Unit 1

LOS 1. Differentiate between linear and nonlinear PDEs

Characteristic of linear PDE:

1. $\mathcal{L}(u+v) = \mathcal{L}u + \mathcal{L}v$
2. $\mathcal{L}(cu) = c\mathcal{L}u$

Consequence of linearity:
Solution to homogeneous + solution to inhomogeneous = another solution to inhomogeneous

LOS 2. Distinguish between ODEs and PDEs

In general, ODE involves one dependent variable y which is a function of x. On the other hand, PDE involves one dependent u which is a function multiple independent variables $x, t, ...$.

Example of ODE:

$$\frac{\partial y}{\partial x} + y = 0$$

Example of PDE:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$
$$u_x + u_t = 0$$

LOS 3. Identify the degree of a given PDEs

Degree of PDE is equal to the highest degree of derivative in the PDE.

Degree 1: $u_x + u_t = 0$
Degree 2: $u_{xx} + u_t = 0$
Degree 3: $u_{xxx} + u_t = 0$
...

LOS 4. Solve first order PDE using geometric method for constant coefficients

First order PDE with constant coefficients:

$$u_x + cu_y = 0$$

General Solution:

$$u(x, y) = f(cx - y)$$

LOS 5. Solve first order PDE using geometric method for variable coefficients

First order PDE with variable coefficients:

$$u_x + yu_y = 0$$

General Solution:

$$u(x, y) = \phi(ye^x)$$

Example:

$$\begin{gathered} u_x + yu_y = 0, u(x, 1) = x^2 \\ u(x, y) = \phi(ye^{-x}) \\ u(x, 1) = \phi(e^{-x}) = x^2 \\ x = -\ln t \to \phi(t) = (-\ln t)^2 \\ \therefore u(x, y) = (-\ln ye^{-x})^2 = (-\ln y + x)^2 \end{gathered}$$

LOS 6. Distinguish between parabolic, hyperbolic and elliptic PDEs

In this course, at maximum, we will have second order PDE. General formula of second order PDE:

$$\begin{gathered} a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0 \\ a_{12} = a_{21} \\ u_{xy} = u_{yx} \end{gathered}$$

Type of PDE depends on determinant $\mathscr{D} = a_{12}^2 - a_{11}a_{22}$:

1. Elliptic: $\mathscr{D} < 0$, all eigenvalues are positive

2. Hyperbolic: $\mathscr{D} > 0$, none of the eigenvalues vanish, one eigenvalue has the opposite sign from the $(n-1)$ others

3. Parabolic: $\mathscr{D} = 0$, one zero eigenvalues and the rest have the same signs

LOS 7. Identify different types of initial and boundary conditions

Types of initial and boundary conditions:

- (D) Dirichlet: $u(0) = \alpha, u(L) = \beta$
- (I) Initial: $u(0) = \alpha, u'(0) = \beta$
- (N) Neumann: $u_x(0) = \alpha, u_x(L) = \beta$
- (R) Robin: $\beta u_x + \alpha u(0) = 0$, linear combination of other initial or boundary conditions

LOS 8. Analyze a PDE for being well-posed: existence, uniqueness and stability

Well-posedness includes:

- Existence: there exist a solution that can be expressed explicitly, solution provided by boundary conditions satisfy the PDE
- Uniqueness: the solution is not dependent on auxillary variables and the same across the defined range
- Stability: when data are changed very little, the corresponding solution also change very little

LOS 9. Solve wave equation using the Fourier method and the operator method

Wave equation:

$$u_{xx} - u_{tt} = 0$$

Fourier Method:

$$u_{\xi,\eta} = e^{i\xi t} e^{i\eta x} \qquad \xi, \eta \in \mathbb{R}$$
$$\mathcal{L} u_{\xi,\eta} = (-\xi^2 + \eta^2) u_{\xi,\eta}$$
$$u(x,t) = \phi(x+t) - \psi(x-t)$$
$$\phi(x) = \sum a_k e^{i\xi_k x} \qquad \psi(x) = \sum b_j e^{i\xi_j x} \qquad a_k, b_j \in \mathbb{C}$$

Operator Method:

- Expand the PDE:

$$\partial_{tt} - \partial_{xx} = (\partial_t - \partial_x)(\partial_t + \partial_x)$$
$$y = (\partial_t - \partial_x)(\partial_t + \partial_x)$$

- Assume for solution $u(x,t)$ there is a function v which satisfies the following:

$$v \mid v(x,t) = (\partial_t + \partial_x)u(x,t)$$
$$(\partial_t - \partial_x)v(x,t) = 0$$
$$v_x - v_t = 0$$
$$v(x,t) = f(x+t)$$
$$(\partial_t + \partial_x)u(x,t) = f(x+t)$$

- By linearity:

$$u = u^0 + w$$
$$(\partial_t + \partial_x)u^0 = f(x+t) \qquad (\partial_t + \partial_x)w = 0$$

- By method of characteristic line:

$$(\partial_t + \partial_x)w = 0$$
$$w_t + w_x = 0$$
$$w(x,t) = g(x-t)$$

- Substitute $z = x + t$ and assume $h' = f$:

$$(\partial_t + \partial_x)u^0 = f(x+t)$$
$$\frac{d}{dz}u^0(z) = f(z)$$
$$u^0(z) = h(z)$$
$$u^0(x+t) = \frac{h(x+t)}{2} \because (\partial_t + \partial_x)u^0 = \frac{h'(x+t)}{2} + \frac{h'(x+t)}{2} = f(x,t)$$

- Substitute to the original equation:

$$u(x,t) = \frac{h(x+t)}{2} + g(x-t)$$

LOS 10. Learn the principle of causality

General solution to wave equation:

$$u_{xx} - u_{tt} = 0$$
$$u(x,0) = \phi(x)$$
$$u_t(x,0) = \psi(x)$$
$$u(x,t) = f(x+t) + g(x-t) \qquad f, g \in C^2$$

Example with Dirichlet condition:

- Problem statement:

$$\begin{aligned} u_{xx} - u_{tt} &= 0 \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= 0 \end{aligned}$$

- Evaluate the boundary conditions:

$$\begin{aligned} u(x,t) &= f(x+t) + g(x-t) \\ u(x,0) &= f(x) + g(x) = \phi(x) \qquad (1) \\ u_t(x,0) &= f'(x+t) - g'(x-t) = f'(x) - g'(x) = 0 \qquad (2) \end{aligned}$$

- Integrate (2):

$$\begin{aligned} f'(x) + g'(x) &= 0 \\ f(x) &= g(x) + c \end{aligned}$$

- From (1) and assume $c = 0$:

$$\begin{aligned} f(x) + g(x) &= \phi(x) \\ f(x) + f(x) - c &= \phi(x) \\ 2f(x) &= \phi(x) + c \\ f(x) &= \phi(x)/2 = g(x) \\ \therefore u(x,t) &= \frac{\phi(x+t) + \phi(x-t)}{2} \end{aligned}$$

- From (1) and assume $c = 0$:

$$\begin{aligned} \phi(x) &= u(x,0) \\ \therefore u(x,t) &= \frac{u(x+t,0) + u(x-t,0)}{2} \end{aligned}$$

The above equation implies causality. The value of u at point t can be predicted if we know exactly the points at $t = 0$, $u(x+t,0)$ and $u(x-t,0)$.

LOS 11. Theorem on solution for wave equation

Example with Neumann condition:

- Problem statement:

$$
\begin{gathered}
u_{xx} - u_{tt} = 0 \\
u(x,0) = 0 \\
u_t(x,0) = \psi(x)
\end{gathered}
$$

- Evaluate the boundary conditions:

$$
\begin{gathered}
u(x,t) = f(x+t) + g(x-t) \\
u(x,0) = f(x) + g(x) = 0 \implies f = -g \\
u_t(x,0) = f'(x+t) - g'(x-t) \to 2f'(x) = \psi(x) \\
f(x) = \frac{1}{2}\int_{-\infty}^{x} \psi(y)dy \\
\therefore u(x,t) = \frac{1}{2}\left(\int_{-\infty}^{x+t} \psi(y)dy + \int_{-\infty}^{x-t} \psi(y)dy\right) = \frac{1}{2}\int_{[x+t,x-t]} \psi(y)dy
\end{gathered}
$$

Therefore, the general solution for the wave equation:

$$
\begin{gathered}
u_{xx} - u_{tt} = 0 \\
u(x,0) = \phi(x) \\
u_t(x,0) = \psi(x) \\
u(x,t) = \frac{\phi(x+t) + \phi(x-t)}{2} + \frac{1}{2}\int_{[x+t,x-t]} \psi(y)dy
\end{gathered}
$$

Alternative formulation (S^1 is solution operator for Dirichlet condition and S^2 is solution operator for Neumann condition and $\dot{S}$ is the derivative):

$$
\begin{gathered}
S^1(\phi)(x,t) = \frac{\phi(x+t) + \phi(x-t)}{2} \\
S^1(\phi)(x,0) = \phi(x) = Id \\
\dot{S}^1(\phi)(x,0) = 0
\end{gathered}
$$

$$
\begin{gathered}
S^2(\psi)(x,t) = \frac{1}{2}\int_{[x+t,x-t]} \psi(y)dy \\
S^2(\psi)(x,0) = 0 \\
\dot{S}^2(\psi)(x,0) = \psi(x) = Id
\end{gathered}
$$

$$
\therefore S(\phi,\psi) = S^1(\phi) + S^2(\psi)
$$