CIS 621 Assignment 4

Steven Walton

March 10, 2019

Problem 1

For the graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of edges, we define the following nonlinear integer program, where $w_{i,j} \geq 0, \forall (i,j) \in \mathcal{E}$ and k is a nonnegative integer:

$$\begin{aligned}
&\sup \sum_{(i,j)\in\mathcal{E}} w_{i,j}(x_i + x_j - 2x_i x_j) \\
&s.t. \sum_{i\in\mathcal{V}} x_i = k \\
&\quad x_i \in \{0,1\}, \forall i \in \mathcal{V}
\end{aligned}$$

Show that the following linear program is a relaxation of the above problem:

$$\begin{aligned}
&\sup \sum_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j} \\
s.t. \quad & z_{i,j} \leq x_i + x_j, \forall (i,j) \in \mathcal{E} \\
& z_{i,j} \leq 2 - x_i - x_j, \forall (i,j) \in \mathcal{E} \\
& \sum_{i\in\mathcal{V}} x_i = k \\
& 0 \leq x_i \leq 1, \forall i \in \mathcal{V} \\
& 0 \leq z_{i,j} \leq 1, \forall (i,j) \in \mathcal{E}
\end{aligned}$$

Also, let $F(x) = \sum_{(i,j)\in\mathcal{E}} w_{i,j}(x_i + x_j - 2x_i x_j)$ be the objective function of the nonlinear integer program. Show that for any (x, z) that is feasible to the linear program, $F(x) \geq \frac{1}{2} \sum_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j}$

Part 1)

If we remember the definition of linear relaxation we see that it is

$$x_i \in \{0,1\} \mapsto 0 \leq x_i \leq 1$$

We can see from the given problem that

$$z_{i,j} = x_i + x_j - 2x_i x_j$$

We may also notice that $z_{i,j} = 0 \forall x_i, x_j \in \{0, 1\}$. We can also notice that $z_{i,j} = 1 - \delta_{ij}$. Clearly this needs a relaxation so we can approximate a solution. From here we can easily check the given linear program is a relaxation of the initial integer program.
It is trivial to see that $z_{i,j} \leq x_i + x_j \because$ the negative term ensures that $z_{i,j}$ is smaller. Similarly we can see that if we remove the $x_i x_j$ term from the 2 that we obtain a maximal value and thus our $z_{i,j}$ has to be smaller than that. Finally, we see that we have the relaxation terms where $x_{i,j}$ and $z_{i,j}$ are not limited to integer values. Therefore we can conclude that this is a relaxation of the initial problem.

Part 2)

Looking at the objective function, we'll need to solve for the expectation value.

$$\begin{aligned}
\langle F(x) \rangle &= \left\langle \sum_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j} \right\rangle \\
&= \sum_{(i,j)\in\mathcal{E}} w_{i,j} \mathbf{Pr}(\textsf{vertex within cut}) \\
&= \sum_{(i,j)\in\mathcal{E}} w_{i,j} \frac{1}{2} \\
&= \frac{1}{2} \cancelto{1}{\sum_{(i,j)\in\mathcal{E}} w_{i,j}} \\
&= \frac{1}{2}
\end{aligned}$$

We can determine that $\mathbf{Pr}(\textsf{vertx within cut}) = \frac{1}{2}$ if we better define this as the probability that a vertex is on one side of a cut. This is clearly a bifrication problem and thus the value is $\frac{1}{2}$.
Since we can see that the expectation value, $\langle F(x) \rangle = \frac{1}{2}$, and thus we can conclude that $F(x) \geq \frac{1}{2} \sum_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j}$

Problem 2

For the directed graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of directed edges, we want to partition $\mathcal{V}$ into two sets, $\mathcal{U}$ and $\mathcal{W} = \mathcal{V}/\mathcal{U}$, in order to maximize the total weight of the edges going from $\mathcal{U}$ to $\mathcal{W}$ (edges (i, j) with $i \in \mathcal{U}$ and $j \in \mathcal{W}$

- Give a randomized $\frac{1}{4}$-approximation algorithm for this problem.
- Show that the following linear program is a relaxation of this problem.

$$
\begin{aligned}
max \sum_{(i,j)\in\mathcal{E}} & w_{i,j} z_{i,j} \\
s.t. \quad & z_{i,j} \leq x_i, \forall (i,j) \in \mathcal{E} \\
& z_{i,j} \leq 1 - x_j, \forall (i,j) \in \mathcal{E} \\
& 0 \leq x_i \leq 1, \forall i \in \mathcal{V} \\
& 0 \leq z_{i,j} \leq 1, \forall (i,j) \in \mathcal{E}
\end{aligned}
$$

- For the above linear program, give a randomized $\frac{1}{2}$-approximation algorithm based on rounding $x_i \forall i \in \mathcal{V}$ to 1, with the probability of $\frac{1}{2}x_i + \frac{1}{4}$

Part 1)

Part 2)

Part 3)