Transit Durations

1 The duration of a central transit

To compute duration compared to Earth-Sun (= about 13 hours) we need to compare the speed of the planet with the speed of the Earth, assuming a circular orbit.

1) When moving in a circle, the thing is accelerated with acceration

$$a=\frac{v^2}{r}$$

where v is the speed, r is the radius of the circle (orbit).

2) Gravitational acceleration is

$$a=\frac{G\,M}{r^2}$$

whre G is the gravitational constant and M is the mass of the star.

So for a circular orbit around a star with $M=1$ (has the mass of the Sun) at 1AU we have

$$\frac{v^2}{r}=\frac{G\,M}{r^2}$$

$$v=\sqrt{\frac{G\,M}{r}}$$

but we want speed relative to Earth $\oplus$, orbiting the Sun $\odot$, so compute $v/v_\oplus$:

$$\frac{v}{v_\oplus}=\frac{\sqrt{\frac{G\,M}{r}}}{\sqrt{\frac{G\,M_\odot}{r_\oplus}}}=\left(\frac{M}{M_\odot}\right)^{\frac{1}{2}}\left(\frac{r}{r_\oplus}\right)^{-\frac{1}{2}} \tag{1}$$

where M is the mass of the star, and r is the radius of the orbit. This is almost what we want, but we don't know r.

Now convert the radius of the orbit into period. The circumference of the orbit is $2\,\pi\,r$, and the period of the Earth's orbit is $P_\oplus=365$ days. From above, the speed of the orbit is $v=\sqrt{\frac{G\,M}{r}}$. The time it takes to do one orbit (= the period) is the distance $2\,\pi\,r$ divided by the speed v, so the period is

$$P=\frac{2\,\pi\,r}{\sqrt{\frac{G\,M}{r}}}=2\,\pi\,r\,\sqrt{\frac{r}{G\,M}}=2\,\pi\,\sqrt{\frac{r^3}{G\,M}}.$$

We want the period of the planet relative to the Earth, or

$$\frac{P}{P_\oplus}=\frac{\frac{2\,\pi\,r}{v}}{\frac{2\,\pi\,r_\oplus}{v_\oplus}}=\frac{r}{r_\oplus}\,\frac{v_\oplus}{v}$$

But

$$\frac{P}{P_{\oplus}} = \frac{2\,\pi\,\sqrt{\frac{r^3}{G\,M}}}{2\,\pi\,\sqrt{\frac{r_{\oplus}^3}{G\,M_{\odot}}}} = \sqrt{\frac{\left(\frac{r}{r_{\oplus}}\right)^3}{\frac{M}{M_{\odot}}}}$$

so

$$\left(\frac{r}{r_{\oplus}}\right)^3 = \frac{M}{M_{\odot}}\left(\frac{P}{P_{\oplus}}\right)^2 \Rightarrow \frac{r}{r_{\oplus}} = \left(\frac{M}{M_{\odot}}\right)^{\frac{1}{3}}\left(\frac{P}{P_{\oplus}}\right)^{\frac{2}{3}} \tag{2}$$

(this is Kepler's third law of planetary motion) so combining equations (1) and (2)

$$\begin{aligned}
\frac{v}{v_{\oplus}} &= \left(\frac{M}{M_{\odot}}\right)^{\frac{1}{2}}\left(\frac{r}{r_{\oplus}}\right)^{-\frac{1}{2}} \\
&= \left(\frac{M}{M_{\odot}}\right)^{\frac{1}{2}}\left(\left(\frac{M}{M_{\odot}}\right)^{\frac{1}{3}}\left(\frac{P}{P_{\oplus}}\right)^{\frac{2}{3}}\right)^{-\frac{1}{2}} \\
&= \left(\frac{M}{M_{\odot}}\right)^{\frac{1}{2}}\left(\frac{M}{M_{\odot}}\right)^{-\frac{1}{6}}\left(\frac{P}{P_{\oplus}}\right)^{-\frac{2}{6}} \\
&= \left(\frac{M}{M_{\odot}}\right)^{\frac{1}{2}-\frac{1}{6}}\left(\frac{P}{P_{\oplus}}\right)^{-\frac{1}{3}} \\
&= \left(\frac{M}{M_{\odot}}\right)^{\frac{1}{3}}\left(\frac{P}{P_{\oplus}}\right)^{-\frac{1}{3}}.
\end{aligned}$$

Assuming a central transit, a circular orbit, and that the planet is much smaller than the star, the transit duration D is just the diameter of the star $=2R$ divided by the speed of the planet: $D = 2\,R/v$. So the duration relative to the Earth-Sun system is

$$\begin{aligned}
\frac{D}{D_{\oplus}} &= \frac{2R}{2R_{\odot}}\left(\frac{v}{v_{\oplus}}\right)^{-1} \\
&= \frac{R}{R_{\odot}}\left(\frac{M}{M_{\odot}}\right)^{-\frac{1}{3}}\left(\frac{P}{P_{\oplus}}\right)^{\frac{1}{3}}.
\end{aligned}$$

Let's see how well this works for Kepler-11, which has 6 planets. Since these all orbit the same star, which is about the same size and mass as the Sun, the durations should be approximately related to the periods by $\frac{D}{D_{\oplus}} = \left(\frac{P}{P_{\oplus}}\right)^{\frac{1}{3}}$. The periods are 10.3, 13.0, 22.7, 32.0, 46.7 and 118.4 days. Dividing them all by 365 days, we get 0.028, 0.036, 0.062, 0.088, 0.128 and 0.324. Taking the cube root, we get 0.304, 0.330, 0.396, 0.445, 0.504, and 0.687. Multiplying by the Earth transit duration of $D_{\oplus} = 13$ hours, we get 3.95, 4.29, 5.148, 5.782, 6.552, and 8.93 hours. The measured durations are 4.2, 4.6, 5.5, 4.3, 6.4, and 9.6 hours.

We can be a little more precise by using the mass of the Kepler-11 star $\frac{M}{M_{\odot}} = 0.921$ and radius $\frac{R}{R_{\odot}} = 1.05$, so $\frac{R}{R_{\odot}}\left(\frac{M}{M_{\odot}}\right)^{-\frac{1}{3}} = 1.08$, which makes the predicted durations 4.26, 4.63, 5.56, 6.24, 7.08, and 9.64 hours. We seem to be overestimating some of the durations, but others are right on!

2 Accounting for impact parameter

One reason our predictions are different from the measurements is because the Kepler-11 planets are not transiting across the center of the star: their impact parameters are 0.017, 0.004, 0.023, 0.69, 0.217 and 0.011. The higher the impact parameter, the less of the star the planet has to cross and the shorter the duration.

Let's account for the impact parameter b. b is the ratio of closest distance the planet gets to the center of the star's disk as it transits, to the star's radius R. So by Pythagoras' theorem, the distance the planet travels across the star's disk from where it first hits the star to the closest point to the center is $\sqrt{R^2 - b^2 R^2}$ (draw a picture to see this). So instead of the distance being $2R$ above, the distance is actually $2R\sqrt{1-b^2}$, and the transit duration is

$$\frac{D}{D_\oplus} = \frac{R\sqrt{1-b^2}}{R_\odot}\left(\frac{M}{M_\odot}\right)^{-\frac{1}{3}}\left(\frac{P}{P_\oplus}\right)^{\frac{1}{3}}.$$

Using the impact parameters, we get the impact parameter correction factor $\sqrt{1-b^2} = 0.99986$, 0.999992, 0.99974, 0.7238, 0.9762 and 0.99993. This correction can be ignored because it is very close to 1, except for the 4th and 5th planet, where we get the predicted durations: 4th planet duration = 4.52 hours and 5th planet duration = 6.91 hours. A little hight but not bad!