2, y_2, 0 \le t \le 1$, we need to show $f(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \le tf(x_1, y_1) + (1-t)f(x_2, y_2)$. We can have the following:

$$\begin{aligned}
& \|tx_1 + (1-t)x_2\|_p^p \\
&\le t^p\|x_1\|_p^p + (1-t)^p\|x_2\|_p^p \\
&\le t\|x_1\|_p^p(t + (1-t)\tfrac{y_2}{y_1})^{p-1} + (1-t)\|x_2\|_p^p(t\tfrac{y_1}{y_2} + (1-t))^{p-1}.
\end{aligned}$$

Dividing both sides of the inequality by $(ty_1 + (1-t)y_2)^{p-1}$, we get the following and also complete the proof:

$$\frac{\|tx_1 + (1-t)x_2\|_p^p}{(ty_1 + (1-t)y_2)^{p-1}} \le t\frac{\|x_1\|_p^p}{y_1^{p-1}} + (1-t)\frac{\|x_2\|_p^p}{y_2^{p-1}}.$$

Solution 5.4 Note that f is twice differentiable. We have $\frac{\partial^2 f}{\partial x_{ij}^2} = \frac{1}{x_{ij}+1} > 0$, $\forall i, j$ and $\frac{\partial^2 f}{\partial x_{ij} x_{kh}} = 0$, $\forall i, j, k, h$, where $i \ne k$ and $j \ne h$. That said,

$$\nabla^2 f = \begin{bmatrix}
\frac{1}{x_{11}+1} & 0 & ... & 0 \\
0 & \frac{1}{x_{12}+1} & ... & 0 \\
... & ... & \ddots & 0 \\
0 & 0 & 0 & \frac{1}{x_{mn}+1}
\end{bmatrix}.$$

We can verify that $\forall x \in \mathbb{R}^{mn}$, $x^T \nabla^2 f x \ge 0$, i.e., $\nabla^2 f \succeq 0$. Thus f is convex.

Problem 6 (1 point) Consider the function $f(x) = \max\{|a^T x + b|, \ln\frac{1}{c^T x + d}\}$, where $a, c, x \in \mathbb{R}^n$ and $b, d \in \mathbb{R}$. Is this a convex function? Explain why.

Solution We know that $|x|$ and $\ln\frac{1}{x}$ are convex functions. We also know that for any convex function $f(x)$, $f(a^T x + b)$ is also a convex function, and thus $|a^T x + b|$ and $\ln\frac{1}{c^T x + d}$ are convex. Besides, we know that $f(x) = \max\{f_1(x), f_2(x), ..., f_m(x)\}$ is convex if $f_i(x), i = 1, 2, ..., m$ are all convex, and thus $\max\{|a^T x + b|, \ln\frac{1}{c^T x + d}\}$ is convex.