Unit 10

LOS 1. Learn how to find solution operator for PDEs using orthonormal basis

Problem (try solving for (3) for practice):

$$u_t = -Au \tag{1}$$
$$u_t = iAu \tag{2}$$
$$u_{tt} = -Au \tag{3}$$

where A is a differential operator

Assumption: $\exists$ ONB for V_n such that $AV_n = \lambda_n V_n$ where $\lambda_n \geq 0$ Solution for (1):

$$u(t,x) = \sum_n C_n e^{-t\lambda_n} V_n(x) \qquad x \in \mathbb{R}^n$$

$$C_n = (V_n, u_0(x))_\mu = \int_{\mathbb{R}} \overline{V_n(y)} u_0(y) d\mu(y) \qquad C_n \text{ is a Fourier coefficient}$$

Solution operator for (1):

$$\text{Kernel } K_t(x,y) = \sum_n e^{-t\lambda_n} V_n(x) \overline{V_n(y)}$$

$$S(t)(u_0)(x) = \int_{\text{space}} K_t(x,y) u_0(y) d\mu(y$$

Proof:

$$\begin{aligned}
u(t,x) &= \sum_n C_n e^{-t\lambda_n} V_n(x) \\
&= \sum_n \int_{\mathbb{R}} \overline{V_n(y)} u_0(y) d\mu(y) e^{-t\lambda_n} V_n(x) \\
&= \int_{\mathbb{R}} \left[\sum_n e^{-t\lambda_n} V_n(x) \overline{V_n(y)} \right] u_0(y) d\mu(y) \\
&= \int_{\mathbb{R}} K_t(x,y) u_0(y) d\mu(y)
\end{aligned}$$

Kernel converges when eigenfunction is decaying at infinity

Solution for (2):

$$u(t,x) = \sum_n C_n e^{it\lambda_n} V_n(x)$$
$$K_t(x,y) = \sum_n e^{it\lambda_n} V_n(x)\overline{V_n(y)}$$
$$\int_{\mathbb{R}} \overline{V_k(x)} K_t(x,y) V_j(y) d\mu(x) d\mu(y) = \sum_n (V_k, V_n)(V_n, V_j) e^{it\lambda_n}$$

LOS 2. Solve the differential operator heat problem along with Dirichlet boundary conditions

Example 1:

$$D[-\pi, \pi]$$
$$A = -\frac{d^2}{dx^2}$$
$$u_t = u_{xx}$$
$$u(x,0) = u(x+2\pi, 0)$$

$$\text{Then } \lambda_n = n^2$$
$$V_n = e^{inx}$$
$$V_n = e^{-inx}$$
$$u(t,x) = \sum_{u\in\mathbb{Z}} \hat{u}_0(n) e^{-n^2 t} e^{inx}$$
$$K_t(x,y) = \sum_n e^{-iny} e^{inx} e^{-n^2 t}$$

LOS 3. Solve harmonic oscillator differential operator PDE using orthonormal basis

Example 2:

$$A = -\frac{d^2}{dx^2} + x\frac{d}{dx}$$
$$u_t = -Au$$
$$\text{ONB } \tilde{P}_n = \frac{P_n(x)}{\sqrt{x}}$$
$$e^{xz-\frac{z^2}{2}} = \sum_{n=0}^{\infty} P_n(x)\frac{z^n}{n!}$$

$$u(t,x) = \sum_n C_n e^{tn} P_n(x)$$
$$C_n = \frac{1}{n!}(P_n, u_0)$$
$$d\mu(x) = e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}$$

$$K_t(x,y) = \sum_n e^{-tn} P_n(x)P_n(y)\frac{1}{n!}$$
$$K_t(x,y) = \sum_n e^{itn} P_n(x)P_n(y)\frac{1}{n!} \qquad \text{(for)} u_t = iAu$$

Harmonic oscillator:

$$A(v) = -v_{xx} + x^2 v$$
$$Av = \lambda v$$
$$v_{xx} - x^2 v - \lambda v$$
$$v_x x + (\lambda - x^2)v = 0$$

$$w(x) = e^{-\frac{x^2}{2}} w(x) \qquad \text{(By change of variable)}$$
$$w_k(x) = P_k(\sqrt{2}x)$$
$$\lambda_k = 2k+1$$

$$u(t,x) = \sum_k C_k e^{-t(2k+1)} P_k(\sqrt{2}x) e^{-\frac{x^2}{2}}$$

Facts:

$$\int_{\mathbb{R}} P_n(x)P_m(x)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} = n!\delta_{nm}$$

$$x = \sqrt{2}y,\ \frac{x^2}{2} = y \qquad \text{(By change of variable)}$$
$$\int_{\mathbb{R}} P_n(x)P_m(x)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} = \int_{\mathbb{R}} P_n(\sqrt{2}y)P_m(\sqrt{2}y)e^{-y^2}\frac{dy}{\sqrt{\pi}}$$

$Q_k(y) = P_k(\sqrt{2}x) \qquad$ (By change of variable)

Q_k is orthogonal in $L_2\left(\frac{e^{-y^2}dy}{\sqrt{\pi}}\right)$

$$u_0(x) = \sum_k C_k P_k(\sqrt{2}x)e^{-\frac{x^2}{2}}$$

$$C_k = \frac{(Q_k, u_0 e^{y^2})}{(Q_k, Q_k)}$$

Condition: $u_0 e^{y^2} \in L_2(e^{-y^2}dy)$

C_k implies:

$$\int_{\mathbb{R}} |u_0(x)e^{\frac{x^2}{2}}|^2 e^{-x^2}\frac{dx}{\sqrt{\pi}} < \infty$$
$$\int_{\mathbb{R}} |u_0(x)|^2 e^{x^2}e^{-x^2}\frac{dx}{\sqrt{\pi}} < \infty$$

$\therefore u_0 \in L_x(dx)$

Therefore:

$$\begin{aligned}
A(v) &= -v_{xx} + x^2 v \\
u_t &= -Au \\
u(t,x) &= \sum_k C_k Q_k(x) e^{-y^2} e^{-t(2k+1)} \\
K_t(x,y) &= \sum_{k=0}^{\infty} \frac{1}{k!} e^{-\frac{x^2}{2}} Q_k(x) e^{\frac{y^2}{2}} Q_k(y) e^{-y^2} e^{-t(2k+1)} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} Q_k(x) Q_k(y) e^{-\frac{x^2}{2}-\frac{y^2}{2}} e^{-t(2k+1)}
\end{aligned}$$

Summary:

$$u_t = -Au \qquad K_t(x,y) = \frac{1}{\pi}\sum_n \frac{1}{k!}e^{-(2n+1)t}P_n(\sqrt{2}x)P_n(\sqrt{2}y)e^{-\frac{x^2}{2}-\frac{y^2}{2}}$$

$$u_t = iAu \qquad K_t(x,y) = \frac{1}{\pi}\sum_n \frac{1}{k!}e^{-i(2n+1)t}P_n(\sqrt{2}x)P_n(\sqrt{2}y)$$

LOS 4. Learn how to tackle two-dimensional space differential operator PDEs

Problem:

$$\begin{aligned}
&D \to \mathbb{R}\\
&B(v) = -\frac{d^2v}{dx^2} - \frac{dv}{dy^2} + (x^2+y^2)v = A_x(v) + A_y(v)
\end{aligned}$$

Observation (product of two eigenfunctions is also an eigenfunction):

$$\begin{aligned}
BV_n &= \lambda_n V_n\\
&= (A_x + A_y)(V_n(x_1), V_k(x_2))\\
&= (A_x)(V_n(x_1), V_k(x_2)) + (A_y)(V_n(x_1), V_k(x_2))\\
&= (\lambda_n)(V_n(x_1), V_k(x_2)) + (\lambda_k)(V_n(x_1), V_k(x_2))\\
&= (\lambda_n + \lambda_k)(V_n(x_1), V_k(x_2))
\end{aligned}$$

Solution (Green's function):

$$\begin{aligned}
G_t(x_1x_2, y_1y_2) = \frac{1}{\pi^2}\sum_{n,m}\frac{1}{n!m!}&e^{-[(2n+1)+(2m+1)]t}e^{-\frac{x_1^2+x_2^2}{2}-\frac{y_1^2+y_2^2}{2}}\\
&\times P_n(\sqrt{2}x_1)P_n(\sqrt{2}x_2)P_m(\sqrt{2}y_1)P_m(\sqrt{2}y_2)
\end{aligned}$$

LOS 5. Solve a PDE representing hydrogen atom using separation ansatz

Problem:

$$\begin{aligned}
&u_t = \frac{\Delta}{2}u - \frac{u}{r}\\
&u \in L_2(\mathbb{R}^3)\\
&u \text{ radial}
\end{aligned}$$

Separation Ansatz:

$$\begin{aligned}
&u(t,x) = T(t)V(r) && r = \sqrt{x_1^2 + x_2^2 + x_3^2} \\
&\frac{iT'}{T} = \frac{-\frac{\Delta}{2} - \frac{v}{r}}{v} = \frac{\lambda}{2} \\
&T' = \frac{\lambda}{2i}T \\
&T(t) = e^{-i\frac{\lambda}{2}t}T(0)
\end{aligned}$$

$$-\Delta V - \frac{2V}{r} = \lambda V$$

Assume $V = R$:

$$\begin{aligned}
&-R_{rr} - \frac{2}{r}R_r - \frac{2}{r}R = \lambda R \\
&-R_{rr} = \lambda R + \frac{2}{r}R_r + \frac{2}{r}R \qquad (1)
\end{aligned}$$

Trick:

$$\begin{aligned}
&\beta^2 = -\lambda \\
&\beta = \sqrt{-\lambda} \\
&R(r) = e^{-\beta r}w(r) \\
&R_r = -\beta e^{-\beta r}w + e^{-\beta r}w' \\
&R_{rr} = \beta^2 e^{-\beta r}w - 2\beta e^{-\beta r}w' + e^{-\beta r}w''
\end{aligned}$$

Substitute to (1):

$$\begin{aligned}
&\beta^2 e^{-\beta r}w - 2\beta e^{-\beta r}w' + e^{-\beta r}w'' = \lambda(e^{-\beta r}w) + \frac{2}{r}(-\beta e^{-\beta r}w + e^{-\beta r}w') + \frac{2}{r}(e^{-\beta r}w) \\
&0 = w'' - 2\left(\frac{1}{r} - \beta\right)w' + \frac{2(1-\beta)}{r}w \\
&0 = rw'' - 2(1-\beta r)w' + 2(1-\beta)w \qquad (2)
\end{aligned}$$

Trick:

$$\begin{aligned}
&\varphi(r) = w(\gamma r) && \gamma = \frac{1}{2\beta} \\
&\varphi'(r) = \gamma w'(\gamma r) \\
&\varphi''(r) = \gamma^2 w''(\gamma r)
\end{aligned}$$

Substitute to (2):

$$0 = \gamma r w''(\gamma r) + 2w'(\gamma r) - 2\beta r w'(\gamma r) + 2(1-\beta)w$$
$$0 = r\varphi'' + (2-r)\varphi' + \left(\frac{1-\beta}{\beta}\right)\varphi \qquad (3)$$

$\left(\frac{1-\beta}{\beta}\right)$ has to be an integer:

$$\beta = \frac{1}{k},\ \lambda = -\frac{1}{k^2}$$
$$\left(\frac{1-\beta}{\beta}\right) = k\left(1-\frac{1}{k}\right) = (k-1) \in \mathbb{N}$$

LOS 6. Learn theory about Laguerre differential equation as a solution of hydrogen atom PDE

Laguerre polynomials:

$$xy'' + (\alpha + 1 - x)y'' + ny = 0$$

Solution: $L_n^\alpha(x)$

where $\sum_n t^n L_n^\alpha(x) = \frac{1}{(1-t)^{\alpha+1}} e^{-\frac{tx}{1-t}}$

$L_n^\alpha(x)$ are orthogonal polynomials:

$$\int_0^\alpha r^\alpha L_n^\alpha(x) L_m^\alpha(x) e^{-r} dr = \delta_{nm} \frac{\Gamma(n+\alpha+1)}{n!}$$

Therefore, solution to hydrogen problem:

$$u(t,r) = \sum_k a_k e^{\frac{it}{2k^2}} e^{-\frac{r}{k}} L_{k-1}^1(r)$$

LOS 7. Understand theory of harmonic functions related to PDEs

For u harmonic:

- Theorem:

  $\Delta u = 0$

  If $\Omega \subseteq \mathbb{R}^2 (\Omega$ connected and compact, u continuous on the boundary)

  Then u achieves max. on the boundary $\partial\Omega$

- Proof:

  $\epsilon > 0 \qquad v_\epsilon(x) = u + \epsilon(x^2 + y^2)$

  Let x_0 be such that $\sup_x v_\epsilon(x) = v_\epsilon(x_0)$

  Assume x_0 is in interior:

  $\frac{\partial^2}{\partial x^2} v_\epsilon \leq 0, \ \frac{\partial^2}{\partial r^2} v_\epsilon \leq 0$

  $\underbrace{\Delta(v_3)(x_0)}_{\leq 0} = \underbrace{\Delta u(x_0)}_{0} + \underbrace{4\epsilon}_{>0}$

- There is a contradiction between LHS and RHS, therefore maximum cannot be an interior.

Ball:

- Problem:

$$\Delta u = 0 \text{ on Ball with } r = 1$$

- Separation Ansatz:

$$u(r,\theta) = R(r)\Theta(\theta)$$
$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

$$\frac{r^2 R_{rr} + rR_r}{R} = -\frac{\Theta''}{\Theta} = \lambda = n^2$$

$$\frac{\Theta''}{\Theta} = -n^2 \qquad \rightarrow \Theta(\theta) = e^{in\theta}$$

$$r^2 R_{rr} + rR_r = n^2 R$$
Assume $R(r) = r^n$
$$n(n+1)r^n + nr^n = n^2 r^n$$

- Solution:

$$
\begin{aligned}
u(r,\theta) &= \sum_{n\in\mathbb{Z}} a_n r^{|n|} e^{in\theta} \\
u(e^{i\theta}) &= \sum a_n e^{in\theta} \\
a_n &= \int e^{-in\theta} u(e^{i\theta}) \frac{d\theta}{2\pi}
\end{aligned}
$$

LOS 8. Some remarks on harmonic extensions and annulus circular region

Theorem 1:

$$
\begin{aligned}
&\text{Let be } u|_{\partial D} \text{ a continuous function} \\
&\exists \text{ a unique function on } D \text{ such that} \\
&u|_{\partial\Omega} = u \\
&\Delta u = 0 \text{ in } D
\end{aligned}
$$

Lemma 1:

$$
f_r(\theta) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} = \frac{1-r^2}{(1-r\cos(\theta)) + r^2\sin^2(\theta)} \geq 0
$$

Proof:

$$
\begin{aligned}
\text{LHS} &= 1 + \sum_{i}^{\infty} r^n e^{in\theta} + \sum_{i}^{\infty} r^n e^{-in\theta} \\
&= 1 + \frac{re^{i\theta}}{1-re^{i\theta}} + \frac{re^{-i\theta}}{1-re^{-i\theta}} \\
&= \frac{(1-re^{-i\theta})(1-re^{i\theta}) + re^{i\theta}(1-re^{-i\theta}) + re^{-i\theta}(1-re^{i\theta})}{(1-re^{-i\theta}(1-re^{i\theta}} \\
&= \frac{1+r^2-2r^2}{(1-r\cos(\theta)) + r^2\sin^2(\theta)} \geq 0
\end{aligned}
$$

Lemma 2:

$$
P_r(g) = \int_0^{2\pi} f_r(\theta-\eta) g(\eta) \frac{d\eta}{2\pi}
$$

1. P_r is linear

2. $g \geq 0 \to P_r(g) \geq 0$

3. $\|P_r(g)\|_\infty \leq \|g\|_\infty$

   Proof:

   $$|P_r(g)(\theta)| \leq \int |f_r(\theta-\eta)||g(\eta)|\frac{d\eta}{2\pi} \leq \|g\|_\infty \int_0^{2\pi} f_r(\theta-\eta)\frac{d\eta}{2\pi} = \|g\|_\infty$$

Theorem 2:

- Let u be a continuous on ∂D, then u has a unique extension in the interior with $\Delta u = 0$

- Proof:

$$\begin{aligned} g(\theta) &= u(e^{i\theta}) \\ u(re^{i\theta}) &= \sum_{n\in\mathbb{Z}} \hat{g}(n) r^{|n|} e^{in\theta} \qquad \to \Delta u = 0 \end{aligned}$$

- Claim:

$$lim_{r\to 1} u(re^{i\theta}) = g(\theta) \qquad \text{where } u_0(re^{i\theta}) = g(\theta)$$

- Remarks: claim is obvious if $g(\theta) = \sum_{-m}^{m} \hat{g}(n) r^{|n|} e^{in\theta}$ is a trigonometric polynomials (has finitely many Fourier coefficients)

- By Weierstrass approximation:

$$\forall \epsilon \; \exists q \text{ trigonometric polynomial such that } \|g-q\| \leq \frac{\epsilon}{3}$$

  Let r_0 so that $\forall r > r_0$:

$$\sup_\theta \|P_r(q)(\theta) - q(\theta)\| \leq \frac{\epsilon}{3}$$
$$\sup_\theta \|q(rei\theta) - q(\theta)\| \leq \frac{\epsilon}{3}$$

  Then:
$$\|P_r(g) - g\| \leq \underbrace{\|P_r(g) - q\|_\infty}_{<\frac{\epsilon}{3}} + \underbrace{\|g-q\|_\infty}_{<\frac{\epsilon}{3}} + \underbrace{\|P_r(q) - g\|_\infty}_{<\frac{\epsilon}{3}} \leq \epsilon$$

- Therefore, $P_r(g)$ converges uniformly to g

Solution operator:

- Problem:

$$u_t = -\sqrt{-\Delta}u$$

- Solution:

$$\begin{aligned}
&u(t,x) = S_t(u_0)(x) \\
&S_t = P_{e^{-t}} \qquad \text{Solution operator on } [-\pi,\pi] \\
&S_t(g) = \sum_{n\in\mathbb{Z}} \hat{g}(n)\underbrace{(e^{-t})}_{\text{radius}}{}^{|n|}e^{in\theta}
\end{aligned}$$

- Remarks: $S_t(u_0)$ satisfies wave equation:

$$\begin{aligned}
u_t &= -\sqrt{-\Delta}u \\
u_{tt} &= -\sqrt{-\Delta}u_t \\
u_{tt} &= -\sqrt{-\Delta}(-\sqrt{-\Delta}u) \\
u_{tt} &= -\Delta u
\end{aligned}$$

- Corollary (mean value property):

$$\begin{aligned}
&\text{Let } u \text{ be harmonic} \\
&\text{Then } u(z_0) = \int_0^{2\pi} u(z_0 + re^{i\theta})\frac{d\theta}{2\pi}
\end{aligned}$$

- Without loss of generality:

$$\begin{aligned}
&z_0 = 0,\ r = 1 \quad \text{where } z_0 \subseteq \Omega \\
&g(\theta) = u(e^{i\theta}) \text{ is continuous}
\end{aligned}$$

By theorem 2, we only have one unique extension:

$$\tilde{u}(re^{i\theta}) = \sum_{n\in\mathbb{Z}} \hat{g}(n)r^{|n|}e^{in\theta}$$

By uniqueness $\tilde{u} = u$:

$$\tilde{u}(0) = \hat{g}(0) = \int_0^{2\pi} g(\theta)\frac{d\theta}{2\pi} = \int_0^{2\pi} u(e^{i\theta})\frac{d\theta}{2\pi}$$

- Conclusion: condition in the interior can be derived from condition on the boundary

Annulus (Donut):

- Problem:

$$u_t = \Delta u$$
$$u|_{\partial \text{ annulus}} = u_0$$

- Solution:

$$u(0,x) = u_0(x)$$
$$u(t, re^{i\theta}) = \sum_n a_n e^{-t|n|^2} r^{|n|} e^{in\theta} + \sum_n a_n e^{-t|n|^2} r^{-|n|} e^{in\theta} + C + D\log r$$

log term is comes from descent method

- Steps:

1. Pretend to solve initial value problem
2. Solve with solution formula
3. Find $u(0,x)$ by unique harmonic extension