# CIS 621 Assignment 1

Steven Walton

February 8, 2019

## Problem 1

For the linear program below, where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, d_3, e_1, e_2, e_3$ are non-zero constants, derive (1) its dual linear program and (2) the KKT conditions for the dual linear program.

$$\inf_{x_1 \geq 0, x_2 \leq 0, x3} a_1x_1 + a_2x_2 + a_3x_3$$

$$\begin{aligned} s.t. \qquad b_1x_1 + b_2x_2 + b_3x_3 &\leq e_1 \\ c_1x_1 + c_2x_2 &= e_3 \\ d_3x_3 &\geq e_3 \end{aligned}$$

### Part 1)

Solving for the dual problem

First we rewrite as a sup function and rearrange our constraints

$$\sup_{x_1 \geq 0, x_2 \leq 0, x_3} -a_1x_1 - a_2x_2 - a_3x_3$$

$$\begin{aligned} s.t. \qquad b_1x_1 + b_2x_2 + b_3x_3 - e_1 &\leq 0 \\ c_1x_1 + c_2x_2 - e_2 &= 0 \\ e_3 - d_3x_3 &\leq 0 \end{aligned}$$

Now we rewrite and regroup with the associated x's

$$\sup_{\lambda_1 \geq 0, \lambda_2, \lambda_3 \geq 0} \sup_{x_1 \geq 0, x_2 \leq 0, x_3} x_1(\lambda_1 b_1 + \lambda_2 c_1 - a_1) + x_2(\lambda_1 b_2 + c_2 - a_2) + x_3(b_3 - a_3 - \lambda_3 d_3) - \lambda_1 e_1 - \lambda_2 e_2 + \lambda_3 e_3$$

Rewriting

$$
\begin{aligned}
\sup_{\lambda_1 \geq 0, \lambda_2, \lambda_3 \geq 0} \quad & -\lambda_1 e_1 - \lambda_2 e_2 + \lambda_3 e_3 \\
s.t. \quad & \lambda_1 b_1 + \lambda_2 c_1 \geq a_1 \\
& \lambda_1 b_2 \leq a_2 - c_2 \\
& \lambda_3 d_3 = b_3 - a_3
\end{aligned}
$$

Resulting in the dual function

$$
\begin{aligned}
\inf_{\lambda_1 \geq 0, \lambda_2, \lambda_3 \geq 0} \quad & \lambda_1 e_1 + \lambda_2 e_2 - \lambda_3 e_3 \\
s.t. \quad & \lambda_1 b_1 + \lambda_2 c_1 \geq a_1 \\
& \lambda_1 b_2 \leq a_2 - c_2 \\
& \lambda_3 d_3 = b_3 - a_3
\end{aligned}
$$

## Part 2)

**Stationary:**

$$f(x) = x_1(a_1 + b_1 + c_1) + x_2(a_2 + b_2 + c_2) + x_3(a_3 + b_3 - d_3) - e_1 - e_2 + e_3$$

$$
\begin{aligned}
\frac{\partial L}{\partial x_1} &= a_1 + b_2 + c_1 \\
\frac{\partial L}{\partial x_2} &= a_2 + b_2 + c_2 \\
\frac{\partial L}{\partial x_3} &= a_3 + b_3 - d_3
\end{aligned}
$$

From here we can see that for a stationary solution we need

$$-e_1 - e_2 + e_3 = 0$$

**Complementary Slackness:**

We know that the conditions that have constraints need to result in 0. But because the constraints cannot be zero we must conclude that

$$
\begin{aligned}
b_1 x_1 + b_2 x_2 + b_3 x_3 - e_1 &= 0 \\
e_3 - d_3 x_3 &= 0
\end{aligned}
$$

**Primal Feasibility:**

Our conditions established with the problem satisfy the primal feasibility condition.

**Dual Feasibility:**

Our conditions given with the dual problem satisfy the dual feasibility condition.

## Problem 2

Let $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ where $i = 1, 2, \cdots, m$ $j = 1, 2, \cdots, n$ $i, j \in \mathbb{R}^+$, (1) show that it is a convex optimization problem (2) derive its KKT conditions

$$\begin{aligned} \inf \sum_{i=1,j=1}^{m,n} a_{ij}x_{ij} + \sum_{i=1,j=1}^{m,n} b_{ij}((x_{ij}+1)\log(x_{ij}+1) - x_{ij}) \\ s.t. \qquad \sum_{i,j}^{m,n} x_{ij} \geq c_j, \forall j \\ \sum_{i,j}^{m,n} x_{ij} + d_i \geq \sum_{j}^{n} x_{ij} + \sum_{j}^{n} c_j, \forall i \\ x_{ij} \geq 0 \ \forall i, j \end{aligned}$$

### Part 1)

We know that this is a convex optimization problem because it is written in the standard form. Writing it more conveniently we have

$$\begin{aligned} \inf \sum_{i=1,j=1}^{m,n} a_{ij}x_{ij} + \sum_{i=1,j=1}^{m,n} b_{ij}((x_{ij}+1)\log(x_{ij}+1) - x_{ij}) \\ s.t. \qquad c_j - \sum_{i=1}^{m} x_{ij} \leq 0, j = 1, \cdots, n \\ \sum_{j=1}^{n} x_{ij} + \sum_{j=1}^{n} c_j - \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} - d_i \leq 0, i = 1, \cdots, m \\ -x_{ij} \leq 0, i = 1, \cdots, m, j = 1, \cdots, n \end{aligned}$$

We also notice that all $x_{ij}$ are constrained and that the last constraint can be written as two, one where $x_{ij} = 0$

Next we need to show that the objective function is convex. This is easy to do because we can see that in the summation $x_{ij}ln(x_{ij}+1)$ is the dominant term, and thus $(x_{ij}+1)\log(x_{ij}+1)$ is convex. The other terms are linear, and thus trivially convex.

### Part 2)

**Stationary:**

Taking the partial derivative with respect to $x_{ij}$ of the summation of the above terms we get

$$a_{ij} + b_{ij}\log(x_{ij}+1) - 2$$

Setting this equal to 0, to find the stationary solution, we find that

$$x_{ij} = e^{\frac{2-a_{ij}}{b_{ij}}} - 1$$

**Complementary Slackness:**

To determine complementary slackness we need to set $h(x_{ij}) = 0$. Where

$$h(x_{ij}) = c_j - \sum_{i=1}^{m} x_{ij} + \sum_{j=1}^{n}(x_{ij} + c_j) - \sum_{i=1,j=1}^{m,n} x_{ij} - d_i - x_{ij}$$

**Primal Feasibility:**

These are seen from part 1

**Dual Feasibility:**

We don't need to find the dual solution, but rather can determine that $\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0$, where each $\lambda$ is associated with its respective condition in the primal problem.

# Problem 3

Let $a, x \in \mathbb{R}^n, B \in \mathbb{R}^{mxn}, c \in \mathbb{R}^m$

$P_1$

$$\begin{aligned} \inf a^T x \qquad & s.t. Bx \preceq c \\ & x_i \in \{0, 1\}, i = 1, \cdots, n \end{aligned}$$

$P_2$

$$\begin{aligned} \inf a^T x \qquad & s.t. Bx \preceq c \\ & 0 \leq x_i \leq 1, i = 1, \cdots, n \end{aligned}$$

$P_3$

$$\begin{aligned} \inf a^T x \qquad & s.t. Bx \preceq c \\ & x_i(1 - x_i) = 0, i = 1, \cdots, n \end{aligned}$$

## Part 1)

Derive the dual problem, $P_4$, of $P_3$

First let's rewrite

$$\begin{aligned} \inf a^T x \qquad & s.t. Bx - c \preceq 0 \\ & x_i(1 - x_i) = 0, i = 1, \cdots, n \end{aligned}$$

Then we want to make it a max problem

$$\begin{aligned} \sup -a^T x \qquad & s.t. Bx - c \preceq 0 \\ & x_i(1 - x_i) = 0, i = 1, \cdots, n \end{aligned}$$

Then we introduce $\lambda_1 \geq 0$ that will be associated with the first condition and $\lambda_2$ (no constraint) that is associated with the $x_i$ constraint. Next we need to find the max value and solve for x

$$\sup_{\lambda_1 \geq 0, \lambda_2} \sup -a^T x + \lambda_1 Bx - \lambda_1 c + \lambda_2 \sum_{x=1}^{n} x_i(1-x_i)$$

We now need to minimize over x (We're going to use Einsteinian notation)

$$\begin{aligned}
&- a^T x_i + (B_i x_i - c_i)\lambda_{i1} + x_i(1-x_i)\lambda_{2i} \\
&- a^T x_i + B_i x_i \lambda_{i1} - c_i \lambda_{i1} + x_i \lambda_{i2} - x_i \lambda_{i2} x_i \\
\nabla_x &= -a^T + B_i \lambda_{i1} + \lambda_{i2} - 2x_i \lambda_{i2} \\
2x_i \lambda_{i2} &= -a^T + B_i \lambda_{i1} + \lambda_{i2} \\
x_i &= \frac{-a^T + B_i \lambda_{i1} + \lambda_{i2}}{2\lambda_{i2}}
\end{aligned}$$

First we need to recognize that $\lambda_{i2}$ could be 0. That would result in a $g(\lambda_1, \lambda_2) = -\infty$.Substituting back in

$$\begin{aligned}
-a_i^T \left(\frac{-a^T + B_i \lambda_{i1} + \lambda_{i2}}{2\lambda_{i2}}\right) + B_i \left(\frac{-a^T + B_i \lambda_{i1} + \lambda_{i2}}{2\lambda_{i2}}\right) \lambda_{i1} - c_i \lambda_{i1} \\
+ \left(\frac{-a^T + B_i \lambda_{i1} + \lambda_{i2}}{2\cancel{\lambda_{i2}}}\right) \cancel{\lambda_{i2}} - \left(\frac{(-a^T + B_i \lambda_{i1} + \lambda_{i2})^2}{2\cancel{\lambda_{i2}}\lambda_{i2}}\right) \cancel{\lambda_{i2}}
\end{aligned}$$

$$\begin{aligned}
-a_i^T \left(\frac{-a^T + B_i \lambda_{i1} + \lambda_{i2}}{2\lambda_{i2}}\right) + B_i \left(\frac{-a^T + B_i \lambda_{i1} + \lambda_{i2}}{2\lambda_{i2}}\right) \lambda_{i1} - c_i \lambda_{i1} \\
+ \frac{-a^T + B_i \lambda_{i1} + \lambda_{i2}}{2} - \frac{(-a^T + B_i \lambda_{i1} + \lambda_{i2})^2}{2\lambda_{i2}}
\end{aligned}$$

Now we need to maximize our $\lambda$'s

$$\begin{aligned}
&B_i \left(\frac{-a^T + B_i \lambda_{i1} + \lambda_{i2}}{2\lambda_{i2}}\right) \lambda_{i1} - c_i \lambda_{i1} + \frac{-a^T + B_i \lambda_{i1} + \lambda_{i2}}{2} \\
&- c_i \lambda_{i1} + (B_i \lambda_{i1} + \lambda_{i2}) x_i
\end{aligned}$$

$$\begin{aligned}
\inf c\lambda_1 \qquad & s.t. (B_i \lambda_{i1} + \lambda_{i2}) \geq 0 \\
& \lambda_{i2} > 0
\end{aligned}$$

## Part 2)

Are $L_1$ and $L_2$ equal?

If we carefully look at the three problems we will notice that they are in fact the same ones. It is clear that $P_1$ and $P_3$ are the same, because they trivially have the same solution to the $x_i$ condition, those being $\{0, 1\}$. We can rewrite $P_2$ in a more convenient way to show that the constraints are the same.

$$
\begin{aligned}
\inf a^T x \qquad & s.t. Bx \preceq c \\
& \quad x_i \geq 0, i = 1, \cdots, n \\
& \quad x_i \leq 1, i = 1, \cdots, n \\
\inf a^T x \qquad & s.t. Bx \preceq c \\
& \quad -x_i \leq 0, i = 1, \cdots, n \\
& \quad x_i - 1 \leq 0, i = 1, \cdots, n
\end{aligned}
$$

From here we can see that stationary solutions are, again, when $x_i = \{0, 1\}$. With these primal conditions and our clear dual conditions, we can tell that the KKT conditions are the same as well. $\therefore$ they must have the same optimal solution. $\therefore$ they must be the same problem.