Contents

Introduction

Occasional references are made throughout these notes to exercises in the textbooks by Zill and Fowles. These books are:

- Zill, D. G., *A First Course in Differential Equations*, PWS Publishing Company, fifth edition, 1993.
- Fowles, G. R., *Analytical Mechanics*, Holt Rinehart Winston, third edition, 1977.

While the exact page references in these notes refer to the editions listed above, earlier editions cover much the same material with slightly different page and section numbering.

Chapter 1

Ordinary Differential Equations

An equation of the form

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$$

involving $y(x)$ and its derivatives is an **ordinary differential equation** (o.d.e.). The **order** of an o.d.e. is the order of the highest derivative that appears. An o.d.e. is **linear** if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x)$$

A **solution** of an o.d.e. is a function $y(x)$ which when substituted into the o.d.e. renders it an identity. An **integral** of an o.d.e. is an implicit relationship connecting y and x which when substituted into the o.d.e. renders it an identity (this is sometimes called an **implicit solution**).

EXAMPLE 1.1
An integral or implicit solution of the o.d.e.

$$\frac{dy}{dx} = -\frac{x}{y}$$

is

$$y^2 + x^2 = C$$

■

1.1 First Order O.D.E.s

The general **first order o.d.e.** is written

$$\frac{dy}{dx} = f(x, y)$$

Note that $F(x, y, \frac{dy}{dx}) = 0$ may not be able to be inverted to obtain $\frac{dy}{dx}$ in terms of x or y, or there may be a multiplicity of solutions, for example, as occurs in

$$\left(\frac{dy}{dx}\right)^2 - g(x, y) = 0$$

The **initial value problem** is to find the solution of an o.d.e. subject to the **initial condition**

$$y(x_0) = y_0$$

1.1.1 Separable Equations

An o.d.e. of the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

is called **separable**. Separable o.d.e.s are solved by considering the indefinite integral

$$G(x) = \int^x g(x)\, dx$$

From the fundamental theorem of calculus,

$$\frac{dG}{dx} = g(x)$$

Similarly, the function

$$H(x) = \int^x h(x)\, dx$$

obeys

$$\frac{dH}{dx} = h(x)$$

Thus, using chain rule,

$$\frac{d}{dx} H\left(y(x)\right) = \frac{dH}{dy}\frac{dy}{dx} = h(y)\frac{dy}{dx} = g(x)$$

Using $g(x) = \frac{dG}{dx}$ shows that

$$\frac{d}{dx} H\left(y(x)\right) = \frac{d}{dx} G(x)$$

Integrating both sides with respect to x shows that

$$H\left(y(x)\right) = G(x) + C$$

Therefore the general solution is

$$\int^{y} h(y)\, dy = \int^{x} g(x)\, dx + C$$

This is in the form of an integral or implicit solution of the o.d.e..

EXAMPLE 1.2
For these examples, $h(y) = 1$ so that the separable equation has the form

$$\frac{dy}{dx} = g(x)$$

$$\begin{aligned}
\frac{dy}{dx} = x^{n} \quad &\Longrightarrow \quad y(x) = \frac{x^{n+1}}{n+1} + C \\
\frac{dy}{dx} = \sin ax \quad &\Longrightarrow \quad y(x) = -\frac{1}{a}\cos ax + C \\
\frac{dy}{dx} = B\, e^{ax} \quad &\Longrightarrow \quad y(x) = \frac{B}{a}\, e^{ax} + C \\
\frac{dy}{dx} = \tan ax \quad &\Longrightarrow \quad y(x) = -\frac{1}{a}\ln|\cos ax| + C
\end{aligned}$$

*These are all examples of **explicit solutions** because the solution $y(x)$ is given as a function of x.* ■

Note that C, the **constant of integration**, is determined by the problem's initial conditions.

EXAMPLE 1.3
To solve

$$\frac{dy}{dx} = \frac{y}{1+x}$$

note that $h(y) = 1/y$ and $g(x) = 1/(1+x)$. Then

$$\frac{dy}{y} = \frac{dx}{1+x}$$

Integrating both sides gives

$$\int^{y} \frac{dy}{y} = \int^{x} \frac{dx}{1+x} + C$$

which means that

$$\ln|y| = \ln|1+x| + C$$

Then by writing $C = \ln|A|$ *and removing the logs, the solution is*

$$y = A(1+x)$$

■

EXAMPLE 1.4
To solve

$$\frac{dy}{dx} = -\frac{x}{y}$$

rewrite the o.d.e. as

$$y\,dy = -x\,dx$$

Integrating both sides gives the solution

$$y^2 = -x^2 + C$$

which is the equation for a circle of radius $\sqrt{C}$

$$x^2 + y^2 = C$$

■

EXAMPLE 1.5
To solve

$$\frac{dy}{dx} = y^2 - a^2$$

rewrite it as

$$\frac{dy}{y^2 - a^2} = dx$$

Integrating both sides gives

$$\int^y \frac{dy}{y^2 - a^2} = x + C$$

Using partial fractions, the integrand is equal to

$$\frac{1}{y^2 - a^2} = \frac{1}{2a}\left(\frac{1}{y-a} - \frac{1}{y+a}\right)$$

Therefore

$$\begin{aligned}\int^y \frac{dy}{y^2-a^2} &= \frac{1}{2a}\int^y \frac{1}{y-a}-\frac{1}{y+a}\,dy \\ &= \frac{1}{2a}\ln\left|\frac{y-a}{y+a}\right| \\ &= x+C\end{aligned}$$

Now writing $|A| = e^{2aC}$ *gives the solution*

$$\frac{y-a}{y+a} = A\,e^{2ax}$$

which as a function $y(x)$ *is*

$$y = a\frac{1+A\,e^{2ax}}{1-A\,e^{2ax}}$$

■

EXAMPLE 1.6
The ***Doomsday Model*** *is*

$$\frac{dP}{dt} = kP$$

with initial condition $P(0) = P_0$. *The general solution is*

$$\begin{aligned}\int^P \frac{dP}{P} &= k\int^t dt + C \\ \therefore\ \ln P &= kt + C\end{aligned}$$

At $t = 0$, *the initial condition can be used to determine* C

$$\ln P_0 = 0 + C$$

so that

$$\ln\frac{P}{P_0} = kt$$

or

$$P(t) = P_0\,e^{kt}$$

■

EXAMPLE 1.7
The ***logistic equation*** *is*

$$\frac{dP}{dt} = kP(1 - bP)$$

with initial condition $P(0) = P_0$. The solution is found using partial fractions

$$\begin{aligned} \int^P \frac{dP}{P(1-bP)} &= k\int^t dt + C \\ \therefore \ \ln\frac{P}{1-bP} &= kt + C \end{aligned}$$

At $t = 0$, the initial condition can be used to determine C, giving the solution

$$\frac{P(t)}{1 - bP(t)} = \frac{P_0}{1 - bP_0}\, e^{kt}$$

which can easily be rearranged to give an explicit solution for $P(t)$ in terms of t. ■

Exercise 2.2 of Zill, pp. 44–46, has more examples of separable o.d.e.s.

1.1.2 Homogeneous Equations

If a function $f(x, y)$ satisfies

$$f(tx, ty) = t^n f(x, y)$$

for all x and y and for some constant real number n, then $f(x, y)$ is said to be a **homogeneous** function of degree n.

EXAMPLE 1.8
Here are some examples of functions that are homogeneous or inhomogeneous:

$$\begin{array}{rl} ax^2 + bxy + cy^2 & \textit{homogenous } (n = 2) \\ \dfrac{ax + by}{cx + dy} & \textit{homogenous } (n = 0) \\ \cos(ax^2 + by^2) & \textit{inhomogenous} \\ \exp\left(\dfrac{ax^2 + by^2}{cx^2 + dy^2}\right) & \textit{homogenous } (n = 0) \end{array}$$

■

An o.d.e. of the form

$$N(x,y)\,\frac{dy}{dx} + M(x,y) = 0$$

where $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree n is called a **homogeneous** o.d.e.. Homogeneous o.d.e.s can be turned into separable o.d.e.s with the substitution

$$u = \frac{y}{x}$$

When using this substitution, note what happens to $M(x,y)$, $N(x,y)$ and $\frac{dy}{dx}$:

$$\begin{aligned} M(x,y) &= M(x,xu) = x^n\,M(1,u) \\ N(x,y) &= N(x,xu) = x^n\,N(1,u) \\ \frac{dy}{dx} &= \frac{d(xu)}{dx} = x\frac{du}{dx} + u \end{aligned}$$

Now, substituting these into the o.d.e., we have

$$0 = x^n\,M(1,u) + x^n\,N(1,u)\left(x\frac{du}{dx} + u\right)$$

which can be rearranged to give

$$\left(\frac{N(1,u)}{M(1,u) + uN(1,u)}\right)\frac{du}{dx} = -\frac{1}{x}$$

which is a separable o.d.e. in u and x.

Note that sometimes it may be simpler to use the alternative substitution

$$v = \frac{x}{y}$$

which gives a different separable equation.

Never try to remember these final equations; learn the principle and rederive each time starting with the substitution $u = y/x$ or $v = x/y$.

EXAMPLE 1.9

To solve

$$\frac{dy}{dx} = \frac{x+3y}{3x+y} = \frac{1+3y/x}{3+y/x}$$

put $u = y/x$ *so that* $y = xu$ *and*

$$x\,\frac{du}{dx} + u = \frac{dy}{dx}$$

Therefore

$$x\,\frac{du}{dx}+u=\frac{1+3u}{3+u}$$

so that

$$x\,\frac{du}{dx}=\frac{1+3u}{3+u}-u=\frac{1-u^2}{3+u}$$

Separating the terms involving x and u,

$$\int^u \frac{3+u}{1-u^2}\,du=\int^x\frac{dx}{x}+C$$

Using partial fractions,

$$\begin{aligned}\int^u \frac{1}{1+u}+\frac{2}{1-u}\,du &= \ln|x|+C\\ \therefore\ \ln\left|\frac{1+u}{(1-u)^2}\right| &= \ln|x|+C\end{aligned}$$

Writing the constant of integration as $C=\ln|A|$ gives the solution

$$\frac{1+u}{(1-u)^2}=Ax$$

■

Exercise 2.3 of Zill, pp. 52–53, has more examples of homogeneous o.d.e.s.

1.1.3 Linear First Order O.D.E.s

An o.d.e. of the form

$$\frac{dy}{dx}+p(x)\,y=f(x)$$

is a **linear** first order o.d.e.. When $f(x)=0$, the o.d.e. is **homogeneous**. When $f(x)\neq 0$, the o.d.e. is **inhomogeneous**.

One **trivial solution** of an homogeneous linear o.d.e. is always $y=0$. This usually need not be considered since the initial conditions will usually be non-zero.

To solve a linear first order o.d.e., use the **integrating factor**, $I(x)$. $I(x)$ is defined to be any solution of

$$\frac{dI}{dx}=p(x)\,I$$

This is a separable equation which is solved using

$$\int^I \frac{dI}{I} = \int^x p(x)\,dx + C$$

so that

$$\ln I = \int^x p(x)\,dx + C$$

Set $C = 0$ so that the integrating factor is

$$I(x) = \exp\left(\int^x p(x)\,dx\right)$$

Note that $y = 1/I(x)$ is the solution of the homogeneous equation

$$\frac{dy}{dx} + p(x)\,y = 0$$

To prove this, substitute $y = 1/I(x)$ so that

$$\frac{dy}{dx} = -\frac{1}{I^2}\frac{dI}{dx}$$

Now substituting this into the homogeneous equation shows that

$$\frac{dy}{dx} + py = -\frac{1}{I^2}\left(\frac{dI}{dx} - pI\right) = 0$$

where the term in parentheses is zero from the definition of the integrating factor.

Now, to complete the solution of the o.d.e.

$$\frac{dy}{dx} + p(x)\,y = f(x)$$

multiply both sides by $I(x)$ to give

$$I(x)\frac{dy}{dx} + p(x)I(x)\,y = I(x)f(x)$$

But from the definition of the integrating factor, $p(x)I(x) = \frac{dI}{dx}$, so that

$$I(x)\frac{dy}{dx} + \frac{dI}{dx}\,y = I(x)f(x)$$

Using product rule, the left hand side is seen to be $\frac{d}{dx}(Iy)$, hence

$$\frac{d}{dx}\left[I(x)\,y\right] = I(x)f(x)$$

Integrating to remove the derivative gives the solution

$$I(x)\, y = \int^x I(x) f(x)\, dx + C$$

which is

$$y = C\frac{1}{I(x)} + \frac{1}{I(x)} \int^x I(x) f(x)\, dx$$

The first part of this solution is C times the solution of the homogeneous o.d.e., and the second part is a **particular solution** which exists because $f(x) \neq 0$.

EXAMPLE 1.10

To solve the o.d.e.

$$x^2 \frac{dy}{dx} + xy = 1$$

divide by x^2 *to turn it into standard form*

$$\frac{dy}{dx} + \frac{1}{x}\, y = \frac{1}{x^2}$$

The integrating factor is

$$I(x) = x$$

found by solving

$$\frac{dI}{dx} = \frac{1}{x}\, I$$

and setting the constant in

$$\int^x \frac{dI}{I} = \int^x \frac{dx}{x} + C$$

to zero.

Now solve the o.d.e. by multiplying it by the integrating factor

$$x\, \frac{dy}{dx} + y = \frac{1}{x}$$

Using product rule, the left hand side is

$$x\, \frac{dy}{dx} + y = \frac{d}{dx}\,(xy)$$

so that

$$xy = \ln|x| + C$$

Therefore the solution is

$$y(x) = \frac{\ln|x|}{x} + \frac{C}{x}$$

■

EXAMPLE 1.11
To solve the o.d.e.

$$(1-\cos x)\frac{dy}{dx}+2y\sin x-\tan x=0$$

put it in standard form, so that

$$\frac{dy}{dx}+\left(\frac{2\sin x}{1-\cos x}\right)y=\frac{\tan x}{1-\cos x}$$

To find the integrating factor, solve $\frac{dI}{dx}=p(x)I$.

$$\begin{aligned}\int^I\frac{dI}{I} &= \int^x\frac{2\sin x}{1-\cos x}\,dx\\ \therefore\ \ln I &= 2\ln(1-\cos x)\end{aligned}$$

where the constant of integration has been set to zero. Therefore

$$I(x)=(1-\cos x)^2$$

Now, solve the o.d.e. by multiplying by the integrating factor. This gives

$$\frac{d}{dx}(Iy)=(1-\cos x)^2\frac{\tan x}{1-\cos x}=(1-\cos x)\tan x$$

Integrate with respect to x

$$\begin{aligned}Iy &= \int^x(1-\cos x)\tan x\,dx+C\\ &= \int^x(\tan x-\sin x)\,dx+C\\ &= -\int^x\frac{d(\cos x)}{\cos x}+\cos x+C\\ &= -\ln|\cos x|+\cos x+C\end{aligned}$$

Dividing both sides by $I(x)=(1-\cos x)^2$ *gives the solution*

$$y(x)=\frac{C}{(1-\cos x)^2}+\frac{\cos x-\ln|\cos x|}{(1-\cos x)^2}$$

which is of the form of C *times the homogeneous solution plus the particular solution due to* $f(x)$. ■

EXAMPLE 1.12
Solving

$$\frac{dy}{dx} = \frac{y}{y^3 + x}$$

is a little harder because the o.d.e. is neither linear, homogeneous nor separable. But by inverting it and rearranging, it becomes a linear o.d.e. in y

$$\frac{dx}{dy} = \frac{y^3 + x}{y} \implies \frac{dx}{dy} - \frac{1}{y}x = y^2$$

The solution gives x *as a function of* y*. The integrating factor* $I(y)$ *satisfies*

$$\frac{dI}{dy} = -\frac{1}{y}I$$

so that

$$I(y) = \frac{1}{y}$$

Multiply both sides of the o.d.e. by $I(y)$ *and use product rule on the left hand side to give*

$$\frac{d}{dy}\left(\frac{x}{y}\right) = y$$

Integrate with respect to y

$$\frac{x}{y} = \int^y y\,dy + C$$

to show that the solution is

$$x = Cy + \frac{y^3}{2}$$

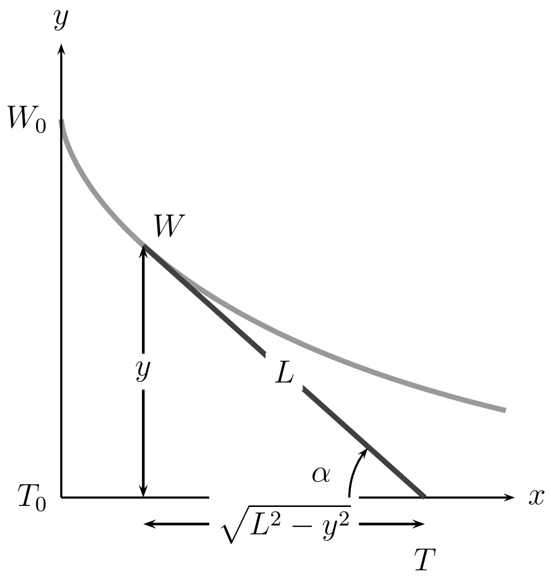
Once again, this takes the form of C *times the homogeneous solution plus the particular solution due to* $f(x)$*.* ■

Exercise 2.5 of Zill, pp. 69–71, has more examples of linear o.d.e.s.

1.2 Applications of First Order O.D.E.s

EXAMPLE 1.13 (The Tractrix)
A tractor is connected by a taut chain of length L to a weight. The tractor begins to move in a direction at right angles to the line joining the tractor

Figure 1.1: The curve $y(x)$ is traced out by the weight W dragged by a tractor T in the tractrix problem of example 1.13.



and the weight. It proceeds in that direction in a straight line dragging the weight. What curve does the weight trace out?

Solution: *The geometry of this problem is illustrated in figure 1.1. The tractor is the point T and the weight the point W. At time $t = 0$, the tractor is at the origin, $(0, 0)$, and it moves along the x axis. Since the weight is initially at right angles to the direction of the tractor's motion, the weight is at $(0, L)$ at time $t = 0$. Let the curve traced out by the weight be given by $y(x)$, where $y(0) = L$.*

The key observation is that the weight moves in the direction of the chain. Therefore, the chain is always a tangent to the curve $y(x)$. So if the weight is at the point (x, y) when the chain is at an angle α to the x axis, the gradient of the tangent of $y(x)$ satisfies

$$\frac{dy}{dx} = -\tan\alpha = -\frac{y}{\sqrt{L^2 - y^2}}$$

This has been obtained by using figure 1.1 and remembering that the chain's length is L.

Therefore the differential equation to be solved is

$$\frac{dy}{dx} = -\frac{y}{\sqrt{L^2 - y^2}}$$

with the initial condition

$$y(0) = L$$

The o.d.e. is separable, so

$$\int^y \frac{\sqrt{L^2 - y^2}}{y}\, dy = -\int^x dx + C = -x + C$$

To solve this integral, make the substitution $z^2 = L^2 - y^2$. *Then* $z\, dz = -y\, dy$.

$$\begin{aligned}
x &= C - \int^y \frac{\sqrt{L^2 - y^2}}{y}\, dy \\
&= C + \int^{\sqrt{L^2-y^2}} \frac{z^2}{L^2 - y^2}\, dz \\
&= C + \int^{\sqrt{L^2-y^2}} \left(-1 + \frac{L^2}{L^2 - z^2}\right) dz \\
&= C + \int^{\sqrt{L^2-y^2}} \left[-1 + \frac{L}{2}\left(\frac{1}{L-z} + \frac{1}{L+z}\right)\right] dz \\
&= C - \sqrt{L^2 - y^2} + \frac{L}{2}\left(-\ln\left(L - \sqrt{L^2 - y^2}\right) + \ln\left(L + \sqrt{L^2 - y^2}\right)\right) \\
&= C - \sqrt{L^2 - y^2} + \frac{L}{2}\ln\left(\frac{L + \sqrt{L^2 - y^2}}{L - \sqrt{L^2 - y^2}}\right)
\end{aligned}$$

Applying the initial conditions of $y(0) = L$ *shows that*

$$0 = C - 0 + \frac{L}{2}\ln\left(\frac{L}{L}\right) = C + 0$$

so that $C = 0$.

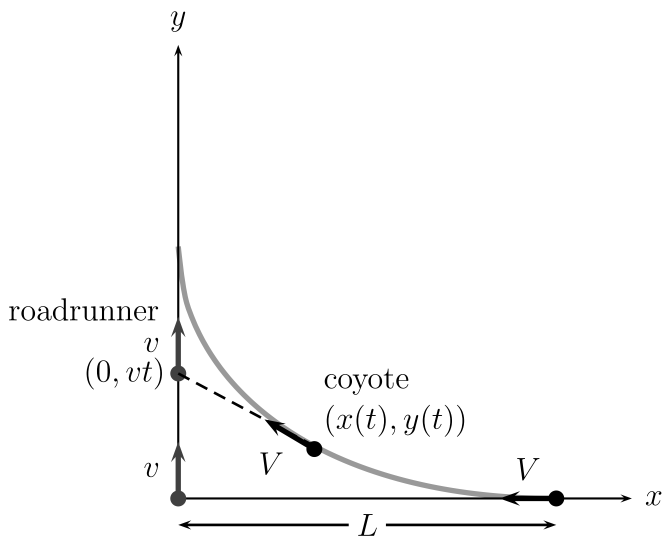
Therefore the equation of the tractrix is

$$x = \frac{L}{2}\ln\left(\frac{L + \sqrt{L^2 - y^2}}{L - \sqrt{L^2 - y^2}}\right) - \sqrt{L^2 - y^2}$$

■

Now try the tractrix problem again with the tractor heading off at an angle θ (which is greater than $\pi/2$) to the initial direction of the chain.

Figure 1.2: The curve $y(x)$ is traced out by the coyote as it runs towards the roadrunner in the pursuit curve problem of example 1.14. (For this example, $\lambda = 1/2$ so the coyote catches the roadrunner)



EXAMPLE 1.14 (The Pursuit Curve)
A roadrunner commences running at constant speed v in a northerly direction. A coyote a distance L away to the east scents the roadrunner and immediately gives chase at constant speed V, always running directly at the roadrunner. What path does the coyote chase out?

Solution: *The diagram in figure 1.2 shows the roadrunner running up the y axis at speed v, starting at the origin at time $t = 0$. At time $t = 0$, the coyote is on the x axis at the point $(L, 0)$, and moves towards the roadrunner with speed V. Let the function $y(x)$ describe the path of the coyote.*

Since the coyote always runs directly towards the roadrunner, the coyote's velocity vector, which is the tangent to the pursuit curve, intercepts the y axis at the point $(0, vt)$. Therefore

$$\frac{dy}{dx} = \frac{y - vt}{x}$$

so that

$$x\,\frac{dy}{dx} = y - vt$$

The time variable t is eliminated by considering the arc length s of the pursuit *curve. The length of the curve at time t is*

$$s = Vt$$

which is the total distance travelled by the coyote at time t.

Using $ds^2 = dx^2 + dy^2$, $\frac{ds}{dx}$ is

$$\frac{ds}{dx} = -\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

The negative square root has been chosen because s increases as x decreases.

From $s = Vt$, write $t = s/V$ and eliminate t to give

$$x\,\frac{dy}{dx} = y - \frac{v}{V}s$$

For convenience, write the ratio of the roadrunner's and coyote's speeds as

$$\lambda = \frac{v}{V}$$

so that

$$x\,\frac{dy}{dx} = y - \lambda s$$

Differentiate both sides with respect to x

$$\frac{d}{dx}\left(x\,\frac{dy}{dx}\right) = \frac{dy}{dx} - \lambda\,\frac{ds}{dx}$$

Now substitute for $\frac{ds}{dx}$

$$\frac{dy}{dx} + x\frac{d^2y}{dx^2} = \frac{dy}{dx} + \lambda\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Therefore the equation of the pursuit curve is

$$x\frac{d^2y}{dx^2} = \lambda\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

This equation for the pursuit curve appears to be a second order o.d.e.. However make the substitution

$$w = \frac{dy}{dx}$$

so that the o.d.e. becomes a first order separable o.d.e.

$$x\frac{dw}{dx} = \lambda\sqrt{1+w^2}$$

This is solved by separating and integrating to give w as a function of x. Then integrating a second time gives y in terms of x, as required.

Separating the separable equation in x and w gives

$$\int^w \frac{dw}{\sqrt{1+w^2}} = \lambda\int^x \frac{dx}{x} + C$$

Make the substitution $w = \sinh u$ so that $1 + w^2 = \cosh^2 u$ and

$$du = \frac{dw}{\sqrt{1+w^2}}$$

Therefore

$$\int^{\operatorname{arcsinh} w} du = \operatorname{arcsinh} w = \lambda \ln x + C$$

To find the value of C, note that at time $t = 0$ when $x = L$,

$$w = \frac{dy}{dx} = 0$$

Therefore $0 = \lambda \ln L + C$ so

$$\operatorname{arcsinh} w = \ln\left(\frac{x}{L}\right)^{\lambda}$$

Invert the equation to give w as an explicit function of x

$$w = \sinh\left(\ln\left(\frac{x}{L}\right)^{\lambda}\right)$$

Then use the expression

$$\sinh z = \frac{1}{2}\left(e^z - e^{-z}\right)$$

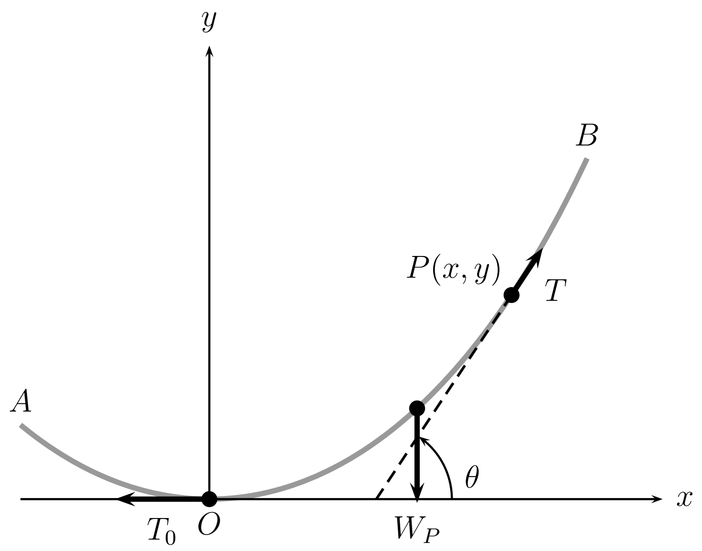
to write $w = \frac{dy}{dx}$ as

$$\frac{dy}{dx} = \frac{1}{2}\left[\left(\frac{x}{L}\right)^{\lambda} - \left(\frac{L}{x}\right)^{\lambda}\right]$$

Integrating $\frac{dy}{dx}$ with respect to x gives the solution

$$y = \frac{1}{2(\lambda+1)}\frac{x^{\lambda+1}}{L^{\lambda}} + \frac{1}{2(\lambda-1)}\frac{L^{\lambda}}{x^{\lambda-1}} + C$$

Figure 1.3: The catenary is the shape made by a hanging chain, as in example 1.15.



provided $\lambda \neq 1$. The constant C is determined by the initial condition $y = 0$ at $x = L$ so that

$$C = -\frac{L\lambda}{\lambda^2 - 1}$$

■

EXAMPLE 1.15 (The Catenary)
A flexible chain of mass ρ per unit length hangs under gravity from points A and B. Find the shape of the chain.
Solution: *At the lowest point in the chain, its slope is zero. Take this point as the origin. If the chain takes the shape given by $y(x)$, consider the slope of the chain at some point $P(x, y)$, as shown in figure 1.3.*

Define T as the tension in the chain at $P = (x, y)$, T_0 as the tension in the chain at the origin $O = (0, 0)$, and W_P as the weight of the segment of chain from the origin to the point P. The weight W_P is

$$W_P = \rho g s$$

where ρ is the known mass per unit length of the chain, g is the acceleration due to gravity and s is the length of the arc OP. The tangent to the chain at P makes an angle θ with the x axis.

The derivative of s is given in terms of the chain's shape $y(x)$ by

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Here the positive square root has been chosen because s increases with x if P is on the right hand side of the y axis. If P were on the other side of the y axis, the negative square root would have been chosen.

Resolving the horizontal forces on the chain segment OP shows that

$$T_0 = T\cos\theta$$

and resolving vertically shows that

$$W_P = T\sin\theta$$

Dividing these two gives

$$\tan\theta = \frac{W_P}{T_0} = \frac{\rho g}{T_0}\, s$$

But θ is the angle of the tangent to the curve so

$$\frac{dy}{dx} = \tan\theta = \frac{\rho g}{T_0}\, s$$

Write $\lambda = \rho g/T_0$ to simplify the notation, and differentiate both sides with respect to x so that

$$\frac{d^2y}{dx^2} = \lambda\frac{ds}{dx} = \lambda\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

which is the equation of the catenary.

The solution of the catenary is found by putting $p = \frac{dy}{dx}$ and solving the first order separable equation in p, then integrating p with respect to x to give y as a function of x.

With $p = \frac{dy}{dx}$, differentiating gives $\frac{dp}{dx} = \frac{d^2y}{dx^2}$ so that the catenary's differential equation becomes

$$\frac{dp}{dx} = \lambda\sqrt{1+p^2}$$

which is separable with

$$\int^p \frac{dp}{\sqrt{1+p^2}} = \lambda\int^x dx + C$$

Make the substitution $p = \sinh u$ *so that* $1+p^2 = \cosh^2 u$ *and* $dp = \cosh u\, du$. *Integrating gives*

$$\operatorname{arcsinh} p = \lambda x + C$$

which is

$$\frac{dy}{dx} = p = \sinh(\lambda x + C)$$

The constant of integration has to be chosen so that $\frac{dy}{dx} = 0$ *at the origin, because the origin was placed at the lowest point of the catenary. Substituting in* $x = 0$ *and* $\frac{dy}{dx} = 0$ *shows that the constant is* $C = 0$. *Therefore*

$$\frac{dy}{dx} = \sinh \lambda x$$

Integrating with respect to x *gives*

$$y = \frac{1}{\lambda}\cosh \lambda x + C$$

Once again, the catenary passes through the origin, so $y(0) = 0$. *The constant is* $C = -1/\lambda$, *giving the equation of the catenary as*

$$y = \frac{1}{\lambda}\left[\cosh(\lambda x) - 1\right]$$

This is still not quite the complete solution because $\lambda = \rho g/T_0$, *the tension at the origin* T_0, *has not yet been determined. This depends on the length of the chain and the position of the endpoints relative to the origin.*

Another way of solving the catenary starts with the equation

$$\frac{d^2y}{dx^2} = \lambda\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

and uses the identity

$$\frac{d}{dy}\left[\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right] = \frac{d^2y}{dx^2}$$

to write the catenary's differential equation as

$$\frac{d}{dy}\left[\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right] = \lambda\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Now make the substitution

$$Q = 1 + \left(\frac{dy}{dx}\right)^2$$

so that the catenary's equation becomes

$$\frac{dQ}{dy} = 2\lambda\sqrt{Q}$$

This is separable, so

$$\begin{aligned}\int^Q \frac{dQ}{\sqrt{Q}} &= 2\lambda \int^y y\,dy + C \\ \therefore\ 2\sqrt{Q} &= 2\lambda y + C\end{aligned}$$

At the origin, $Q = 1$ because $\frac{dy}{dx} = 0$. Therefore $C = 2$ and

$$\sqrt{Q} = \lambda y + 1$$

Squaring both sides gives

$$1 + \left(\frac{dy}{dx}\right)^2 = (1 + \lambda y)^2$$

Move the 1 to the right-hand side and take the square root, giving

$$\frac{dy}{dx} = \sqrt{(1 + \lambda y)^2 - 1}$$

The positive square root has been chosen so that $\frac{dy}{dx} > 0$ for $x > 0$ (the negative root would have been chosen for the part of the catenary with $x < 0$). This is separable, so separate and integrate

$$\int^y \frac{dy}{\sqrt{(1 + \lambda y)^2 - 1}} = \int^x dx + C$$

Make the substitution $1 + \lambda y = \cosh u$ to show that the solution is

$$\frac{1}{\lambda}\operatorname{arccosh}(1 + \lambda y) = x + C$$

which is

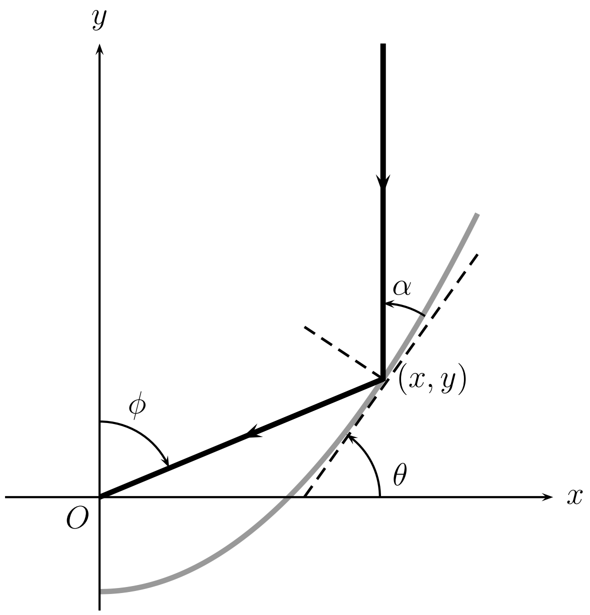
$$1 + \lambda y = \cosh \lambda(x + C)$$

Using the boundary condition that $y(0) = 0$ shows that $C = 0$ so the equation of the catenary is

$$y = \frac{1}{\lambda}\left[\cosh(\lambda x) - 1\right]$$

which is the same as the first solution. ■

Figure 1.4: The mirror $y(x)$ is shaped so that beams of light parallel to the y axis are reflected to the origin, as described in example 1.16.



EXAMPLE 1.16 (Perfect Mirror Focus)
Light strikes a plane curve in such a manner that all beams parallel to the y axis are reflected to a single point O. Determine the shape of the curve.
Solution: *This problem is described in figure 1.4, where the point O has been placed at the origin. Using the property of a mirror that the angle of incidence is equal to the angle of reflection,*

$$(\pi - \phi) + 2\alpha = \pi$$

Thus $\phi = 2\alpha$.

With θ the angle the tangent to the mirror makes with the x axis, the geometry requires that

$$(\pi - \phi) + \alpha + \left(\frac{\pi}{2} - \theta\right) = \pi$$

which shows that $\theta = \frac{\pi}{2} - \alpha$.

The mirror's equation is given by $y(x)$, so that from figure 1.4,

$$\tan\left(\frac{\pi}{2} - \phi\right) = \frac{y}{x}$$

Writing ϕ in terms of θ gives the condition

$$\frac{y}{x} = \tan\left(2\theta - \frac{\pi}{2}\right)$$

Now

$$\tan\left(2\theta - \frac{\pi}{2}\right) = -\cot 2\theta = -\frac{1-\tan^2\theta}{2\tan\theta}$$

Since this is equal to y/x *and using* $\tan\theta = \frac{dy}{dx}$*, the differential equation describing the mirror is*

$$\frac{y}{x} = -\frac{1-\left(\frac{dy}{dx}\right)^2}{2\frac{dy}{dx}}$$

Rearrange this to give

$$2\frac{y}{x}\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2 = -1$$

The solution is found by putting $w = x^2$ *so that*

$$\frac{dy}{dw} = \frac{dy}{dx}\frac{dx}{dw} = \frac{1}{2x}\frac{dy}{dx}$$

Therefore the differential equation becomes

$$4y\frac{dy}{dw} - 4w\left(\frac{dy}{dw}\right)^2 = -1$$

Divide both sides by $4\left(\frac{dy}{dw}\right)^2$ *to give*

$$y\frac{dw}{dy} - w = -\frac{1}{4}\left(\frac{dw}{dy}\right)^2$$

This is an example of Clairaut's equation

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$$

which is described in detail in section 1.3.3 and chapter 2.6 of Zill. The solution is

$$y = mx + f(m)$$

where m *is an arbitrary constant. Applying this to the mirror's differential equation, the solution is*

$$w = my + \frac{1}{4}m^2$$

Since $w = x^2$*, this is*

$$y = \frac{x^2}{m} - \frac{m}{4}$$

which is the equation of a parabola.

The shape of the mirror can also be found without using the substitution leading to Clairaut's equation. Starting with

$$2\frac{y}{x}\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2 = -1$$

divide by $\frac{1}{x}\left(\frac{dy}{dx}\right)^2$ *to obtain*

$$x\left(\frac{dx}{dy}\right)^2 + 2y\frac{dx}{dy} - x = 0$$

This is a quadratic in $\frac{dx}{dy}$ *with roots*

$$\frac{dx}{dy} = \frac{-y \pm \sqrt{x^2+y^2}}{x}$$

Rearrange this to give

$$\pm\frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = dy$$

Now $x\,dx + y\,dy = \frac{1}{2}d(x^2+y^2)$ *so*

$$\pm\frac{\frac{1}{2}d(x^2+y^2)}{\sqrt{x^2+y^2}} = dy$$

Integrate both sides

$$\pm\sqrt{x^2+y^2} = y + C$$

and square so that

$$x^2 + y^2 = (y+C)^2 = y^2 + 2yC + C^2$$

Make y *the subject*

$$y = \frac{x^2}{2C} - \frac{C}{2}$$

and put $m = 2C$ *to give the answer in the same form as before*

$$y = \frac{x^2}{m} - \frac{m}{4}$$

■

Chapters 1 and 3 of Zill have problems involving applications of first order o.d.e.s.

1.3 Special Nonlinear First Order O.D.E.s

1.3.1 Bernoulli's Equation

An o.d.e. of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

where n is any real number, is called **Bernoulli's equation** (named after James Bernoulli (1654–1705)).

Note that we can already solve this equation for $n = 0$ and $n = 1$. When $n = 0$, the equation is an inhomogeneous linear o.d.e.. When $n = 1$, the equation is a homogeneous linear o.d.e..

To solve Bernoulli's equation, make the substitution

$$w = y^{1-n}$$

to obtain a linear expression. Since

$$y = w^{\frac{1}{1-n}}$$

differentiating y with respect to x gives

$$\frac{dy}{dx} = \frac{1}{1-n} w^{\frac{1}{1-n}-1} \frac{dw}{dx}$$

Substitute this into the differential equation

$$\frac{1}{1-n} w^{\frac{n}{1-n}} \frac{dw}{dx} + P(x) w^{\frac{1}{1-n}} = f(x) w^{\frac{n}{1-n}}$$

then multiply each side by $1-n$ and divide by $w^{\frac{n}{1-n}}$. This results in the following linear o.d.e.

$$\frac{dw}{dx} + (1-n)P(x)w = f(x)(1-n)$$

which can easily be solved.

EXAMPLE 1.17

To solve

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2$$

note that this is Bernoulli's equation with $P(x) = 1/x$, $f(x) = x$ and $n = 2$. Try the substitution $w = y^{1-n} = y^{1-2} = 1/y$ so that

$$y = \frac{1}{w}$$

Then

$$\frac{dy}{dx} = -\frac{1}{w^2}\frac{dw}{dx}$$

which gets substituted into the o.d.e. giving

$$-\frac{1}{w^2}\frac{dw}{dx} + \frac{1}{xw} = \frac{x}{w^2}$$

Upon rearranging, we have

$$\frac{dw}{dx} - \frac{1}{x}w = -x$$

The integrating factor is found by solving

$$\frac{dI}{dx} = -\frac{1}{x}I$$

so that $\ln I = -\ln x$ *or* $I = 1/x$*. Multiply the o.d.e. by the integrating factor to obtain*

$$\frac{d}{dx}\left[\frac{w}{x}\right] = -1$$

whence

$$\frac{w}{x} = -x + c$$

so that

$$w = cx - x^2$$

Expressed in terms of y *rather than* w*, this is*

$$y = \frac{1}{cx - x^2}$$

■

Exercise 2.6 of Zill, pp. 75–76, has more examples of Bernoulli's equation.

1.3.2 Ricatti's Equation

An o.d.e. of the form

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$$

is called **Ricatti's equation** (named after Count Jacobo Francesco Ricatti (1676–1754)). Note that

- For many $P(x)$, $Q(x)$ and $R(x)$ the solution cannot be expressed in terms of elementary functions.
- If $y_1(x)$ is a known solution of Ricatti's equation, then a family of solutions is
$$y(x) = y_1(x) + u(x)$$
where $u(x)$ satisfies
$$\frac{du}{dx} - (Q + 2y_1 R)u = Ru^2$$
which is Bernoulli's equation with $n = 2$.
- When $P(x) = 0$, Ricatti's equation reduces to a Bernoulli equation with $n = 2$.

EXAMPLE 1.18
To solve
$$\frac{dy}{dx} = 2 - 2xy + y^2$$
given that $y_1 = 2x$ is one solution, put
$$y = 2x + u$$
Then
$$\frac{dy}{dx} = 2 + \frac{du}{dx}$$
so that the differential equation becomes
$$2 + \frac{du}{dx} = 2 - 2x(2x + u) + (2x + u)^2$$
which simplifies to
$$\frac{du}{dx} = 2xu + u^2$$
To solve this Bernoulli equation, put $w = u^{1-2} = 1/u$ so that $u = 1/w$. Substitute this into the o.d.e. to obtain
$$-\frac{1}{w^2}\frac{dw}{dx} = \frac{2x}{w} + \frac{1}{w^2}$$
This is the first order linear o.d.e.
$$\frac{dw}{dx} + 2xw = -1$$

The integrating factor is found by solving

$$\frac{dI}{dx} = 2xI$$

hence

$$I(x) = e^{x^2}$$

Multiply the o.d.e. by the integrating factor to give

$$\frac{d}{dx}\left[e^{x^2}w\right] = -e^{x^2}$$

Integrating gives

$$w = -e^{-x^2}\int^x e^{t^2}\,dt + Ce^{-x^2}$$

Use $u = 1/w$ and $y = 2x + u$ to give the final solution

$$y(x) = 2x + \frac{e^{x^2}}{C - \int^x e^{t^2}\,dt}$$

which cannot be represented in terms of elementary functions without the integral. ■

Exercise 2.6 of Zill, pp. 75–76, has more examples of Ricatti's equation.

1.3.3 Clairaut's Equation

An o.d.e. of the form

$$F\left(y - x\frac{dy}{dx}, \frac{dy}{dx}\right) = 0$$

or alternatively

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$$

is a **Clairaut equation** (named after Alexis Claude Clairaut (1713–1765)).

There is a family of solutions that are straight lines, and there may also be a second solution that is singular.

The family of solutions that are straight lines have the form

$$y = mx + f(m)$$

where each value of m gives a different solution. To see that these are solutions of the Clairaut equation, substitute into the o.d.e.. From $y = mx + f(m)$,

$$\frac{dy}{dx} = m$$

so that

$$x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right) = xm + f(m) = y$$

as required.

Clairaut's equation can possess a second solution, the **singular solution**, which is not obtainable from the general solution $y = mx + f(m)$. It is not in general a straight line. The singular solution expresses the relationship between x and y as a pair of parametric equations in t

$$\begin{aligned} x(t) &= -f'(t) \\ y(t) &= f(t) - tf'(t) \end{aligned}$$

To prove that this does give a solution to Clairaut's equation, differentiate the parametric equations for x and y with respect to t

$$\begin{aligned} \frac{dx}{dt} &= -f''(t) \\ \frac{dy}{dt} &= f' - tf'' - f' = -tf'' \end{aligned}$$

Therefore

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = t$$

provided $f''(t) \neq 0$.

Then since

$$y(t) = f(t) - tf'(t) = (-f'(t))t + f(t)$$

substituting $t = \frac{dy}{dx}$ gives

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$$

which is indeed Clairaut's equation.

EXAMPLE 1.19

To solve

$$y = xy' + \frac{1}{(y')^2}$$

note that this is Clairaut's equation with the general solution

$$y = mx + \frac{1}{m^2}$$

Since $f(t) = 1/t^2$, the singular solution is given parametrically by

$$\begin{aligned} x(t) &= \frac{2}{t^3} \\ y(t) &= \frac{1}{t^2} - t\left(-\frac{2}{t^3}\right) = \frac{3}{t^2} \end{aligned}$$

To eliminate t, express both x and y in terms of $1/t^2$

$$\left(\frac{x}{2}\right)^{2/3} = \frac{1}{t^2} = \frac{y}{3}$$

which is simply

$$y = 3\left(\frac{x}{2}\right)^{2/3}$$

■

Note that if you are asked to solve a Clairaut equation, you should always give both solutions.

Exercise 2.6 of Zill, pp. 75–76, has more examples of Clairaut's equation.

1.4 Second Order Ordinary Differential Equations

An o.d.e. of the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

is a **second order linear o.d.e.**. If $R(x) = 0$, the o.d.e. is **homogeneous**. If $R(x) \neq 0$, the o.d.e. is **inhomogeneous**.

A solution to a second order linear o.d.e. ought to require two additional pieces of information to specify it uniquely. In an **initial value problem**, two initial conditions on the same point are specified

$$y(a) = c \qquad \frac{dy}{dx}(a) = d$$

provided P, Q and R are defined at $x = a$.

There are two types of **boundary value problems**, simple and mixed. A **simple boundary condition** specifies the value of the solution at two different points

$$y(a) = c \qquad y(b) = d$$

A **mixed boundary condition** specifies the value of the solution and its derivative at two different points

$$y(a) = c \qquad \frac{dy}{dx}(b) = d$$

Boundary value problems may have a unique solution, several solutions, or no solution.

EXAMPLE 1.20
Solve

$$y'' + a^2 y = 0$$

subject to

$$y(0) = 0 \qquad y'(0) = 1$$

Solution: *This is an initial value problem. The solution is*

$$y(x) = \frac{1}{a} \sin ax$$

To prove that this is the solution, $y(0) = 0$ *and*

$$y'(x) = \cos ax$$

so $y'(0) = 1$, *therefore the initial conditions are satisfied. Now*

$$y''(x) = -a \sin ax = -a^2\, y(x)$$

so that $y'' + a^2 y = 0$ *as required.*

Note that the second solution to the differential equation is

$$y(x) = C \cos ax$$

but this does not satisfy the initial conditions. ■

EXAMPLE 1.21
Solve

$$y'' + a^2 y = 0$$

subject to

$$y(0) = 0 \qquad y\left(\frac{2\pi}{a}\right) = 0$$

Solution: *This is a boundary value problem. Note that $y = 0$ is a solution. To see whether there are any others, the general solution to the differential equation is*

$$y(x) = C_1 \sin ax + C_2 \cos ax$$

Now C_1 and C_2 have to be chosen to satisfy the boundary conditions. $y(0) = C_2$ so the boundary condition $y(0) = 0$ means that $C_2 = 0$. The second boundary condition is always satisfied because $y\,(2\pi/a) = C_2 = 0$. Therefore, writing $C = C_1$, there are infinitely many solutions satisfying the o.d.e. and both boundary conditions

$$y(x) = C \sin ax$$

■

1.4.1 Superposition of Solutions

Two functions $y_1(x)$ and $y_2(x)$ are **linearly independent** on the range $a \leq x \leq b$ if no non-zero constants C_1 and C_2 can be found such that

$$C_1 y_1(x) + C_2 y_2(x) = 0$$

for all x in the interval $[a, b]$.

THEOREM 1.1
If $y_1(x)$ and $y_2(x)$ are two solutions of the homogeneous equation

$$y'' + Py' + Qy = 0$$

then $C_1 y_1(x) + C_2 y_2(x)$ is also a solution.

Proof:
$y_1(x)$ and $y_2(x)$ are both solutions so

$$y_1'' + Py_1' + Qy_1 = 0$$

$$y_2'' + Py_2' + Qy_2 = 0$$

Now multiply the first by C_1 and the second by C_2 and add, showing that

$$\frac{d^2}{dx^2}(C_1 y_1 + C_2 y_2) + P\frac{d}{dx}(C_1 y_1 + C_2 y_2) + Q(C_1 y_1 + C_2 y_2) = 0$$

Therefore $C_1 y_1 + C_2 y_2$ satisifes the o.d.e., so it is also a solution. ■

THEOREM 1.2
If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous equation

$$y'' + Py' + Qy = 0$$

then $C_1y_1(x) + C_2y_2(x)$ is the most general solution.

EXAMPLE 1.22
The second order homogeneous o.d.e.

$$y'' + a^2y = 0$$

has solutions

$$y_1(x) = \sin ax$$

and

$$y_2(x) = \cos ax$$

Now $y_1/y_2 = \tan ax$, which is not constant over any interval. Therefore y_1 and y_2 are linearly independent and the general solution is

$$y(x) = C_1 \sin ax + C_2 \cos ax$$

where C_1 and C_2 can be determined by initial or boundary conditions. ■

1.4.2 Inhomogeneous Second Order Linear O.D.E.s

THEOREM 1.3
If $y_p(x)$ is any solution of the inhomogeneous o.d.e.

$$y'' + Py' + Qy = R$$

then the general solution is

$$y(x) = C_1y_1(x) + C_2y_2(x) + y_p(x)$$

where $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous o.d.e.

$$y'' + Py' + Qy = 0$$

Proof:

Let $y(x)$ be the general solution of the inhomogeneous o.d.e. and $y_p(x)$ be any particular solution of the inhomogeneous o.d.e.. Form the function $Y = y - y_p$. Then

$$\begin{aligned} Y'' + PY' + QY &= (y'' + Py' + Qy) - (y_p'' + Py_p' + Qy_p) \\ &= R - R \\ &= 0 \end{aligned}$$

So Y satisfies the homogeneous o.d.e.. By theorem 1.2, the general form of Y is

$$Y = C_1 y_1 + C_2 y_2$$

where y_1 and y_2 are linearly independent solutions of the homogeneous o.d.e.. Therefore

$$y = Y + y_p = C_1 y_1 + C_2 y_2 + y_p$$

■

This shows that any solution of an inhomogeneous linear o.d.e. can be written as the sum of the particular solution to the inhomogeneous o.d.e. and the general solution of the homogeneous o.d.e..

EXAMPLE 1.23
To solve

$$y'' + a^2 y = 1$$

note that a particular solution of this o.d.e. is

$$y_p(x) = \frac{1}{a^2}$$

We already know that $y_1 = \sin ax$ *and* $y_2 = \cos ax$ *are linearly independent solutions of*

$$y'' + a^2 y = 0$$

Therefore the general solution of the inhomogeneous o.d.e. is

$$y(x) = C_1 \sin ax + C_2 \cos ax + \frac{1}{a^2}$$

■

Repeat this example with $R(x)$ equal to $1 + x$, $1 + x^2$ and $e^{\lambda x}$.

1.4.3 Reduction of Order

The method of **reduction of order** is a way of obtaining a second solution $y_2(x)$ of

$$y'' + Py' + Qy = 0$$

given a first solution $y_1(x)$.

To derive the method, put $y_2 = uy_1$ and substitute into the o.d.e. Then using product rule

$$y_2' = u'y_1 + uy_1'$$

and

$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

These show that

$$y_2'' + Py_2' + Qy_2 = (u'' + Pu')y_1 + 2u'y_1' + u(y_1'' + Py_1' + Qy_1)$$

Now $y_1'' + Py_1' + Qy_1 = 0$ because y_1 is a solution of the o.d.e.. If y_2 is to be a solution of the o.d.e. too, then y_2 must satisfy

$$y_2'' + Py_2' + Qy_2 = 0$$

But

$$y_2'' + Py_2' + Qy_2 = (u'' + Pu')y_1 + 2u'y_1'$$

so for y_2 to be a solution of the o.d.e.,

$$(u'' + Pu')y_1 + 2u'y_1' = 0$$

Rearranging this shows that u must satisfy

$$u''y_1 + (Py_1 + 2y_1')u' = 0$$

Divide by y_1 and put $w = u'$. Then

$$w' + \left(P + 2\frac{y_1'}{y_1}\right)w = 0$$

This is a linear first order o.d.e. which can be solved by finding the integrating factor to give $w(x)$. Then $u'(x) = w(x)$ so determine $u(x)$ by integrating $w(x)$. Finally construct the second solution

$$y_2(x) = u(x)y_1(x)$$

Note that one solution of the second order homogeneous o.d.e. is u equal to any constant. This can always be neglected because it would not produce a linearly independent $y_2(x)$.

EXAMPLE 1.24
Find a second solution of

$$x^2y'' + 2xy' - 6y = 0$$

given that $y_1 = x^2$ is a solution.
Solution: *Look for a second solution of the form* $y_2 = ux^2$. *Then*

$$y_2' = 2xu + x^2u'$$

and

$$y_2'' = 2u + 4xu' + x^2u''$$

Substituting y_2 *into the o.d.e. gives*

$$0 = x^2y_2'' + 2xy_2' - 6y_2 = x^4u'' + 6x^3u' + u(2x^2 + 4x^2 - 6x^2)$$

As expected, the last term is zero, leaving

$$x^4u'' + 6x^3u' = 0$$

if y_2 *is a second solution. Introduce* $w = u'$ *so that*

$$x^4w' + 6x^3w = 0$$

So

$$w' + \frac{6}{x}w = 0$$

which means that

$$u' = w = \frac{C}{x^6}$$

Therefore

$$u = \int^x \frac{C}{x^6}\,dx + A = \frac{D}{x^5} + A$$

where the constant $D = -\frac{C}{5}$. *Thus*

$$y_2 = ux^2 = \frac{D}{x^3} + Ax^2$$

The Ax^2 *term is an arbitrary multiple of* y_1 *so the second linearly independent solution is*

$$y_2(x) = \frac{1}{x^3}$$

■

Exercise 4.2 of Zill, pp. 158–159, has more examples of reduction of order.

1.4.4 Homogeneous Second Order O.D.E.s with Constant Coefficients

A homogeneous second order linear o.d.e. with **constant coefficients** has the form

$$ay'' + by' + cy = 0$$

where a, b and c are constants.

The method of solution involves substituting

$$y = e^{mx}$$

to obtain

$$(am^2 + bm + c)e^{mx} = 0$$

Thus e^{mx} is a solution provided m is chosen so that

$$am^2 + bm + c = 0$$

or, using the general quadratic formula,

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There are three cases that can arise, depending on whether $b^2 - 4ac < 0$, $b^2 - 4ac = 0$ or $b^2 - 4ac > 0$.

If $b^2 - 4ac > 0$, there are two real roots, m_1 and m_2, and the general solution is the sum of two exponentials

$$y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

If $b^2 - 4ac = 0$, there is only one real root, $m_1 = -b/2a$ so

$$y_1(x) = e^{-\frac{b}{2a}x}$$

The second solution can be found using reduction of order. Put

$$y_2 = ue^{-\frac{b}{2a}x}$$

so that

$$y_2' = \left(u' - \frac{b}{2a}u\right) e^{-\frac{b}{2a}x}$$

and

$$y_2'' = \left(u'' - \frac{b}{a}u' + \frac{b^2}{4a^2}u\right) e^{-\frac{b}{2a}x}$$

Therefore

$$ay_2'' + by_2' + cy_2 = \left(au'' + u\left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)\right) e^{-\frac{b}{2a}x}$$

The coefficient of u can be written

$$\frac{b^2}{4a} - \frac{b^2}{2a} + c = -\frac{b^2 - 4ac}{4a}$$

which is zero because $b^2 - 4ac = 0$. Therefore

$$u'' = 0$$

so that $u = x$, which means

$$y_2(x) = x\, e^{-\frac{b}{2a}x}$$

This gives the complete solution as

$$y = (C_1 + C_2 x)\, e^{-\frac{b}{2a}x}$$

for the case when $b^2 - 4ac = 0$.

If $b^2 - 4ac < 0$, there are two imaginary roots m_1 and m_2 that are complex conjugates, where

$$m_1 = \frac{-b + i\sqrt{4ac - b^2}}{2a}$$

Write m_1 in the form

$$m_1 = \alpha + i\beta$$

so that the second root is

$$m_2 = \alpha - i\beta$$

The general solution is

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} = e^{\alpha x}\left(C_1 e^{i\beta x} + C_2 e^{-i\beta x}\right)$$

Now $e^{i\theta} = \cos\theta + i\sin\theta$ so that

$$y(x) = e^{\alpha x}\left((C_1 + C_2)\cos\beta x + i(C_1 - C_2)\sin\beta x\right)$$

Write $A = C_1 + C_2$ and $B = i(C_1 - C_2)$, giving an alternative form of the solution

$$y(x) = e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right)$$

EXAMPLE 1.25
To solve

$$y'' - y' - 6y = 0$$

try e^{mx} *in the o.d.e. to get*

$$(m^2 - m - 6)e^{mx} = 0$$

Therefore e^{mx} *is a solution if*

$$m^2 - m - 6 = (m - 3)(m + 2) = 0$$

The two roots are $m_1 = 3$ *and* $m_2 = -2$ *so the general solution is*

$$y(x) = C_1 e^{3x} + C_2 e^{-2x}$$

■

EXAMPLE 1.26
To solve

$$y'' + 2y' + y = 0$$

try e^{mx} *in the o.d.e. to get*

$$(m^2 + 2m + 1)e^{mx} = 0$$

Therefore e^{mx} *is a solution if*

$$m^2 + 2m + 1 = 0$$

There is only one root, $m = -1$*, so the two solutions are*

$$y_1 = e^{-x} \qquad \text{and} \qquad y_2 = xe^{-x}$$

giving the general solution

$$y(x) = (C_1 + C_2 x)e^{-x}$$

■

EXAMPLE 1.27
To solve

$$y'' + a^2 y = 0$$

try e^{mx} *in the o.d.e. to get*

$$(m^2 + a^2)e^{mx} = 0$$

Therefore e^{mx} is a solution if

$$m = \pm ia$$

The general solution is

$$y(x) = C_1 e^{iax} + C_2 e^{-iax}$$

or alternatively

$$y(x) = A \sin ax + B \cos ax$$

■

EXAMPLE 1.28
To solve

$$y'' + 4y' - y = 0$$

try e^{mx} in the o.d.e. to get

$$(m^2 + 4m - 1)e^{mx} = 0$$

Therefore e^{mx} is a solution if

$$m^2 + 4m - 1 = 0$$

The two roots are

$$m = \frac{-4 \pm \sqrt{4^2 - 4(-1)}}{2} = -2 \pm \sqrt{5}$$

so the general solution is

$$y(x) = C_1 e^{-(2+\sqrt{5})x} + C_2 e^{-(2-\sqrt{5})x} = e^{-2x}\left(C_1 e^{-\sqrt{5}x} + C_2 e^{\sqrt{5}x}\right)$$

■

EXAMPLE 1.29
To solve

$$2y'' + 2y' + y = 0$$

try e^{mx} in the o.d.e. to get

$$(2m^2 + 2m + 1)e^{mx} = 0$$

Therefore e^{mx} is a solution if

$$m = \frac{-2 \pm \sqrt{4 - 8}}{4} = -\frac{1}{2}(1 \pm i)$$

The general solution is

$$y(x) = e^{-x/2}\left(A \cos \frac{x}{2} + B \sin \frac{x}{2}\right)$$

■

Figure 1.5: This table shows the form of the particular solution $y_p(x)$ for a given $R(x)$ in the method of undetermined coefficients.

$R(x)$	$y_p(x)$
constant	A
$e^{\lambda x}$	$Ae^{\lambda x}$
$e^{\lambda x}\left[E\cos\mu x + F\sin\mu x\right]$	$e^{\lambda x}\left[A\cos\mu x + B\sin\mu x\right]$
$a_0 + a_1 x + \cdots + a_n x^n$	$A_0 + A_1 x + \cdots + A_n x^n$
$e^{\lambda x}[a_0 + a_1 x + \cdots + a_n x^n]$	$e^{\lambda x}[A_0 + A_1 x + \cdots + A_n x^n]$

Exercise 4.3 of Zill, pp. 167–168, has more examples of constant coefficient o.d.e.s.

1.4.5 Inhomogeneous Constant Coefficient O.D.E.s

The solution of an o.d.e. of the form

$$ay'' + by' + cy = R(x)$$

is the sum of the solutions of the homogeneous equation and the particular solution due to $R(x)$.

In the **method of undetermined coefficients**, the particular solution $y_p(x)$ is found by assuming a general expression for $y_p(x)$ depending on the form of $R(x)$ and solving for any unknowns so that the o.d.e. is satisfied. The appropriate forms of $y_p(x)$ for $R(x)$ are given in figure 1.5.

EXAMPLE 1.30

The solution of

$$y'' + a^2 y = R(x)$$

depends on the form of $R(x)$.

- *When* $R(x) = 1$, *try* $y_p = C$. *Then* $y_p'' = 0$ *so* $C = 1/a^2$. *Thus*

$$y_p(x) = \frac{1}{a^2}$$

- *When* $R(x) = 1 + x$, *try* $y_p = A + Bx$ *so that* $y_p'' = 0$. *Therefore*

$$a^2(A + Bx) = 1 + x$$

Equating coefficients of x^0 *and* x^1 *shows that* $a^2A = 1$ *and* $a^2B = 1$. *Thus*

$$y_p(x) = \frac{1+x}{a^2}$$

- *When* $R(x) = 1+x^2$, *try* $y_p = A+Bx+Cx^2$. *Then* $y_p'' = 2C$. *Therefore*

  $$2C + a^2(A + Bx + Cx^2) = 1 + x^2$$

  Equating coefficients of x^0, x^1 *and* x^2 *shows that*

  $$2C + a^2A = 1 \qquad a^2B = 0 \qquad a^2C = 1$$

  Therefore

  $$A = \frac{1}{a^2} - \frac{2}{a^4} \qquad B = 0 \qquad C = \frac{1}{a^2}$$

  so the particular solution is

  $$y_p(x) = \frac{1}{a^2} - \frac{2}{a^4} + \frac{x^2}{a^2}$$

- *When* $R(x) = e^{\lambda x}$, *try* $y_p = Ae^{\lambda x}$. *Then* $y_p'' = A\lambda^2 e^{\lambda x}$. *Therefore*

  $$(\lambda^2 + a^2)Ae^{\lambda x} = e^{\lambda x}$$

  so that

  $$A = \frac{1}{\lambda^2 + a^2}$$

  giving the particular solution

  $$y_p = \frac{e^{\lambda x}}{\lambda^2 + a^2}$$

- *When* $R(x) = \cos\lambda x$, *try* $y_p = A\cos\lambda x + B\sin\lambda x$. *Therefore*

  $$y_p'' = -\lambda^2(A\cos\lambda x + B\sin\lambda x) = -\lambda^2 y_p$$

  so that

  $$(a^2 - \lambda^2)(A\cos\lambda x + B\sin\lambda x) = \cos\lambda x$$

  Equating coefficients of $\cos\lambda x$ *and* $\sin\lambda x$ *shows that*

  $$(a^2 - \lambda^2)A = 1 \qquad (a^2 - \lambda^2)B = 0$$

  so that

  $$A = \frac{1}{a^2 - \lambda^2} \qquad B = 0$$

and the particular solution is

$$y_p = \frac{\cos \lambda x}{a^2 - \lambda^2}$$

(What happens here as $a \to \lambda$?)

■

Note that this method fails whenever

$$R(x) = e^{m_1 x} \text{ or } e^{m_2 x}$$

where m_1 and m_2 are the roots of

$$am^2 + bm + c = 0$$

There is a general method of obtaining $y_p(x)$ when $y_1(x)$ and $y_2(x)$ are known. Section 4.7 of Zill (pp. 194–203) describes this method, called **variation of parameters**.

EXAMPLE 1.31
When solving

$$y'' + a^2 y = \cos ax$$

note that $R(x) = \cos ax$ is a solution of the homogeneous equation, whose general solution is

$$y(x) = A \cos ax + B \sin ax$$

In cases like this, the method of undetermined coefficients has to be modified to try a particular solution of the form

$$y_p(x) = u(x) \cos ax$$

Then

$$y_p'(x) = u'(x) \cos ax - au(x) \sin ax$$

and

$$y_p''(x) = u''(x) \cos ax - 2au'(x) \sin ax - a^2 u(x) \cos ax$$

so that

$$\cos ax = y_p'' + a^2 y_p = u''(x) \cos ax - 2au'(x) \sin ax$$

which gives the differential equation for $u(x)$

$$u'' - 2au' \tan ax = 1$$

which can be solved to determine u, hence determine y_p. ■

Exercise 4.4 of Zill, pp. 179–181, has more examples of the method of undetermined coefficients.

1.4.6 Superposition of Inhomogeneous Equations

If $y_{p1}(x)$ is a particular solution of

$$y'' + Py' + Qy = R_1$$

and $y_{p2}(x)$ is a particular solution of

$$y'' + Py' + Qy = R_2$$

then $C_1 y_{p1}(x) + C_2 y_{p2}(x)$ is a particular solution of

$$y'' + Py' + Qy = C_1 R_1 + C_2 R_2$$

EXAMPLE 1.32
The equation

$$y_{p1}(x) = \frac{e^{\lambda x}}{\lambda^2 + a^2}$$

is a particular solution of

$$y'' + a^2 y = e^{\lambda x}$$

and

$$y_{p2}(x) = \frac{1}{a^2} - \frac{2}{a^4} + \frac{x^2}{a^2}$$

is a particular solution of

$$y'' + a^2 y = 1 + x^2$$

Thus

$$y_p(x) = 2\left(\frac{1}{a^2} - \frac{2}{a^4} + \frac{x^2}{a^2}\right) + \frac{3e^{\lambda x}}{\lambda^2 + a^2}$$

is a particular solution of

$$y'' + a^2 y = 2(1 + x^2) + 3e^{\lambda x}$$

■

1.5 Free and Forced Vibrations

Let $x(t)$ describe the response of an oscillating system over time, and write

$$\dot{x} = \frac{dx}{dt} \qquad \ddot{x} = \frac{d\dot{x}}{dt}$$

By Newton's second law, the system's acceleration $\ddot{x}$ multiplied by its mass m is equal to the sum of the forces acting on it. The three forces acting on the system are a restoring force of magnitude $-kx$, resistance or drag of $-\lambda\dot{x}$ and an externally applied force $F(t)$. This gives the second order differential equation

$$m\ddot{x} = -kx - \lambda\dot{x} + F(t)$$

This can be rewritten in the standard form for a second order differential equation as

$$\ddot{x} + 2p\dot{x} + \omega_0^2 x = f(t)$$

where $p = \lambda/2m$ is the friction term ($p \geq 0$), $\omega_0 = \sqrt{k/m}$ is the natural frequency ($\omega_0 > 0$) and $f(t) = F(t)/m$ is the forcing function which drives the system.

This is a second order linear o.d.e. with constant coefficients. When $f(t) = 0$, this equation describes **free vibration**. When $f(t) \neq 0$, there is an external force driving the system, so the equation describes **forced vibration**.

1.5.1 Free Vibration

Since $f(t) = 0$ for free vibration, the general equation of motion becomes

$$\ddot{x} + 2p\dot{x} + \omega_0^2 x = 0$$

The nature of the solution depends on the value of the friction constant p.

Case 1: $p = 0$

The general solution of the differential equation

$$\ddot{x} + \omega_0^2 x = 0$$

is

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

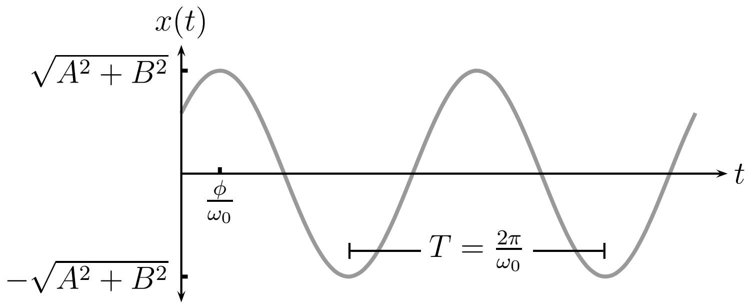
where A and B are arbitrary constants. This can be written

$$x(t) = \sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}}\cos\omega_0 t + \frac{B}{\sqrt{A^2 + B^2}}\sin\omega_0 t\right)$$

Since the magnitudes of $A/\sqrt{A^2 + B^2}$ and $B/\sqrt{A^2 + B^2}$ are less than or equal to one, and the sum of their squares is one, write

$$\frac{A}{\sqrt{A^2 + B^2}} = \cos\phi \qquad \text{and} \qquad \frac{B}{\sqrt{A^2 + B^2}} = \sin\phi$$

Figure 1.6: Typical time response $x(t)$ for free vibrations in an undamped system ($p = 0$).



so the angle ϕ is given by $\tan\phi = B/A$. Then

$$\begin{aligned}\frac{A}{\sqrt{A^2+B^2}}\cos\omega_0 t + \frac{B}{\sqrt{A^2+B^2}}\sin\omega_0 t &= \cos\phi\cos\omega_0 t + \sin\phi\sin\omega_0 t \\ &= \cos(\omega_0 t - \phi)\end{aligned}$$

This gives the simpler expression for $x(t)$

$$x(t) = \sqrt{A^2+B^2}\cos\left(\omega_0 t - \phi\right)$$

which is a sinusoid with amplitude $\sqrt{A^2+B^2}$, period $T = 2\pi/\omega_0$ and a phase shift of ϕ.

The system's behaviour is **undamped** because there is no friction to reduce $x(t)$'s amplitude over time. This is shown in figure 1.6.

Case 2: p is small and positive

When p is small and positive, the system is damped because friction reduces the vibration's amplitude over time. The differential equation

$$\ddot{x} + 2p\dot{x} + \omega_0^2 x = 0$$

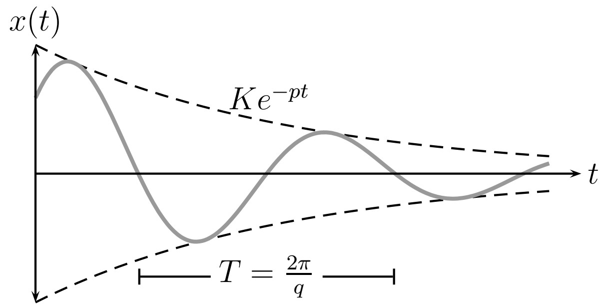
is solved by assuming $x(t)$ has the form $x(t) = e^{\alpha t}$. This gives the characteristic equation

$$\alpha^2 + 2p\alpha + \omega_0^2 = 0$$

whose roots are

$$\alpha = -p \pm \sqrt{p^2 - \omega_0^2} = -p \pm i\sqrt{\omega_0^2 - p^2}$$

Figure 1.7: Typical time response $x(t)$ for free vibrations in an underdamped system $(0 < p < \omega_0)$.



Since we are considering the case when p is small and positive, the square root $\sqrt{p^2 - \omega_0^2}$ is imaginary for $0 < p < \omega_0$. Writing

$$q = \omega_0 \sqrt{1 - \left(\frac{p}{\omega_0}\right)^2}$$

gives the general solution

$$\begin{aligned} x(t) &= e^{-pt}\left(A \cos qt + B \sin qt\right) \\ &= K e^{-pt} \cos\left(qt - \phi\right) \end{aligned}$$

where the constants ϕ and K are related to the arbitrary constants of integration by

$$\begin{aligned} \tan\phi &= \frac{B}{A} \\ K &= \sqrt{A^2 + B^2} \end{aligned}$$

A typical time response $x(t)$ for underdamped free vibration is shown in figure 1.7.

Case 3: $p = \omega_0$

When $p = \omega_0$, the characteristic equation has only one solution, and the system is said to be **critically damped**.

This gives $x_1(t) = e^{-pt}$ as one of the solutions of the homogeneous differential equation. The second solution has the form $x_2(t) = t x_1(t)$ so the general solution for critically damped free vibration is

$$x(t) = (A + Bt)\, e^{-pt}$$

A typical $x(t)$ for critically damped free vibration is shown in figure 1.8.

Figure 1.8: Typical time response $x(t)$ for free vibrations in a critically damped system ($p = \omega_0$).

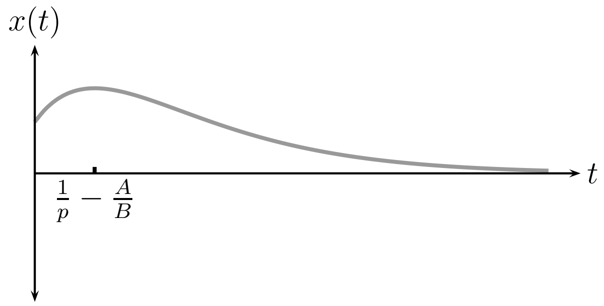
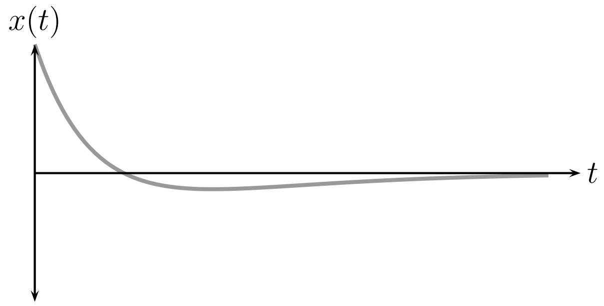


Figure 1.9: Typical time response $x(t)$ for free vibrations in an overdamped system ($p > \omega_0$).



Case 4: $p > \omega_0$

When $p > \omega_0$, the system is said to be **overdamped**. The characteristic equation has two solutions

$$\begin{aligned}\alpha_1 &= -p + \sqrt{p^2 - \omega_0^2} \\ \alpha_2 &= -p - \sqrt{p^2 - \omega_0^2}\end{aligned}$$

These are both real and negative, giving solutions which decay with time without oscillating

$$x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$$

Since α_2 is more negative than α_1, the $e^{\alpha_2 t}$ term decays to zero more quickly than the $e^{\alpha_1 t}$ term. So for large t, $x(t)$ goes asymptotically like $Ae^{\alpha_1 t}$. A typical $x(t)$ for overdamped free vibration is shown in figure 1.9.

Note that for all three cases when damping is present (when $p > 0$), $x(t) \to 0$ as $t \to \infty$. That is, the solutions are **transient**, decaying to zero over time.

1.5.2 Forced Vibration

Since $f(t) \neq 0$ for forced vibration, solutions of the differential equation

$$\ddot{x} + 2p\dot{x} + \omega_0^2 x = f(t)$$

have the form

$$x(t) = x_h(t) + x_p(t)$$

The solution to the homogeneous equation, $x_h(t)$, is given by one of the four cases of free vibration. For $p > 0$, the homogeneous solution is transient and tends to zero as $t \to \infty$. The particular solution $x_p(t)$ depends on the forcing function $f(t)$, and this is the dominant contribution to $x(t)$ as $t \to \infty$.

Resonance

When the forcing function is sinusoidal, for example $f(t) = \sin \omega t$, the differential equation is

$$\ddot{x} + 2p\dot{x} + \omega_0^2 x = \sin \omega t$$

When $p > 0$, the homogeneous solution is transient and tends to zero for large t. The particular solution $x_p(t)$ is non-transient, and has the general form

$$x_p(t) = A \cos \omega t + B \sin \omega t$$

so that the first and second derivatives of $x_p(t)$ are

$$\begin{aligned} \dot{x}_p(t) &= \omega \left(-A \sin \omega t + B \cos \omega t\right) \\ \ddot{x}_p(t) &= -\omega^2 \left(A \cos \omega t + B \sin \omega t\right) \end{aligned}$$

Substituting these into the differential equation shows that

$$\left(\omega_0^2 - \omega^2\right) \left(A \cos \omega t + B \sin \omega t\right) + 2p\omega \left(-A \sin \omega t + B \cos \omega t\right) = \sin \omega t$$

Equating coefficients of $\sin \omega t$ and $\cos \omega t$ gives two simultaneous equations

$$\begin{aligned} \left(\omega_0^2 - \omega^2\right) A + 2p\omega B &= 0 \\ \left(\omega_0^2 - \omega^2\right) B - 2p\omega A &= 1 \end{aligned}$$

whose solutions are

$$\begin{aligned}
A &= \frac{-2p\omega}{\left(\omega_0^2-\omega^2\right)^2+4p^2\omega^2} \\
B &= \frac{\omega_0^2-\omega^2}{\left(\omega_0^2-\omega^2\right)^2+4p^2\omega^2}
\end{aligned}$$

The $\cos\omega t$ and $\sin\omega t$ parts of $x_p(t)$ can be combined into a single sinusoid using the same trick as before. Written this way, the particular solution is

$$\begin{aligned}
x_p(t) &= \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos\omega t+\frac{B}{\sqrt{A^2+B^2}}\sin\omega t\right) \\
&= \alpha\left(-\sin\theta\cos\omega t+\cos\theta\sin\omega t\right) \\
&= \alpha\sin\left(\omega t-\theta\right)
\end{aligned}$$

The amplitude α is called the **amplification factor**

$$\begin{aligned}
\alpha &= \sqrt{A^2+B^2} \\
&= \frac{1}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4p^2\omega^2}}
\end{aligned}$$

and the phase shift θ is called the **phase-lag**

$$\tan\theta=-\frac{A}{B}=\frac{2p\omega}{\omega_0^2-\omega^2}$$

Note that this definition of θ is slightly different from the corresponding definition of ϕ in the equation for undamped free vibration.

Amplification Factor

The amplitude of the system's response to a sinusoidal forcing function depends on the relationship between the natural frequency of the undamped free vibration, ω_0, the frequency of the forcing function, ω, and the damping coefficient, p.

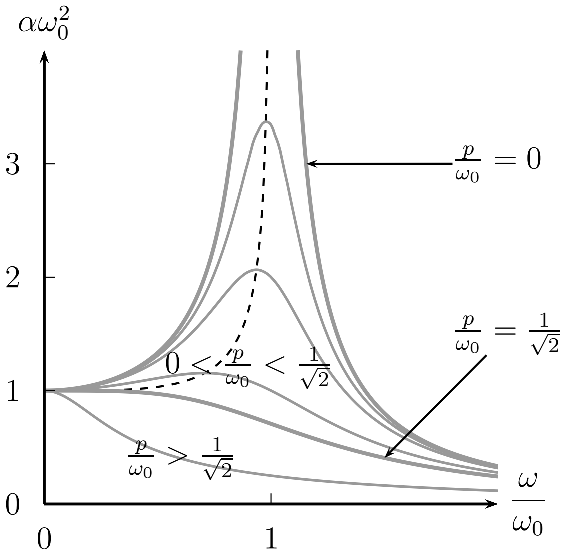
For constant ω_0 and p, the amplitude factor reaches its maximum when $d\alpha/d\omega=0$. Since

$$\frac{d\alpha}{d\omega}=-\frac{-2\omega\left(\omega_0^2-\omega^2\right)+4p^2\omega}{\left(\left(\omega_0^2-\omega^2\right)^2+4p^2\omega^2\right)^{3/2}}$$

α is at its maximum when the numerator of $d\alpha/d\omega$ is zero. This occurs when

$$\omega=0 \qquad \text{and} \qquad \omega^2-\omega_0^2+2p^2=0$$

Figure 1.10: Resonance curves showing the amplitude factor α as a function of frequency ω. The maxima of the resonance curves occur when $\omega/\omega_0 = \sqrt{1-2\left(p/\omega_0\right)^2}$ for $p/\omega_0 < 1/\sqrt{2}$.



If $p < \omega_0/\sqrt{2}$, the local maxima of α occur when $\omega = 0$ and when

$$\omega = \omega_0\sqrt{1-2\left(\frac{p}{\omega_0}\right)^2}$$

If $p > \omega_0/\sqrt{2}$, the only peak in the resonance curve is at $\omega = 0$.

The amplification factor is plotted as a function of ω for a number of different values of p in figure 1.10. The ω axis has been scaled by $1/\omega_0$ and the α axis has been scaled by ω_0^2 to make the curves dimensionless. Note that as p increases from zero to $\omega_0/\sqrt{2}$, the peak of the resonance curve moves from $\omega/\omega_0 = 1$ back towards the origin along the dashed line. Also note that the peak of the resonance curve tends to ∞ as $p \to 0$.

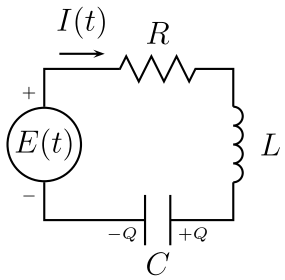
EXAMPLE 1.33

A series LRC circuit consists of a resistor R, inductor L and capacitor C in series across a time-varying voltage source $E(t)$. This is illustrated in figure 1.11.

Summing voltage drops around the loop gives the equation

$$-E(t) + I(t)R + L\frac{dI(t)}{dt} + \frac{Q(t)}{C} = 0$$

Figure 1.11: LRC series resonant circuit.



Since $I(t) = \frac{d}{dt}Q(t)$, differentiating with respect to t gives

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}$$

Dividing by L gives the standard second order differential equation for forced vibration

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = \frac{1}{L}\frac{dE}{dt}$$

which is the force vibration differential equation with $p = R/2L$, $\omega_0 = 1/\sqrt{LC}$ and $f(t) = \frac{1}{L}\frac{dE}{dt}$.

If a voltage of frequency ω

$$E(t) = V\cos\omega t$$

is applied, the current $I(t)$ reaches its maximum when the LRC circuit is tuned to ω, provided that $p/\omega_0 < 1/\sqrt{2}$. If so, ω is given by

$$\omega = \omega_0\sqrt{1 - 2\left(\frac{p}{\omega_0}\right)^2} = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{R^2C}{2L}}$$

■

Exercises 5.1 to 5.4 and the review exercises of Zill, pp. 219–251, have more examples of free and forced vibration.

Chapter 2

Vector Functions

A vector function $\boldsymbol{F}(t)$

$$\boldsymbol{F}(t) = F_x(t)\hat{\boldsymbol{\imath}} + F_y(t)\hat{\boldsymbol{\jmath}} + F_z(t)\hat{\boldsymbol{k}}$$

where $F_x(t)$, $F_y(t)$ and $F_z(t)$ are scalar functions of t, describes a trajectory in space as the parameter t is varied.

Since the three unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ are constant vectors, the first and second derivatives of $\boldsymbol{F}(t)$ are

$$\begin{aligned} \frac{d\boldsymbol{F}}{dt} = \dot{\boldsymbol{F}} &= \lim_{\delta t \to 0} \frac{\boldsymbol{F}(t+\delta t) - \boldsymbol{F}(t)}{\delta t} \\ &= \dot{F}_x\hat{\boldsymbol{\imath}} + \dot{F}_y\hat{\boldsymbol{\jmath}} + \dot{F}_z\hat{\boldsymbol{k}} \end{aligned}$$

and

$$\begin{aligned} \frac{d^2\boldsymbol{F}}{dt^2} = \ddot{\boldsymbol{F}} &= \lim_{\delta t \to 0} \frac{\dot{\boldsymbol{F}}(t+\delta t) - \dot{\boldsymbol{F}}(t)}{\delta t} \\ &= \ddot{F}_x\hat{\boldsymbol{\imath}} + \ddot{F}_y\hat{\boldsymbol{\jmath}} + \ddot{F}_z\hat{\boldsymbol{k}} \end{aligned}$$

In the limit as $\delta t \to 0$, $\boldsymbol{F}(t+\delta t) - \boldsymbol{F}(t)$ becomes tangential to the trajectory curve so that $\dot{\boldsymbol{F}}(t)$ is the tangent at t.

If t is time and $\boldsymbol{r}(t)$ is the position of a particle at time t

$$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}$$

where $x(t)$, $y(t)$ and $z(t)$ are the particle's Cartesian coordinates, the velocity of the particle is

$$\dot{\boldsymbol{r}}(t) = \dot{x}(t)\hat{\boldsymbol{\imath}} + \dot{y}(t)\hat{\boldsymbol{\jmath}} + \dot{z}(t)\hat{\boldsymbol{k}}$$

and the acceleration is

$$\ddot{\boldsymbol{r}}(t) = \ddot{x}(t)\hat{\boldsymbol{\imath}} + \ddot{y}(t)\hat{\boldsymbol{\jmath}} + \ddot{z}(t)\hat{\boldsymbol{k}}$$

2.1 Differentiation of Sums and Products

Let $\boldsymbol{F}(t)$ and $\boldsymbol{G}(t)$ be vector functions and $f(t)$ be a scalar function. Then

$$\frac{d}{dt}(\boldsymbol{F}+\boldsymbol{G}) = \frac{d\boldsymbol{F}}{dt} + \frac{d\boldsymbol{G}}{dt}$$

$$\frac{d}{dt}(\boldsymbol{F}\cdot\boldsymbol{G}) = \frac{d\boldsymbol{F}}{dt}\cdot\boldsymbol{G} + \boldsymbol{F}\cdot\frac{d\boldsymbol{G}}{dt}$$

$$\frac{d}{dt}(\boldsymbol{F}\times\boldsymbol{G}) = \frac{d\boldsymbol{F}}{dt}\times\boldsymbol{G} + \boldsymbol{F}\times\frac{d\boldsymbol{G}}{dt}$$

$$\frac{d}{dt}(f\boldsymbol{F}) = \frac{df}{dt}\boldsymbol{F} + f\frac{d\boldsymbol{F}}{dt}$$

EXAMPLE 2.1
*A particle moving with **constant rectilinear motion** has Cartesian coordinates*

$$\begin{aligned} x(t) &= at+b \\ y(t) &= ct+d \\ z(t) &= et+f \end{aligned}$$

so the position of the particle can be written

$$\boldsymbol{r}(t) = (at+b)\hat{\boldsymbol{i}} + (ct+d)\hat{\boldsymbol{j}} + (et+f)\hat{\boldsymbol{k}} = \boldsymbol{r}_0 + \boldsymbol{v}t$$

where

$$\begin{aligned} \boldsymbol{r}_0 &= b\hat{\boldsymbol{i}} + d\hat{\boldsymbol{j}} + f\hat{\boldsymbol{k}} \\ \boldsymbol{v} &= a\hat{\boldsymbol{i}} + c\hat{\boldsymbol{j}} + e\hat{\boldsymbol{k}} \end{aligned}$$

are constant vectors.

Then the velocity is

$$\dot{\boldsymbol{r}} = \dot{x}\hat{\boldsymbol{i}} + \dot{y}\hat{\boldsymbol{j}} + \dot{z}\hat{\boldsymbol{k}} = a\hat{\boldsymbol{i}} + c\hat{\boldsymbol{j}} + e\hat{\boldsymbol{k}} = \boldsymbol{v}$$

and the acceleration is

$$\ddot{\boldsymbol{r}} = \ddot{x}\hat{\boldsymbol{i}} + \ddot{y}\hat{\boldsymbol{j}} + \ddot{z}\hat{\boldsymbol{k}} = \boldsymbol{0}$$

■

EXAMPLE 2.2
*A particle moving with **uniform circular motion** has Cartesian coordinates*

$$\begin{aligned} x(t) &= x_0 + a\cos\omega t \\ y(t) &= y_0 + a\sin\omega t \\ z(t) &= 0 \end{aligned}$$

so the position of the particle can be written

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + a\cos\omega t\hat{\boldsymbol{\imath}} + a\sin\omega t\hat{\boldsymbol{\jmath}}$$

where

$$\boldsymbol{r}_0 = x_0\hat{\boldsymbol{\imath}} + y_0\hat{\boldsymbol{\jmath}}$$

Note that

$$|\boldsymbol{r} - \boldsymbol{r}_0| = \left|a\cos\omega t\hat{\boldsymbol{\imath}} + a\sin\omega t\hat{\boldsymbol{\jmath}}\right| = a$$

The velocity is

$$\dot{\boldsymbol{r}}(t) = -a\omega\sin\omega t\hat{\boldsymbol{\imath}} + a\omega\cos\omega t\hat{\boldsymbol{\jmath}}$$

so the speed is constant

$$|\dot{\boldsymbol{r}}| = a\omega$$

and that the velocity is orthogonal to $\boldsymbol{r} - \boldsymbol{r}_0$

$$\dot{\boldsymbol{r}} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0$$

This orthogonality can be proved by direct substitution, or alternatively, by using

$$(\boldsymbol{r} - \boldsymbol{r}_0) \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = |\boldsymbol{r} - \boldsymbol{r}_0|^2 = a^2$$

Then differentiating with respect to time,

$$0 = \frac{d}{dt}a^2 = \frac{d}{dt}|\boldsymbol{r} - \boldsymbol{r}_0|^2 = 2\left(\frac{d}{dt}(\boldsymbol{r} - \boldsymbol{r}_0)\right) \cdot (\boldsymbol{r} - \boldsymbol{r}_0)$$

Since $\dot{\boldsymbol{r}}_0 = 0$,

$$\dot{\boldsymbol{r}} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0$$

as required.

The acceleration is

$$\ddot{\boldsymbol{r}}(t) = -a\omega^2\cos\omega t\hat{\boldsymbol{\imath}} - a\omega^2\sin\omega t\hat{\boldsymbol{\jmath}} = -\omega^2(\boldsymbol{r} - \boldsymbol{r}_0)$$

Note that the acceleration is inwards along the radius of the circle and that the magnitude of the acceleration is constant

$$|\ddot{\boldsymbol{r}}| = \omega^2 a$$

Also, the acceleration and velocity are at right angles

$$\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} = 0$$

This can be shown by direct substitution or by differentiating

$$\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = |\dot{\boldsymbol{r}}|^2 = a^2\omega^2$$

with respect to t to give

$$\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} + \ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = 0$$

so that

$$\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} = 0$$

■

Problems and more examples can be found in chapter 1 of Fowles.

2.2 Polar Coordinates

Consider a point P at (x, y) so that $\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$. As shown in figure 2.1, $\boldsymbol{r}$ can be written in terms of r and θ, where r is the particle's distance from the origin and θ is the angle that $\boldsymbol{r}$ makes with the positive x axis.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

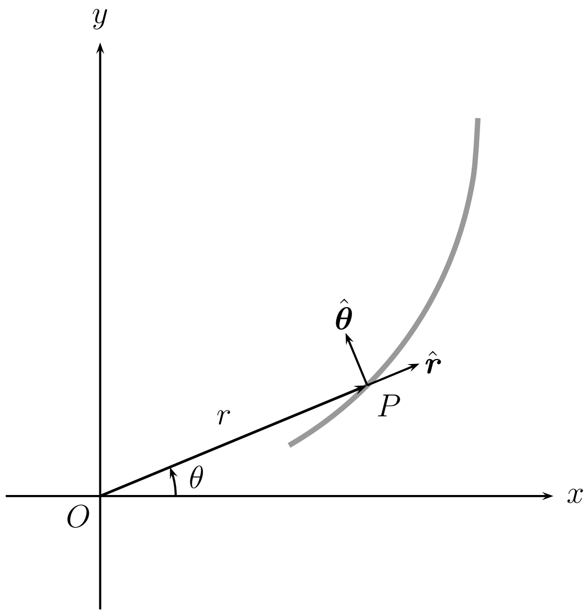
Then the velocity can also be written in terms of r and θ

$$\begin{aligned} \dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{aligned}$$

so that

$$\begin{aligned} \dot{\boldsymbol{r}} &= \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} \\ &= (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\hat{\boldsymbol{\imath}} + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\hat{\boldsymbol{\jmath}} \\ &= \dot{r}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) + r\dot{\theta}(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}) \end{aligned}$$

Figure 2.1: The unit vectors for polar coordinates (r, θ) are $\hat{\boldsymbol{r}}$ in the radial direction and $\hat{\boldsymbol{\theta}}$ in the transverse direction.



Therefore

$$\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

where $\hat{\boldsymbol{r}}$ is the **radial unit vector** in the direction of r increasing

$$\hat{\boldsymbol{r}} = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}} = \frac{\boldsymbol{r}}{r}$$

Note that $|\hat{\boldsymbol{r}}| = 1$ because $\hat{\boldsymbol{r}}$ is a unit vector. $\hat{\boldsymbol{r}}$ is not a constant vector (like $\hat{\boldsymbol{\imath}}$) but changes direction as θ changes.

$\hat{\boldsymbol{\theta}}$ is the **transverse unit vector** in the direction of θ increasing

$$\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$$

which is orthogonal to $\hat{\boldsymbol{r}}$

$$\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{r}} = 0$$

Note that $\left|\hat{\theta}\right| = 1$ because $\hat{\theta}$ is a unit vector. $\hat{\theta}$ is not a constant vector but changes direction as θ changes.

In the expression for velocity in polar coordinates,

$$\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

$\dot{r}$ is called the **radial velocity** and $r\dot{\theta}$ is called the **transverse velocity**.

The derivative of the radial unit vector is

$$\frac{d\hat{\boldsymbol{r}}}{dt} = \frac{d}{dt}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) = -\dot{\theta}\sin\theta\hat{\boldsymbol{\imath}} + \dot{\theta}\cos\theta\hat{\boldsymbol{\jmath}}$$

which can be written

$$\dot{\hat{\boldsymbol{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}}$$

The derivative of the transverse unit vector is

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = \frac{d}{dt}(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}) = -\dot{\theta}\cos\theta\hat{\boldsymbol{\imath}} - \dot{\theta}\sin\theta\hat{\boldsymbol{\jmath}}$$

which can be written

$$\dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\hat{\boldsymbol{r}}$$

Taking the derivative of the polar form of the particle's velocity gives

$$\begin{aligned}
\ddot{\boldsymbol{r}} &= \frac{d}{dt}\left(\dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}\right) \\
&= \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\dot{\hat{\boldsymbol{r}}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\dot{\hat{\boldsymbol{\theta}}} \\
&= \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\boldsymbol{r}}
\end{aligned}$$

Gathering terms shows that acceleration in polar coordinates is

$$\ddot{\boldsymbol{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}}$$

where $\ddot{r} - r\dot{\theta}^2$ is called the **radial acceleration** and $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ is called the **transverse acceleration**.

EXAMPLE 2.3

For a particle moving in a circular orbit around the origin, $r = a$ so

$$\dot{\boldsymbol{r}} = a\dot{\theta}\hat{\boldsymbol{\theta}}$$

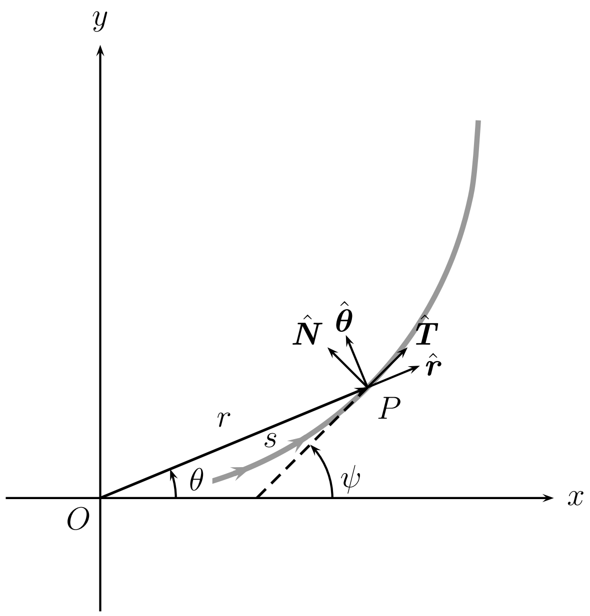
so that the velocity is always transverse, and

$$\ddot{\boldsymbol{r}} = -a\dot{\theta}^2\hat{\boldsymbol{r}} + a\ddot{\theta}\hat{\boldsymbol{\theta}}$$

*where $\dot{\theta}$ is the **instantaneous angular velocity** and $\ddot{\theta}$ is the **instantaneous angular acceleration**.*

If we write $v = a\dot{\theta} = |\dot{\boldsymbol{r}}|$ as the speed of the particle, then the radial acceleration is $-a\dot{\theta}^2 = -v^2/a$. ■

Figure 2.2: The unit vectors for intrinsic coordinates (s, ψ) are $\hat{\boldsymbol{T}}$ in the tangential direction and $\hat{\boldsymbol{N}}$ in the normal direction. s is the distance along the curve and ψ the angle the tangent makes with the x axis.



EXAMPLE 2.4

For a particle moving with rectilinear motion through the origin, θ is constant so that $\hat{\boldsymbol{r}}$ is constant. Therefore

$$\begin{aligned}
\boldsymbol{r} &= r\hat{\boldsymbol{r}} \\
\dot{\boldsymbol{r}} &= \dot{r}\hat{\boldsymbol{r}} \\
\ddot{\boldsymbol{r}} &= \ddot{r}\hat{\boldsymbol{r}}
\end{aligned}$$

These equations for rectilinear motion in polar coordinates are only simple if the particle's trajectory includes the origin, otherwise θ changes along the trajectory. ■

2.3 Intrinsic Coordinates

2.3.1 Two-Dimensional Intrinsic Coordinates

The point P at (x, y) in Cartesian coordinates or (r, θ) is polar coordinates can also be described by the **intrinsic coordinates** (s, ψ) shown in figure 2.2.

s is arc length measured along the trajectory from some specified starting point and ψ is the angle the tangent to the trajectory makes with the positive x axis.

The associated unit vectors are $\hat{\boldsymbol{T}}$, the unit vector along the tangent in the direction of increasing s

$$\hat{\boldsymbol{T}} = \cos\psi\hat{\boldsymbol{\imath}} + \sin\psi\hat{\boldsymbol{\jmath}}$$

and $\hat{\boldsymbol{N}}$, the unit vector transverse to $\hat{\boldsymbol{T}}$ in the direction of increasing ψ

$$\hat{\boldsymbol{N}} = -\sin\psi\hat{\boldsymbol{\imath}} + \cos\psi\hat{\boldsymbol{\jmath}}$$

To find expressions for velocity and acceleration in intrinsic coordinates, let $\delta\boldsymbol{r}$ be the difference in $\boldsymbol{r}$ as the particle moves a distance δs along the curve

$$\delta\boldsymbol{r} = \boldsymbol{r}(s+\delta s) - \boldsymbol{r}(s)$$

As $\delta s \to 0$, $\delta\boldsymbol{r}$ becomes tangential to the curve so that the magnitude of $\delta\boldsymbol{r}$ tends to δs

$$\frac{|\delta\boldsymbol{r}|}{\delta s} \to 1$$

and $\delta\boldsymbol{r}$ becomes a tangent to the curve

$$\frac{\delta\boldsymbol{r}}{\delta s} \to \hat{\boldsymbol{T}}$$

This implies that

$$\frac{d\boldsymbol{r}}{ds} = \hat{\boldsymbol{T}}$$

Therefore

$$\frac{d}{dt}\left(\boldsymbol{r}\left(s(t)\right)\right) = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} = \dot{s}\hat{\boldsymbol{T}}$$

so the expression for velocity in intrinsic coordinates is

$$\dot{\boldsymbol{r}} = \dot{s}\hat{\boldsymbol{T}}$$

To find the expression for acceleration,

$$\ddot{\boldsymbol{r}} = \frac{d}{dt}\left(\dot{s}\hat{\boldsymbol{T}}\right) = \ddot{s}\hat{\boldsymbol{T}} + \dot{s}\dot{\hat{\boldsymbol{T}}}$$

Now $\hat{\boldsymbol{T}} = \cos\psi\hat{\boldsymbol{\imath}} + \sin\psi\hat{\boldsymbol{\jmath}}$ so

$$\dot{\hat{\boldsymbol{T}}} = \dot{\psi}\left(-\sin\psi\hat{\boldsymbol{\imath}} + \cos\psi\hat{\boldsymbol{\jmath}}\right) = \dot{\psi}\hat{\boldsymbol{N}}$$

Therefore the expression for acceleration in intrinsic coordinates is

$$\ddot{\boldsymbol{r}} = \ddot{s}\hat{\boldsymbol{T}} + \dot{s}\dot{\psi}\hat{\boldsymbol{N}}$$

Note that from the definitions of $\hat{\boldsymbol{T}}$ and $\hat{\boldsymbol{N}}$,

$$\frac{d\hat{\boldsymbol{T}}}{ds} = \frac{d\psi}{ds}\hat{\boldsymbol{N}}$$

on the curve $\psi(s)$. $\frac{d\psi}{ds}$ is the **curvature** of the curve at $P(s)$ and

$$\frac{d\psi}{ds} = \frac{1}{\rho(s)}$$

where $\rho(s)$ is the **radius of curvature** of the curve at the point $P(s)$.

2.3.2 Three-Dimensional Intrinsic Coordinates

When the particle follows a trajectory through three-dimensional space, s is the arc length measured along the curve from some specified point. As for two-dimensional intrinsic coordinates,

$$\frac{d\boldsymbol{r}}{ds} = \hat{\boldsymbol{T}}$$

and

$$\frac{d\hat{\boldsymbol{T}}}{ds} = \kappa\hat{\boldsymbol{N}} = \frac{1}{\rho}\hat{\boldsymbol{N}}$$

where $\hat{\boldsymbol{N}}$ is the **principal normal** to the curve (transverse to $\hat{\boldsymbol{T}}$ in the plane of the local curve). Here κ is the **curvature** of the curve at $P(s)$

$$\kappa = \left|\frac{d\hat{\boldsymbol{T}}}{ds}\right|$$

and ρ is the **radius of curvature** at $P(s)$.

Velocity in three-dimensional intrinsic coordinates is given by

$$\dot{\boldsymbol{r}} = \frac{d}{dt}\boldsymbol{r}(s(t)) = \frac{d\boldsymbol{r}}{ds}\dot{s}$$

so that

$$\dot{\boldsymbol{r}} = \dot{s}\hat{\boldsymbol{T}}$$

Acceleration in three-dimensional intrinsic coordinates is

$$\ddot{\boldsymbol{r}} = \frac{d}{dt}(\dot{s}\hat{\boldsymbol{T}}) = \ddot{s}\hat{\boldsymbol{T}} + \dot{s}\frac{d\hat{\boldsymbol{T}}}{dt}$$

Now $\frac{d}{dt}\hat{\boldsymbol{T}} = \frac{d\hat{\boldsymbol{T}}}{ds}\dot{s}$ so that acceleration can be written

$$\ddot{\boldsymbol{r}} = \ddot{s}\hat{\boldsymbol{T}} + \frac{\dot{s}^2}{\rho(s)}\hat{\boldsymbol{N}}$$

Chapter 3

Single Particle Dynamics

A **particle** is a moving geometric point at which matter is concentrated. **Newton's Laws of Motion** describe how particles behave:

First Law A particle moves at constant velocity $\dot{\boldsymbol{r}}$ relative to an inertial frame of reference unless acted upon by a force.

Second Law If a particle is acted upon by a force $\boldsymbol{F}$, it will accelerate relative to an inertial frame of reference such that

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}}$$

The constant of proportionality m is the **inertial mass** of the particle.

Third Law When two particles act on one another, the two forces acting on the particles are equal in magnitude and opposite in direction and along the line of their centres

$$\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$$

Particles are also affected by **Newton's postulate of gravitation**, which says that any two particles in the universe attract each other with a gravitational force of magnitude

$$\frac{Gm_1m_2}{r^2}$$

along the line of their centres, where r is the distance between the particles, m_1 and m_2 are their inertial masses and G is the **gravitational constant**

$$G = 6.67 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$$

The gravitational force between a uniform sphere of mass M and a particle of mass m is

$$\boldsymbol{F}_p = -\frac{GmM}{r^2}\hat{\boldsymbol{r}}$$

This is derived in Fowles, pp. 134–136.

The near gravitational field of the Earth can be found using

$$\boldsymbol{F}_p = -\frac{GmM}{r^2}\hat{\boldsymbol{r}}$$

where M is the mass of the Earth and r is the distance of the particle from the centre of the Earth. Write $r = R + h$ where R is the radius of the Earth and h is the height of the particle above the surface of the Earth. Then

$$\boldsymbol{F}_p = -\frac{GmM}{(R+h)^2}\hat{\boldsymbol{r}}$$

If the particle is close to the Earth's surface,

$$\boldsymbol{F}_p = -\frac{GM}{R^2}m\hat{\boldsymbol{r}} + O\left(\frac{h}{R}\right)$$

Since $h \ll R$, the h/R terms can be neglected so that

$$\boldsymbol{F}_p = -\frac{GM}{R^2}m\hat{\boldsymbol{r}} = -gm\hat{\boldsymbol{r}}$$

where g is gravitational acceleration at the Earth's surface

$$g = \frac{GM}{R^2} = 9.81\,\mathrm{ms}^{-2}$$

This simplification is valid as long as $h \ll R$ and it assumes that the Earth is composed of uniform spherical shells.

3.1 Rectilinear Motion

If the particle moves in a straight line, the force must always be parallel to the direction of its motion. Suppose that the force is constant

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}$$

Choose the x axis to lie in the direction of $\boldsymbol{F}$. Then $\boldsymbol{F} = F\hat{\boldsymbol{\imath}}$ and

$$\begin{aligned} m\ddot{x} &= F \\ m\ddot{y} &= 0 \\ m\ddot{z} &= 0 \end{aligned}$$

Integrating gives the particle's velocities

$$\begin{aligned}\dot{x}(t) &= \frac{F}{m}t + \dot{x}(0)\\ \dot{y}(t) &= \dot{y}(0) = 0\\ \dot{z}(t) &= \dot{z}(0) = 0\end{aligned}$$

because there is no motion in the y and z directions.

Integrating again gives the particle's position

$$\begin{aligned}x(t) &= \frac{1}{2}\frac{F}{m}t^2 + \dot{x}(0)t + x(0)\\ y(t) &= y(0)\\ z(t) &= z(0)\end{aligned}$$

Using this, note that

$$\begin{aligned}2\frac{F}{m}\left[x(t) - x(0)\right] &= \left(\frac{F}{m}t\right)^2 + 2\frac{F}{m}t\dot{x}(0)\\ &= \left(\dot{x}(t) - \dot{x}(0)\right)^2 + 2\dot{x}(0)\left(\dot{x}(t) - \dot{x}(0)\right)\\ &= \dot{x}^2(t) - \dot{x}^2(0)\end{aligned}$$

This can be written as

$$F\left[x(t) - x(0)\right] = \frac{1}{2}m\dot{x}^2(t) - \frac{1}{2}m\dot{x}^2(0)$$

which shows that the work done is equal to the change in kinetic energy.

Alternatively, this can be obtained using

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx}\frac{dx}{dt} = \dot{x}\frac{d\dot{x}}{dx} = \frac{1}{2}\frac{d}{dx}\left(\dot{x}^2\right)$$

so that

$$\frac{1}{2}\frac{d}{dx}\left(\dot{x}^2\right) = \frac{F}{m}$$

Integrating both sides with respect to x shows that

$$\dot{x}^2(t) - \dot{x}^2(0) = \frac{2F}{m}\left(x(t) - x(0)\right)$$

In the **general rectilinear problem**, only one coordinate is important. If the z axis lies along the line of motion,

$$\boldsymbol{r} = z\hat{\boldsymbol{k}}$$

and

$$\boldsymbol{F} = F\hat{\boldsymbol{k}}$$

The equation of motion becomes a scalar differential equation in z

$$m\ddot{z} = F(z, \dot{z}, t)$$

EXAMPLE 3.1
A particle is dropped from rest at a height z_0 close to the Earth's surface, with air resistance proportional to the particle's speed. Find the particle's height $z(t)$ and its terminal velocity $\dot{z}_{\text{ter}}$.
Solution: *Air resistance is proportional to speed so the force on the particle is*

$$\boldsymbol{F} = -mg\hat{\boldsymbol{k}} - \lambda\dot{z}\hat{\boldsymbol{k}}$$

Therefore the equation of motion is a second order linear o.d.e. with constant coefficients

$$m\ddot{z} = -mg - \lambda\dot{z}$$

with $\dot{z} = 0$ and $z = z_0$ when $t = 0$.

In standard form, this is

$$\ddot{z} + \frac{\lambda}{m}\dot{z} = -g$$

Try a homogeneous solution of the form $z = e^{\alpha t}$. This gives

$$\ddot{z} + \frac{\lambda}{m}\dot{z} = \left[\alpha^2 + \frac{\lambda}{m}\alpha\right] e^{\alpha t} = 0$$

so that

$$z = A + Be^{-\frac{\lambda}{m}t} + z_p$$

where the particular solution z_p is yet to be determined.

The homogeneous solution with $\alpha = 0$ has the same form as the inhomogeneous term in the differential equation, thus a constant is not an acceptable particular solution. Instead try

$$z_p(t) = ct$$

Substituting into the differential equation shows that

$$c = -\frac{mg}{\lambda}$$

so that

$$z_p(t) = -\frac{mg}{\lambda}t$$

giving the general solution

$$z = A + Be^{-\frac{\lambda}{m}t} - \frac{mg}{\lambda}t$$

When $t = 0$, $z_0 = A + B$ and $\dot{z} = 0 = -\lambda B/m - mg/\lambda$, so the constants are

$$B = -\left(\frac{m}{\lambda}\right)^2 g$$

and

$$A = z_0 - B = z_0 + \left(\frac{m}{\lambda}\right)^2 g$$

giving the solution

$$z(t) = z_0 + \left(\frac{m}{\lambda}\right)^2 g\left[1 - e^{-\frac{\lambda}{m}t} - \frac{\lambda}{m}t\right]$$

The speed of the particle is

$$\dot{z}(t) = -\frac{mg}{\lambda}\left[1 - e^{-\frac{\lambda}{m}t}\right]$$

which reaches a terminal velocity of

$$\dot{z}_{\text{ter}} = -\frac{mg}{\lambda}$$

as $t \to \infty$.

Alternatively the terminal velocity can be obtained directly from the o.d.e. by noting that as the terminal velocity is approached, $\dot{z}$ becomes constant so that $\ddot{z}$ tends to zero. Therefore the differential equation

$$m\ddot{z} = -mg - \lambda\dot{z}$$

becomes

$$0 = -mg - \lambda\dot{z}_{\text{ter}}$$

so that the terminal velocity is

$$\dot{z}_{\text{ter}} = -\frac{mg}{\lambda}$$

To see what happens in the absence of air resistance, take the limit as $\lambda \to 0$. *Now*

$$\begin{aligned}
\lim_{\lambda\to 0} z(t) &= z_0 + \lim_{\lambda\to 0}\left(\frac{m}{\lambda}\right)^2 g\left(1 - e^{-\frac{\lambda}{m}t} - \frac{\lambda}{m}t\right) \\
&= z_0 + g\lim_{\alpha\to 0}\frac{1}{\alpha^2}\left(1 - e^{-\alpha t} - \alpha t\right) \\
&= z_0 + g\lim_{\alpha\to 0}\frac{1}{\alpha^2}\left[(1-\alpha t) - \left(1 - \alpha t + \frac{\alpha^2 t^2}{2} - \cdots\right)\right] \\
&= z_0 - \frac{gt^2}{2}
\end{aligned}$$

which is the free-fall limit. ■

EXAMPLE 3.2
Find the height reached by a particle launched vertically upwards with speed v from the Earth's surface with air resistance proportional to the square of the particle's speed.
Solution: *The equation of motion is*

$$m\ddot{z} = -mg - \lambda\dot{z}^2$$

subject to $z = 0$ *and* $\dot{z} = v$ *at* $t = 0$.
The differential equation is

$$\ddot{z} = -g - \frac{\lambda}{m}\dot{z}^2$$

But $\ddot{z} = \frac{d}{dz}\left(\frac{1}{2}\dot{z}^2\right)$ *so put* $w = \dot{z}^2/2$ *in the equation of motion to give*

$$\frac{dw}{dz} = -g - \frac{2\lambda}{m}w$$

or

$$\frac{dw}{dz} + \frac{2\lambda}{m}w = -g$$

$w(z)$ *is the sum of the homogeneous solution and a particular solution. For the homogeneous solution, trying* $w = e^{\alpha z}$ *shows that* $\alpha = -2\lambda/m$. *For the particular solution, try*

$$w_p = A$$

Substituting into the o.d.e. shows that

$$\frac{2\lambda}{m}A = -g$$

so that the general solution is

$$w(z) = Ce^{-\frac{2\lambda}{m}z} - \frac{mg}{2\lambda}$$

At $z = 0$, $w = v^2/2$ *so*

$$\frac{1}{2}v^2 = C - \frac{mg}{2\lambda}$$

Therefore

$$w(z) = \left(\frac{v^2}{2} + \frac{mg}{2\lambda}\right) e^{-\frac{2\lambda}{m}z} - \frac{mg}{2\lambda}$$

At the greatest height reached, the velocity is zero, hence

$$w(z_{\max}) = 0$$

Therefore

$$0 = \left(\frac{v^2}{2} + \frac{mg}{2\lambda}\right) e^{-\frac{2\lambda}{m}z_{\max}} - \frac{mg}{2\lambda}$$

so that

$$z_{\max} = \frac{m}{2\lambda} \ln\left[1 + \frac{\lambda v^2}{mg}\right]$$

To find the greatest height reached when there is no air resistance, take the limit as $\lambda \to 0$. *Since, for small* x,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

integrating term-by-term gives

$$\ln(1+x) = \int^x \frac{dx}{1+x} = x - \frac{x^2}{x} + \frac{x^3}{3} - \cdots$$

Therefore

$$z_{\max} = \frac{m}{2\lambda}\left[\frac{\lambda v^2}{mg} - \frac{1}{2}\left(\frac{\lambda v^2}{mg}\right)^2 + \frac{1}{3}\left(\frac{\lambda v^2}{mg}\right)^3 - \cdots\right]$$

This can be written in the form

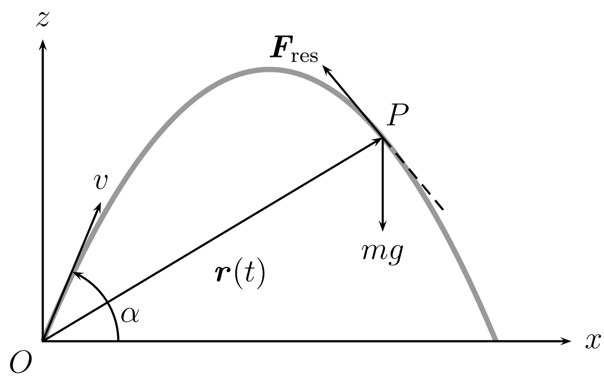
$$z_{\max} = \frac{v^2}{2g} + O(\lambda)$$

where $O(\lambda)$ *indicates terms of order* λ *and higher. In the limit as* $\lambda \to 0$, *the* $O(\lambda)$ *terms tend to zero, leaving*

$$z_{\max} = \frac{v^2}{2g}$$

■

Figure 3.1: Motion of a projectile with air resistance $\boldsymbol{F}_{\text{res}}$. The projectile's initial velocity is v at an angle α to the horizontal.



3.2 Motion in a Plane

For this analysis of projectile motion, ignore the Earth's rotation and assume that the height of the projectile is always much smaller than the radius of the Earth.

From figure 3.1, the equation of motion is

$$m\ddot{\boldsymbol{r}} = -mg\hat{\boldsymbol{k}} + \boldsymbol{F}_{\text{res}}$$

where the air resistance has magnitude $\phi(v)$ and is directed in the opposite direction to the projectile's motion

$$\boldsymbol{F}_{\text{res}} = -\phi(v)\hat{\dot{\boldsymbol{r}}} = -\frac{\phi(v)}{v}\dot{\boldsymbol{r}}$$

where the projectile's speed is

$$v = |\dot{\boldsymbol{r}}| = \sqrt{\dot{x}^2 + \dot{z}^2}$$

The equation of motion can be written as separate equations for the x and z directions

$$\ddot{x} = -\frac{\phi(v)}{mv}\dot{x}$$

and

$$\ddot{z} = -g - \frac{\phi(v)}{mv}\dot{z}$$

The solution of these depend on the form of the air resistance function $\phi(v)$.

EXAMPLE 3.3
When $\phi(v) = 0$ and there is no air resistance, the equations of motion become

$$\ddot{x} = 0 \qquad \text{and} \qquad \ddot{z} = -g$$

subject to the initial conditions $x(0) = z(0) = 0$, $\dot{x}(0) = v\cos\alpha$ and $\dot{z}(0) = v\sin\alpha$.

The solution is

$$x(t) = vt\cos\alpha$$
$$z(t) = vt\sin\alpha - \frac{1}{2}gt^2$$

Eliminating t from these two equations gives the equation of the projectile's trajectory

$$z = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha}$$

which is a parabola. Factorising z as

$$z = \frac{x}{\cos\alpha}\left(\sin\alpha - \frac{gx}{2v^2\cos\alpha}\right)$$

shows that $z = 0$ when $x = 0$ and when $x = 2v^2\sin\alpha\cos\alpha/g$ so that the horizontal range is

$$x_{\max} = \frac{v^2}{g}\sin 2\alpha$$

This range is maximised when $\sin 2\alpha = 1$, which is an elevation of $\alpha = \pi/4$.

The maximum height occurs when $\dot{z}(t) = 0$, which occurs when

$$t = \frac{v\sin\alpha}{g}$$

Therefore the maximum height as a function of the projectile's initial elevation is

$$z_{\max}(\alpha) = \frac{v^2\sin^2\alpha}{2g}$$

which occurs when x is equal to half the range. Note that

$$z_{\max}\left(\frac{\pi}{4}\right) = \frac{v^2}{4g}$$

which is equal to one-quarter of the maximum range. ∎

Now try the problem again assuming that the ground is at an angle β so that when the projectile lands, x and z must satisfy

$$z = x \tan \beta$$

Combine this with the equation for the projectile's motion to find the x and z coordinates of the point when the projectile lands.

EXAMPLE 3.4

When $\phi(v) = \lambda v$ and the air resistance is proportional to the projectile's speed, the equations of motion become

$$\ddot{x} = -\frac{\lambda}{m}\dot{x}$$

and

$$\ddot{z} = -g - \frac{\lambda}{m}\dot{z}$$

subject to the initial conditions $x(0) = z(0) = 0$, $\dot{x}(0) = v\cos\alpha$ and $\dot{z}(0) = v\sin\alpha$.

This has already been solved for z with different boundary conditions in example 3.1, so the solution is

$$z(t) = A + Be^{-\frac{\lambda}{m}t} - \frac{m}{\lambda}gt$$

The boundary conditions are

$$z(0) = 0 = A + B$$

and

$$\dot{z}(0) = v\sin\alpha = -\frac{\lambda}{m}B - \frac{mg}{\lambda}$$

Therefore

$$A = -B = \left(\frac{m}{\lambda}\right)^2 g + \frac{m}{\lambda}v\sin\alpha$$

so the z solution is

$$z(t) = \left[\left(\frac{m}{\lambda}\right)^2 g + \frac{m}{\lambda}v\sin\alpha\right]\left(1 - e^{-\frac{\lambda}{m}t}\right) - \frac{m}{\lambda}gt$$

The general solution for x is

$$x(t) = C + De^{-\frac{\lambda}{m}t}$$

The boundary conditions are

$$x(0) = 0 = C + D$$

and

$$\dot{x}(0) = v\cos\alpha = -\frac{\lambda}{m}D$$

Therefore

$$C = -D = \frac{m}{\lambda}v\cos\alpha$$

so the x solution is

$$x(t) = \frac{m}{\lambda}v\cos\alpha\left(1 - e^{-\frac{\lambda}{m}t}\right)$$

Eliminating t from the equations for $x(t)$ and $z(t)$ gives

$$z = \left(\frac{v\sin\alpha + mg/\lambda}{v\cos\alpha}\right)x + \left(\frac{m}{\lambda}\right)^2 g\ln\left(1 - \frac{\lambda x}{mv\cos\alpha}\right)$$

which is plotted in figure 3.2. ∎

Find expressions for the points x_1 and x_3 in figure 3.2.

EXAMPLE 3.5

The general expression for air resistance is $\phi(v) = \lambda v^m$ for some power m. The equation of motion is

$$m\ddot{\boldsymbol{r}} = -mg\hat{\boldsymbol{k}} - \phi(v)\,\hat{\boldsymbol{T}}$$

The velocity of the projectile is just the speed in the tangential direction $\dot{\boldsymbol{r}} = \dot{s}\,\hat{\boldsymbol{T}}$ so

$$|\dot{\boldsymbol{r}}| = v = \dot{s}$$

From

$$\ddot{\boldsymbol{r}} = \ddot{s}\,\hat{\boldsymbol{T}} + \dot{s}\dot{\psi}\hat{\boldsymbol{N}}$$

the equation of motion in the $\hat{\boldsymbol{T}}$ direction is

$$\begin{aligned}\ddot{s} &= -g\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{T}} - \frac{\phi(\dot{s})}{m}\\ &= -g\sin\psi - \frac{\phi(\dot{s})}{m}\end{aligned}$$

and in the $\hat{\boldsymbol{N}}$ direction,

$$\begin{aligned}\dot{s}\dot{\psi} &= -g\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{N}}\\ &= -g\cos\psi\end{aligned}$$

The remainder of the solution depends on the form of $\phi(\dot{s})$. ∎

Figure 3.2: For projectile motion with air resistance, three points x_1, x_2 and x_3 can be defined. x_1 is the horizontal position of the projectile's greatest elevation, x_2 the point at which the projectile hits the ground, and x_3 the position of the vertical asymptote if the projectile were to continue falling through the ground.

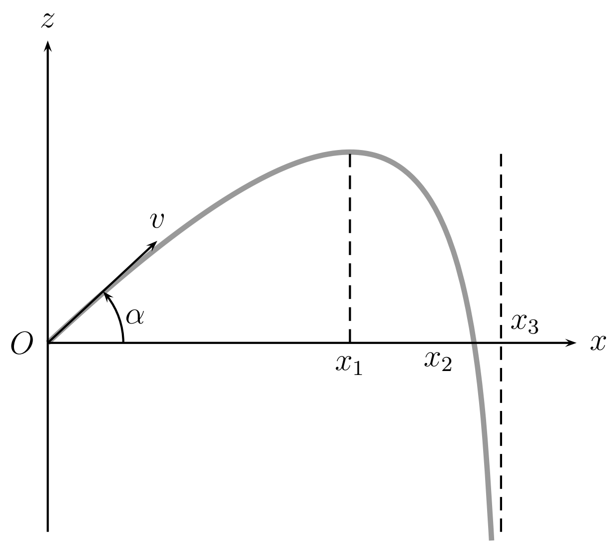
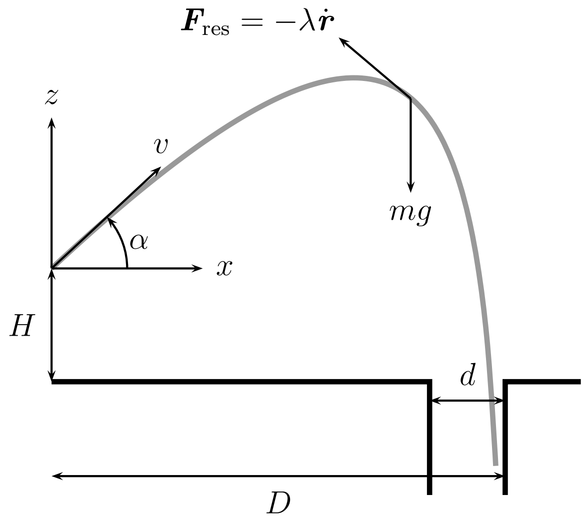


Figure 3.3: Motion of the missile for the Death Star problem.



EXAMPLE 3.6 (The Death Star)
From the diagram in figure 3.3, you have to fire a missile that falls down a narrow chute of width d. The missile is launched at an initial angle α with speed v from a height H and a distance D from the far edge of the chute. Given that air resistance is proportional to speed, calculate the maximum value of D. Also calculate the minimum height H from which the missile can be released at this maximum distance D and still fall down the chute.
Solution: *With air resistance proportional to speed, the equation of motion is*

$$m\ddot{\boldsymbol{r}} = -mg\hat{\boldsymbol{k}} - \lambda\dot{\boldsymbol{r}}$$

This is identical to the situation considered in example 3.4 with solution

$$z = \left(\frac{v\sin\alpha + mg/\lambda}{v\cos\alpha}\right)x + \left(\frac{m}{\lambda}\right)^2 g\ln\left(1 - \frac{\lambda x}{mv\cos\alpha}\right)$$

The maximum horizontal range is the furthest x distance reached as $z \to -\infty$. This occurs when

$$1 - \frac{\lambda x}{mv\cos\alpha} = 0$$

or when

$$x = \frac{mv}{\lambda}\cos\alpha$$

The maximum range is maximised when $\alpha = 0$, so this is achieved when the missile is fired horizontally, and

$$D = x_{\max} = \frac{mv}{\lambda}$$

Putting $\alpha = 0$, the trajectory becomes

$$z = \left(\frac{mg}{\lambda v}\right)x + \left(\frac{m}{\lambda}\right)^2 g\ln\left(1 - \frac{\lambda x}{mv}\right)$$

The trajectory with minimum height just grazes the inner lip of the chute and the asymptote is the chute's back wall. Therefore the trajectory passes through

$$z = -H \qquad \text{at} \qquad x = D - d$$

Substituting this into the trajectory with $\alpha = 0$ gives

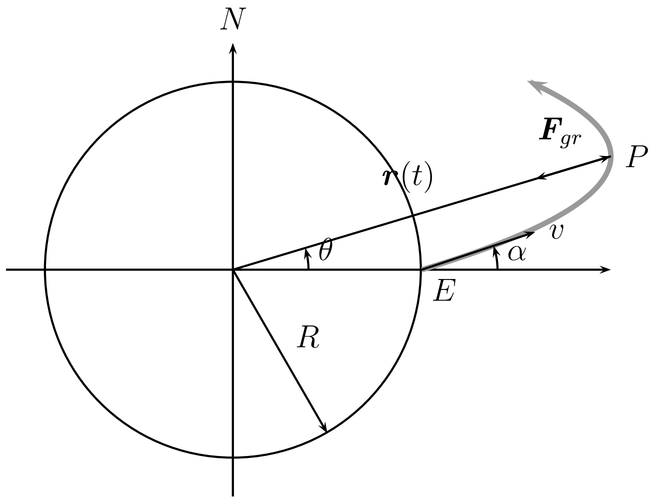
$$-H = \left(\frac{mg}{\lambda v}\right)(D-d) + \left(\frac{m}{\lambda}\right)^2 g\ln\left(1 - \frac{\lambda(D-d)}{mv}\right)$$

which gives

$$H = \frac{gD^2}{v^2}\left[-\ln\left(\frac{d}{D}\right) + \frac{d}{D} - 1\right]$$

■

Figure 3.4: This shows a projectile P launched from the equator E of an airless planet with speed v at an angle α to the vertical towards the north. $\boldsymbol{F}_{\mathrm{gr}}$ is the gravitational attraction.



EXAMPLE 3.7
A projectile is fired from the equator E of an airless planet of radius R northwards with initial speed v and at an angle α to the vertical. The acceleration due to gravity at the surface is g. This is illustrated in figure 3.4.

(a) Show that the projectile's motion satisfies

$$\ddot{r} - r\dot{\theta}^2 = -\frac{gR^2}{r^2}$$

and

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

(b) Hence show that

$$\dot{\theta} = \frac{Rv\sin\alpha}{r^2}$$

(c) Also show that the maximum height of the projectile is given by

$$\frac{r}{R} = \frac{1}{2(\lambda - 1)}\left(\sqrt{\lambda^2 - 4(\lambda - 1)\sin^2\alpha} + \lambda\right)$$

where $\lambda = 2gR/v^2$ provided $\lambda > 1$.

Solution: *With m the projectile's mass and M the planet's mass, the force due to gravity is*

$$\boldsymbol{F} = -\frac{GmM}{r^2}\hat{\boldsymbol{r}}$$

At the surface, when $r = R$, the force due to gravity is

$$\boldsymbol{F} = -mg\hat{\boldsymbol{r}} = -\frac{GmM}{R^2}\hat{\boldsymbol{r}}$$

so that

$$\boldsymbol{F} = -\frac{mgR^2}{r^2}\hat{\boldsymbol{r}}$$

There is no air resistance, so from Newton's second law,

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}$$

Choosing a polar coordinate system,

$$\ddot{\boldsymbol{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} = -\frac{gR^2}{r^2}\hat{\boldsymbol{r}}$$

Equating radial and polar components yields

$$\ddot{r} - r\dot{\theta}^2 = -\frac{gR^2}{r^2}$$

and

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

as required.

From the initial conditions, $r(0) = R$ and $\theta(0) = 0$. Initially,

$$\dot{\boldsymbol{r}}(0) = v\cos\alpha\hat{\boldsymbol{r}} + v\sin\alpha\hat{\boldsymbol{\theta}}$$

Using

$$\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

shows that

$$\dot{r}(0) = v\cos\alpha$$

and

$$R\dot{\theta}(0) = v\sin\alpha$$

To show that

$$\dot{\theta} = \frac{Rv\sin\alpha}{r^2}$$

multiply

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

by r *and use product rule to show that*

$$\frac{d}{dt}(r^2\dot{\theta}) = 0$$

Integrating shows that

$$r^2\dot{\theta} = C$$

for some constant C. *At* $t = 0$, $r = R$ *and* $\dot{\theta} = v \sin\alpha/R$. *Therefore*

$$r^2\dot{\theta} = R^2 \frac{v}{R} \sin\alpha$$

so that

$$\dot{\theta} = \frac{Rv\sin\alpha}{r^2}$$

To show that the maximum height of the projectile is given by

$$\frac{r}{R} = \frac{1}{2(\lambda - 1)} \left(\sqrt{\lambda^2 - 4(\lambda - 1)\sin^2\alpha} + \lambda \right)$$

where $\lambda = 2gR/v^2$ *provided* $\lambda > 1$, *we know that*

$$\dot{\theta} = \frac{Rv\sin\alpha}{r^2}$$

and

$$\ddot{r} - r\dot{\theta}^2 = -\frac{gR^2}{r^2}$$

Substituting for $\dot{\theta}$ *shows that*

$$\ddot{r} = \frac{v^2R^2\sin^2\alpha}{r^3} - \frac{gR^2}{r^2}$$

Using the usual identity,

$$\ddot{r} = \frac{d}{dr}\left(\frac{1}{2}\dot{r}^2\right) = \frac{v^2R^2\sin^2\alpha}{r^3} - \frac{gR^2}{r^2}$$

Integrating with respect to r *gives*

$$\frac{1}{2}\dot{r}^2 = -\frac{1}{2}\frac{v^2R^2\sin^2\alpha}{r^2} + \frac{gR^2}{r} + C$$

At time $t = 0$, $r = R$ and $\dot{r} = v\cos\alpha$ so

$$\frac{1}{2}v^2\cos^2\alpha = -\frac{1}{2}\frac{v^2R^2\sin^2\alpha}{R^2} + \frac{gR^2}{R} + C$$

which determines the constant

$$C = \frac{1}{2}v^2 - gR$$

Therefore, a first integral of the radial equation is

$$\frac{1}{2}\dot{r}^2 = \frac{1}{2}v^2 - gR - \frac{1}{2}\frac{v^2R^2\sin^2\alpha}{r^2} + \frac{gR^2}{r}$$

The maximum height occurs when $\dot{r} = 0$, thus

$$0 = \frac{1}{2}v^2 - gR - \frac{1}{2}\frac{v^2R^2\sin^2\alpha}{r^2} + \frac{gR^2}{r}$$

Divide by $v^2/2$ and rearrange to give the quadratic

$$\left(\frac{r}{R}\right)^2(\lambda - 1) - \lambda\frac{r}{R} + \sin^2\alpha = 0$$

Therefore

$$\frac{r}{R} = \frac{1}{2(\lambda - 1)}\left(\lambda \pm \sqrt{\lambda^2 - 4(\lambda - 1)\sin^2\alpha}\right)$$

To determine which of the two solutions is appropriate, note that as $v \to 0$, r/R must tend to 1. As $v \to 0$, $\lambda \to \infty$, showing that the positive solution is needed. This gives the maximum height as

$$\frac{r}{R} = \frac{1}{2(\lambda - 1)}\left(\sqrt{\lambda^2 - 4(\lambda - 1)\sin^2\alpha} + \lambda\right)$$

as required.

Note that if $\lambda < 1$, then $\sqrt{\lambda^2 - 4(\lambda - 1)\sin^2\alpha} > \lambda$ so the negative solution must be chosen to ensure that $r/R > 0$. In this case,

$$\frac{r}{R} = \frac{1}{2(1 - \lambda)}\left(\sqrt{\lambda^2 - 4(\lambda - 1)\sin^2\alpha} - \lambda\right) < 1$$

so no maximum height is reached. When $\lambda = 1$, the speed of the projectile is exactly equal to the escape velocity of the planet. ■

3.3 Momentum and Torque

Linear momentum $\boldsymbol{p}$ is defined as

$$\boldsymbol{p} = m\dot{\boldsymbol{r}}$$

From Newton's second law, in an inertial reference frame,

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}} = \frac{d}{dt}(m\dot{\boldsymbol{r}})$$

so that

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}$$

When $\boldsymbol{F} = 0$, $\boldsymbol{p}$ is constant, so linear momentum is conserved when no net force acts on the particle.

The **angular momentum $\boldsymbol{L}$** of a particle about the point O is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

where $\boldsymbol{r}$ is the particle's position relative to O and $\boldsymbol{p}$ is the particle's linear momentum. Note that

$$\frac{d\boldsymbol{L}}{dt} = \frac{d}{dt}(\boldsymbol{r} \times (m\dot{\boldsymbol{r}})) = \dot{\boldsymbol{r}} \times (m\dot{\boldsymbol{r}}) + \boldsymbol{r} \times (m\ddot{\boldsymbol{r}})$$

from product rule. The first cross-product is zero because $\dot{\boldsymbol{r}}$ and $m\dot{\boldsymbol{r}}$ are parallel. Writing $\boldsymbol{F} = m\ddot{\boldsymbol{r}}$ shows that

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \boldsymbol{F}$$

provided the reference frame is inertial.

The **torque** of a force $\boldsymbol{F}$ about a point O is $\boldsymbol{r} \times \boldsymbol{F}$ where $\boldsymbol{r}$ is the position vector from O to the point of application of the force.

Therefore, torque is equal to the rate of change of angular momentum. Angular momentum is conserved when no torque acts.

3.4 Central Forces

A force $\boldsymbol{F}$ which always acts along the position vector $\boldsymbol{r}$ is called a **central force**.

$$\boldsymbol{F} = f(r)\hat{\boldsymbol{r}}$$

THEOREM 3.1
The trajectory of a particle acted on by a central force lies in a plane.

Proof:
At any instant t, the two vectors $\boldsymbol{r}(t)$ and $\dot{\boldsymbol{r}}(t)$ define a plane. The acceleration is
$$\ddot{\boldsymbol{r}} = \frac{1}{m}\boldsymbol{F} = \frac{1}{m}f(r)\hat{\boldsymbol{r}}$$
which lies in the plane. Thus the particle's subsequent motion is confined to that plane. ■

Therefore central forces imply motion in a plane.

THEOREM 3.2
The angular momentum of a particle in a central force field is constant.

Proof:
The rate of change is angular momentum is
$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r} \times f(r)\hat{\boldsymbol{r}} = 0$$
because $\boldsymbol{F}$ is parallel to $\boldsymbol{r}$. Therefore $\boldsymbol{L}$ is constant. ■

The physical significance of constant angular momentum is related to the area swept out by the particle's position vector $\boldsymbol{r}(t)$. Suppose that $\boldsymbol{r}(t)$ sweeps out an area $A(t)$ as it moves in the plane. In a short time δt, the area swept out is
$$\delta A \approx \frac{1}{2}\left|\boldsymbol{r}(t+\delta t) \times \boldsymbol{r}(t)\right|$$
Now $\boldsymbol{r}(t+\delta t) = \boldsymbol{r}(t) + \delta\boldsymbol{r}$ so
$$\boldsymbol{r}(t+\delta t) \times \boldsymbol{r}(t) = \delta\boldsymbol{r} \times \boldsymbol{r}(t)$$
Therefore
$$\frac{dA}{dt} = \lim_{\delta t \to 0}\frac{\delta A}{\delta t} = \lim_{\delta t \to 0}\frac{1}{2}\left|\boldsymbol{r} \times \frac{\delta\boldsymbol{r}}{\delta t}\right| = \frac{1}{2}\left|\boldsymbol{r} \times \dot{\boldsymbol{r}}\right|$$
Writing
$$\frac{1}{2}\left|\boldsymbol{r} \times \dot{\boldsymbol{r}}\right| = \frac{1}{2m}\left|\boldsymbol{r} \times (m\dot{\boldsymbol{r}})\right|$$
shows that
$$\frac{dA}{dt} = \frac{1}{2m}\left|\boldsymbol{L}\right|$$

Since angular momentum is constant in a central force field, we have

$$\frac{dA}{dt} = \text{constant}$$

This is Kepler's second rule of planetary motion (1609) which was explained by Newton as a property of central forces in his *Principia* of 1687.

In polar coordinates,

$$\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

so that

$$\boldsymbol{L} = \boldsymbol{r} \times (m\dot{\boldsymbol{r}}) = m(r\hat{\boldsymbol{r}}) \times (\dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}})$$

Therefore

$$\boldsymbol{L} = mr^2\dot{\theta}(\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}})$$

Now $\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}}$ is the unit vector normal to the plane in which the particle moves, so the magnitude of the angular momentum is

$$|\boldsymbol{L}| = mr^2 \left|\dot{\theta}\right|$$

Substituting this back into the equation for $\frac{dA}{dt}$ shows that

$$\frac{dA}{dt} = \frac{1}{2}r^2 \left|\dot{\theta}\right|$$

3.5 Orbits in a Central Force Field

The equation of motion is

$$m\ddot{\boldsymbol{r}} = f(r)\hat{\boldsymbol{r}}$$

where the acceleration can be expressed as

$$\ddot{\boldsymbol{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}}$$

in polar coordinates.

The radial component is

$$\ddot{r} - r\dot{\theta}^2 = \frac{f(r)}{m}$$

and the transverse component is

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

This can be solved for $\dot{\theta}$ by writing it as

$$\frac{d}{dt}\dot{\theta} + 2\frac{\dot{r}}{r}\dot{\theta} = 0$$

The integrating factor is r^2 which shows that

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

so that $r^2\dot{\theta}$ is a constant. Let this constant be h so that

$$r^2\dot{\theta} = h$$

Note that $\frac{1}{2}|h| = \frac{dA}{dt} = \frac{1}{2m}|\boldsymbol{L}|$.

To obtain the equation of the orbit in the form $r(\theta)$, use

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{h}{r^2}\frac{dr}{d\theta}$$

and

$$\ddot{r} = \frac{h}{r^2}\frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right)$$

The radial equation of motion can then be written as

$$\frac{h^2}{r^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - r\left(\frac{h}{r^2}\right)^2 = \frac{f(r)}{m}$$

Rearranging this gives

$$\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{1}{r} = \frac{r^2 f(r)}{mh^2}$$

which can be solved to give $r(\theta)$.

3.5.1 Motion in a Gravitational Field

Let the central force be

$$f(r) = -\frac{GMm}{r^2}$$

where M is the mass of the Sun at the origin and m is the mass of the body. We will asssume that $M \gg m$ so that the Sun may be assumed to be fixed at the origin.

The equation for the orbit becomes

$$\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{1}{r} = -\frac{GM}{h^2}$$

Since

$$\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{d}{d\theta}\left(\frac{1}{r}\right)$$

this can be written

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{GM}{h^2}$$

The general solution is

$$\frac{1}{r} = B\cos\theta + C\sin\theta + \frac{GM}{h^2} = A\cos(\theta - \theta_0) + \frac{GM}{h^2}$$

where A and θ_0 are arbitrary constants giving the orbit's amplitude and phase angle. In terms of r, this is

$$r = \frac{1}{A\cos(\theta - \theta_0) + GM/h^2}$$

A and θ_0 are determined by the initial conditions. Since θ_0 determines only the orientation of the orbit in the plane, we can set it to zero when discussing the orbit's shape.

Define the orbit's **eccentricity** to be

$$e = \frac{Ah^2}{GM}$$

and r_0 to be the **perihelion distance**

$$r_0 = \frac{h^2/GM}{1+e}$$

which is the smallest distance between the body and the Sun during the orbit. Then the orbit equation can be written

$$r = r_0\frac{1+e}{1+e\cos\theta}$$

so that $r = r_0$ when $\theta = 0$ and $r > r_0$ for all other θ during each orbit.

The shape of the orbit depends on the eccentricity e. If $e = 0$, the orbit is a circle, and for $0 < e < 1$, the orbit is an ellipse. Both of these are closed, periodic orbits. When $e = 1$, the orbit is a parabola and for $e > 1$, the orbit is a hyperbola. Both of these are open, single encounter orbits.

3.5.2 Cartesian Equation of an Orbit

The orbit equation can be written

$$r(1 + e\cos\theta) = r_0(1 + e)$$

Writing $L = r_0(1 + e)$ and putting $x = r\cos\theta$, we have

$$r = L - ex$$

Squaring both sides so that $r^2 = x^2 + y^2$,

$$x^2 + y^2 = (L - ex)^2$$

thus

$$(1 - e^2)x^2 + 2Lex + y^2 = L^2$$

Completing the square gives

$$\frac{(x - x_0)^2}{L^2/(1 - e^2)^2} + \frac{y^2}{L^2/(1 - e^2)} = 1$$

where

$$x_0 = -\frac{Le}{1 - e^2}$$

When $e = 0$, this simplifies to the equation of a circle

$$x^2 + y^2 = L^2$$

When $0 < e < 1$, this simplifies to the equation of an ellipse

$$\frac{(x - x_0)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a = L/(1 - e^2)$ and $b = L/\sqrt{1 - e^2}$.

When $e = 1$, the equation becomes that of a parabola

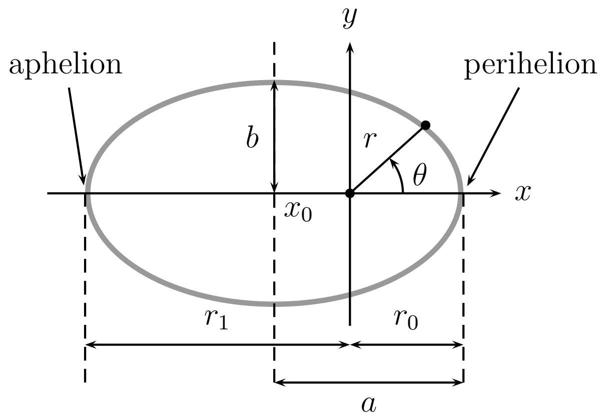
$$y^2 = L^2 - 2Lx$$

and when $e > 1$, it becomes the equation of a hyperbola

$$\frac{(x - x_0)^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $a = L/(e^2 - 1)$ and $b = L/\sqrt{e^2 - 1}$.

Figure 3.5: Planets move in an ellipse around the Sun. Their closest approach is at perihelion and their furthest distance is at aphelion.



3.5.3 Kepler's Laws

Centuries of increasingly accurate astronomical observations culminated in Kepler's laws of planetary motion (1609):

1. Each planet moves in an ellipse with the Sun as a focus.
2. The radius vector sweeps out equal areas in equal times ($\frac{dA}{dt}$ is constant).
3. The square of the period of revolution about the Sun is proportional to the cube of the major axis of the orbit.

Law 2 is a consequence of gravity being a central force. Law 1 is a consequence of gravity being an inverse square attractive force.

3.5.4 Elliptical Planetary Orbits

The equations for a planet's orbit are

$$r = r_0 \frac{1+e}{1+e\cos\theta}$$

or

$$\frac{(x-x_0)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a = L/(1-e^2)$ and $b = L/\sqrt{1-e^2}$ (so $a > b$) and $x_0 = -Le/(1-e^2)$.

This elliptic orbit is shown in figure 3.5. The **aphelion distance**, r_1, is related to the **perihelion distance**, r_0, according to

$$r_1 = r_0 \frac{1+e}{1-e}$$

The area of the ellipse is

$$A = \pi ab = \frac{\pi L^2}{(1 - e^2)^{3/2}}$$

The rate at which the ellipse's area is swept out is

$$\frac{dA}{dt} = \frac{h}{2} = \frac{1}{2} r^2 \dot{\theta}$$

The period T of the orbit is thus

$$T = \frac{A}{\frac{dA}{dt}} = \frac{2\pi L^2}{h(1 - e^2)^{3/2}}$$

so that T^2 is

$$T^2 = \frac{4\pi^2 L^4}{h^2(1 - e^2)^3} = \frac{4\pi^2 a^3 L}{h^2}$$

where $a = L/(1 - e^2)$. But $L = h^2/GM$ so

$$\frac{a^3}{T^2} = \frac{GM}{4\pi^2}$$

which is the same for all planets. This is the explanation of Kepler's third law given by Newton.

3.5.5 Orbit in Terms of Perihelion Parameters

From the definition of r_0,

$$e = \frac{h^2}{GMr_0} - 1$$

At perihelion, $r = r_0$ and $\dot{r} = 0$. The speed at perihelion v_0 is

$$v_0 = r_0 \dot{\theta}_{\text{per}}$$

But $h = r^2 \dot{\theta}$ so at perihelion,

$$h = r_0^2 \dot{\theta}_{\text{per}} = r_0 v_0$$

Therefore the eccentricity can be written

$$e = \frac{r_0 v_0^2}{GM} - 1$$

Define v_c as the perihelion speed necessary to give a circular orbit

$$v_c = \sqrt{\frac{GM}{r_0}}$$

so that another way of writing the eccentricity is

$$e = \left(\frac{v_0}{v_c}\right)^2 - 1$$

Then the orbit equation can be written as

$$r = \frac{r_0 \left(v_0/v_c\right)^2}{1 + \left[\left(v_0/v_c\right)^2 - 1\right] \cos\theta}$$

At aphelion, $r = r_1$ and $\theta = \pi$ so that

$$r_1 = \frac{r_0 \left(v_0/v_c\right)^2}{2 - \left(v_0/v_c\right)^2}$$

EXAMPLE 3.8

Find the speed of a satellite executing a circular orbit around the Earth.

Solution: *Assuming that the Earth's gravitational field dominates over the Sun's field, the satellite is moving in a central force field with*

$$f(r) = -\frac{GM_e m}{r^2}$$

where M_e is the mass of the Earth.

The critical speed for a circular orbit at $r = r_0$ is

$$v_c = \sqrt{\frac{GM_e}{r_0}}$$

Note that $g = GM_e/R_e^2$ where R_e is the Earth's radius so this critical velocity can also be written

$$v_c = \sqrt{\frac{gR_e^2}{r_0}}$$

For a satellite close to the Earth, $r_0 \approx R_e$ so

$$v_c \approx \sqrt{gR_e} = 7920 \text{ ms}^{-1}$$

■

EXAMPLE 3.9
Calculate the escape velocity at perihelion from the solar system.
Solution: *When* $e = 1$*, the orbit is open and so the orbiting body may escape. Since* $e = (v_0/v_c)^2 - 1$*,* $e = 1$ *when* $v_0 = \sqrt{2}v_c$*. Therefore the escape velocity is*

$$v_c = \sqrt{\frac{2gR_e^2}{r_0}}$$

■

EXAMPLE 3.10
A satellite in a circular orbit r_0 fires its rocket to increase its speed suddenly by a factor of α. Calculate the new apogee distance.
Solution: *The old speed was*

$$v_c = \sqrt{\frac{GM_e}{r_0}}$$

The new speed

$$v_0 = (1 + \alpha)v_c$$

is the perihelion speed since the satellite will immediately move to increase r*. The new orbit is*

$$r = \frac{r_0\,(v_0/v_c)^2}{1 + \left[(v_0/v_c)^2 - 1\right]\cos\theta} = \frac{r_0(1+\alpha)^2}{1 + [(1+\alpha)^2 - 1]\cos\theta}$$

The apogee distance is

$$r = \frac{r_0(1+\alpha)^2}{2 - (1+\alpha)^2}$$

Note that when $\alpha = 0.15$*,* $r_1 = 1.95r_0$*. When* $\alpha = \sqrt{2} - 1 \approx 0.4$*,* $r_1 = \infty$ *and the satellite escapes.* ■

3.5.6 Inverse Cubic Attraction

In general, the equation of orbital motion is

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{r^2 f(r)}{mh^2}$$

When the attraction follows an inverse cubic law,

$$f(r) = -\frac{k}{r^3}$$

the orbital motion with inverse cubic attraction is found by solving

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{k}{mh^2}\frac{1}{r}$$

which may be written as

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)-\left(\frac{k}{mh^2}-1\right)\left(\frac{1}{r}\right)=0$$

When $k/mh^2 > 1$, the solution is

$$\frac{1}{r}=Ae^{-\alpha\theta}+Be^{\alpha\theta}$$

where

$$\alpha=\sqrt{\frac{k}{mh^2}-1}$$

and A and B are determined by the initial conditions. If $B > 0$, $1/r \to Be^{\alpha\theta} \to \infty$ and the satellite follows a path that spirals into the planet. If $B < 0$, $1/r$ will vanish (which means that $r \to \infty$) at some finite value of θ. If $B = 0$, $1/r = Ae^{-\alpha\theta}$ so $r \to \infty$ and the satellite escapes from the planet.

When $k/mh^2 = 1$, the solution is

$$\frac{1}{r}=A+B\theta$$

If $B > 0$, $1/r \to \infty$ as θ increases, so the satellite is captured in a decaying spiralling orbit. If $B < 0$, $1/r \to 0$ as θ tends to some finite θ_∞ and the satellite escapes. If $B = 0$, $1/r = A$ and the satellite is in a closed circular orbit.

When $k/mh^2 < 1$, the solution is

$$\frac{1}{r}=A\cos\alpha\theta+B\sin\alpha\theta=C\cos\alpha(\theta-\theta_0)$$

where $\alpha=\sqrt{1-k/mh^2}<1$, $C=\sqrt{A^2+B^2}$ and $\tan\alpha\theta_0 = B/A$. Since $1/r \to 0$ as $\theta \to \theta_0+\pi/2\alpha$, the satellite escapes.

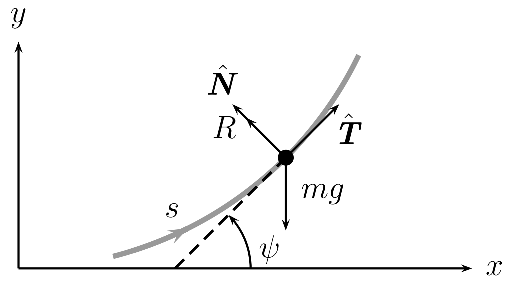
3.6 Constrained Motion

3.6.1 Frictionless Constrained Motion

Suppose a bead of mass m slides smoothly on a wire whose shape is given by the function $\boldsymbol{r}(s)$. From section 2.3.1, the bead's velocity and acceleration in intrinsic coordinates are

$$\dot{\boldsymbol{r}}=\dot{s}\,\hat{\boldsymbol{T}}$$

Figure 3.6: This shows a bead moving on a smooth wire. There is no friction, so the only forces on the bead are gravity mg and the reaction R.



$$\ddot{\boldsymbol{r}} = \ddot{s}\hat{\boldsymbol{T}} + \dot{s}\dot{\psi}\hat{\boldsymbol{N}}$$

The intrinsic coordinates are illustrated in figure 3.6.

For a smooth wire, the only force the wire can exert on the bead is a **reaction** that is **normal**, $R\hat{\boldsymbol{N}}$ (R can be either positive or negative).

The only other force is gravity, giving the total force

$$\boldsymbol{F} = -mg\hat{\boldsymbol{j}} + R\hat{\boldsymbol{N}}$$

From Newton's second law, the equation of motion is

$$m\ddot{\boldsymbol{r}} = -mg\hat{\boldsymbol{j}} + R\hat{\boldsymbol{N}}$$

The tangential component is

$$m\ddot{s} = -mg\hat{\boldsymbol{T}} \cdot \hat{\boldsymbol{j}}$$

and the normal component is

$$m\dot{s}\dot{\psi} = -mg\hat{\boldsymbol{N}} \cdot \hat{\boldsymbol{j}} + R$$

Since $\hat{\boldsymbol{T}} \cdot \hat{\boldsymbol{j}} = \sin\psi$ and $\hat{\boldsymbol{N}} \cdot \hat{\boldsymbol{j}} = \cos\psi$,

$$\ddot{s} = -g\sin\psi$$

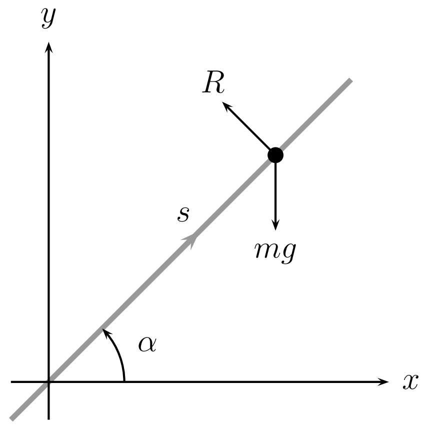
and

$$\dot{s}\dot{\psi} = -g\cos\psi + \frac{R}{m}$$

Suppose the curve can be specified in terms of $\psi = \psi(s)$. Then

$$\dot{\psi} = \frac{d\psi}{ds}\dot{s}$$

Figure 3.7: This shows a bead moving on a straight wire at an angle α to the horizontal. The distance s is measured from the origin.



where $\frac{d\psi}{ds}$ is a known function of s. Substituting this into the normal equation of motion gives

$$R = m\left[\dot{s}^2\frac{d\psi}{ds} + g\cos\psi\right]$$

Therefore, once $\dot{s}$ is known as a function of s, $R(s)$ is known at any point s on the wire. The tangential component can then be written

$$\ddot{s} = \frac{1}{2}\frac{d}{ds}\left(\dot{s}^2\right) = -g\sin\psi$$

Since $\psi(s)$ is specified, integrate the tangential equation with respect to s to obtain

$$\dot{s}^2 = -2g\int_0^s \sin\psi(s)\,ds + \left.\dot{s}^2\right|_{s=0}$$

Once $\dot{s}(s)$ is known, we can integrate again to obtain $s(t)$.

Note that it may be more convenient to solve with ψ rather than with s as the variable. For example, the tangential equation can be written

$$\frac{d\dot{s}^2}{d\psi} = \frac{d(\dot{s}^2)}{ds}\frac{ds}{d\psi} = -2g\sin\psi\frac{ds}{d\psi}$$

where $\frac{ds}{d\psi}$ is a known function of ψ.

EXAMPLE 3.11

Consider a wire at an angle α to the horizontal with the bead on the wire released from rest at a height H, as in figure 3.7. Calculate the equation of motion of the bead.

Solution: *In this case, $\psi = \alpha$ and*

$$\frac{d\dot{s}^2}{ds} = -2g \sin\alpha$$

Integrate to show that

$$\dot{s}^2 = -2gs \sin\alpha + C$$

The bead is released from a height H so when $t = 0$

$$s = \frac{H}{\sin\alpha}$$

and

$$\dot{s} = 0$$

Therefore $C = 2gH$ and

$$\dot{s}^2 = 2g\,(H - s\sin\alpha)$$

and the normal equation is

$$R = m\left(\dot{s}^2 \frac{d\psi}{ds} + g\cos\psi\right) = mg\cos\alpha$$

because ψ is independent of s.

Taking the square root of the expression for $\dot{s}^2$ gives

$$\dot{s} = -\sqrt{2g(H - s\sin\alpha)}$$

where the negative root has been selected because the motion is in the direction of s decreasing. This is a separable equation

$$\frac{ds}{\sqrt{H - s\sin\alpha}} = -\sqrt{2g}\,dt$$

Integrating gives

$$\frac{-2\sqrt{H - s\sin\alpha}}{\sin\alpha} = -\sqrt{2g}t + C$$

At $t = 0$, $s = H/\sin\alpha$ so $C = 0$ and

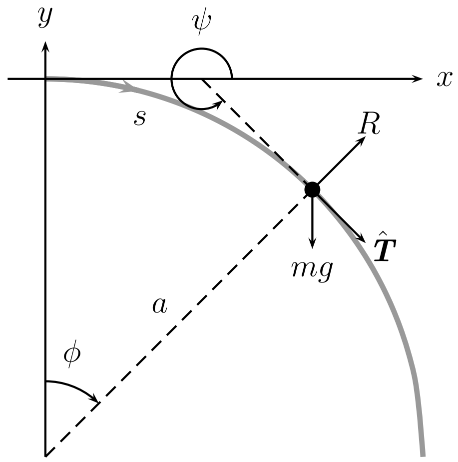
$$H - s\sin\alpha = \frac{gt^2}{2}\sin^2\alpha$$

which can be rearranged to give

$$s = \frac{H}{\sin\alpha} - \frac{1}{2}gt^2\sin\alpha$$

∎

Figure 3.8: For the particle sliding down the sphere in problem 3.12, place the top of the sphere at the origin.



EXAMPLE 3.12
A particle slides down the surface of a sphere of radius a. Initially, the particle is at the top of the sphere and is moving so slowly that it is nearly at rest. When does the particle leave the sphere?
Solution: *This is illustrated in figure 3.8, from which it can be seen that*

$$\psi = 2\pi - \phi$$

The tangential equation of motion is

$$\frac{d}{ds}(\dot{s}^2) = -2g \sin\psi = 2g \sin\phi$$

The normal equation of motion is

$$\dot{s}\dot{\psi} = -g\cos\psi + \frac{R}{m}$$

which can be written in terms of ϕ *as*

$$\dot{s}\dot{\phi} = g\cos\phi - \frac{R}{m}$$

The equation of the surface is

$$s = a\phi$$

where s is measured from the top. This shows that

$$\dot{\phi} = \frac{\dot{s}}{a}$$

and

$$R = m\left(g\cos\phi - \frac{\dot{s}^2}{a}\right)$$

The tangential equation becomes

$$\frac{d}{ds}(\dot{s}^2) = 2g\sin\phi = 2g\sin\frac{s}{a}$$

Integrating gives

$$\dot{s}^2 = -2ag\cos\frac{s}{a} + C$$

But when $t = 0$, $s = 0$ and $\dot{s} = 0$ so that $C = 2ag$. Therefore

$$\dot{s}^2 = 2ag\left(1 - \cos\frac{s}{a}\right) = 2ag\left(1 - \cos\phi\right)$$

Substituting this into the equation for the reaction R shows that

$$R = m\left(g\cos\phi - \frac{2ag(1-\cos\phi)}{a}\right) = mg(3\cos\phi - 2)$$

From this, the reaction is zero when

$$R = mg(3\cos\phi - 2) = 0$$

which occurs when

$$\cos\phi = \frac{2}{3}$$

which is point at which the particle leaves the sphere.

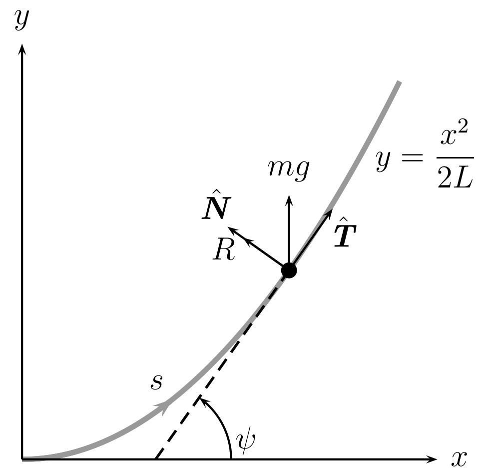
Note that in problems like this, there is nothing to sustain negative reactions, unlike the bead on the wire problem. ∎

EXAMPLE 3.13

Find the motion of a bead under gravity starting with speed u at the origin, threaded on a frictionless wire that has been bent into a parabolic shape

$$y = -\frac{x^2}{2L}$$

Figure 3.9: This diagram shows a bead on a parabolic wire with equation $y = -x^2/2L$. It has been drawn upside down so that gravity points upwards.



Solution: *This is illustrated upside down (so gravity points upwards) in figure 3.9. From this, the tangential equation of motion is*

$$m\ddot{s} = mg\sin\psi$$

and the normal equation is

$$m\dot{s}\frac{d\psi}{ds} = mg\cos\psi + R$$

To find the equation of the wire in intrinsic coordinates, use

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where the positive square root has been chosen as s increases with x.

Since $y = x^2/2L$ in the inverted diagram,

$$\frac{dy}{dx} = \frac{x}{L} = \tan\psi$$

Therefore

$$\frac{dx}{d\psi} = \frac{L}{\cos^2\psi}$$

and

$$\frac{ds}{d\psi} = \frac{ds}{dx}\frac{dx}{d\psi} = \sqrt{1 + tan^2\psi}\frac{L}{\cos^2\psi} = \frac{L}{\cos^3\psi}$$

The tangential equation of motion can be written as

$$\frac{d\dot{s}^2}{d\psi} = \frac{d\dot{s}^2}{ds}\frac{ds}{d\psi} = 2\ddot{s}\frac{ds}{d\psi} = 2Lg\frac{\sin\psi}{\cos^3\psi}$$

Integrating gives

$$\dot{s}^2 = 2gL\int_0^{\psi}\frac{sin\psi}{\cos^3\psi}\,d\psi + \dot{s}^2\big|_{\psi=0} = \left.\frac{gL}{\cos^2\psi}\right|_0^{\psi} + u^2$$

Therefore

$$\dot{s}^2 = gL\left[\frac{1}{\cos^2\psi} - 1\right] + u^2$$

Now

$$\frac{1}{\cos^2\psi} - 1 = \tan^2\psi = \frac{x^2}{L^2}$$

so

$$\dot{s}^2 = \frac{g}{L}x^2 + u^2$$

From the normal equation of motion,

$$\begin{aligned} R &= m\dot{s}^2\frac{d\psi}{ds} - mg\cos\psi \\ &= m\left[gL\left(\frac{1}{\cos^2\psi} - 1\right) + u^2\right]\frac{\cos^3\psi}{L} - mg\cos\psi \\ &= m(u^2 - gL)\cos^3\psi \end{aligned}$$

Therefore the reaction is

$$R = \frac{m(u^2 - gL)}{(1 + x^2/L^2)^{3/2}}$$

Note that the reaction is positive for $u > \sqrt{gL}$ and negative for $u < \sqrt{gL}$. ■

3.6.2 Friction

Friction is a force that acts to oppose the relative motion of rough surfaces. As shown in figure 3.10, f_{app} is the force applied to the body, R is the normal reaction force of the surface on the body and F is the frictional force on the body acting along the surface in a direction which opposes the motion that would occur if friction were absent.

Figure 3.10: The three forces on a stationary mass on a rough surface are the reaction R, an applied force f_{app} and friction F. The direction of the frictional force F opposes the motion that would occur if friction were absent.

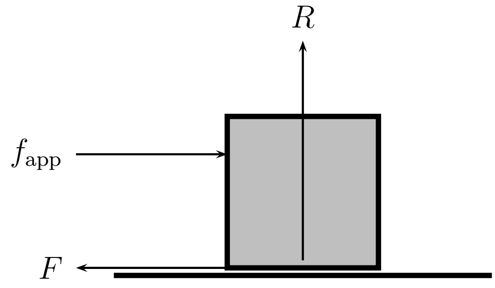
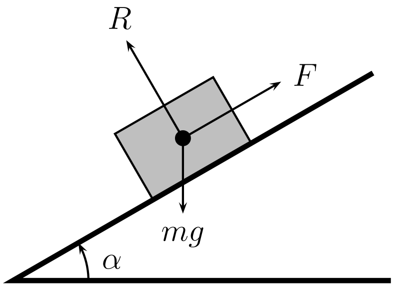


Figure 3.11: The three forces on a stationary mass on a rough plane inclined at an angle α are the reaction R, friction F and weight mg.



Static friction occurs provided there is no motion, so that

$$F = f_{\text{app}}$$

As f_{app} is increased, F increases. Experiments show that there is a critical f_{app} at which the body starts to move, which is proportional to the reaction R. The friction force at this point can be written

$$F = \mu_s R$$

where μ_s is the coefficient of limiting static friction.

EXAMPLE 3.14

For a stationary mass on an inclined plane, such as in figure 3.11, the forces must add to zero. The tangential force component is

$$F - mg \sin\alpha = 0$$

and the normal force component is

$$R - mg\cos\alpha = 0$$

At a critical angle α_c when the mass is just about to slip, we know that

$$F = \mu_s R$$

Therefore, eliminating R from the force equations at this point,

$$\mu_s = \tan\alpha_c$$

So if $\alpha < \alpha_c$, $F < \mu_s R$ and there is no motion. If $\alpha \geq \alpha_c$, the mass slides down the incline. ■

Once motion has started, the frictional force still acts to oppose the motion. Experimentally, it is observed that

$$F = \mu_k R$$

for low speeds, where μ_k is the coefficient of **kinetic friction**.

Note that $\mu_k \leq \mu_s$. Friction in problems involving rolling is slightly different; this is dealt with later.

The kinetic friction acts to oppose the motion, so it can be described by

$$\boldsymbol{F} = -\mu_k\,|R|\,\hat{\dot{\boldsymbol{r}}}$$

where $\hat{\dot{\boldsymbol{r}}}$ is the unit vector in the direction of motion.

EXAMPLE 3.15

A floor polisher of weight w has a circular brush of radius a which is rotating with angular speed ω and slides over a floor with speed v. Calculate the frictional force, where μ is the coefficient of sliding friction.

Solution: *Consider the element of the brush between r to $r + dr$ and θ to $\theta + d\theta$, as shown in figure 3.12. The area of this element is*

$$dA = r\,dr\,d\theta$$

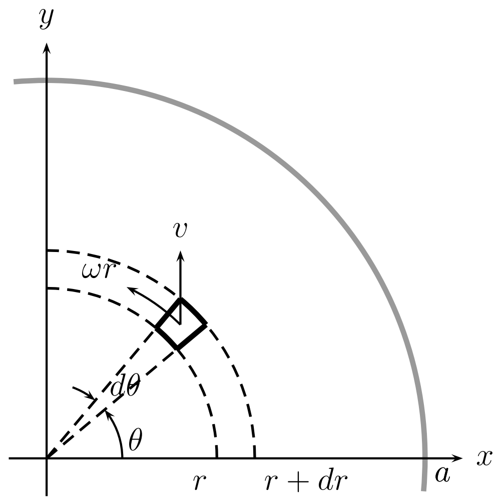
Assume that the weight is distributed uniformly over all area elements, so that the reaction force on this element is

$$R_e = \frac{w}{\pi a^2} r\,dr\,d\theta$$

The velocity of the element is

$$\dot{\boldsymbol{r}} = v\hat{\boldsymbol{j}} + \omega r(-\sin\theta\hat{\boldsymbol{i}} + \cos\theta\hat{\boldsymbol{j}})$$

Figure 3.12: This shows one quarter of a rotating floor-polisher, which is rotating with angular speed ω and moving with velocity v in the y direction. The elemental unit of area dA extends from r to $r + dr$ and from θ to $\theta + d\theta$.



and the unit vector in this direction is

$$\hat{\boldsymbol{r}} = \frac{v\hat{\boldsymbol{j}} + \omega r(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{j}})}{\sqrt{v^2 + 2\omega r v\cos\theta + \omega^2 r^2}}$$

The frictional force on the element is

$$-\frac{\mu w r\, dr\, d\theta}{\pi a^2}\hat{\boldsymbol{r}}$$

The frictional force on the whole annulus from r to $r + dr$ is found by integrating the frictional force on the element for θ from 0 to 2π

$$-\int_0^{2\pi} \hat{\boldsymbol{r}}\, d\theta\, \frac{\mu w r\, dr}{\pi a^2}$$

This is difficult to solve exactly, so suppose that $\omega r \gg v$. Then

$$\frac{1}{\sqrt{v^2 + 2\omega r v\cos\theta + \omega^2 r^2}} \approx \frac{1}{\omega r}\left[1 - \frac{v\cos\theta}{\omega r}\right]$$

So the frictional force on the annulus is

$$-\hat{\boldsymbol{j}}\frac{\mu w r\, dr}{a^2}\frac{v}{\omega r}$$

and the frictional force on the whole brush is

$$-\hat{\boldsymbol{j}}\,\frac{\mu w v}{a\omega}$$

If the brush is not rotating, $\omega = 0$, and the frictional force must be

$$\boldsymbol{F}_0 = -\hat{\boldsymbol{j}}\,\mu w$$

Therefore, when the brush is rotating, the friction can be written

$$|\boldsymbol{F}| = |\boldsymbol{F}_0|\,\frac{v}{aw}$$

which tends to zero as $\omega \to \infty$ and the polisher becomes essentially frictionless as the rotational speed is increased. ■

3.6.3 Constrained Motion with Friction

The equation of motion in intrinsic coordinates with friction is

$$m(\ddot{s}\,\hat{\boldsymbol{T}} + \dot{s}\dot{\psi}\hat{\boldsymbol{N}}) = m\boldsymbol{g} + R\hat{\boldsymbol{N}} - F\,\hat{\boldsymbol{T}}$$

where the friction term is

$$F = \mu\,|R|\,\frac{\dot{s}}{|\dot{s}|}$$

EXAMPLE 3.16
Consider the bead on the parabolic wire from example 3.13 where the wire is no longer smooth, so friction must be included. This is illustrated in figure 3.13. Calculate the bead's equation of motion.
Solution: *Recall from example 3.13 that*

$$\frac{ds}{d\psi} = \frac{L}{\cos^3\psi}$$

and

$$1 + \left(\frac{x}{L}\right)^2 = \frac{1}{\cos^2\psi}$$

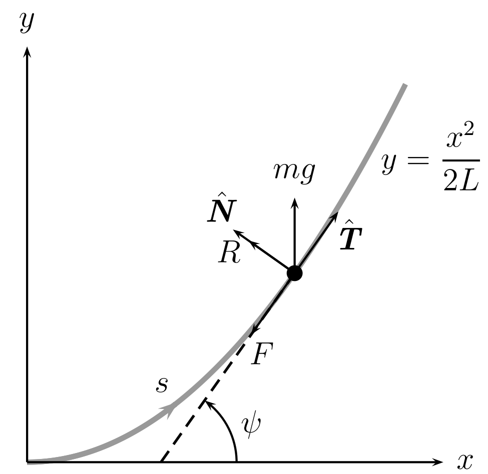
Then tangential equation of motion is

$$m\ddot{s} = mg\sin\psi - F$$

and the normal equation of motion is

$$m\dot{s}\dot{\psi} = mg\cos\psi + R$$

Figure 3.13: This shows a bead on a rough parabolic wire, so friction F opposes the direction of motion s. As in figure 3.9, the diagram has been drawn upside down so that gravity points upwards.



Assume that the initial speed is sufficiently small that $R < 0$ at the start. We can then write

$$F = -\mu R = -\mu(m\dot{s}\dot{\psi} - mg\cos\psi)$$

Therefore

$$\frac{1}{2}\frac{d}{ds}(\dot{s}^2) = g\sin\psi + \mu(\dot{s}\dot{\psi} - g\cos\psi)$$

so that

$$\frac{d}{ds}(\dot{s}^2) - 2\mu\frac{d\psi}{ds}(\dot{s}^2) = 2g(\sin\psi - \mu\cos\psi)$$

This is more conveniently expressed with respect to ψ rather than s. So multiply by $\frac{ds}{d\psi}$ to give

$$\frac{d}{d\psi}(\dot{s}^2) - 2\mu(\dot{s}^2) = 2g\frac{ds}{d\psi}(\sin\psi - \mu\cos\psi)$$

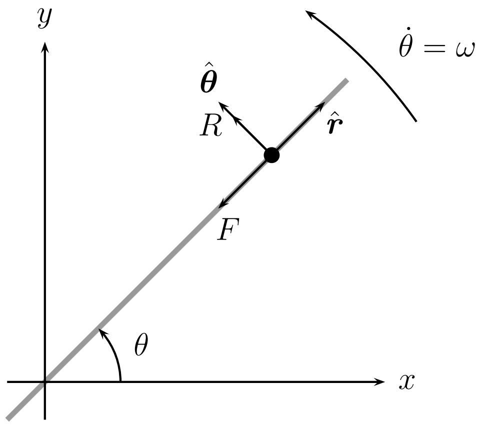
The integrating factor is

$$I(\psi) = e^{-2\mu\psi}$$

so that

$$\frac{d}{d\psi}\left(e^{-2\mu\psi}\dot{s}^2\right) = \frac{2gLe^{-2\mu\psi}}{\cos^3\psi}(\sin\psi - \mu\cos\psi)$$

Figure 3.14: A light bead on a rough wire which is rotating in a horizontal plane.



which gives

$$\dot{s}^2 e^{-2\mu\psi} = u^2 + 2gL \int_0^{\psi} \frac{e^{-2\mu\psi}}{\cos^3\psi}(\sin\psi - \mu\cos\psi)\, d\psi$$

Therefore $\dot{s}^2$ *is*

$$\dot{s}^2 = u^2 e^{2\mu\psi} + 2gLe^{2\mu\psi} \int_0^{\psi} \frac{e^{-2\mu\psi}}{\cos^3\psi}(\sin\psi - \mu\cos\psi)\, d\psi$$

which is substantially harder to solve than when there was no friction. ■

EXAMPLE 3.17
Consider a light bead on a rough wire that is rotating horizontally, as shown in figure 3.14. Since the bead is light, its weight can be ignored. Calculate the bead's position on the wire, $r(t)$.
Solution: *The equations of motion in polar coordinates are*

$$m(\ddot{r} - r\dot{\theta}^2) = -F$$

and

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = R$$

Since the wire is rotating with angular speed ω *and the bead is constained to the wire,*

$$\dot{\theta} = \omega$$

and

$$\ddot{\theta} = 0$$

Thus

$$m(\ddot{r} - r\omega^2) = -F$$

and

$$m(2\dot{r}\omega) = R$$

To find the critical rotation rate for the bead to move, at the point of incipient motion

$$F = \mu_s R$$

with $\dot{r} = \ddot{r} = 0$. *Then from the equations of motion, initially* $R = 0$ *and therefore* $F = 0$ *initially.*

For any ω, *the initial acceleration is*

$$\ddot{r} = r\omega^2$$

Friction only starts to act once $\dot{r}$, *and hence* R, *becomes non-zero. Therefore*

$$\omega_{\text{crit}} = 0$$

although this has to be modified if the weight of the bead is included.

When $\omega \neq 0$ *and the bead is sliding, the transverse equation shows that*

$$F = \mu_k R = 2m\omega\mu_k\dot{r}$$

The radial equation is

$$\ddot{r} - r\omega^2 = -\frac{F}{m} = -2\omega\mu_k\dot{r}$$

which is

$$\ddot{r} + 2\omega\mu_k\dot{r} - \omega^2 r = 0$$

Try $r = e^{\alpha t}$ *in the differential equation to get*

$$\alpha^2 + 2\omega\mu_k\alpha - \omega^2 = 0$$

which can be expressed as

$$\left(\frac{\alpha}{\omega}\right)^2 + 2\mu_k\left(\frac{\alpha}{\omega}\right) - 1 = 0$$

Thus

$$\alpha = \omega\left(\mu_k \pm \sqrt{\mu_k^2 + 1}\right)$$

This gives the two solutions

$$\alpha_1 = -\omega\left(\mu_k + \sqrt{\mu_k^2 + 1}\right) < 0$$

and

$$\alpha_2 = \omega\left(\sqrt{\mu_k^2 + 1} - \mu_k\right) > 0$$

so

$$r(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$$

At $t = 0$, $r = r_0$ *and* $\dot{r} = 0$ *so that*

$$r_0 = A + B \qquad \text{and} \qquad 0 = \alpha_1 A + \alpha_2 B$$

Therefore

$$A = -\frac{\alpha_2}{\alpha_1}\left(\frac{r_0}{1 - \alpha_2/\alpha_1}\right) > 0$$

and

$$B = \frac{r_0}{1 - \alpha_2/\alpha_1} > 0$$

This gives the solution

$$r(t) = \frac{r_0}{1 - \alpha_2/\alpha_1}\left(-\frac{\alpha_2}{\alpha_1}e^{\alpha_1 t} + e^{\alpha_2 t}\right)$$

■

Chapter 4

Dynamics of Systems of Particles

4.1 Centre of Mass and Momentum

Consider a system of particles where $\boldsymbol{r}_i$ is the position vector of the i^{th} particle whose mass is m_i. Then the position vector of the **centre of mass** of the system of particles is

$$\boldsymbol{r}_{\text{cm}} = \frac{\sum_i m_i \boldsymbol{r}_i}{\sum_i m_i}$$

The total mass of the system of particles is

$$M = \sum_i m_i$$

EXAMPLE 4.1
The centre of mass of a two particle system is

$$\boldsymbol{r}_{\text{cm}} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2}$$

If, for example, the two particles are the Sun and a planet,

$$\boldsymbol{r}_{\text{cm}} = \frac{M_s \boldsymbol{r}_s + m_p \boldsymbol{r}_p}{M_s + m_p}$$

■

The linear momentum of the i^{th} particle is

$$\boldsymbol{p}_i = m_i \dot{\boldsymbol{r}}_i$$

so the total linear momentum of the system of particles is

$$\boldsymbol{P} = \sum_i \boldsymbol{p}_i = \sum_i m_i \dot{\boldsymbol{r}}_i$$

If the total mass of the system is constant with time, this can be written as the total mass times the velocity of the centre of mass

$$\boldsymbol{P} = \frac{d}{dt} \sum_i m_i \boldsymbol{r}_i = M \frac{d}{dt} \left(\frac{1}{M} \sum_i m_i \boldsymbol{r}_i \right) = M \dot{\boldsymbol{r}}_{\text{cm}}$$

Newton's second law applies to the system as a whole. Newton's second law for an individual particle is

$$m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i + \sum_{j \neq i} \boldsymbol{F}_{ij}$$

where $\boldsymbol{F}_i$ is the total external force on the i^{th} particle and $\boldsymbol{F}_{ij}$ is the force exerted on the i^{th} particle by the j^{th} particle (note that by Newton's third law, $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$).

THEOREM 4.1
Newton's second law for the system of particles is

$$M \ddot{\boldsymbol{r}}_{\text{cm}} = \sum_i \boldsymbol{F}_i$$

Proof:
Start with

$$M \boldsymbol{r}_{\text{cm}} = \sum_i m_i \boldsymbol{r}_i$$

and differentiate twice with respect to time to give

$$M \ddot{\boldsymbol{r}}_{\text{cm}} = \sum_i m_i \ddot{\boldsymbol{r}}_i$$

Then from Newton's second law for the i^{th} particle,

$$M \ddot{\boldsymbol{r}}_{\text{cm}} = \sum_i \left(\boldsymbol{F}_i + \sum_{j \neq i} \boldsymbol{F}_{ij} \right) = \sum_i \boldsymbol{F}_i + \sum_i \sum_{j \neq i} \boldsymbol{F}_{ij}$$

The second summation is zero because $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$, leaving

$$M\ddot{\boldsymbol{r}}_{\mathrm{cm}} = \sum_i \boldsymbol{F}_i$$

as required. ∎

Note that when no net external force acts on the system

$$\sum_i \boldsymbol{F}_i = 0$$

and

$$M\ddot{\boldsymbol{r}}_{\mathrm{cm}} = 0$$

so that $\boldsymbol{P} = M\dot{\boldsymbol{r}}_{\mathrm{cm}}$ is constant. Therefore a system of particles under no net external force has constant linear momentum $\boldsymbol{P}$.

4.2 Kinetic and Potential Energy

The **kinetic energy** of a system is the sum of the kinetic energies of the individual particles

$$KE = \sum_i \frac{1}{2} m_i \dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i$$

The **potential energy** of external conservative forces is

$$PE_{\mathrm{ext}} = \sum_i V_{\mathrm{ext}}(\boldsymbol{r}_i)$$

where the potential fuction $V_{\mathrm{ext}}(\boldsymbol{r})$ is defined in terms of the external force by

$$-\nabla V_{\mathrm{ext}}(\boldsymbol{r}) = F(\boldsymbol{r})$$

Internal forces are usually central forces between each pair of particles so that there is associated with each pair of particles an internal potential energy $V_{\mathrm{int}}(r_{ij})$ which is a function of separation distance $r_{ij} = |\boldsymbol{r}_i - \boldsymbol{r}_j|$ between the pair. The internal potential energy is

$$PE_{\mathrm{int}} = \frac{1}{2} \sum_i \sum_j V_{\mathrm{int}}(r_{ij})$$

Here the summation counts each pair of particles twice, hence the factor of 1/2.

The **total mechanical energy** of the system is

$$\begin{aligned} E &= KE + PE_{\text{ext}} + PE_{\text{int}} \\ &= \sum_i \frac{1}{2} m_i \dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i + \sum_i V_{\text{ext}}(\boldsymbol{r}_i) + \frac{1}{2} \sum_i \sum_j V_{\text{int}}(r_{ij}) \end{aligned}$$

THEOREM 4.2
In the absence of external extraneous forces

$$\frac{dE}{dt} = 0$$

so E is a constant of the motion of the system and mechanical energy is conserved.

EXAMPLE 4.2 (Reduced Mass of a Two Body System)
When no external force acts, $\dot{\boldsymbol{r}}_{\text{cm}}$ is constant. A coordinate system with the centre of mass at the origin would therefore be an inertial frame of reference.

If $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the positions of two particles relative to this coordinate system, then

$$m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2 = 0$$

Define the position vector of particle 1 with respect to particle 2 as

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 = \boldsymbol{r}_1 \left(1 + \frac{m_1}{m_2} \right)$$

Newton's second law holds in inertial frames of reference, so that

$$m_1 \ddot{\boldsymbol{r}}_1 = \boldsymbol{F}_{12}$$

If the force that particle 2 exerts on particle 1 is a function of the distance between them and directed along the line of their centres

$$\boldsymbol{F}_{12} = f(r) \hat{\boldsymbol{r}}$$

then

$$m_1 \ddot{\boldsymbol{r}}_1 = f(r) \hat{\boldsymbol{r}}$$

Substituting

$$\ddot{\boldsymbol{r}}_1 = \frac{m_2}{m_1 + m_2} \boldsymbol{r}$$

we have

$$\mu \ddot{\boldsymbol{r}} = f(r) \hat{\boldsymbol{r}}$$

where μ is the ***reduced mass*** *of the system*

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

This is equivalent to motion in a central force field of a particle of mass μ. ■

EXAMPLE 4.3 (Sun-Planet System)
It was incorrect to assume that the Sun stays fixed at the origin as a planet orbits it. Instead we should use the centre of mass of the Sun/planet system as the origin. In this case, the position vector of the planet relative to the Sun is

$$\boldsymbol{r} = \boldsymbol{r}_p - \boldsymbol{r}_s$$

and therefore

$$\mu \ddot{\boldsymbol{r}} = -\frac{GM_s m_p}{r^2}\hat{\boldsymbol{r}}$$

where

$$\mu = \frac{M_s m_p}{M_s + m_p} \approx m_p$$

when $M_s \gg m_p$.

Our previous calculations are all valid if we replace GM_s *by* $G(M_s + m_p)$.

■

4.3 Torque

The **angular momentum** of the i^{th} particle about the origin is

$$\boldsymbol{L}_i = \boldsymbol{r}_i \times \boldsymbol{p}_i = \boldsymbol{r}_i \times (m_i \dot{\boldsymbol{r}}_i)$$

The angular momentum of the system of particles about the origin is

$$\boldsymbol{L} = \sum_i \boldsymbol{L}_i = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i$$

THEOREM 4.3
In an inertial frame of reference

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{N}$$

where

$$\boldsymbol{N} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i$$

is the ***total external torque*** *on the system of particles about the origin.*

Proof:

From the definition of $\boldsymbol{L}$

$$\frac{d\boldsymbol{L}}{dt} = \frac{d}{dt}\sum_i \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i$$

If the masses m_i are constant with time, $\frac{dm_i}{dt} = 0$ so

$$\frac{d\boldsymbol{L}}{dt} = \sum_i \boldsymbol{r}_i \times m_i \ddot{\boldsymbol{r}}_i$$

Now $m_i \ddot{\boldsymbol{r}}_i$ can be written in terms of the internal and external forces to give

$$\frac{d\boldsymbol{L}}{dt} = \sum_i \boldsymbol{r}_i \times \left(\boldsymbol{F}_i + \sum_j \boldsymbol{F}_{ij} \right)$$

The second summation is zero because it is composed of pairs of the form

$$\boldsymbol{r}_i \times \boldsymbol{F}_{ij} + \boldsymbol{r}_j \times \boldsymbol{F}_{ji} = (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ij} = 0$$

which are zero if $\boldsymbol{F}_{ij}$ lies along the line of the two particles' centres.

Therefore

$$\frac{d\boldsymbol{L}}{dt} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i = \boldsymbol{N}$$

■

Note that when there is no external torque, $\boldsymbol{N} = 0$ so $\boldsymbol{L}$ is constant and angular momentum is conserved. $\boldsymbol{N} = 0$ when all of the external forces are zero, hence angular momentum is conserved when no external forces act on the system.

EXAMPLE 4.4

Angular momentum and torque depend on the origin of the coordinate system.

Define $\bar{\boldsymbol{r}}_i$ to be the position of the i^{th} particle relative to a coordinate system fixed on the centre of mass.

$$\bar{\boldsymbol{r}}_i = \boldsymbol{r}_i - \boldsymbol{r}_{\text{cm}}$$

Therefore

$$\boldsymbol{r}_i = \bar{\boldsymbol{r}}_i + \boldsymbol{r}_{\text{cm}}$$

and

$$\dot{\boldsymbol{r}}_i = \dot{\bar{\boldsymbol{r}}}_i + \dot{\boldsymbol{r}}_{\text{cm}}$$

Now the angular momentum is

$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times (m_i \dot{\boldsymbol{r}}_i) = \sum_i m_i (\bar{\boldsymbol{r}}_i + \boldsymbol{r}_{\text{cm}}) \times (\dot{\bar{\boldsymbol{r}}}_i + \dot{\boldsymbol{r}}_{\text{cm}})$$

Expanding the cross-product of the sums gives the sum of four cross-products.

$$\begin{aligned}\boldsymbol{L} &= \left(\sum_i m_i\right)\boldsymbol{r}_{\mathrm{cm}} \times \dot{\boldsymbol{r}}_{\mathrm{cm}} + \boldsymbol{r}_{\mathrm{cm}} \times \left(\sum_i m_i \dot{\bar{\boldsymbol{r}}}_i\right) \\ &\quad + \left(\sum_i m_i \bar{\boldsymbol{r}}_i\right) \times \dot{\boldsymbol{r}}_{\mathrm{cm}} + \sum_i \bar{\boldsymbol{r}}_i \times m_i \dot{\bar{\boldsymbol{r}}}_i\end{aligned}$$

Now $\sum_i m_i \bar{\boldsymbol{r}}_i = 0$ from the definition of centre of mass

$$\sum_i m_i \bar{\boldsymbol{r}}_i = \sum_i m_i \boldsymbol{r}_i - \sum_i m_i \boldsymbol{r}_{\mathrm{cm}} = M\boldsymbol{r}_{\mathrm{cm}} - M\boldsymbol{r}_{\mathrm{cm}} = 0$$

Differentiating shows that $\sum_i m_i \dot{\bar{\boldsymbol{r}}}_i = 0$, which leaves

$$\boldsymbol{L} = \left(\sum_i m_i\right)\boldsymbol{r}_{\mathrm{cm}} \times \dot{\boldsymbol{r}}_{\mathrm{cm}} + \sum_i \bar{\boldsymbol{r}}_i \times m_i \dot{\bar{\boldsymbol{r}}}_i$$

which is the sum of the angular momentum of the centre of mass about the origin and the angular momentum of the system about the centre of mass

$$\boldsymbol{L} = \boldsymbol{r}_{\mathrm{cm}} \times (M\dot{\boldsymbol{r}}_{\mathrm{cm}}) + \sum_i \bar{\boldsymbol{r}}_i \times (m_i \dot{\bar{\boldsymbol{r}}}_i)$$

■

The **kinetic energy** of the system T is the sum of the kinetic energies of the individual particles

$$T = \frac{1}{2}\sum_i m_i \dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i = \sum_i T_i$$

where $T_i = \frac{1}{2} m_i \dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i$ is the kinetic energy of the i^{th} particle.

EXAMPLE 4.5

The kinetic energy of a system is the sum of the kinetic energy due to the translation of the centre of mass plus the kinetic energy due to the system's motion about the centre of mass.

To see this, write

$$\dot{\boldsymbol{r}}_i = \dot{\bar{\boldsymbol{r}}}_i + \dot{\boldsymbol{r}}_{\mathrm{cm}}$$

Now the kinetic energy is

$$T = \frac{1}{2}\sum_i m_i(\dot{\bar{\boldsymbol{r}}}_i + \dot{\boldsymbol{r}}_{\mathrm{cm}}) \cdot (\dot{\bar{\boldsymbol{r}}}_i + \dot{\boldsymbol{r}}_{\mathrm{cm}})$$

Expanding the dot-product of the sums gives the sum of four dot-products.

$$\boldsymbol{T} = \frac{1}{2}\left(\sum_i m_i\right)\dot{\boldsymbol{r}}_{\rm cm}\cdot\dot{\boldsymbol{r}}_{\rm cm} + \frac{1}{2}\dot{\boldsymbol{r}}_{\rm cm}\cdot\left(\sum_i m_i\dot{\bar{\boldsymbol{r}}}_i\right) \tag{4.1}$$

$$+\frac{1}{2}\left(\sum_i m_i\dot{\bar{\boldsymbol{r}}}_i\right)\cdot\dot{\boldsymbol{r}}_{\rm cm} + \frac{1}{2}\sum_i \dot{\bar{\boldsymbol{r}}}_i\cdot m_i\dot{\bar{\boldsymbol{r}}}_i \tag{4.2}$$

As in the previous example, $\sum_i m_i\dot{\bar{\boldsymbol{r}}}_i = 0$, *which leaves*

$$\boldsymbol{T} = \frac{1}{2}\left(\sum_i m_i\right)\dot{\boldsymbol{r}}_{\rm cm}\cdot\dot{\boldsymbol{r}}_{\rm cm} + \frac{1}{2}\sum_i \dot{\bar{\boldsymbol{r}}}_i\cdot m_i\dot{\bar{\boldsymbol{r}}}_i$$

which is the sum of the kinetic energy of translation of the system as a whole and the kinetic energy of the motion of the system about the centre of mass

$$\boldsymbol{T} = \frac{1}{2}M\dot{\boldsymbol{r}}_{\rm cm}\cdot\dot{\boldsymbol{r}}_{\rm cm} + \frac{1}{2}\sum_i m_i\dot{\bar{\boldsymbol{r}}}_i\cdot\dot{\bar{\boldsymbol{r}}}_i$$

■

From example 4.4, the angular momentum $\boldsymbol{L}$ relative to an inertial origin can be written as

$$\boldsymbol{L} = \boldsymbol{L}_{\rm cm} + \boldsymbol{L}_{\rm r}$$

where $\boldsymbol{L}_{\rm r}$ is the system's angular momentum relative to the centre of mass. $\boldsymbol{L}$ is related to the total external torque relative to the origin according to

$$\boldsymbol{N} = \frac{d\boldsymbol{L}}{dt} = \sum_i \boldsymbol{r}_i\times\boldsymbol{F}_i$$

The following theorem shows that a similar result applies to the angular momentum about the centre of mass, even though the centre of mass may not be an inertial reference frame.

THEOREM 4.4
The total external torque relative to the centre of mass is

$$\boldsymbol{N}_{\rm r} = \frac{d\boldsymbol{L}_{\rm r}}{dt} = \sum_i \bar{\boldsymbol{r}}_i\times\boldsymbol{F}_i$$

Proof:

From the definition of

$$\boldsymbol{L}_{\mathrm{cm}} = \boldsymbol{r}_{\mathrm{cm}} \times M\dot{\boldsymbol{r}}_{\mathrm{cm}}$$

differentiate to give

$$\frac{d\boldsymbol{L}_{\mathrm{cm}}}{dt} = \boldsymbol{r}_{\mathrm{cm}} \times M\ddot{\boldsymbol{r}}_{\mathrm{cm}} = \boldsymbol{r}_{\mathrm{cm}} \times \sum_i \boldsymbol{F}_i$$

But

$$\boldsymbol{L} = \boldsymbol{L}_{\mathrm{cm}} + \boldsymbol{L}_{\mathrm{r}}$$

so

$$\frac{d\boldsymbol{L}}{dt} = \frac{d\boldsymbol{L}_{\mathrm{cm}}}{dt} + \frac{d\boldsymbol{L}_{\mathrm{r}}}{dt}$$

Therefore

$$\frac{d\boldsymbol{L}_{\mathrm{r}}}{dt} = \frac{d\boldsymbol{L}}{dt} - \frac{d\boldsymbol{L}_{\mathrm{cm}}}{dt}$$

This is

$$\frac{d\boldsymbol{L}_{\mathrm{r}}}{dt} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i - \boldsymbol{r}_{\mathrm{cm}} \times \sum_i \boldsymbol{F}_i = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_{\mathrm{cm}}) \times \boldsymbol{F}_i$$

which gives

$$\frac{d\boldsymbol{L}_{\mathrm{r}}}{dt} = \sum_i \bar{\boldsymbol{r}}_i \times \boldsymbol{F}_i$$

as required. ■

Chapter 5

Rigid Body Motion

A system of particles behaves as a rigid body if, during the motion of the system, the particles do not change their relative internal positions. Even in solids, the particles vibrate about their equilibrium positions, so there is always internal vibrational kinetic energy.

5.1 Continuous Rigid Bodies

If the body is large compared with the internal particle separations, we can regard it as a continuum with negligible error. The mass density of the material in the body then becomes the **mass per unit volume**, which is denoted by the symbol ρ.

This is a meaningful concept provided we are interested in large volumes containing many particles.

The position of the centre of mass of a system of particles

$$\boldsymbol{r}_{\text{cm}} = \frac{1}{M}\sum_i m_i \boldsymbol{r}_i$$

becomes an integral

$$\boldsymbol{r}_{\text{cm}} = \frac{1}{M}\int_V \rho\, \boldsymbol{r}\, dV$$

where the integral is taken over the whole of the volume of the rigid body, and $\rho(\boldsymbol{r})$ is the function giving the body's mass per unit volume as a function of position $\boldsymbol{r}$. The total mass of the body is

$$M = \int_V \rho\, dV$$

If the body is uniform so that ρ is constant, the formula for the centre of mass becomes

$$\boldsymbol{r}_{\text{cm}} = \frac{1}{V}\int_V \boldsymbol{r}\, dV$$

In a Cartesian coordinate system, $dV = dx\, dy\, dz$ and

$$\boldsymbol{r}_{\text{cm}} = \frac{1}{M}\iiint \rho(x,y,z)\boldsymbol{r}(x,y,z)\, dx\, dy\, dz$$

or, for a uniform body

$$\boldsymbol{r}_{\text{cm}} = \frac{1}{V}\iiint \boldsymbol{r}(x,y,z)\, dx\, dy\, dz$$

The centres of mass of the individual coordinates for a uniform body are

$$x_{\text{cm}} = \frac{1}{V}\int_V x\, dV$$

with similar expressions for y_{cm} and z_{cm}.

For a thin sheet, called a **lamina**, the volume element is $dV = dA\, t$ where dA is the element of area on the sheet and t is the thickness of the sheet. Then

$$\boldsymbol{r}_{\text{cm}} = \frac{\displaystyle\int_A \bar{\rho}\boldsymbol{r}\, dA}{\displaystyle\int_A \bar{\rho}\, dA}$$

where $\bar{\rho} = \rho t$ is the mass per unit area of the lamina. The total mass of the sheet is

$$M = \int_A \bar{\rho}\, dA$$

If the rigid body has a plane of symmetry, $\boldsymbol{r}_{\text{cm}}$ lies in that plane. If the rigid body has a line of symmetry, $\boldsymbol{r}_{\text{cm}}$ lies on that line.

EXAMPLE 5.1

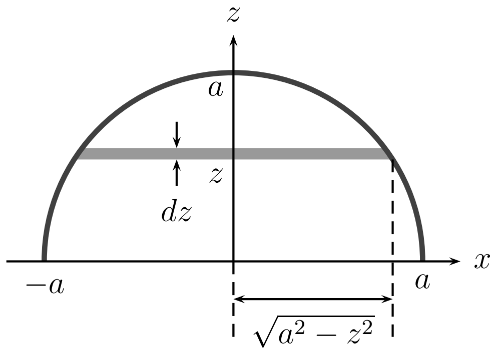
For a solid hemisphere with its centre at the origin, radius a and $z \geq 0$

$$x_{\text{cm}} = y_{\text{cm}} = 0 \qquad z_{\text{cm}} = \frac{3a}{8}$$

For a hemispherical shell with its centre at the origin, radius a and $z \geq 0$

$$x_{\text{cm}} = y_{\text{cm}} = 0 \qquad z_{\text{cm}} = \frac{a}{2}$$

Figure 5.1: This semicircular lamina in the xz plane has its centre at the origin and has radius a. The shaded region is the strip from z to $z + dz$.



For a semicircular line of radius a

$$x_{\text{cm}} = y_{\text{cm}} = 0 \qquad z_{\text{cm}} = \frac{2a}{\pi}$$

For a semicircular lamina of radius a

$$x_{\text{cm}} = y_{\text{cm}} = 0 \qquad z_{\text{cm}} = \frac{4a}{3\pi}$$

For an isosceles triangular lamina with height h

$$x_{\text{cm}} = y_{\text{cm}} = 0 \qquad z_{\text{cm}} = \frac{h}{3}$$

■

EXAMPLE 5.2 (Semicircular Lamina)
Show that the centre of mass of the semicircular lamina of radius a in the half-plane $z > 0$ is

$$z_{\text{cm}} = \frac{4a}{3\pi}$$

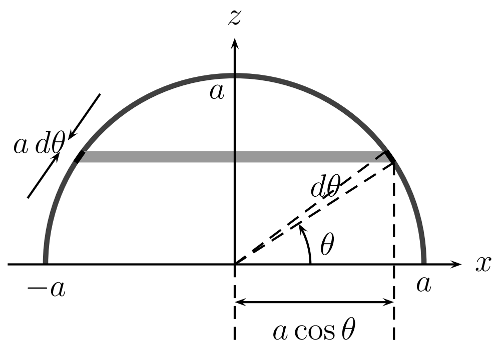
Solution: *As shown in figure 5.1, the mass of the strip of width dz of the semicircular lamina is*

$$w(z) = \bar{\rho} \cdot 2\sqrt{a^2 - z^2}\, dz$$

The z axis is a line of symmetry, so the centre of mass of the strip is on the z axis at z ($x = y = 0$). Therefore the centre of mass of the whole body is

$$z_{\text{cm}} = \frac{1}{M} \int_0^a 2\bar{\rho} z \sqrt{a^2 - z^2}\, dz$$

Figure 5.2: The hemispherical shell has its centre at the origin and has radius a. The shaded region is the circumferential strip from θ to $\theta + d\theta$.



The mass of the lamina is

$$M = \bar{\rho} \cdot \frac{\pi a^2}{2}$$

so the location of the centre of mass becomes

$$z_{\rm cm} = \frac{4}{\pi a^2} \int_0^a z\sqrt{a^2 - z^2}\, dz$$

Make the substitution

$$u^2 = 1 - \frac{z^2}{a^2}$$

so that

$$z_{\rm cm} = \frac{4a}{\pi} \int_0^1 u^2\, du = \frac{4a}{3\pi}$$

■

EXAMPLE 5.3 (Hemispherical Shell)
Show that the centre of mass of a hemispherical shell of radius a with $z > 0$ is

$$z_{\rm cm} = \frac{a}{2}$$

Solution: *As shown in figure 5.2, the strip from an angle θ to $\theta + d\theta$ has z coordinate*

$$z = a \sin\theta$$

The surface area of this strip is

$$dA = 2\pi(a\cos\theta)\, a\, d\theta = 2\pi a^2 \cos\theta\, d\theta$$

From symmetry, the centre of mass of this strip is at $z = a\sin\theta$. Therefore, the centre of mass of the hemispherical shell has $x_{\text{cm}} = y_{\text{cm}} = 0$ and

$$z_{\text{cm}} = \frac{1}{M}\int_A \bar{\rho} z \, dA = \frac{1}{M}\int_0^{\pi/2} 2\pi a^3 \bar{\rho} \cos\theta \sin\theta \, d\theta$$

The mass of the hemispherical shell is

$$M = \bar{\rho} \cdot 2\pi a^2$$

so the location of the centre of mass becomes

$$z_{\text{cm}} = a\int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$$

Make the substitution

$$u = \sin\theta$$

so that

$$z_{\text{cm}} = a\int_0^1 u \, du = \frac{a}{2}$$

■

5.2 Rotation About a Fixed Axis

Consider a body rotating about the z axis through a fixed origin O with angular speed

$$\omega = \dot{\phi}$$

The mass at point $\boldsymbol{r} = (x, y, z)$ is moving with velocity $\boldsymbol{v}$ in the xy plane through the point, where $\boldsymbol{v}$ satisfies

$$|\boldsymbol{v}| = \omega\sqrt{x^2 + y^2}$$

Now the point is moving in uniform circular motion, so $\boldsymbol{v}$ is also given by

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{x}\hat{\boldsymbol{i}} + \dot{y}\hat{\boldsymbol{j}}$$

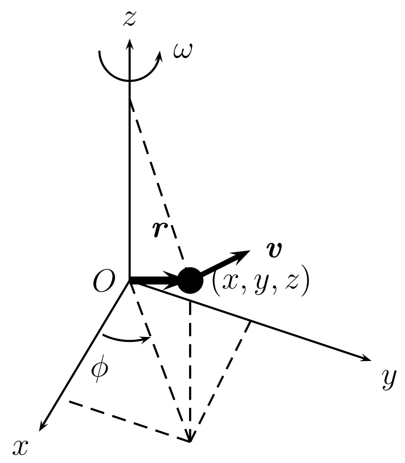
Using

$$\dot{x} = -|\boldsymbol{v}|\sin\phi = -\omega y$$

and

$$\dot{y} = |\boldsymbol{v}|\cos\phi = \omega x$$

Figure 5.3: Here is the point at (x, y, z) rotating at constant speed $\omega = \dot{\phi}$ about the z-axis.



the velocity can be written

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$$

where the angular velocity vector is

$$\boldsymbol{\omega} = \omega \hat{\boldsymbol{k}}$$

For a discrete body, the rotational kinetic energy is

$$T_{\text{rot}} = \sum_i \frac{1}{2} m_i \left|\boldsymbol{v}_i\right|^2 = \frac{1}{2} \sum_i m_i \left(\omega \sqrt{x_i^2 + y_i^2}\right)^2$$

This can be written as

$$T_{\text{rot}} = \frac{1}{2} I_z \omega^2$$

where I_z is the **moment of inertia** of the body about the z axis through O

$$I_z = \sum_i m_i \left(x_i^2 + y_i^2\right)$$

For a continuum solid, the same formula holds

$$T_{\text{rot}} = \frac{1}{2} I_z \omega^2$$

when now the moment of inertia is

$$I_z = \int_V \rho\,(x^2 + y^2)\,dV$$

When the body is rotating about the z axis, the velocity is

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}}$$

so the angular momentum about O

$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times (m_i \dot{\boldsymbol{r}}_i)$$

has only a component in the z direction

$$L_z = \sum_i m_i (x_i \dot{y}_i - y_i \dot{x}_i)$$

Using $\dot{x} = -\omega y$ and $\dot{y} = \omega x$, this becomes

$$L_z = \sum_i m_i (x_i^2 + y_i^2)\omega = I_z \omega$$

The **angular equation of motion** for a rigid body can be found from the z component of $\frac{d}{dt}\boldsymbol{L} = \boldsymbol{N}$

$$\frac{d}{dt}(I_z \omega) = N_z$$

For a rigid body, I_z is constant so that

$$I_z \frac{d\omega}{dt} = N_z$$

This leads to simple relations between the equations of motion for translation along the z axis and rotation around the z axis. Making the connections

$$\left.\begin{matrix} m \\ v_z \\ \dot{v}_z \end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix} I_z \\ \omega \\ \dot{\omega} \end{matrix}\right.$$

shows how linear and angular momentum obey similar relationships

$$p_z = m v_z \qquad \Longleftrightarrow \qquad L_z = I_z \omega$$

as do force and torque

$$F_z = m \dot{v}_z \qquad \Longleftrightarrow \qquad N_z = I_z \dot{\omega}$$

and translational and rotational kinetic energy

$$\frac{1}{2} m v_z^2 \qquad \Longleftrightarrow \qquad \frac{1}{2} I_z \omega^2$$

EXAMPLE 5.4 (Thin Rod about Centre of Mass)
The z component of the moment of inertia of a thin rod of length l about its centre with the rod lying along the x axis is given by

$$I_z = \int_{-l/2}^{l/2} \iint \rho \, (x^2 + y^2) \, dz \, dy \, dx$$

Since the rod is thin and lies along the x axis, this becomes

$$I_z = \int_{-l/2}^{l/2} \iint \rho x^2 \, dz \, dy \, dx = \bar{\rho} \int_{-l/2}^{l/2} x^2 \, dx$$

where $\bar{\rho}$ is the rod's mass per unit length

$$\bar{\rho} = \rho \iint dz \, dy = \frac{M}{l}$$

Therefore

$$I_z = \frac{M}{l} \int_{-l/2}^{l/2} x^2 \, dx = \frac{M}{l} \left. \frac{x^3}{3} \right|_{-l/2}^{l/2}$$

so that

$$I_z = \frac{1}{12} M l^2$$

■

EXAMPLE 5.5 (Laminar Annulus about Centre of Mass)
Consider an annulus of radius R and thickness h with $h \ll R$, lying in the xy plane with the z axis at its centre. The element of unit area is

$$dA = Rh \, d\theta$$

and the mass per unit area is

$$\bar{\rho} = \frac{M}{2\pi Rh}$$

Therefore the moment of inertia about the z axis is

$$I_z = \int \bar{\rho} \, (x^2 + y^2) \, dA = \bar{\rho} \int_0^{2\pi} Rh(R^2) \, d\theta = (2\pi Rh\bar{\rho}) R^2$$

Since $M = 2\pi Rh\bar{\rho}$, the moment of inertia is

$$I_z = MR^2$$

■

EXAMPLE 5.6 (Circular Disc about the Centre of Mass)
From the previous example, the moment of inertia of the annulus from r to $r + dr$ is

$$dI_z = (2\pi r\, dr\, \bar{\rho})r^2$$

Therefore

$$I_z = 2\pi\bar{\rho}\int_0^a r^3\, dr = 2\pi\bar{\rho}\frac{a^4}{4}$$

But $\bar{\rho} = M/\pi a^2$ so the moment of inertia becomes

$$I_z = \frac{1}{2}Ma^2$$

■

Show that the moments of inertia of a sphere and a thin spherical shell, both of radius a, are respectively

$$I_z = \frac{2}{5}Ma^2 \qquad \text{and} \qquad I_z = \frac{2}{3}Ma^2$$

5.2.1 Moments of Inertia about Other Axes

The moments of inertia of a body about the three Cartesian axes are

$$I_z = \sum_i m_i(x_i^2 + y_i^2)$$

$$I_x = \sum_i m_i(y_i^2 + z_i^2)$$

$$I_y = \sum_i m_i(x_i^2 + z_i^2)$$

for bodies composed of discrete particles, and

$$I_z = \int_V \rho\,(x^2 + y^2)\, dV$$

$$I_x = \int_V \rho\,(y^2 + z^2)\, dV$$

$$I_y = \int_V \rho\,(x^2 + z^2)\, dV$$

for continuous bodies.

For a laminar body lying in the xy plane

$$I_z = \int_V \bar{\rho}\,(x^2 + y^2)\,dV$$

$$I_x = \int_V \bar{\rho}\,y^2\,dV$$

$$I_y = \int_V \bar{\rho}\,x^2\,dV$$

because the z component is zero. Adding these shows that

$$I_z = I_x + I_y$$

which is known as the **perpendicular axis theorem**. In other words, the moment of inertia of any plane body about an axis normal to the body is equal to the sum of the moments of inertia about any two perpendicular axes passing through the given axis and lying in the plane.

EXAMPLE 5.7

The moment of inertia in the z direction of a circular plate lying in the xy plane is

$$I_z = \frac{1}{2}Ma^2$$

if the origin is at the centre of the circular plate.

By symmetry, $I_x = I_y$ and by the perpendicular axis theorem,

$$I_z = I_x + I_y$$

so that

$$I_x = I_y = \frac{1}{2}I_z = \frac{1}{4}Ma^2$$

■

EXAMPLE 5.8

To find the moment of inertia I_z of a flat square lying in the xy plane, whose size is l by l and mass is M, note that looking along the x or the y axes, the plate looks like a rod of length l. Therefore I_x and I_y are the moments of inertia of a rod of length l and mass M measured about the centre of the rod

$$I_x = I_y = \frac{1}{12}Ml^2$$

Then by the perpendicular axis theorem,

$$I_z = I_x + I_y = \frac{1}{6}Ml^2$$

■

The following theorem, the **parallel axis theorem** describes how the moment of inertia changes when measured relative to an axis that is parallel to the original axis.

THEOREM 5.1 (Parallel Axis Theorem)
Suppose a body has a moment of inertia I_z measured through the origin and $I_{z,\mathrm{cm}}$ measured relative to the centre of mass. If the perpendicular distance from the centre of mass to the z axis is l, the two moments of inertia are related by

$$I_z = Ml^2 + I_{z,\mathrm{cm}}$$

Proof:
The moment of inertia through the origin is

$$I_z = \sum_i m_i(x_i^2 + y_i^2)$$

and the moment of inertia through the centre of mass is

$$I_{z,\mathrm{cm}} = \sum_i m_i(\bar{x}_i^2 + \bar{y}_i^2)$$

where the coordinates relative to the origin $\boldsymbol{r}_i$ are related to the coordinates relative to the centre of mass $\bar{\boldsymbol{r}}_i$ according to

$$\boldsymbol{r}_i = \boldsymbol{r}_{\mathrm{cm}} + \bar{\boldsymbol{r}}_i$$

where $\boldsymbol{r}_{\mathrm{cm}}$ is the position of the centre of mass relative to the origin. Therefore

$$I_z = \sum_i m_i(x_i^2 + y_i^2) = \sum_i m_i \left((x_{\mathrm{cm}} + \bar{x}_i)^2 + (y_{\mathrm{cm}} + \bar{y}_i)^2\right)$$

Expanding the squares shows that

$$\begin{aligned} I_z &= \sum_i m_i(x_{\mathrm{cm}}^2 + y_{\mathrm{cm}}^2) + 2\left(\sum_i m_i\bar{x}_i\right)x_{\mathrm{cm}} \\ &\quad + 2\left(\sum_i m_i\bar{y}_i\right)y_{\mathrm{cm}} + \sum_i m_i(\bar{x}_i^2 + \bar{y}_i^2) \end{aligned}$$

The second and third summations are zero. Writing $l^2 = x_{\mathrm{cm}}^2 + y_{\mathrm{cm}}^2$, $M = \sum_i m_i$ and noting that the fourth summation is the moment of inertia about the centre of mass shows that

$$I_z = Ml^2 + I_{z,\mathrm{cm}}$$

as required. ■

EXAMPLE 5.9
To find the moment of inertia I_z of a circular plate in the xy plane relative to an axis through the edge of the plate, note that the moment of inertia through the centre of the plate is

$$I_{z,\mathrm{cm}} = \frac{1}{2}Ma^2$$

Measuring the moment of inertia relative to an axis through the edge of the plate shifts the axis a distance $l = a$. Therefore the moment of inertia is

$$I_z = Ml^2 + I_{z,\mathrm{cm}} = Ma^2 + \frac{1}{2}Ma^2 = \frac{3}{2}Ma^2$$

■

EXAMPLE 5.10
The moment of inertia I_x for the plate in the previous example can be found by noting that the moment of inertia through the centre of mass of the plate is

$$I_{x,\mathrm{cm}} = \frac{1}{4}Ma^2$$

Measuring the moment of inertia relative to an axis through the edge of the plate shifts the axis a distance $l = a$. Therefore the moment of inertia is

$$I_x = Ml^2 + I_{x,\mathrm{cm}} = Ma^2 + \frac{1}{4}Ma^2 = \frac{5}{4}Ma^2$$

■

5.3 Gravitational Torque

If the force $\boldsymbol{F}_i$ on the i^{th} particle is the gravitational force $m_i\boldsymbol{g}$, then the total external gravitational torque about the origin is

$$\boldsymbol{N} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i$$

This can be written

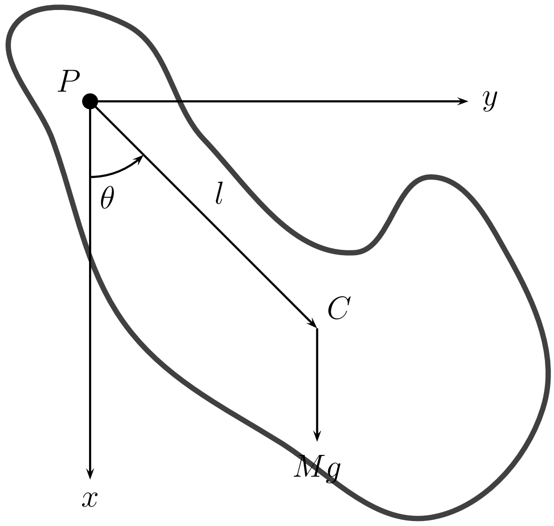
$$\boldsymbol{N} = \sum_i \boldsymbol{r}_i \times (m_i\boldsymbol{g}) = \sum_i (m_i\boldsymbol{r}_i) \times \boldsymbol{g}$$

In terms of the centre of mass, this becomes

$$\boldsymbol{N} = \sum_i (m_i\boldsymbol{r}_i) \times \boldsymbol{g} = M\boldsymbol{r}_{\mathrm{cm}} \times \boldsymbol{g}$$

Therefore the gravitional torque is equal to the torque acting on the total weight of the body concentrated at the centre of mass.

Figure 5.4: This is a compound pendulum of mass M with centre of mass at C. It pivots about the point P which is a distance l from the centre of mass. The z axis points out of the page.



EXAMPLE 5.11 (The Compound Pendulum)
The equation of motion of the compound pendulum about the pivot, shown in figure 5.4, is

$$I_z \dot{\omega} = N_z$$

where $\dot{\omega} = \ddot{\theta}$, *the torque is* $\boldsymbol{N} = \boldsymbol{l} \times M\boldsymbol{g}$ *so that*

$$N_z = -Mgl \sin\theta$$

and the moment of inertia about the pivot is

$$I_z = Ml^2 + I_{z,\mathrm{cm}}$$

Therefore

$$(Ml^2 + I_{z,\mathrm{cm}})\ddot{\theta} = -Mgl \sin\theta$$

which gives

$$\ddot{\theta} = -\frac{g}{l + I_{z,\mathrm{cm}}/Ml} \sin\theta$$

For small amplitude oscillations, $|\theta| \ll 1$ *so that* $\sin\theta \approx \theta$. *The equation of motion is approximately*

$$\ddot{\theta} = -\frac{g}{l + I_{z,\mathrm{cm}}/Ml}\theta$$

The solution is

$$\theta(t) = \theta_0 \cos \omega t$$

where $\theta(0) = \theta_0$ *and* $\dot{\theta}(0) = 0$. *The angular frequency of oscillation is*

$$\omega = \sqrt{\frac{g}{l + I_{z,\mathrm{cm}}/Ml}}$$

and the period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l + I_{z,\mathrm{cm}}/Ml}{g}}$$

which is independent of the initial conditions.

For large oscillations, the equation of motion is

$$\ddot{\theta} = -\omega_0^2 \sin\theta$$

where

$$\omega_0^2 = \frac{g}{l + I_{z,\mathrm{cm}}/Ml}$$

Using $\ddot{\theta} = \frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right)$, *integrate to obtain*

$$\frac{1}{2}\dot{\theta}^2 = \omega_0^2 \cos\theta + C$$

With $\theta(0) = \theta_0$ *and the pendulum starting from rest so that* $\dot{\theta}(0) = 0$, *the solution is*

$$\dot{\theta}^2 = 2\omega_0^2(\cos\theta - \cos\theta_0)$$

The integral of this is a bit complicated and cannot be obtained in terms of ordinary functions. However an expression for the period T *can be obtained.*

Taking the square root of the equation for $\dot{\theta}^2$

$$\dot{\theta} = \pm\sqrt{2}\omega_0\sqrt{\cos\theta - \cos\theta_0}$$

During the period $0 < t < \frac{T}{4}$, *the pendulum swings from* $\theta = \theta_0$ *to* $\theta = 0$ *so* $\dot{\theta} < 0$. *Thus for these times, the negative square root is appropriate and*

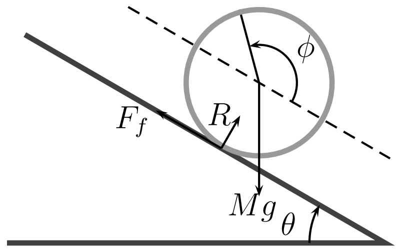
$$\dot{\theta} = -\sqrt{2}\omega_0\sqrt{\cos\theta - \cos\theta_0}$$

which is a separable equation.

The solution is

$$T = \frac{\sqrt{8}}{\omega_0}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

Figure 5.5: This diagram shows a body rolling on a plane inclined at an angle θ. The dashed line indicates the path taken by the rolling body's centre of mass.



which is an elliptic integral. Denoting the elliptic integral as $f(\theta_0)$, the solution is

$$T = \frac{\sqrt{8}}{\omega_0} f(\theta_0)$$

which is a function of the initial conditions. ■

5.4 Rolling

For the body rolling down the slope illustrated in figure 5.5, the forces on the body lead to the following equations of motion

$$M\ddot{x}_{\text{cm}} = Mg\sin\theta - F_f$$

$$M\ddot{y}_{\text{cm}} = -Mg\cos\theta + R$$

The body stays in contact with the slope so that

$$y_{\text{cm}} = a$$

and

$$\ddot{y}_{\text{cm}} = 0$$

so that the reaction is

$$R = Mg\cos\theta$$

The torque on the body about the centre of mass is

$$I_{\text{cm}}\dot{\omega} = N_z = -F_f a$$

since all other forces have zero torque about the centre of mass. The velocity of the point of the rolling body in contact with the slope is

$$\dot{x}_{\text{cm}} = -a\omega$$

so that

$$\omega = -\frac{\dot{x}_{\text{cm}}}{a} \qquad \Longrightarrow \qquad \dot{\omega} = -\frac{\ddot{x}_{\text{cm}}}{a}$$

Substituting into the torque equation gives

$$I_{\text{cm}}\left(-\frac{\ddot{x}_{\text{cm}}}{a}\right) = -F_f a$$

so that

$$F_f = \frac{I_{\text{cm}}}{a^2}\ddot{x}_{\text{cm}}$$

Substituting into the force equation shows that

$$\ddot{x}_{\text{cm}} = \frac{g\sin\theta}{1 + I_{\text{cm}}/Ma^2}$$

which is a constant. Thus

$$\dot{\omega} = -\frac{\ddot{x}_{\text{cm}}}{a} = \frac{g\sin\theta}{a\left(1 + I_{\text{cm}}/Ma^2\right)}$$

which is also a constant.

If the point of contact slips on the inclined plane, then

$$F_f = \mu R$$

where μ is the coefficient of sliding friction. Therefore the force equation becomes

$$M\ddot{x}_{\text{cm}} = Mg\sin\theta - \mu Mg\cos\theta$$

or

$$\ddot{x}_{\text{cm}} = g\left(\sin\theta - \mu\cos\theta\right)$$

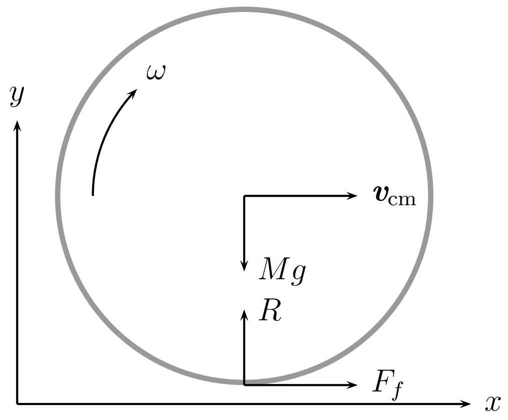
which is a constant. To calculate the angular velocity, use the torque equation

$$I_{\text{cm}}\dot{\omega} = -F_f a$$

so

$$\dot{\omega} = -\frac{a\mu Mg\cos\theta}{I_{\text{cm}}}$$

Figure 5.6: Initially, this billiard ball is spinning with an angular velocity of ω_0 and is released with zero forward velocity. At the time shown in this figure, the ball is spinning with angular velocity ω and has a forward velocity of $\boldsymbol{v}_{\rm cm}$.



EXAMPLE 5.12 (Slipping Billiard Ball)
A billiard ball rotating with angular speed ω_0 is released with zero forward velocity such that it slips on the table (as shown in figure 5.6), where the coefficient of friction is μ. When does it start to roll?
Solution: *The reaction is $R = Mg$ and initially, the ball is slipping so that*

$$F_f = \mu R$$

The force equation gives

$$M\ddot{x}_{\rm cm} = F_f = \mu Mg$$

so that

$$\ddot{x}_{\rm cm} = \mu g \qquad \dot{x}_{\rm cm} = \mu g t \qquad x_{\rm cm} = \frac{\mu g t^2}{2}$$

which is valid up to the point when rolling starts and slipping stops as then $F_f \neq \mu R$.

The torque equation about the centre of mass is

$$I_{\rm cm}\dot{\omega} = +\mu Mga$$

so that

$$\omega = \frac{\mu Mga}{I_{\rm cm}}t - \omega_0$$

Slipping stops when ω slows down to the point when

$$-\frac{\dot{x}_{\rm cm}}{a} = \omega$$

At this time, $t_{\rm roll}$,

$$-\frac{\mu g t_{\rm roll}}{a} = \frac{\mu M g a}{I_{\rm cm}} t_{\rm roll} - \omega_0$$

so the time when pure rolling starts is

$$t_{\rm roll} = \frac{a\omega_0}{\mu g\left(1 + Ma^2/I_{\rm cm}\right)}$$

which is a distance

$$x_{\rm roll} = \frac{a^2\omega_0^2}{2\mu g\left(1 + Ma^2/I_{\rm cm}\right)^2}$$

from the point when it is first released.

After this point, we have

$$\ddot{x}_{\rm cm} = \frac{F_f}{M}$$

Now

$$I_{\rm cm}\dot{\omega} = F_f a$$

but the ball is rotating without slipping so

$$\dot{\omega} = -\frac{\ddot{x}_{\rm cm}}{a}$$

so that

$$F_f = -\frac{I_{\rm cm}\ddot{x}_{\rm cm}}{a^2}$$

This implies that

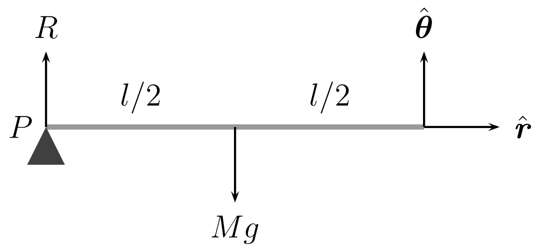
$$\ddot{x}_{\rm cm}\left(1 + \frac{I_{\rm cm}}{Ma^2}\right) = 0$$

which means that the ball's velocity is constant after pure rolling has started. This implies that F_f does no work on a rolling body. ■

EXAMPLE 5.13

Two people hold a plank of mass M and length l at each end. One person lets their end go. Show that the load supported by the other person drops from $Mg/2$ to $Mg/4$ instantly. Show that the acceleration of the free end is $3g/2$.

Figure 5.7: This shows the forces on a horizontal plank of length l and mass M at the instant that the right-hand end is dropped. The dropped plank pivots about the point P.



Solution: *A diagram of the force is shown in figure 5.7. The plank rotates about P according to*

$$I_P\dot{\omega} = -Mg\frac{l}{2}$$

With R the reaction from the person holding the plank at P, the equation of rotation about the plank's centre of mass is

$$I_{\mathrm{cm}}\dot{\omega} = -R\frac{l}{2}$$

so that

$$\frac{I_{\mathrm{cm}}}{I_P} = \frac{R}{Mg}$$

Using the parallel axis theorem to write $I_P = I_{\mathrm{cm}} + M\left(l/2\right)^2$ gives

$$R = Mg\frac{I_{\mathrm{cm}}}{I_{\mathrm{cm}} + M\left(l/2\right)^2}$$

But the moment of inertia about the centre of mass is

$$I_{\mathrm{cm}} = \frac{1}{12}Ml^2$$

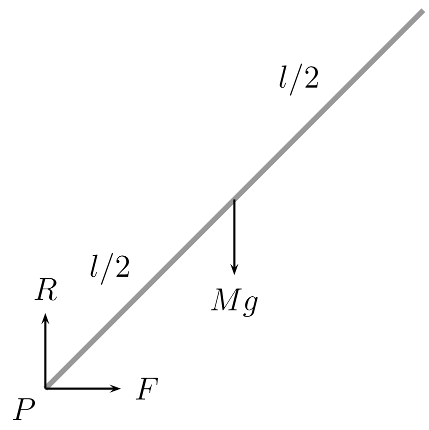
so the reaction is

$$R = Mg\frac{\frac{1}{12}}{\frac{1}{12}+\frac{1}{4}} = \frac{Mg}{4}$$

To work out the acceleration of the free end, use

$$\ddot{\boldsymbol{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}}$$

Figure 5.8: This shows the forces on a thin rod of mass M and length l falling over on a rough table.



Here $r = l$ *so* $\dot{r} = \ddot{r} = 0$. *Initially,* $\omega = \dot{\theta} = 0$ *so the acceleration is*

$$\ddot{\boldsymbol{r}} = l\ddot{\theta}\hat{\boldsymbol{\theta}} = l\dot{\omega}\hat{\boldsymbol{\theta}}$$

Using

$$I_{\text{cm}}\dot{\omega} = -R\frac{l}{2}$$

shows that

$$\ddot{\boldsymbol{r}} = -l\frac{R\frac{l}{2}}{I_{\text{cm}}}\hat{\boldsymbol{\theta}} = -\frac{Mgl^2}{8I_{\text{cm}}}\hat{\boldsymbol{\theta}} = -\frac{3}{2}g\hat{\boldsymbol{\theta}}$$

so the initial acceleration of the free end is $3g/2$ *as required.* ■

EXAMPLE 5.14

A thin rod of mass M and length l falls over from an initial vertical position on a rough table. Solve the subsequent motion.

Solution: *A diagram of the falling rod is given in figure 5.8. The rod's equation of angular motion about* P *is*

$$I_P\ddot{\theta} = -Mg\frac{l}{2}\cos\theta$$

which can be written as

$$I_P \frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right) = -Mg\frac{l}{2}\cos\theta$$

Integrating and using the initial condition $\dot{\theta} = 0$ at $\theta = \pi/2$ gives

$$\dot{\theta}^2 = \frac{Mgl}{I_P}(1 - \sin\theta)$$

Now from the parallel axes theorem,

$$I_P = M\left(\frac{l}{2}\right)^2 + \frac{1}{12}Ml^2 = \frac{1}{3}Ml^2$$

so

$$\dot{\theta}^2 = \left(\frac{3g}{l}\right)(1 - \sin\theta)$$

Also, substituting for I_P in the initial equation of angular motion about P gives

$$\ddot{\theta} = -\left(\frac{3g}{2l}\right)\cos\theta$$

The equations of motion of the centre of mass are

$$\begin{aligned} M\ddot{x}_{cm} &= F \\ M\ddot{y}_{cm} &= R - Mg \end{aligned}$$

If no slipping initially, then we have

$$\begin{aligned} x_{cm} &= \frac{l}{2}\cos\theta \\ y_{cm} &= \frac{l}{2}\sin\theta \end{aligned}$$

Therefore

$$\dot{x}_{cm} = -\frac{l}{2}\sin\theta\dot{\theta}$$

and

$$\begin{aligned} \ddot{x}_{cm} &= -\frac{l}{2}\sin\theta\ddot{\theta} - \frac{l}{2}\cos\theta\dot{\theta}^2 \\ &= -\frac{l}{2}\left[\sin\theta\left(-\frac{3g}{2l}\cos\theta\right) + \cos\theta\left(\frac{3g}{l}(1 - \sin\theta)\right)\right] \\ &= \frac{3g}{4}\cos\theta(3\sin\theta - 2) \end{aligned}$$

Similarly for y_{cm},

$$\dot{y}_{cm} = \frac{l}{2}\cos\theta\dot{\theta}$$

and

$$\begin{aligned}
\ddot{y}_{cm} &= -\frac{l}{2}\sin\theta\dot{\theta}^2 + \frac{l}{2}\cos\theta\ddot{\theta} \\
&= -\frac{l}{2}\left[\sin\theta\left(\frac{3g}{l}(1-\sin\theta)\right) - \cos\theta\left(-\frac{3g}{2l}\cos\theta\right)\right] \\
&= -\frac{3g}{4}(1+2\sin\theta-3\sin^2\theta)
\end{aligned}$$

Substituting into the equations of motion,

$$\begin{aligned}
F &= M\ddot{x}_{cm} \\
&= \frac{3Mg}{4}\cos\theta(3\sin\theta-2)
\end{aligned}$$

and

$$\begin{aligned}
R &= Mg - M\ddot{y}_{cm} \\
&= \frac{Mg}{4}(1-6\sin\theta+9\sin^2\theta)
\end{aligned}$$

Slipping starts at the angle θ^* *where*

$$F = \mu R$$

so that μ *is given in terms of* θ^* *as*

$$\mu = \frac{3\cos\theta^*(3\sin\theta^*-2)}{1-6\sin\theta^*+9\sin^2\theta^*}$$

■

Chapter 6

Systems of Differential Equations

Systems of differential equations arise when a higher order differential equation is written as a set of first order differential equations or when a system is modelled as a set of coupled differential equations.

EXAMPLE 6.1
The second order differential equation

$$\ddot{y} = f(t, y, \dot{y})$$

can be rewritten with $y_1 = y$ *and* $y_2 = \dot{y}$ *as a set of coupled first order o.d.e.s.*

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = f(t, y_1, y_2)$$

■

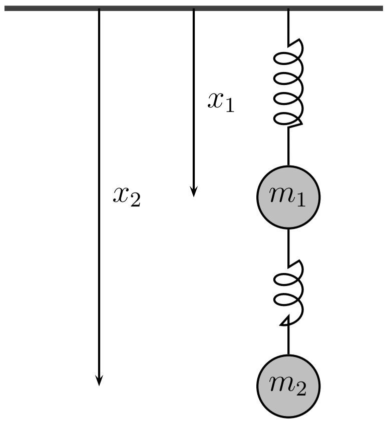
EXAMPLE 6.2 (Volterra-Lotka Systems)
Volterra-Lotka systems are models of populations. Let y_1 *be the number of foxes (the predators) and* y_2 *the number of rabbits (the prey). Then one model for the rabbits' and foxes' populations is*

$$\dot{y}_1 = -ay_1 + by_1y_2$$

$$\dot{y}_2 = cy_2 - dy_1y_2$$

where a *sets the rate at which foxes die of starvation,* b *the fox birth rate in the presence of a food supply,* c *the rabbit birth rate and* d *the rate of rabbit deaths due to predation. All* a*,* b*,* c *and* d *are positive.*

Figure 6.1: These coupled springs give the system of coupled differential equations in example 6.3. The two springs have spring constants k_1 and k_2 respectively, and their unstretched lengths are l_1 and l_2.



In matrix form, the system becomes

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix} = \begin{bmatrix}-a & by_1\\ -dy_2 & c\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}$$

■

EXAMPLE 6.3 (Coupled Springs I)
For the system of coupled springs shown in figure 6.1, whose unstretched lengths are l_1 and l_2, and whose spring constants are k_1 and k_2, from Newton's laws

$$m_1\ddot{x}_1 = m_1 g - k_1(x_1 - l_1) + k_2(x_2 - x_1 - l_2)$$

$$m_2\ddot{x}_2 = m_2 g - k_2(x_2 - x_1 - l_2)$$

In matrix form, this is

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} = \begin{bmatrix} g + \frac{k_1 l_1 - k_2 l_2}{m_1}\\ g + \frac{k_2 l_2}{m_2}\end{bmatrix} - \begin{bmatrix}\frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1}\\ -\frac{k_2}{m_2} & \frac{k_2}{m_2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

■

EXAMPLE 6.4 (Chemical Reaction)
The following four chemical reactions occur between four chemicals, species 1, species 2, A and B:

1. *species 1* $\xrightarrow{k_1}$ *products*
2. *species* $1 + A \xrightarrow{k_2}$ *species* $2+$ *products*
3. *species 2* $\xrightarrow{k_3}$ *products*
4. *species* $2 + B \xrightarrow{k_4}$ *species* $1+$ *products*

Then the concentrations of species 1, c_1 *and of species 2,* c_2*, follow*

$$\dot{c}_1 = -k_1c_1 - k_2[A]c_1 + k_4[B]c_2$$

$$\dot{c}_2 = -k_3c_2 + k_2[A]c_1 - k_4[B]c_2$$

where $[A]$ *and* $[B]$ *are the concentrations of* A *and* B*.*

Suppose that $[A], [B] \gg c_1, c_2$ *so that* $[A]$ *and* $[B]$ *hardly change during the reaction. Then*

$$K_1 = k_2[A] \qquad \text{and} \qquad K_2 = k_4[B]$$

can be regarded as constant. In matrix form

$$\begin{bmatrix} \dot{c}_1 \\ \dot{c}_2 \end{bmatrix} = -\begin{bmatrix} k_1 + K_1 & -K_2 \\ -K_1 & k_3 + K_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

■

6.1 Systems of Linear First Order O.D.E.s

The general form is

$$\dot{y}_1 = a_{11}y_1 + a_{12}y_2 + f_1(t)$$

$$\dot{y}_2 = a_{21}y_1 + a_{22}y_2 + f_2(t)$$

Expressed as matrices, this is

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

or

$$\dot{\boldsymbol{Y}} = \boldsymbol{AY} + \boldsymbol{F}$$

We will only consider systems where $\boldsymbol{A}$ is a constant matrix independent of t.

6.2 Homogeneous Systems

The homogeneous system is

$$\dot{\boldsymbol{Y}} = \boldsymbol{A}\boldsymbol{Y}$$

where

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \boldsymbol{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

EXAMPLE 6.5

If $a_{12} = a_{21} = 0$, these equations decouple

$$\dot{y}_1 = a_{11}y_1$$

$$\dot{y}_2 = a_{22}y_2$$

with solution

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 e^{a_{11}t} \\ C_2 e^{a_{22}t} \end{bmatrix}$$

■

The general solution is found by trying an exponential solution of the form

$$\boldsymbol{Y}(t) = \boldsymbol{K}e^{\alpha t}$$

where $\boldsymbol{K}$ and α are to be determined. Now

$$\dot{\boldsymbol{Y}} = \alpha\boldsymbol{K}e^{\alpha t}$$

so substituting into the o.d.e. system gives

$$\alpha\boldsymbol{K}e^{\alpha t} = \boldsymbol{A}\boldsymbol{K}e^{\alpha t}$$

Therefore we require

$$\boldsymbol{A}\boldsymbol{K} = \alpha\boldsymbol{K}$$

so α is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{K}$ is the corresponding eigenvector.

So the general method of solution is

1. Find the two eigenvalues of $\boldsymbol{A}$ and their corresponding eigenvectors (which are linearly independent if $\alpha_1 \neq \alpha_2$).

2. This gives two linearly independent solutions

$$\boldsymbol{Y}_1 = \boldsymbol{K}_1 e^{\alpha_1 t}$$

$$\boldsymbol{Y}_2 = \boldsymbol{K}_2 e^{\alpha_2 t}$$

3. The general solution is then

$$\boldsymbol{Y} = C_1 \boldsymbol{K}_1 e^{\alpha_1 t} + C_2 \boldsymbol{K}_2 e^{\alpha_2 t}$$

EXAMPLE 6.6
To show that the same solution is obtained from a second order o.d.e. as from the corresponding system, let

$$\dot{y}_1 = a_{11} y_1 + a_{12} y_2$$

$$\dot{y}_2 = a_{21} y_1 + a_{22} y_2$$

Then differentiate the first and substitute the second

$$\ddot{y}_1 = a_{11} \dot{y}_1 + a_{12} \dot{y}_2 = a_{11} \dot{y}_1 + a_{12}(a_{21} y_1 + a_{22} y_2)$$

Then rearranging and using the first equation to eliminate y_2 gives

$$\ddot{y}_1 = a_{11} \dot{y}_1 + a_{12} a_{21} y_1 + a_{22}(\dot{y}_1 - a_{11} y_1)$$

This is the second order differential equation

$$\ddot{y}_1 - (a_{11} + a_{22}) \dot{y}_1 + (a_{11} a_{22} - a_{12} a_{21}) y_1 = 0$$

whose solution is

$$y_1 = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}$$

where α_1 and α_2 are the roots of

$$\alpha^2 - (a_{11} + a_{22})\alpha + (a_{11} a_{22} - a_{12} a_{21}) = 0$$

■

EXAMPLE 6.7
The system

$$\dot{y}_1 = y_1 + 4y_2$$

$$\dot{y}_2 = 4y_1 + y_2$$

gives

$$\dot{\boldsymbol{Y}} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \boldsymbol{Y}$$

The eigenvalues of $\boldsymbol{A}$ satisfy

$$\det \begin{bmatrix} 1-\alpha & 4 \\ 4 & 1-\alpha \end{bmatrix} = 0$$

This gives the quadratic

$$(1-\alpha)^2 - 4^2 = 0$$

whose roots are $\alpha_1 = 5$ *and* $\alpha_2 = -3$.

The eigenvectors are found by solving

$$(\boldsymbol{A} - \boldsymbol{I}\alpha)\boldsymbol{K} = 0$$

When $\alpha_1 = 5$, *this gives*

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

so that $k_1 = k_2$ *and the first eigenvector is*

$$\boldsymbol{K}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

When $\alpha_2 = -3$, *this gives*

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

so that $k_1 = -k_2$ *and the second eigenvector is*

$$\boldsymbol{K}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore the general solution is

$$\boldsymbol{Y} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}$$

The constants C_1 *and* C_2 *may be determined by the initial conditions on* y_1 *and* y_2. ■

This can be generalized to n equations. If $\boldsymbol{Y}$ has n elements and $\boldsymbol{A}$ is $n \times n$, the general solution is

$$\boldsymbol{Y} = C_1 \boldsymbol{K}_1 e^{\alpha_1 t} + C_2 \boldsymbol{K}_2 e^{\alpha_2 t} + \cdots + C_n \boldsymbol{K}_n e^{\alpha_n t}$$

where

$$\boldsymbol{A}\boldsymbol{K}_i = \alpha_i \boldsymbol{K}_i$$

6.2.1 Repeated Factors of the Characteristic Polynomial

When two eigenvalues are equal, so that $\alpha_1 = \alpha_2$, then one solution is

$$\boldsymbol{Y}_1 = \boldsymbol{K}_1 e^{\alpha_1 t}$$

A second solution can always be found of the form

$$\boldsymbol{Y}_2 = \boldsymbol{K}_1 t e^{\alpha_1 t} + \boldsymbol{P} e^{\alpha_1 t}$$

To see this, differentiate to give

$$\dot{\boldsymbol{Y}}_2 = \boldsymbol{K}_1 e^{\alpha_1 t} + \alpha_1 \boldsymbol{K}_1 t e^{\alpha_1 t} + \alpha_1 \boldsymbol{P} e^{\alpha_1 t}$$

and substitute into $\dot{\boldsymbol{Y}} = \boldsymbol{A}\boldsymbol{Y}$, giving

$$e^{\alpha_1 t}(\boldsymbol{K}_1 + \alpha_1 \boldsymbol{P} + \alpha_1 \boldsymbol{K}_1 t) = e^{\alpha_1 t}(\boldsymbol{A}\boldsymbol{K}_1 t + \boldsymbol{A}\boldsymbol{P})$$

Equating coefficients of $e^{\alpha_1 t}$ and $te^{\alpha_1 t}$ shows that

$$\boldsymbol{A}\boldsymbol{K}_1 = \alpha_1 \boldsymbol{K}_1$$

which is just the eigenvalue/eigenvector equation, and

$$\boldsymbol{A}\boldsymbol{P} = \alpha_1 \boldsymbol{P} + \boldsymbol{K}_1$$

which is equivalent to

$$(\boldsymbol{A} - \alpha_1 \boldsymbol{I})\boldsymbol{P} = \boldsymbol{K}_1$$

This equation does not completely determine $\boldsymbol{P}$ because the determinant of $\boldsymbol{A} - \alpha_1 \boldsymbol{I}$ is zero.

EXAMPLE 6.8
The system

$$\dot{\boldsymbol{Y}} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \boldsymbol{Y}$$

has eigenvalues which satisfy

$$\det \begin{bmatrix} 1-\alpha & 0 \\ 2 & 1-\alpha \end{bmatrix} = 0$$

which gives the quadratic

$$(1-\alpha)^2 = 0$$

so there is a single repeated eigenvalue $\alpha = 1$.

The eigenvector for $\alpha = 1$ satisfies

$$\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

so that $k_1 = 0$ and the eigenvector is

$$\boldsymbol{K}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus the first solution is

$$\boldsymbol{Y}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t$$

The second solution has the form

$$\boldsymbol{Y}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} te^t + \boldsymbol{P}e^t$$

where $\boldsymbol{P}$ satisfies

$$(\boldsymbol{A} - \alpha_1 \boldsymbol{I})\boldsymbol{P} = \boldsymbol{K}_1$$

In this case,

$$\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This shows that $p_1 = 1/2$ but p_2 is not determined. Choosing any value of p_2, set $p_2 = 0$ so that the second solution is

$$\boldsymbol{Y}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} e^t$$

Therefore the general solution is

$$\boldsymbol{Y} = C_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t + C_2 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} e^t \right)$$

Note that choosing another value for $\boldsymbol{P}$ would just change the constant C_1. Suppose that $\boldsymbol{P}$ were

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{2} \\ a \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + a\boldsymbol{K}_1$$

This gives the general solution

$$\boldsymbol{Y} = (C_1 + aC_2) \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t + C_2 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} e^t \right)$$

which is the same as before except for the constant C_1. ■

6.2.2 Higher Order Degeneracy

For $n \geq 3$, we have the possibility of 3 degenerate eigenvalues. We have already found two solutions

$$\boldsymbol{Y}_1 = \boldsymbol{K}_1 e^{\alpha_1 t}$$

$$\boldsymbol{Y}_2 = \boldsymbol{K}_1 t e^{\alpha_1 t} + \boldsymbol{P} e^{\alpha_1 t}$$

where $(\boldsymbol{A} - \alpha_1 \boldsymbol{I})\boldsymbol{P} = \boldsymbol{K}_1$.

A third linearly independent solution is

$$\boldsymbol{Y}_3 = \boldsymbol{K}_1 \frac{t^2}{2} e^{\alpha_1 t} + \boldsymbol{P} t e^{\alpha_1 t} + \boldsymbol{Q} e^{\alpha_1 t}$$

where $(\boldsymbol{A} - \alpha_1 \boldsymbol{I})\boldsymbol{Q} = \boldsymbol{P}$.

EXAMPLE 6.9
The system

$$\dot{\boldsymbol{Y}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{Y}$$

has a single eigenvalue $\alpha = 1$ of multiplicity three.

The eigenvector $\boldsymbol{K}_1$ satisfies

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so that

$$\boldsymbol{K}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The vector $\boldsymbol{P}$ satisfies

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

so that choosing $p_1 = 0$ for simplicity gives

$$\boldsymbol{P} = \begin{bmatrix} p_1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The vector $\boldsymbol{Q}$ satisfies

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

so that choosing $q_1 = 0$ for simplicity gives

$$\boldsymbol{Q} = \begin{bmatrix} q_1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Therefore the general solution is

$$\boldsymbol{Y} = C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + C_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t \right) + C_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} e^t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^t \right)$$

■

Exercise 8.6 of Zill, pp. 466–469, has more examples of coupled linear systems.

6.3 Inhomogeneous Simultaneous Systems

An inhomogeneous simultaneous system has the form

$$\dot{\boldsymbol{Y}} = \boldsymbol{A}\boldsymbol{Y} + \boldsymbol{F}(t)$$

where $\boldsymbol{F}(t)$ is the inhomogeneous term.

Then general solution is

$$\boldsymbol{Y} = C_1 \boldsymbol{Y}_1 + C_2 \boldsymbol{Y}_2 + \cdots + C_n \boldsymbol{Y}_n + \boldsymbol{Y}_{\text{ps}}$$

where the $\boldsymbol{Y}_i$ are the n linearly independent solutions of $\dot{\boldsymbol{Y}} = \boldsymbol{A}\boldsymbol{Y}$ and $\boldsymbol{Y}_{\text{ps}}$ is any particular solution of the inhomogeneous problem.

$\boldsymbol{Y}_{\text{ps}}$ is found using the method of undetermined coefficients. If $\boldsymbol{F}(t)$ is a polynomial of degree n, try a polynomial of degree n for $\boldsymbol{Y}_{\text{ps}}$. If $\boldsymbol{F}(t) = \boldsymbol{C}e^{kt}$, try a solution of the form $\boldsymbol{Y}_{\text{ps}} = \boldsymbol{D}e^{kt}$. Finally, if $\boldsymbol{F}(t) = \boldsymbol{A}\sin\omega t + \boldsymbol{B}\cos\omega t$, try a solution of the form $\boldsymbol{Y}_{\text{ps}} = \boldsymbol{D}\sin\omega t + \boldsymbol{E}\cos\omega t$.

EXAMPLE 6.10
The coupled differential equations

$$\dot{y}_1 = y_1 + 4y_2 + \cos t$$

$$\dot{y}_2 = 4y_1 + y_2 + \sin t$$

gives the matrix system

$$\dot{\boldsymbol{Y}} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \boldsymbol{Y} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t$$

The homogeneous problem has already been solved, so that

$$\boldsymbol{Y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}$$

$$\boldsymbol{Y}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}$$

Try a particular solution of the form

$$\boldsymbol{Y}_{\mathrm{ps}} = \boldsymbol{D} \cos t + \boldsymbol{E} \sin t$$

whose derivative is

$$\dot{\boldsymbol{Y}}_{\mathrm{ps}} = -\boldsymbol{D} \sin t + \boldsymbol{E} \cos t$$

Substituting into the differential equation gives

$$-\boldsymbol{D} \sin t + \boldsymbol{E} \cos t = \boldsymbol{A}\boldsymbol{D} \cos t + \boldsymbol{A}\boldsymbol{E} \sin t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t$$

where $\boldsymbol{A} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$. *Equating coefficients of* $\cos t$ *and* $\sin t$ *gives*

$$-\boldsymbol{D} = \boldsymbol{A}\boldsymbol{E} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{E} = \boldsymbol{A}\boldsymbol{D} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This is four equations in four unknowns

$$-d_1 = e_1 + 4e_2$$

$$-d_2 = 4e_1 + e_2 + 1$$

$$e_1 = d_1 + 4d_2 + 1$$

$$e_2 = 4d_1 + e_1$$

where $\boldsymbol{D} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ *and* $\boldsymbol{E} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$. *Solving for* d_1, d_2, e_1 *and* e_2 *gives the general solution*

$$\boldsymbol{Y} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \cos t + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \sin t$$

■

Exercise 8.7 of Zill, pp. 472–473, has more examples of inhomogeneous simultaneous systems.

6.4 Coupled Oscillators

A **coupled second order linear system** is one such as

$$\ddot{\boldsymbol{Y}} = -\boldsymbol{A}\boldsymbol{Y}$$

where

$$\boldsymbol{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix} \qquad \ddot{\boldsymbol{Y}} = \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \vdots \end{bmatrix}$$

If the solution has the form

$$\boldsymbol{Y} = \boldsymbol{K}e^{i\lambda t}$$

then

$$\ddot{\boldsymbol{Y}} = -\lambda^2\boldsymbol{K}e^{i\lambda t}$$

Substituting in the original differential equation shows that

$$\boldsymbol{A}\boldsymbol{K} = \lambda^2\boldsymbol{K}$$

Thus λ^2 is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{K}$ is its corresponding eigenvector. Taking the square root to recover λ gives both a positive and a negative solution. Therefore a 2×2 second order system may have two eigenvalues and four values of λ. The general solution is

$$\boldsymbol{Y} = \boldsymbol{K}_1\left(C_1e^{i\lambda_1 t} + C_2e^{-i\lambda_1 t}\right) + \boldsymbol{K}_2\left(C_3e^{i\lambda_2 t} + C_4e^{-i\lambda_2 t}\right)$$

The four constants of integration are determined by the initial or boundary conditions on y_1, $\dot{y}_1$, y_2 and $\dot{y}_2$. Alternatively, the solution can be expressed as

$$\boldsymbol{Y} = \boldsymbol{K}_1 A\cos(\lambda_1 t + \phi_1) + \boldsymbol{K}_2 B\cos(\lambda_2 t + \phi_2)$$

where A, B, ϕ_1 and ϕ_2 are the constants to be determined.

EXAMPLE 6.11
To find the general solution of

$$\ddot{\boldsymbol{Y}} = -\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}\boldsymbol{Y}$$

the eigenvalues are found from

$$\det \begin{bmatrix} 3-\alpha & -1 \\ -2 & 2-\alpha \end{bmatrix} = 0$$

which is the quadratic

$$(3-\alpha)(2-\alpha) - 2 = 0$$

This gives $\alpha = 1, 4$.

The eigenvector $\boldsymbol{K}_1$ *corresponding to* $\alpha = 4$ *satisfies*

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that

$$\boldsymbol{K}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The eigenvector $\boldsymbol{K}_2$ *corresponding to* $\alpha = 1$ *satisfies*

$$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that

$$\boldsymbol{K}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore the general solution is

$$\boldsymbol{Y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} A\cos(2t + \phi_1) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} B\cos(t + \phi_2)$$

■

Note that if the coupling terms in the previous example are turned off, we would have

$$\ddot{\boldsymbol{Y}} = -\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \boldsymbol{Y}$$

with solution

$$y_1 = A_1 \cos(\sqrt{3}t + \phi_1)$$
$$y_2 = A_2 \cos(\sqrt{2}t + \phi_2)$$

so that the uncoupled frequencies are $\sqrt{3}$ and $\sqrt{2}$. When the coupling is turned on, the system's frequencies are 1 and 2. Note that it is not just y_1 which oscillates with frequency 2; both components y_1 and y_2 exhibit a frequency 2 contribution.

Figure 6.2: The rapid mode $\omega = \omega_+$ for the coupled springs has the two masses moving in opposite directions.

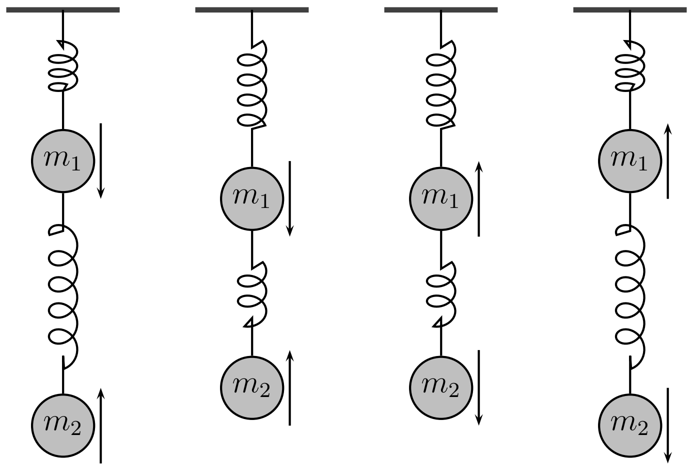
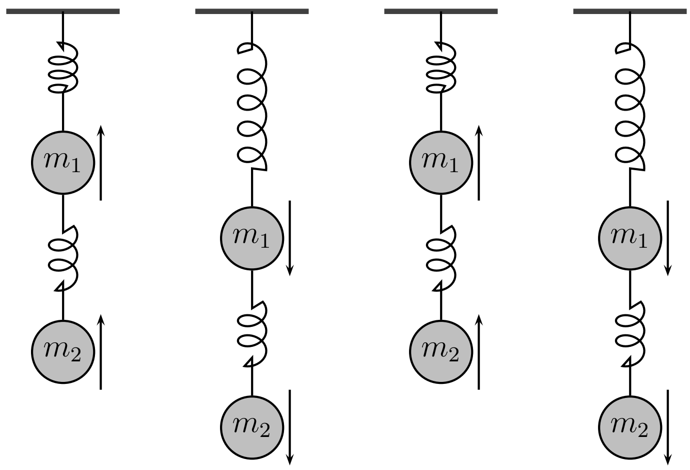


Figure 6.3: The slow mode $\omega = \omega_-$ for the coupled springs has the two masses moving in the same direction.



EXAMPLE 6.12 (Coupled Springs II)
The coupled spring system of example 6.3 can be written as

$$\ddot{\boldsymbol{X}} = -\boldsymbol{A}\boldsymbol{X} + \boldsymbol{F}$$

where $\boldsymbol{A}$ is

$$\boldsymbol{A} = \begin{bmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_2}{m_2} \end{bmatrix}$$

and $\boldsymbol{F}$ is a constant

$$\boldsymbol{F} = \begin{bmatrix} g + \frac{k_1 l_1 - k_2 l_2}{m_1} \\ g + \frac{k_2 l_2}{m_2} \end{bmatrix}$$

The eigenvalues of $\boldsymbol{A}$ are found by solving

$$\det \begin{bmatrix} \frac{k_1+k_2}{m_1} - \alpha & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_2}{m_2} - \alpha \end{bmatrix} = 0$$

which gives the quadratic

$$\alpha^2 - \left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2}\right)\alpha + \frac{k_1 k_2}{m_1 m_2} = 0$$

The solutions are

$$\alpha = \frac{1}{2}\left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2} \pm \sqrt{\left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - 4\frac{k_1 k_2}{m_1 m_2}}\right)$$

Denote these two solutions ω_+^2 and ω_-^2.

Therefore the system of coupled springs can oscillate in two distinct ***modes***

$$\boldsymbol{X}_1 = \boldsymbol{K}_1 \cos(\omega_+ t + \phi_1)$$

and

$$\boldsymbol{X}_2 = \boldsymbol{K}_2 \cos(\omega_- t + \phi_2)$$

and the general motion is the linear combination

$$\boldsymbol{X} = A_1 \boldsymbol{K}_1 \cos(\omega_+ t + \phi_1) + A_2 \boldsymbol{K}_2 \cos(\omega_- t + \phi_2)$$

where the values of A_1, A_2, ϕ_1 and ϕ_2 depend on how the system is started off.

For simplicity, consider equal spring constants $k_1 = k_2 = k$ and equal masses $m_1 = m_2 = m$. Then the system is described by

$$\ddot{\boldsymbol{X}} = -\omega_0^2 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \boldsymbol{X}$$

where $\omega_0 = \sqrt{k/m}$ is the natural frequency of the single spring and mass system.

The eigenvalues are

$$\omega_\pm^2 = \frac{3k}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{3k}{m}\right)^2 - 4\frac{k^2}{m^2}} = \frac{1}{2}(3 \pm \sqrt{5})\frac{k}{m}$$

The eigenvector for ω_+ satisfies

$$\begin{bmatrix} 2 - \frac{1}{2}(3+\sqrt{5}) & -1 \\ -1 & 1 - \frac{1}{2}(3+\sqrt{5}) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that

$$\boldsymbol{K}_+ = \begin{bmatrix} 1 - \frac{1}{2}(3+\sqrt{5}) \\ 1 \end{bmatrix}$$

where $1 - \frac{1}{2}(3+\sqrt{5}) < 0$. In this mode,

$$x_1(t) = -\left(\frac{1}{2}(3+\sqrt{5}) - 1\right) x_2(t)$$

so when the second spring is extended ($x_2(t) > 0$), the first spring is compressed ($x_1(t) < 0$), and vice versa. This is shown in figure 6.2.

The eigenvector for ω_- satisfies

$$\begin{bmatrix} 2 - \frac{1}{2}(3-\sqrt{5}) & -1 \\ -1 & 1 - \frac{1}{2}(3-\sqrt{5}) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that

$$\boldsymbol{K}_- = \begin{bmatrix} 1 - \frac{1}{2}(3-\sqrt{5}) \\ 1 \end{bmatrix}$$

where $1 - \frac{1}{2}(3-\sqrt{5}) > 0$. In this mode,

$$x_1(t) = -\left(\frac{1}{2}(3-\sqrt{5}) - 1\right) x_2(t)$$

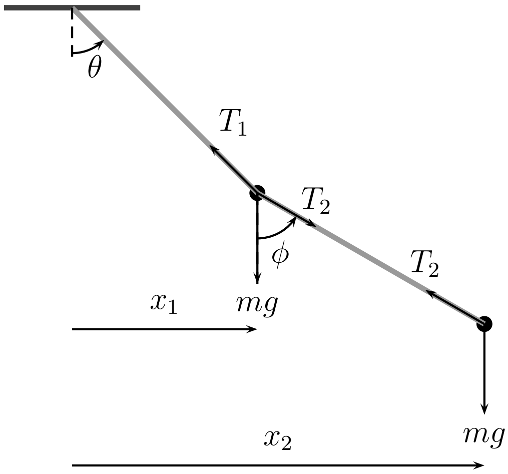
so the two springs move in the same direction. This is shown in figure 6.3. ■

EXAMPLE 6.13 (Double Pendulum)

In the diagram of the double pendulum in figure 6.4, assume that ϕ and θ are small so that the masses hardly move vertically. Therefore the vertical forces on each mass should balance, giving

$$T_1 \cos\theta = mg + T_2 \cos\phi$$

Figure 6.4: The double pendulum has two segments, each of length l with weights of mass m at the end.



$$T_2 \cos\phi = mg$$

If ϕ and θ are small, their cosines are approximately unity, so that

$$T_1 = mg + T_2$$

$$T_2 = mg$$

which shows that $T_1 = 2mg$. Now

$$x_1 = l\sin\theta \approx l\theta$$

and

$$x_2 = x_1 + l\sin\phi \approx l(\theta + \phi)$$

In the horizontal direction on particle 1,

$$m\ddot{x}_1 = -T_1 \sin\theta + T_2 \sin\phi$$

Thus

$$\ddot{\theta} = -\frac{2g}{l}\theta + \frac{g}{l}\phi$$

In the horizontal direction on particle 2,

$$m\ddot{x}_2 = -T_2 \sin\phi$$

Thus

$$\ddot{\phi} = \frac{2g}{l}\theta - \frac{2g}{l}\phi$$

These can be written as the system

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = -\omega_0^2 \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \theta \\ \phi \end{bmatrix}$$

where $\omega_0 = \sqrt{g/l}$ *is the natural frequency of a single pendulum.*

The general solution is

$$\begin{bmatrix} \theta \\ \phi \end{bmatrix} = C_1 \boldsymbol{K}_1 \cos(\omega_1 t + \phi_1) + C_2 \boldsymbol{K}_2 \cos(\omega_2 t + \phi_2)$$

with

$$\omega_1 = \omega_0 \sqrt{\alpha_1}$$
$$\omega_2 = \omega_0 \sqrt{\alpha_2}$$

where α_1 *and* α_2 *are the eigenvalues of* $\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$ *and* $\boldsymbol{K}_1$ *and* $\boldsymbol{K}_2$ *are the corresponding eigenvectors.*

The eigenvalues are found by solving

$$\det \begin{bmatrix} 2-\alpha & -1 \\ -2 & 2-\alpha \end{bmatrix} = 0$$

which gives the quadratic

$$(2-\alpha)^2 - 2 = 0$$

The solutions are $\alpha_1 = 2 + \sqrt{2}$ *and* $\alpha_2 = 2 - \sqrt{2}$.

The eigenvector for $\alpha_1 = 2 + \sqrt{2}$ *satisfies*

$$\begin{bmatrix} -\sqrt{2} & -1 \\ -2 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that

$$\boldsymbol{K}_1 = \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$$

The eigenvector for $\alpha_2 = 2 - \sqrt{2}$ *satisfies*

$$\begin{bmatrix} \sqrt{2} & -1 \\ -2 & \sqrt{2} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that

$$\boldsymbol{K}_2 = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$$

This gives two modes of operation of the double pendulum. The high frequency mode is $\omega_1 = \omega_0\sqrt{2+\sqrt{2}}$ *so that*

$$\begin{bmatrix} \theta \\ \phi \end{bmatrix}_1 = \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \cos(\omega_1 t + \phi_1)$$

Here ϕ *and* θ *have opposite signs, so the two halves of the double pendulum move in opposite directions.*

The low frequency mode is $\omega_2 = \omega_0\sqrt{2-\sqrt{2}}$ *so that*

$$\begin{bmatrix} \theta \\ \phi \end{bmatrix}_2 = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \cos(\omega_2 t + \phi_1)$$

Here ϕ *and* θ *have the same sign, so the two halves of the double pendulum move in the same direction.* ■

Chapter 7

Non-Linear Coupled First Order Equations

All o.d.e.s of any order can be written in the autonomous form

$$\dot{\boldsymbol{Y}} = \boldsymbol{F}(\boldsymbol{Y})$$

To see why, consider the general $(n-1)$th order o.d.e.

$$0 = F(t, Y, Y', Y'', \ldots, Y^{(n-1)})$$

This can, in principle, be written in the alternative form

$$t = G(Y, Y', Y'', \ldots, Y^{(n-1)})$$

Now differentiate with respect to t, giving

$$1 = \sum_{i=0}^{n-1} \frac{\partial G}{\partial Y^{(i)}} \frac{dY^{(i)}}{dt}$$

Since $\frac{dY^{(i)}}{dt} = Y^{(i+1)}$, this can be written

$$1 = \sum_{i=1}^{n} \frac{\partial G}{\partial Y^{(i-1)}} Y^{(i)}$$

by changing the limits of i in the summation.

Then rearranging this equation gives $Y^{(n)}$ in terms of the lower-order derivatives

$$Y^{(n)} = \Phi(Y, Y', Y'', \ldots, Y^{(n-1)})$$

This is referred to as an **autonomous system** because there is now no explicit dependence on time t.

Finally, define the set of variables

$$y_i = Y^{(i-1)} \qquad \text{for } i = 1, \ldots, n$$

so that

$$\begin{aligned} \dot{y}_1 &= \quad Y' &&= y_2 \\ \dot{y}_2 &= \quad Y'' &&= y_3 \\ &\quad\;\vdots \\ \dot{y}_n &= \quad Y^{(n)} &&= \Phi(y_1, y_2, \ldots, y_{n-1}) \end{aligned}$$

These relations can written more generally as

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, y_2, \ldots, y_n) \\ \dot{y}_2 &= f_2(y_1, y_2, \ldots, y_n) \\ &\;\vdots \\ \dot{y}_n &= f_n(y_1, y_2, \ldots, y_n) \end{aligned}$$

which can be expressed more concisely as a set of n coupled first-order differential equations of a vector argument

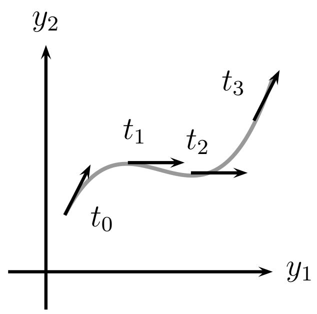
$$\dot{\boldsymbol{Y}} = \boldsymbol{F}(\boldsymbol{Y})$$

7.1 Phase Space and Trajectories

Consider the coupled system for $n = 2$:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{bmatrix}$$

Solve this to obtain expressions for $y_1(t)$ and $y_2(t)$. Then, for each time t, plot (y_1, y_2). This gives a **trajectory** of points $(y_1(t), y_2(t))$ through the (y_1, y_2) **phase space**.



The diagram at the left shows a trajectory through phase space starting at the initial position $(y_1(t_0), y_2(t_0))$. The arrows indicate the direction of motion along the trajectory at different times. Note that there is always only one trajectory for any set of initial conditions $(y_1(t_0), y_2(t_0))$. This follows from the uniqueness of the solution of the differential equations. In this chapter, we will restrict ourselves to two-dimensional systems ($n = 2$) for ease of graphical interpretation. However the concept of phase space and trajectories holds in higher dimensions.

EXAMPLE 7.1
The coupled system for $y'' + y = 0$ is found by setting $y_1 = y$ and $y_2 = \dot{y}$. Then

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \ddot{y} = -y_1 \end{aligned}$$

so that the coupled system is

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -y_1 \end{aligned}$$

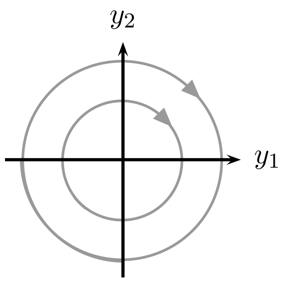
The solutions have the form

$$\begin{aligned} y_1 &= A\sin(t+\phi) \\ y_2 &= A\cos(t+\phi) \end{aligned}$$

which gives

$$y_1^2 + y_2^2 = A^2$$

when t is eliminated. Therefore the phase portrait consists of concentric circles about the origin starting at the initial value $(y_1(t_0), y_2(t_0))$. The movement of each trajectory is clockwise because when $y_1 > 0$ and $y_2 = 0$, $\dot{y}_2 < 0$.



■

7.1.1 Critical Points in Phase Space

A point $(y_1, y_2, \ldots, y_n)$ where the n equations

$$f_k(y_1, y_2, \ldots, y_n) = 0 \qquad \text{for } k = 1, 2, \ldots, n$$

are simultaneously satisfied is a **critical point** of the system. The trajectory which passes through a critical point is just the point itself since the phase point has zero velocity.

A trajectory can approach a critical point arbitrarily closely as $t \to \pm\infty$ but can never reach the critical point in finite time.

EXAMPLE 7.2
Find the critical points of the Volterra-Lotka system

$$\begin{aligned} \dot{y}_1 &= -ay_1 + by_1y_2 \\ \dot{y}_2 &= cy_2 - dy_1y_2 \end{aligned}$$

Solution: *The critical points are found by setting* $(\dot{y}_1, \dot{y}_2) = (0,0)$. *Now*

$$\begin{aligned} \dot{y}_1 = 0 &\implies y_1 = 0 \text{ or } y_2 = a/b \\ \dot{y}_2 = 0 &\implies y_2 = 0 \text{ or } y_1 = c/d \end{aligned}$$

so the two critical points are $(0,0)$ *and* $(c/d, a/b)$. ■

7.2 Classification of Critical Points

Critical points are named according to the trajectory's behaviour in the neighbourhood of the critical point. For $n = 2$, there are six basic behaviours:

Stable node: Trajectories approach the critical point along an asymptotically straight line.

Stable spiral: Trajectories approach the critical point along a spiral.

Unstable node: Trajectories move away from the critical point along a straight line.

Unstable spiral: Trajectories move away from the critical point along a spiral.

Saddle Point: Some trajectories approach; some move away.

Centre: Trajectories orbit around the critical point.

7.3 Classification for Linear Systems

For the linear system

$$\dot{\boldsymbol{Y}} = \boldsymbol{A}\boldsymbol{Y}$$

there is one critical point, at $(0, 0)$. The general solution for distinct eigenvalues is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} e^{\alpha_1 t} + C_2 \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix} e^{\alpha_2 t}$$

where α_1 and α_2 are the eigenvalues of $\boldsymbol{A}$ whose corresponding eigenvectors are

$$\boldsymbol{K}_{\alpha_1} = \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} \qquad \boldsymbol{K}_{\alpha_2} = \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix}$$

The fundamental behaviour of the critical point depends on the values of α_1 and α_2.

7.3.1 Case 1: $\alpha_1 > \alpha_2 > 0$

If $C_2 = 0$, the solution is

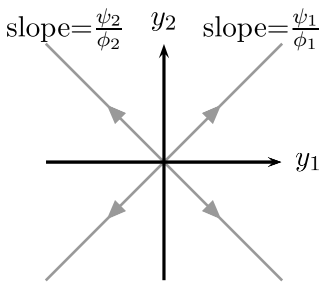
$$\boldsymbol{Y} = C_1 \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} e^{\alpha_1 t}$$

so that

$$y_2 = \frac{\psi_1}{\phi_1} y_1$$

This trajectory is a line with slope ψ_1/ϕ_1 moving away from the origin.

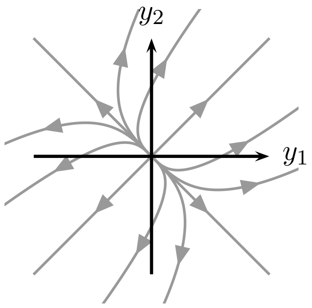
Similarly, if $C_1 = 0$, the trajectory is a line with slope ψ_2/ϕ_2 moving away from the origin.



In the general case, the gradient of the trajectory is

$$\frac{dy_2}{dy_1} = \frac{\dot{y}_2}{\dot{y}_1} = \frac{\alpha_1 C_1 \psi_1 e^{\alpha_1 t} + \alpha_2 C_2 \psi_2 e^{\alpha_2 t}}{\alpha_1 C_1 \phi_1 e^{\alpha_1 t} + \alpha_2 C_2 \phi_2 e^{\alpha_2 t}}$$

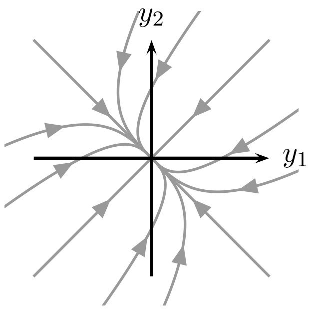
As $t \to +\infty$, the gradient tends to ψ_1/ϕ_1. As $t \to -\infty$, the gradient tends to ψ_2/ϕ_2.



Since the trajectories move away from the origin asymptotically along a straight line, this is an **unstable node**.

7.3.2 Case 2: $\alpha_1 < \alpha_2 < 0$

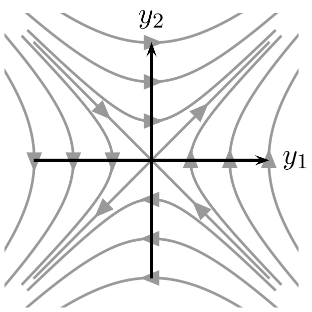
This is similar to case 1, except that the arrows point the other direction. Since the trajectories move towards the origin, this is a **stable node**.



7.3.3 Case 3: $\alpha_1 > 0 > \alpha_2$

As $t \to \infty$, the trajectories' gradients tend to ψ_1/ϕ_1, and as $t \to -\infty$, their gradients tend to ψ_2/ϕ_2, similar to cases 1 and 2. However, as $t \to \pm\infty$, y_1 and y_2 also tend to $\pm\infty$.

This gives a **saddle point**.



7.3.4 Case 4: α_1 and α_2 are imaginary

When $\alpha_1 = i\eta$ and $\alpha_2 = -i\eta$, the solution is

$$\boldsymbol{Y} = C_1 \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} e^{i\eta t} + C_2 \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix} e^{-i\eta t}$$

For real solutions, we require that $C_1 = C_2^*$. Therefore, there are only two arbitrary constants—the real and imaginary parts of C_1. Then the solution is

$$\boldsymbol{Y} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \cos \eta t + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \sin \eta t$$

where

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 2\Re \begin{bmatrix} C_1\phi_1 \\ C_1\psi_1 \end{bmatrix}$$

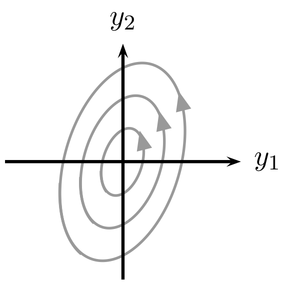
$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = -2\Im \begin{bmatrix} C_1\phi_1 \\ C_1\psi_1 \end{bmatrix}$$

Upon eliminating t from the solution, we have

$$(B_2 y_1 - B_1 y_2)^2 + (A_2 y_1 - A_1 y_2)^2 = (A_1 B_2 - A_2 B_1)^2$$

This is the equation of an ellipse. The orientation of the ellipse's axes are determined by ϕ_1, ψ_1, ϕ_2 and ψ_2.

Since the trajectories orbit the origin, this is a **centre**.



To obtain a rough idea of the ellipse's shape, examine $\frac{dy_2}{dy_1}$ for $y_1 = 0$ and for $y_2 = 0$. Determine the sense of the rotation (clockwise or anticlockwise) by examining $\dot{y}_2$ for $y_2 = 0$ and $y_1 > 0$.

7.3.5 Case 5: α_1 and α_2 are complex conjugates

When the eigenvalues are complex conjugates,

$$\alpha_1 = \mu + i\eta \qquad \text{and} \qquad \alpha_2 = \mu - i\eta$$

the eigenvectors are also complex conjugates

$$\begin{bmatrix}\phi_1\\ \psi_1\end{bmatrix} = \begin{bmatrix}\phi_2\\ \psi_2\end{bmatrix}^*$$

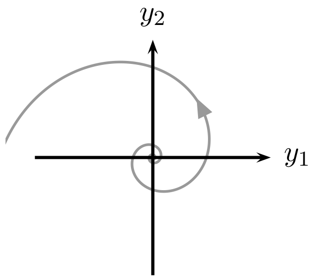
So for a real solution, the arbitrary constants must be complex conjugates too

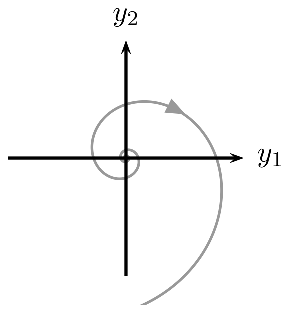
$$C_1 = {C_2}^*$$

Thus the solution is

$$\boldsymbol{Y} = e^{\mu t}\left(\begin{bmatrix}A_1\\ A_2\end{bmatrix}\cos\eta t + \begin{bmatrix}B_1\\ B_2\end{bmatrix}\sin\eta t\right)$$

This is a spiral about the origin. If $\mu > 0$, $\|\boldsymbol{Y}\| \to \infty$ as $t \to \infty$, so we have an **unstable spiral**. If $\mu < 0$, $\|\boldsymbol{Y}\| \to 0$ as $t \to \infty$, so we have a **stable spiral**.





To obtain a rough idea of the spiral's shape, examine $\frac{dy_2}{dy_1}$ for $y_1 = 0$ and for $y_2 = 0$. Determine the sense of the rotation (clockwise or anticlockwise) by examining $\dot{y}_2$ for $y_2 = 0$ and $y_1 > 0$.

7.3.6 Case 6: α_1 and α_2 are equal and non-zero

When $\alpha_1 = \alpha_2 = \alpha \neq 0$, there are two subcases, depending on whether the coefficient matrix $\boldsymbol{A}$ has one or an infinite number of eigenvectors.

When there is only a single eigenvector, the solution is

$$\boldsymbol{Y} = C_1\begin{bmatrix}\phi_1\\ \psi_1\end{bmatrix}e^{\alpha t} + C_2\left(\begin{bmatrix}\phi_1\\ \psi_1\end{bmatrix}t + \begin{bmatrix}p_1\\ p_2\end{bmatrix}\right)e^{\alpha t}$$

where

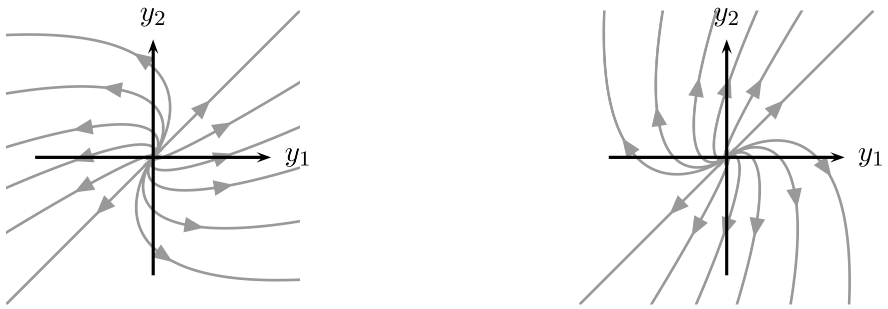
$$(\boldsymbol{A} - \boldsymbol{I}\alpha)\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}$$

A little algebra shows that

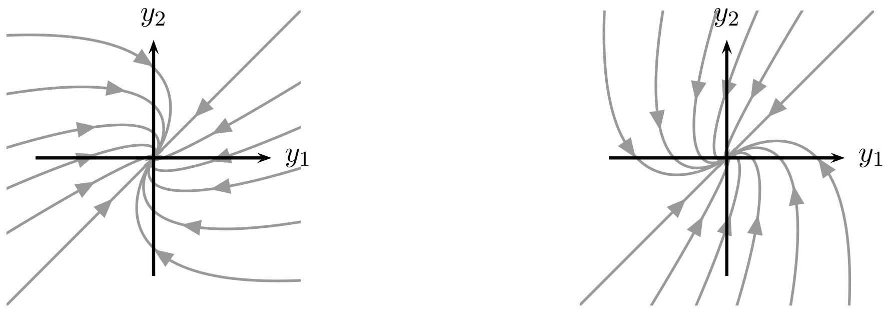
$$\frac{dy_2}{dy_1} \to \frac{\psi_1}{\phi_1}$$

as $t \to \pm\infty$. Therefore, the trajectories become parallel to $\begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}$ as $t \to \pm\infty$. This is a **degenerate node**. It is **unstable** if $\alpha > 0$ or **stable** if $\alpha < 0$.

unstable degenerate nodes



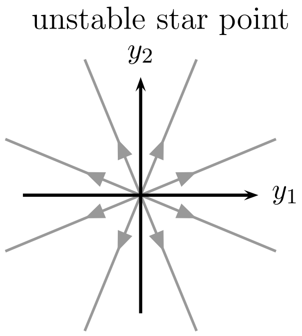
stable degenerate nodes

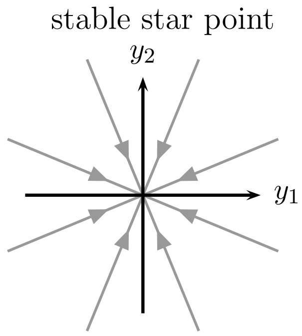


When there is an infinite number of independent eigenvectors, the equations have become decoupled and the solution is

$$\boldsymbol{Y} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} e^{\alpha t}$$

This gives an **unstable star point** if $\alpha > 0$ or a **stable star point** if $\alpha < 0$.





7.3.7 Case 7: $\alpha_1 = 0$

The case of $\alpha_1 = 0$ is only possible if the coefficient matrix $\boldsymbol{A}$ has zero determinant. Therefore there is an infinite number of critical points which satisfy

$$\begin{aligned} a_{11}y_1 + a_{12}y_2 &= 0 \\ a_{21}y_1 + a_{22}y_2 &= 0 \end{aligned}$$

These two equations are equivalent because $a_{11}a_{22} - a_{12}a_{21} = 0$. Therefore any point on the line

$$y_2 = -\frac{a_{11}}{a_{12}}y_1 = -\frac{a_{21}}{a_{22}}y_1$$

is a critical point. Any point on this line is a constant solution to the differential equation system.

For any point away from this line,

$$\frac{dy_2}{dy_1} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2} = \frac{a_{21}}{a_{11}} = \frac{a_{22}}{a_{12}}$$

Therefore the trajectory is

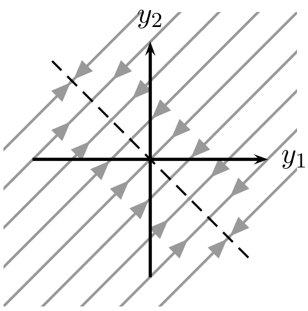
$$y_2 = \frac{a_{21}}{a_{11}}y_1 + c$$

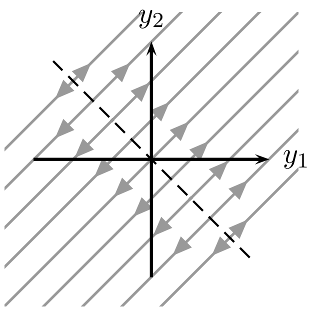
There are two subcases, depending on whether α_2 is non-zero or zero.

If α_2 is non-zero, the trace of $\boldsymbol{A}$ is non-zero and

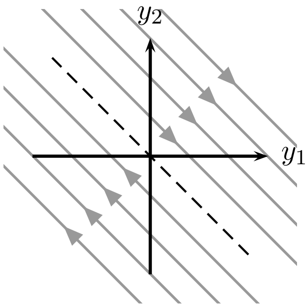
$$a_{11} + a_{22} \neq 0$$

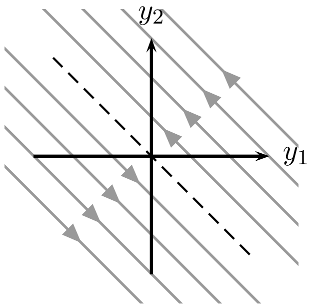
Therefore the slope of the trajectory, a_{21}/a_{11}, is not equal to $-a_{21}/a_{22}$, the slope of the line of critical points. Thus the trajectories are not parallel to the line of critical points.





When $\alpha_2 = 0$, the trace of $\boldsymbol{A}$ is $a_{11} + a_{22} = 0$. Thus the slope of the trajectory, a_{21}/a_{11}, is equal to $-a_{21}/a_{22}$, the slope of the line of critical points, and the trajectories are parallel to the line of critical points.





7.3.8 Summary of Classification of Critical Points

Define p and q in terms of the coefficient matrix $\boldsymbol{A}$ as

$$\begin{aligned} p &= \frac{1}{2}\,\mathrm{trace}(\boldsymbol{A}) \\ q &= |\boldsymbol{A}| \end{aligned}$$

Then the eigenvalues of $\boldsymbol{A}$ are

$$\alpha_1, \alpha_2 = p \pm \sqrt{p^2 - q}$$

The critical points can be classified in terms of p and q as shown in figure 7.1.

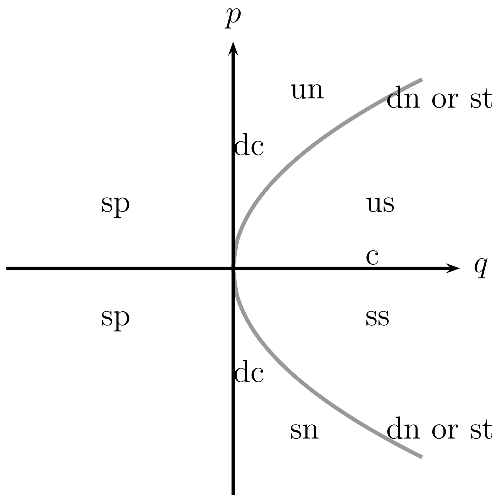
7.3.9 Examples

The following examples illustrate phase portraits of the system

$$\dot{\boldsymbol{Y}} = \boldsymbol{A}\boldsymbol{Y}$$

for various matrices $\boldsymbol{A}$.

Figure 7.1: This illustrates the classification of critical points in terms of p and q. The abbreviations are: un - unstable node; us - unstable spiral; c - centre; ss - stable spiral; sn - stable node; dn - degenerate node; st - star point; dc - degenerate case; sp - saddle point.

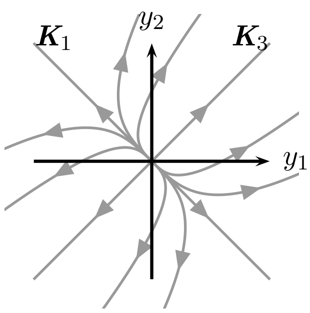


EXAMPLE 7.3

When

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$p = 2$ *and* $q = 3$ *so the eigenvalues are* $2 \pm \sqrt{4-3}$ *or* 1 *and* 3. *The corresponding eigenvectors are* $\boldsymbol{K}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $\boldsymbol{K}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. *As* $t \to \infty$, *the trajectories become parallel to the eigenvector with the largest eigenvalue, hence parallel to* $\boldsymbol{K}_3$. *Since* $p > 0$, *this is an unstable node.*



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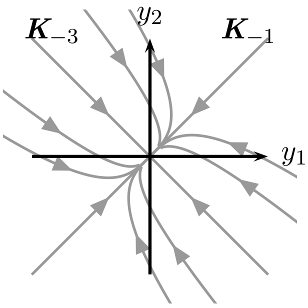
EXAMPLE 7.4
When

$$\boldsymbol{A} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$p = -2$ and $q = 3$ so the eigenvalues are $-2 \pm \sqrt{4-3}$ or -1 and -3. The corresponding eigenvectors are $\boldsymbol{K}_{-3} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\boldsymbol{K}_{-1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Since $p < 0$, this is a stable node.



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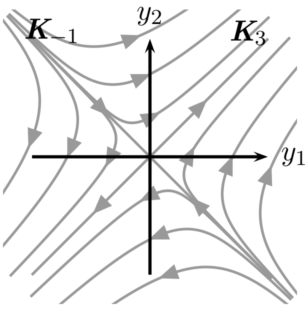
EXAMPLE 7.5
When

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$p = 1$ and $q = -3$ so the eigenvalues are $1 \pm \sqrt{1+3}$ or 3 and -1. The corresponding eigenvectors are $\boldsymbol{K}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{K}_{-1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Since $q < 0$, this is a saddle point.



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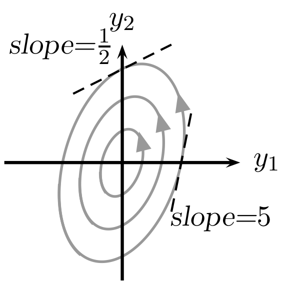
EXAMPLE 7.6
When

$$\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ 5 & -1 \end{bmatrix}$$

$p = 0$ and $q = 9$ so the eigenvalues are $0 \pm \sqrt{0-9}$ or $\pm i3$. The eigenvector corresponding to $\alpha = i3$ is $\boldsymbol{K}_{i3} = \begin{bmatrix} 1 \\ (1-i3)/2 \end{bmatrix}$.

Therefore this is a centre, and the trajectories are ellipses about the origin. Since $\dot{y}_2 > 0$ when $y_2 = 0$ and $y_1 > 0$, the trajectories are anticlockwise. The ellipses can be sketched by noting that

$$\begin{aligned} \frac{dy_2}{dy_1} &= 5 \qquad \text{when } y_2 = 0 \\ \frac{dy_2}{dy_1} &= \frac{1}{2} \qquad \text{when } y_1 = 0 \end{aligned}$$



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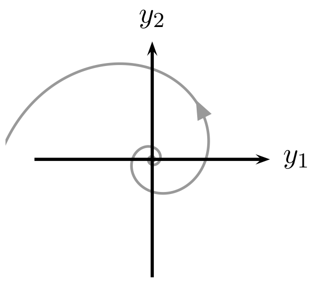
EXAMPLE 7.7
When

$$\boldsymbol{A} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$$

$p = 1$ and $q = 10$ so the eigenvalues are $1 \pm \sqrt{-9}$ or $1 \pm i3$. Therefore the trajectories are unstable spirals. To find out whether the arrow on the spiral trajectories is pointing clockwise or anticlockwise, consider $\dot{y}_2 = 3y_1$. Thus when $y_2 = 0$ and $y_1 > 0$, the trajectory's gradient is positive. Hence the spiral has anticlockwise sense. The shape of the spiral is found by considering

$$\begin{aligned} \frac{dy_2}{dy_1} &= 3 \quad \text{when } y_2 = 0 \\ \frac{dy_2}{dy_1} &= -\frac{1}{3} \quad \text{when } y_1 = 0 \end{aligned}$$



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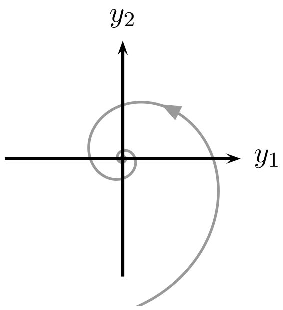
EXAMPLE 7.8
When

$$A = \begin{bmatrix} -1 & -3 \\ 3 & -1 \end{bmatrix}$$

$p = -1$ and $q = 10$ so the eigenvalues are $-1 \pm \sqrt{-9}$ or $-1 \pm i3$. Therefore the trajectories are stable spirals. To find out whether the arrow on the spiral trajectories is pointing clockwise or anticlockwise, consider $\dot{y}_2 = 3y_1$. Thus when $y_2 = 0$ and $y_1 > 0$, the trajectory's gradient is positive. Hence the spiral has anticlockwise sense. The shape of the spiral is found by considering

$$\begin{aligned} \frac{dy_2}{dy_1} &= -3 \quad \text{when } y_2 = 0 \\ \frac{dy_2}{dy_1} &= \frac{1}{3} \quad \text{when } y_1 = 0 \end{aligned}$$



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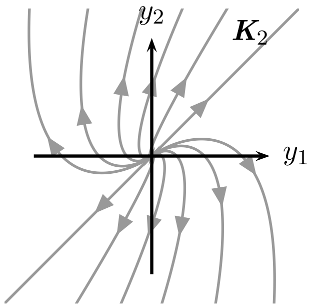
EXAMPLE 7.9
When

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

$p = 2$ and $q = 4$ so there is a repeated eigenvalue of 2. This is therefore an unstable degenerate node. The eigenvector is $\boldsymbol{K}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The form of the phase portrait can be sketched once $\dot{y}_2 = -y_1$ has been determined for $y_2 = 0$ and $y_1 > 0$.



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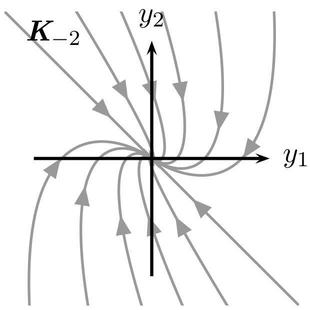
EXAMPLE 7.10

When

$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}$$

$p = -2$ and $q = 4$ so there is a repeated eigenvalue of -2. This is therefore a stable degenerate node. The eigenvector is $\boldsymbol{K}_{-2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The form of the phase portrait can be sketched once $\dot{y}_2 = -y_1$ has been determined for $y_2 = 0$ and $y_1 > 0$.



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EXAMPLE 7.11

When

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

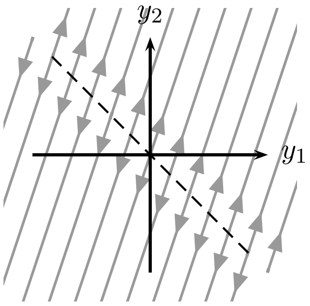
$p = 2$ and $q = 0$ so the eigenvalues are $2 \pm \sqrt{4}$ or 0 and 4. $\alpha = 0$ shows that this is a degenerate case, and $\alpha = 4$ shows that the critical points are unstable.

The line of critical points satisfies $y_1 + y_2 = 0$ hence is given by $y_1 = -y_2$. For points away from this line, the trajectories are straight lines with gradients given by

$$\frac{dy_2}{dy_1} = \frac{3y_1 + 3y_2}{y_1 + y_2} = 3$$

hence the equation of the trajectories is

$$y_2 = 3y_1 + c$$



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7.4 Non-Linear Critical Point Analysis

Suppose the non-linear coupled system

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, y_2) \\ \dot{y}_2 &= f_2(y_1, y_2) \end{aligned}$$

has a critical point at (y_1^*, y_2^*) so that

$$f_1(y_1^*, y_2^*) = f_2(y_1^*, y_2^*) = 0$$

Then using Taylor's theorem, we can expand the function $f(y_1, y_2)$ about (y_1^*, y_2^*)

$$\begin{aligned} f(y_1, y_2) &= f(y_1^*, y_2^*) + \left.\frac{\partial f}{\partial y_1}\right|_{(y_1^*, y_2^*)} (y_1 - y_1^*) \\ &\quad + \left.\frac{\partial f}{\partial y_2}\right|_{(y_1^*, y_2^*)} (y_2 - y_2^*) + \cdots \end{aligned}$$

where the "$\cdots$" indicates terms that are quadratic and higher powers of $(y_1 - y_1^*)$ and $(y_2 - y_2^*)$.

Close to the critical point, the quadratic and higher order terms may be neglected, leading to the approximate expression for the coupled system

$$\begin{aligned}
\dot{y}_1 &= \left.\frac{\partial f_1}{\partial y_1}\right|_{(y_1^*,y_2^*)} (y_1 - y_1^*) + \left.\frac{\partial f_1}{\partial y_2}\right|_{(y_1^*,y_2^*)} (y_2 - y_2^*) \\
\dot{y}_2 &= \left.\frac{\partial f_2}{\partial y_1}\right|_{(y_1^*,y_2^*)} (y_1 - y_1^*) + \left.\frac{\partial f_2}{\partial y_2}\right|_{(y_1^*,y_2^*)} (y_2 - y_2^*)
\end{aligned}$$

Define new local variables

$$\begin{aligned}
z_1 &= y_1 - y_1^* \\
z_2 &= y_2 - y_2^*
\end{aligned}$$

Then for sufficiently small (z_1, z_2), we have

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

where $\boldsymbol{A}$ is the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix}$$

evaluated at (y_1^*, y_2^*).

Then locally in the neighbourhood of the critical point, the trajectories of the non-linear system will be the same as those of the local linear system

$$\dot{\boldsymbol{Z}} = \boldsymbol{A}\boldsymbol{Z}$$

This gives the following procedure for classifying the critical points of a general non-linear coupled system:

- Classify the critical points of

$$\begin{aligned}
\dot{y}_1 &= f_1(y_1, y_2) \\
\dot{y}_2 &= f_2(y_1, y_2)
\end{aligned}$$

  by classifying the local linear problem

$$\dot{\boldsymbol{Z}} = \boldsymbol{A}\boldsymbol{Z}$$

  for each critical point in turn. Note that $\boldsymbol{A}$ will be different for each critical point.

- Construct the global phase portrait by blending the local linear phase portraits around each critical point.

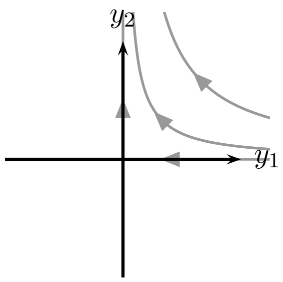
EXAMPLE 7.12 (Volterra-Lotka System)

As previously shown in example 7.2, the critical points of the Volterra-Lotka system are at $(0,0)$ *and at* $(c/d, a/b)$.

In the neighbourhood of $(0,0)$, y_1 *and* y_2 *are both small so the quadratic terms* $y_1 y_2$ *may be neglected. This gives the linearised system*

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -a & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The eigenvalues are $-a$ *and* c, *hence the origin is a saddle point. The eigenvectors are* $\boldsymbol{K}_{-a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $\boldsymbol{K}_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. *We are only interested in the system's behaviour for positive* y_1 *and* y_2, *so the portion of the phase portrait near the origin is*



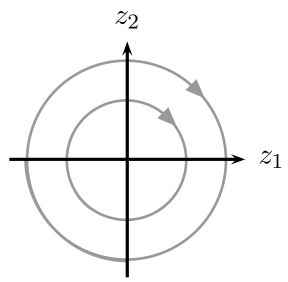
In the neighbourhood of $(c/d, a/b)$, *shift the origin by writing*

$$\begin{aligned} y_1 &= \frac{c}{d} + z_1 \\ y_2 &= \frac{a}{b} + z_2 \end{aligned}$$

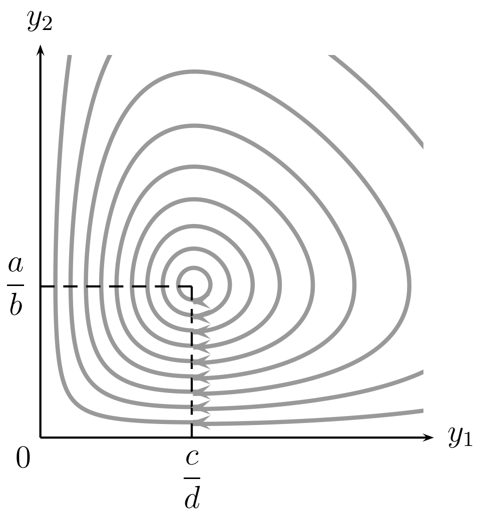
and substitute into the Volterra-Lotka equations. After neglecting quadratic terms, the linearised system is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & cb/d \\ -ad/b & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

The eigenvalues are $\pm i\sqrt{ac}$, *hence this critical point is a centre. Looking at the signs of the coefficients shows that the direction of rotation about the critical point is clockwise.*



Putting these two critical points together and smoothly combining their trajectories gives the global phase portrait shown below.



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EXAMPLE 7.13

For the system

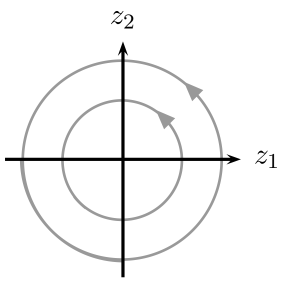
$$\begin{aligned}\dot{y}_1 &= 1 - y_1^2 - y_2^2 \\ \dot{y}_2 &= 2y_1\end{aligned}$$

the critical points are $(0, 1)$ *and* $(0, -1)$.

For the critical point at $(0, 1)$, *let* $y_1 = z_1$ *and* $y_2 = 1 + z_2$ *and substitute into the differential equations. Neglecting the quadratic terms gives the linearised system*

$$\begin{bmatrix}\dot{z}_1 \\ \dot{z}_2\end{bmatrix} = \begin{bmatrix}0 & -2 \\ 2 & 0\end{bmatrix}\begin{bmatrix}z_1 \\ z_2\end{bmatrix}$$

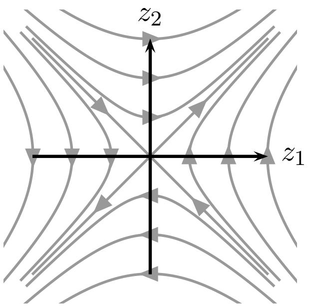
The eigenvalues are $\pm i\sqrt{ac}$, *hence this critical point is a centre. Looking at the signs of the coefficients shows that the direction of rotation about the critical point is anticlockwise.*



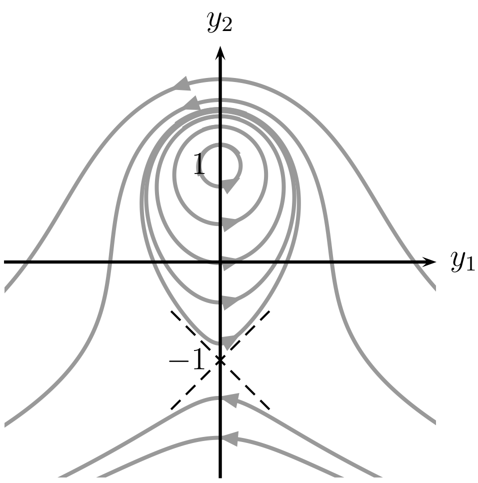
For the critical point at $(0, -1)$, let $y_1 = z_1$ and $y_2 = -1 + z_2$ and substitute into the differential equations. Neglecting the quadratic terms gives the linearised system

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

This gives a saddle point.



Putting these two critical points together and smoothly combining their trajectories gives the global phase portrait shown below.



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