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A note on the Mooney–Rivlin material model

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Abstract In finite elasticity, the Mooney–Rivlin material model for the Cauchy stress tensor T in terms of the left Cauchy–Green strain tensor B is given by

$$T = -pI + s_1 B + s_2 B^{-1},$$

where p is the pressure and s_1, s_2 are two material constants. It is usually assumed that $s_1 > 0$ and $s_2 \leq 0$, known as E-inequalities, based on the assumption that the free energy function be positive definite for *any* deformation. In this note, we shall relax this assumption and with a thermodynamic stability analysis, prove that s_2 need not be negative so that some typical behavior of materials under contraction can also be modeled.

Keywords Mooney–Rivlin material · E-inequalities · Uniaxial extension · Stability analysis · Stress–strain curve

1 Introduction

In finite elasticity, the Mooney–Rivlin material model for incompressible isotropic elastic solids is given by the Cauchy stress tensor T in terms of the left Cauchy–Green strain tensor B,

$$T = -pI + s_1 B + s_2 B^{-1},$$

where p is the pressure and s_1, s_2 are two material constants. Based on the assumption that the free energy function be positive definite for *any* deformation, it is usually assumed that $s_1 > 0$ and $s_2 \leq 0$, known as E-inequalities, which is strongly supported from experimental evidence for rubber-like materials.

However, the assumption $s_2 \leq 0$ gives rise to a very restrictive behavior in bodies under contraction, that the compressive stress–strain curves are increasing and concave upward, which means that it is getting harder and harder to contract as the compressive strain increases. This behavior may seem to be reasonable for rubber-like materials, and it is also known in soil mechanics as the *locking* behavior observed in dry sand under confined compression. Nevertheless, the locking behavior is not typical for most materials, such as soil and rock, that sustain essentially compressive stresses.

Dedicated to Ingo Müller on the occasion of his 75th birthday.

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In this note, we shall relax the assumption that leads to the E-inequalities and with a thermodynamic stability analysis for an isothermal uniaxial experiment, prove that s_2 need not be negative, so that some typical behavior of materials under contraction can also be modeled for moderate compressive strains as Mooney–Rivlin materials.

2 Mooney–Rivlin thermoelastic materials

We shall briefly re-derive a constitutive equation for moderate strains of an incompressible isotropic thermoelastic solid, starting start with the free energy function ψ and the Cauchy stress T given by

$$\psi = \psi(\theta, \mathrm{I}_B, \mathrm{II}_B), \quad T = -p\,I + 2\rho\frac{\partial\psi}{\partial B}B, \tag{1}$$

where θ is the temperature, ρ the mass density, and p the indeterminate pressure for incompressibility, while $\{\mathrm{I}_B, \mathrm{II}_B, \mathrm{III}_B = 1\}$ are the three principal invariants of the left Cauchy–Green tensor B and I is the identity tensor.

By Taylor series expansion, we can write

$$\psi(\theta, \mathrm{I}_B, \mathrm{II}_B) = \psi_0 + \psi_1(\mathrm{I}_B - 3) + \psi_2(\mathrm{II}_B - 3) + o(\delta_1^2, \delta_2^2, \delta_1\delta_2), \tag{2}$$

where $\psi_k = \psi_k(\theta)$ for $k = 0, 1, 2$, and $o(\delta_1^2, \delta_2^2, \delta_1\delta_2)$ stands for the second- order terms in $\mathrm{I}_B - 3 = o(\delta_1)$ and $\mathrm{II}_B - 3 = o(\delta_2)$, say, respectively of the order of $\delta_1 < 1$ and $\delta_2 < 1$.

Since the deformation gradient $F = I + H$, we have

$$B = FF^T = I + A, \text{ where } A = H + H^T + HH^T.$$

Therefore, if we assume that the displacement gradient H is of the order of $\delta < 1$, write, $H = o(\delta)$, it follows that A is of the same order, $A = o(\delta)$.

Note that for $B = I + A$, one can show that

$$\begin{aligned} \mathrm{I}_B &= 3 + \mathrm{I}_A, \\ \mathrm{II}_B &= 3 + 2\,\mathrm{I}_A + \mathrm{II}_A, \\ \mathrm{III}_B &= 1 + \mathrm{I}_A + \mathrm{II}_A + \mathrm{III}_A. \end{aligned}$$

In particular, for $\det B = \mathrm{III}_B = 1$, we have

$$\mathrm{I}_A = -\mathrm{II}_A - \mathrm{III}_A.$$

Therefore, since $\mathrm{II}_A = o(\delta^2)$, of the order δ^2, so is $\mathrm{I}_A = o(\delta^2)$, and we have

$$\begin{aligned} \mathrm{I}_B - 3 &= \mathrm{I}_A = o(\delta_1) = o(\delta^2), \\ \mathrm{II}_B - 3 &= 2\,\mathrm{I}_A + \mathrm{II}_A = o(\delta_2) = o(\delta^2), \end{aligned}$$

which implies $o(\delta_1^2, \delta_2^2, \delta_1\delta_2) = o(\delta^4)$. Consequently, from (2) we have

$$\psi(\theta, \mathrm{I}_B, \mathrm{II}_B) = \psi_0 + \psi_1(\mathrm{I}_B - 3) + \psi_2(\mathrm{II}_B - 3) + o(\delta^4).$$

In other words, the free energy function,

$$\psi(\theta, \mathrm{I}_B, \mathrm{II}_B) = \psi_0 + \psi_1(\mathrm{I}_B - 3) + \psi_2(\mathrm{II}_B - 3), \tag{3}$$

is a general representation with an error in the fourth order of the displacement gradient.

By the use of the relations

$$\frac{\partial\,\mathrm{I}_B}{\partial B} = I, \qquad \frac{\partial\,\mathrm{II}_B}{\partial B} = \mathrm{I}_B I - B,$$

from (1), we obtain

$$T + p\,I = s_1 B + s_2(B^2 - \mathrm{I}_B B), \tag{4}$$

where

$$s_1(\theta) = 2\rho\psi_1(\theta), \quad s_2(\theta) = -2\rho\psi_2(\theta). \tag{5}$$

Note that the constitutive equation (4) can be written as

$$T = -p\,I + t_1 B + t_2 B^2,$$

with

$$t_1 = s_1 - s_2 \mathrm{I}_B, \quad t_2 = -s_2.$$

Consequently, t_1 is not a function of θ only. However, by the use of the Cayley-Hamilton theorem, with $\mathrm{III}_B = 1$,

$$B^2 - \mathrm{I}_B B = B^{-1} - \mathrm{II}_B I,$$

it gives

$$T = -p\,I + s_1 B + s_2 B^{-1},$$

in which the term $\mathrm{II}_B I$ is absorbed into the indeterminate pressure $p\,I$. In this case, the material parameters s_1 and s_2 are functions of the temperature only. Therefore, we conclude that:

Mooney–Rivlin material. *The constitutive equation*

$$T = -p\,I + s_1 B + s_2 B^{-1}, \tag{6}$$

with $s_1(\theta)$ and $s_2(\theta)$, is general enough to account for incompressible thermoelastic isotropic solids at moderate strain with an error in the fourth order of the displacement gradient in free energy.

We claim no originality in the derivation of this result, since similar derivation can be found elsewhere (for example, see [1]). However, we have included it here to facilitate later discussions on the model.

3 Isothermal uniaxial experiment

In order to examine some possible restrictions on the values of the material parameters s_1 and s_2, we shall consider a simple isothermal uniaxial experiment: For the uniaxial deformation $\lambda > 0$,

$$x = \lambda X, \quad y = \frac{1}{\sqrt{\lambda}}\,Y, \quad z = \frac{1}{\sqrt{\lambda}}\,Z,$$

of a cylinder $\mathcal{V}$, with cross-section area A and length $0 \leq X \leq L$, in the reference state κ, we have

$$B = \lambda^2\,\boldsymbol{e}_x \otimes \boldsymbol{e}_x + \frac{1}{\lambda}\,(\boldsymbol{e}_y \otimes \boldsymbol{e}_y + \boldsymbol{e}_z \otimes \boldsymbol{e}_z),$$
$$\mathrm{I}_B = \lambda^2 + \frac{2}{\lambda}, \qquad \mathrm{II}_B = \frac{1}{\lambda^2} + 2\lambda,$$

where $\{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$ are the base vectors in the Cartesian coordinate system. The free energy function (3) in this case takes the following form,

$$\psi(\lambda) = \psi_0 + \psi_1\Big(\lambda^2 + \frac{2}{\lambda} - 3\Big) + \psi_2\Big(\frac{1}{\lambda^2} + 2\lambda - 3\Big). \tag{7}$$

One can easily show that both $(\lambda^2 + \frac{2}{\lambda} - 3)$ and $(\frac{1}{\lambda^2} + 2\lambda - 3)$ are non-negative for any $\lambda > 0$. Therefore, the free energy function $\psi(\lambda)$ is positive definite if

$$\psi_0 > 0, \quad \psi_1 \geq 0, \quad \psi_2 \geq 0,$$

which, from (5), justifies the well-known (empirical) E-inequalities,

$$s_1 > 0, \quad s_2 \leq 0, \tag{8}$$

strongly supported by experimental evidence for rubber-like materials (see [2] and Sect. 55 of [3]).

3.1 Thermodynamic stability analysis

To analyze thermodynamic stability of an isothermal uniaxial experiment, we start with the energy equation and the entropy inequality,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \rho\left(\varepsilon+\frac{1}{2}\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}\right)\mathrm{d}v+\int_{\partial\mathcal{V}}(\boldsymbol{q}-T\dot{\boldsymbol{x}})\cdot\boldsymbol{n}\,\mathrm{d}a=0,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \rho\eta\,\mathrm{d}v+\frac{1}{\theta}\int_{\partial\mathcal{V}}\boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}a\geq 0.
\end{aligned}$$

Elimination of the heat flux $\boldsymbol{q}$ between these relations leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \rho\left(\psi+\frac{1}{2}\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}\right)\mathrm{d}v-\int_{\partial\mathcal{V}}\dot{\boldsymbol{x}}\cdot T\boldsymbol{n}\,\mathrm{d}a\leq 0,$$

where the free energy function $\psi=\varepsilon-\theta\eta$.

Assuming the experiment is quasi-static by neglecting the kinetic energy, we can rewrite the above inequality in the reference state as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}_\kappa} \rho\psi\,\mathrm{d}v_\kappa-\int_{\partial\mathcal{V}_\kappa}\dot{\boldsymbol{x}}\cdot T_\kappa\boldsymbol{n}_\kappa\,\mathrm{d}a_\kappa\leq 0,$$

where T_κ is the Piola Kirchhoff stress on the reference boundary $\partial\mathcal{V}_\kappa$. In this case, we have the following boundary conditions: $T_\kappa=0$ on the lateral boundary of the cylinder $\mathcal{V}_\kappa$ and is constant on the top and the bottom, where the velocities are given by

$$\dot{\boldsymbol{x}}|_{X=0}=0,\quad \dot{\boldsymbol{x}}|_{X=L}=\dot{\lambda}L\,\boldsymbol{e}_x.$$

Therefore, it follows after integration that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\rho\psi-T_{\kappa\langle xx\rangle}\lambda\Big)AL\leq 0.$$

If we call $\Psi(\lambda)=\rho\psi(\lambda)-T_{\kappa\langle xx\rangle}\lambda$ the *available energy*, as the sum of the free energy and the potential energy, then we have

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t}\leq 0.$$

This allows us to establish a stability criterion. If we assume the equilibrium state at the deformation λ is a stable state, then any small perturbation from this state will eventually return to this state as time tends to infinity. Suppose that such a perturbation is represented by a dynamic deformation process $\lambda(t)$ under the same boundary conditions, it follows that

$$\lim_{t\to\infty}\lambda(t)=\lambda,$$

and since $\Psi(\lambda(t))$ is a decreasing function of time, it implies that

$$\Psi(\lambda(t))\geq\Psi(\lambda)\quad\forall t>0.$$

In other words, the available energy $\Psi(\lambda(t))$ attains its minimum at λ. Therefore, we arrive at the following criterion:

A thermodynamic stability criterion. *A state at the uniaxial deformation λ is stable (as $t \to \infty$) if the value of the available energy function $\Psi(\lambda)$ attains its minimum at this state.*

The minimization of $\Psi(\lambda)$ at given λ requires that

$$\frac{\mathrm{d}\Psi(\lambda)}{\mathrm{d}\lambda} = 0, \quad \text{and} \quad \frac{\mathrm{d}^2\Psi(\lambda)}{\mathrm{d}\lambda^2} \geq 0.$$

From the expression (7), the first condition

$$\frac{\mathrm{d}\Psi(\lambda)}{\mathrm{d}\lambda} = 2\rho\psi_1\Big(\lambda - \frac{1}{\lambda^2}\Big) - 2\rho\psi_2\Big(\frac{1}{\lambda^3} - 1\Big) - T_{\kappa\langle xx\rangle} = 0, \tag{9}$$

implies from (5) that

$$T_{\kappa\langle xx\rangle} = s_1\Big(\lambda - \frac{1}{\lambda^2}\Big) + s_2\Big(\frac{1}{\lambda^3} - 1\Big).$$

This is the axial stress per unit original area of the cylinder in the experiment, which can also be obtained directly from the constitutive equation (6) for the uniaxial deformation λ.

The second condition, on the other hand, is more interesting, we have

$$\frac{\mathrm{d}^2\Psi(\lambda)}{\mathrm{d}\lambda^2} = 2\rho\psi_1\Big(1 + \frac{2}{\lambda^3}\Big) + 2\rho\psi_2\,\frac{3}{\lambda^4} \geq 0. \tag{10}$$

Since $\psi_2/\psi_1 = -s_2/s_1$ by assuming $s_1 > 0$, it follows that

$$\frac{s_2}{s_1} \leq \frac{1}{3}\,(2\lambda + \lambda^4). \tag{11}$$

Obviously, this is a restriction for possible values of the material parameters s_1 and s_2. In particular, for $\lambda = 1$, it gives $s_1 \geq s_2$, which, in turns, implies that the requirement is always satisfied for uniaxial extension, $\lambda > 1$. On the other hand, for uniaxial contraction as $\lambda \to 0$, it requires $s_2 \leq 0$. Therefore, the E-inequalities satisfies the condition (11) for any deformation $\lambda > 0$.

However, for most materials, it is almost impossible to reach the limit at $\lambda \to 0$ in uniaxial contraction without structure failure, and since the Mooney–Rivlin material is a good material model for thermoelastic solid with moderate strain, say, $\lambda \in (\lambda_L, \lambda_U)$, for some $\lambda_U > 1$ and $1 > \lambda_L > 0$, the condition (11) requires that

$$s_2 \leq \gamma\, s_1, \quad \text{for}\ \gamma = \frac{1}{3}\,\big(2\lambda_L + \lambda_L^4\big) < 1. \tag{12}$$

In other words, for a material body that has a finite lower limit in uniaxial contraction, it is *not necessary* to require that the material parameter s_2 be negative as proposed by the E-inequalities, but rather only $s_2 < s_1$.

3.2 Stress–strain curves

In the uniaxial experiment, let the axial stress $\sigma = T_{\kappa\langle xx\rangle}$ be the force per unit area and the axial strain $\epsilon = \lambda - 1$ be the elongation per unit length in the reference state κ. Then from the first condition of thermodynamic stability (9), we have the stress–strain relation,

$$\sigma(\epsilon) = s_1\left((1+\epsilon) - \frac{1}{(1+\epsilon)^2}\right) + s_2\left(\frac{1}{(1+\epsilon)^3} - 1\right),$$

and the second condition (10) becomes

$$\frac{\mathrm{d}\sigma(\epsilon)}{\mathrm{d}\epsilon} \geq 0.$$

Consequently, for thermodynamic stability, the stress–strain curve must be non-decreasing in uniaxial experiments.

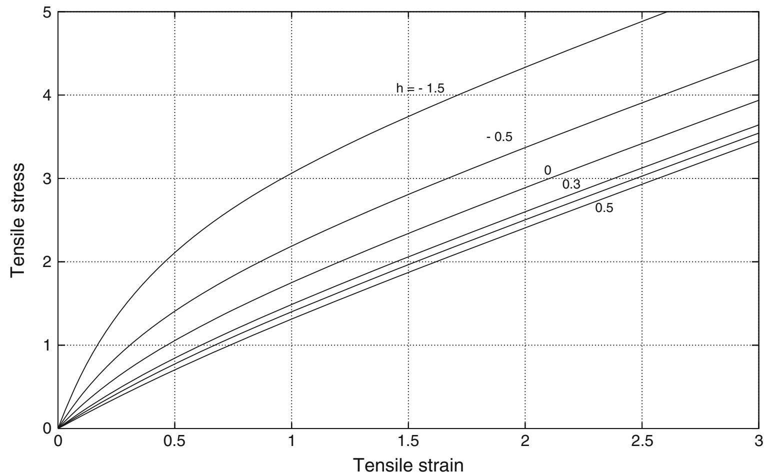

Fig. 1 Stress–strain curve under tension, σ/s_1 versus ϵ, for $h = -1.5$ up to $h = 0.5$

From (12), let

$$h = \frac{s_2}{s_1} < 1,$$

and express the stress–strain curve as

$$\frac{\sigma}{s_1} = \left((1+\epsilon) - \frac{1}{(1+\epsilon)^2}\right) + h\left(\frac{1}{(1+\epsilon)^3} - 1\right). \tag{13}$$

In Fig. 1, for tensile test, we plot the stress–strain curve, σ/s_1 versus ϵ, for $h = -1.5, -0.5, 0$, and for some positive values $h = 0.3, 0.4, 0.5$. Note that all these curves are increasing, therefore, thermodynamically stable, including the positive values of s_2.

For compressive test, it is customary to plot the axial compressive stress $(-\sigma)$ versus the axial compressive strain $(-\epsilon)$. Therefore, let

$$\bar{\sigma} = -\sigma, \quad \bar{\epsilon} = -\epsilon,$$

and the compressive stress–strain relation can be written as

$$\bar{\sigma}(\bar{\epsilon}) = -\sigma(\epsilon).$$

In Fig. 2, the compressive stress–strain curve is plotted as $(\bar{\sigma}/s_1)$ versus $(\bar{\epsilon})$ for several values of h. It is clear that for $h \le 0$, so that E-inequalities hold, the curves are increasing, therefore, thermodynamically stable. However, they are concave upward, or mathematically, we have

$$\frac{\mathrm{d}^2\bar{\sigma}(\bar{\epsilon})}{\mathrm{d}\bar{\epsilon}^2} = -\frac{\mathrm{d}^2\sigma(\epsilon)}{\mathrm{d}\bar{\epsilon}^2} = -\frac{\mathrm{d}^2\sigma(\epsilon)}{\mathrm{d}\epsilon^2} > 0 \quad \text{for } h \le 0, \ \epsilon < 0.$$

Physically, a concave-upward compressive stress–strain curve means that the material is getting harder to contract as the compressive strain increases. This kind of behavior may actually be reasonable for rubber-like materials but may not be appropriate for some other kind of materials, like soil and rock, which might exhibit concave-downward compressive stress–strain curves (for example, see Fig. 2 of [4]).

On the other hand, Fig. 2 also shows that for $h \ge 0.3$, the curves are indeed concave downward for moderate compressive strains, which assures that by allowing s_2 to be positive, it is possible to model some kind of materials exhibiting such behavior.

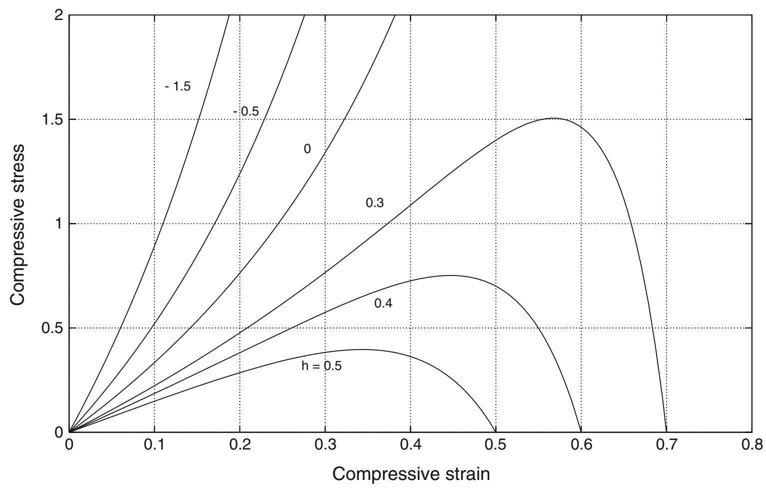

Fig. 2 Stress–strain curve under compression, $\bar{\sigma}/s_1$ versus $\bar{\epsilon}$, for $h = 0.5$ down to $h = -1.5$

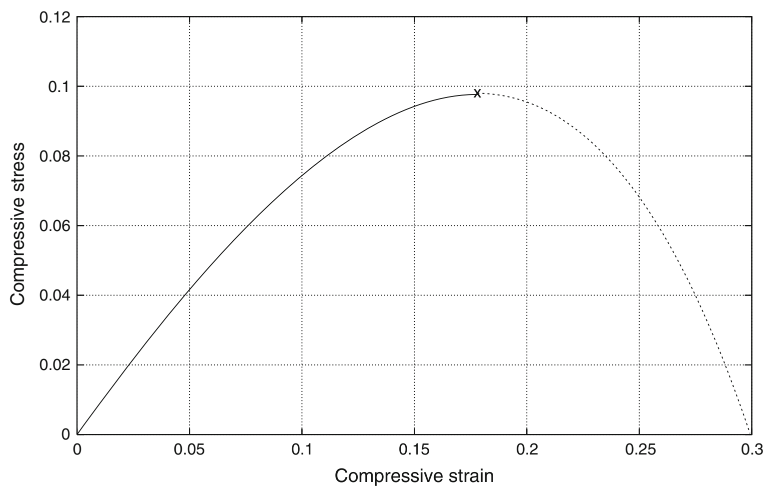

Fig. 3 Stress–strain curve under compression, $\bar{\sigma}/s_1$ versus $\bar{\epsilon}$, for $h = 0.7$. The *point* marked at the maximum stress corresponds to $\bar{\epsilon} = 0.1781$. The *dotted part* of the curve is thermodynamically unstable

To analyze the qualitative behavior of such materials [4], we consider the compressive stress–strain curve for $h = 0.7$ shown in Fig. 3. It is a concave-downward curve with a maximum compressive stress marked at the compressive strain $\bar{\epsilon} = 0.1781$. Note that thermodynamic stability requires that

$$\frac{\mathrm{d}\bar{\sigma}(\bar{\epsilon})}{\mathrm{d}\bar{\epsilon}} = -\frac{\mathrm{d}\sigma(\epsilon)}{\mathrm{d}\bar{\epsilon}} = \frac{\mathrm{d}\sigma(\epsilon)}{\mathrm{d}\epsilon} \geq 0.$$

Therefore, the dotted part of the curve with negative slope is thermodynamically unstable. Likewise, the part of the curves with negative slope for $h \geq 0.3$ in Fig. 2 are unstable.

Moreover, at the maximum compressive stress, the compressive strain $\bar{\epsilon} = 0.1781$ corresponds to $\lambda = 1 - \bar{\epsilon} = 0.8219$. One can easily check from the condition (12) that, for $\gamma = h = 0.7$, the lower contraction limit λ_L gives

$$\frac{1}{3}\left(2\lambda_L + \lambda_L^4\right) = 0.7 \Rightarrow \lambda_L = 0.8219.$$

Therefore, $\bar{\epsilon} = 0.1781$ is the maximum compressive strain for thermodynamic stability, and contraction beyond the lower limit would possibly mean a certain structural failure of the experimental specimen, such as crack and shear bend.

From the above observation, there is a simply way to determine the values of the material constants s_1 and s_2 from a concave-downward compressive stress–strain curve of an uniaxial experiment. From the experimental curve, one can determines the maximum compressive strain $\bar{\epsilon}_{\max}$ at the maximum compressive stress $\bar{\sigma}_{\max}$, and hence, obtain

$$\lambda_L = 1 - \bar{\epsilon}_{\max}, \quad h = \frac{s_2}{s_1} = \frac{1}{3}\,(2\lambda_L + \lambda_L^4).$$

After that, from (13), it is easy to determine the value of s_1 with the calculated value of h,

$$\frac{-\bar{\sigma}_{\max}}{s_1} = \left((1 - \bar{\epsilon}_{\max}) - \frac{1}{(1 - \bar{\epsilon}_{\max})^2}\right) + h\left(\frac{1}{(1 - \bar{\epsilon}_{\max})^3} - 1\right).$$

One can then decide whether the Mooney–Rivlin model is an acceptable material model by comparison of the fitted curve with the experimental one.

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