Assignment 1

CIS 621: Algorithms and Complexity

Problem 1 (2 points) Prove the NP-completeness of the following problem by reduction: given n positive integers x_1, x_2, ..., x_n, and another positive integer w, is there a subset of the n integers that add up to exactly w?

Problem 2 Given $a, x \in \mathbb{R}^2$, where $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $b_1, b_2 \in \mathbb{R}$,

- **(1 point)** prove $a^T x = \|a\|\|x\| \cos\theta$ using the definition of inner product and the law of cosine, where θ is the angle between a and x;
- **(1 point)** calculate the distance between the two parallel hyperplanes $\{x|a^T x = b_1\}$ and $\{x|a^T x = b_2\}$.

Problem 3 (2 points) Given $x_0, x_1, ..., x_k \in \mathbb{R}^n$. Consider the set of points that are closer to x_0 than any other x_i, i.e., $S = \{x \in \mathbb{R}^n | \|x - x_0\| \leq \|x - x_i\|, i = 1, 2, ..., k\}$. Transform S and express it in the form of $S = \{x|Ax \preceq b\}$.

Problem 4 (2 points) Let f be a twice differentiable function, with $dom(f)$ convex. Prove f is convex if and only if $(\nabla f(x) - \nabla f(y))^T (x - y) \geq 0$. (Hint: prove the “necessity” and the “sufficiency” separately.)

Problem 5 Prove the following functions are convex:

- **(1 point)** $f(x) = \max\{f_1(x), f_2(x), ..., f_m(x)\}$, where $f_i(x), i = 1, 2, ..., m$ are convex;
- **(1 point)** $f(x) = \min_{y \in C} g(x, y)$, where $g(x, y)$ is convex in both x and y, and C is a convex set;
- **(2 points)** $f(x, y) = \frac{|x_1|^p + |x_2|^p + ... + |x_n|^p}{y^{p-1}}$, where $p > 1, x \in \mathbb{R}^n, y \in \mathbb{R}_{++}$;
- **(2 points)** $f(x_{11}, ..., x_{1n}, x_{21}, ..., x_{2n}, ..., x_{m1}, ..., x_{mn}) = \sum_{i=1}^{m} \sum_{j=1}^{n} ((x_{ij} + 1) \ln (x_{ij} + 1) - x_{ij})$, where $x_{ij} \in \mathbb{R}_{++}$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$.

Problem 6 (1 point) Consider the function $f(x) = \max\{|a^T x + b|, \ln \frac{1}{c^T x + d}\}$, where $a, c, x \in \mathbb{R}^n$ and $b, d \in \mathbb{R}$. Is this a convex function? Explain why. (Hint: just use the rules that preserve convexity to explain.)