Assignment 2

CIS 621: Algorithms and Complexity

Problem 1 (5 points) For the linear program below, where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, d_3, e_1, e_2, e_3$ are non-zero constants, derive (1) its dual linear program and (2) the KKT conditions for the dual linear program.

$$\begin{aligned} \min_{x_1 \geq 0, x_2 \leq 0, x_3} \quad & a_1x_1 + a_2x_2 + a_3x_3 \\ s.t. \quad & b_1x_1 + b_2x_2 + b_3x_3 \leq e_1 \,, \\ & c_1x_1 + c_2x_2 = e_2 \,, \\ & d_3x_3 \geq e_3 \,. \end{aligned}$$

Solution Assume α, β, and γ are the dual variables for the three constraints, respectively; also assume δ is the dual variable for $x_1 \geq 0$, and θ is the dual variable for $x_2 \leq 0$. Skipping the intermediate steps of derivation, we show the following dual problem:

$$\begin{aligned} \max_{\alpha \geq 0, \beta \geq 0, \gamma \geq 0, \delta \geq 0, \theta \geq 0} \quad & -e_1\alpha - e_2\beta + e_3\gamma \\ s.t. \quad & a_1 + b_1\alpha + c_1\beta - \delta = 0 \,, \\ & a_2 + b_2\alpha + c_2\beta + \theta = 0 \,, \\ & a_3 + b_3\alpha - d_3\gamma = 0 \,. \end{aligned} \tag{1}$$

Note that this problem can be equivalently rewritten as

$$\begin{aligned} \max_{\alpha \geq 0, \beta \geq 0, \gamma \geq 0} \quad & -e_1\alpha - e_2\beta + e_3\gamma \\ s.t. \quad & a_1 + b_1\alpha + c_1\beta \geq 0 \,, \\ & a_2 + b_2\alpha + c_2\beta \leq 0 \,, \\ & a_3 + b_3\alpha - d_3\gamma = 0 \,. \end{aligned} \tag{2}$$

For this new version of the problem, let's assume μ, η, ρ are the dual variables for the three constraints, respectively, and also assume π, ϕ, ϵ are the dual variables for $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, respectively. Skipping the intermediate steps of derivation, we have the following KKT conditions.

Stationarity:

$$\begin{aligned} & e_1 - b_1\mu + b_2\eta + b_3\rho - \pi = 0 \\ & e_2 - c_1\mu + c_2\eta - \phi = 0 \\ & -e_3 - d_3\rho - \epsilon = 0 \end{aligned}$$

Complementary slackness:

$$\begin{aligned} & \mu(a_1 + b_1\alpha + c_1\beta) = 0 \\ & \eta(a_2 + b_2\alpha + c_2\beta) = 0 \\ & \pi\alpha = 0 \\ & \phi\beta = 0 \\ & \epsilon\gamma = 0 \end{aligned}$$

Primal feasibility:

$$\begin{aligned} & a_1 + b_1\alpha + c_1\beta \geq 0 \\ & a_2 + b_2\alpha + c_2\beta \leq 0 \\ & a_3 + b_3\alpha - d_3\gamma = 0 \\ & \alpha \geq 0, \beta \geq 0, \gamma \geq 0 \end{aligned}$$

Dual feasibility:

$$\mu \geq 0, \eta \geq 0, \rho \geq 0, \pi \geq 0, \phi \geq 0, \epsilon \geq 0$$

Problem 2 (5 points) For the following problem, where a_{ij}, b_{ij}, c_j, d_i, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ are positive constants: (1) show that it is a convex optimization problem; (2) derive its KKT conditions.

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_{ij} + \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} \left((x_{ij} + 1) \ln(x_{ij} + 1) - x_{ij} \right) \\ s.t. \quad & \sum_{i=1}^{m} x_{ij} \geq c_j \,, j = 1, 2, ..., n \,, \\ & \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} + d_i \geq \sum_{j=1}^{n} x_{ij} + \sum_{j=1}^{n} c_j \,, i = 1, 2, ..., m \,, \\ & x_{ij} \geq 0 \,, i = 1, 2, ..., m \,, j = 1, 2, ..., n \,. \end{aligned}$$

Solution (1): Omitted. Use Hessian matrix. All constraints are affine (Slater's condition).

(2): Assume α_j, $j = 1, 2, ..., n$ is the dual variable for the first constraint, β_i, $i = 1, 2, ..., m$ is the dual variable for the second constraint, and γ_{ij}, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ is the dual variable for the third constraint. Skipping the intermediate steps of derivation, we have the following KKT conditions.

Stationarity:

$$a_{ij} + b_{ij} \ln(x_{ij} + 1) - \alpha_j - \sum_{i=1}^{m} \beta_i + \beta_i - \gamma_{ij} = 0, i = 1, 2, ..., m \,, j = 1, 2, ..., n$$

Complementary slackness:

$$\begin{aligned} & \alpha_j (c_j - \sum_{i=1}^{m} x_{ij}) = 0, j = 1, 2, ..., n \\ & \beta_i (\sum_{j=1}^{n} x_{ij} + \sum_{j=1}^{n} c_j - \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} - d_i) = 0, i = 1, 2, ..., m \\ & \gamma_{ij} x_{ij} = 0, i = 1, 2, ..., m, j = 1, 2, ..., n \end{aligned}$$

Primal feasibility:

$$\begin{aligned} & \sum_{i=1}^{m} x_{ij} \geq c_j \,, j = 1, 2, ..., n \\ & \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} + d_i \geq \sum_{j=1}^{n} x_{ij} + \sum_{j=1}^{n} c_j \,, i = 1, 2, ..., m \\ & x_{ij} \geq 0 \,, i = 1, 2, ..., m \,, j = 1, 2, ..., n \end{aligned}$$

Dual feasibility:

$$\begin{aligned} & \alpha_j \geq 0, j = 1, 2, ..., n \\ & \beta_i \geq 0, i = 1, 2, ..., m \\ & \gamma_{ij} \geq 0, i = 1, 2, ..., m, j = 1, 2, ..., n \end{aligned}$$

Problem 3 (10 points) Given $c, x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, we have Problem P_1

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax \preceq b \,, \\ & x_i \in \{0, 1\} \,, i = 1, 2, ..., n \,, \end{aligned}$$

Problem P_2

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax \preceq b \,, \end{aligned}$$

$$0 \le x_i \le 1 \,, i = 1, 2, ..., n \,,$$

and Problem P_3 (which is an equivalent reformulation of P_1)

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax \preceq b \,, \\ & x_i(1 - x_i) = 0 \,, i = 1, 2, ..., n \,. \end{aligned}$$

Note that the optimal value of P_2 is a lower bound, denoted as L_1, for the optimal value of P_1. Now, derive the dual problem, denoted as P_4, for P_3. The optimal value of P_4 is also a lower bound, denoted as L_2, for the optimal value of P_1. Are L_1 and L_2 equal? Explain why. (Hint: derive the dual problem, denoted as P_5, for P_2.)

Solution

(a) The Lagrangian is

$$\begin{aligned} L(x, \mu, \nu) &= c^T x + \mu^T (Ax - b) - \nu^T x + x^T \mathbf{diag}(\nu) x \\ &= x^T \mathbf{diag}(\nu) x + (c + A^T \mu - \nu)^T x - b^T \mu. \end{aligned}$$

Minimizing over x gives the dual function

$$g(\mu, \nu) = \begin{cases} -b^T \mu - (1/4) \sum_{i=1}^n (c_i + a_i^T \mu - \nu_i)^2 / \nu_i & \nu \succeq 0 \\ -\infty & \text{otherwise} \end{cases}$$

where a_i is the ith column of A, and we adopt the convention that $a^2/0 = \infty$ if $a \neq 0$, and $a^2/0 = 0$ if $a = 0$.
The resulting dual problem is

$$\begin{array}{ll} \text{maximize} & -b^T \mu - (1/4) \sum_{i=1}^n (c_i + a_i^T \mu - \nu_i)^2 / \nu_i \\ \text{subject to} & \nu \succeq 0. \end{array}$$

In order to simplify this dual, we optimize analytically over ν, by noting that

$$\begin{aligned} \sup_{\nu_i \ge 0} \left(-\frac{(c_i + a_i^T \mu - \nu_i)^2}{\nu_i} \right) &= \begin{cases} (c_i + a_i^T \mu) & c_i + a_i^T \mu \le 0 \\ 0 & c_i + a_i^T \mu \ge 0 \end{cases} \\ &= \min\{0, (c_i + a_i^T \mu)\}. \end{aligned}$$

This allows us to eliminate ν from the dual problem, and simplify it as

$$\begin{array}{ll} \text{maximize} & -b^T \mu + \sum_{i=1}^n \min\{0, c_i + a_i^T \mu\} \\ \text{subject to} & \mu \succeq 0. \end{array}$$

(b) We follow the hint. The Lagrangian and dual function of the LP relaxation re

$$\begin{aligned} L(x, u, v, w) &= c^T x + u^T (Ax - b) - v^T x + w^T (x - \mathbf{1}) \\ &= (c + A^T u - v + w)^T x - b^T u - \mathbf{1}^T w \\ g(u, v, w) &= \begin{cases} -b^T u - \mathbf{1}^T w & A^T u - v + w + c = 0 \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

The dual problem is

$$\begin{array}{ll} \text{maximize} & -b^T u - \mathbf{1}^T w \\ \text{subject to} & A^T u - v + w + c = 0 \\ & u \succeq 0, v \succeq 0, w \succeq 0, \end{array}$$

which is equivalent to the Lagrange relaxation problem derived above. We conclude that the two relaxations give the same value.