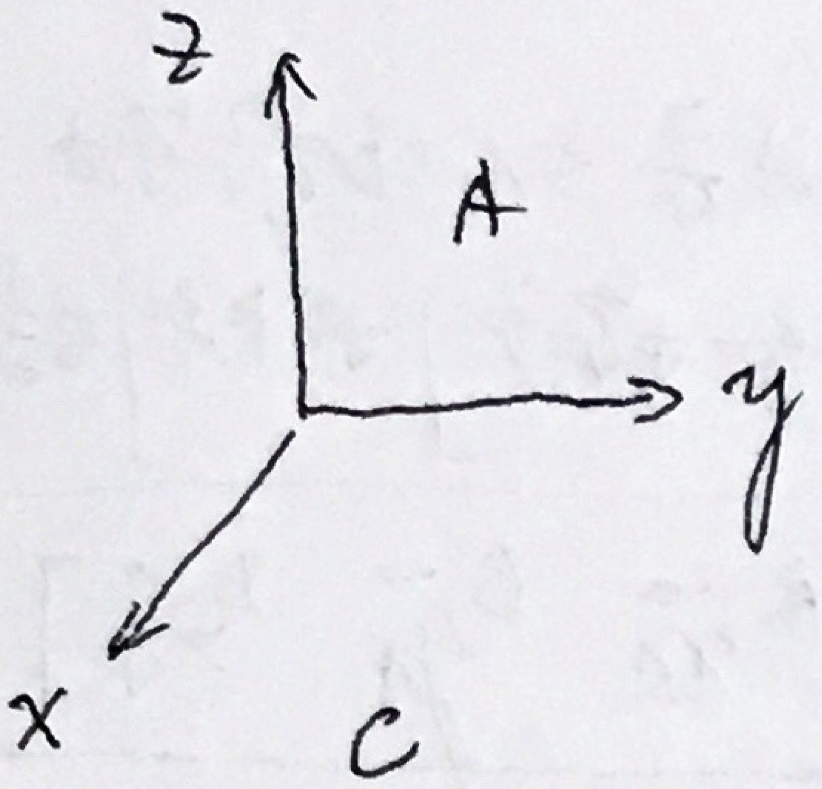
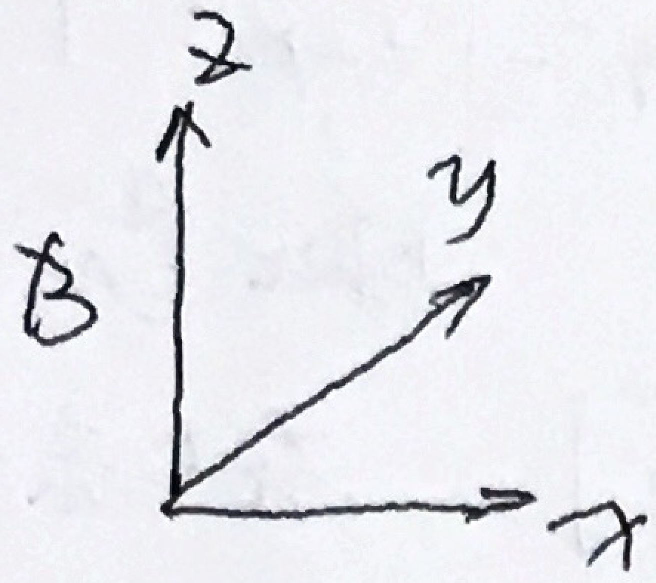
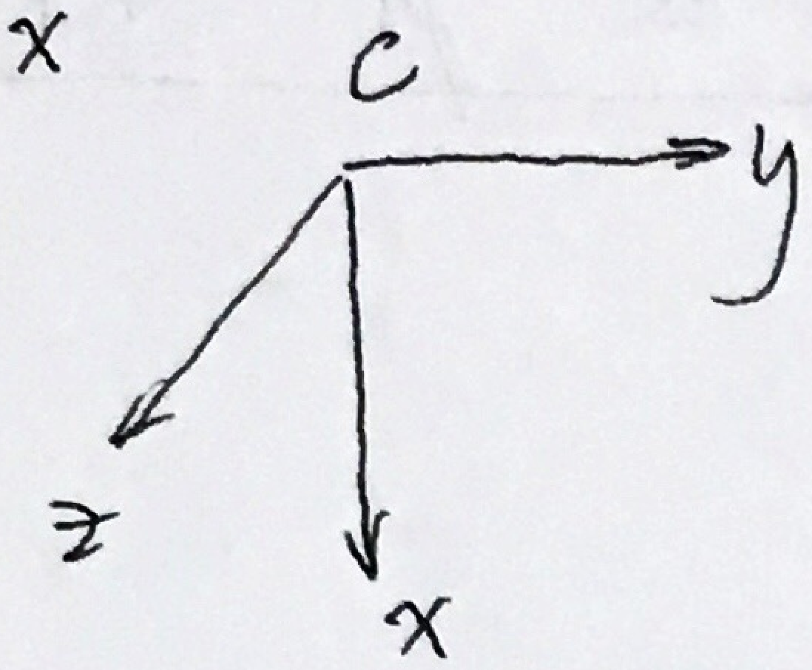
四元数与坐标转换基本运算：

给定坐标系A、B、C如下，B由A绕其z轴转90°得到，C由A绕其y轴旋转90°得到。







① 四元数的构造

B由A绕 ${}^A\vec{v}$ 旋转 θ_A 得到，则定义

$$\boxed{{}^A_B q = \left(\cos\frac{\theta_A}{2}, -\sin\frac{\theta_A}{2}\,{}^A\vec{v}\right)}$$

表示A相对于B 或 B到A 的旋转。

$${}^A_B q = \left(\cos\frac{90°}{2}, -\sin\frac{90°}{2}\,{}^A z_A\right) = \left(\frac{\sqrt{2}}{2},0,0,-\frac{\sqrt{2}}{2}\right) \quad \{{}^A z_A, 90°\}$$

$${}^A_C q = \left(\cos\frac{90°}{2}, -\sin\frac{90°}{2}\,{}^A y_A\right) = \left(\frac{\sqrt{2}}{2},0,-\frac{\sqrt{2}}{2},0\right) \quad \{{}^A y_A, 90°\}$$

② 坐标转换 $\boxed{{}^B\vec{v} = {}^A_B q \otimes {}^A\vec{v} \otimes ({}^A_B q)^*}$ 简记为 ${}^B\vec{v} = {}^A_B q \otimes {}^A\vec{v}$.

取 ${}^A\vec{v} = {}^A\vec{x}_A$，则A系x轴在B系表示 ${}^B\vec{x}_A$：

$${}^B\vec{x}_A = {}^A_B q \otimes {}^A\vec{x}_A \otimes ({}^A_B q)^* = \left(\frac{\sqrt{2}}{2},0,0,-\frac{\sqrt{2}}{2}\right)\otimes(0,1,0,0)\otimes\left(\frac{\sqrt{2}}{2},0,0,\frac{\sqrt{2}}{2}\right) = (0,0,-1,0)$$

$= (0,-1,0)^T = -{}^B\vec{y}_B$，即为B系y轴负方向 ■

同理，

$${}^C\vec{x}_A = {}^A_C q \otimes {}^A\vec{x}_A \otimes ({}^A_C q)^* = \left(\frac{\sqrt{2}}{2},0,-\frac{\sqrt{2}}{2},0\right)\otimes(0,1,0,0)\otimes\left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2},0\right) = (0,0,0,1)$$

$= (0,0,1)^T = {}^C\vec{z}_C$，即为C系z轴方向 ■

考虑B与C之间的关系，根据四元数乘法

$$\boxed{{}^B_C q = {}^A_C q \otimes {}^B_A q} = {}^A_C q \otimes ({}^A_B q)^{-1}$$

$$= \left(\frac{\sqrt{2}}{2},0,-\frac{\sqrt{2}}{2},0\right)\otimes\left(\frac{\sqrt{2}}{2},0,0,\frac{\sqrt{2}}{2}\right) = \left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right).$$

则B系x轴在C系表示为

$${}^C\vec{x}_B = {}^B_C q \otimes {}^B\vec{x}_B \otimes ({}^B_C q)^*$$

$$= \left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)\otimes(0,1,0,0)\otimes\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$$

$= (0,1,0)^T = {}^C\vec{y}_C$，即为C系y轴方向 ■

四元数乘法

$a = [a_1, a_2, a_3, a_4]$

$b = [b_1, b_2, b_3, b_4]$

$$a \otimes b = \begin{bmatrix} a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 \\ a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3 \\ a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2 \\ a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1 \end{bmatrix}^T$$

③ 四元数与旋转矩阵的转换. 已知 ${}^A_B q = [q_1, q_2, q_3, q_4]$.

$$
{}^A_B M = \begin{bmatrix} 2q_1^2-1+2q_2^2 & 2(q_2q_3-q_1q_4) & 2(q_2q_4+q_1q_3) \\ 2(q_2q_3+q_1q_4) & 2q_1^2-1+2q_3^2 & 2(q_3q_4-q_1q_2) \\ 2(q_2q_4-q_1q_3) & 2(q_3q_4+q_1q_2) & 2q_1^2-1+2q_4^2 \end{bmatrix}.
$$

对于系 A, B, C.

$$
{}^A_B M = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [{}^B\vec{x}_A \,\vdots\, {}^B\vec{y}_A \,\vdots\, {}^B\vec{z}_A].
$$

$$
{}^A_C M = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = [{}^C\vec{x}_A \,\vdots\, {}^C\vec{y}_A \,\vdots\, {}^C\vec{z}_A]. \quad ∎
$$

因此 ${}^A_B M$ 即为 A 系 3 个坐标轴在 B 系的表示按列排列

$$
\boxed{{}^A_B M = [{}^B\vec{x}_A \,\vdots\, {}^B\vec{y}_A \,\vdots\, {}^B\vec{z}_A]}
$$

④ 用旋转矩阵进行坐标转换.

$$
\boxed{{}^B\vec{x}_A = {}^A_B M \cdot {}^A\vec{x}_A} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = -{}^B\vec{y}_B
$$

$$
{}^C\vec{x}_A = {}^A_C M \cdot {}^A\vec{x}_A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = {}^C\vec{z}_C.
$$

$$
\boxed{{}^B_C M = {}^A_C M \cdot {}^B_A M} = {}^A_C M \cdot {}^A_B M^T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = [{}^C\vec{x}_B \,\vdots\, {}^C\vec{y}_B \,\vdots\, {}^C\vec{z}_B] \quad ∎
$$

⑤ 四元数、旋转矩阵与欧拉角的转换.

基本单位旋转矩阵

$$
R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \quad R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

若 A 系绕其自身 z 轴转 γ 角度, 再绕新的 y、x 轴分别转 β、α, 则表示为 A 按照 zyx 顺序转 (γ, β, α) 得到 B.

$$
\boxed{{}^B_A M = R_z(\gamma) R_y(\beta) R_x(\alpha)}
$$

上面定义的 B、C 中, B 系按照 zyx 顺序转 $(-90°, 90°, 0)$ 得到 C 系, 则

$$
{}^C_B M = R_z(-90°) R_y(90°).
$$

$$
{}^B_C M = [R_z(-90°) R_y(90°)]^T = R_y(90°)^T \cdot R_z(-90°)^T = \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{{}^A_C M} \cdot \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{{}^B_A M} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad ∎
$$

⑥ 两个姿态四元数求相对旋转轴与角度.

例如：以 C 为全局系，已知 A 与 B 相对于 C 的姿态 ${}^{A}_{C}q$、${}^{B}_{C}q$，求从 A 旋转到 B 的旋转轴及转角.

方法一：由四元数构造的定义. ${}^{B}_{A}q = (\cos\frac{\theta_A}{2}, \sin\frac{\theta_A}{2} \cdot {}^{A}\vec{r}) = (\frac{\sqrt{2}}{2}, 0, 0, \frac{\sqrt{2}}{2})$.

$\begin{cases} {}^{A}\vec{r} = (0,0,1) \\ \theta_A = 90^\circ \end{cases}$ ${}^{C}\vec{r} = {}^{A}_{C}q \cdot {}^{A}\vec{r} = (-1,0,0)$ 即绕 C 系中的一个轴 / A 系中的 z 轴转 90°.

方法二：定义 $q = {}^{B}_{C}q \otimes {}^{C}_{A}q = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) \otimes (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0) = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0, 0)$.

取 q 的向量部分 ${}^{C}\vec{r} = (-1,0,0)$，$\theta_C = a\cos(\frac{\sqrt{2}}{2}) \times 2 = 90^\circ = \theta_A$.

即 $\boxed{{}^{B}_{C}q \otimes {}^{C}_{A}q = (\cos\frac{\theta_C}{2}, \sin\frac{\theta_C}{2}\,{}^{C}\vec{r})}$ → 可由证明得到，证略.