Exercise 2.4.2

Consider the undrestricted multiple regression model $y = X_1\beta_1 + X_2\beta_2 + \epsilon$. If we impose the null hypothesis that $\beta_2 = 0$, we get the restricted model $y = X_1\beta_1 + \epsilon$

(a) Suppose that both the restricted and the unrestricted model contain a constant term. Then prove that $F = \frac{(R_1^2 - R_0^2)/g}{(1-R_1^2)/(n-k)}$ where R_0^2 and R_1^2 are the R-squared of respectively the restricted and unrestricted model.

$$F = \frac{(e_0^T e_0 - e_1^T e_1)/g}{e_1^T e_1/(n-k)}$$
$$R^2 = 1 - \frac{e^T e}{\sum_{i=1}^n (y_i - \bar{y})^2}$$
$$e^T e = (1 - R^2) \sum_{i=1}^n (y_i - \bar{y})^2$$

$$F = \frac{[(1 - R_0^2)\sum_{i=1}^n (y_i - \bar{y})^2 - (1 - R_1^2)\sum_{i=1}^n (y_i - \bar{y})^2]/g}{[(1 - R_1^2)\sum_{i=1}^n (y_i - \bar{y})^2]/(n-k)}$$
$$F = \frac{(R_1^2 - R_0^2)/g}{(1 - R_1^2)/(n-k)}$$

(b) Suppose that we test of a single restriction $H_0 : \beta_j = 0$, so that $g = 1$. Then prove that $F = t^2$.

$$t = \frac{b_j}{s\sqrt{a_{jj}}}$$

Generally:

$H_0 : R\beta = r$

$F = \frac{1}{s^2}(Rb - r)^T V^{-1}(Rb - r)/g$

$V = R(X^T X)^{-1} R^T$

For $\beta_j = 0$:

$H_0 : R\beta = r$

$R = (0\ 0\ ...\ 1\ ...\ 0),\ r = 0$

$V = R(X^T X)^{-1} R^T = a_{jj}$

$F = \frac{1}{s^2}(b_j - 0)\frac{1}{a_{jj}}(b_j - 0) = \frac{b_j^2}{s^2 a_{jj}} = t^2$