Decomposition Methods

- separable problems, complicating variables
- primal decomposition
- dual decomposition
- complicating constraints
- general decomposition structures

Separable problem

$$\begin{array}{ll} \text{minimize} & f_1(x_1) + f_2(x_2) \\ \text{subject to} & x_1 \in \mathcal{C}_1, \quad x_2 \in \mathcal{C}_2 \end{array}$$

- we can solve for x_1 and x_2 separately (in parallel)
- even if they are solved sequentially, this gives advantage if computational effort is superlinear in problem size
- called **separable** or **trivially parallelizable**
- generalizes to any objective of form $\Psi(f_1, f_2)$ with Ψ nondecreasing ($e.g.$, $\max$)

Complicating variable

consider unconstrained problem,

$$\text{minimize} \quad f(x) = f_1(x_1, y) + f_2(x_2, y)$$

$x = (x_1, x_2, y)$

- y is the **complicating variable** or **coupling variable**; when it is fixed the problem is separable in x_1 and x_2

- x_1, x_2 are **private** or **local** variables; y is a **public** or **interface** or **boundary** variable between the two subproblems

Primal decomposition

fix y and define

$$\begin{array}{ll} \text{subproblem 1:} & \text{minimize}_{x_1} \quad f_1(x_1, y) \\ \text{subproblem 2:} & \text{minimize}_{x_2} \quad f_2(x_2, y) \end{array}$$

with optimal values $\phi_1(y)$ and $\phi_2(y)$

original problem is equivalent to **master problem**

$$\text{minimize}_y \quad \phi_1(y) + \phi_2(y)$$

with variable y

called **primal decomposition** since master problem manipulates primal (complicating) variables

- if original problem is convex, so is master problem
- can solve master problem using
  - bisection (if y is scalar)
  - gradient or Newton method (if ϕ_i differentiable)
  - subgradient, cutting-plane, or ellipsoid method
- each iteration of master problem requires solving the two subproblems (in parallel)
- if master algorithm converges fast enough and subproblems are sufficiently easier to solve than original problem, we get savings

Primal decomposition algorithm

(using subgradient algorithm for master)

repeat

1. Solve the subproblems (in parallel).
   Find x_1 that minimizes $f_1(x_1, y)$, and a subgradient $g_1 \in \partial\phi_1(y)$.
   Find x_2 that minimizes $f_2(x_2, y)$, and a subgradient $g_2 \in \partial\phi_2(y)$.
2. Update complicating variable.
   $y := y - \alpha_k(g_1 + g_2)$.

step length α_k can be chosen in any of the standard ways

Dual decomposition

Step 1: introduce new variables y_1, y_2

$$\begin{array}{ll} \text{minimize} & f(x) = f_1(x_1, y_1) + f_2(x_2, y_2) \\ \text{subject to} & y_1 = y_2 \end{array}$$

- y_1, y_2 are **local** versions of complicating variable y
- $y_1 = y_2$ is **consensus constraint**

Step 2: form dual problem

$$L(x_1, y_1, x_2, y_2) = f_1(x_1, y_1) + f_2(x_2, y_2) + \nu^T(y_1 - y_2)$$

separable; can minimize over (x_1, y_1) and (x_2, y_2) separately

$$\begin{aligned} g_1(\nu) &= \inf_{x_1, y_1} \left(f_1(x_1, y_1) + \nu^T y_1 \right) = -f_1^*(0, -\nu) \\ g_2(\nu) &= \inf_{x_2, y_2} \left(f_2(x_2, y_2) - \nu^T y_2 \right) = -f_2^*(0, \nu) \end{aligned}$$

dual problem is: maximize $g(\nu) = g_1(\nu) + g_2(\nu)$

- computing $g_i(\nu)$ are the **dual subproblems**
- can be done in parallel
- a subgradient of $-g$ is $y_2 - y_1$ (from solutions of subproblems)

Dual decomposition algorithm

(using subgradient algorithm for master)

repeat

1. Solve the dual subproblems (in parallel).
   Find x_1, y_1 that minimize $f_1(x_1, y_1) + \nu^T y_1$.
   Find x_2, y_2 that minimize $f_2(x_2, y_2) - \nu^T y_2$.
2. Update dual variables (prices).
   $\nu := \nu - \alpha_k(y_2 - y_1)$.

- step length α_k can be chosen in standard ways
- at each step we have a lower bound $g(\nu)$ on $p^\star$
- iterates are generally infeasible, *i.e.*, $y_1 \neq y_2$

Finding feasible iterates

- reasonable guess of feasible point from (x_1, y_1), (x_2, y_2):

$$(x_1, \bar{y}), \qquad (x_2, \bar{y}), \qquad \bar{y} = (y_1 + y_2)/2$$

  - projection onto feasible set $y_1 = y_2$
  - gives upper bound $p^\star \le f_1(x_1, \bar{y}) + f_2(x_2, \bar{y})$

- a better feasible point: replace y_1, y_2 with $\bar{y}$ and solve *primal* subproblems minimize$_{x_1} f_1(x_1, \bar{y})$, minimize$_{x_2} f_2(x_2, \bar{y})$
  - gives (better) upper bound $p^\star \le \phi_1(\bar{y}) + \phi_2(\bar{y})$

Interpretation

- y_1 is resources consumed by first unit, y_2 is resources generated by second unit
- $y_1 = y_2$ is **consistency** condition: supply equals demand
- ν is a set of resource prices
- master algorithm adjusts prices at each step, rather than allocating resources directly (primal decomposition)

Recovering the primal solution from the dual

- iterates in dual decomposition:

$$\nu^{(k)}, \qquad (x_1^{(k)}, y_1^{(k)}), \qquad (x_2^{(k)}, y_2^{(k)})$$

  – $x_1^{(k)}, y_1^{(k)}$ is minimizer of $f_1(x_1, y_1) + \nu^{(k)T} y_1$ found in subproblem 1
  – $x_2^{(k)}, y_2^{(k)}$ is minimizer of $f_2(x_2, y_2) - \nu^{(k)T} y_2$ found in subproblem 2

- $\nu^{(k)} \rightarrow \nu^\star$ ($i.e.$, we have price convergence)

- subtlety: we need not have $y_1^{(k)} - y_2^{(k)} \rightarrow 0$

- the hammer: if f_i strictly convex, we have $y_1^{(k)} - y_2^{(k)} \rightarrow 0$

- can fix allocation ($i.e.$, compute ϕ_i), or add regularization terms $\epsilon \|y_i\|^2$

Decomposition with constraints

can also have **complicating constraints**, as in

$$\begin{array}{ll} \text{minimize} & f_1(x_1) + f_2(x_2) \\ \text{subject to} & x_1 \in \mathcal{C}_1, \quad x_2 \in \mathcal{C}_2 \\ & h_1(x_1) + h_2(x_2) \preceq 0 \end{array}$$

- f_i, h_i, $\mathcal{C}_i$ convex
- $h_1(x_1) + h_2(x_2) \preceq 0$ is a set of p complicating or coupling constraints, involving both x_1 and x_2
- can interpret coupling constraints as limits on resources shared between two subproblems

Primal decomposition

fix $t \in \mathbf{R}^p$ and define

subproblem 1: $$\begin{array}{ll} \text{minimize} & f_1(x_1) \\ \text{subject to} & x_1 \in \mathcal{C}_1, \quad h_1(x_1) \preceq t \end{array}$$

subproblem 2: $$\begin{array}{ll} \text{minimize} & f_2(x_2) \\ \text{subject to} & x_2 \in \mathcal{C}_2, \quad h_2(x_2) \preceq -t \end{array}$$

- t is the quantity of resources allocated to first subproblem ($-t$ is allocated to second subproblem)
- master problem: minimize $\phi_1(t) + \phi_2(t)$ (optimal values of subproblems) over t
- subproblems can be solved separately (in parallel) when t is fixed

Primal decomposition algorithm

repeat

1. Solve the subproblems (in parallel).
   Solve subproblem 1, finding x_1 and λ_1.
   Solve subproblem 2, finding x_2 and λ_2.
2. Update resource allocation.
   $t := t - \alpha_k(\lambda_2 - \lambda_1)$.

- λ_i is an optimal Lagrange multiplier associated with resource constraint in subproblem i
- $\lambda_2 - \lambda_1 \in \partial(\phi_1 + \phi_2)(t)$
- α_k is an appropriate step size
- all iterates are feasible (when subproblems are feasible)

Dual decomposition

form (separable) partial Lagrangian

$$\begin{aligned} L(x_1, x_2, \lambda) &= f_1(x_1) + f_2(x_2) + \lambda^T(h_1(x_1) + h_2(x_2)) \\ &= \big(f_1(x_1) + \lambda^T h_1(x_1)\big) + \big(f_2(x_2) + \lambda^T h_2(x_2)\big) \end{aligned}$$

fix dual variable λ and define

subproblem 1: $\begin{array}{ll} \text{minimize} & f_1(x_1) + \lambda^T h_1(x_1) \\ \text{subject to} & x_1 \in \mathcal{C}_1 \end{array}$

subproblem 2: $\begin{array}{ll} \text{minimize} & f_2(x_2) + \lambda^T h_2(x_2) \\ \text{subject to} & x_2 \in \mathcal{C}_2 \end{array}$

with optimal values $g_1(\lambda)$, $g_2(\lambda)$

- $-h_i(\bar{x}_i) \in \partial(-g_i)(\lambda)$, where $\bar{x}_i$ is any solution to subproblem i

- $-h_1(\bar{x}_1) - h_2(\bar{x}_2) \in \partial(-g)(\lambda)$

- the master algorithm updates λ using this subgradient

Dual decomposition algorithm

(using projected subgradient method)

repeat

1. Solve the subproblems (in parallel).
   Solve subproblem 1, finding an optimal $\bar{x}_1$.
   Solve subproblem 2, finding an optimal $\bar{x}_2$.
2. Update dual variables (prices).
   $\lambda := (\lambda + \alpha_k(h_1(\bar{x}_1) + h_2(\bar{x}_2)))_+$.

- α_k is an appropriate step size
- iterates need not be feasible
- can again construct feasible primal variables using projection

Interpretation

- λ gives prices of resources

- subproblems are solved separately, taking income/expense from resource usage into account

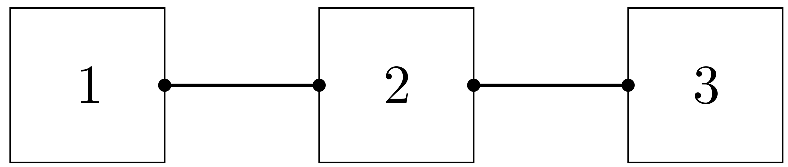
- master algorithm adjusts prices

- prices on over-subscribed resources are increased; prices on undersubscribed resources are reduced, but never made negative

General decomposition structures

- multiple subsystems
- (variable and/or constraint) coupling constraints between subsets of subsystems
- represent as hypergraph with subsystems as vertices, coupling as hyperedges or nets
- without loss of generality, can assume all coupling is via consistency constraints

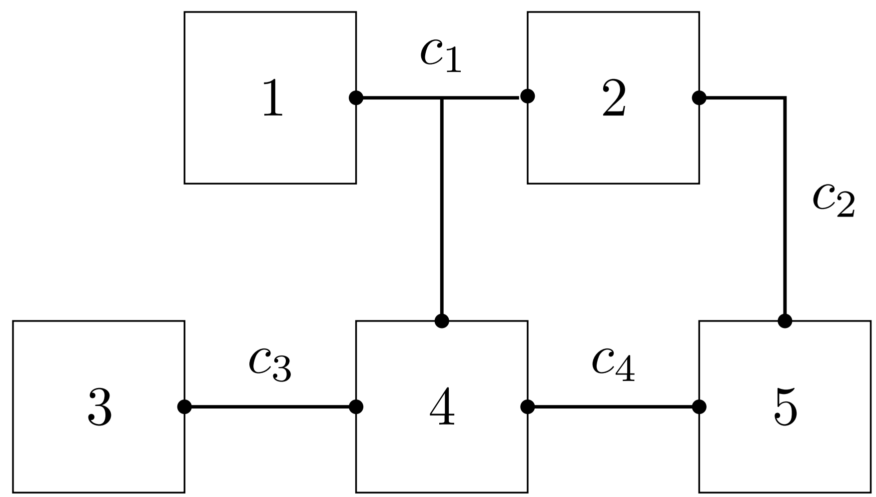
Simple example



- 3 subsystems, with private variables x_1, x_2, x_3, and public variables y_1, (y_2, y_3), and y_4
- 2 (simple) edges

$$
\begin{array}{ll}
\text{minimize} & f_1(x_1, y_1) + f_2(x_2, y_2, y_3) + f_3(x_3, y_4) \\
\text{subject to} & (x_1, y_1) \in \mathcal{C}_1, \quad (x_2, y_2, y_3) \in \mathcal{C}_2, \quad (x_3, y_4) \in \mathcal{C}_3 \\
& y_1 = y_2, \quad y_3 = y_4
\end{array}
$$

A more complex example



General form

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^K f_i(x_i, y_i) \\ \text{subject to} & (x_i, y_i) \in \mathcal{C}_i, \quad i = 1, \ldots, K \\ & y_i = E_i z, \quad i = 1, \ldots, K \end{array}$$

- private variables x_i, public variables y_i
- net (hyperedge) variables $z \in \mathbf{R}^N$; z_i is common value of public variables in net i
- matrices E_i give **netlist** or **hypergraph**
  row k is e_p, where kth entry of y_i is in net p

Primal decomposition

$\phi_i(y_i)$ is optimal value of subproblem

$$\begin{array}{ll} \text{minimize} & f_i(x_i, y_i) \\ \text{subject to} & (x_i, y_i) \in \mathcal{C}_i \end{array}$$

repeat

1. Distribute net variables to subsystems.
   $y_i := E_i z, \quad i = 1, \ldots, K.$
2. Optimize subsystems (separately).
   Solve subproblems to find optimal x_i, $g_i \in \partial \phi_i(y_i), \quad i = 1, \ldots, K.$
3. Collect and sum subgradients for each net.
   $g := \sum_{i=1}^{K} E_i^T g_i$.
4. Update net variables.
   $z := z - \alpha_k g$.

Dual decomposition

$g_i(\nu_i)$ is optimal value of subproblem

$$\begin{array}{ll} \text{minimize} & f_i(x_i, y_i) + \nu_i^T y_i \\ \text{subject to} & (x_i, y_i) \in \mathcal{C}_i \end{array}$$

given initial price vector ν that satisfies $E^T\nu = 0$ ($e.g.$, $\nu = 0$).

repeat

1. Optimize subsystems (separately).
   Solve subproblems to obtain x_i, y_i.
2. Compute average value of public variables over each net.
   $\hat{z} := (E^T E)^{-1} E^T y$.
3. Update prices on public variables.
   $\nu := \nu + \alpha_k (y - E\hat{z})$.