Unit 4

LOS 1. Determinants

- 2 by 2 matrix determinants
  - $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
  - If $\det A \neq 0$, then A^{-1} exists and therefore, $Ax = b \Rightarrow x = A^{-1}b$
  - Additionally, $Ax = 0$ only when $x = 0$ if $\det A \neq 0$
- 3 by 3 matrix determinants:
  - pneumonic for 3 by 3 matrix determinants
  - $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, $\det A = aei - afh - bdi + bfg + cdh - ceg$

LOS 2. Laplace Expansion:

- used to find the determinant for $n \times n$ matrices
- example for 3×3 matrix:
  - $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$
  - $\det A = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = aei - afh - bdi + bfg + cdh - ceg$
- the above method for all columns and row i.e. we can traverse through d, e, f to calculate the determinants, usually we traverse along columns or rows with the most number of zeros
- sign for traversing:

$$\begin{vmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}$$

LOS 3. Liebniz Formula

- used to find determinant using permutations of the terms in the matrix
- example for 3×3 matrix:
  - $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$
  - $\det A == aei - afh - bdi + bfg + cdh - ceg$
  - number of terms: $3!$ terms

term	column	sign	no. of flips	type
aei	1, 2, 3	$+$	0	even
afh	1, 3, 2	$-$	1	odd
bfg	2, 3, 1	$+$	2	even
bdi	2, 1, 3	$-$	1	odd
cdh	3, 1, 2	$+$	2	even
ceg	3, 2, 1	$-$	1	odd

  - column: column in which each of the value resides
  - no. of flips: number of swaps to rearrange the column into 1, 2, 3
  - for even number flips, the term is positive. for odd number flips, the term is negative
- for $n \times n$ matrix: $n!$ terms

LOS 4. Properties of a determinant

- property 1: $\det I = 1$
- property 2: $\det$ changes sign under row interchange
  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$
- property 3: $\det$ is a linear function of the first row
  $\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}, \begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$
- corollary properties
  - $\det$ is a linear function of any row (does not have to be the first row), $\det$ does not change when we multiply a row by a constant and when add it to another row (the original row itself is not changed) $\rightarrow$ Gaussian elimination does not change the $\det$, except when rows are exchanged in order or rows are multiplied by a constant
  - $\det = 0$ if there are two identical rows
  - $\det = 0$ if there is a row of zeros
  - $\det = 0$, then the matrix is not invertible
  - $\det D, \det L, \det U$ determinant of a diagonal, lower triangular and upper triangular matrix is equal to the product of the diagonal elements
  - $\det AB = \det A \det B$
  - $\det\left(A^{-1}\right) = \frac{1}{\det A}$
  - $\det A^T = \det A$
- we can use Gaussian elimination to convert the matrix to a triangular matrix and then calculate the determinant

LOS 5. Eigenvalue problem

- A $n \times n$ matrix: $Ax = \lambda x$ where λ is the eigenvalue and x is the eigenvectors
- steps:
  - $Ax = \lambda Ix$

- $Ax - \lambda I x = 0$
- $(A - \lambda I)x = 0$
- if $\det(A - \lambda I) \neq 0$, then $(A - \lambda I)$ has an inverse and therefore $(A - \lambda I)^{-1}(A - \lambda I)x = 0$ and we get the trivial solution of 0
- to avoid getting the trivial solution, we need $\det(A - \lambda I) = 0$, which is called the characteristic of A
- $\det(A - \lambda I)$ is an n-th order polynomial equation in λ which has n roots

* example for 2 $\times$ 2 matrix:
  - $\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + ad - bc = 0$
  - $\det(A - \lambda I) = \lambda^2 - \mathrm{Tr}A + \det A = 0$
* cases for λ:
  - 2 real values of λ
  - complex λ i.e. $\det(A - \lambda I) = \lambda^2 + 1 = 0$ where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
  - 1 real λ (the eigenvalues are degenerate) i.e. $\det(A - \lambda I) = \lambda^2 = 0$ where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

LOS 6. Finding eigenvalues and eigenvectors

* all symmetric matrices have real eigenvalues
* example:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1$$

$$\lambda_1 = -1 \quad \text{OR} \quad \lambda_2 = 1$$

$$\lambda_1 = -1 : (A + I)x_1 = 0$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_1 + v_2 = 0 \Rightarrow v_2 = -v_1$$

$$\lambda_1 = 1 : (A - I)x_2 = 0$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-v_1 + v_2 = 0 \Rightarrow v_2 = v_1$$

$$\lambda_1 = -1 \quad x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 1 \quad x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

LOS 7. Matrix diagonalization

* if A is diagonalizable, formula:
  - $A = S \Lambda S^{-1}$

    - $\Lambda = S^{-1}AS$
- in matrix diagonalization, we need to normalize the eigenvectors, therefore S is orthogonal
- example for 2 $\times$ 2 matrix:

$$
\begin{gathered}
A \in 2 \times 2 \\
\lambda_1, x_1 = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} \\
\lambda_2, x_2 = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} \\
A = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}
\end{gathered}
$$

- two eigenvectors corresponding to distinct eigenvalues are linearly independent
- if columns of an $n \times n$ matrix are linearly independent, the matrix is invertible

LOS 8. Powers of a matrix

- derivation:

$$
\begin{gathered}
A = S\Lambda S^{-1} \\
A^2 = (S\Lambda S^{-1})(S\Lambda S^{-1}) = S\Lambda^2 S^{-1} \\
A^p = S\Lambda^p S^{-1}
\end{gathered}
$$

- if A is diagonalizable, we can compute its power easily as the eigenvalue matrix Λ is a diagonal matrix
- matrix exponential:

$$
\begin{gathered}
e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \\
e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots \\
e^A = Se^\Lambda S^{-1} \\
e^A = \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{pmatrix}
\end{gathered}
$$