The exact solution of plate with

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November 15, 2011

1 Introduction

The lateral deflection in plain-plate problem is described by the equilibrium equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = p(x,y) \tag{1}$$

where $p(x,y)$ is a continuously distributed lateral load. The general solution of the partial Equation (1) is given by

$$w(x,y) = w_0(x,y) + \varphi_1(x+iy) + \varphi_2(x-iy) + x[\psi_1(x+iy) + \psi_2(x-iy)] \tag{2}$$

where $w_0(x,y)$ is a particular solution of Equation (1) and φ_1,φ_2,ψ_1,ψ_2 are four arbitrary functions.

Theorem 1.1. *(Four boundaries are clamped) The solution of the following plate problem*

$$\left\{\begin{array}{c} \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = p(x,y) \\ w(x,0) = w(x,b) = \frac{\partial w}{\partial y}(x,0) = \frac{\partial w}{\partial y}(x,b) = 0 \\ w(0,y) = w(a,y) = \frac{\partial w}{\partial x}(0,y) = \frac{\partial w}{\partial x}(a,y) = 0 \end{array}\right. \tag{3}$$

is given by

Proof.

Step 1. To make the solution (2) satisfy the boundary conditions $w(x,0)=0$ and $\frac{\partial w}{\partial y}(x,0)=0$, we must put

$$\begin{array}{c} \varphi_1(x) + \varphi_2(x) + x[\psi_1(x) + \psi_2(x)] = A(x) \\ \varphi_1'(x) - \varphi_2'(x) + x[\psi_1'(x) - \psi_2'(x)] = B(x) \end{array} \tag{4}$$

where $A(x) = -w_0(x,0)$ and $B(x) = i\frac{\partial w_0}{\partial y}(x,0)$. Letting

$$\varphi_1(x) + x\psi_1(x) = \alpha(x) \tag{5}$$

where $\alpha(x)$ is an arbitrary function. By (5) and the first equation of Eqs.(4), we have

$$\varphi_2(x) + x\psi_2(x) = A(x) - \alpha(x) \tag{6}$$

(5) and (6) imply that

$$\left\{\begin{array}{l} \varphi_1(x) = -x\psi_1(x) + \alpha(x) \\ \varphi_2(x) = -x\psi_2(x) + A(x) - \alpha(x) \end{array}\right. \tag{7}$$

Substituting (7) into the second equation of Eqs. (4) yields

$$\psi_2(x) = \psi_1(x) - 2\alpha'(x) + A_1(x) \tag{8}$$

where $A_1(x) = A'(x) + B(x)$. By (7) and (8), we get

$$\begin{cases} \varphi_1(x) = -x\psi_1(x) + \alpha(x) \\ \varphi_2(x) = -x\psi_1(x) + 2x\alpha'(x) - \alpha(x) + A_2(x) \\ \psi_2(x) = \psi_1(x) - 2\alpha'(x) + A_1(x) \end{cases} \tag{9}$$

where $A_2(x) = A(x) - xA_1(x)$.

Step 2. Similarly, to make the solution (2) satisfy the boundary conditions $w(x,b) = 0$ and $\frac{\partial w}{\partial y}(x,b) = 0$, we must put

$$\begin{cases} \varphi_1(x+ib) + \varphi_2(x-ib) + x[\psi_1(x+ib) + \psi_2(x-ib)] = C(x) \\ \varphi_1'(x+ib) - \varphi_2'(x-ib) + x[\psi_1'(x+ib) - \psi_2'(x-ib)] = D(x) \end{cases} \tag{10}$$

where $C(x) = -w_0(x,b)$ and $D(x) = i\frac{\partial w_0}{\partial y}(x,b)$. Letting

$$\varphi_1(x+ib) = -x\psi_1(x+ib) + \beta(x+ib) \tag{11}$$

where $\beta(x)$ is an arbitrary function. By (11) and the first equation of Eqs. (10), we have

$$\varphi_2(x-ib) = -x\psi_2(x-ib) + C(x) - \beta(x+ib) \tag{12}$$

Using (11), (12) and the second equation of Eqs. (10), we obtain

$$\psi_2(x-ib) = \psi_1(x+ib) - 2\beta'(x+ib) + C'(x) + D(x) \tag{13}$$

where $C_1(x) = C'(x+ib) + D(x+ib)$ and $C_2(x) = C(x+ib) - (x+2ib)C_1(x)$. By (11), (12) and (13), we have

$$\begin{cases} \varphi_1(x) = -(x-ib)\psi_1(x) + \beta(x) \\ \varphi_2(x) = -(x+ib)\psi_1(x+2ib) + 2(x+ib)\beta'(x+2ib) - \beta(x+2ib) + C_2(x) \\ \psi_2(x) = \psi_1(x+2ib) - 2\beta'(x+2ib) + C_1(x) \end{cases} \tag{14}$$

By Eqs. (9) and Eqs. (14), we obtain

$$-x\psi_1(x) + \alpha(x) = -(x-ib)\psi_1(x) + \beta(x) \tag{15}$$

$$-x\psi_1(x) + 2x\alpha'(x) - \alpha(x) + A_2(x) = -(x+ib)\psi_1(x+2ib) + 2(x+ib)\beta'(x+2ib) - \beta(x+2ib) + C_2(x) \tag{16}$$

$$\psi_1(x) - 2\alpha'(x) + A_1(x) = \psi_1(x+2ib) - 2\beta'(x+2ib) + C_1(x) \tag{17}$$

By (15), we get

$$\alpha(x) = ib\psi_1(x) + \beta(x) \tag{18}$$

By (16) $+ x \times$ (17), we have

$$-\alpha(x) = -ib\psi_1(x+2ib) + 2ib\beta'(x+2ib) - \beta(x+2ib) + E(x) \tag{19}$$

where $E(x) = C_2(x) + xC_1(x) - A_2(x) - xA_1(x)$. By (15) $+ (x+ib) \times$ (16), we obtain

$$ib\psi_1(x) - 2ib\alpha'(x) - \alpha(x) = -\beta(x+2ib) + E_1(x) \tag{20}$$

where $E_1(x) = C_2(x) + (x+ib)C_1(x) - A_2(x) - (x+ib)A_1(x)$. By (18) and (19), we get

$$\alpha(x+2ib) - \alpha(x) = 2ib\beta'(x+2ib) + E(x) \tag{21}$$

By (18) and (20), we get

$$\beta(x+2ib) - \beta(x) = 2ib\alpha'(x) + E_1(x) \tag{22}$$

By (21) and (22), we get

$$(2ib)^2\beta''(x+2ib) = \beta(x+4ib) - 2\beta(x+2ib) + \beta(x) + E_2(x) \tag{23}$$

where $E_2(x) = E_1(x) - E_1(x+2ib) - 2ibE'(x)$. □

Lemma 1.2. *The general solution of the function equation (23) can be written as*

$$\begin{aligned}\beta(x) = \beta_0(x) + \sum_{n=1}^{\infty} &[a_n exp(\tfrac{\lambda_n x}{2ib}) + b_n exp(-\tfrac{\lambda_n x}{2ib}) + c_n exp(\tfrac{\delta_n x}{2ib}) + d_n exp(-\tfrac{\delta_n x}{2ib}) \\ &+ e_n exp(\tfrac{\bar{\lambda}_n x}{2ib}) + f_n exp(-\tfrac{\bar{\lambda}_n x}{2ib}) + g_n exp(\tfrac{\bar{\delta}_n x}{2ib}) + h_n exp(-\tfrac{\bar{\delta}_n x}{2ib})]\end{aligned} \tag{24}$$

where

$$exp(\frac{\lambda}{2}) - exp(-\frac{\lambda}{2}) = \lambda \tag{25}$$

$$exp(\frac{\delta}{2}) - exp(-\frac{\delta}{2}) = -\delta \tag{26}$$

By (21) and (22), we can also get

$$(2ib)^2\alpha''(x) = \alpha(x+2ib) - 2\alpha(x) + \alpha(x-2ib) + E_3(x) \tag{27}$$

where $E_3(x) = -E(x) + E(x-2ib) - 2ibE_1'(x)$.

Similar to lemma 1.2, we can get the general solution of equation (27)

$$\begin{aligned}\alpha(x) = \alpha_0(x) + \sum_{n=1}^{\infty} &[a'_n exp(\tfrac{\lambda_n x}{2ib}) + b'_n exp(-\tfrac{\lambda_n x}{2ib}) + c'_n exp(\tfrac{\delta_n x}{2ib}) + d'_n exp(-\tfrac{\delta_n x}{2ib}) \\ &+ e'_n exp(\tfrac{\bar{\lambda}_n x}{2ib}) + f'_n exp(-\tfrac{\bar{\lambda}_n x}{2ib}) + g'_n exp(\tfrac{\bar{\delta}_n x}{2ib}) + h'_n exp(-\tfrac{\bar{\delta}_n x}{2ib})]\end{aligned} \tag{28}$$

Substituting (24) and (28) to equation (21) or equation(22), we can get the relationship between the coefficients in the two solutions

$$\begin{aligned} a'_n &= a_n exp(\frac{\lambda_n}{2}) \\ b'_n &= b_n exp(-\frac{\lambda_n}{2}) \\ c'_n &= -c_n exp(\frac{\delta_n}{2}) \\ d'_n &= -d_n exp(-\frac{\delta_n}{2}) \\ e'_n &= e_n exp(\frac{\bar{\lambda}_n}{2}) \\ f'_n &= f_n exp(-\frac{\bar{\lambda}_n}{2}) \\ g'_n &= -g_n exp(\frac{\bar{\delta}_n}{2}) \\ h'_n &= -h_n exp(-\frac{\bar{\delta}_n}{2}) \end{aligned} \tag{29}$$

Now we get the expressions of $\alpha(x)$ and $\beta(x)$, then we substitute them back to the Equations (18) and (9), we can get the expressions of the functions $\varphi_1(x)$, $\varphi_2(x)$, $\psi_1(x)$, and $\psi_2(x)$.

$$\begin{cases} \psi_1(x) = \frac{\alpha(x)-\beta(x)}{ib} \\ \varphi_1(x) = -x\psi_1(x) + \alpha(x) \\ \varphi_2(x) = -x\psi_1(x) + 2x\alpha'(x) - \alpha(x) + A_2(x) \\ \psi_2(x) = \psi_1(x) - 2\alpha'(x) + A_1(x) \end{cases} \tag{30}$$

In order to express $w(x,y)$, we should rewrite the Eqs. (30) as follow:

$$\begin{cases} \psi_1(x+iy) = \frac{\alpha(x+iy)-\beta(x+iy)}{ib} \\ \varphi_1(x+iy) = -(x+iy)\psi_1(x+iy) + \alpha(x+iy) \\ \varphi_2(x-iy) = -(x-iy)\psi_1(x-iy) + 2(x-iy)\alpha'(x-iy) - \alpha(x-iy) + A_2(x-iy) \\ \psi_2(x-iy) = \psi_1(x-iy) - 2\alpha'(x-iy) + A_1(x-iy) \end{cases} \tag{31}$$

then we can get

$$\begin{aligned} w(x,y) &= w_0(x,y) + \varphi_1(x+iy) + \varphi_2(x-iy) + x[\psi_1(x+iy) + \psi_2(x-iy)] \\ &= w_0(x,y) - (x+iy)\psi_1(x+iy) + \alpha(x+iy) - (x-iy)\psi_1(x-iy) + 2(x-iy)\alpha'(x-iy) - \alpha(x-iy) \\ &+ A_2(x-iy) + x[\psi_1(x+iy) + \psi_1(x-iy) - 2\alpha'(x-iy) + A_1(x-iy)] \end{aligned} \tag{32}$$

We also know that $\psi_1(x-iy) = \frac{\alpha(x-iy)-\beta(x-iy)}{ib}$ and $A_2(x-iy) = A(x-iy) - (x-iy)A_1(x-iy)$. Substitute them to equation (32), we get

$$\begin{aligned} w(x,y) = w_0(x,y) + (-\frac{y}{b} + 1)[\alpha(x+iy) - \alpha(x-iy)] + \frac{y}{b}[\beta(x+iy) - \beta(x-iy)] \\ - 2iy\alpha'(x-iy) + A(x-iy) + iyA_1(x-iy) \end{aligned} \tag{33}$$