$$w_1(x,y) = w_0(x,y) + A(x-iy) + iyA_1(x-iy)$$
$$w(x,y) = w_1(x,y) + (1-\frac{y}{b})[\alpha(x+iy) - \alpha(x-iy)] + \frac{y}{b}[\beta(x+iy) - \beta(x-iy)] - 2iy\alpha'(x-iy) \tag{1}$$

# 1 $w(0,y) = 0$

Since we have the boundary condition $w(0,y) = 0$, we get

$$(1-\frac{y}{b})[\alpha(iy) - \alpha(-iy)] + \frac{y}{b}[\beta(iy) - \beta(-iy)] - 2iy\alpha'(-iy) = -w_1(0,y) \tag{2}$$

Assume that

$$(1-\frac{y}{b})\alpha(iy) + \frac{y}{b}\beta(iy) = \phi(iy) \tag{3}$$

we have

$$(1-\frac{y}{b})\alpha(-iy) + \frac{y}{b}\beta(-iy) + 2iy\alpha'(-iy) = w_1(0,y) + \phi(iy) \tag{4}$$

Replace $y$ by $-y$ in the previous equation, then we can get

$$(1+\frac{y}{b})\alpha(iy) - \frac{y}{b}\beta(iy) - 2iy\alpha'(iy) = w_1(0,-y) + \phi(-iy) \tag{5}$$

equation (3) + (5), we can get

$$2\alpha(iy) - 2iy\alpha'(iy) = w_1(0,-y) + \phi(iy) + \phi(-iy) \tag{6}$$

we set $iy = z$, then we can get

$$2\alpha(z) - 2z\alpha'(z) = w_1(0,iz) + \phi(z) + \phi(-z) \tag{7}$$

Rewrite it as

$$\alpha'(z) - \frac{1}{z}\alpha(z) = -\frac{1}{2z}w_1(0,iz) - \frac{1}{2z}\phi(z) - \frac{1}{2z}\phi(-z) \tag{8}$$

Because it is a nonhomogeneous ODE, we can use the method of variation of parameters, then we get

$$\alpha(z) = z\left[c - \frac{1}{2}\int\left(\frac{1}{z^2}w_1(0,iz) + \frac{1}{z^2}\phi(z) + \frac{1}{z^2}\phi(-z)\right)dz\right] \tag{9}$$

Letting $\phi(z) = z^2\varphi'(z)$ , then we get

$$\alpha(z) = z\left[c - \frac{1}{2}\int\frac{1}{z^2}w_1(0,iz)dz - \frac{1}{2}\varphi(z) + \frac{1}{2}\varphi(-z)\right] \tag{10}$$

Then we can get the expression of $\beta(z)$ .

$$\beta(z) = ibz\varphi'(z) - (ib-z)\big[c - \frac{1}{2}\int\frac{1}{z^2}w_1(0,iz)dz\big] - (ib-z)[-\frac{1}{2}\varphi(z) + \frac{1}{2}\varphi(-z)] \tag{11}$$

## 2 solution

Assuming $w_2(z) = \frac{1}{2}\int \frac{1}{z^2} w_1(0, iz)dz$ then equations (10) and (11) will be written as

$$\begin{cases} \alpha(z) = z\big[c - w_2(z) - \frac{1}{2}\varphi(z) + \frac{1}{2}\varphi(-z)\big] \\ \beta(z) = ibz\varphi'(z) - (ib - z)\big[c - w_2(z)\big] - (ib - z)[-\frac{1}{2}\varphi(z) + \frac{1}{2}\varphi(-z)] \end{cases} \tag{12}$$

We know that $\alpha'(z) = \frac{1}{z}\alpha(z) - \frac{1}{2z}w_1(0, iz) - \frac{1}{2z}\phi(z) - \frac{1}{2z}\phi(-z)$

equations (1) and (12) imply that (making a careful simplification)

$$\begin{aligned} w(x, y) = {} & w_1(x, y) + x\big(c - w_2(x + iy)\big) - x\big(c - w_2(x - iy)\big) + \frac{iy}{x - iy} w_1(0, i(x - iy)) \\ & - \frac{x}{2}\big(\varphi(x + iy) - \varphi(-x - iy)\big) + \frac{x}{2}\big(\varphi(x - iy) - \varphi(-x + iy)\big) \\ & + iy(x + iy)\varphi'(x + iy) + iy(x - iy)\varphi'(-x + iy) \end{aligned} \tag{13}$$