Problem Set 1
ECON 236B Advanced Macroeconomics
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1 Problem 1

A time series process is deterministic if its stochastic nature has no relation with time so that it can be predicted with certainty. The simplest example of a deterministic process is $x_t = x$ for all t. Consider $x_t = \alpha * \cos(t) + \beta * \sin(t)$ with $\alpha \sim N(0, 1)$ and $\beta \sim N(0, 1)$

1.1 Problem 1a

What is the mean and variance of x_t?

From class, we have the mean

$$\begin{aligned} x_t &= \alpha \cos(t) + \beta \sin(t) \\ \implies \mathbb{E}[x_t] &= \mathbb{E}[\alpha] \cos(t) + \mathbb{E}[\beta] \sin(t) \\ &= 0 \end{aligned}$$

and the variance

$$\begin{aligned} \mathrm{Var}[x_t] &= \mathbb{E}[x_t^2] \\ &= \mathbb{E}[\alpha^2] \cos^2(t) + \mathbb{E}[\beta^2] \sin^2(t) \\ &= \cos^2(t) + \sin^2(t) \\ &= 1. \end{aligned}$$

1.2 Problem 1b

Does $\mathbb{E}(x_t x_{t-k})$ depend on time t? Prove.

We have the covariance

$$\begin{aligned} \mathbb{E}[x_t x_{t-k}] &= \mathbb{E}[(\alpha \cos(t) + \beta \sin(t))(\alpha \cos(t-k) + \beta \sin(t-k))] \\ &= \mathbb{E}\left[\alpha^2 \cos(t) \cos(t-k) + \alpha\beta \sin(t-k) \cos(t) + \alpha\beta \sin(t) \cos(t-k) + \beta^2 \sin(t) \sin(t-k)\right] \\ &= \mathbb{E}\left[\alpha^2\right] \cos(t) \cos(t-k) + \mathbb{E}[\alpha\beta] \sin(t-k) \cos(t) + \mathbb{E}[\alpha\beta] \sin(t) \cos(t-k) + \mathbb{E}\left[\beta^2\right] \sin(t) \sin(t-k) \\ &= \cos(t) \cos(t-k) + \sin(t) \sin(t-k) \\ &= \cos(k) \end{aligned}$$

so $\mathbb{E}[x_t x_{t-k}]$ does not depend on time t.

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2 Problem 2

Suppose $e_t = u_t\sqrt{b + \alpha e_{t-1}^2}$ where $u_t \sim iid(0,1)$. The process e_t is called an ARCH process.

2.1 Problem 2a

Is e_t serially correlated?

Note that u_t is IID so $\mathbb{E}[u_t u_{t-k}] = 0$. Then we have

$$\begin{aligned}
\mathbb{E}[e_t e_{t-k}] &= \mathbb{E}\left[u_t\sqrt{b+\alpha e_{t-1}^2}\,u_{t-k}\sqrt{b+\alpha e_{t-k}^2}\right] \\
&= \mathbb{E}\left[u_t u_{t-k}\right]\mathbb{E}\left[\sqrt{b+\alpha e_{t-1}^2}\sqrt{b+\alpha e_{t-k}^2}\right], \text{ by independence} \\
&= 0
\end{aligned}$$

so e_t is not serially correlated.

2.2 Problem 2b

What is variance of e_t conditional on e_{t-1}? Is the conditional variance serially correlated?

Note that $\mathbb{E}_{t-1}[e_t] = 0$ since $v_t \sim IID(0,1)$. Then we have

$$\begin{aligned}
\mathbb{E}_{t-1}[e_t^2] &= \mathbb{E}_{t-1}\left[u_t^2\left(b+\alpha e_{t-1}^2\right)\right] \\
&= \mathbb{E}_{t-1}\left[u_t^2\right]\mathbb{E}_{t-1}\left[b+\alpha e_{t-1}^2\right], \text{ by independence} \\
&= \mathbb{E}_{t-1}\left[b+\alpha e_{t-1}^2\right] \\
&= b+\alpha e_{t-1}^2
\end{aligned}$$

so the conditional variance of e_t is dependent on the realized value of e_{t-1}^2. Note that the unconditional variance is

$$\begin{aligned}
\mathbb{E}[e_t^2] &= b + \alpha\mathbb{E}[e_{t-1}^2] \\
&= b + \alpha\mathbb{E}[e_t^2] \\
\iff (1-\alpha)\mathbb{E}[e_t^2] &= b \\
\iff \mathbb{E}[e_t^2] &= \frac{b}{1-\alpha}.
\end{aligned}$$

We have $\mathbb{E}_{t-2}[e_{t-1}^2] = b + \alpha e_{t-2}^2$ so

$$\begin{aligned}
\operatorname{Cov}\left\{\mathbb{E}_{t-1}[e_t^2], \mathbb{E}_{t-2}[e_{t-1}^2]\right\} &= \mathbb{E}\left\{\mathbb{E}_{t-1}[e_t^2]\mathbb{E}_{t-2}[e_{t-1}^2]\right\} - \mathbb{E}\left\{\mathbb{E}_{t-1}[e_t^2]\right\}\mathbb{E}\left\{\mathbb{E}_{t-2}[e_{t-1}^2]\right\} \\
&= \mathbb{E}\left\{\left(b+\alpha e_{t-1}^2\right)\left(b+\alpha e_{t-2}^2\right)\right\} - \mathbb{E}\left\{e_t^2\right\}\mathbb{E}\left\{e_{t-1}^2\right\} \\
&= \mathbb{E}\left\{b^2 + b\alpha e_{t-2}^2 + b\alpha e_{t-1}^2 + \alpha^2 e_{t-1}^2 e_{t-2}^2\right\} - \left(\frac{b}{1-\alpha}\right)^2 \\
&= b^2 + b\alpha\mathbb{E}\left\{e_{t-2}^2\right\} + b\alpha\mathbb{E}\left\{e_{t-1}^2\right\} + \alpha^2\mathbb{E}\left\{e_{t-1}^2 e_{t-2}^2\right\} - \left(\frac{b}{1-\alpha}\right)^2 \\
&= b^2 + 2b\alpha\frac{b}{1-\alpha} + \alpha^2\mathbb{E}\left\{e_{t-1}^2 e_{t-2}^2\right\} - \left(\frac{b}{1-\alpha}\right)^2
\end{aligned}$$

so there is some serial correlation in the conditional variances.

2.3 Problem 2c

Let $x_t = e_t^2 - \sigma^2$ where $\sigma^2 = E\left(e_t^2 \mid e_{t-1}, e_{t-2}, \ldots\right)$. Prove that x_t is mds.

WTS: $\mathbb{E}_{t-1}[x_t] = 0$.
We have

$$\begin{aligned}
\mathbb{E}_{t-1}[x_t] &= \mathbb{E}_{t-1}\left[e_t^2 - \sigma^2\right] \\
&= \mathbb{E}_{t-1}\left[e_t^2\right] - \mathbb{E}_{t-1}\left[\sigma^2\right] \\
&= \sigma^2 - \sigma^2 \\
&= 0.
\end{aligned}$$

2.4 Problem 2d

Prove that x_t is white noise.

From Enders, we need to show that $\{x_t\}$ is mean zero, constant variance, and uncorrelated with other realizations.

We have the mean

$$\begin{aligned}
\mathbb{E}[x_t] &= \mathbb{E}\left[e_t^2 - \sigma^2\right] \\
&= 0, \text{ by law of iterated expectations}
\end{aligned}$$

and variance

$$\begin{aligned}
\mathbb{E}[x_t^2] &= \mathbb{E}\left[\left(e_t^2 - \sigma^2\right)^2\right] \\
&= \mathbb{E}\left[\left(e_t^2\right)^2 - 2e_t^2\sigma^2 + \left(\sigma^2\right)^2\right] \\
&= \mathbb{E}\left[e_t^2\right]\mathbb{E}\left[e_t^2\right] - 2\sigma^2\mathbb{E}\left[e_t^2\right] + \left(\sigma^2\right)^2 \\
&= \left[\frac{b}{1-\alpha}\right]^2 - 2\sigma^2\frac{b}{1-\alpha} + \left(\sigma^2\right)^2
\end{aligned}$$

And the covariance

$$\begin{aligned}
\mathbb{E}\left[x_t x_{t-k}\right] &= \mathbb{E}\left[\left(e_t^2 - \sigma^2\right)\left(e_{t-k}^2 - \sigma^2\right)\right] \\
&= \mathbb{E}\left[e_t^2 e_{t-k}^2 - \sigma^2 e_t^2 - \sigma^2 e_{t-k}^2 + \left(\sigma^2\right)^2\right] \\
&= \mathbb{E}\left[e_t^2\right]\mathbb{E}\left[e_{t-k}^2\right] - \sigma^2\mathbb{E}\left[e_t^2\right] - \sigma^2\mathbb{E}\left[e_{t-k}^2\right] + \left(\sigma^2\right)^2 \\
&= \left(\sigma^2\right)^2 - \left(\sigma^2\right)^2 - \left(\sigma^2\right)^2 + \left(\sigma^2\right)^2 \\
&= 0
\end{aligned}$$

for all j.

So it's white noise.

3 Problem 3

ARMA + ARMA arithmetic

3.1 Problem 3a

Suppose that $\mathrm{y_{1t}} \sim ARMA(2,1)$ and $y_{2t} \sim ARMA(1,2)$ are two independent processes. Process $x_t = y_{1t} + y_{2t}$ is also an $ARMA(p,q)$ process. What p and q describe x? Prove.

See part b for the general case proof. Applying that result, we have $x_t \sim ARMA(3,4)$.

3.2 Problem 3b

Suppose that $y_{1t} \sim ARMA(p_1, q_1)$ and $y_{2t} \sim ARMA(p_2, q_2)$ are two independent processes. Again, $x_t = y_{1t} + y_{2t}$ is also an A R M A(p,q) process. What p and q describe x in this general case? Prove.

Claim: x_t is a ARMA(p,q) process where $p \leq p_1 + p_2$ and $q \leq \max\{p_1 + q_2, p_2 + q_1\}$, and this holds with equality when there's no common factor.

Proof: We can write y_{1t} and y_{2t} as

$$\alpha_1(L)y_{1t} = \beta_1(L)e_{1t}$$
$$\alpha_2(L)y_{2t} = \beta_2(L)e_{2t}$$

where e_{it} for $i = 1, 2$ is a white noise process, and α_i is a lag polynomial of order p_i and β_i is a lag polynomial of order q_i. Then we have

$$\begin{aligned}
\alpha_1(L)\alpha_2(L)x_t &= \alpha_1(L)\alpha_2(L)(y_{1t} + y_2) \\
&= \alpha_1(L)\alpha_2(L)y_{1t} + \alpha_1(L)\alpha_2(L)y_{2t} \\
&= \alpha_1(L)\alpha_2(L)\frac{\beta_1(L)}{\alpha_1(L)}e_{1t} + \alpha_1(L)\alpha_2(L)\frac{\beta_2(L)}{\alpha_2(L)}e_{2t} \\
&= \alpha_2(L)\beta_1(L)e_{1t} + \alpha_1(L)\beta_2(L)e_{2t}
\end{aligned}$$

where the lag polynomial $\alpha_1(L)\alpha_2(L)$ is of order $p = p_1 + p_2$, the lag polynomial $\alpha_2(L)\beta_1(L)$ is of order $p_2 + q_1$ and the lag polynomial is of order $p_1 + q_2$. So we have $q = \max\{p_2 + q_1, p_1 + q_2\}$ and we have a ARMA(p,q).

4 Problem 4

Working problem. Problem 8.3 in Romer's textbook 4th edition; also see problem 8.12 for derivations; scans for these are on p. 3). Contrast ACF/PACF for actual and measured consumption. Download quarterly US series for consumption of non-durables (FRED database). Plot ACF/PACF. Compare your results with results and predictions in Mankiw (1982).

8.3. The time-averaging problem. (Working, 1960.) Actual data give not consumption at a point in time, but average consumption over an extended period, such as a quarter. This problem asks you to examine the effects of this fact.

Suppose that consumption follows a random walk: $C_t = C_{t-1} + e_t$, where e is white noise. Suppose, however, that the data provide average consumption over two-period intervals; that is, one observes $(C_t + C_{t+1})/2, (C_{t+2} + C_{t+3})/2$, and so on.

4.1 Problem 4a

Find an expression for the change in measured consumption from one two-period interval to the next in terms of the e's.

Let's clear up some notation. True consumption $\{C_t\}$ follows a random walk

$$C_t = C_{t-1} + e_t$$

but measured (or observed) consumption $\{\hat{C}_t\}$ is a two-period average of the current and last period's true consumption

$$\hat{C}_t = \frac{C_t + C_{t-1}}{2}.$$

Then the change in measured consumption is after substituting in repeatedly

$$\begin{aligned}
\Delta\hat{C}_t &= \hat{C}_t - \hat{C}_{t-1} \\
&= \frac{C_t + C_{t-1}}{2} - \frac{C_{t-1} + C_{t-2}}{2} \\
&= \frac{1}{2}\left[C_t - C_{t-2}\right] \\
&= \frac{1}{2}\left[C_{t-1} + e_t - C_{t-2}\right] \\
&= \frac{1}{2}\left[C_{t-2} + e_{t-1} + e_t - C_{t-2}\right] \\
&= \frac{1}{2}\left[e_t + e_{t-1}\right] \\
&= \frac{1}{2}\Delta e_t.
\end{aligned}$$

4.2 Problem 4b

Is the change in measured consumption uncorrelated with the previous value of the change in measured consumption? In light of this, is measured consumption a random walk?

Note that $\mathbb{E}\left[\Delta\hat{C}_t\right] = \frac{1}{2}\mathbb{E}\left[\Delta e_t\right] = 0$. So then

$$\begin{aligned}
\text{Cov}\left[\Delta\hat{C}_t, \Delta\hat{C}_{t-1}\right] &= \mathbb{E}\left[\Delta\hat{C}_t\Delta\hat{C}_{t-1}\right] \\
&= \frac{1}{2}\mathbb{E}\left[\Delta e_t \Delta e_{t-1}\right] \\
&= \frac{1}{2}\mathbb{E}\left[(e_t - e_{t-1})(e_{t-1} - e_{t-2})\right] \\
&= \frac{1}{2}\mathbb{E}\left[e_t e_{t-1} - e_t e_{t-2} - e_{t-1}^2 + e_{t-1}e_{t-2}\right] \\
&= -\frac{1}{2}\sigma_e^2
\end{aligned}$$

where $\sigma_e^2 = \mathbb{E}\left[e_t^2\right]$. So then there is some correlation and measured consumption is not a random walk.

4.3 Problem 4c

Given your result in part (a), is the change in consumption from one two period interval to the next necessarily uncorrelated with anything known as of the first of these two-period intervals? Is it necessarily uncorrelated with anything known as of the two-period interval immediately preceding the first of the two-period intervals?

From part a we have $\Delta\hat{C}_t = \frac{1}{2}\Delta e_t$ so the question is if $\text{Cov}_{t-1}\left[\Delta\hat{C}_t\right] = 0$? We have

$$\begin{aligned}
\Delta\hat{C}_t &= \frac{1}{2}\Delta e_t \\
&= \frac{1}{2}\left[e_t - e_{t-1}\right].
\end{aligned}$$

So if consumption is truly random walk, that is, e_{t-1} is effectively a forecast error realized at $t-1$, the no $\Delta\hat{C}_t$ is not correlated with anything known at $t-1$. Similarly if we shift the reference period to $t-2$ since consumption is assumed to be a random walk here.

4.4 Problem 4d

Suppose that measured consumption for a two-period interval is not the average over the interval, but consumption in the second of the two periods. That is, one observes C_{t+1}, C_{t+3}, and so on. In this case, is measured consumption a random walk?

Lets write it out

$$\begin{aligned}
\Delta\hat{C}_t &= \hat{C}_t - \hat{C}_{t-1} \\
&= C_t - C_{t-2} \\
&= C_{t-1} + e_t - C_{t-2} \\
&= \Delta C_{t-1} + e_t \\
&= e_{t-1} + e_t \\
\iff \hat{C}_t &= \alpha_{t-1} + \hat{C}_{t-1} + e_t
\end{aligned}$$

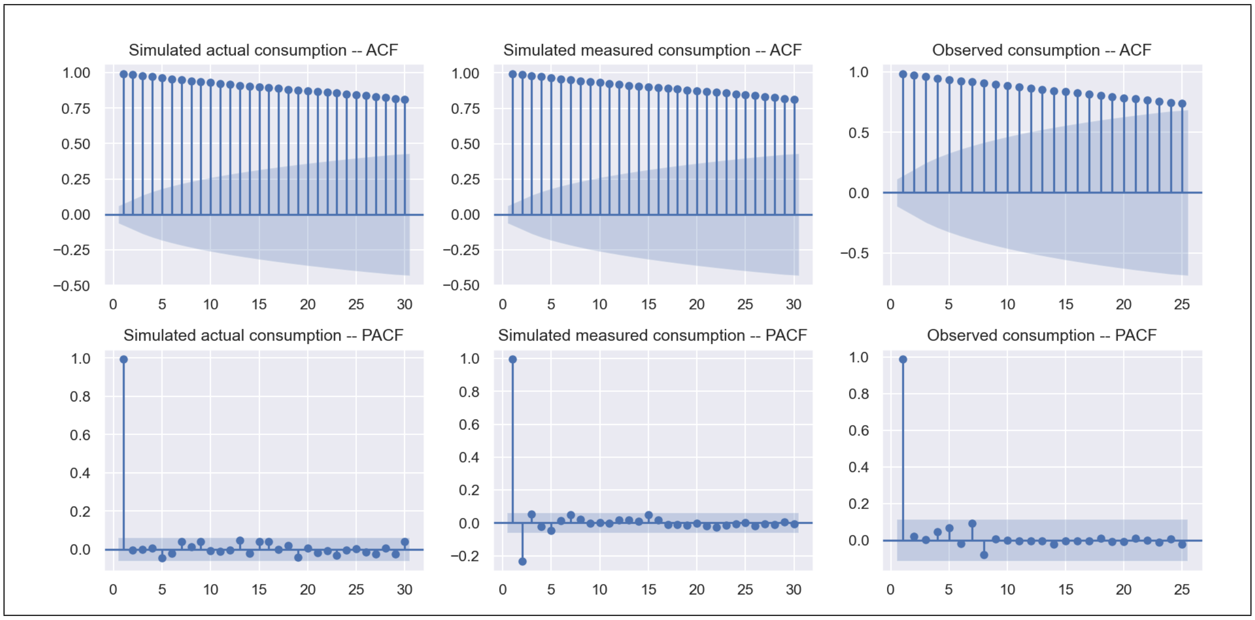
where we defined $\alpha_{t-1} \equiv e_{t-1}$, it's a random walk with drift where the drift α_{t-1} is random, but known as of $t-1$.

4.5 Problem 4e

Contrast ACF/PACF for actual and measured consumption. Download quarterly US series for consumption of non-durables (FRED database). Plot ACF/PACF.

This is in the notebook “q4.ipynb”.

Here’s the ACF and PCF for the simulated actual and measured consumption along with the observed non-durables consumption.



5 Problem 5

Download U.S. series for CPI inflation rate, real GDP growth rate, civilian unemployment rate, USD/UK pound exchange rate (You may want to use St. Louis Fed data base FRED). All series are at quarterly frequency (aggregate [average] to quarterly frequency if necessary). Exclude data after December 2019 (i.e., do not cover the COVID19 period).

All code are in the notebook "q5.ipynb".

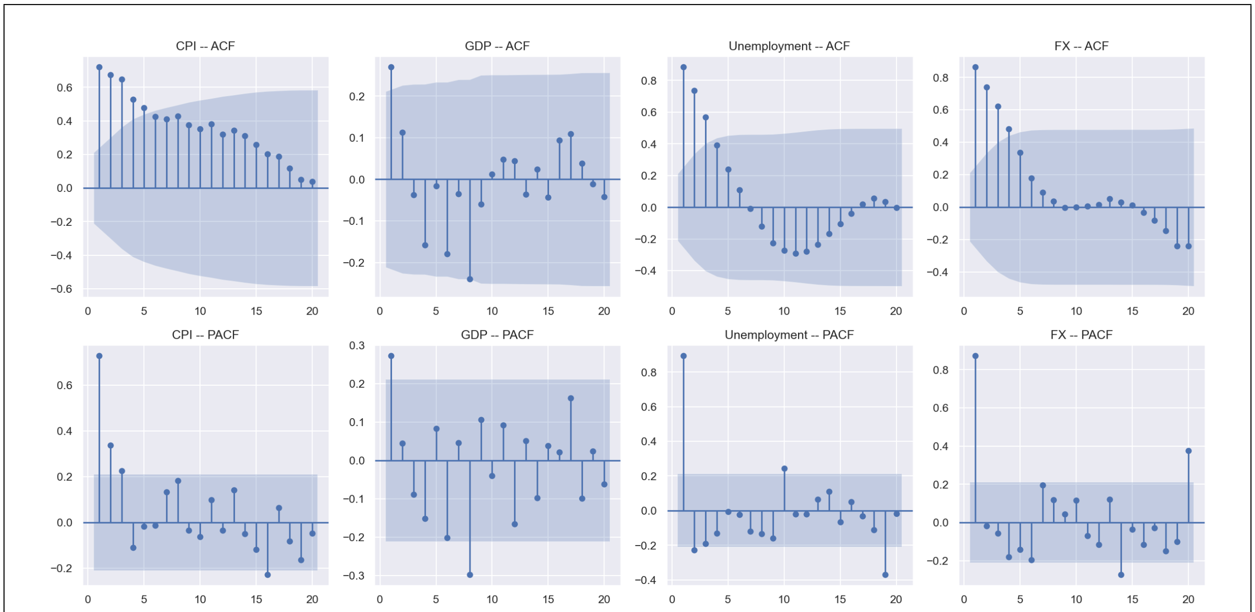
Note: for the CPI and GDP, I converted to QoQ growth rates, not annualizing. Then I construct 10 lags, and finally drop all missing observations (including those created by constructing growth rates and lagging) in order to have a super nice and clean balanced panel from 1973Q3 to 2019Q4.

5.1 Problem 5a

Estimate and plot ACF and PACF. Describe common patterns.

Here's the ACF and PACF. For inflation, the ACF dies out quickly and the first lag of the PACF, and maybe the next two, stick out. For GDP, the first lag sticks out but seems unstable. And for both unemployment and FX, the first lag of the PACF sticks out, but has some later lags too.

For all four series, the it looks like the first autocorrelation sticks out, so maybe a AR(1) works well.



5.2 Problem 5b

Using Box-Jenkins approach, what ARMA(p,q) model may be appropriate for these series? Discuss.

Looking at the ACF and PACF plots from part a, inflation looks like a AR(3), GDP looks like a ARMA(1,1), unemployment looks like a AR(1) and FX looks like a AR(1). Although this approach seems pretty subjective and with room for error.

5.3 Problem 5c

Using BIC, choose a parsimonious parametric $AR(p)$ model. Report the values of BIC for p $= 0, \ldots, 10$.

The BIC suggests lags of 3, 1, 1, and 2 for inflation, FX, GDP and unemployment, respectively.

Lag	1	2	3	4	5	6	7	8	9	10
CPI	166.36	165.98	160.43	162.04	164.47	168.93	172.69	175.93	179.89	183.36
GDP	208.39	211.86	215.48	218.94	223.39	225.44	228.22	224.66	223.44	227.27
Unemployment	41.51	-2	2.44	4.67	6.46	10.81	15.23	18.22	21.71	26.08
FX	-173.91	-169.48	-165.6	-161.42	-156.96	-152.57	-148.44	-144.44	-140.05	-135.64

5.4 Problem 5d

Conduct the diagnostic checking of the estimated optimal AR model: serial correlation of residuals (e.g., Box-Ljung; you may get the code for it at http://ideas.repec.org/c/boc/bocode/t961403.html); ACF/PACF for estimated residuals; normality of the error term (Jarque-Bera test; "jbtest" in Matlab).

I use the Jarque-Bera test. The null is that the residuals are normally distributed, and we can see that the null is rejected quite strongly (almost all at the 0.01 level) for all specifications. Here's the table, with the p-value in parentheses, for every specification.

Lag	1	2	3	4	5	6	7	8	9	10
CPI	996.86	1150.2	1504.3	1637.28	1400.92	1393.83	1345.24	854.55	716.78	673.31
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
GDP	84.42	106.45	81.01	64.7	64.71	60.88	58.28	65.56	23.05	19.72
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Unemployment	37.32	12.61	13.02	9.69	7.91	7.46	7.22	6.23	5.4	5.68
	(0.00)	(0.00)	(0.00)	(0.01)	(0.02)	(0.02)	(0.03)	(0.04)	(0.07)	(0.06)
FX	31.3	31.49	31.81	37.46	37.47	35.72	35.4	37.04	38.34	39
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)

5.5 Problem 5e

Estimate the model using $1, \ldots, \mathrm{T}-10$ observations and compute point forecasts and forecast errors for $T-9, \ldots, T$ (for now, ignore sampling uncertainty in parameter estimates).

The file "q5e.txt" includes the point forecasts and forecast errors. The suffix "_e" denotes forecast error and the suffix "_yhat" denotes forecast. Below reports the RMSE for the optimal models. I prefer the RMSE as it is in our original units for interpretability.

Variable	CPI	FX	GDP	Unemployment
RMSE	0.32	0.05	0.22	0.24

5.6 Problem 5f

Estimate the same model using $T-50, \ldots, T-10$ observations and compute point forecasts and forecast errors for $T-9, \ldots, T$ (for now, ignore sampling uncertainty in parameter estimates). Compare with forecasts and forecast errors in e). Discuss.

The file "q5f.txt" includes the point forecasts and forecast errors. The suffix "_e" denotes forecast error and the suffix "_yhat" denotes forecast. Below reports the RMSE for the optimal models. I prefer the RMSE as it is in our original units for interpretability.

Variable	CPI	FX	GDP	Unemployment
RMSE	0.29	0.05	0.23	0.19

So it looks like the models perform roughly similar using the two training periods. Perhaps CPI and unemployment perform a bit better using the recent sample. This is the trade off between using longer training samples (more data and more precision in the estimated model coefficients) vs. using shorter training samples (better reflect any changes in the underlying data generating processes).