Exercise 2.3

(a) Prove that $\hat{y}, e, s^2$ and R^2 do not depend on A (that is, are invariant under linear transformations).

Let $\tilde{X} = XA$ where A is a $(k \times k)$ invertible matrix

Matrix Properties:
$(AB)^T = B^T A^T$
$(A^T)^{-1} = (A^{-1})^T$
$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ for A, B, C $(k \times k)$ invertible matrix

$$\tilde{H} = \tilde{X}(\tilde{X}^T\tilde{X})^{-1}\tilde{X}^T$$
$$\tilde{H} = XA((XA)^T(XA))^{-1}(XA)^T$$
$$\tilde{H} = XA(A^TX^TXA)^{-1}A^TX^T$$
$$\tilde{H} = XAA^{-1}(X^TX)^{-1}(A^T)^{-1}A^TX^T$$
$$\tilde{H} = X(X^TX)^{-1}X^T = H$$

$$\tilde{\hat{y}} = \tilde{X}b = \tilde{H}y = Hy = \hat{y}$$
$$\tilde{e} = \tilde{M}y = (1 - \tilde{H})y = (1 - H)y = e$$
$$\tilde{s}^2 = \frac{\tilde{e}^T\tilde{e}}{n-k} = \frac{e^Te}{n-k} = s^2$$
$$\tilde{R}^2 = cor(y, \tilde{\hat{y}})^2 = cor(y, \hat{y})^2 = R^2$$

(b) Prove that $\tilde{b} = A^{-1}b$ and provide an intuitive interpretation.

$$\begin{aligned}
\tilde{b} &= (\tilde{X}^T\tilde{X})^{-1}\tilde{X}^Ty \\
&= ((XA)^T(XA))^{-1}(XA)^Ty \\
&= (A^TX^TXA)^{-1}A^TX^Ty \\
&= A^{-1}(X^TX)^{-1}(A^T)^{-1}A^TX^Ty \\
&= A^{-1}(X^TX)^{-1}X^Ty \\
&= A^{-1}b
\end{aligned}$$

Alternative explanation:

$$
\begin{aligned}
y &= X\beta + \epsilon \\
y &= XAA^{-1}\beta + \epsilon \\
y &= \tilde{X}(A^{-1}\beta) + \epsilon
\end{aligned}
$$

$$
\begin{aligned}
\therefore \tilde{\beta} &= A^{-1}\beta \\
\therefore \tilde{b} &= A^{-1}b
\end{aligned}
$$