Prior for spline coefficients $\boldsymbol{\beta} \sim \sqrt{\tau^{-1}}AR2(2\rho, -\rho), \quad \rho \in (0,1)$, where $AR2$ is standardised with marginal variances $= 1$.

PC priors for ρ

Assuming $\boldsymbol{\beta}$ follows an AR2 process with $\boldsymbol{\rho} = (2\rho, -\rho)$, with $\rho \in (0,1)$ and let $\boldsymbol{\Sigma}$ be the corresponding correlation matrix, i.e. marginal variances of $\boldsymbol{\beta} = 1$. Then the KLD for the construction of the PC prior for ρ, assuming base model is $\rho = 0$ (i.e. an i.i.d Normal), is:

$$KLD(f_1 \| f_0) = -\frac{1}{2}\left(\text{tr}(\boldsymbol{\Sigma_0^{-1}\Sigma_1}) - n - \log\left(\frac{|\boldsymbol{\Sigma_1}|}{|\boldsymbol{\Sigma_0}|}\right)\right)$$

Since the base model is $\rho = 0$, $\boldsymbol{\Sigma_0} = \boldsymbol{I}$ and hence $\text{tr}(\boldsymbol{\Sigma_0^{-1}\Sigma_1}) = \text{tr}(\boldsymbol{\Sigma_1}) = n$, since $\boldsymbol{\Sigma_1}$ is a correlation matrix.

$$\begin{aligned}
\implies KLD(f_1 \| f_0) &= -\frac{1}{2}\log(|\boldsymbol{\Sigma_1}|) \\
&= -\frac{1}{2}\log\left[(1-\psi_1^2)^{n-1}(1-\psi_2^2)^{n-2}\right] \\
&= -\frac{1}{2}\log\left[(1-(\frac{2\rho}{1+\rho})^2)^{n-1}(1-\rho^2)^{n-2}\right] \\
&= -\frac{1}{2}\log\left[(1+3\rho)^{n-1}(1-\rho)^{2n-3}(1+\rho)^{-n}\right]
\end{aligned}$$

The PC prior is defined as an exponential prior on $d(\rho) = \sqrt{2KLD(f_1 \| f_0)}$ with rate λ.
Consider

$$\begin{aligned}
d(\rho) &= \sqrt{2KLD(f_1 \| f_0)} \\
&= \sqrt{-\log\left[(1-(\frac{2\rho}{1+\rho})^2)^{n-1}(1-\rho^2)^{n-2}\right]} \\
&= \sqrt{(1-n)\log(1+3\rho) + (3-2n)\log(1-\rho) + n\log(1+\rho)} \\
&= \sqrt{f(\rho)},
\end{aligned} \tag{1}$$

then

$$\begin{aligned}
\frac{\partial d(\rho)}{\partial \rho} &= \frac{1}{2\sqrt{f(\rho)}}\left[\frac{3-3n}{1+3\rho} + \frac{2n-3}{1-\rho} + \frac{n}{1+\rho}\right] \\
&= \frac{1}{2\sqrt{f(\rho)}}\left[\frac{(3-3n)(1-\rho)(1+\rho) + (2n-3)(1+3\rho)(1+\rho) + n(1+3\rho)(1-\rho)}{(1+3\rho)(1-\rho)(1+\rho)}\right] \\
&= \frac{1}{2\sqrt{f(\rho)}}\left[\frac{\rho^2(6n-12) + \rho(10n-12)}{(1+3\rho)(1-\rho)(1+\rho)}\right] > 0 \qquad \forall \rho \in (0,1),
\end{aligned}$$

i.e. $d(\rho)$ is a monotonically increasing positive function in ρ, which is intuitive as $\rho \to 1$ means higher deviation from the base model $\rho = 0$ and hence higher KLD.

Thus, PC prior for ρ is

$$\begin{aligned}
\pi(\rho) &= \pi_d(d(\rho))\left|\frac{\partial d(\rho)}{\partial \rho}\right| \\
&= \lambda_\rho e^{-\lambda_\rho d(\rho)}\frac{\partial d(\rho)}{\partial \rho}
\end{aligned}$$

To determine the decay rate λ_ρ, the authors suggested inferring from a interpretable probability statement $P(Q(\rho) > U) = \alpha$ for some α as the tail probability. Here I plan to directly work on the ρ scale, e.g. specifying $P(\rho > 0.999) = 0.01$ as a loose PC prior. Since $d(\rho)$ is a monotonically increasing one-to-one mapping, the upper bound specified on ρ can be translated into an upper for $d(\rho)$ as $P(d(\rho) > d(U)) = 0.01$. Knowing the tail probability of an exponential R.V $P(X > x) = 1 - \text{c.d.f}(x) = e^{-\lambda x}$, λ_ρ can be derived from $e^{-\lambda_\rho d(U)} = 0.01 \implies \lambda_\rho = -\log(0.01)/d(U)$.

PC prior for τ

If information is available on the variability on the spline coefficients, the PC prior for τ can be inferred directly by a similar procedure of setting an upper bound for τ, however the scale of τ often depends on several factors, e.g. the spacing of the knots, the scale of the problem, etc. To see this, consider $y = \boldsymbol{B\beta}$ where $\boldsymbol{B}$ is the design matrix, then $\text{VAR}(y) = \tau^{-1}\boldsymbol{B\Sigma_1(\rho)B'}$. Therefore the scale of τ depends on ρ. The diagonal of the matrix $\boldsymbol{B\Sigma_1(\rho)B'}$ is relatively uniform, due to the stationarity of the AR2 process assumed and the equal knot spacing and even data. To simplify the construction, we approximate the marginal variance of each $y_i = \text{VAR}(y_1) = \tau^{-1}\boldsymbol{B_{1\cdot}}\ \boldsymbol{\Sigma_1(\rho)}\ \boldsymbol{B'_{1\cdot}}$, where $\boldsymbol{B_{1\cdot}}$ is the first row of the design matrix. It is usually easier to give probability statements on the standard deviation of $\boldsymbol{y}$ instead of $\boldsymbol{\beta}$. Hence, the PC prior for τ is the type-2 Gumbel prior with decay rate $\lambda_\tau = -\log(0.01)\sqrt{\text{VAR}(y_1)}/U_y$, where U_y is an upper bound of the standard deviation of y.

Therefore

$$
\begin{aligned}
f(\tau,\rho) &= f(\tau|\rho)f(\rho) \\
&= \text{type-2 Gumbel}(\tau, \lambda_\tau|\rho) \cdot \pi(\rho) \\
&= \text{type-2 Gumbel}(\tau, \lambda_\tau|\rho) \cdot \lambda_\rho e^{-\lambda_\rho d(\rho)} \frac{\partial d(\rho)}{\partial \rho}
\end{aligned}
$$

The dependence of λ_τ on ρ is through $\text{VAR}(y_1)$.