The exact solution of right-angled triangular plate problem by solving functional equations

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1 Introduction

The lateral deflection in plain-plate problem is described by the equilibrium equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = p(x, y) \tag{1}$$

where $p(x, y)$ is a continuously distributed lateral load. The general solution of the partial Equation (1) is given by

$$w(x, y) = w_0(x, y) + \varphi_1(x + iy) + \varphi_2(x - iy) + x[\psi_1(x + iy) + \psi_2(x - iy)] \tag{2}$$

where $w_0(x, y)$ is a particular solution of Equation (1) and φ_1,φ_2,ψ_1,ψ_2 are four arbitrary functions. The problem of a clamped right-angled triangular plate can be expressed as follow:

$$\left\{\begin{array}{c} \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = p(x, y) \\ w(x, 0) = \frac{\partial w}{\partial y}(x, 0) = 0 \\ w(0, y) = \frac{\partial w}{\partial x}(0, y) = 0 \\ w(x, y)_l = \frac{\partial w}{\partial n}(x, y)_l = 0 \end{array}\right. \tag{3}$$

l means the hypotenuse of the triangle. $\frac{\partial w}{\partial n}$ means the directional derivative of w along $\vec{n}$. $\vec{n}$ is perpendicular to l. The triangular plate can be shown as Figure (1).

Theorem 1.1. *(All boundaries are clamped) The solution of the equations (3) is given by*

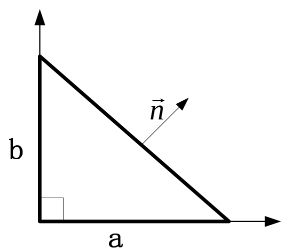

Figure 1: The clamped right-angled triangular plate

Proof.

Step 1. To make the solution (2) satisfies the boundary conditions $w(x,0)=0$ and $\frac{\partial w}{\partial y}(x,0)=0$, we must put

$$\begin{aligned} \varphi_1(x)+\varphi_2(x)+x[\psi_1(x)+\psi_2(x)] &= A(x) \\ \varphi_1'(x)-\varphi_2'(x)+x[\psi_1'(x)-\psi_2'(x)] &= B(x) \end{aligned} \tag{4}$$

where $A(x)=-w_0(x,0)$ and $B(x)=i\dfrac{\partial w_0}{\partial y}(x,0)$. Letting

$$\varphi_1(x)+x\psi_1(x)=\alpha(x) \tag{5}$$

where $\alpha(x)$ is an arbitrary function. By (5) and the first equation of Eqs.(4), we have

$$\varphi_2(x)+x\psi_2(x)=A(x)-\alpha(x) \tag{6}$$

(5) and (6) imply that

$$\begin{cases} \varphi_1(x)=-x\psi_1(x)+\alpha(x) \\ \varphi_2(x)=-x\psi_2(x)+A(x)-\alpha(x) \end{cases} \tag{7}$$

Substituting (7) into the second equation of Eqs. (4) yields

$$\psi_2(x)=\psi_1(x)-2\alpha'(x)+A_1(x) \tag{8}$$

where $A_1(x)=A'(x)+B(x)$. By (7) and (8), we get

$$\begin{cases} \varphi_1(x)=-x\psi_1(x)+\alpha(x) \\ \varphi_2(x)=-x\psi_1(x)+2x\alpha'(x)-\alpha(x)+A_2(x) \\ \psi_2(x)=\psi_1(x)-2\alpha'(x)+A_1(x) \end{cases} \tag{9}$$

where $A_2(x)=A(x)-xA_1(x)$.

Then the solution can be expressed as

$$w(x,y)=w_1(x,y)-iy\psi_1(x+iy)+iy\psi_1(x-iy)+\alpha(x+iy)-\alpha(x-iy)-2iy\alpha'(x-iy) \tag{10}$$

where $w_1(x,y)=w_0(x,y)+A_2(x-iy)+xA_1(x-iy)$.

Step 2. Similarly, to make the solution (2) satisfies the boundary conditions $w(0,y)=0$ and $\frac{\partial w}{\partial x}(0,y)=0$, we must put

$$\begin{cases} -iy\psi_1(iy)+iy\psi_1(-iy)+\alpha(iy)-\alpha(-iy)-2iy\alpha'(-iy)=C(y) \\ -iy\psi_1'(iy)+iy\psi_1'(-iy)+\alpha'(iy)-\alpha'(-iy)-2iy\alpha''(-iy)=D(y) \end{cases} \tag{11}$$

where $C(y)=-w_1(0,y)$, $D(y)=-\dfrac{\partial w_1}{\partial x}(0,y)$.

Now, we differentiate the first equation of Eqs.(11) with respect to y, then we get

$$-i\psi_1(iy)+i\psi_1(-iy)+y\psi_1'(iy)+y\psi_1'(-iy)+i\alpha'(iy)-i\alpha'(-iy)-2y\alpha''(-iy)=C'(y) \tag{12}$$

It is easy to rewrite the first equation of Eqs.(11) as follow:

$$\psi_1(iy)-\psi_1(-iy)=\frac{\alpha(iy)-\alpha(-iy)}{iy}-2\alpha'(-iy)-\frac{C(y)}{iy} \tag{13}$$

Substituting equation(13) to equation(12) yields

$$-\frac{\alpha(iy)-\alpha(-iy)}{y}+i\alpha'(iy)+i\alpha'(-iy)-2y\alpha''(-iy)+y\psi_1'(iy)+y\psi_1'(-iy)=-\frac{C(y)}{y}+C'(y) \tag{14}$$

By the second equation of Eqs.(11) and equation(14), we can get

$$y\psi_1'(iy) - y\psi_1'(-iy) = -i\alpha'(iy) + i\alpha'(-iy) - 2y\alpha''(-iy) + iD(y) \tag{15}$$

$$y\psi_1'(iy) + y\psi_1'(-iy) = \frac{\alpha(iy) - \alpha(-iy)}{y} - i\alpha'(iy) - i\alpha'(-iy) + 2y\alpha''(-iy) - \frac{C(y)}{y} + C'(y) \tag{16}$$

By (16)+(15), we obtain

$$2y\psi_1'(iy) = \frac{\alpha(iy) - \alpha(-iy)}{y} - 2i\alpha'(iy) + iD(y) - \frac{C(y)}{y} + C'(y) \tag{17}$$

By (16)−(15), we obtain

$$2y\psi_1'(-iy) = \frac{\alpha(iy) - \alpha(-iy)}{y} - 2i\alpha'(-iy) + 4y\alpha''(-iy) - iD(y) - \frac{C(y)}{y} + C'(y) \tag{18}$$

Let $y \to -y$ in equation(18), we have

$$-2y\psi_1'(iy) = \frac{\alpha(-iy) - \alpha(iy)}{-y} - 2i\alpha'(iy) - 4y\alpha''(iy) - iD(-y) - \frac{C(-y)}{-y} + C'(-y) \tag{19}$$

By (17)+(19), we obtain

$$2\frac{\alpha(iy) - \alpha(-iy)}{y} - 4i\alpha'(iy) - 4y\alpha''(iy) + iD(y) - iD(-y) - \frac{C(y)}{y} + \frac{C(-y)}{y} + C'(y) + C'(-y) = 0 \tag{20}$$

Simplify equation(20), we get

$$2\alpha(iy) - 2\alpha(-iy) - 4iy\alpha'(iy) - 4y^2\alpha''(iy) + iyD(y) - iyD(-y) - C(y) + C(-y) + yC'(y) + yC'(-y) = 0 \tag{21}$$

□

Lemma 1.2. *The general solution of the function equation(21) can be written as*

$$\alpha(x) = \alpha_0(x) + \sum_{n=1}^{\infty} a_n x^{\lambda_n} \tag{22}$$

where

$$2\lambda^2 - 4\lambda + 1 = (-1)^{\lambda} \tag{23}$$

λ can be a complex.

By equation(17), we can derive that

$$\psi_1'(iy) = \frac{\alpha(iy) - \alpha(-iy)}{2y^2} - \frac{2i\alpha'(iy)}{2y} + E(iy) \tag{24}$$

where $E(iy) = \frac{iD(y)}{2y} - \frac{C(y)}{2y^2} + \frac{C'(y)}{2y}$. Let $iy = z$, then equation(24) can be converted to:

$$\psi_1'(z) = \frac{\alpha(z) - \alpha(-z)}{-2z^2} + \frac{\alpha'(z)}{z} + E(z) \tag{25}$$

Substituting equation(22) into equation(25), we obtain:

$$\psi_1'(z) = -\frac{\alpha_0(z) - \alpha_0(-z)}{2z^2} + \frac{\alpha_0'(z)}{z} + E(z) + \sum_{n=1}^{\infty} \big(-1 + (-1)^{\lambda_n} + 2\lambda_n\big)\frac{a_n}{2} z^{\lambda_n - 2}$$

Integrate the above equation we can get:

$$\psi_1(z) = \psi_{10}(z) + \sum_{n=1}^{\infty} \big(-1 + (-1)^{\lambda_n} + 2\lambda_n \big) \frac{a_n}{2(\lambda_n - 1)} z^{\lambda_n - 1} \tag{26}$$

where $\psi_{10}(z) = \int \big(-\frac{\alpha_0(z)-\alpha_0(-z)}{2z^2} + \frac{\alpha_0'(z)}{z} + E(z)\big)\mathrm{d}z$.

Using the equation(23), we can simplify the equation(26):

$$\begin{aligned}\psi_1(z) &= \psi_{10}(z) + \sum_{n=1}^{\infty} \big(2\lambda_n^2 - 2\lambda_n\big) \frac{a_n}{2(\lambda_n - 1)} z^{\lambda_n - 1} \\ &= \psi_{10}(z) + \sum_{n=1}^{\infty} a_n \lambda_n z^{\lambda_n - 1}\end{aligned} \tag{27}$$

Substituting equation(27) and equation(22) to Eqs. (9), we obtain:

$$\begin{cases} \varphi_1(x) = -x\psi_{10}(x) + \alpha_0(x) + \sum_{n=1}^{\infty} a_n (1-\lambda_n) x^{\lambda_n} \\ \varphi_2(x) = -x\psi_{10}(x) + 2x\alpha_0'(x) - \alpha_0(x) + A_2(x) - \sum_{n=1}^{\infty} a_n (1-\lambda_n) x^{\lambda_n} \\ \psi_2(x) = \psi_{10}(x) - 2\alpha_0'(x) + A_1(x) - \sum_{n=1}^{\infty} a_n \lambda_n x^{\lambda_n} \end{cases} \tag{28}$$

Then the solution $w(x, y)$ can be rewritten as:

$$w(x,y) = w_2(x,y) + \sum_{n=1}^{\infty} \big(a_n x + i(1-\lambda_n) a_n y\big)(x+iy)^{\lambda_n - 1} + \sum_{n=1}^{\infty} \big(-a_n x + i(1-\lambda_n) a_n y\big)(x-iy)^{\lambda_n - 1} \tag{29}$$

where

$$w_2(x,y) = w_0(x,y) - iy\psi_{10}(x+iy) + iy\psi_{10}(x-iy) - 2iy\alpha_0'(x-iy) + \alpha_0(x+iy) - \alpha_0(x-iy) + A_2(x-iy) + xA_1(x-iy)$$

Simplify the equation(29):

$$\begin{cases} w(x,y) = w_2(x,y) + \sum_{n=1}^{\infty} a_n f_n(x,y) \\ f_n(x,y) = (1-\lambda_n)(x+iy)^{\lambda_n} - (1-\lambda_n)(x-iy)^{\lambda_n} + \lambda_n x (x+iy)^{\lambda_n - 1} - \lambda_n x (x-iy)^{\lambda_n - 1} \end{cases} \tag{30}$$

By equation(30), we can get the partial differentials of $w(x, y)$:

$$\begin{cases} \frac{\partial w}{\partial x} = \frac{\partial w_2}{\partial x} + \sum_{n=1}^{\infty} a_n \frac{\partial f_n}{\partial x} \\ \frac{\partial f_n}{\partial x} = -x\left(\lambda_n - 1\right)\lambda_n (x-iy)^{\lambda_n - 2} - \left(1-\lambda_n\right)\lambda_n (x-iy)^{\lambda_n - 1} - \lambda_n (x-iy)^{\lambda_n - 1} \\ \qquad + \lambda_n (x+iy)^{\lambda_n - 1} + \left(1-\lambda_n\right)\lambda_n (x+iy)^{\lambda_n - 1} + x\left(\lambda_n - 1\right)\lambda_n (x+iy)^{\lambda_n - 2} \end{cases} \tag{31}$$

$$\begin{cases} \frac{\partial w}{\partial y} = \frac{\partial w_2}{\partial y} + \sum_{n=1}^{\infty} a_n \frac{\partial f_n}{\partial y} \\ \frac{\partial f_n}{\partial y} = ix\left(\lambda_n - 1\right)\lambda_n (x-iy)^{\lambda_n - 2} + i\left(1-\lambda_n\right)\lambda_n (x-iy)^{\lambda_n - 1} \\ \qquad + i\left(1-\lambda_n\right)\lambda_n (x+iy)^{\lambda_n - 1} + ix\left(\lambda_n - 1\right)\lambda_n (x+iy)^{\lambda_n - 2} \end{cases} \tag{32}$$

The parametrical form of the hypotenuse line is:

$$\begin{cases} x = a - at \\ y = bt \end{cases}, t \in [0, 1] \tag{33}$$

Then we can change the boundary conditions of the hypotenuse line to:

$$\begin{cases} w(a-at, bt) = 0 \\ \frac{\partial w}{\partial n}(a-at, bt) = b\frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = 0 \end{cases} \tag{34}$$

In order to derive the solution of equation(29), we need to determine the coefficient a_n. In this paper we use the Least Square Method. Consider a function $\Pi(a_1, a_2, \cdots, a_n, \cdots)$:

$$\Pi = \int_0^1 w^2(a - at, bt) + (b\frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y})^2 \mathrm{d}t \tag{35}$$

We need to choose a_n to minimize Π.