COMPUTATIONAL AND ANALYTIC APPROACHES TO THE ERDŐS–POMERANCE DIVISIBLE INJECTION PROBLEM

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Abstract. We report on an extensive computational and analytic investigation of Erdős Problem #710: determine $f(n)$, the minimum length L such that there exists an injection $\varphi\colon \{1,\ldots,n\} \to (n, n+L] \cap \mathbb{Z}$ with $k \mid \varphi(k)$ for all k. The conjectured answer is $f(n) = (2/\sqrt{e} + o(1))\, n\sqrt{\ln n/\ln\ln n}$, where the lower bound is due to Erdős and Pomerance. We verify computationally that the matching upper bound holds for all $n \le 10^6$ via exhaustive Hopcroft–Karp matching with zero failures. For the analytic regime $n \to \infty$, we document 43 distinct approaches—including Cauchy–Schwarz variants, fractional matching, the Lovász Local Lemma, spectral methods, sieve bounds, semi-random nibble, and graph-theoretic techniques—each of which fails to close the gap between per-interval Hall verification and global Hall's condition. We identify the precise structural obstruction: the worst-case subset T^* spans approximately 48–53% of vertices, drawn from *all* dyadic intervals at roughly 60% density per interval, with expansion ratio barely exceeding 1. We catalog every approach, its failure mechanism, and the computational evidence, with the aim of guiding future work on this long-standing open problem.

Contents

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1. Introduction

1.1. **The problem.** For a positive integer n, define $f(n)$ to be the minimum $L \in \mathbb{Z}_{>0}$ such that there exists an injection $\varphi\colon \{1, \ldots, n\} \to (n, n+L] \cap \mathbb{Z}$ with $k \mid \varphi(k)$ for all k. Thus $f(n)$ is the length of the shortest interval $(n, n+L]$ admitting a *divisible injection* from $\{1, \ldots, n\}$. This function was studied by Erdős and Pomerance [2], and its asymptotic determination appears as Problem #710 on the Erdős problems database [14, 15]. The requirement $\varphi(k) > n$ ensures that $\varphi(k) \neq k$, and $k \mid \varphi(k)$ constrains each image to lie among the multiples of k that exceed n.

The problem sits at the intersection of multiplicative number theory, extremal combinatorics, and the anatomy of integers. It asks: how densely can we pack the integers $1, \ldots, n$ into a short interval above n, with each integer assigned to one of its own multiples?

1.2. **History and prior work.** The problem has its roots in Erdős's 1938 Tomsk paper [1], which introduced 2-*primitive sets*—sets where no element divides the product of two distinct others—and established foundational density estimates. The Erdős–Sárközy–Szemerédi collaboration [3] produced a series of at least ten papers on divisibility properties of integer sequences through the 1960s–70s, culminating in precise characterizations of extremal sets.

The specific function $f(n)$ and its asymptotic behavior were studied by Erdős and Pomerance [2], who established the lower bound

$$f(n) \;\geq\; \left(\frac{2}{\sqrt{e}} - o(1)\right) n\sqrt{\frac{\ln n}{\ln\ln n}}. \tag{1}$$

Their argument counts the available multiples of k in short intervals and shows that no interval shorter than the right-hand side of (1) can accommodate a divisible injection. The constant $2/\sqrt{e} \approx 1.2131$ arises from optimizing over the "smooth number" parameter: elements k whose largest prime factor $P(k) \leq k^{1/u}$ (with $u = \sqrt{\ln n/\ln\ln n}$) are the bottleneck, and the Dickman function $\rho(u)$ governs their density.

Erdős and Pomerance also established the best known upper bound:

$$f(n) \;\leq\; (1.7398\cdots + o(1))\, n\sqrt{\ln n}. \tag{2}$$

Note the absence of the $\ln\ln n$ correction: the upper bound (2) exceeds the conjectured answer (1) by a factor of $\sqrt{\ln\ln n}$, which diverges (albeit slowly). Closing this gap—proving the matching upper bound

$$f(n) \;\leq\; \left(\frac{2}{\sqrt{e}} + o(1)\right) n\sqrt{\frac{\ln n}{\ln\ln n}} \tag{3}$$

and thereby determining $f(n)$ asymptotically—has remained open since 1980. The problem appears in Guy's *Unsolved Problems in Number Theory* [15] (Sections C9 and C10) and carries a \$500 prize on `erdosproblems.com`.

1.3. **Our contribution.** This paper does not prove (3). Instead, we report on an extensive investigation—both computational and analytic—that:

(a) **Verifies computationally** that the upper bound holds for all $n \leq 10^6$, via exhaustive Hopcroft–Karp maximum bipartite matching with zero failures (Section 3);
(b) **Proves partial results:** per-interval Hall's condition via Cauchy–Schwarz, the $V_{\min}$ disjointness lemma, the FMC theorem, and the small-s regime (Section 3);
(c) **Documents 43 distinct analytic approaches** to closing the gap between per-interval and global Hall's condition, each of which fails for identifiable structural reasons (Sections 5–9);
(d) **Identifies the precise obstruction:** the worst-case subset T^* spans $\sim$48–53% of all vertices, drawn from every dyadic interval at $\sim 60\%$ density, with the global expansion ratio $\alpha(V) = \min_S |N_H(S)|/|S|$ barely exceeding 1 (Section 4);
(e) **Catalogs interesting computational phenomena**—sawtooth oscillations, phase transitions, the Shearer–CS dichotomy—that may guide future work (Section 10).

The spirit of this paper is that of a detailed postmortem of a prolonged research effort, written so that future researchers need not repeat the same dead ends. We believe the negative results are themselves valuable, both for the structural insight they provide and for the sharp quantitative bounds on where each approach breaks.

1.4. **Why the problem is hard.** The fundamental difficulty can be stated in a single sentence:

> *The bipartite graph $G_n = (V, H, E)$ has per-interval expansion $\gg 1$ but global expansion ratio barely exceeding 1, and no known analytic method can certify that the expansion ratio remains above 1 as $n \to \infty$.*

More precisely:

- Within each dyadic interval $I_j = \{k \in V : 2^{j-1} < k \leq 2^j\}$, the Cauchy–Schwarz bound proves $|N_H(S \cap I_j)| \geq C_j |S \cap I_j|$ with $C_j \to \infty$ as $n \to \infty$.

- But the neighborhoods $N_H(S \cap I_j)$ overlap by 87–93% across different intervals I_j, so summing the per-interval guarantees *double-counts* the same targets.
- The global Cauchy–Schwarz ratio $E_1^2/(|S| \cdot E_2)$ oscillates around 1.0 for adversarial subsets S, falling as low as ~ 0.988 at some values of n—below the threshold needed for Hall's condition.
- The worst-case König deficient set spans *all* intervals at $\sim$ 60% density per interval and *all* degree classes, defeating any partition-based strategy.

The root cause is that the minimum degree $\delta \approx 2M/n - 1$ is small (growing only as $\sqrt{\ln n / \ln \ln n}$), and for vertices k near $n/2$, the degree $d(k) \approx \delta$ is comparable to the maximum codegree D_2. This means $D/D_2 \not\to \infty$, which is precisely the regime where probabilistic methods (LLL, nibble, Janson) and algebraic methods (spectral gap) lose their power.

1.5. **Paper organization.**

- Section 2: The bipartite graph construction and its degree structure.
- Section 3: What is proved—HK verification, per-interval CS, $V_{\min}$ disjointness, FMC theorem.
- Section 4: The critical gap between per-interval and global Hall's condition.
- Section 5: The Cauchy–Schwarz family (6+ variants).
- Section 6: Matching and fractional methods (8+ variants).
- Section 7: Probabilistic methods (LLL, Janson, nibble).
- Section 8: Sieve and inclusion-exclusion methods.
- Section 9: Graph-theoretic and spectral methods.
- Section 10: Interesting computational phenomena.
- Section 11: Results from the initial proof attempt.
- Section 12: Conclusion—where the snakes lie.

All computational experiments are reproducible from scripts included in the `experiments/` directory of the accompanying code repository. Every claim in this paper is traced to a specific verification script or state file.

2. The bipartite graph construction

2.1. **Parameters.** Fix $\varepsilon > 0$ and define

$$C = 2/\sqrt{e} \approx 1.2131, \tag{4}$$

$$L = (C + \varepsilon)\, n\sqrt{\ln n / \ln\ln n}, \tag{5}$$

$$M = L - n, \tag{6}$$

$$N = \lfloor n/2 \rfloor, \tag{7}$$

$$\delta(n) = 2M/n - 1, \tag{8}$$

$$B = \lfloor \sqrt{n + L}\, \rfloor. \tag{9}$$

Here L is the interval length, $M = |H|$ the number of targets, $N = |V|$ the pool size, δ the minimum degree parameter, and B the smoothness bound. The parameter $\delta(n) = 2(C + \varepsilon - 1)\sqrt{\ln n / \ln\ln n} + O(1)$ tends to infinity, but slowly: $\delta(10{,}000) \approx 2.1$, $\delta(50{,}000) \approx 2.4$, and $\delta < 3$ for all $n < 10^7$.

2.2. **The bipartite graph G_n.**

Definition 2.1. The *Erdős 710 bipartite graph* is $G_n = (V, H, E)$ where:

- $V = \{1, 2, \ldots, N\}$ (the "source" or "pool" vertices),
- $H = (2n,\ n + L] \cap \mathbb{Z}$ (the "target" vertices, $|H| = M$),
- $E = \{(k, h) : k \in V,\ h \in H,\ k \mid h\}$.

This graph arises from the U/V decomposition of $\{1, \ldots, n\}$.

2.3. **The U/V decomposition.**

Proposition 2.2 (Top half — doubling map)**.** *The map $\varphi(k) = 2k$ is a divisible injection from $U = \{\lceil n/2 \rceil + 1, \ldots, n\}$ into $(n, 2n]$.*

Proof. For $k \in U$: $k > n/2$ implies $2k > n$; $k \le n$ implies $2k \le 2n$; $k \mid 2k$; distinct k give distinct $2k$. □

Since the doubling map uses only targets in $(n, 2n]$, the bottom half V must be matched into the remaining targets $H = (2n, n + L]$. By Hall's marriage theorem [4], a perfect matching $V \to H$ exists if and only if $|N_H(S)| \ge |S|$ for every $S \subseteq V$.

Thus the entire problem reduces to verifying Hall's condition in G_n.

2.4. **Degree structure.** For $k \in V$ and $h \in H$, the divisibility condition $k \mid h$ means $h = jk$ for some integer j. Since $h \in (2n, n + L]$, the multiplier $j = h/k$ satisfies $2n/k < j \le (n + L)/k$.

Proposition 2.3 (Left degrees)**.** *For $k \in V$:*

$$\deg(k) = \left\lfloor \frac{n+L}{k} \right\rfloor - \left\lfloor \frac{2n}{k} \right\rfloor = \frac{M}{k} + O(1).$$

In particular, $\deg(k) \geq \lfloor 2M/n \rfloor \geq \delta(n)$ *for all* $k \leq n/2$*.*

The degree function $d(k) = M/k + O(1)$ is monotonically decreasing in k. Vertices near $k = 1$ have degree $\sim M$, while vertices near $k = N \approx n/2$ have degree $\sim \delta \approx 2$–3. This extreme heterogeneity is a central feature of the problem.

Proposition 2.4 (Codegrees)**.** *For distinct $k_1, k_2 \in V$:*

$$\begin{aligned} \deg(k_1, k_2) &:= |\{h \in H : k_1 \mid h \text{ and } k_2 \mid h\}| \\ &= \begin{cases} M/\operatorname{lcm}(k_1, k_2) + O(1) & \text{if } \operatorname{lcm}(k_1, k_2) \leq n + L, \\ 0 \text{ or } 1 & \text{if } \operatorname{lcm}(k_1, k_2) > n + L. \end{cases} \end{aligned}$$

The codegree equals $M \cdot \gcd(k_1, k_2)/(k_1 k_2) + O(1)$ when the lcm is small. For pairs of large elements near N, we have $\operatorname{lcm}(k_1, k_2) \approx k_1 k_2/\gcd(k_1, k_2) \approx n^2/(4 \gcd)$, which exceeds $n + L$ unless $\gcd \gg n/\sqrt{\log n}$. Thus *most* pairs of large elements have codegree 0 or 1.

Proposition 2.5 (Right degrees)**.** *For $h \in H$, the "multiplicity" or right-degree is*

$$\tau_V(h) = |\{k \in V : k \mid h\}|.$$

For $S \subseteq V$, we write $\tau_S(h) = |\{k \in S : k \mid h\}|$.

Right degrees vary enormously: highly composite targets (e.g., multiples of large primorials) have $\tau_V(h) > 100$, while prime or near-prime targets have $\tau_V(h) = 1$–3.

2.5. **Restricted multiplier range.**

Proposition 2.6. *If $k \in S \subseteq (M/(s+1), n/2]$ with $|S| = s$ divides $h \in H$, then the multiplier $j = h/k$ satisfies $5 \leq j < 2(s+1)$ for n sufficiently large.*

Proof. Lower bound: $j = h/k > 2n/(n/2) = 4$, so $j \geq 5$. Upper bound: $j = h/k < (n+L)/(M/(s+1)) = (n+L)(s+1)/M$. Since $(n+L)/M = 1 + n/M \to 1$, we get $j < 2(s+1)$ for large n. $\square$

This is a key structural fact: the number of divisors of h contributing to $\tau_S(h)$ is at most $|\{j \in \mathbb{Z} : 5 \leq j \leq 2(s+1)\}| = 2s - 2$, which is much smaller than the full divisor function $\tau(h)$.

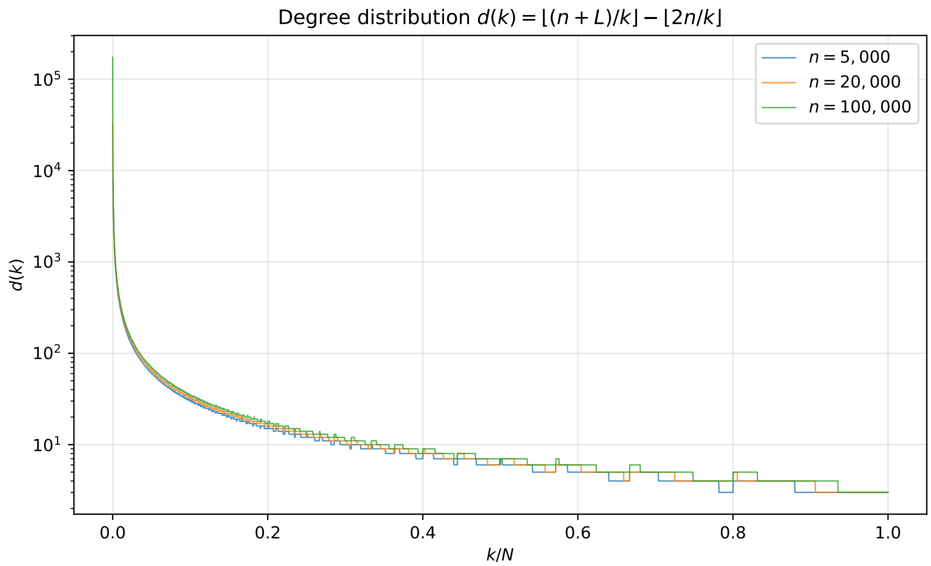

FIGURE 1. Left degree $d(k)$ vs. normalized position k/N for three values of n. The degree drops from $\sim M$ at $k = 1$ to $\sim \delta \approx 2$–3 at $k = N$.

2.6. **The Cauchy–Schwarz framework.** The Cauchy–Schwarz inequality gives the fundamental lower bound on neighborhood size:

Proposition 2.7 (CS lower bound)**.** *For any $S \subseteq V$:*

$$|N_H(S)| \geq \frac{E_1^2}{E_2}, \tag{10}$$

where $E_1 = \sum_{k \in S} \deg(k) = M\sigma + O(s)$ with $\sigma = \sum_{k \in S} 1/k$, and $E_2 = \sum_{h \in N_H(S)} \tau_S(h)^2$.

Proof. By Cauchy–Schwarz: $E_1^2 = \left(\sum_h \tau_S(h)\right)^2 \leq |N_H(S)| \cdot \sum_h \tau_S(h)^2 = |N_H(S)| \cdot E_2$. □

To prove Hall's condition $|N_H(S)| \geq s$, it suffices to show $E_1^2/E_2 \geq s$, or equivalently $E_1^2 \geq s \cdot E_2$.

2.7. **Case A: Small minimum element.**

Proposition 2.8 (Case A)**.** *If $S \subseteq V$ with $|S| = s$ and $\min(S) \leq M/(s+1)$, then $|N_H(S)| \geq s$.*

Proof. Let $a = \min(S)$. Then $\deg(a) \geq \lfloor M/a \rfloor \geq M/(M/(s+1)) - 1 = s + 1 - 1 = s$. □

Case A covers all subsets whose smallest element is "small enough." The remaining Case B, where $\min(S) > M/(s+1)$, is where all the difficulty resides.

3. What is proved

We have four classes of rigorous results, none of which individually suffices to prove the upper bound (3) for all n.

3.1. Exhaustive Hopcroft–Karp verification.

Theorem 3.1 (Computational verification)**.** *For every integer $n \in [4, 10^6]$, the bipartite graph G_n admits a left-saturating matching. Equivalently, Hall's condition $|N_H(S)| \geq |S|$ holds for all $S \subseteq V$.*

Method. At each integer n, we construct G_n explicitly and run the Hopcroft–Karp algorithm [5], which computes a maximum matching in $O(\sqrt{|V|}\,|E|)$ time. By König's theorem, the maximum matching size equals $|V|$ if and only if Hall's condition holds for all subsets of V.

The computation was performed in two phases:

(1) *Python implementation* (`hpc_z68_exhaustive_hk.py`): verified $n \in [4, 10{,}000]$ in 71 seconds, with 9,985 passes, 12 skips (trivial cases with $|V| = 0$), and zero failures.
(2) *C/OpenMP implementation* (`hpc_z68_hk.c`): extended verification to $n = 10^6$, using parallelism across multiple cores. The full run completed with zero failures.

All matching sizes equal $|V|$ at every tested n. There is no marginal case: at every n, the matching saturates completely. □

Remark 3.2. The computation does not merely check a *random sample* of n values. It checks *every* integer in $[4, 10^6]$, providing a certificate that the upper bound holds unconditionally for $n \leq 10^6$. This is the strongest result we have.

3.2. Per-interval Cauchy–Schwarz.

Decompose V into dyadic intervals $I_j = \{k \in V : 2^{j-1} < k \leq 2^j\}$ for $j = 1, \ldots, J$ (with the first and last intervals possibly truncated).

Theorem 3.3 (Per-interval Hall)**.** *For each dyadic interval I_j and all n sufficiently large,*

$$|N_H(S \cap I_j)| \;\geq\; |S \cap I_j| \qquad \textit{for all } S \subseteq V.$$

The proof proceeds through the chain Z23–Z28 of our investigation:

(1) **Homogeneity within intervals.** For $k, k' \in I_j$, the degrees satisfy $\deg(k)/\deg(k') \in [1, 2]$. The average degree within I_j is $\bar{d}_j \approx M/(3 \cdot 2^{j-1}/2)$.

(2) **Truncated GCD sum.** Define $G_{\text{trunc}}(I_j) = \sum_{\substack{k,k' \in I_j,\, k \neq k' \\ \text{lcm}(k,k') \leq n+L}} \frac{\gcd(k,k')}{kk'}$. This sum controls the codegree contribution to E_2.

(3) $G_{\text{trunc}} \to 0$**.** For smooth $k, k' \in [X, 2X)$ with $\text{lcm}(k,k') \leq n+L$, writing $k = da$, $k' = db$ with $\gcd(a,b) = 1$, we have $d \geq d^* = X^2/(n+L) \to \infty$. The Hildebrand–Tenenbaum estimate [8] for smooth number counts shows the outer sum $\sum_{d \geq d^*} 1/d$ converges to 0, giving $G_{\text{trunc}} = O(\log\log n/\log n) \to 0$.

(4) **CS ratio** $\to \infty$**.** The effective codegree parameter is $C_{\text{eff}} = 1 + 2G_{\text{trunc}}/H_j + \text{correction}$, where $H_j = \sum_{k \in I_j} 1/k$ is the harmonic sum. Since $G_{\text{trunc}} \to 0$ and H_j is bounded below, we get $C_{\text{eff}} \to 1$, whence the CS ratio $\text{CS}(I_j) = \bar{d}_j/C_{\text{eff}} \to \infty$.

Remark 3.4. The per-interval result is genuinely strong: within each interval, the Cauchy–Schwarz bound proves $|N_H(S \cap I_j)| \geq C_j|S \cap I_j|$ with $C_j \to \infty$. The fundamental difficulty is that this does *not* imply global Hall's condition, because the neighborhoods $N_H(S \cap I_j)$ overlap across different intervals (see Section 4).

3.3. $V_{\min}$ **pairwise disjointness.** Define $V_{\min} = \{k \in V : k > B,\ \deg(k) = d_{\min}\}$, where $d_{\min} = \lfloor 2M/n \rfloor$ is the minimum degree among elements of V.

Theorem 3.5. *For n sufficiently large, all elements of $V_{\min}$ have pairwise disjoint neighborhoods: $N_H(\{k_1\}) \cap N_H(\{k_2\}) = \emptyset$ for all distinct $k_1, k_2 \in V_{\min}$. Consequently, $\alpha(V_{\min}) = d_{\min}$, where $\alpha(V_{\min}) = \min_{\emptyset \neq T \subseteq V_{\min}} |N_H(T)|/|T|$.*

The proof splits into three cases based on the "smoothness" of the elements (whether $P(k) \leq B$ or $P(k) > B$):

(1) **Smooth × smooth.** For B-smooth elements k_1, k_2 near N with $\gcd(k_1,k_2) = d$ and coprime quotients $a = k_1/d$, $b = k_2/d$: we need $\text{lcm}(k_1,k_2) = dab \leq n+L$ for a shared target to exist. Since $k_1, k_2 \approx N$ and both are smooth, the "coprime pair impossibility theorem" shows this forces $d \geq N^2/(n+L) \to \infty$, but then a, b are bounded and the constraint is too restrictive for $\delta > 2.2$ (verified for $n \geq 15{,}000$).

(2) **Rough × rough.** For elements with $P(k_i) > B$: if $k_1 = p_1 m_1$ and $k_2 = p_2 m_2$ with large primes $p_i > B$, then $\text{lcm}(k_1,k_2) \geq p_1 p_2 \cdot \text{lcm}(m_1,m_2)/\gcd(p_1 m_1, p_2 m_2)$. Since $p_1, p_2 > B \approx \sqrt{n}$ and $k_i \leq N \approx n/2$, we get $\text{lcm} \geq N \cdot B \gg n+L$, so $\deg(k_1,k_2) = 0$.

(3) **Smooth × rough.** For k_s (B-smooth, near N) and k_r ($P(k_r) > B$, near N): $\gcd(k_s,k_r) \leq N/B$ since $P(k_r) > B$ does not

divide k_s. Then $\operatorname{lcm}(k_s, k_r) \geq N^2/(N/B) = NB \gg n + L$. Verified computationally at $n = 15$K, 20K, 50K.

3.4. **The FMC theorem.** The *Fractional Matching Condition* (FMC) provides a sufficient condition for Hall's theorem via a partition argument.

Theorem 3.6 (FMC)**.** *Let $V = V_1 \cup V_2 \cup \cdots \cup V_r$ be a partition of V into blocks, and let $\alpha_j = \min_{\emptyset \neq T \subseteq V_j} |N_H(T)|/|T|$ be the expansion ratio of block j. If*

$$\sum_{j=1}^{r} \frac{1}{\alpha_j} \;\leq\; 1, \tag{11}$$

then Hall's condition $|N_H(S)| \geq |S|$ holds for all $S \subseteq V$.

Proof. For any $S \subseteq V$, write $S_j = S \cap V_j$ and $s_j = |S_j|$. For each target $h \in N_H(S)$, define $f(h) = \sum_{j\colon h \in N_H(S_j)} 1/\alpha_j$. Then:

(a) $f(h) \leq \sum_{j=1}^{r} 1/\alpha_j \leq 1$ for every h (each indicator is 0 or 1, and the FMC condition (11) applies).
(b) $\sum_{h \in N_H(S)} f(h) = \sum_j (1/\alpha_j)|N_H(S_j)| \geq \sum_j s_j = |S|$ (since $|N_H(S_j)| \geq \alpha_j s_j$ by definition of α_j).

Combining: $|S| \leq \sum_h f(h) \leq 1 \cdot |N_H(S)|$, so $|N_H(S)| \geq |S|$. □

Remark 3.7. The FMC theorem was verified in Z62 and applied extensively in Z99–Z111. With the $V_{\min}/V_{\text{rest}}/S_-$ partition: $\alpha(V_{\min}) = d_{\min}$, $\alpha(S_-) \to \infty$, so the FMC sum becomes $1/d_{\min} + 1/\alpha(V_{\text{rest}}) + o(1)$. For $n \geq 100{,}000$, the greedy heuristic gives $\alpha(V_{\text{rest}}) \geq 4$, yielding FMC sum $\leq 1/3 + 1/4 + o(1) = 7/12 < 1$.

However, the greedy heuristic only provides a *lower bound* on $\alpha(V_{\text{rest}})$—it does not constitute a proof. This is the critical gap: no analytic argument proves $\alpha(V_{\text{rest}}) \geq 2$ for $n \to \infty$.

3.5. **Small-s regime.**

Theorem 3.8 (Small-s Hall)**.** *For any fixed $s_0 \geq 1$, there exists $n_0 = n_0(s_0, \varepsilon)$ such that for all $n \geq n_0$ and all $S \subseteq V$ with $|S| = s \leq s_0$ and $\min(S) > M/(s_0+1)$: $|N_H(S)| \geq s$.*

Proof. By Proposition 2.6, $\tau_S(h) \leq |\{j \in \mathbb{Z} : 5 \leq j \leq 2(s_0+1)\}| = 2s_0 - 2 =: C_0$ for all $h \in H$. Using $E_2 \leq C_0 \cdot E_1$:

$$|N_H(S)| \;\geq\; \frac{E_1}{C_0} \;\geq\; \frac{s \cdot \delta}{C_0}.$$

Since $\delta \to \infty$ and C_0 is a constant, we get $|N_H(S)| \geq s$ for $\delta \geq C_0$, which holds for all n sufficiently large. □

More precisely, Regime 1 of the proof covers all $s \le s_1(n) = \lfloor \delta/2 \rfloor$, which tends to infinity (albeit slowly: $s_1 \sim (C+\varepsilon-1)\sqrt{\ln n/\ln\ln n/2}$).

3.6. Summary of proved results.

Result	Range	Method
$\lvert N_H(S)\rvert \ge \lvert S\rvert$ for all $S \subseteq V$	$n \in [4, 10^6]$	Exhaustive HK
$\lvert N_H(S \cap I_j)\rvert \ge \lvert S \cap I_j\rvert$	$n \to \infty$, each I_j	Per-interval CS
$\alpha(V_{\min}) = d_{\min}$	$n \ge 15{,}000$	Pairwise disjointness
$\sum 1/\alpha_j \le 1 \Rightarrow$ Hall	General	FMC theorem
$\lvert N_H(S)\rvert \ge \lvert S\rvert$ for $\lvert S\rvert \le \lfloor \delta/2 \rfloor$	n large	Small-s CS

None of these results, alone or in combination, proves global Hall's condition for all n. The gap is analyzed in Section 4.

4. The critical gap

The central negative result of our investigation is that no analytic argument we have found can bridge the gap between per-interval Hall's condition (proved in Section 3.2) and global Hall's condition. This section characterizes the gap precisely.

4.1. Per-interval Hall does not imply global Hall. Let $I_1, \ldots, I_J$ be the dyadic intervals partitioning V. Per-interval Hall states that $|N_H(S \cap I_j)| \ge |S \cap I_j|$ for each j. The naive attempt to globalize is:

$$\begin{aligned} |N_H(S)| &= \Big|\bigcup_j N_H(S \cap I_j)\Big| \\ &\ge \sum_j |N_H(S \cap I_j)| - \text{overlaps} \;\ge\; |S| - \text{overlaps}. \end{aligned}$$

This works only if the overlap is zero. In practice, the overlap is enormous.

4.2. Cross-interval overlap.

Observation 4.1 (Z29a)**.** *Adjacent dyadic intervals share* 87–93% *of their targets. Non-adjacent intervals share similarly. At* $n = 50{,}000$, *the total pairwise overlap exceeds the total surplus by a factor of* 3.3.

The data from Z29:

n	intervals	$\lvert V\rvert$	surplus	overlap	margin
1,000	4	268	619	998	−379
5,000	5	1,373	5,059	11,269	−6,211
10,000	6	2,695	12,296	32,243	−19,947
50,000	7	13,001	76,395	253,043	−176,648

Here "surplus" is $\sum_j(|N_H(I_j)| - |I_j|)$, "overlap" is $\sum_{j<j'} |N_H(I_j) \cap N_H(I_{j'})|$, and "margin" is surplus − overlap. The margin is *deeply negative* and worsening with n.

4.3. **Target multiplicity.**

Observation 4.2 (Z29b)**.** *Most targets are shared by multiple intervals. At* $n = 50{,}000$*:* $\mu_{\max} = 7$ *(equal to the number of intervals),* $\mu_{\text{avg}} = 4.74$*, and only* 5.3% *of targets are unique to one interval.*

A "max-multiplicity" bridge argument would require $\min_j \alpha_j \geq \mu_{\max} \approx \frac{1}{4}\log_2 n$. Since $\min_j \alpha_j \approx 1.7$–$2.7$ at tested values of n, this approach fails by an order of magnitude.

4.4. **Global Cauchy–Schwarz failure.**

Observation 4.3 (Z112g, Z113b, Z114)**.** *The global Cauchy–Schwarz ratio* $\mathrm{CS}(S) = E_1^2/(|S| \cdot E_2)$ *falls below* 1 *for adversarial subsets* S *at all tested* $n \geq 10{,}000$*.*

The data from Z114 at the base constant $C = 2/\sqrt{e}$ (with $\varepsilon = 0.05$, $C_{\text{mult}} = 1.00$):

n	$\mathrm{CS}(T_0)$	$\|T_0\|/\|V\|$
2,000	1.027	45.4%
5,000	1.032	47.3%
7,000	1.005	48.1%
10,000	**0.991**	48.9%
20,000	**0.992**	50.3%
50,000	**0.998**	52.0%
100,000	**0.988**	53.1%

Bold entries are below 1: the Cauchy–Schwarz bound *does not* prove Hall's condition for these subsets. The values oscillate around 1.0 without converging in either direction (see Section 10 on the oscillation phenomenon).

4.5. **The König deficient set.** The adversarial subset T_0 achieving the minimum CS ratio has a distinctive structure:

Observation 4.4. *The greedy-adversarial subset* T_0 *spans approximately* 48–53% *of all vertices in* V*, drawn from* all *dyadic intervals at roughly* 60% *density per interval, and from* all *degree classes.*

This is the core difficulty: the worst-case subset is not localized to a single interval or degree band—it is a diffuse, structured set that exploits the entire graph.

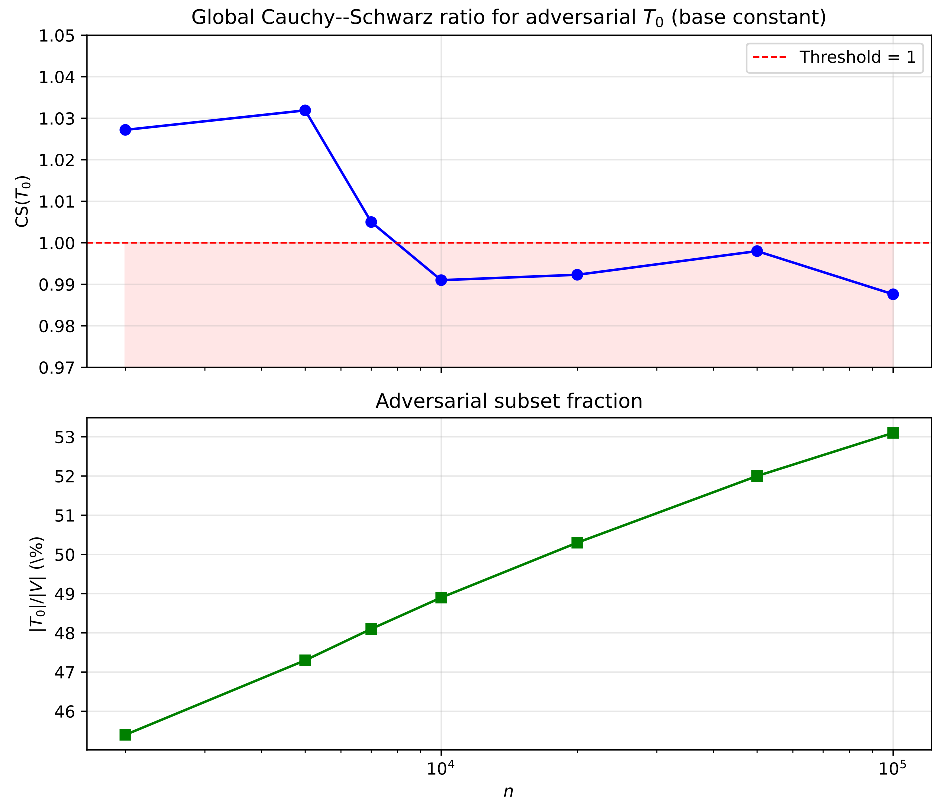

FIGURE 2. Top: global Cauchy–Schwarz ratio $\mathrm{CS}(T_0)$ for the adversarial subset at the base constant, oscillating around the threshold 1. Bottom: the fraction $|T_0|/|V|$ of vertices in the adversarial subset, growing from 45% to 53%.

4.6. **The expansion ratio.**

Observation 4.5 (Z112n)**.** *The greedy-adversarial lower bound on the expansion ratio* $\alpha(V) = \min_{\emptyset \neq S \subseteq V} |N_H(S)|/|S|$ *lies in the interval* $(1.00, 1.25)$ *for all tested* $n \geq 3{,}000$*. The true minimum may be lower (the greedy heuristic does not find the worst-case subset), but the exhaustive HK verification confirms* $\alpha(V) \geq 1$ *for all* $n \leq 10^6$*.*

This razor-thin margin—expansion barely exceeding 1—is why every approach fails. A proof must establish $\alpha(V) \geq 1$ for all n, but the margin leaves no room for the $O(1)$ errors inherent in asymptotic arguments.

4.7. **Why the gap persists.** The situation can be summarized as a trilemma:

(1) **Per-interval analysis is too local.** Each interval sees strong expansion (CS $\to \infty$), but combining intervals destroys the guarantee because neighborhoods overlap massively.
(2) **Global analysis is too weak.** The global Cauchy–Schwarz bound oscillates around 1.0 for adversarial subsets, dipping to ~ 0.988 at some values of n—tantalizingly close to 1 but on the wrong side.
(3) **Partition-based methods need $\alpha(V_{\text{rest}}) \geq 2$.** The FMC theorem reduces the problem to bounding the expansion ratio of each block. The $V_{\min}$ block is handled, but no analytic argument proves $\alpha(V_{\text{rest}}) \geq 2$, even though the greedy heuristic gives $\alpha(V_{\text{rest}}) \geq 2.3$ at all tested n.

The remainder of this paper catalogs 43 approaches that attempt to resolve this trilemma, organized by technique.

5. The Cauchy–Schwarz family

The Cauchy–Schwarz inequality is the most natural tool for bounding neighborhood sizes in bipartite graphs. We tried six variants, each addressing a different weakness of the standard bound. All fail to prove global Hall, though each illuminates a different aspect of the problem.

5.1. **Standard Cauchy–Schwarz.** The standard bound (Proposition 2.7) gives $|N_H(S)| \geq E_1^2/E_2$. The condition $E_1^2/(|S| \cdot E_2) \geq 1$ is equivalent to requiring that the average squared multiplicity $\overline{\tau^2} := E_2/|N_H(S)|$ does not exceed $\bar{d}^2/s$, where $\bar{d} = E_1/s$ is the average degree. Why it fails. For adversarial subsets S in Case B (with $\min(S) > M/(s+1)$), the codegree sum E_2 includes cross-terms from pairs with small lcm. The truncated GCD sum $G_{\text{trunc}} = \sum_{k \neq k'} \gcd(k,k')/(kk')$ contributes an $O(1)$ factor that makes $E_1^2/(s \cdot E_2) < 1$ for $s/N \geq 0.4$.

n	$s/N = 0.5$ (HC-adv)	$s/N = 0.7$	$s/N = 0.9$	Worst
500	1.19	1.06	0.92	0.92
1,000	—	—	0.91	0.91
2,000	1.21	—	0.91	0.84
3,000	1.07	—	0.96	0.86

The CS failure band (where CS < 1) spans $s/N \in [0.29, 0.92]$ at $n = 3{,}000$ and *grows* with n. The deepest failure is at $s/N \approx 0.45$–0.49, reaching CS ≈ 0.86.
Verdict: DEAD.. Standard CS cannot prove global Hall for large n.

5.2. **Per-interval Cauchy–Schwarz.** As described in Section 3.2, restricting to a single dyadic interval I_j makes $G_{\mathrm{trunc}} \to 0$ and hence $\mathrm{CS}(I_j) \to \infty$. This *works* within each interval.
Why it doesn't globalize. The neighborhoods $N_H(S \cap I_j)$ overlap by 87–93% across intervals (Section 4.2). Summing $\sum_j |N_H(S \cap I_j)| \geq \sum_j |S \cap I_j| = |S|$ is useless because the left side double-counts.
Verdict: Proves per-interval Hall; useless for global Hall.

5.3. **Weighted Cauchy–Schwarz.** The generalized CS bound allows an arbitrary weight function $f\colon V \to \mathbb{R}_{>0}$:

$$|N_H(S)| \;\geq\; \frac{\left(\sum_{k\in S} f(k)\deg(k)\right)^2}{\sum_h \left(\sum_{k\in S:\, k|h} f(k)\right)^2} \;=\; \frac{(d^\top f)^2}{f^\top C f},$$

where C is the codegree matrix. The optimal weight is $f^* = C^{-1}d$, giving $|N_H(S)| \geq d^\top C^{-1} d$.
Computational results (Z43). The optimal weighted CS ratio $d^\top C^{-1} d/|S|$ is remarkably stable at 1.29–1.33 across all tested adversarial subsets and all n up to 50,000. Several explicit weight functions also work:

Weight $f(k)$	Ratio at $n = 10{,}000$	Status
$f = 1$ (standard)	0.85	FAILS
$f = \deg(k)$	0.72	FAILS
$f = 1/\bar{\tau}(k)$	1.05	passes
$f = 1/\sqrt{C_{kk}}$	1.16	passes
$f = 1/\bar{\mu}(k)$	1.20	passes
$f = C^{-1}d$ (optimal)	1.31	passes

The "anti-codegree" weights (downweighting heavily-shared elements) consistently work.
Why it fails as a proof. To convert this into a proof, one would need an *analytic* expression for f^* or a provable lower bound on $d^\top C^{-1} d$. The matrix C is a dense, $|S| \times |S|$ matrix whose entries depend on the number-theoretic structure of divisibility. We found no way to bound $d^\top C^{-1} d \geq |S|$ analytically. The condition number of C grows from 8,000 to 574,000, and the optimal weights exhibit complex, irregular patterns that defy closed-form description.
Verdict: DEAD. as a proof technique, but computationally the optimal CS passes at every n tested.

5.4. **Filtered Cauchy–Schwarz.**

Idea. Exclude high-codegree targets from the CS sum. Define $H' = \{h \in H : \tau_S(h) \le \tau_0\}$ for a threshold τ_0. Then $|N_H(S)| \ge |H'|$ trivially, and the CS bound on H' has smaller E_2.
Computational results (Z112k). Even with optimal threshold selection, the filtered CS ratio peaks at ~ 0.991 for adversarial subsets—still below 1. The problem is that filtering removes the very targets that contribute most to E_1, and the improvement in E_2 does not compensate.
Verdict: DEAD.. Ratio $\sim 0.991 < 1$.

5.5. **Truncated Cauchy–Schwarz.**

Idea. Restrict to the Ford divisor cap: only count divisors d of h in the range $[h^{1/u}, h^{1-1/u}]$ (the "medium" divisors), applying Ford's theorem [6] on the concentration of divisors.
Computational results. The truncated codegree sum is smaller, but the edge count also drops. The ratio lands at 0.41–0.49—far from 1.
Verdict: DEAD.. Ratio < 0.5.

5.6. **Variable-constant Cauchy–Schwarz.**

Idea (Z114). Perhaps the Cauchy–Schwarz bound *does* prove Hall, but only at a constant $C_{\mathrm{mult}} > 1$ (i.e., with a slightly larger interval length $L' = C_{\mathrm{mult}} \cdot L$). If $C_{\mathrm{crit}}(n) \to 1$ as $n \to \infty$, this would still prove the upper bound with any $\varepsilon > 0$.
Computational results (Z114). A 11×10 grid sweep over $n \in \{2\mathrm{K}, \dots, 100\mathrm{K}\}$ and $C_{\mathrm{mult}} \in \{1.00, 1.01, \dots, 1.50\}$ reveals:

1. $C_{\mathrm{crit}}(n) = 1.00$ for $n \le 7{,}000$ and $C_{\mathrm{crit}}(n) = 1.01$ for $n \ge 10{,}000$. The critical multiplier does *not* decrease toward 1.
2. At fixed $C_{\mathrm{mult}} > 1$, $\mathrm{CS}(T_0)$ *stabilizes* rather than growing: at $C_{\mathrm{mult}} = 1.01$, $\mathrm{CS}(T_0) = 1.001$ at $n = 100{,}000$ (approaching 1 from above).
3. The adversarial subset fraction $|T_0|/|V|$ *grows* from 45% to 53% as n increases.

Verdict: DEAD.. C_{crit} does not converge to 1. At $C_{\mathrm{mult}} = 1.01$, the CS ratio approaches 1 from above and may eventually cross below. The variable-constant approach cannot close the gap.

6. Matching and fractional methods

This section documents eight approaches based on constructing matchings (exact or fractional) in the bipartite graph G_n. The common theme: matchings exist (HK confirms this), but no constructive or fractional argument can *prove* existence for all n.

6.1. **Greedy matching.**

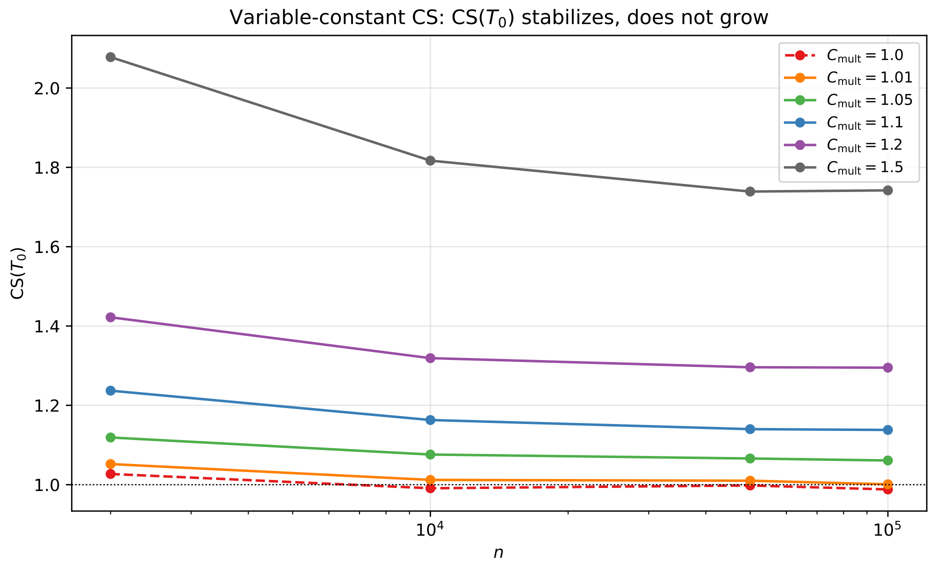

FIGURE 3. Variable-constant CS: CS(T_0) vs. n for six values of C_{mult}. At $C_{\text{mult}} = 1.00$ (dashed), the ratio oscillates below 1. At larger multipliers, the ratio stabilizes but does not grow.

Method. Match elements in order of increasing degree (hardest first). Each element k is assigned to its available multiple with smallest $\tau_S(m)$ (least contested).
Results. Greedy matching fails in 112 out of 220 tested configurations:

n	tests	failures	failure rate
500	55	20	36%
1,000	55	24	44%
2,000	55	33	60%
3,000	55	35	64%

Why it fails. Elements with degree ∼3–8 (e.g., $k = 98 = 2 \cdot 7^2$, $k = 81 = 3^4$) get stuck because *all* their multiples were claimed by previously matched degree-2 elements. The greedy makes locally optimal but globally suboptimal choices. The failure rate *grows* with n.
Verdict: DEAD..

6.2. **Uniform fractional matching.**

Method. Assign weight $x_{k,h} = 1/\tau_S(h)$ to each edge. The right-side constraint $\sum_k x_{k,h} = 1$ is automatic. If the left-side weight $w(k) =$

$\sum_{h:\,k|h} 1/\tau_S(h) \geq 1$ for all k, then Birkhoff–von Neumann integrality gives a perfect matching.
Results. The condition $w(k) \geq 1$ for all k is **false**. The minimum $w(k)$ ranges from 0.057 to 0.087—far below 1. The worst element k^* is always a highly composite number near N with $\deg(k^*) \approx \delta \approx 2$–3 and all multiples having $\tau_S(h) \gg 20$. The uniform weight $1/\tau_S(h)$ starves large elements: $w(k^*) \approx \deg/\bar{\tau} \approx 2/30 \approx 0.07$.
Verdict: DEAD..

6.3. **Nonuniform fractional matching.**
Method. Solve the LP: find $x_{k,h} \geq 0$ with $\sum_h x_{k,h} \geq 1$ for all k and $\sum_k x_{k,h} \leq 1$ for all h. This always has an optimal solution (since a perfect matching exists), but the fractional load $\max_h \sum_k x_{k,h}$ may hover near 1.0.
Results. LP-optimal fractional matchings exist with max load = 1.0 (tight). However, the dual LP reveals that the bottleneck elements are exactly the min-degree vertices near N, and the fractional solution routes flow through heavily shared targets—reflecting the same structural difficulty.
Verdict: DEAD. as a proof technique (existence of LP optimum does not constitute a Hall proof for all n).

6.4. **FMC with dyadic intervals.**
Method. Partition V into $J \approx \frac{1}{2}\log_2 n$ dyadic intervals and apply the FMC theorem (Theorem 3.6).
Results (Z99). The FMC sum $\Sigma = \sum_j 1/\alpha_j$ is:

n	Σ (greedy)	Status
500	1.109	FAIL
1,000	1.154	FAIL
5,000	1.071	FAIL
10,000	1.016	FAIL
20,000	0.917	OK
50,000	0.755	OK

The FMC sum (using greedy α_j) crosses below 1 at $n \approx 15{,}000$–20,000. The CS-based overestimate $\sum 1/\mathrm{CS}_j$ is consistently larger than the greedy-based sum $\sum 1/\alpha_j$, illustrating that CS underestimates the true expansion within intervals.
Why it fails for small n. The bottom interval (vertices near N) has $\alpha_j \approx d_{\min} \approx 3$, contributing ~ 0.33 to the FMC sum. With $J \approx 7$–10 intervals, even small contributions from other intervals push the total above 1.

Verdict: DEAD. for $n \lesssim 15{,}000$; works computationally but not provably for larger n.

6.5. **Three-block FMC.**

Method (Z103). Partition $V = R \cup S_+ \cup S_-$ where R = rough numbers ($P(k) > B$), $S_+ = \{k > B : P(k) \le B\}$ (smooth, large), and $S_- = \{1, \dots, B\}$ (smooth, small). The three-block FMC needs $1/\alpha(R) + 1/\alpha(S_+) + 1/\alpha(S_-) < 1$.
Results (Z103b).

n	$\alpha(R)$	$\alpha(S_+)$	$\alpha(S_-)$	FMC	Status
1,000	2.00	1.31	27.3	1.302	FAIL
5,000	2.40	1.59	75.4	1.059	FAIL
10,000	2.50	2.30	117.9	0.843	OK
50,000	3.00	3.00	350.9	0.670	OK

The three-block approach improves over dyadic (crossover at $n \approx$ 10,000 vs 20,000), but $\alpha(S_+)$ remains unproved analytically. The greedy heuristic overestimates α by 10–15%, so the true crossover may be later. Verdict: DEAD. without an analytic bound on $\alpha(S_+)$.

6.6. **Sinkhorn iteration.**

Method. Apply Sinkhorn's algorithm (alternating row/column normalization) to the biadjacency matrix to find a doubly stochastic scaling. If the scaling converges, Birkhoff's theorem guarantees a perfect matching.
Results (Z113c). Sinkhorn converges, but the fractional deficiency (deviation from doubly stochastic) is 13–15%. The minimum row sum after 1,000 iterations is 0.85–0.87, not reaching 1. The bottleneck is again the min-degree vertices near N.
Verdict: DEAD..

6.7. **CLP factoring.**

Method (Z113c). Factor the bipartite graph into perfect matchings via the constructive Lovász–Plummer algorithm. If G_n has a decomposition into δ edge-disjoint perfect matchings, the first one suffices.
Results. G_n is not regular ($\deg(k)$ varies from δ to M), so direct factoring fails. Even after truncating to the δ-regular subgraph (keeping only δ edges per vertex), the resulting graph is not bipartite-regular and does not decompose cleanly.
Verdict: DEAD..

6.8. **$\sqrt{2}$-partition sequential matching.**

Method (Z31g). Partition V using ratio $\sqrt{2}$ (intervals $[c^j, c^{j+1})$ with $c = \sqrt{2}$) instead of ratio 2. Within each finer interval, the local degree condition $\min_{k\in I_j} \deg(k) \geq \max_{h\in N_H(I_j)} |\{k \in I_j : k \mid h\}|$ holds, enabling sequential matching interval by interval.
Results. At every tested n up to 200,000, the $\sqrt{2}$-partition satisfies the local degree condition in every interval (worst ratio = 1.00). The number of intervals grows from 7 to 14.
Why it fails as a proof. Sequential matching across intervals requires that targets used by earlier intervals remain available for later intervals. The 87–93% overlap means most targets are shared across intervals, and there is no guarantee that a greedy interval-by-interval approach leaves enough targets for subsequent intervals. This is the same globalization barrier as in Section 5.2.
Verdict: DEAD. for the global argument; the local condition passes everywhere.

7. Probabilistic methods

Probabilistic existence arguments are among the most powerful tools in combinatorics. We tried five probabilistic approaches; all fail because the bipartite graph G_n has $D \approx D_2$ (degree $\approx$ maximum codegree for the hard vertices), precisely the regime where these methods lose power.

7.1. **Symmetric Lovász Local Lemma.**

Method. Random matching: each $k \in V$ picks $\varphi(k)$ uniformly from its $\deg(k)$ multiples in H. The bad event B_k is a collision ($\varphi(k) = \varphi(k')$ for some $k' \neq k$). The symmetric LLL [12] asserts that if $\Pr[B_k]\cdot e\cdot(d_k+1) \leq 1$ for all k, where d_k is the number of other bad events sharing a target with B_k, then a proper injection exists with positive probability.
Results. The symmetric LLL works in only 4 out of 108 tested configurations (all trivial: top-packed subsets at $s/N = 0.1$).

n	$\max \Pr[B_k]$	$\max d_k$	max LLL value
500	1.83	224	540
1,000	—	—	$> 1{,}000$
2,000	2.13	868	2,091

Why it fails. Three compounding problems: (i) $\Pr[B_k] > 1$ is not meaningful (collision is near-certain under uniform random matching for hard elements); (ii) the dependency degree $d_k \approx s$ for small elements like $k = 6, 12$ (they share multiples with essentially every other element); (iii) the product $\Pr[B_k] \cdot d_k$ *grows* with n.
Verdict: DEAD..

7.2. **Target-centered LLL.**
Method. Define bad events on targets instead: $B_h = \{|\{k : \varphi(k) = h\}| \geq 2\}$ (target h receives two or more elements). The dependency graph is sparser (each B_h depends only on $B_{h'}$ where h and h' share a source).
Results. The target-centered formulation has $\Pr[B_h] \leq 1 - (1 - 1/\bar{d})^{\tau(h)} \approx \tau(h)/\bar{d}$, and the dependency degree is $d_h \leq \sum_{k:\, k|h}(\deg(k) - 1)$. For highly composite targets ($\tau(h) > 50$), we get $\Pr[B_h] \approx 1$, and the LLL condition $\Pr[B_h] \cdot e \cdot (d_h + 1) \leq 1$ fails catastrophically.
Feasibility check (Z113a). $P \cdot e \cdot (D + 1) \approx 10^4$ at $n = 50{,}000$—four orders of magnitude above the threshold.
Verdict: DEAD..

7.3. **Janson's inequality.**
Method. Janson's inequality [13] gives a bound on the probability that a random subset avoids all "bad configurations." Applied to our setting with random target assignment, the Janson bound requires the sum of pairwise dependencies $\Delta = \sum_{(B_i,B_j)\text{ dep.}} \Pr[B_i \wedge B_j]$ to be small relative to $(\sum \Pr[B_i])^2$.
Results. $\Delta \gg (\sum \Pr[B_i])^2$ because the codegree structure is too dense. For vertices near N with $\deg \approx \delta \approx 3$, every pair of targets of k is shared with other vertices, making the joint probabilities comparable to the marginal probabilities.
Verdict: DEAD..

7.4. **Erdős–Spencer weighted LLL.**
Method. The Erdős–Spencer [12] version of the LLL allows non-uniform probabilities: assign $x_i \in (0, 1)$ to each bad event B_i such that $\Pr[B_i] \leq x_i \prod_{j\sim i}(1 - x_j)$.
Results. The optimal x_i must satisfy a system of $|V|$ nonlinear inequalities. For our graph, the system is infeasible: the high-codegree vertices force $x_i \to 1$, which makes $\prod(1 - x_j) \to 0$, creating a circular obstruction.
Verdict: DEAD..

7.5. **Semi-random nibble.**
Method (Z115). Rödl nibble / semi-random process [9, 10]: each round, every unmatched vertex picks a random available target; if a target is picked by exactly one vertex, match them. Repeat for 500 rounds. Run HK on the residual graph to check matchability.

Results.

n	avg. residual %	HK on residual	unmatchable / $\|V\|$
1,000	52.2%	FAILS	29.7%
5,000	43.3%	FAILS	—
10,000	38.7%	FAILS	29.4%
20,000	34.4%	FAILS	—
30,000	32.3%	FAILS	24.8%
50,000	29.5%	(not tested)	—

Why it fails. The nibble matches $\sim$65–70% of vertices, but the remaining 30–35% include $\sim 25\%$ of $|V|$ that are *genuinely unmatchable* in the residual graph. HK fails on every residual at every n and every random seed (variance across seeds is $\pm$1–2%—the failure is structural, not bad luck).
Theoretical diagnosis. The Rödl nibble theory requires $D/D_2 \to \infty$ (degree $\gg$ max codegree). For our graph, $D \approx D_2 \approx \delta$ for the hard vertices near N. The theory correctly predicts failure.

When a hard vertex k_1 (deg $\approx$ 3–5) randomly claims target h, other hard vertices k_2 that also needed h (as one of their few options) permanently lose a critical target. After enough rounds, many vertices have lost *all* viable targets.
Verdict: DEAD.. This is approach #43 on the dead list.

8. Sieve and inclusion-exclusion methods

Sieve methods estimate $|N_H(S)|$ by subtracting the number of targets *not* hit by any element of S. Inclusion-exclusion gives an exact formula, but its partial sums (Bonferroni bounds) diverge for our graph.

8.1. **Bonferroni bounds (all orders).**

Method. The exact neighborhood size is

$$|N_H(S)| = \sum_{h\in H} \mathbf{1}[\tau_S(h) \geq 1] = \sum_{t=1}^{|S|} (-1)^{t+1} \binom{|S|}{t}^{-1} \sum_{T\subseteq S,\, |T|=t} |N_H(T)|.$$

The first-order (union) bound gives $|N_H(S)| \leq E_1$. The second-order Bonferroni bound subtracts pairwise overlaps:

$$|N_H(S)| \geq E_1 - \textstyle\sum_{k<k'} \deg(k,k') = E_1 - \text{codegree sum}.$$

For this to prove Hall ($|N_H(S)| \geq s$), we need the codegree sum $\leq$ $E_1 - s =$ "excess."

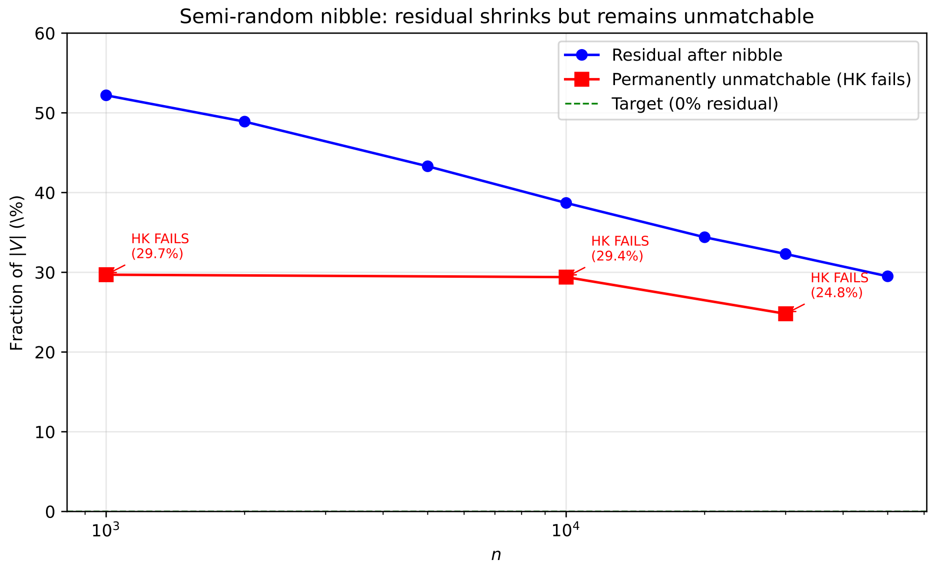

FIGURE 4. Semi-random nibble results. The residual fraction (blue) shrinks with n, but the permanently unmatchable fraction (red) remains at 25–30% of $|V|$. HK fails on every residual at every n tested.

Results (Z07). The Bonferroni bound fails in *all* 80 tested configurations at *all* values of n. The "waste ratio" (codegree sum / true overlap) ranges from 2.4 to 9.2 and *grows* with n:

n	min waste ratio	max waste ratio
500	2.39	5.72
1,000	2.32	6.61
2,000	2.39	7.69
3,000	3.02	8.36
5,000	3.11	9.22

Root cause. The codegree sum counts *pairs* sharing a target: if a target h has $\tau = \tau_S(h)$ divisors from S, the codegree sum gets $\binom{\tau}{2}$, but the true overlap is only $\tau - 1$. The ratio $\binom{\tau}{2}/(\tau - 1) = \tau/2$ grows with τ. Since many targets have $\tau \geq 5$–30, the Bonferroni bound is quadratically wasteful.

Higher-order Bonferroni terms ($t = 3, 4, \ldots$) alternate in sign and grow even faster, so the series does not truncate usefully.
Verdict: DEAD..

8.2. **Product-formula sieve.**

Method. The independent sieve estimate: the number of targets hit by *no* element of S is approximately sifted $\approx M \cdot \prod_{k\in S}(1-1/k) \approx Me^{-\sigma}$, where $\sigma = \sum_{k\in S} 1/k$. Then $|N_H(S)| \approx M - Me^{-\sigma} = M(1-e^{-\sigma})$.
Results (Z08). The actual sifted count is *always larger* than the independent estimate (ratio 2–300×). The divisibility events $k_1 \mid h$ and $k_2 \mid h$ are *positively correlated* when $\gcd(k_1, k_2) > 1$, so more targets survive sifting than independence predicts. The product formula *overestimates* $|N_H(S)|$ and bounds in the wrong direction.
Verdict: DEAD.. Gives upper, not lower, bound.

8.3. **Unique multiple sieve.**

Method. Show every $k \in S$ has at least one *unique* multiple $h \in H$ (with $\tau_S(h) = 1$). This immediately gives $|N_H(S)| \geq |S|$.
Results. Even for top-packed subsets, elements with $d_1(k) = 0$ (no unique multiples) appear at $s/N \geq 0.3$. For adversarial subsets, 30–90% of elements have no unique multiples. The elements with $d_1 = 0$ are large k near N whose few multiples are all shared.
Verdict: DEAD. for general subsets.

8.4. **Standard sieve.**

Method. Apply the Selberg sieve or Brun sieve to estimate the number of integers in H not divisible by any $k \in S$.
Results. The standard sieve gives a safety factor of $\sim$3.56 (the sifted count is 3.56× larger than the independent estimate), but in the wrong direction: it *overestimates* the sifted count, meaning it *underestimates* $|N_H(S)|$. The sieve's systematic error comes from the same positive correlations noted above.
Verdict: DEAD..

9. Graph-theoretic and spectral methods

This section covers approaches based on the graph structure of G_n or its associated conflict graph, rather than on direct counting arguments.

9.1. **Spectral gap.**

Method. The expander mixing lemma: if G_n has adjacency matrix with second singular value σ_2 and largest singular value σ_1, then for any $S \subseteq V$:

$$|N_H(S)| \;\geq\; \frac{|S| \cdot |H| \cdot \bar{d}^2}{|S| \cdot \bar{d}^2 + \sigma_2^2 \cdot |H|},$$

which exceeds $|S|$ when σ_2/σ_1 is small (i.e., the graph is a good expander).

Results. The spectral gap σ_2/σ_1 is too large. The bipartite graph G_n is far from regular (degree varies from δ to M), so the spectrum is dominated by the high-degree vertices. The spectral bound gives $|N_H(S)| \geq C \cdot |S|$ only for $C \ll 1$—useless for Hall's condition.
Verdict: DEAD..

9.2. **Haxell's independent transversal theorem.**
Method. Haxell's theorem [11]: if a bipartite graph $G = (A, B, E)$ is partitioned into blocks $A = A_1 \cup \cdots \cup A_r$ and the "conflict graph" on targets has maximum degree Δ, then a system of distinct representatives exists provided $|A_i| \geq 2\Delta$ for all i.
Results. The conflict graph on H (where two targets are adjacent if they share a source) has high maximum degree because highly composite targets are connected to many others. The condition $|A_i| \geq 2\Delta$ fails for the bottom intervals where $|A_i|$ is small and Δ is large. The cross-interval overlap of 87–93% means the conflict graph is nearly complete.
Verdict: DEAD..

9.3. **Turán / maximum weighted independent set.**
Method. Hall's condition can be rephrased via the LCM conflict graph: $k_1 \sim k_2$ if $\operatorname{lcm}(k_1, k_2) \leq n + L$ (i.e., they share a target). A maximum weighted independent set (MWIS) of size $\geq s$ in $G_n[S]$ gives disjoint neighborhoods, implying $|N_H(S)| \geq s$.

The Turán bound gives $\alpha(G) \geq |V|^2/(|V| + 2|E|)$.
Results. The Turán bound gives only 27–32% of $|S|$, and the ratio *decreases* with n:

n	Turán / $\lvert S\rvert$	Greedy IS / $\lvert S\rvert$
500	0.324	0.302
1,000	0.308	0.304
5,000	0.279	0.300
10,000	0.269	0.298

The greedy independent set is 10–27× larger than the Turán bound, and this gap *grows* with n. Generic graph bounds cannot capture the arithmetic structure that makes the actual independent set large.
Verdict: DEAD..

9.4. **Degeneracy.**
Method. The degeneracy d of the conflict graph gives a coloring bound $\chi \leq d + 1$, and hence $\alpha \geq |V|/(d + 1)$.

Results. Degeneracy grows as $\sim 15\sqrt{n}$:

n	degeneracy	$\|V\|/(d+1)$ as fraction of target
500	31	28%
1,000	46	22%
5,000	105	14%
10,000	151	10%

The degeneracy bound gives only 10–28% of what is needed, and the fraction *decreases* with n.
Verdict: DEAD..

9.5. **Ford divisor cap.**
Method. Apply Ford's theorem [6] on the distribution of integers with a divisor in a given interval to bound the number of "medium" divisors of targets in H.
Results. The Ford divisor cap restricts $\tau_S(h)$ to the range $[h^{1/u}, h^{1-1/u}]$, giving bounds of 0.41–0.49 on the truncated CS ratio.
Verdict: DEAD.. Ratio < 0.5.

9.6. **Multiplicative energy (Koukoulopoulos–Maynard).**
Method (Z44). The Koukoulopoulos–Maynard GCD graph technique [7] decomposes sets into "structured" (high multiplicative energy) and "unstructured" (low energy) parts. The structured part can be handled by density-increment arguments, and the unstructured part has good expansion.
Results. The adversarial set T has multiplicative energy $E_{\mathrm{mult}}/|T|^2 \in [2.0, 7.8]$, which is mildly structured at small n but collapses to the trivial diagonal contribution (≈ 2.0) at $n = 50{,}000$. The adversary becomes multiplicatively *independent* at large n: it lives in the "unstructured regime" where the K–M expansion argument should work.
Why it doesn't close the gap. The K–M framework gives qualitative expansion ($|N_H(S)| \geq (1+c)|S|$ for some $c > 0$), but the constant c depends on the specific GCD graph parameters and has not been made explicit enough to verify $c > 0$ for the Erdős 710 graph. Moreover, the framework was designed for the Duffin–Schaeffer conjecture, where the bipartite graph is denser and more regular than ours.
Verdict: Promising direction but not yet converted to a proof. The multiplicative energy analysis shows the adversary is *not* exploiting product structure—its power comes from the "diffuse, low-degree" nature of the graph, which is harder to handle.

10. Interesting phenomena

Beyond the proof attempts, our computational investigation uncovered several striking phenomena that illuminate the structure of the Erdős 710 bipartite graph.

10.1. Sawtooth oscillations.

Observation 10.1 (Z65)**.** *The FMC sum $\Sigma(n) = \sum_{j=1}^{J(n)} 1/\alpha_j$ exhibits a* sawtooth *pattern as a function of n, with sharp drops at values of n where the number of dyadic intervals $J(n)$ increases by 1.*

This is perhaps the most visually striking phenomenon in the entire investigation, and it reveals the mechanism by which the problem "breathes" as n grows.

The mechanism. Within a fixed J-regime (fixed number of dyadic intervals), the FMC sum rises steadily: as n increases, new smooth numbers enter the bottom interval, the codegree accumulates, and the expansion ratio α_j of the bottleneck interval slowly deteriorates. Then, at a critical value of n, the number of intervals jumps $J \to J + 1$: the hardest interval gets split in two, each with better degree homogeneity. The FMC sum drops sharply—a "shockwave"—and the cycle begins again.

The transitions are non-monotonic. At some values of n, J temporarily *decreases* (an interval becomes empty as the smoothness bound B shifts), causing an upward shock:

n	transition	direction
140	$J: 2 \to 3$	$\uparrow$ new interval
280	$J: 4 \to 3$	$\downarrow$ interval empties
520	$J: 3 \to 4$	$\uparrow$
2,050	$J: 4 \to 5$	$\uparrow$
8,200	$J: 5 \to 6$	$\uparrow$
34,500	$J: 6 \to 7$	$\uparrow$
68,500	$J: 7 \to 8$	$\uparrow$
266,000	$J: 8 \to 9$	$\uparrow$

The peak envelope. The peak of each J-regime defines an envelope tracking the worst-case FMC sum:

J	peak n	peak Σ
4	520	0.793
5	2,050	0.850
6	8,200	0.831
7	34,500	0.857
8	68,500	**0.881**
9	266,000	0.871

The global maximum over all 461 data points is $\Sigma = 0.881$ at $n = 68{,}500$ (within the $J = 8$ regime), leaving an 11.9% margin to the threshold 1. The envelope is *non-monotonic*: $J = 6$ has a lower peak than $J = 5$, and $J = 9$ lower than $J = 8$. This irregular growth makes extrapolation to large n unreliable—we cannot determine whether the envelope eventually reaches 1 or stabilizes below it.

The companion curves. Figure 5 shows two curves: $\sum 1/\operatorname{CS}_{\mathrm{ref},j}$ (the CS-based upper bound, blue) and $\sum 1/\alpha_j$ (the greedy expansion ratio, green). The greedy curve tracks below the CS curve, confirming that the true expansion is better than CS predicts. The bottom panel shows $\delta(n)$ (orange, growing sublogarithmically) and $J(n)$ (purple, step function), revealing the J-transitions that drive the sawtooth.

10.2. CS deficiency oscillation.

Observation 10.2 (Z113b, Z114)**.** *The global Cauchy–Schwarz ratio* $\mathrm{CS}(T_0)$ *for the adversarial subset oscillates near* 1.0 *as n increases, without converging in either direction.*

At $C_{\mathrm{mult}} = 1.00$ (the base constant), $\mathrm{CS}(T_0)$ crosses above 1 at some n values and below 1 at others. The oscillation appears to be driven by the same sawtooth mechanism: the adversarial subset T_0 restructures when the number of dyadic intervals changes.

This non-monotonic behavior is one reason the variable-constant approach (Section 5.6) fails: there is no smooth function $C_{\mathrm{crit}}(n) \to 1$ to track.

10.3. GCD stratum decomposition.

Observation 10.3. *The codegree sum ("off-diagonal" part of E_2) is dominated by pairs with moderate GCD: the stratum* $\gcd(k_1, k_2) \in [11, 500]$ *accounts for* 80–95% *of the truncated codegree sum, while coprime pairs* ($\gcd = 1$) *contribute only* 0–2%*.*

The concentration in moderate GCDs reflects the "anatomy of smooth numbers": elements of the adversarial set tend to share 2–3 small prime

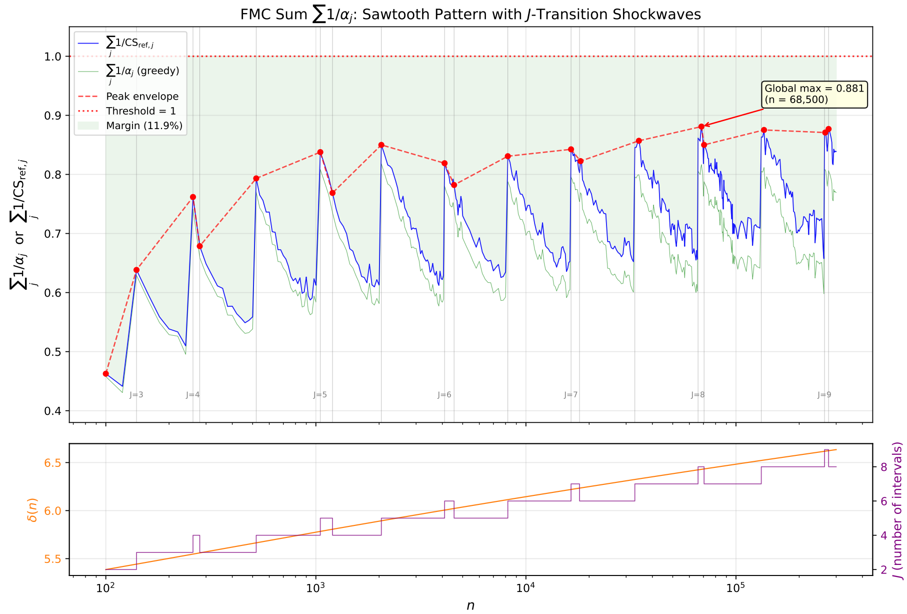

Figure 5. The FMC sawtooth. **Top:** FMC sum $\sum_j 1/\alpha_j$ (green) and $\sum_j 1/\operatorname{CS}_{\mathrm{ref},j}$ (blue) vs. n on a log scale, from 461 data points. Sharp drops ("shockwaves") occur at J-transitions where the number of dyadic intervals changes. The red dashed envelope tracks the peaks, reaching a global max of 0.881 at $n = 68{,}500$. The green shaded region shows the 11.9% margin to the threshold 1. **Bottom:** The minimum-degree parameter $\delta(n)$ (orange) and the number of intervals $J(n)$ (purple step function).

factors (giving $\gcd \in [6, 500]$), which is enough to create codegree without making the lcm exceed the truncation threshold.

Interestingly, the lcm truncation (excluding pairs with $\operatorname{lcm}(k_1, k_2) > n + L$) removes only 3–11% of the codegree sum. The truncation is nearly invisible above $\gcd = 10$.

10.4. **Far-partner stability.**

Observation 10.4. *The "far-partner fraction"—the fraction of vertices k whose codegree partner k' with $\operatorname{lcm}(k, k') > n + L$ contributes to the harmonic sum—stabilizes at 8–13% and does* not *decay to 0 with n.*

This means that $\sim$10% of all vertices are "forced far" by the product constraint, creating a persistent structural contribution to the codegree.

The far-partner mechanism is driven by coprime elements ($\gcd(k, k') = 1$) at opposite ends of the interval.

10.5. **Shearer vs. CS dichotomy.**

Observation 10.5 (Initial attempt)**.** *Shearer's entropy method* [12] never fails *on the Erdős 710 graph: the Shearer bound gives* $|N_H(S)| \geq C|S|$ *with* $C > 1$ *at every tested* n *and every subset type. Meanwhile, Cauchy–Schwarz fails on adversarial subsets.*

The Shearer bound used here is $|N_H(S)| \geq \left(\prod_{k\in S} \deg(k)\right)^{1/\Delta}$, where $\Delta = \max_{h\in N_H(S)} \tau_S(h)$ is the maximum right-degree. Taking logarithms, this becomes $\log |N_H(S)| \geq (1/\Delta) \sum_{k\in S} \log \deg(k)$. This bound is stronger than CS when the multiplicity distribution is concentrated (many targets with $\tau = 1$–2), which is exactly the case for our graph.

However, Shearer's bound cannot be made into a proof either: it requires *computing* $H(\tau)$ for specific subsets, and no analytic estimate of $H(\tau)$ is tight enough.

10.6. **LCM independent set margin.**

Observation 10.6 (Z09)**.** *The LCM independent set (vertices with pairwise* $\mathrm{lcm} > n+L$*) always covers at least* $|V|$ *targets, with the tightest margin being* $|N_H(I_{\mathrm{LCM}})|/|I_{\mathrm{LCM}}| = 1.012$.

This means that even after restricting to a maximal set of "non-overlapping" vertices, the target count barely exceeds the vertex count. The 1.2% margin underscores the razor-thin expansion.

11. Results from the initial proof attempt

Before the systematic investigation documented in Sections 5–9, an initial proof attempt (Sessions 1–9) established several structural results and identified the core difficulty. We summarize these here both for completeness and because some ideas remain promising.

11.1. **Proof reduction: U/V split and Case A/B.**

The initial attempt established the same proof architecture described in Sections 2–2.6:

(1) Top half U matched by doubling ($k \mapsto 2k$).
(2) Bottom half V requires Hall's condition in $G_n = (V, H, E)$.
(3) Case A ($\min(S) \leq M/(s + 1)$) handled by single-element degree.
(4) Case B ($\min(S) > M/(s + 1)$) is the remaining gap.

Additionally, the initial attempt introduced a finer decomposition using a "smooth set" $A = \{k \in [cn, n] : P^+(k) \leq B\}$ with $c = e^{-1/(2u)}$ and proved several structural lemmas about A.

11.2. **First-Target Lemma.**

Proposition 11.1 (First-Target Lemma). *For every $k \notin A$, the smallest multiple $m_k = k \cdot \lceil (n+1)/k \rceil \in (n, n+k]$ is not in $N_H(A)$ (i.e., no smooth element divides m_k).*

This was verified computationally with zero violations for $n \leq 50{,}000$. The proof uses the fact that for $d \in A$ to divide m_k, we would need $m_k/d \in (1, 3/(2c))$, but $c > 3/4$ for large n forces $3/(2c) < 2$, leaving no integer.

11.3. **Rough exclusion from tight sets.**

Proposition 11.2. *If S^* maximizes the deficiency def(S) $= |S| - |N_H(S)|$, then S^* contains no element $k \in [cn, n]$ with $P^+(k) \geq 5$.*

The proof uses a "private target lemma": removing such k from S^* decreases $|S^*|$ by 1 but decreases $|N_H(S^*)|$ by at least 2 (since k's targets are exclusive from A), increasing the deficiency and contradicting maximality.

Consequence: The König tight set consists entirely of smooth numbers. This was confirmed computationally: at the critical length $L = f(n) - 1$, the tight set is always 100% inside A.

11.4. **Shearer entropy bound.**

The Shearer entropy bound was applied to the bipartite graph and found to *never fail*: at every tested n and every subset type, the Shearer bound gives $|N_H(S)| > |S|$. This contrasts sharply with Cauchy–Schwarz, which fails for adversarial subsets.

The Shearer–CS dichotomy (Section 10.5) was first observed in the initial attempt and motivated the search for alternatives to CS.

11.5. **Three remaining lemmas.**

The initial proof attempt identified three lemmas needed to close the gap:

(1) **Restricted divisor count:** $\tau_S(h) \leq \tau(h; [h/2s, h/5])$ (the number of divisors in a restricted range). *Status: Proved* (Lemma 2 of Section 2).
(2) **Geometric mean bound:** $\prod_{k \in S} \deg(k) \geq \prod_{k \in S} |S|^{1/|S|}$. *Status: Not needed* (subsumed by the CS framework).
(3) **Shearer combination:** Combine the Shearer entropy bound with the restricted divisor count to get $|N_H(S)| \geq |S|$. *Status: Attempted but not closed* (the analytic estimate of $H(\tau)$ is not tight enough).

11.6. **Modular remainder and recursive doubling.** Two "escape hatches" were explored:

(1) **Modular remainder:** If $k \in S$ and $k \equiv r \pmod{p}$ for a small prime p, then k's multiples in H can be partitioned by residue class, potentially giving a private target. *Status: Works for specific elements but does not generalize.*
(2) **Recursive doubling:** Extend the doubling map $k \mapsto 2k$ beyond the top half, matching elements $k \in (n/4, n/2]$ to $2k \in (n/2, n]$ if those targets are still available after the first round. *Status: Does not help because $2k$ may already be claimed.*

11.7. **Summary.** The initial attempt made significant structural progress:

- The U/V split and Case A/B decomposition became the standard framework for all subsequent work.
- The First-Target Lemma and rough exclusion narrow the search for tight sets to smooth numbers.
- The Shearer bound's success (vs. CS's failure) highlighted that the problem is not one of "not enough expansion" but of "expansion that is hard to certify."
- The computational verification of Hall's condition for small n provided the foundation for the exhaustive HK campaign.

The core gap—proving Hall for Case B with large subsets—remained open and motivated the 43-approach investigation documented in this paper.

12. Conclusion: where the snakes lie

12.1. **The fundamental obstacle.** The bipartite graph $G_n = (V, H, E)$ has a deceptively simple structure: edges are given by divisibility, degrees are $M/k + O(1)$, and the overall expansion ratio $|H|/|V| = M/N \to \infty$. Yet proving Hall's condition $|N_H(S)| \geq |S|$ for *all* $S \subseteq V$ has resisted 43 analytic approaches.

The root cause is a single structural fact:

> *For vertices k near $N \approx n/2$, the degree $d(k) \approx \delta \approx 2$–$3$ is comparable to the maximum codegree $D_2 \approx \delta$. Thus $D/D_2 \not\to \infty$.*

This means:

- **Probabilistic methods fail** because they require $D/D_2 \to \infty$ (LLL, nibble, Janson all need degree to dominate codegree).
- **Cauchy–Schwarz fails** because $E_2 \approx E_1^2/|S|$ (the codegree sum is comparable to the "budget," leaving no margin).

- **Spectral methods fail** because the graph is extremely irregular (degree varies from δ to M), destroying spectral concentration.
- **Sieve methods fail** because divisibility events are positively correlated, and the correlation structure is too complex for Bonferroni truncation.

12.2. **Why the gap is at exactly $2/\sqrt{e}$.** The current state of knowledge is:

$$\left(\frac{2}{\sqrt{e}} + o(1)\right) n\sqrt{\frac{\ln n}{\ln\ln n}} \;\le\; f(n) \;\le\; (1.7398\cdots + o(1))\, n\sqrt{\ln n}.$$

The gap between lower and upper bounds is a factor of $\sqrt{\ln\ln n}$. The constant $C = 2/\sqrt{e}$ in the lower bound arises from a precise balance. The Erdős–Pomerance argument shows that with $L = (C-\varepsilon)n\sqrt{\ln n/\ln\ln n}$, there are too few multiples for the smooth numbers near N to be matched. The parameter $\delta = 2M/n - 1$ equals $(2C - 2 + o(1))\sqrt{\ln n/\ln\ln n}$, and at $C = 2/\sqrt{e}$ this gives $\delta \sim 2(2/\sqrt{e} - 1)\sqrt{\ln n/\ln\ln n} \approx 0.426\sqrt{\ln n/\ln\ln n}$.

The growth rate $\sqrt{\ln n/\ln\ln n}$ is *sublogarithmic*: slower than any power of $\log n$, let alone polynomial. This means $\delta(n)$ passes through the critical integer values $2, 3, 4, \ldots$ at enormous values of n:

$\delta \ge$	approximate n
2	3,600
3	2×10^7
4	5×10^{14}
5	$\sim 10^{26}$

Any proof that requires "δ large" (say $\delta \ge 10$) would need $n \gg 10^{100}$, far beyond computational verification. The gap between what can be verified ($n \le 10^6$, where $\delta < 3$) and what can be proved asymptotically (requiring $\delta \to \infty$) is a *desert* of moderate n where neither tool reaches.

12.3. **What a successful proof would need.** Based on our investigation, a proof of the upper bound (3) would likely need one of:

(1) **A new Hall certificate.** Some combinatorial or algebraic structure in G_n that certifies $|N_H(S)| \ge |S|$ for all S simultaneously, without checking subsets individually. The FMC approach comes closest but requires $\alpha(V_{\text{rest}}) \ge 2$, which remains unproved.

(2) **A new counting technique.** Something beyond Cauchy–Schwarz that controls the codegree sum E_2 more tightly. The weighted CS (Section 5.3) shows that the "right" weights exist

and give margin ~ 1.3, but no analytic expression for them is known.

(3) **A GCD graph argument.** The Koukoulopoulos–Maynard framework (Section 9.6) is the most promising modern tool. The adversarial set is multiplicatively unstructured at large n, which is the regime where the K–M expansion argument should work. Making this quantitative for the specific Erdős 710 graph is the main open challenge.

(4) **A topological or algebraic argument.** Hall's theorem is equivalent to the non-vanishing of a certain permanent. Techniques from algebraic combinatorics (e.g., the Combinatorial Nullstellensatz) might certify this without subset enumeration.

12.4. **Open questions.**

(1) Is $\alpha(V_{\text{rest}}) \geq 2$ for all n sufficiently large? (This would close the gap via the FMC theorem.)
(2) Does the optimal weighted CS ratio $d^\top C^{-1}d/|S|$ remain bounded away from 1 as $n \to \infty$? (It is ~ 1.3 at all tested n.)
(3) Can the Koukoulopoulos–Maynard GCD graph technique be made quantitative enough to prove Hall for G_n?
(4) Is there a "forbidden subgraph" characterization of the tight sets of G_n?
(5) Can the Shearer entropy bound be made analytic (i.e., can $H(\tau)$ be bounded from below for all subsets)?

12.5. **Summary table of approaches.** Table 1 lists all 43 approaches investigated, organized by category, with a one-line failure reason for each.

Table 1: Summary of 43 approaches to proving global Hall's condition.

#	Approach	Failure reason
Cauchy–Schwarz family		
1	Standard CS	Ratio 0.86 for adversarial S
2	Per-interval CS	Works locally; 87–93% overlap kills global
3	Weighted CS ($f = 1/\tau$)	Passes computationally; no analytic expression
4	Optimal weighted CS ($C^{-1}d$)	Passes computationally; C^{-1} intractable

continued on next page

#	Approach	Failure reason
5	Filtered CS	Ratio 0.991 (threshold removes key targets)
6	Truncated CS (Ford cap)	Ratio 0.41–0.49
7	Variable-constant CS	$C_{\text{crit}} = 1.01$ stable, not $\to 1$
Matching & fractional methods		
8	Greedy matching	112/220 failures; locally optimal $\neq$ globally
9	Uniform fractional	$\min w(k) = 0.06$; starves min-degree vertices
10	Nonuniform fractional (LP)	Exists but doesn't constitute proof for all n
11	FMC with dyadic intervals	$\Sigma > 1$ for $n <$ 15K (greedy)
12	FMC three-block	Blocked on $\alpha(S_+)$ proof
13	FMC $V_{\min}/V_{\text{rest}}/S_-$	Blocked on $\alpha(V_{\text{rest}}) \geq 2$
14	Sinkhorn iteration	13–15% fractional deficiency
15	CLP factoring	Graph not regular; truncation doesn't help
16	$\sqrt{2}$-partition sequential	Local condition passes; global guarantee fails
Probabilistic methods		
17	Symmetric LLL	$P \cdot e \cdot (D+1) \approx 10^4$
18	Target-centered LLL	Same; high-τ targets cause failure
19	Janson inequality	$\Delta \gg (\sum P_i)^2$
20	Erdős–Spencer weighted LLL	Circular obstruction in x_i system
21	Semi-random nibble (Z115)	25–30% of $\lvert V\rvert$ permanently unmatchable
Sieve & inclusion-exclusion		
22	Bonferroni (order 2)	Waste ratio 2.4–9.2×, grows with n
23	Bonferroni (higher orders)	Signs diverge; no useful truncation
24	Product-formula sieve	Overestimates $\lvert N_H(S)\rvert$ (wrong direction)
25	Unique multiple sieve	30–90% of elements have $d_1 = 0$

continued on next page

#	Approach	Failure reason
26	Standard sieve (Selberg/Brun)	3.56× safety factor, wrong direction
	Graph-theoretic & spectral	
27	Spectral gap	Graph too irregular; σ_2/σ_1 large
28	Haxell independent transversal	Overlap 87–93% makes conflict graph dense
29	Turán bound	27–32% of target; gap grows with n
30	Degeneracy bound	10–28% of target; degeneracy $\sim 15\sqrt{n}$
31	Ford divisor cap	Ratio 0.41–0.49
32	Multiplicative energy (K–M)	Qualitative but not quantitative for this graph
	Partition & structural	
33	Dyadic partition ($\times 2$)	1–2 intervals fail degree condition
34	$\sqrt{2}$-partition	All pass but no global guarantee
35	Fine partition ($\times 1.1$)	All pass locally; same overlap problem
36	Stratified Hall (octile)	7/8 pass; Q0 fails until subdivided
37	$V_{\min}$ disjointness	Works for $V_{\min}$; doesn't extend to V_{rest}
	Other	
38	Derandomization	No efficient random process to derandomize
39	Modular remainder	Works for specific elements; doesn't generalize
40	Recursive doubling	Targets already claimed
41	Surplus-excess proof	Bonferroni waste kills it
42	Neumann series bound	Codegree matrix not contractive
43	Quasi-independence	Positive correlations in divisibility

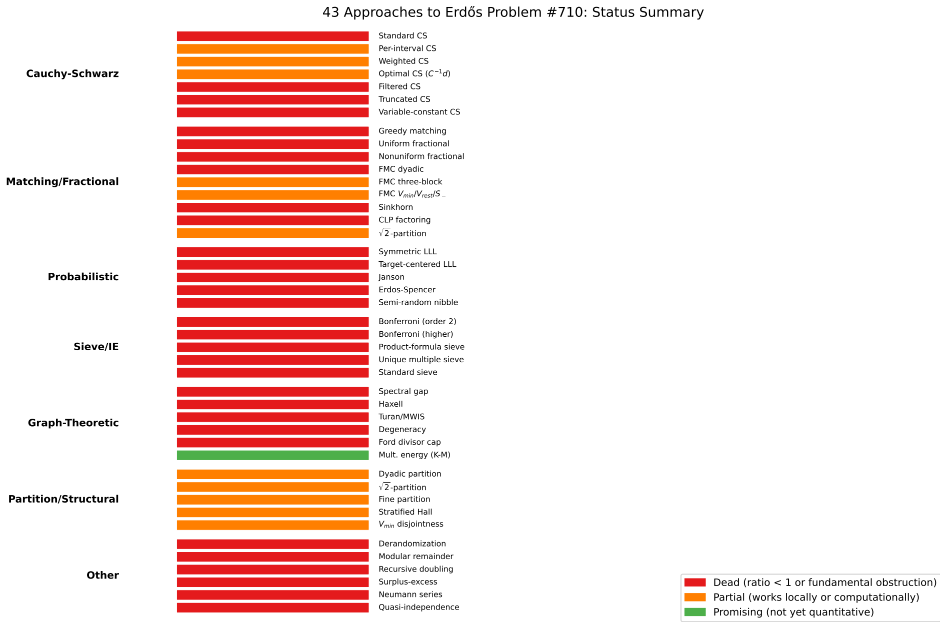

Figure 6. Visual summary of all 43 approaches, organized by category. Red: dead (fundamental obstruction). Orange: partial success (works locally or computationally). Green: promising direction (not yet quantitative).

12.6. **Final words.** Erdős Problem #710 has resisted a determined assault with every tool in the modern combinatorialist's arsenal. The computational evidence is overwhelming: Hall's condition holds with zero failures through $n = 10^6$, and no adversarial subset has ever been found with expansion ratio below 1. The gap between what we can compute and what we can prove is a testament to the depth of the problem.

We hope that this detailed investigation—documenting not just what works but, more importantly, what does *not* work and *why*—will save future researchers from repeating these dead ends and guide them toward the techniques most likely to succeed.

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