Prior for spline coefficients $\boldsymbol{\beta} \sim \sqrt{\tau^{-1}}AR2(2\rho, -\rho), \quad \rho \in (0,1)$, where $AR2$ is standardised with marginal variances $= 1$.

PC priors for ρ

Assuming $\boldsymbol{\beta}$ follows an AR2 process with $\boldsymbol{\rho} = (2\rho, -\rho)$, with $\rho \in (0,1)$ and let $\boldsymbol{\Sigma}$ be the corresponding correlation matrix, i.e. marginal variances of $\boldsymbol{\beta} = 1$. Then the KLD for the construction of the PC prior for ρ, assuming base model is $\rho = 0$ (i.e. an i.i.d Normal), is:

$$KLD(f_1\|f_0) = \frac{1}{2}\left(\text{tr}(\boldsymbol{\Sigma_0^{-1}\Sigma_1}) - n - \log\left(\frac{|\boldsymbol{\Sigma_1}|}{|\boldsymbol{\Sigma_0}|}\right)\right)$$

Since the base model is $\rho = 0$, $\boldsymbol{\Sigma_0} = \boldsymbol{I}$ and hence $\text{tr}(\boldsymbol{\Sigma_0^{-1}\Sigma_1}) = \text{tr}(\boldsymbol{\Sigma_1}) = n$, since $\boldsymbol{\Sigma_1}$ is a correlation matrix.

$$\begin{aligned}
\implies KLD(f_1\|f_0) &= -\frac{1}{2}\log(|\boldsymbol{\Sigma_1}|) \\
&= -\frac{1}{2}\log\left[\,(1-\psi_1^2)^{n-1}(1-\psi_2^2)^{n-2}\,\right] \\
&= -\frac{1}{2}\log\left[\,(1-(\frac{2\rho}{1+\rho})^2)^{n-1}(1-\rho^2)^{n-2}\,\right] \\
&= -\frac{1}{2}\log\left[\,(1+3\rho)^{n-1}(1-\rho)^{2n-3}(1+\rho)^{-n}\,\right]
\end{aligned}$$

The PC prior is defined as an exponential prior on $d(\rho) = \sqrt{2KLD(f_1\|f_0)}$ with rate λ.
Consider

$$\begin{aligned}
d(\rho) &= \sqrt{2KLD(f_1\|f_0)} \\
&= \sqrt{-\log\left[\,(1-(\frac{2\rho}{1+\rho})^2)^{n-1}(1-\rho^2)^{n-2}\,\right]} \\
&= \sqrt{(1-n)\log(1+3\rho) + (3-2n)\log(1-\rho) + n\log(1+\rho)} \\
&= \sqrt{f(\rho)},
\end{aligned} \tag{1}$$

then

$$\begin{aligned}
\frac{\partial d(\rho)}{\partial \rho} &= \frac{1}{2\sqrt{f(\rho)}}\left[\frac{3-3n}{1+3\rho} + \frac{2n-3}{1-\rho} + \frac{n}{1+\rho}\right] \\
&= \frac{1}{2\sqrt{f(\rho)}}\left[\frac{(3-3n)(1-\rho)(1+\rho) + (2n-3)(1+3\rho)(1+\rho) + n(1+3\rho)(1-\rho)}{(1+3\rho)(1-\rho)(1+\rho)}\right] \\
&= \frac{1}{2\sqrt{f(\rho)}}\left[\frac{\rho^2(6n-12) + \rho(10n-12)}{(1+3\rho)(1-\rho)(1+\rho)}\right] > 0 \qquad\qquad \forall \rho \in (0,1),
\end{aligned}$$

i.e. $d(\rho)$ is a monotonically increasing positive function in ρ, which is intuitive as $\rho \to 1$ means higher deviation from the base model $\rho = 0$ and hence higher KLD.

Thus, PC prior for ρ is

$$\begin{aligned}
\pi(\rho) &= \pi_d(d(\rho))\left|\frac{\partial d(\rho)}{\partial \rho}\right| \\
&= \lambda_\rho e^{-\lambda_\rho d(\rho)}\,\frac{\partial d(\rho)}{\partial \rho}
\end{aligned}$$

To determine the decay rate λ_ρ, the authors suggested inferring from a interpretable probability statement $P(Q(\rho) > U) = \alpha$ for some α as the tail probability. Here I plan to directly work on the ρ scale, e.g. specifying $P(\rho > 0.99) = 0.01$ as a loose PC prior. Since $d(\rho)$ is a monotonically increasing one-to-one mapping, the upper bound specified on ρ can be translated into an upper for $d(\rho)$ as $P(d(\rho) > d(U)) = 0.01$. Knowing the tail probability of an exponential R.V $P(X > x) = 1 - \text{c.d.f}(x) = e^{-\lambda x}$, λ_ρ can be derived from $e^{-\lambda_\rho d(U)} = 0.01 \implies \lambda_\rho = -\log(0.01)/d(U)$.

PC prior for τ

If information is available on the variability on the spline coefficients, the PC prior for τ can be inferred directly by a similar procedure of setting an upper bound for τ, however the scale of τ often depends on several factors, e.g. the spacing of the knots, the scale of the problem, etc. To see this, consider $y = \boldsymbol{B\beta}$ where $\boldsymbol{B}$ is the design matrix, then $\text{VAR}(y) = \tau^{-1}\boldsymbol{B\Sigma_1(\rho)B'}$. Therefore the scale of τ depends on ρ. The diagonal of the matrix $\boldsymbol{B\Sigma_1(\rho)B'}$ is relatively uniform, due to the stationarity of the AR2 process assumed and the equal knot spacing and even data. To simplify the construction, we approximate the marginal variance of each $y_i = \text{VAR}(y_1) = \tau^{-1}\boldsymbol{B_{1\cdot}}\ \boldsymbol{\Sigma_1(\rho)}\ \boldsymbol{B'_{1\cdot}}$, where $\boldsymbol{B_{1\cdot}}$ is the first row of the design matrix. It is usually easier to give probability statements on the standard deviation of $\boldsymbol{y}$ instead of $\boldsymbol{\beta}$. Hence, the PC prior for τ is the type-2 Gumbel prior with decay rate $\lambda_\tau = -\log(0.01)\sqrt{\text{VAR}(y_1)}/U_y$, where U_y is an upper bound of the standard deviation of y.

Therefore

$$
\begin{aligned}
f(\tau, \rho) &= f(\tau|\rho)f(\rho) \\
&= \text{type-2 Gumbel}(\tau, \lambda_\tau|\rho) \cdot \pi(\rho) \\
&= \text{type-2 Gumbel}(\tau, \lambda_\tau|\rho) \cdot \lambda_\rho e^{-\lambda_\rho d(\rho)} \frac{\partial d(\rho)}{\partial \rho}
\end{aligned}
$$

The dependence of λ_τ on ρ is through $\text{VAR}(y_1)$.

PC prior for a different variance for the HIV period

It may be more logical to have a different variance, especially for the hump component, during the HIV epidemics. For example, if we have $\boldsymbol{\eta} \sim AR2$, then we can assume $\boldsymbol{\beta} = \sqrt{\boldsymbol{W}}\boldsymbol{\eta} \sim \sqrt{\boldsymbol{W}}AR2$, where $\boldsymbol{W}$ is a diagonal matrix with elements equal to the marginal variances of each β's. For simplicity, I set $\boldsymbol{W} = \text{diag}(w_1, w_1, w_1, \ldots, w_2, w_2, w_2)$, i.e. constant variance before the epidemics and another constant variance after. Obviously, this would destroy the stationarity of the AR2, but it is sensible and hence desirable, as we would not expect the stationarity to hold during the epidemics. Under this formulation, the β's in the pre-HIV era still enjoys stationarity.

It is slightly more difficult to set up the PC prior for w_2. To proceed, I re-parameterise the model such that

$$
\boldsymbol{\beta} = \sqrt{w_1}\begin{pmatrix}
1 & 0 & 0 & 0 & \ldots \\
0 & 1 & 0 & 0 & \ldots \\
\vdots & \ddots & & & \\
 & & \sqrt{\gamma} & 0 & 0 \\
 & & 0 & \sqrt{\gamma} & 0 \\
 & & 0 & 0 & \sqrt{\gamma}
\end{pmatrix}\boldsymbol{\eta},
$$

where $\boldsymbol{\eta}$ follows an AR2$(2\rho, -\rho)$ process with marginal variances $= 1$. Under this parameterisation, we could write the prior $\pi(w_1, \gamma, \rho) = \pi(w_1|\gamma, \rho)\pi(\gamma|\rho)\pi(\rho)$, where $\pi(\rho)$ is the same as above as it does not depend on the variance parameters (since it relates to a correlation matrix), and $\pi(w_1^{-1}|\gamma, \rho)$ has a gumbel density as above, because Simpson et al. showed the PC prior of it is independent on the covariance matrix, after conditioning on γ and ρ.

To calculate the PC prior for γ, again we consider the $KLD(f_1\|f_0)$, where the base model f_0 is now at $\gamma = 1$, i.e. same variance for the whole period. It can also be seen that under this formulation we can set the prior for γ independent of w_1, as the w_1 in the base model and the alternative model cancel out.

$$
\begin{aligned}
KLD(f_1\|f_0) &= \frac{1}{2}\left(\operatorname{tr}(\mathbf{\Sigma_0^{-1}\Sigma_1}) - n - \log\left(\frac{|\mathbf{\Sigma_1}|}{|\mathbf{\Sigma_0}|}\right)\right)\\
&= \frac{1}{2}\left(\operatorname{tr}(\mathbf{\Sigma_0^{-1}}\sqrt{\boldsymbol{W}}\mathbf{\Sigma_0}\sqrt{\boldsymbol{W}}) - n - \log\left(\frac{|\sqrt{\boldsymbol{W}}\mathbf{\Sigma_0}\sqrt{\boldsymbol{W}}|}{|\mathbf{\Sigma_0}|}\right)\right)\\
&= \frac{1}{2}\left(\operatorname{tr}(\mathbf{\Sigma_0^{-1}}\sqrt{\boldsymbol{W}}\mathbf{\Sigma_0}\sqrt{\boldsymbol{W}}) - n - (n_\gamma)\log(\gamma)\right),
\end{aligned}
$$

where n_γ indicates the number of β's with the different variance.

The first term requires more attention. Using the block matrix inversion formula, write:

$$
\begin{aligned}
&\operatorname{tr}\big(\mathbf{\Sigma_0^{-1}}\sqrt{\boldsymbol{W}}\mathbf{\Sigma_0}\sqrt{\boldsymbol{W}}\big) = \operatorname{tr}\left(\begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{C} & \boldsymbol{D}\end{pmatrix}^{-1}\begin{pmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \sqrt{\gamma}\boldsymbol{I}\end{pmatrix}\begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{C} & \boldsymbol{D}\end{pmatrix}\begin{pmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \sqrt{\gamma}\boldsymbol{I}\end{pmatrix}\right)\\
&= \operatorname{tr}\left(\begin{pmatrix}\boldsymbol{A^{-1}} + \boldsymbol{A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1}} & -\boldsymbol{A^{-1}B(D-CA^{-1}B)^{-1}}\\ -\boldsymbol{(D-CA^{-1}B)^{-1}CA^{-1}} & \boldsymbol{(D-CA^{-1}B)^{-1}}\end{pmatrix}\begin{pmatrix}\boldsymbol{A} & \sqrt{\gamma}\boldsymbol{B}\\ \sqrt{\gamma}\boldsymbol{C} & \gamma\boldsymbol{D}\end{pmatrix}\right)\\
&= \operatorname{tr}\big(\boldsymbol{I} + \boldsymbol{A^{-1}B(D-CA^{-1}B)C} - \sqrt{\gamma}\boldsymbol{A^{-1}B(D-CA^{-1}B)C}\big)+\\
&\qquad\qquad \operatorname{tr}\big(-\sqrt{\gamma}\boldsymbol{(D-CA^{-1}B)CA^{-1}B} + \gamma\boldsymbol{(D-CA^{-1}B)D}\big)
\end{aligned}
$$

using the cyclic property,

$$
\begin{aligned}
&= \operatorname{tr}\big(\boldsymbol{I}\big) + (1-2\sqrt{\gamma})\operatorname{tr}\big(\boldsymbol{(D-CA^{-1}B)CA^{-1}B}\big) + \gamma\operatorname{tr}\big(\boldsymbol{(D-CA^{-1}B)^{-1}D}\big)\\
&= \operatorname{tr}\big(\boldsymbol{I}\big) + (1-2\sqrt{\gamma})\operatorname{tr}\big(\boldsymbol{(D-CA^{-1}B)CA^{-1}B}\big) + \gamma\operatorname{tr}\big(\boldsymbol{(D-CA^{-1}B)^{-1}D}\big)+\\
&\qquad\qquad \gamma\operatorname{tr}\big(\boldsymbol{(D-CA^{-1}B)^{-1}CA^{-1}B}\big) - \gamma\operatorname{tr}\big(\boldsymbol{(D-CA^{-1}B)^{-1}CA^{-1}B}\big)\\
&= \operatorname{tr}\big(\boldsymbol{I}\big) + \gamma\operatorname{tr}\big(\boldsymbol{I}_{n_\gamma}\big) + (1-2\sqrt{\gamma}+\gamma)\operatorname{tr}\big(\boldsymbol{(D-CA^{-1}B)CA^{-1}B}\big)\\
&= \operatorname{tr}\big(\boldsymbol{W}\big) + (1-\sqrt{\gamma})^2\operatorname{tr}\big(\boldsymbol{(D-CA^{-1}B)CA^{-1}B}\big)\\
&= \operatorname{tr}\big(\boldsymbol{W}\big) + (1-\sqrt{\gamma})^2\operatorname{tr}\big(\boldsymbol{EB}\big)
\end{aligned}
$$

By comparing term of the block inversion formula, it is easily seen that $-\boldsymbol{E} = -\boldsymbol{(D-CA^{-1}B)CA^{-1}}$ is the bottom left block matrix of the AR2 precision matrix, and $\boldsymbol{B}$ is the upper right block matrix of the AR2 covariance matrix. Since the precision matrix of an AR2 process is a band $2\cdot(2+1)-1$ matrix, $\boldsymbol{E}$ is a sparse right triangular matrix with only three non-zero entries:

$$
-\boldsymbol{E} = \begin{pmatrix}
0 & 0 & 0 & \dots & 0 & -\phi_2\sigma^{-2} & \phi_1(\phi_2-1)\sigma^{-2}\\
0 & 0 & 0 & \dots & 0 & 0 & -\phi_2\sigma^{-2}\\
0 & 0 & 0 & \dots & & &
\end{pmatrix},
$$

where σ^2 is the ratio of adjustment of the innovations at $t > 2$ relative to the first two innovations in order to maintain stationarity. $\sigma^2 = \frac{1+\phi_2}{1-\phi_2}((1-\phi_2)^2 - \phi_1^2)$ for an AR2 process. Given $\phi_1 = 2\rho$ and $\phi_2 = -\rho$, $-\boldsymbol{E}$ reduces to

$$
-\boldsymbol{E} = \begin{pmatrix}
0 & 0 & 0 & \dots & 0 & \frac{\rho(1+\rho)}{(1+3\rho)(1-\rho)^2} & \frac{-2\rho(1+\rho)^2}{(1+3\rho)(1-\rho)^2}\\
0 & 0 & 0 & \dots & 0 & 0 & \frac{\rho(1+\rho)}{(1+3\rho)(1-\rho)^2}\\
0 & 0 & 0 & \dots & & &
\end{pmatrix}.
$$

Therefore,

$$\operatorname{tr}\big(\boldsymbol{E}\boldsymbol{B}\big) = \frac{-\rho(1+\rho)}{(1+3\rho)(1-\rho)^2}\mathrm{ACF}(2) + \frac{2\rho(1+\rho)^2}{(1+3\rho)(1-\rho)^2}\mathrm{ACF}(1) + \frac{-\rho(1+\rho)}{(1+3\rho)(1-\rho)^2}\mathrm{ACF}(2),$$

where $\mathrm{ACF}(\cdot)$ is the auto-correlation function. For AR2,

$$\begin{aligned}
\mathrm{ACF}(1) &= \frac{\phi_1}{1-\phi_2} = \frac{2\rho}{1+\rho} \\
\mathrm{ACF}(2) &= \phi_1\mathrm{ACF}(1) + \phi_2 = \frac{\rho(3\rho-1)}{1+\rho} \\
\Longrightarrow \operatorname{tr}\big(\boldsymbol{E}\boldsymbol{B}\big) &= \frac{2\rho(1+\rho)^2}{(1+3\rho)(1-\rho)^2}\mathrm{ACF}(1) - \frac{2\rho(1+\rho)}{(1+3\rho)(1-\rho)^2}\mathrm{ACF}(2) \\
&= \frac{2\rho(1+\rho)^2}{(1+3\rho)(1-\rho)^2}\Big(\frac{2\rho}{1+\rho}\Big) - \frac{2\rho(1+\rho)}{(1+3\rho)(1-\rho)^2}\Big(\frac{\rho(3\rho-1)}{1+\rho}\Big) \\
&= \frac{2\rho^2(1+\rho)}{(1+3\rho)(1-\rho)^2}\Big((1+\rho)\frac{2}{1+\rho} - \frac{3\rho-1}{1+\rho}\Big) \\
&= \frac{2\rho^2(1+\rho)}{(1+3\rho)(1-\rho)^2}\big(\frac{3-\rho}{1+\rho}\big) \\
&= \frac{2\rho^2(3-\rho)}{(1+3\rho)(1-\rho)^2}
\end{aligned}$$

Hence, given ρ,

$$\begin{aligned}
KLD(f_1\|f_0) &= \frac{1}{2}\Bigg(\operatorname{tr}(\boldsymbol{\Sigma_0^{-1}}\sqrt{\boldsymbol{W}}\boldsymbol{\Sigma_0}\sqrt{\boldsymbol{W}}) - n - (n_\gamma)\log(\gamma)\Bigg) \\
&= \frac{1}{2}\Bigg(n - n_\gamma + \gamma n_\gamma + (1-\sqrt{\gamma})^2\,\frac{2\rho^2(3-\rho)}{(1+3\rho)(1-\rho)^2} - n - n_\gamma\log(\gamma)\Bigg) \\
&= \frac{1}{2}\Bigg(\big(\gamma - \log(\gamma) - 1\big)n_\gamma + (1-\sqrt{\gamma})^2\,\frac{2\rho^2(3-\rho)}{(1+3\rho)(1-\rho)^2}\Bigg) > 0,
\end{aligned}$$

$$\begin{aligned}
d(\gamma|\rho) &= \sqrt{2KLD((f_1\|f_0))} \\
&= \sqrt{\big(\gamma - \log(\gamma) - 1\big)n_\gamma + (1-\sqrt{\gamma})^2\,\frac{2\rho^2(3-\rho)}{(1+3\rho)(1-\rho)^2}},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial d(\gamma|\rho)}{\partial\gamma} &= \frac{1}{2d(\gamma|\rho)}\Big(n_\gamma - \frac{n_\gamma}{\gamma} - (1-\sqrt{\gamma})\frac{1}{\sqrt{\gamma}}\,\frac{2\rho^2(3-\rho)}{(1+3\rho)(1-\rho)^2}\Big) \\
&= \frac{1}{2d(\gamma|\rho)}\Big(n_\gamma(1-\frac{1}{\gamma}) - (1-\sqrt{\gamma})\frac{1}{\sqrt{\gamma}}\,\frac{2\rho^2(3-\rho)}{(1+3\rho)(1-\rho)^2}\Big) > 0 \text{ if } \gamma > 1, \text{ and } < 0 \text{ if } \gamma < 1
\end{aligned}$$

Finally, the PC prior for γ is then

$$\pi(\gamma|\rho) = \pi(d(\gamma|\rho))\Big|\frac{\partial d(\gamma|\rho)}{\partial\gamma}\Big|$$

where $\pi(d(\gamma|\rho)) \sim Exp(\lambda_\gamma)$ with λ_γ set a priori as $-\log(0.01)/d(U_\gamma)$ and U_γ (this is under-stating the tail probability! still working it out) specifies an upper bound for the ratio of the marginal variance in the HIV period to the pre-HIV era.

When fitting to the model, I expect γ to be > 1 most of the time since the HIV epidemics should introduce more variability.

- Relationship to ARCH model?
- Relationship to regime switch model?

Alternative parameterisation of the PC prior for a different variance on the AR2 inoovations
Consider that instead of acting on the marginal variances τ^{-1}, the γ now acts directly on the variances of the AR2 innovations $\boldsymbol{\epsilon}$. This has a cleaner derivation of the PC prior, but of course the interpretation of γ might be tricky. Knowing that the marginal variances will eventually converge to $\gamma\tau^{-1}$ after some time, γ can still be interpreted as the ratio of the marginal variance in the later period to the beginning, but now the marginal variance at the cut-off point is not exactly $\gamma\tau^{-1}$, but some value in between.

This parameterisation is also independent of ρ, hence the PC prior could be set outside TMB (i.e. the decay rate does not depend on ρ).

$$
\begin{aligned}
KLD_\gamma(f_1\|f_0) &= \frac{1}{2}\left(\mathrm{tr}(\boldsymbol{\Sigma_0}^{-1}\boldsymbol{\Sigma_0}) - n - \log\left(\frac{|\boldsymbol{\Sigma_1}|}{|\boldsymbol{\Sigma_0}|}\right)\right) \\
&= \frac{1}{2}\left(\mathrm{tr}\Big(\big(\boldsymbol{P}^{-1}\boldsymbol{V}\boldsymbol{P}^{-1\prime^{-1}}\big)\big(\boldsymbol{P}^{-1}\boldsymbol{W}\boldsymbol{V}\boldsymbol{P}^{-1\prime}\big)\Big) - n - \log\Big(|\boldsymbol{W}|\Big)\right) \\
&= \frac{1}{2}\left(\sum \boldsymbol{W}_{ii} - n - \sum \log(\boldsymbol{W}_{ii})\right)
\end{aligned}
$$

Set

$$
\boldsymbol{W} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \dots \\
0 & 1 & 0 & 0 & 0 & \dots \\
\vdots & & \ddots & & & \\
 & 0 & 0.25\gamma + 0.75 & 0 & 0 & \dots \\
 & 0 & 0 & 0.5\gamma + 0.5 & 0 & \dots \\
 & 0 & 0 & 0 & 0.75\gamma + 0.25 & \dots \\
 & & & \ddots & & \\
 & & & & 0 & \gamma
\end{pmatrix}
$$

Then

$$
\begin{aligned}
d_\gamma &= \sqrt{2KLD_\gamma(f_1\|f_0)} \\
&= \sqrt{(\gamma-1)(n_\gamma - 1.5) - (n_\gamma - 3)\log(\gamma) - \log(0.25\gamma+0.75) - \log(0.5\gamma+0.5) - \log(0.75\gamma+0.25)},
\end{aligned}
$$

where γ is the number of weights not equal to 1, i.e. number of γ's + 3.

$$
\begin{aligned}
\frac{\partial d_\gamma}{\partial\gamma} &= \frac{1}{2d_\gamma}\left(n_\gamma - 1.5 - \frac{n_\gamma - 3}{\gamma} - \frac{0.25}{0.25\gamma+0.75} - \frac{0.5}{0.5\gamma+0.5} - \frac{0.75}{0.75\gamma+0.25}\right), \\
&> 0 \quad \forall\gamma > 1, \qquad\qquad < 0 \quad \forall\gamma < 1
\end{aligned}
$$

Zimbabwe

```
## [1] "Census Females"

## # A tibble: 86 x 6
##      age `1969`  `1982`  `1992`  `2002`  `2012`
##    <dbl>  <dbl>   <dbl>   <dbl>   <dbl>   <dbl>
##  1     0   215. 137025. 169096. 170969. 215352.
##  2     1   201. 134484. 158681. 172089. 210806.
##  3     2   219. 133645. 157913. 168681. 198416.
##  4     3   229. 130954. 157785. 166179. 189947.
##  5     4   241. 128456. 159981. 161303. 180342.
##  6     5   251. 126457. 161906. 157541. 174662.
##  7     6   258. 125796. 163620. 156585. 172448.
##  8     7   261. 125192. 165880. 154780. 170342.
##  9     8   262. 123948. 167305. 154361. 169826.
## 10     9   260. 118228. 164491. 153612. 168733.
## # ... with 76 more rows

## [1] "Census Females 5-year"

## # A tibble: 86 x 2
##      age `1969`
##    <dbl>  <dbl>
##  1     0   219.
##  2     1   217.
##  3     2   218.
##  4     3   223.
##  5     4   231.
##  6     5   244.
##  7     6   253.
##  8     7   258.
##  9     8   257.
## 10     9   250.
## # ... with 76 more rows

## [1] "Census Males"

## # A tibble: 86 x 6
##      age `1969`  `1982`  `1992`  `2002`  `2012`
##    <dbl>  <dbl>   <dbl>   <dbl>   <dbl>   <dbl>
##  1     0   238. 132765. 167346. 170516. 213504.
##  2     1   232. 129179. 157406. 172094. 209755.
##  3     2   231. 128590. 156277. 168314. 197059.
##  4     3   235. 126778. 156372. 166091. 189028.
##  5     4   242. 125240. 158552. 161218. 179354.
##  6     5   248. 124034. 160467. 157608. 173820.
##  7     6   255. 124161. 162006. 156389. 171476.
##  8     7   265. 123595. 163366. 153571. 168424.
##  9     8   271. 122293. 164072. 152514. 167753.
## 10     9   271. 117099. 160915. 151202. 166974.
## # ... with 76 more rows

## [1] "Census Males 5-year"
```

```
## # A tibble: 86 x 2
##      age `1969`
##    <dbl>  <dbl>
##  1     0   231.
##  2     1   229.
##  3     2   230.
##  4     3   234.
##  5     4   240.
##  6     5   251.
##  7     6   259.
##  8     7   262.
##  9     8   261.
## 10     9   255.
## # ... with 76 more rows
```

Thiele log-Normal Hump Spline

```
## [1] "relative convergence (4)"
```

```
##              log_tau2_logpop             log_tau2_logpop
##                  4.965357473                 6.576396862
##              log_tau2_logpop             log_tau2_logpop
##                  4.990709645                 6.510644394
##       log_marginal_lambda_fx      log_marginal_lambda_gx
##                  8.295631036                 6.899631173
##               log_dispersion              log_dispersion
##                  1.239706740                 1.187223274
##                log_lambda_tp                    tp_slope
##                  4.053089638                -0.004206457
##                  tp_params_5                tp_params_10
##                  0.232969902                 0.435940078
##      log_marginal_lambda_phi     log_marginal_lambda_psi
##                  8.625842410                 8.617562883
##   log_marginal_lambda_lambda   log_marginal_lambda_delta
##                  4.010500935                 4.275639308
## log_marginal_lambda_epsilon       log_marginal_lambda_A
##                  5.066083540                 8.600399367
##        log_marginal_lambda_B               logit_rho_phi
##                  4.265930326                -1.657493067
##                logit_rho_psi            logit_rho_lambda
##                 -1.657702314                 1.971076693
##              logit_rho_delta           logit_rho_epsilon
##                  2.992776051                 4.290671956
##                  logit_rho_A                 logit_rho_B
##                 -1.657811604                -1.700191923
##             logit_rho_fx_age           logit_rho_fx_time
##                 -1.663904851                -1.616685578
##             logit_rho_gx_age           logit_rho_gx_time
##                  1.568102198                -1.337095411
##             log_gamma_lambda             log_gamma_delta
##                  3.692463119                 3.414376557
##            log_gamma_epsilon
##                  2.710244437
```

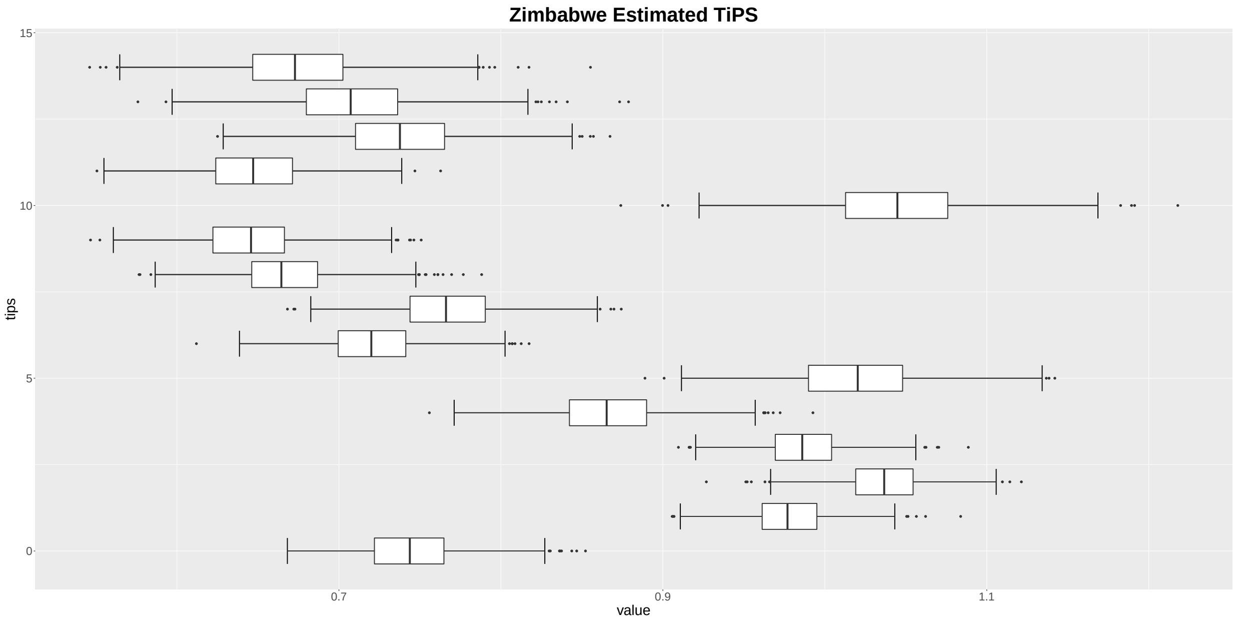

Figure 1: Estimated TiPS

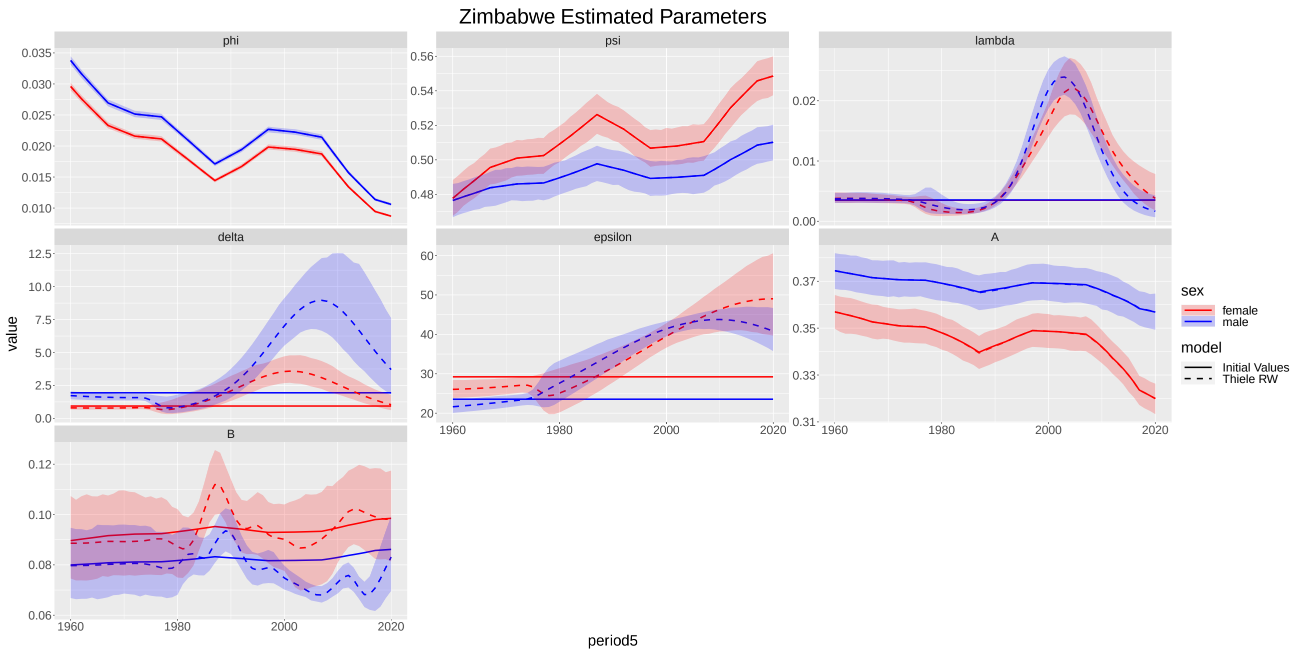

Figure 2: Estimated parameters

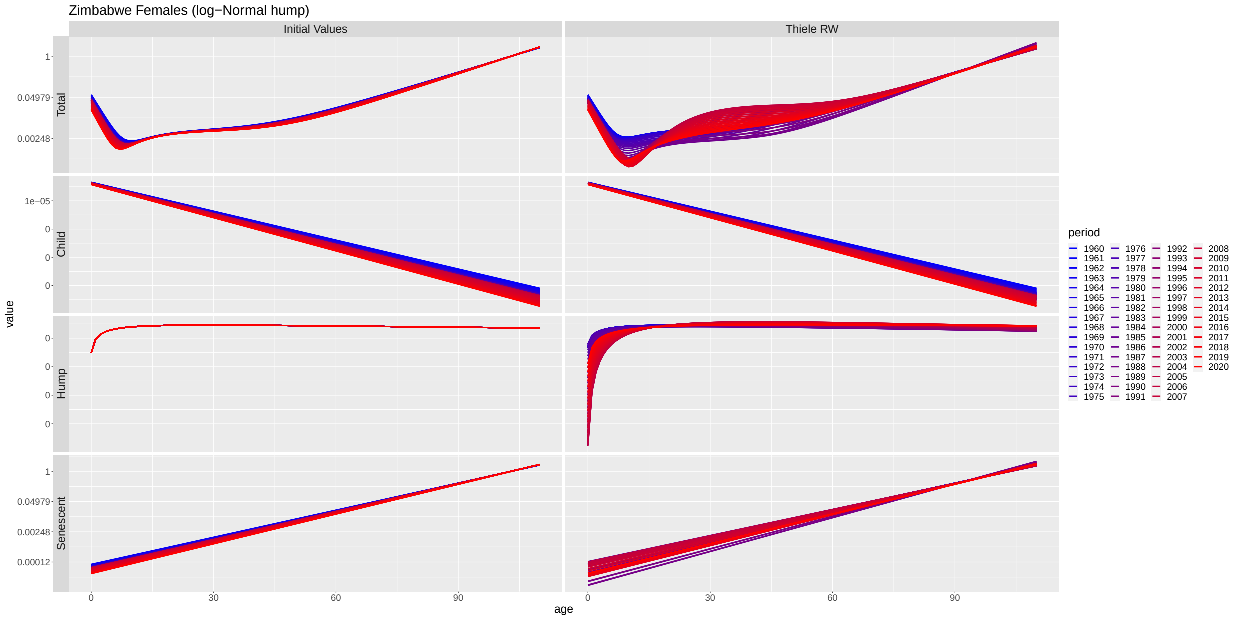

Figure 3: Thiele Decomposed

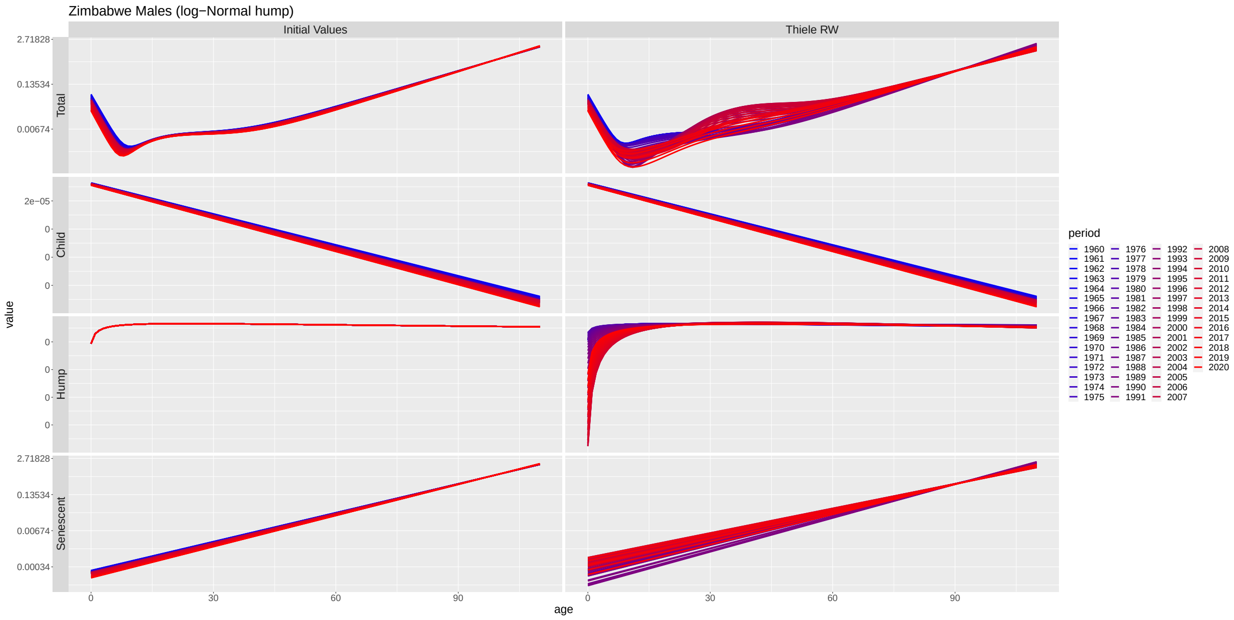

Figure 4: Thiele Decomposed

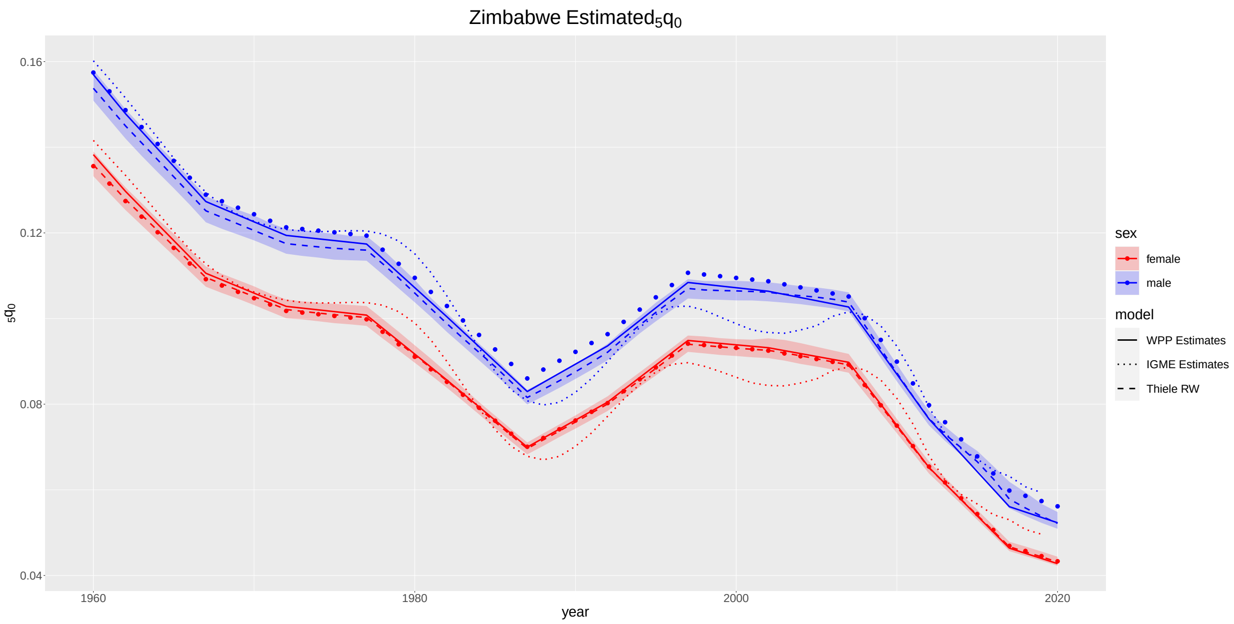

Figure 5: Estimated $_5q_0$

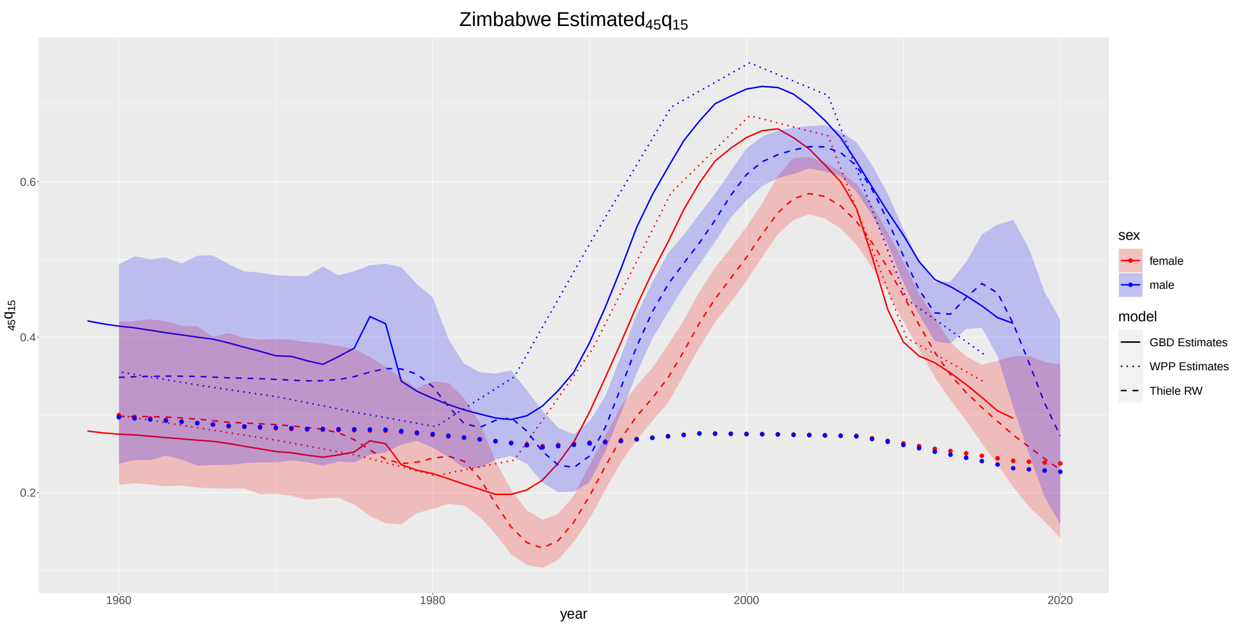

Figure 6: Estimated $_{45}q_{15}$

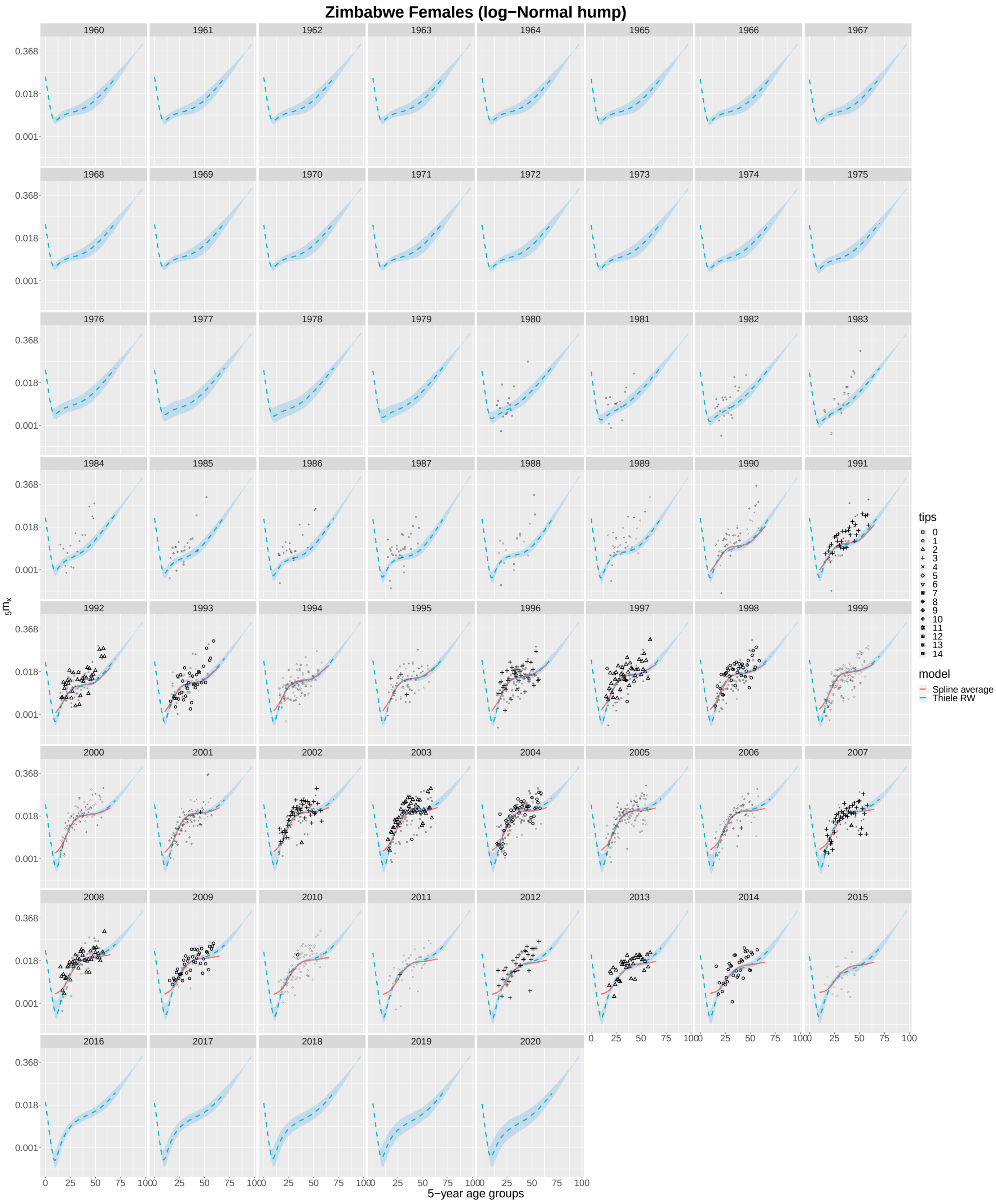

Figure 7: Mortality Schedules

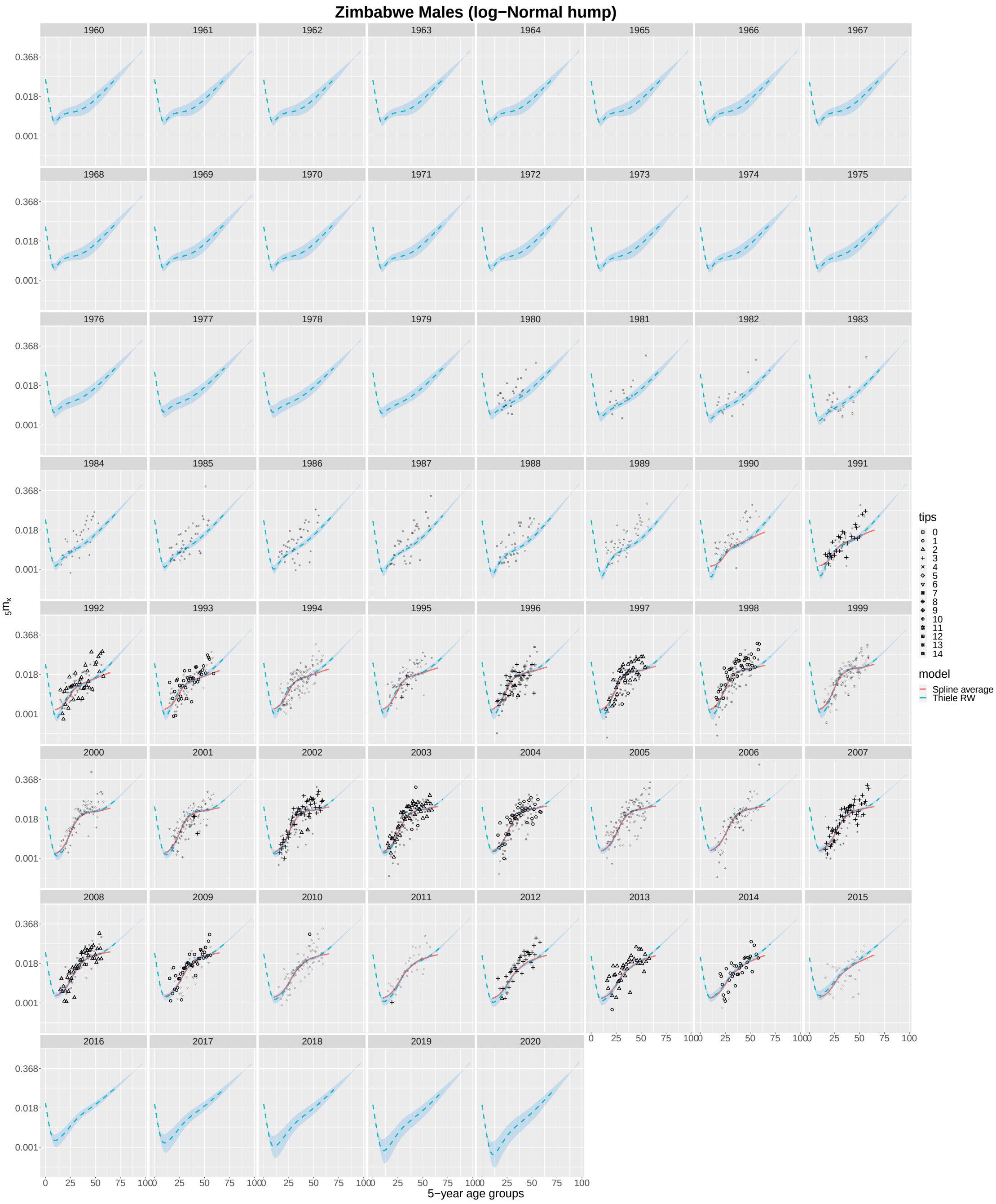

Figure 8: Mortality Schedules

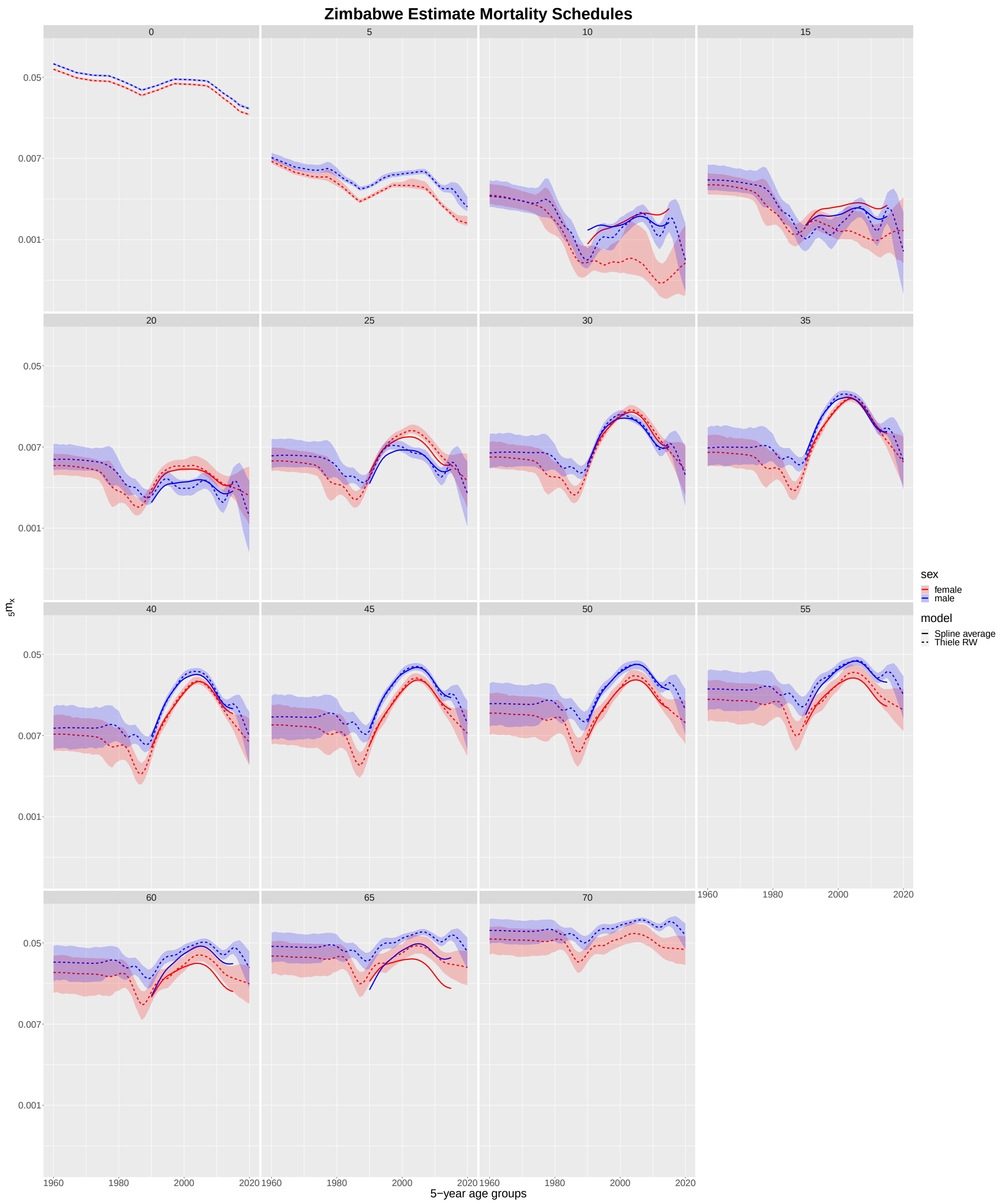

Figure 9: Mortality Schedules

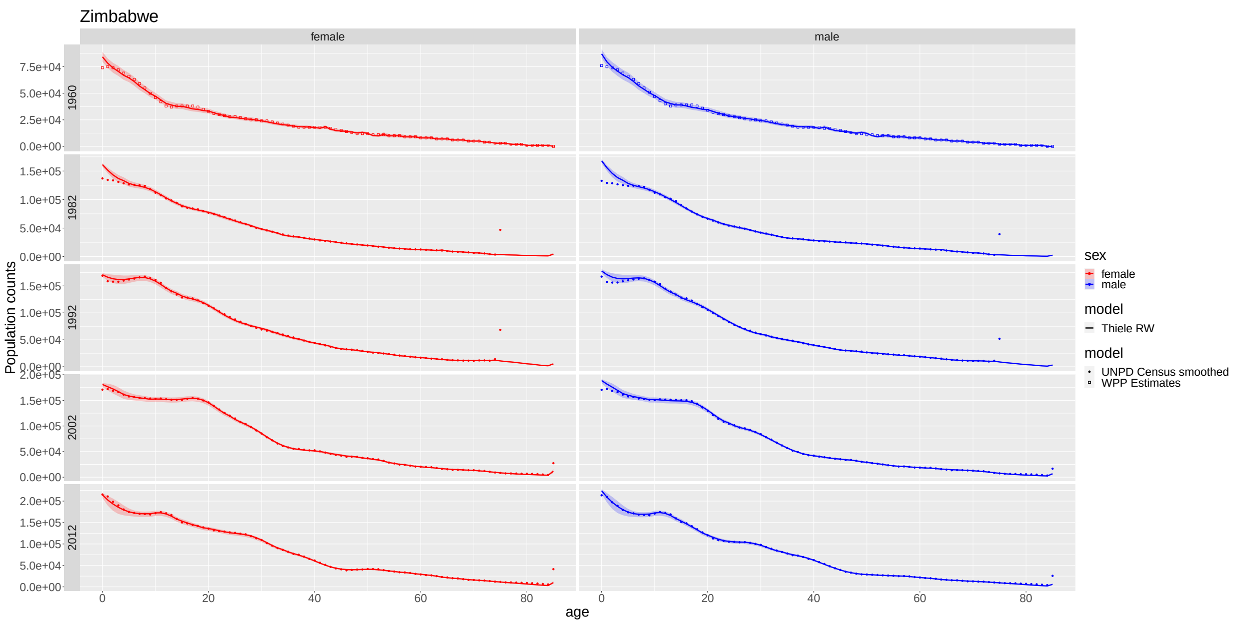

Figure 10: Population

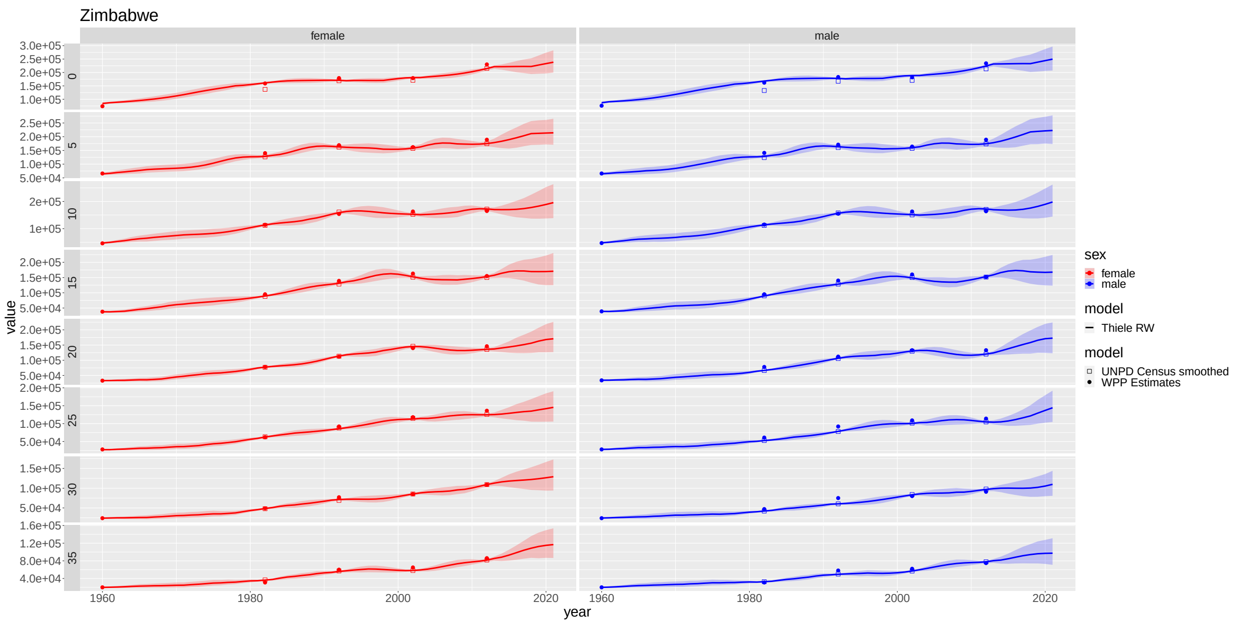

Figure 11: Population

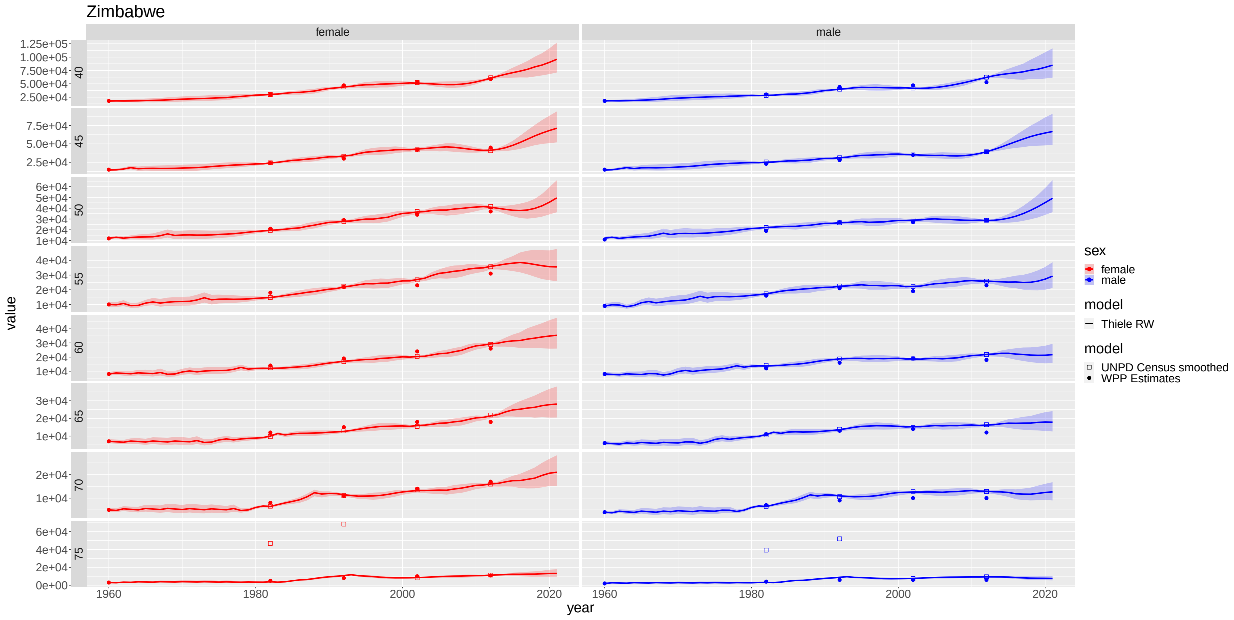

Figure 12: Population

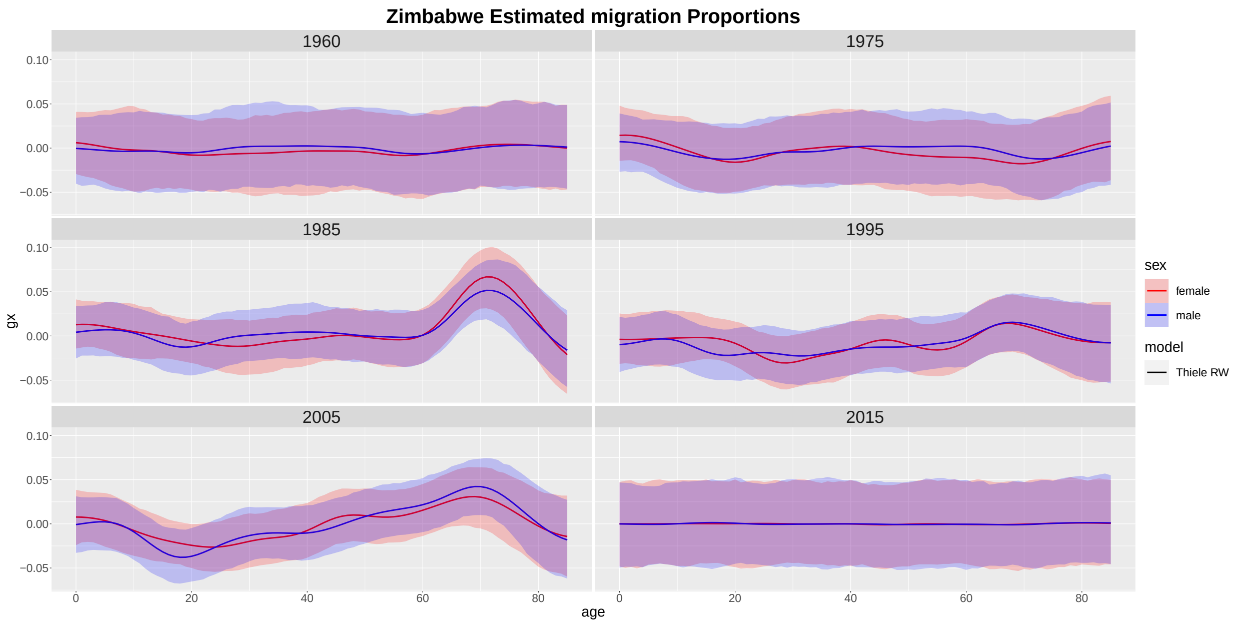

Figure 13: Migration

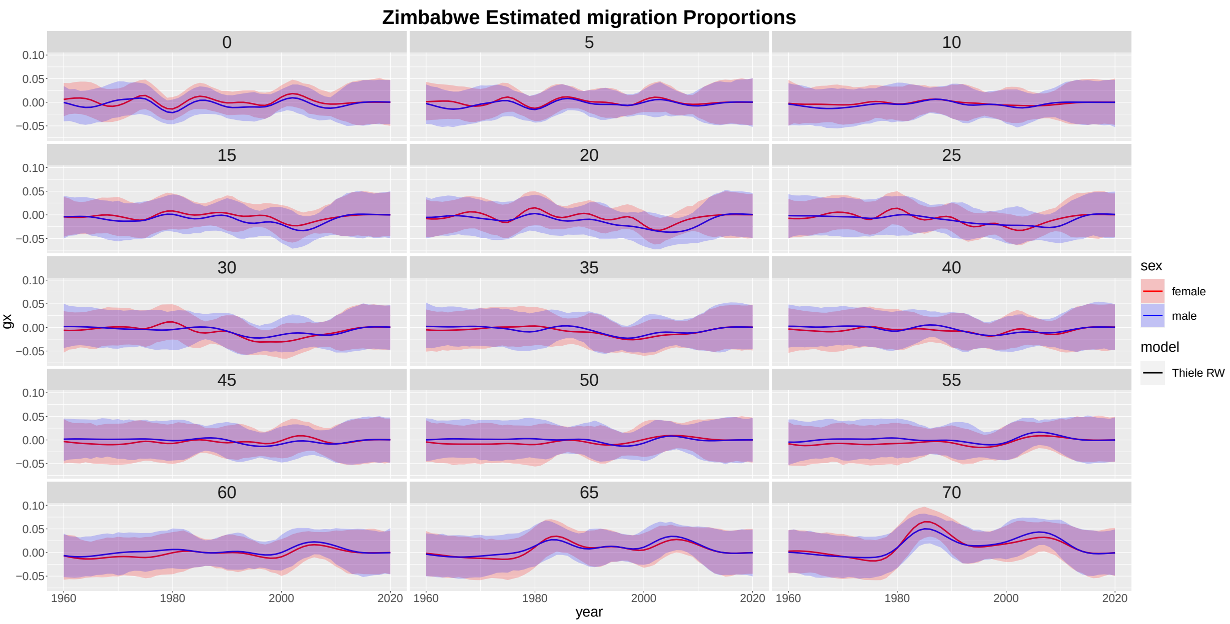

Figure 14: Migration

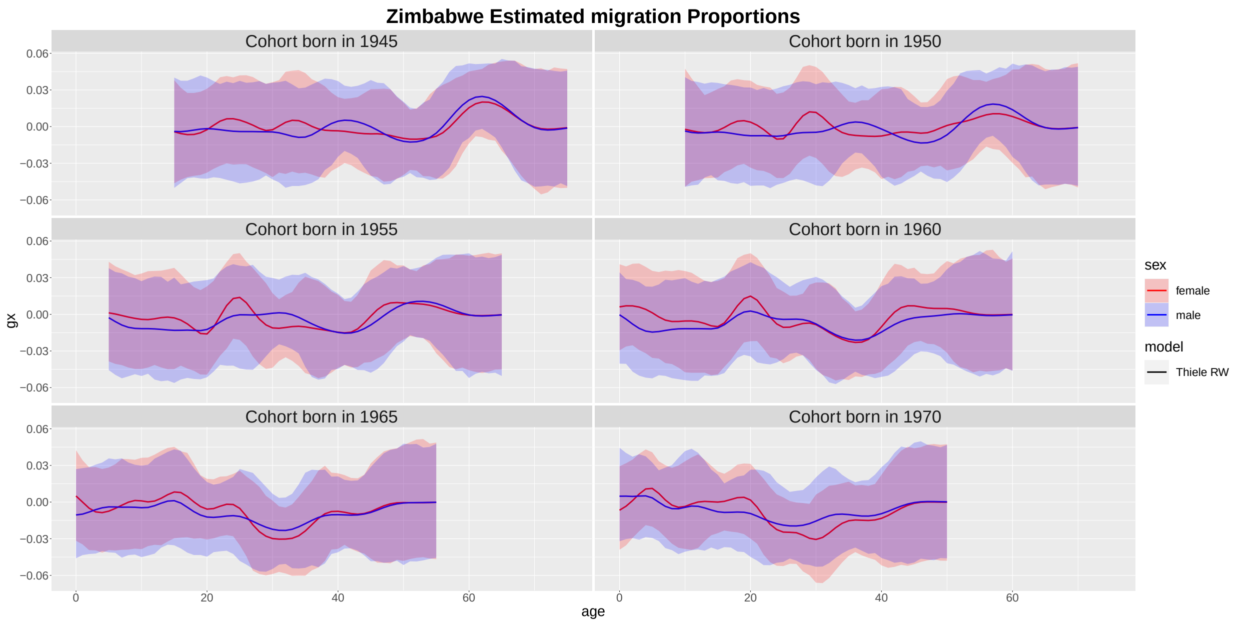

Figure 15: Migration

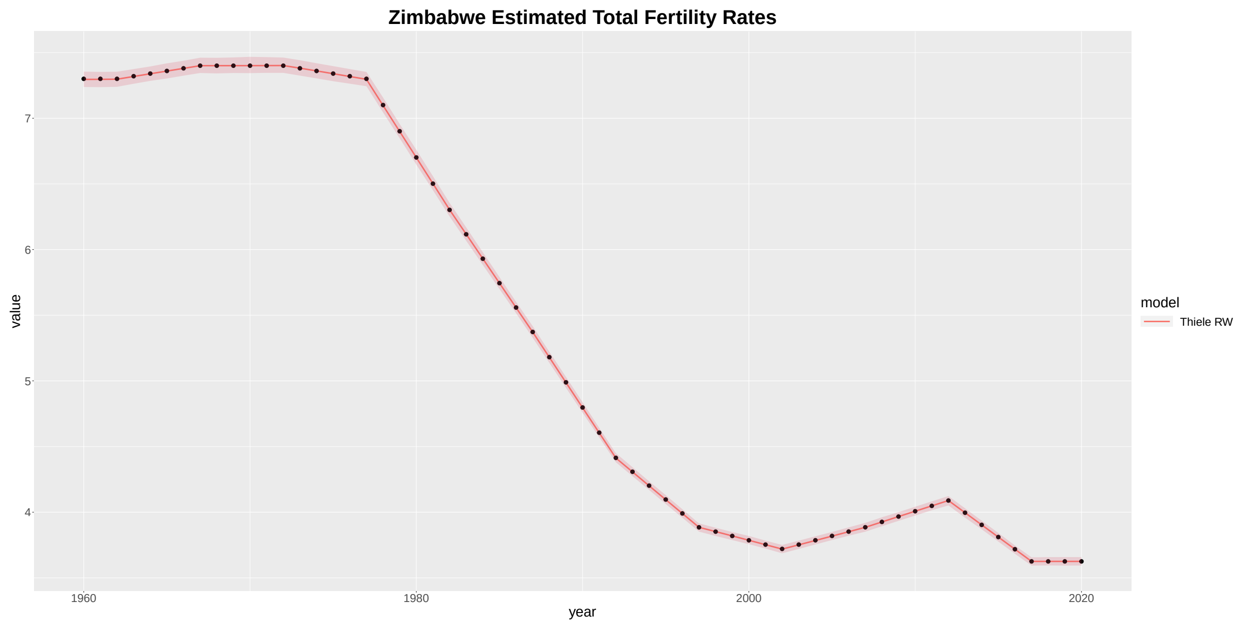

Figure 16: Total Fertility

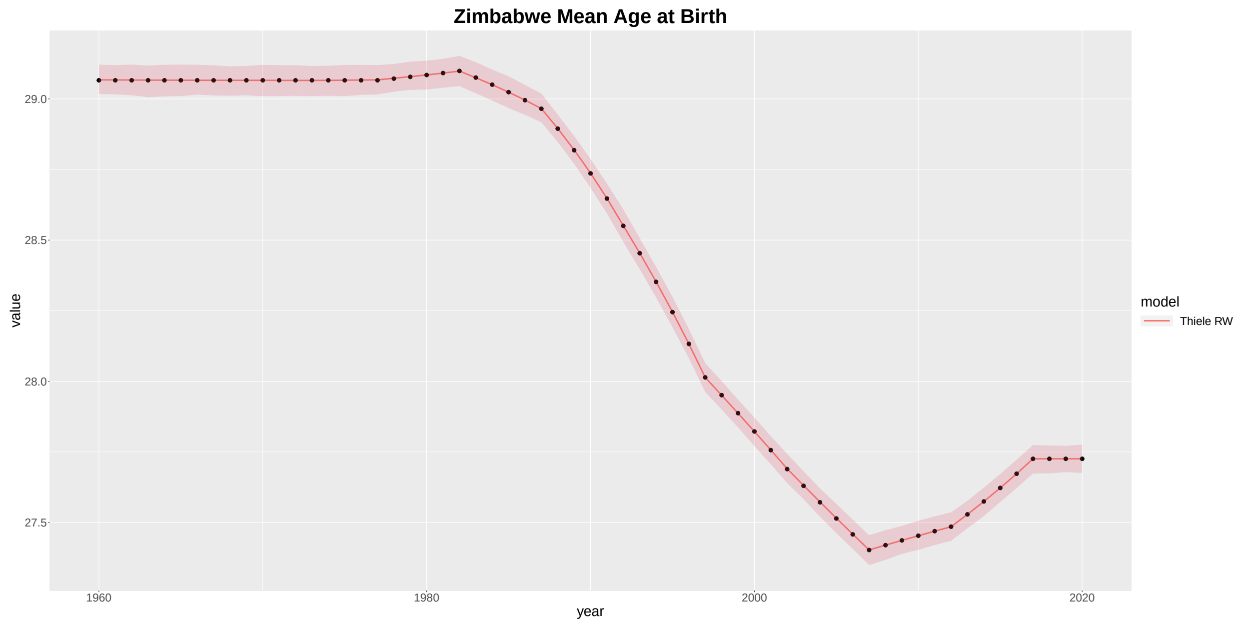

Figure 17: Mean age at births

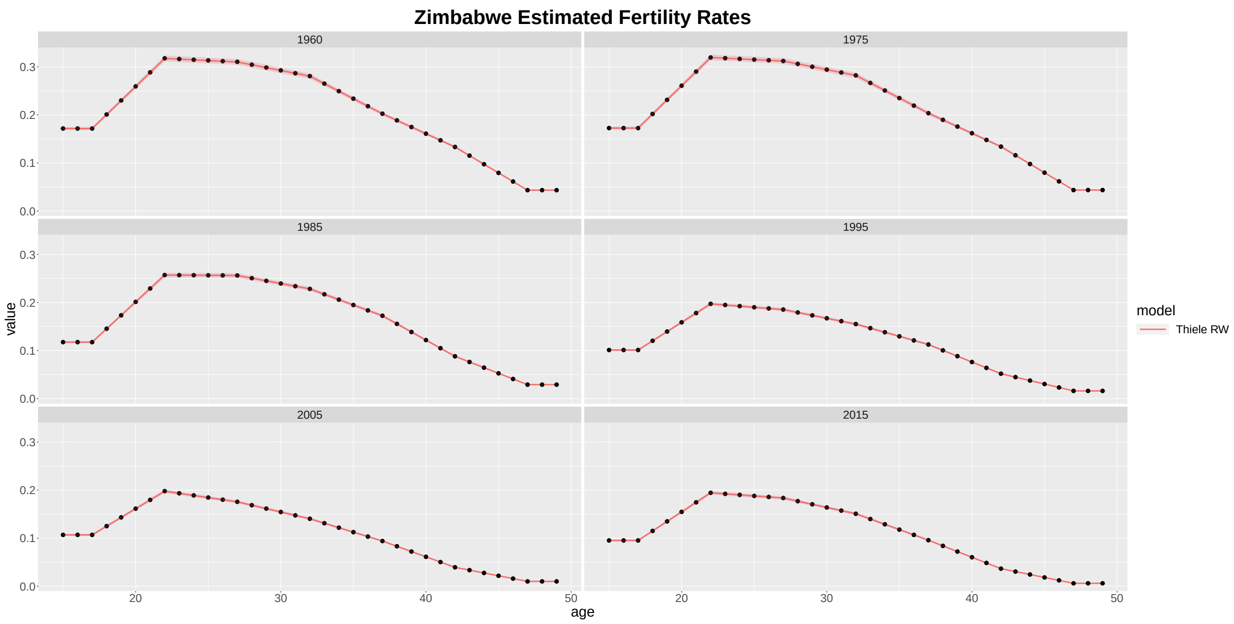

Figure 18: Fertility

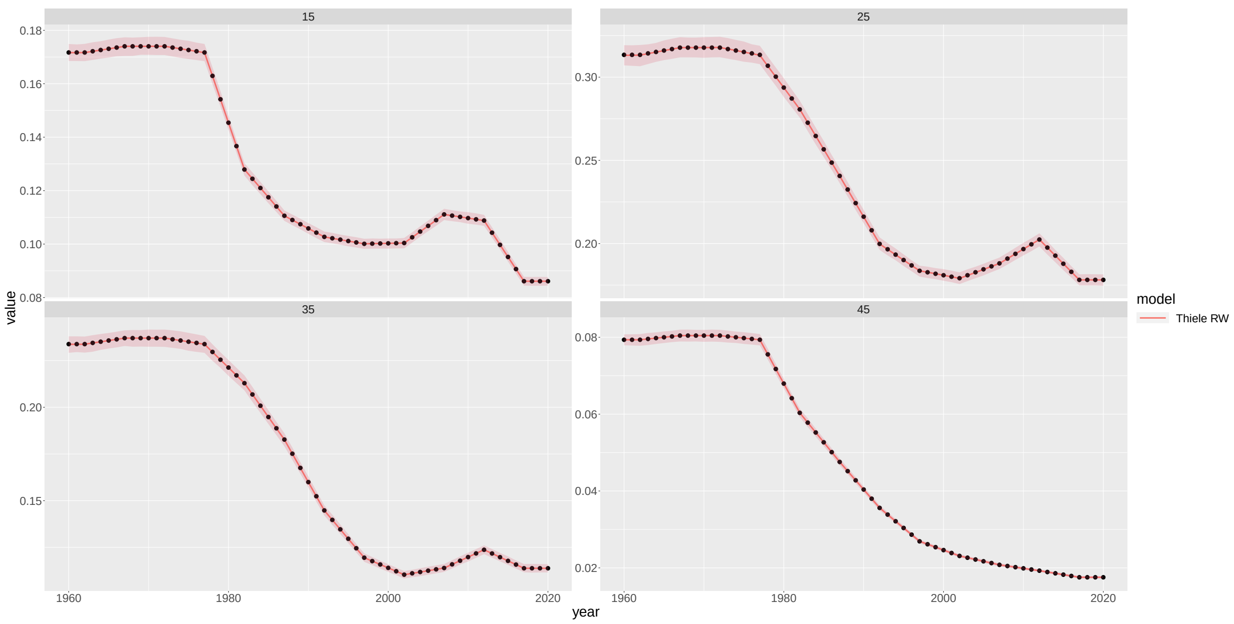

Figure 19: Fertility