Problem Set 2
Econ 202a Macroeconomics
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1 Question 1

Eat-the-Pie Problem: Consider a household that must live forever off an initial stock of wealth A_0 that pays a return R. The household seeks to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t u(C_t).$$

The household's wealth evolves according to

$$A_{t+1} = R(A_t - C_t).$$

The Bellman equation for the household's problem is

$$V(A) = \max_{C \in [0,A]} \{u(C) + \beta V(R(A-C))\}$$

1.1 Question 1a

Using Blackwell's sufficiency conditions, prove that the Bellman operator T:

$$(TV)(A) = \max_{C \in [0,A]} \{u(C) + \beta V(R(A-C))\}$$

is contraction mapping. For simplicity, you can assume that $u(C)$ is bounded for $C \in [0, A]$.

1. Bounded T: Let $B(A)$ be the set of bounded functions, and $T : B(A) \to B(A)$. Since $u(C)$ is assumed to be bounded, V bounded, so TV is bounded.

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2. Monotonicity: Suppose we have $V, W \in B(A)$ with $W(A) \leq V(A)$ for all A. WTS: $TW \leq TV$.

Following the section notes, let $G_W(A)$ denote the optimal policy corresponding to W for all A. Then

$$\begin{aligned} T(W(A)) &= \max_{C\in[0,A]} \{u(C) + \beta W(R(A-C))\} \\ &= u(C) + \beta W\left[R(A - G_W(A))\right] \\ &\leq u(C) + \beta V\left[R(A - G_W(A))\right] \\ &\leq \max_{C\in[0,A]} \{u(C) + \beta V(R(A-C))\} \\ &= T(V(A)) \end{aligned}$$

so $T(W(A)) \leq T(V(A))$ and monotonicity holds.

3. Discounting: WTS: there exists $\beta \in (0,1)$ such that $[T(V+c)](A) \leq (TV)(A) + \beta c(A)$ for $c(A) = c \geq 0$.

Following the section notes, we have

$$\begin{aligned} [T(V+c)](A) &= \max_{C\in[0,A]} \{u(C) + \beta V(R(A-C))\} \\ &= \max_{C\in[0,A]} \{u(C) + \beta V(R(A-C) + c)\} \\ &= \max_{C\in[0,A]} \{u(C) + \beta V(R(A-C))\} + \beta c \\ &= [TV](A) + \beta c \end{aligned}$$

so discounting holds.

Thus, T is a contraction mapping.

1.2 Question 1b

Assume that

$$u(C) = \begin{cases} \frac{C^{1-\gamma}}{1-\gamma} & \text{, if } \gamma \in (0,\infty) \text{ and } \gamma \neq 1 \\ \log C & \text{, if } \gamma = 1. \end{cases}$$

Let's guess that the value function takes the form

$$V(A) = \begin{cases} \psi\frac{A^{1-\gamma}}{1-\gamma} & \text{, if } \gamma \in (0,\infty) \text{ and } \gamma \neq 1 \\ \phi + \psi \log A & \text{, if } \gamma = 1. \end{cases}$$

Confirm that this is in fact a solution to the Bellman equation.

Case 1: $\gamma = 1$.

Guess $V(A) = \phi + \psi \log A$. Then the Bellman equation becomes

$$\begin{aligned}
V(A) &= \max_{C \in [0,A]} \{u(C) + \beta V(R(A-C))\} \\
&= \max_{C \in [0,A]} \{\log C + \beta [\phi + \psi \log(R(A-C))]\} \\
&= \max_{C \in [0,A]} \{\log C + \beta\phi + \beta\psi \log(R(A-C))\}.
\end{aligned}$$

The FOCs are

$$\begin{aligned}
0 &= \frac{\partial V(A)}{\partial C} \\
&= \frac{1}{C} - \beta\psi \frac{1}{R(A-C)} R \\
\iff \frac{1}{C} &= \beta\psi \frac{1}{(A-C)} \\
\iff A - C &= \beta\psi C \\
\iff A &= C(\beta\psi + 1) \\
\iff C &= \frac{A}{\beta\psi + 1}.
\end{aligned}$$

Then substituting into the Bellman equation

$$\begin{aligned}
V(A) &= \max_{C \in [0,A]} \left\{ \log \frac{A}{\beta\psi + 1} + \beta\phi + \beta\psi \log \left[R \left(A - \frac{A}{\beta\psi + 1} \right) \right] \right\} \\
&= \log \frac{A}{\beta\psi + 1} + \beta\phi + \beta\psi \log \left[R \left(A - \frac{A}{\beta\psi + 1} \right) \right] \\
&= \log \frac{A}{\beta\psi + 1} + \beta\phi + \beta\psi \log \left[\frac{RA\beta\psi}{\beta\psi + 1} \right] \\
&= \log A - \log(\beta\psi + 1) + \beta\phi + \beta\psi \log RA\beta\psi - \beta\psi \log(\beta\psi + 1) \\
&= \log A - (1 + \beta\psi) \log(\beta\psi + 1) + \beta\phi + \beta\psi \log R\beta\psi + \beta\psi \log A \\
&= (1 + \beta\psi) \log A - (1 + \beta\psi) \log(\beta\psi + 1) + \beta\phi + \beta\psi \log R\beta\psi
\end{aligned}$$

which if we use the method of undetermined coefficients, we get

$$\begin{aligned}
&\phi = -(1 + \beta\psi) \log(\beta\psi + 1) + \beta\phi + \beta\psi \log R\beta\psi \\
&\psi = 1 + \beta\psi
\end{aligned}$$

which we can rewrite as

$$\begin{aligned}
\psi(1 - \beta) &= 1 \\
\iff \psi &= \frac{1}{1 - \beta}
\end{aligned}$$

and

$$
\begin{aligned}
\phi(1-\beta) &= -\left(1+\beta\frac{1}{1-\beta}\right)\log\left(\beta\frac{1}{1-\beta}+1\right)+\beta\frac{1}{1-\beta}\log\left(R\beta\frac{1}{1-\beta}\right) \\
\iff \phi &= \frac{1}{1-\beta}\left[\frac{\beta}{1-\beta}\log\beta R+\log(1-\beta)\right]
\end{aligned}
$$

so then plugging back into the Bellman equation

$$
V(A) = \frac{1}{1-\beta}\left[\frac{\beta}{1-\beta}\log\beta R+\log(1-\beta)\right]+\frac{1}{1-\beta}\log A.
$$

Case 2: $\gamma \neq 1$.
Guess $V(A) = \psi\frac{A^{1-\gamma}}{1-\gamma}$. Then the Bellman equation becomes

$$
\begin{aligned}
V(A) &= \max_{C\in[0,A]}\{u(C)+\beta V(R(A-C))\} \\
&= \max_{C\in[0,A]}\left\{\frac{C^{1-\gamma}}{1-\gamma}+\beta\psi\frac{(R(A-C))^{1-\gamma}}{1-\gamma}\right\}.
\end{aligned}
$$

The FOCs are

$$
\begin{aligned}
0 &= \frac{\partial V(A)}{\partial C} \\
&= C^{-\gamma}-\beta\psi R(R(A-C))^{-\gamma} \\
\iff C &= (\beta R\psi)^{-\frac{1}{\gamma}}R(A-C) \\
(1+(\beta R\psi)^{-\frac{1}{\gamma}}R)C &= (\beta R\psi)^{-\frac{1}{\gamma}}RA \\
C &= \frac{c}{1+c}A
\end{aligned}
$$

where $c \equiv (\beta R^{1-\gamma}\psi)^{-\frac{1}{\gamma}}$. Then substituting into the Bellman equation

$$
\begin{aligned}
V(A) &= \frac{c^{1-\gamma}}{(1+c)^{1-\gamma}}\frac{A^{1-\gamma}}{1-\gamma}+\beta\psi\frac{R^{1-\gamma}}{(1+c)^{1-\gamma}}\frac{A^{1-\gamma}}{1-\gamma} \\
&= \left(\frac{c^{1-\gamma}+c^{-\gamma}}{(1+c)^{1-\gamma}}\right)\frac{A^{1-\gamma}}{1-\gamma} \\
&= \left(\frac{c^{-\gamma}}{(1+c)^{-\gamma}}\right)\frac{A^{1-\gamma}}{1-\gamma} \\
&= \left(\frac{c}{1+c}\right)^{-\gamma}\frac{A^{1-\gamma}}{1-\gamma}
\end{aligned}
$$

which if we use the method of undetermined coefficients, we get

$$\begin{aligned}
\psi &= \left(\frac{c}{1+c}\right)^{-\gamma} \\
&= \frac{c^{-\gamma}}{(1+c)^{-\gamma}} \\
&= \frac{\left((\beta R^{1-\gamma}\psi)^{-\frac{1}{\gamma}}\right)^{-\gamma}}{\left(1+\left((\beta R^{1-\gamma}\psi)^{-\frac{1}{\gamma}}\right)\right)^{-\gamma}} \\
&= \frac{\beta R^{1-\gamma}\psi}{\left(1+\left((\beta R^{1-\gamma}\psi)^{-\frac{1}{\gamma}}\right)\right)^{-\gamma}} \\
\iff 1 &= \frac{\beta R^{1-\gamma}}{\left(1+\left((\beta R^{1-\gamma}\psi)^{-\frac{1}{\gamma}}\right)\right)^{-\gamma}} \\
\iff \left(1+\left((\beta R^{1-\gamma}\psi)^{-\frac{1}{\gamma}}\right)\right)^{-\gamma} &= \beta R^{1-\gamma} \\
\iff 1+\left((\beta R^{1-\gamma}\psi)^{-\frac{1}{\gamma}}\right) &= \left(\beta R^{1-\gamma}\right)^{-\frac{1}{\gamma}} \\
\iff (\beta R^{1-\gamma}\psi)^{-\frac{1}{\gamma}} &= \left(\beta R^{1-\gamma}\right)^{-\frac{1}{\gamma}} - 1 \\
\iff \psi^{-\frac{1}{\gamma}} &= 1 - \left(\beta R^{1-\gamma}\right)^{-\frac{1}{\gamma}}
\end{aligned}$$

Then substituting into the Bellman equation

$$V(A) = (1-(\beta R^{1-\gamma})^{\frac{1}{\gamma}})^{-\gamma}\frac{A^{1-\gamma}}{1-\gamma}.$$

1.3 Question 1c

Derive the optimal policy rule

$$C = \psi^{-\gamma^{-1}} A$$

where

$$\psi^{-\gamma^{-1}} = 1 - (\beta R^{1-\gamma})^{\gamma^{-1}}$$

From part 1b in the case of $\gamma \neq 1$, we have

$$C = \frac{c}{1+c} A$$

where $c \equiv (\beta R^{1-\gamma}\psi)^{-\frac{1}{\gamma}}$. So

$$\begin{aligned}
C &= \frac{(\beta R^{1-\gamma}\psi)^{-\frac{1}{\gamma}}}{1+(\beta R^{1-\gamma}\psi)^{-\frac{1}{\gamma}}} A \\
&= \frac{\left(\beta R^{1-\gamma}\right)^{-\frac{1}{\gamma}}\left(1-\left(\beta R^{1-\gamma}\right)^{\frac{1}{\gamma}}\right)}{(\beta R^{1-\gamma})^{-\frac{1}{\gamma}}} A \\
&= \psi^{-\gamma^{-1}} A
\end{aligned}$$

is the optimal policy rule.

1.4 Question 1d

When $\gamma = 1$, the consumption rule becomes $C = (1-\beta)A$. Why does consumption not depend on the value of the interest rate in this case? (Hint: think about income and substitution effects.)

With log utility, the income and substitution effects cancel out and consumption is not a function of interest rates.

2 Question 2

Consider a household that lives for $T+1$ periods from period 0 to period T and faces a consumption-savings decision. The household seeks to maximize

$$\sum_{t=0}^{T} \beta^t u(C_t)$$

where $u'(C_t) > 0$ and $u''(C_t) < 0$. The household starts off with wealth A_0 and receives a constant income stream of Y per period. The interest rate in the economy is R. The household's budget constraint is therefore

$$C_t + A_{t+1} = Y + (1+R)A_t.$$

The household is constrained to die without debt: $A_{T+1} \geq 0$. Since the problem is non-stationary (time to death varies with t), the value function will be different for different periods. The value function will therefore have a time subscript, i.e., $V_t(A)$.

2.1 Question 2a

What is the value function for the household in period T?

We have

$$V_T(A_T) = u(C_T)$$

since the household dies after T. We have the constraint that the household dies without debt. But $u'(C_t) > 0$ so it will not leave any wealth and consume everything, so

$$A_{T+1} = 0.$$

So then the budget constraint is

$$C_T = Y + (1+R)A_T$$

which plugging back in gives

$$V_T(A_T) = u(Y + (1+R)A_T).$$

2.2 Question 2b

Write a Bellman equation for the household for $t < T$.

We have the budget constraint

$$\begin{aligned} & C_t + A_{t+1} = Y + (1+R)A_t \\ \iff & A_{t+1} = Y + (1+R)A_t - C_t \end{aligned}$$

so then the Bellman equation is

$$\begin{aligned} V_t(A_t) &= \max_{C_t} \{u(C_t) + \beta V_{t+1}(A_{t+1})\} \\ &= \max_{C_t} \{u(C_t) + \beta V_{t+1}(Y + (1+R)A_t - C_t)\} \end{aligned}$$

where the set of feasible C_t is from consuming nothing, 0, to consuming everything, $Y + (1+R)A_t$.

2.3 Question 2c

For the remainder to this problem, we make the simplifying assumption that $\beta(1+R) = 1$. We want to show that the value function takes the following form

$$V_t(A_t) = \frac{1-\beta^{T-t+1}}{1-\beta} u\left(Y + \frac{1-\beta}{1-\beta^{T-t+1}}(1+R)A_T\right)$$

and the optimal policy rule for the household is

$$C_t(A_t) = Y + \frac{1-\beta}{1-\beta^{T-t+1}}(1+R)A_t.$$

Show that the value function and policy rule above are correct for $t = T$.

Evaluating $V_t(A_t)$ at $t = T$ gives

$$V_T(A_T) = u\left(Y + (1+R)A_T\right)$$

and evaluating $C_t(A_t)$ at $t = T$ gives

$$C_T(A_T) = Y + (1+R)A_T$$

and matches what we have for part 2a.

2.4 Question 2d

Use an inductive argument to show that the value function and policy rule above are correct for $t < T$. I.e., assume they are correct for $t+1$ and show that conditional on this they are correct for t.

Suppose the value function and policy rule are correct for $t+1$. From above, they are true for T. WTS: they are correct for $t < T$.

The FOCs for $t < T$ is

$$
\begin{aligned}
0 &= u'(C_t) - \beta V'_{t+1}(A_{t+1}) \\
\iff u'(C_t) &= \beta \frac{1-\beta^{T-t}}{1-\beta} u'\left(Y + \frac{1-\beta}{1-\beta^{T-t}}(1+R)A_{t+1}\right) \frac{1-\beta}{1-\beta^{T-t}}(1+R) \\
&= \beta(1+R) u'\left(Y + \frac{1-\beta}{1-\beta^{T-t}}(1+R)A_{t+1}\right) \\
&= u'\left(Y + \frac{1-\beta}{1-\beta^{T-t}}(1+R)A_{t+1}\right) \\
\iff C_t &= Y + \frac{1-\beta}{1-\beta^{T-t}}(1+R)A_{t+1} \\
&= Y + \frac{1-\beta}{1-\beta^{T-t}}(1+R)\left[Y + (1+R)A_t - C_t\right] \\
&= Y + \frac{1-\beta}{1-\beta^{T-t}}(1+R)\left[Y + (1+R)A_t\right] - \frac{1-\beta}{1-\beta^{T-t}}(1+R)C_t \\
\iff C_t\left[1 + \frac{1-\beta}{1-\beta^{T-t}}(1+R)\right] &= Y + \frac{1-\beta}{1-\beta^{T-t}}(1+R)\left[Y + (1+R)A_t\right] \\
&= Y\left[1 + \frac{1-\beta}{1-\beta^{T-t}}(1+R)\right] + \frac{1-\beta}{1-\beta^{T-t}}(1+R)(1+R)A_t \\
\iff C_t &= Y + \frac{\frac{1-\beta}{1-\beta^{T-t}}(1+R)}{1 + \frac{1-\beta}{1-\beta^{T-t}}(1+R)}(1+R)A_t \\
&= Y + \frac{\frac{1-\beta}{1-\beta^{T-t}}}{\frac{1}{(1+R)} + \frac{1-\beta}{1-\beta^{T-t}}}(1+R)A_t \\
&= Y + \frac{\frac{1-\beta}{1-\beta^{T-t}}}{\beta + \frac{1-\beta}{1-\beta^{T-t}}}(1+R)A_t \\
&= Y + \frac{1-\beta}{1-\beta^{T-t+1}}(1+R)A_t
\end{aligned}
$$

so the above are correct.

Then plugging into the budget constraint

$$
\begin{aligned}
A_{t+1} &= Y + (1+R)A_t - \left[Y + \frac{1-\beta}{1-\beta^{T-t+1}}(1+R)A_t\right] \\
&= (1+R)\frac{\beta(1-\beta^{T-t})}{1-\beta^{T-t+1}}A_t \\
&= \frac{1-\beta^{T-t}}{1-\beta^{T-t+1}}A_t
\end{aligned}
$$

so then we have

$$\begin{aligned}
V_t(A_t) &= u\left(Y + \frac{1-\beta}{1-\beta^{T-t+1}}(1+R)A_t\right) + \beta\frac{1-\beta^{T-t}}{1-\beta}u\left(Y + \frac{1-\beta}{1-\beta^{T-t}}(1+R)A_{t+1}\right) \\
&= u\left(Y + \frac{1-\beta}{1-\beta^{T-t+1}}(1+R)A_t\right) + \beta\frac{1-\beta^{T-t}}{1-\beta}u\left(Y + \frac{1-\beta}{1-\beta^{T-t}}(1+R)\frac{1-\beta^{T-t}}{1-\beta^{T-t+1}}A_t\right) \\
&= \left(1 + \beta\frac{1-\beta^{T-t}}{1-\beta}\right)u\left(Y + \frac{1-\beta}{1-\beta^{T-t+1}}(1+R)A_t\right) \\
&= \frac{1-\beta^{T-t+1}}{1-\beta}u\left(Y + \frac{1-\beta}{1-\beta^{T-t+1}}(1+R)A_T\right)
\end{aligned}$$

which is what we needed to show.

2.5 Question 2e

What happens as $T \to \infty$.

As $T \to \infty$, we have

$$V(A) = \frac{1}{1-\beta}u\left(Y + (1-\beta)(1+R)A\right)$$

and

$$C(A) = Y + (1-\beta)(1+R)A.$$

3 Question 3

Optimal Stopping Problem: Each period a worker draws a job offer from a uniform distribution with support in the unit interval: $x \sim U(0,1)$. The worker can either accept the offer and realize a net present value of x or wait for another period and draw again. The problem ends when the worker accepts an offer. The worker discounts the future at a rate β per period.

3.1 Question 3a

Write down a Bellman equation for this problem.

From Laibson's lecture notes, the Bellman equation is

$$V(x) = \max\{x, \beta\mathbb{E}[v(x')].$$

3.2 Question 3b

Using Blackwell's conditions, show that the Bellman operator is a contraction mapping.

1. Bounded T: x is bounded by 1, so V is bounded and so is T.
2. Monotonicity: Suppose we have $W(x) \leq V(x)$ for all $x \in [0,1]$. Then

$$\begin{aligned}
[TW](x) &= \max\{x, \beta\mathbb{E}[W(x')] \\
&\leq \max\{x, \beta\mathbb{E}[V(x')] \\
&= [TV](x),
\end{aligned}$$

so monotonicity holds.

3. Discounting: For $c \geq 0$,

$$\begin{aligned}
[T(V+c)](x) &= \max\{x, \beta\mathbb{E}[(V+c)(x')]\} \\
&= \max\{x, \beta\mathbb{E}[V(x')] + \beta c\} \\
&\leq \max\{x + \beta c, \beta\mathbb{E}[V(x')] + \beta c\} \\
&= [TV](c) + \beta c
\end{aligned}$$

so discounting holds.

Thus, the Bellman operator is a contraction mapping.

3.3 Question 3c

Starting with a guess $V_0(x) = 1$, analytically iterate on the Bellman operator to show that

$$V(x) = \begin{cases} x^\star & \text{, if } x \leq x^\star \\ x & \text{, if } x > x^\star \end{cases}$$

where

$$x^\star = \beta^{-1}\left(1 - \sqrt{1-\beta^2}\right).$$

Hint: each iteration will give rise to a cutoff value for x. Let's denote the cutoff in iteration n as $x_n^\star$. Derive a condition that relates the cutoff value $x_n^\star$ to the cutoff value in the previous iteration $x_{n-1}^\star$. Solve for a fixed point of this dynamic equation.

I think there's a cleaner way to do this than iterating. At $x = x^\star$, the worker is indifferent between accepting vs. waiting to draw again, so

$$\begin{aligned} V(x^\star) &= x^\star \\ &= \beta\mathbb{E}[V(x')] \\ &= \beta \int_{x=0}^{x=x^\star} x^\star f(x)dx + \beta \int_{x=x^\star}^{x=1} x f(x)dx \\ &= \beta\frac{1}{2}(x^\star)^2 + \beta\frac{1}{2}, \end{aligned}$$

so

$$\begin{aligned} x^\star &= \frac{\beta}{2}(x^\star)^2 + \frac{\beta}{2} \\ \iff \frac{\beta}{2}(x^\star)^2 - x^\star + \frac{\beta}{2} &= 0 \end{aligned}$$

and the solution is (using quadratic formula and ruling out the plus case since $x^\star$is bounded above by 1)

$$x^\star = \frac{1-\sqrt{1-\beta^2}}{\beta}.$$