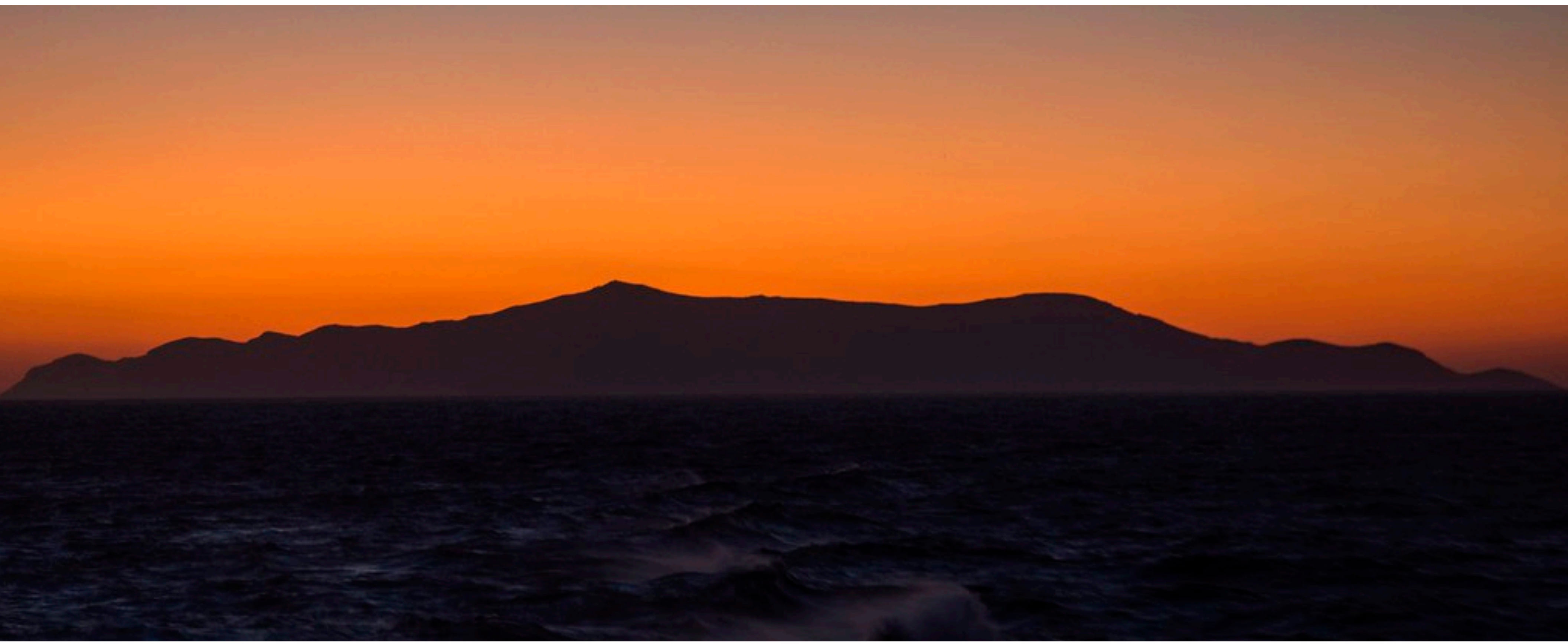
Unit 3, Lecture 2
Quadratic Growth Models
(Part II)



Mild Schedule Restructure

Today (Oct 10): Finish Quadratic Growth

Tuesday (Oct 15): Piecewise Linear Growth

Thursday (Oct 17): Three level models

(in particular, time nested in individuals nested in groups)

Tuesday (Oct 22): Back on track with posted syllabus

Today's Goals

Continue learning about quadratic growth models. In particular:

- ★ Interpret the intercept, slope, and curvature parameters
- ★ Incorporate covariates

★ Talk a bit more about how statistics and modeling can intersect with policy debate

Quadratic Growth

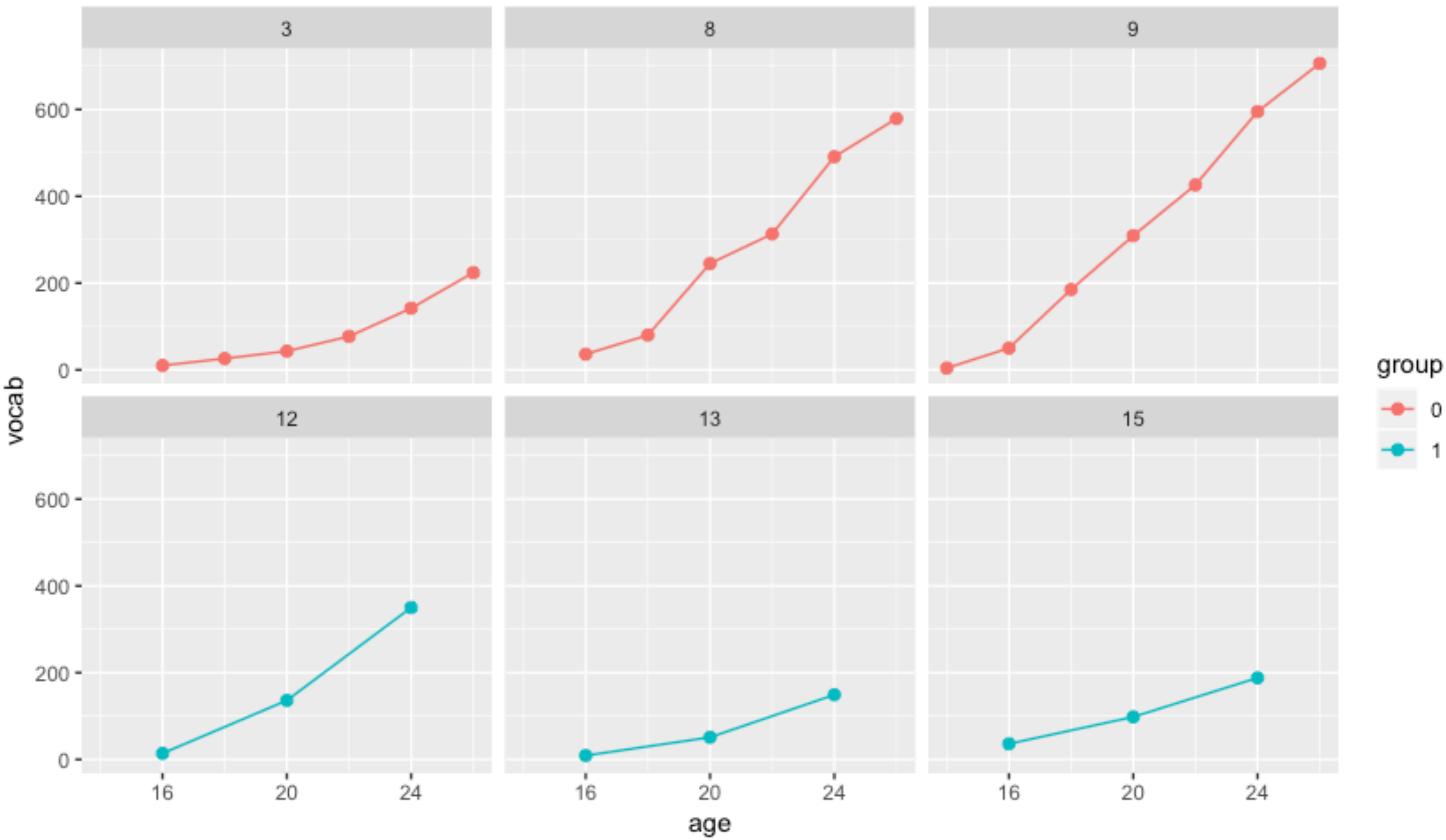
Example from Raudenbush & Bryk, Chapter 6, pgs 169-176

Maternal Speech on Children's Vocabulary

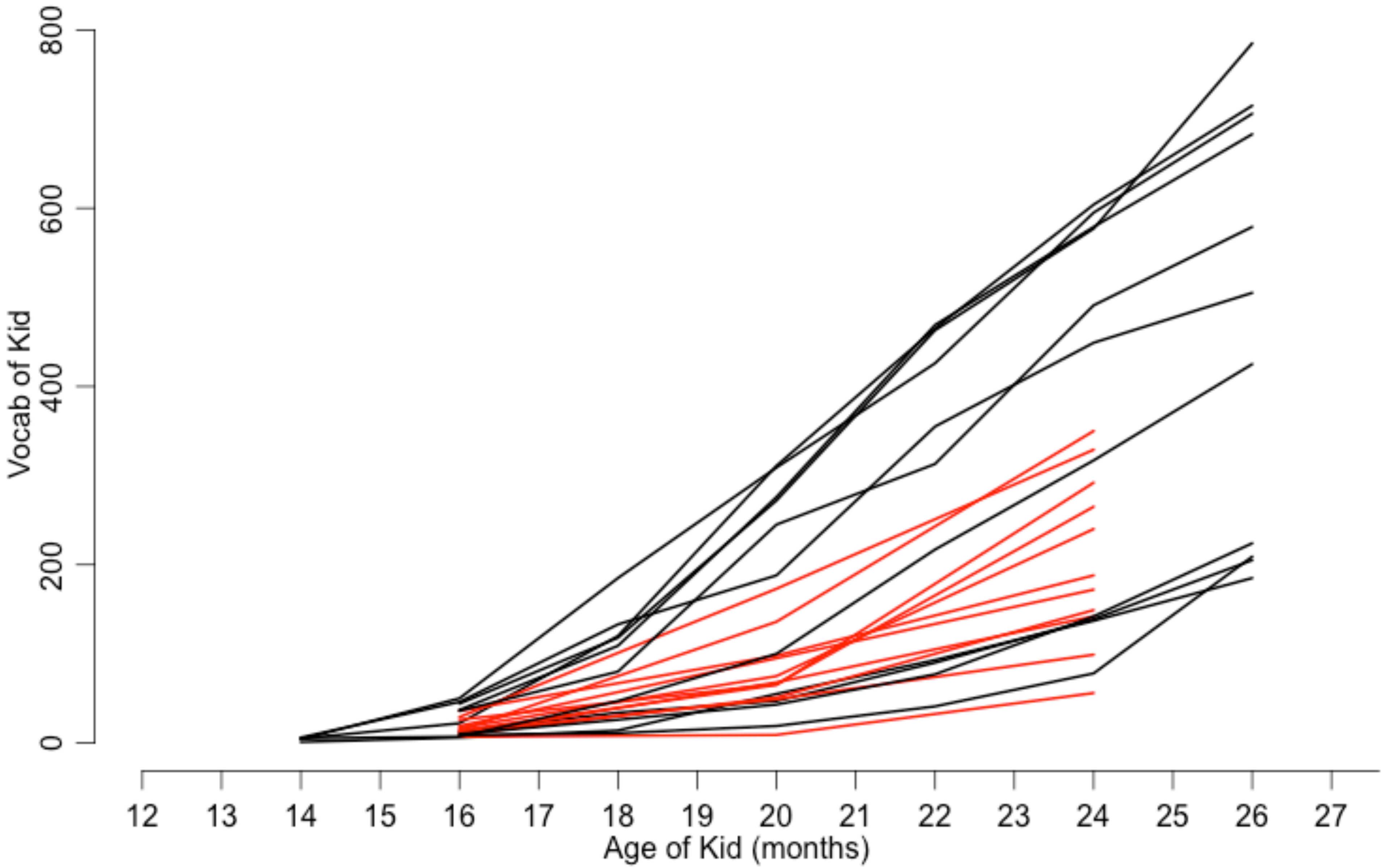
- ★ 22 children observed in the home on differing number of occasions throughout about 1.5 years
- ★ At 16 months, amount of maternal speech was recorded
- ★ Actually two studies with different patterns of observation
  - Study 1 (11 kids) has more home visits (observations)

Research Question:
Describe the association between maternal speech and child's vocabulary

6 of the kids across time



All the raw data (22 children)

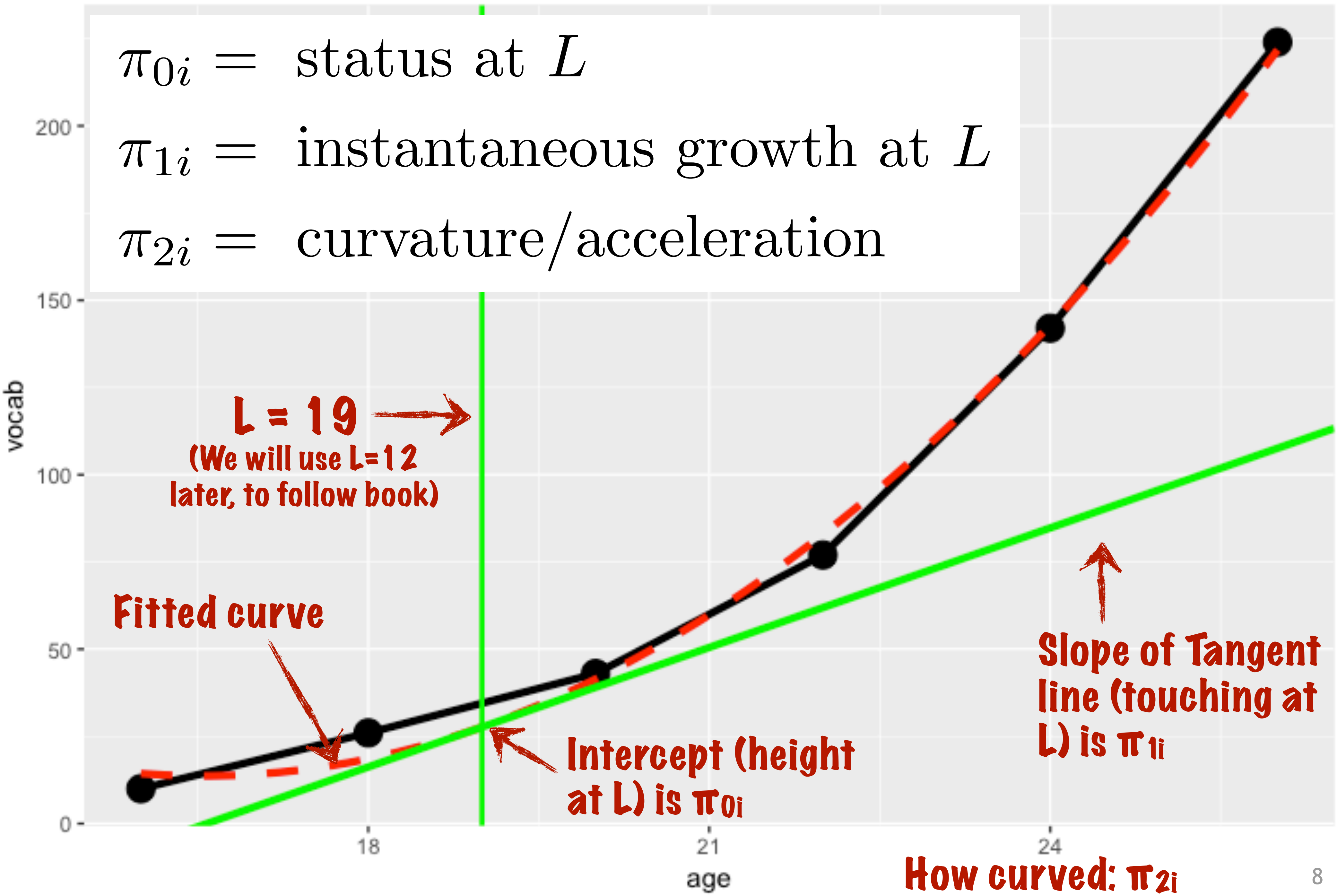


Anatomy of a Quadratic Growth Curve

$$\pi_{0i} = \text{ status at } L$$

$$\pi_{1i} = \text{ instantaneous growth at } L$$

$$\pi_{2i} = \text{ curvature/acceleration}$$



Unconditional Quadratic Growth Model

$$Y_{ti} = \pi_{0i} + \pi_{1i}(a_{ti} - L) + \pi_{2i}(a_{ti} - L)^2 + \varepsilon_{ti}$$

$$\varepsilon_{ti} \sim N(0, \sigma^2)$$

$$\pi_{0i} = \gamma_{00} + u_{0i}$$

$$\pi_{1i} = \gamma_{10} + u_{1i}$$

$$\pi_{2i} = \gamma_{20} + u_{2i}$$

$$(u_{0i}, u_{1i}, u_{2i}) \sim N(0, \Sigma)$$

How many parameters?

This Σ (Sigma) represents all our taus in our matrix. But it is now 3x3 so too big. See next slide.

- ★ *Three* random effects, all correlated.
- ★ We have a fixed centering constant *L*. *(This is a constant picked by you, not a parameter.)*

Our Covariance Matrix Expanded

$$\sum = \begin{bmatrix} \tau_{00} & \text{Symmetric} & \\ \tau_{10} & \tau_{11} & \\ \tau_{20} & \tau_{21} & \tau_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \text{Var}(\pi_{0i}) & & \text{Symmetric} \\ \text{Cov}(\pi_{1i}, \pi_{0i}) & \text{Var}(\pi_{1i}) & \\ \text{Cov}(\pi_{2i}, \pi_{0i}) & \text{Cov}(\pi_{2i}, \pi_{1i}) & \text{Var}(\pi_{2i}) \end{bmatrix}.$$

Picking Model, Setting L

Individual plots from before suggest a curved fit. Quadratic is a first reasonable choice.

L is an important centering decision.

In our case, L=12 months because this is age where words first begin.

(However, picking L in the center of your data is usually preferable without good reason not to.)

We start by looking only at the 11 kids with mostly full data.

Fitting our quadratic unconditional model

We first recenter and make our quadratic predictor

```
> dat$age12 = dat$age - 12
> dat$age12sq = dat$age12 ^ 2
> M1 = lmer( vocab ~ 1 + age12 + age12sq + (1+age12 +
                                  age12sq|pers), data=dat.g0 )
> display( M1 )
             coef.est coef.se
(Intercept) -45.05    27.52
age12        12.14     8.57
age12sq       1.84     0.26

Error terms:
 Groups   Name        Std.Dev. Corr
 pers     (Intercept) 73.76
          age12       24.62    -0.99
          age12sq      0.27     0.45 -0.54
 Residual             26.89
---
number of obs: 71, groups: pers, 11
AIC = 737.1, DIC = 738
deviance = 727.3
```

γ_{00}

γ_{10}

γ_{20}

σ

$\sqrt{\tau_{00}}$

$\sqrt{\tau_{11}}$

$\sqrt{\tau_{22}}$

Results in R&B book different. Different data?

$$Y_{ti} = \pi_{0i} + \pi_{1i}(a_{ti} - L) + \pi_{2i}(a_{ti} - L)^2 + \varepsilon_{ti}$$

Predicting Vocabulary

Using the model, predict vocabulary for a typical kid at

- ★ 20 months
- ★ 12 months

pollev.com/yayMLM

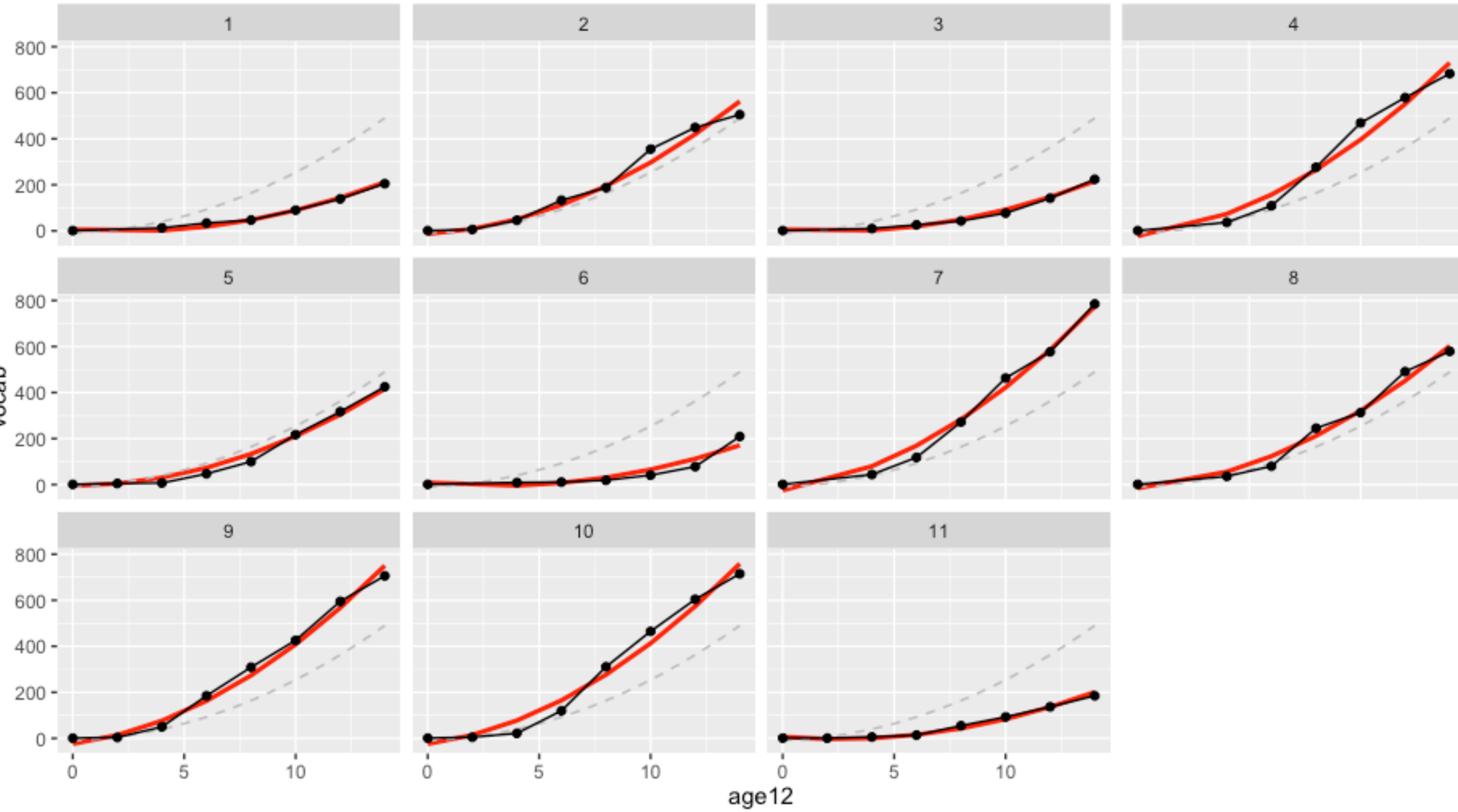
Getting more complicated:

- ★ How can we predict a distribution of vocabularies across kids at 20 months?

```
            coef.est coef.se
(Intercept) -45.05    27.52
age12        12.14     8.57
age12sq       1.84     0.26
```

```
Error terms:
 Groups   Name        Std.Dev.
 pers     (Intercept) 73.76
          age12       24.62
          age12sq      0.27
 Residual             26.89
```

Individual Curves for Each Kid



Getting estimates for the individual growth curves (using Empirical Bayes)

```
> ranef( M1 )$pers
   (Intercept) age12   age12sq
1           81 -26.1   0.06196
2          -22   9.2  -0.22034
3           80 -26.6   0.13738
4          -70  23.1  -0.09955
5           22  -9.1   0.19027
6           93 -31.8   0.22082
7          -82  25.0   0.05896
8          -33  10.3   0.00099
9          -76  27.0  -0.28605
10         -78  26.1  -0.14186
11          83 -27.1   0.07743
```

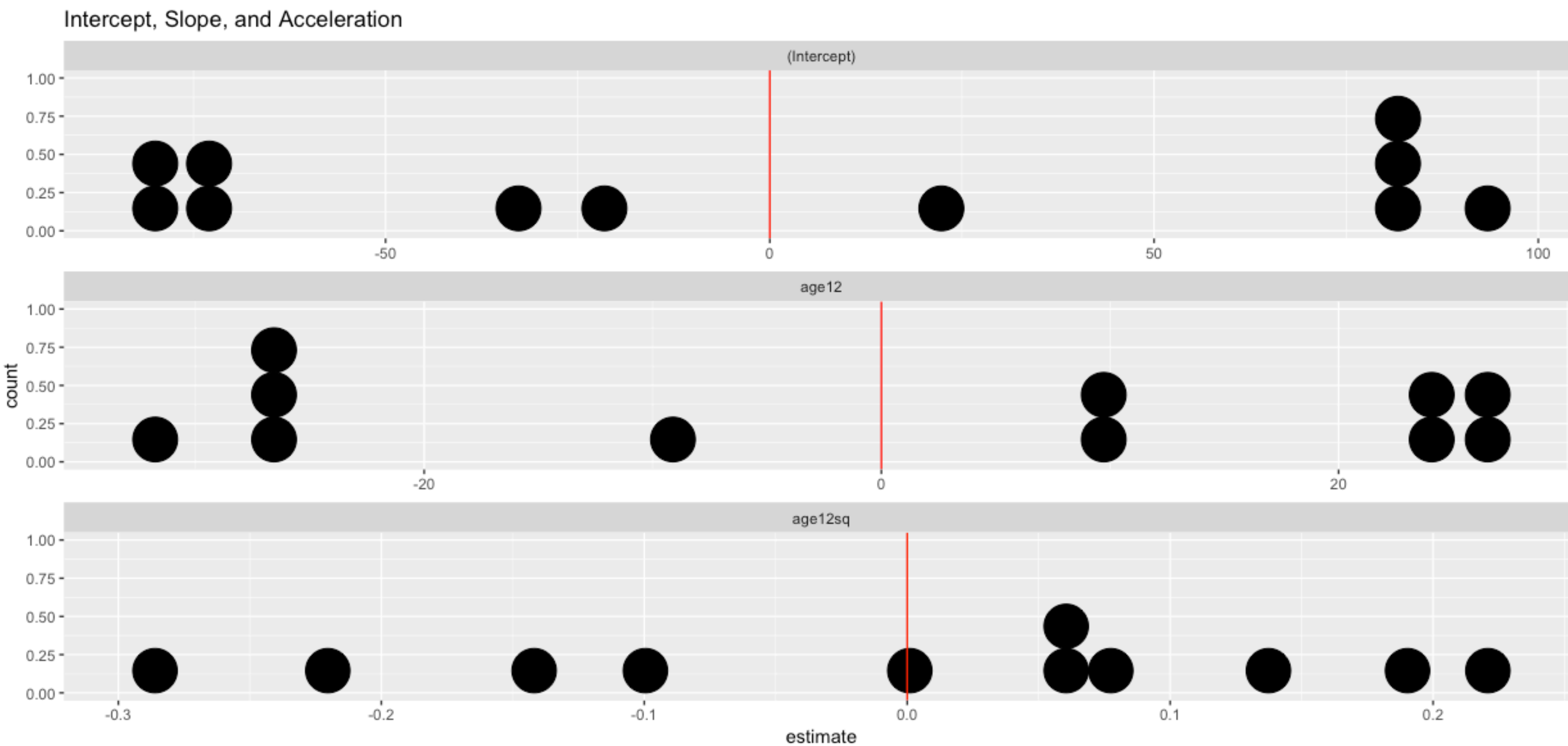
Each row is a single kid.

We have individual coefficients that vary for each kid.

```
> coefs = coef( M1 )$pers
> coefs
   (Intercept)      age12   age12sq
1     35.84194 -13.930663 1.897603
2    -66.57032  21.343986 1.615293
3     34.95373 -14.488695 1.973026
4   -115.17942  35.260103 1.736078
```

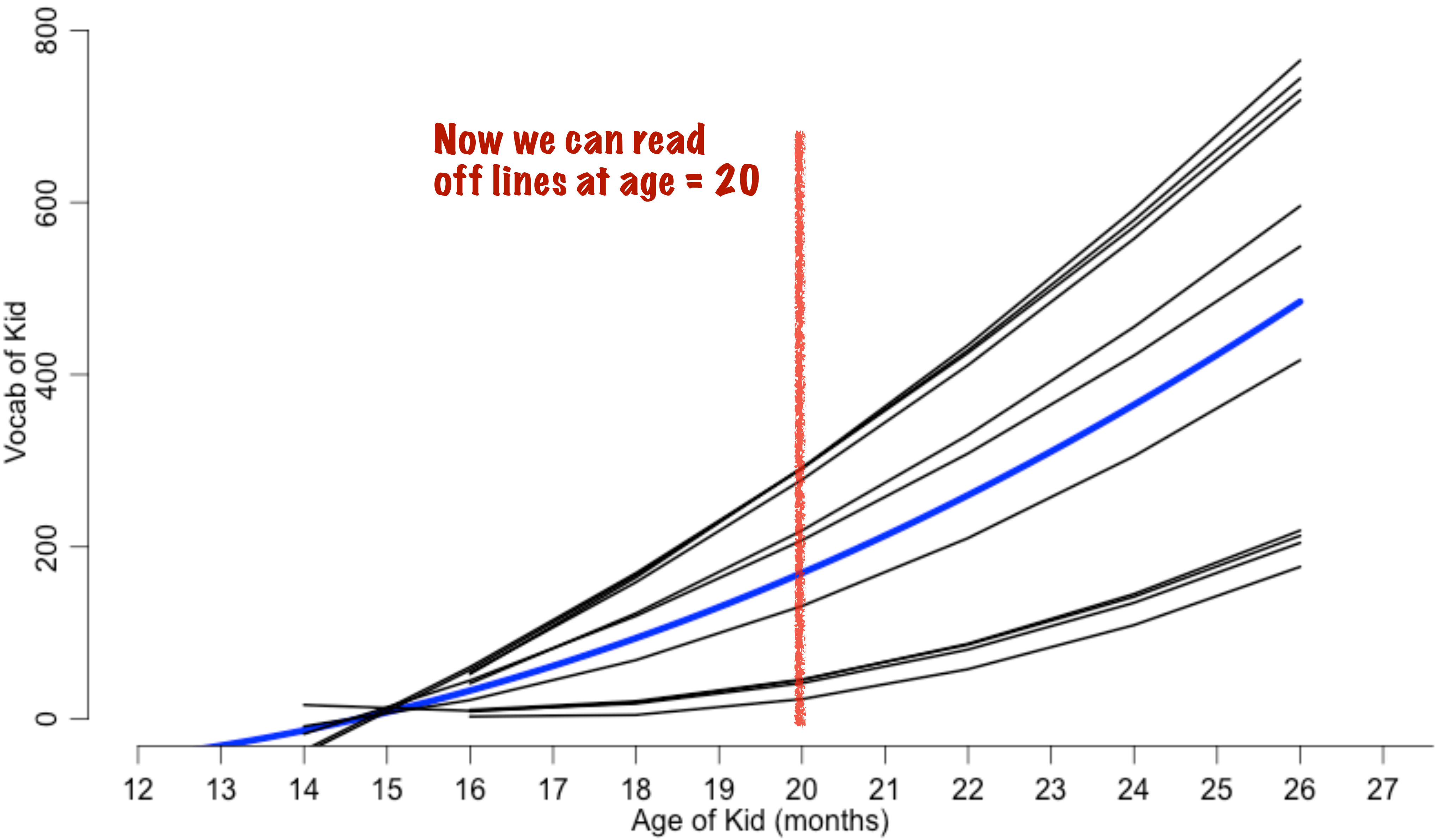
This takes the fixed effects into account as well. We can read off our quadratic curves

Do the distribution of our individual random effects look normal?



We are seeing that the random effects are definitively NOT normally distributed for any of our random effects.

Predicted Median Curve vs Individually Estimated Growth Curves (from Empirical Bayes)



Anatomy of a Quadratic Growth Curve

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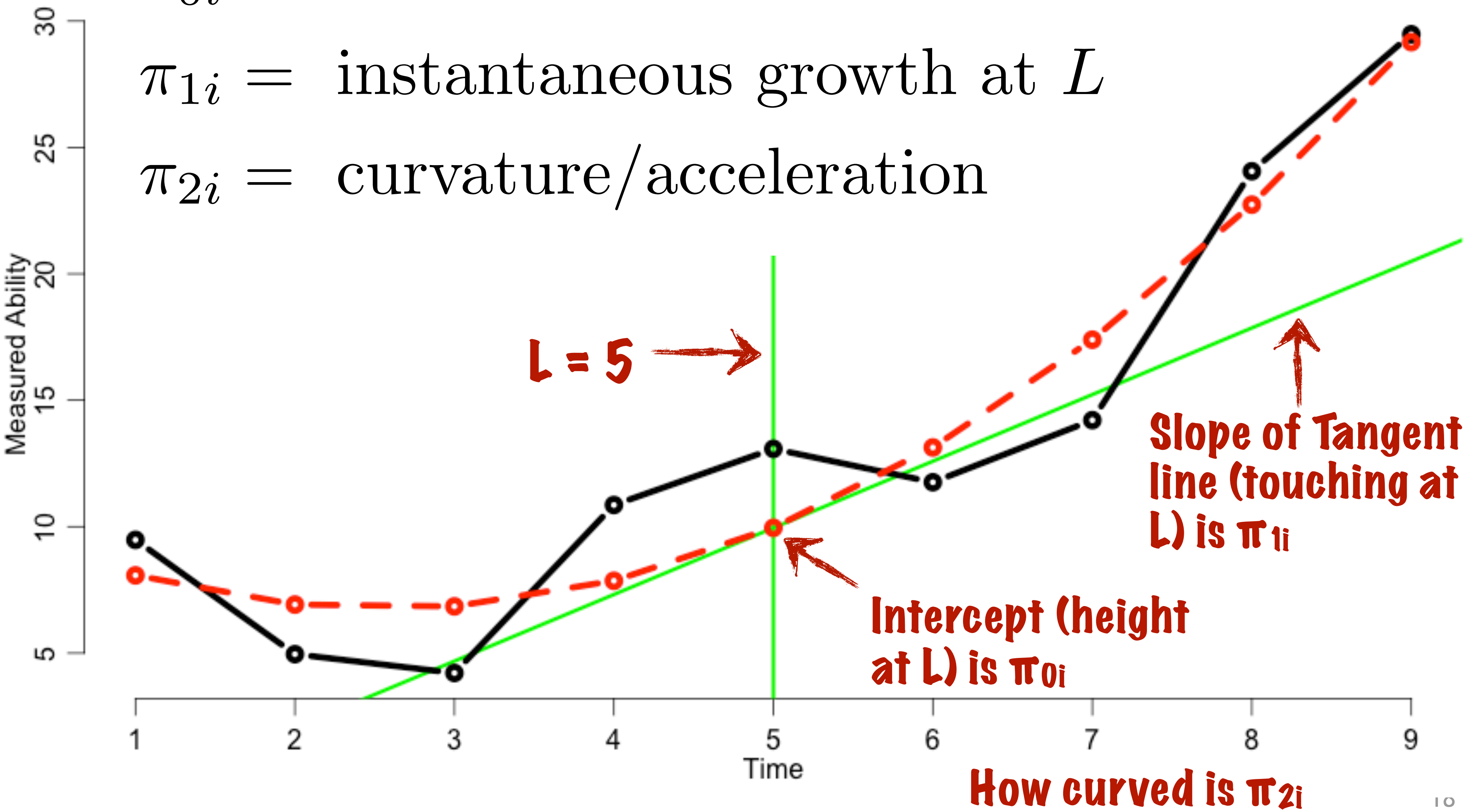
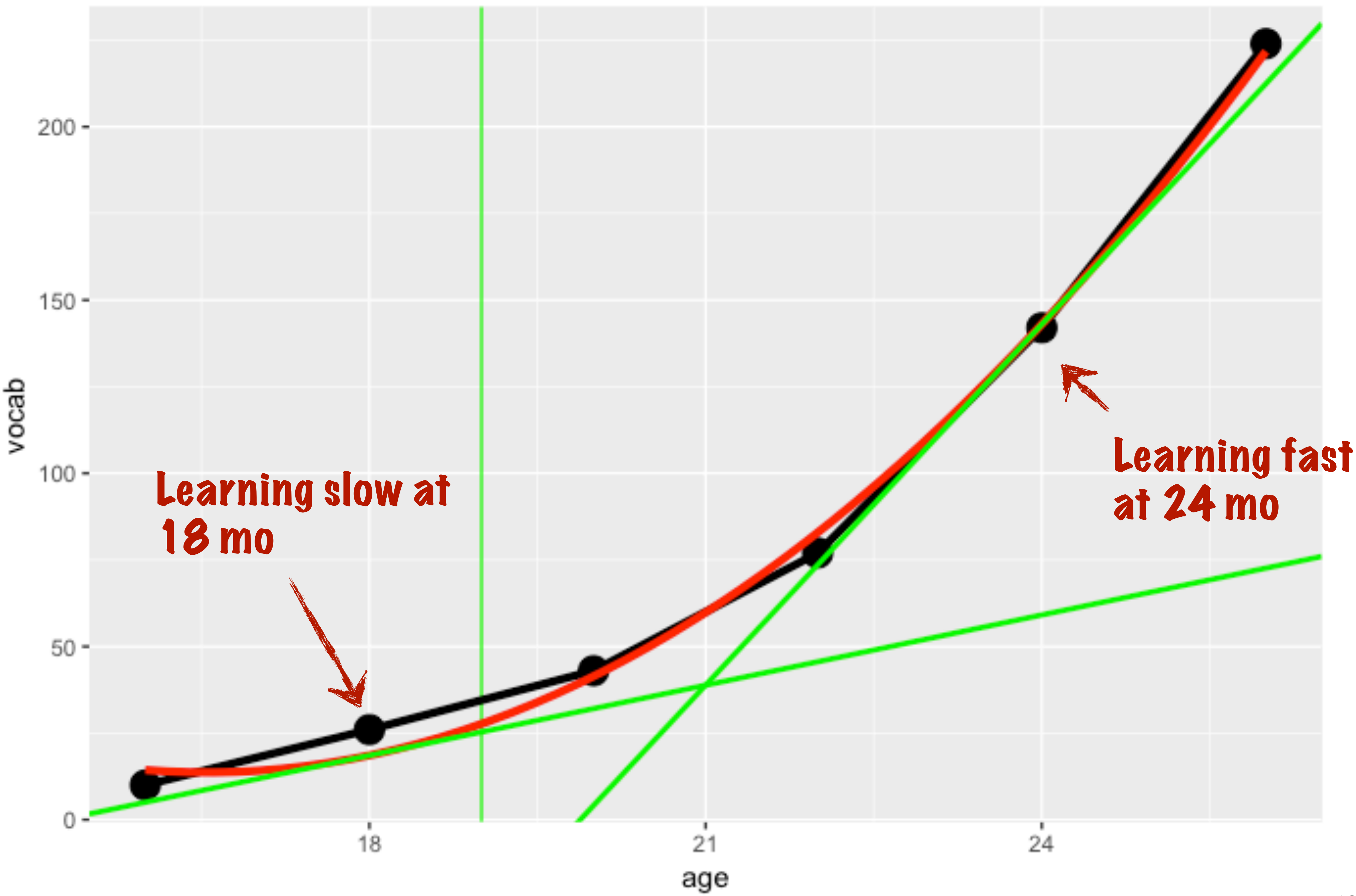


Changing rate of growth



Interpretation: Changing Growth Rate

$$\frac{d}{da}Y_{ti} = \pi_{0i} + \pi_{1i}(a - L) + \pi_{2i}(a - L)^2 + e_{ti}$$

$$= \pi_{1i} + 2\pi_{2i}(a - L)$$

- ★ Rate of growth at age *a* for kid *i* is the derivative of our curve at *a*.
- ★ So, what do we have for our rate of growth at 12 months? 14 months?

```
              coef.est coef.se  Error terms:
(Intercept) -45.05    27.52     Groups   Name        Std.Dev.
age12        12.14     8.57     pers     (Intercept) 73.76
age12sq       1.84     0.26              age12       24.62
                                         age12sq      0.27
                                Residual             26.89
```

Evaluation, modeling and policy

Let's Stop Talking About
the '30 Million Word Gap'

Talking with children matters:
Defending the 30 million word gap

Model building

How can we get a nice, simple model given our data?

Our Original Unconditional Model (repeat slide)

```
> dat$age12 = dat$age - 12
> dat$age12sq = dat$age12 ^ 2
> M1 = lmer( vocab ~ 1 + age12 + age12sq + (1+age12 +
                          age12sq|pers), data=dat.g0 )
> display( M1 )
lmer(formula = vocab ~ 1 + age12 + age12sq + (1 + age12 +
age12sq |
    pers), data = dat.g0)
            coef.est coef.se
(Intercept) -45.05    27.52
age12        12.14     8.57
age12sq       1.84     0.26

Error terms:
 Groups   Name        Std.Dev. Corr
 pers     (Intercept) 73.76
          age12       24.62    -0.99
          age12sq      0.27     0.45 -0.54
 Residual             26.89
---
number of obs: 71, groups: pers, 11
AIC = 737.1, DIC = 738
deviance = 727.3
```

Wildly off our theory

Something to worry about

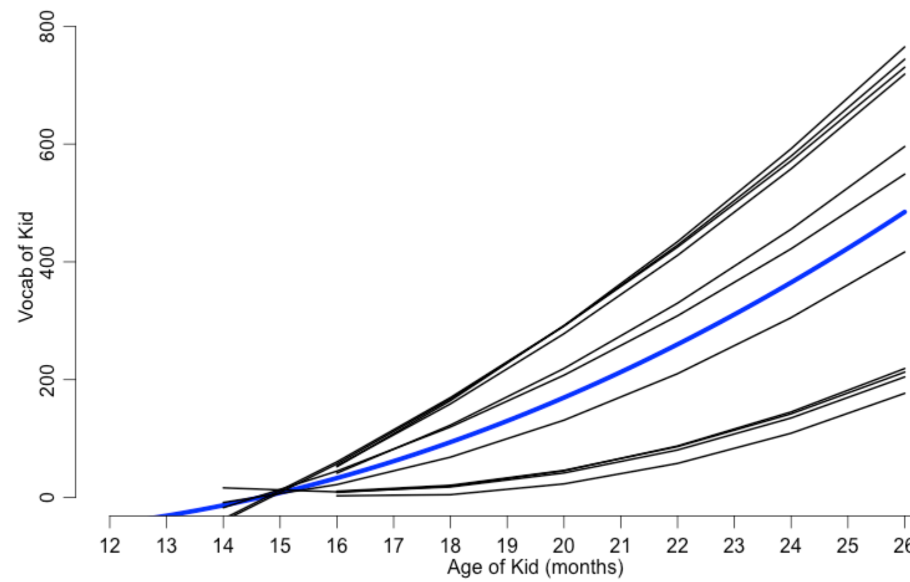
A lot of model for not much data

Theory tells us:

- ★ Initial vocab at 12 mo should be generally very small (i.e., around 0)

Model:

- ★ Our curves seem "overdispersed": possibly missing a variable?



- ★ Intercept is very negative (-45) when it shouldn't be

Next step: Drop our intercept! (Fix it to zero.)

Model Refinement #1: Quadratic Growth with no intercept

```
> M2 = lmer( vocab ~ 0 + age12 + age12sq +
                (0+age12+age12sq|pers),
                data=dat.g0 )
```

No overall intercept or random intercept either!

```
> display( M2 )
        coef.est  coef.se
age12    0.73      2.52
age12sq  2.46      0.30

Error terms:
 Groups   Name     Std.Dev. Corr
 pers     age12     6.09
          age12sq   0.87     1.00
 Residual          32.80
---
number of obs: 71, groups: pers, 11
AIC = 758, DIC = 752
deviance = 749.0
```

SD has gone up from 27. Our constrained model is not hugging the data as tightly, but the difference is not huge

Model Refinement #2
Set **expected** rate of growth at 12mo to 0

```
> M3 = lmer( vocab ~ 0 + age12sq + (0+age12+age12sq|
pers), data=dat.g0 )
> display( M3 )
```

```
coef.est  coef.se
    2.43     0.29
```

Note our SINGLE coefficient is for the age12sq variable.
R is mean and doesn't give us the name

```
Error terms:
 Groups   Name     Std.Dev. Corr
 pers     age12     5.71
          age12sq   0.87    1.00
 Residual          32.68
---
number of obs: 71, groups: pers, 11
AIC = 759.8, DIC = 748
deviance = 749.1
```

Now we have dropped our linear term in our fixed effect, but left it as a random effect. (Hence expected is 0, but not individual.)

Three models to compare

	Model 1	Model 2	Model 3
(Intercept)	-45.05		
	(27.52)		
age12	12.14	0.73	
	(8.57)	(2.52)	
age12sq	1.84 ***	2.46 ***	2.43 ***
	(0.26)	(0.30)	(0.29)
AIC	737.06	758.03	759.76
BIC	759.68	771.60	771.08
Log Likelihood	-358.53	-373.01	-374.88
Num. obs.	71	71	71
Num. groups: pers	11	11	11
Var: pers (Intercept)	5440.07		
Var: pers age12	606.18	37.12	32.60
Var: pers age12sq	0.07	0.76	0.76
Cov: pers (Intercept) age12	-1805.58		
Cov: pers (Intercept) age12sq	8.85		
Cov: pers age12 age12sq	-3.56	5.29	4.97
Var: Residual	722.83	1076.10	1068.27

*** p < 0.001, ** p < 0.01, * p < 0.05

lr test of simpler vs more complete models

```
> anova( M1, M2, M3 )
refitting model(s) with ML (instead of REML)
Data: dat.g0
Models:
M3: vocab ~ 0 + age12sq + (0 + age12 + age12sq | pers)
M2: vocab ~ 0 + age12 + age12sq + (0 + age12 + age12sq | pers)
M1: vocab ~ 1 + age12 + age12sq + (1 + age12 + age12sq | pers)
   Df AIC BIC logLik deviance Chisq Chi Df Pr(>Chisq)
M3  5 759 770   -375      749
M2  6 761 775   -375      749  0.09      1    0.76722
M1 10 747 770   -364      727 21.72      4    0.00023 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

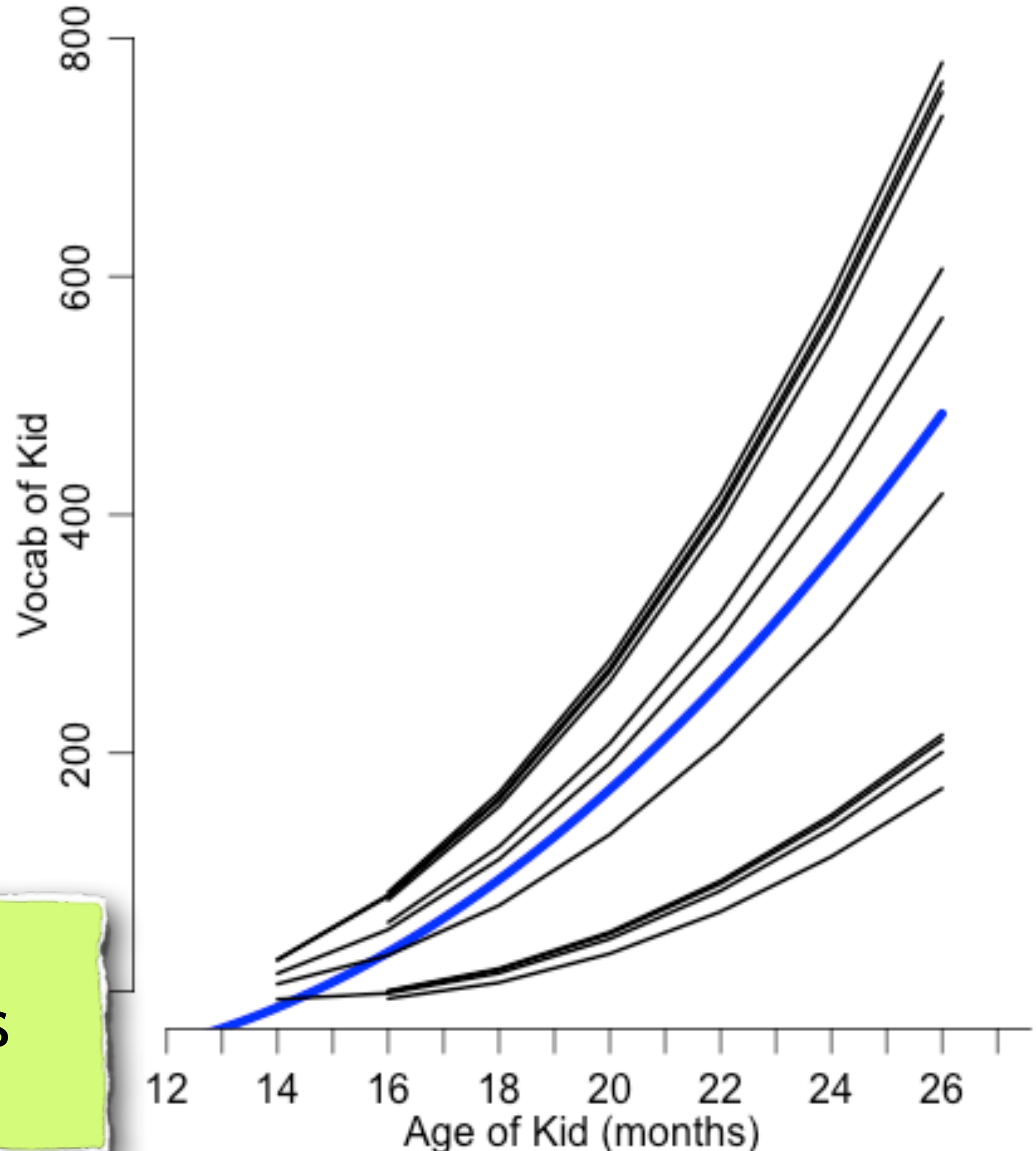
Check out
refitting
automatically

Note how these models
are NESTED
(each is an expansion of the
one above it)

anova() commands does
lrtest and refits models
automatically

New curves for simpler model

```
> coef( M3 )$pers
   age12 age12sq
1   -6.1     1.5
2    2.0     2.7
3   -6.0     1.5
4    5.9     3.3
5   -1.3     2.2
6   -7.0     1.4
7    6.9     3.5
8    3.0     2.9
9    6.3     3.4
10   6.5     3.4
11  -6.3     1.5
```



It looks like we have two clusters. Perhaps covariates explain?

Including covariates in our quadratic growth model

Including covariates: impact of child gender and maternal speech

```
> M4A = lmer( vocab ~ 0 + age12sq * (sex + logmom) +
              (0 + age12 + age12sq|pers),
              data=dat.g0 )
> display( M4A )
   age12sq | pers), data = dat.g0)
               coef.est coef.se
age12sq        -4.93     3.39
sex             2.60    19.19
logmom         -1.28     1.66
age12sq:sex     0.30     0.63
age12sq:logmom  0.88     0.44

Error terms:
 Groups   Name    Std.Dev. Corr
 pers     age12   6.80
          age12sq 0.63     1.00
 Residual         32.68
---
number of obs: 71, groups: pers, 11
AIC = 751.2, DIC = 751.1
deviance = 742.2
```

Can we explain our acceleration?

Is maternal speech predictive of growth?

Dropping main effects
A design decision. Good or bad?

```
> M4A.2 = lmer( vocab ~ 0 + age12sq:sex +
               age12sq:logmom + (0 + age12 + age12sq|pers),
               data=dat.g0 )
> display( M4A.2 )

                coef.est coef.se
age12sq:sex     0.51     0.53
age12sq:logmom  0.28     0.04

Error terms:
 Groups   Name     Std.Dev. Corr
 pers     age12     5.58
          age12sq   0.72    1.00
 Residual          32.80
---
number of obs: 71, groups: pers, 11
AIC = 762.2, DIC = 740.9
deviance = 745.5
```

Including covariates: Our full model on all 22 kids

```
> M4 = lmer( vocab ~ 0 + age12sq + age12sq:group +
          age12sq:sex + age12sq:sex:group +
          age12sq:logmom + age12sq:logmom:group +
         (0 + age12 + age12sq|pers), data=dat )
> display( M4 )
```

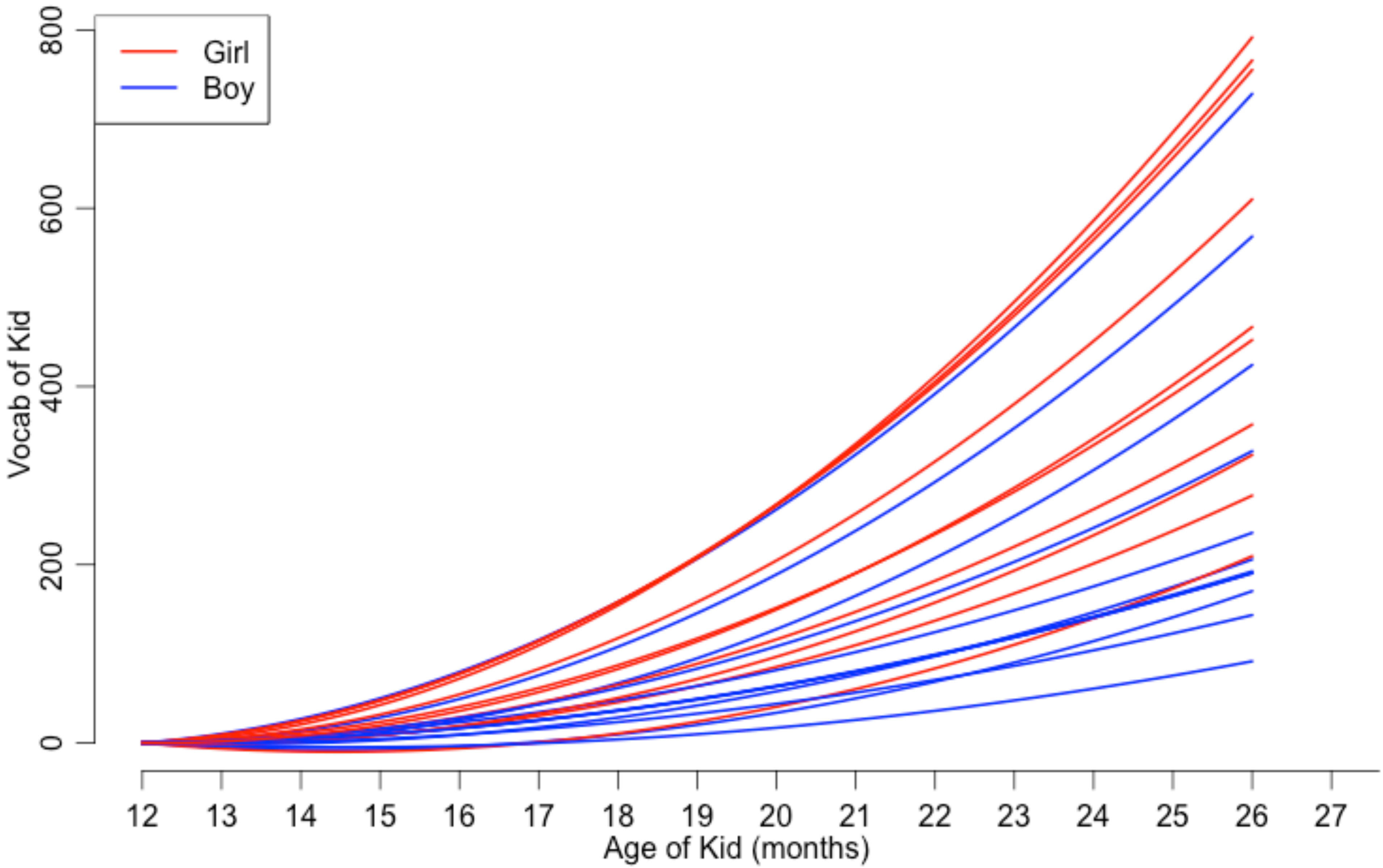
```
                     coef.est coef.se
age12sq              -5.43     2.98
age12sq:group         1.64     5.44
age12sq:sex           0.37     0.46
age12sq:logmom        0.97     0.38
age12sq:group:sex     0.53     0.68
age12sq:group:logmom -0.35     0.69

Error terms:
 Groups   Name    Std.Dev. Corr
 pers     age12    3.98
          age12sq  0.52    1.00
 Residual         30.23
---
number of obs: 104, groups: pers, 22
AIC = 1098.7, DIC = 1075
deviance = 1076.8
```

Can we explain our acceleration?

Is maternal speech predictive of growth?

Curves colored by gender differences



Variance Reduction due to gender and speech?

```
> M3.full = lmer( vocab ~ 0 + age12sq + age12sq:group +
              (0 + age12 + age12sq|pers), data=dat )

> M4 = lmer( vocab ~ 0 + age12sq + age12sq:group +
               age12sq:sex + age12sq:logmom +
               age12sq:sex:group + age12sq:logmom:group +
              (0 + age12 + age12sq|pers), data=dat )


> diag( VarCorr( M3.full )$pers )
  age12 age12sq
  15.70    0.53

> diag( VarCorr( M4 )$pers )
  age12 age12sq
  15.85    0.27
```

How much did we explain the variance in growth rate?

Looking back

Recap

Check-In
http://cs179.org/lec33

★ Quadratic growth allows for a moderately flexible *curved* growth curve

★ We can constrain aspects of a growth curve model to get a simpler model that is still not linear

★ The interpretation of the intercept, slope, and acceleration depend heavily on the choice of centering variable L.

★ Model building is a balance of simple model vs. fitting the data well.