Unit 2

LOS 1. Solve inhomogeneous wave equation

Problem:

$$u_{tt} - u_{xx} = f$$
$$u(x,0) = \phi(x)$$
$$u_t(x,0) = \psi(x)$$

Solution:

$$u(x,t) = \frac{\phi(x+t)+\phi(x-t)}{2} + \frac{1}{2}\int_{[x+t,x-t]} \psi(y)dy + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)dyds$$

LOS 2. Revisit method for solving second order ODEs (homogeneous as well as inhomogeneous)

Solve for first order homogeneous equation:

- Problem statement:

$$f : [a,b] \to \mathbb{R}$$
$$f'(t) - Af(t) = 0$$
$$f(0) = v_0$$

- Solution (S is a solution operator):

$$f(t) = e^{tA}v_0$$
$$f'(t) = Ae^{tA}v_0 = Af(t)$$
$$\therefore S(t) = e^t A$$

Solve for first order inhomogeneous equation:

- Problem statement:

$$f'(t) - Af(t) = g$$
$$f(0) = v_0$$

- Solution:

$$f(t) = \int_0^t e^{(t-s)A}g(s)ds$$

- Using Liebniz integration rule to evaluate $f'(t)$:

$$\frac{d}{dx}\left(\int_{b(x)}^{a(x)} g(s)ds\right) = g(a(x))a'(x) - g(b(x))b'(x) + \int_{b(x)}^{a(x)} \frac{\partial}{\partial x} g(s)ds$$
$$\frac{d}{dx}\left(\int_{0}^{t} e^{(t-s)A} g(s)ds\right) = g(t) + \int_{b(x)}^{a(x)} \frac{\partial}{\partial t} e^{(t-s)A} g(s)ds$$
$$\frac{d}{dx}\left(\int_{0}^{t} e^{(t-s)A} g(s)ds\right) = g(t) + A\int_{b(x)}^{a(x)} e^{(t-s)A} g(s)ds$$
$$\frac{d}{dx}\left(\int_{0}^{t} e^{(t-s)A} g(s)ds\right) = g(t) + Af(t)$$
$$f'(t) = g(t) + Af(t)$$

Solve for second order differential equation:

- Problem statement:

$$f''(t) - Af(t) = 0$$
$$f(0) = v_0$$

- Assume solution takes the form $e^{i\xi t}$:

$$f(t) = e^{i\xi t}$$
$$f''(t) = \xi^2 e^{i\xi t} = Af(t)$$
$$-\xi^2 e^{i\xi t} = Ae^{i\xi t}$$
$$\therefore \xi = \pm i\sqrt{A}$$

- Solutions:

$$f(t) = e^{i\sqrt{A}t} \qquad f(t) = e^{-i\sqrt{A}t}$$
$$S_+(t) = e^{i\sqrt{A}t} \qquad S_-(t) = e^{-i\sqrt{A}t}$$
$$S(t) = C_1 e^{i\sqrt{A}t} + C_2 e^{-i\sqrt{A}t}$$

- In Dirichlet problem:

$$f''(t) - Af(t) = 0$$
$$f(0) = v_0$$
$$f'(0) = 0$$

$$S_D(t) = f(t) = C_1 e^{i\sqrt{A}t} + C_2 e^{-i\sqrt{A}t}$$
$$f(t) = C_1(cos\sqrt{A}t + isin\sqrt{A}t) + C_2(cos\sqrt{A}t - isin\sqrt{A}t)$$
$$f(t) = (C_1 + C_2)(cos\sqrt{A}t) + i(C_1 - C_2)(sin\sqrt{A}t)$$
$$f(0) = (C_1 + C_2)(1) + 0 = v_0$$
$$(C_1 + C_2) = v_0$$

$$\dot{S}_D(t) = f'(t) = v_0\sqrt{a}(sin\sqrt{A}t) + i(C_1 - C_2)\sqrt{A}(sin\sqrt{A}t)$$
$$f'(0) = 0 + i(C_1 - C_2)\sqrt{A} = 0$$
$$(C_1 - C_2) = 0$$

$$\therefore f(t) = v_0cos\sqrt{a}t \to S_D(t) = cos\sqrt{a}t$$

- In Neumann problem:

$$f''(t) - Af(t) = 0$$
$$f(0) = 0$$
$$f'(0) = v_1$$

$$S_N(t) = (C_1 + C_2)(cos\sqrt{A}t) + i(C_1 - C_2)(sin\sqrt{A}t)$$
$$f(0) = (C_1 + C_2)(1) + 0 = 0$$
$$(C_1 + C_2) = 0$$

$$\dot{S}_N(t) = f'(t) = i(C_1 - C_2)\sqrt{A}(sin\sqrt{A}t)$$
$$f'(0) = 0 + i(C_1 - C_2)\sqrt{A} = v_1$$
$$(C_1 - C_2) = \frac{v_1}{\sqrt{A}}$$

$$\therefore f(t) = \frac{v_1 sin\sqrt{A}t}{\sqrt{A}} \to S_N(t) = \frac{sin\sqrt{A}t}{\sqrt{A}}$$

- Solution for inhomogeneous problem:

$$f''(t) - Af(t) = g(t)$$
$$f(0) = v_0$$
$$f'(0) = v_1$$
$$f(t) = S_D(t)v_0 + S_N(t)v_1 + \int_0^t S_N(t-s)g(s)ds$$

LOS 3. Solve inhomogeneous wave equation using Green's formula and the operator method

Solving using Green's formula:

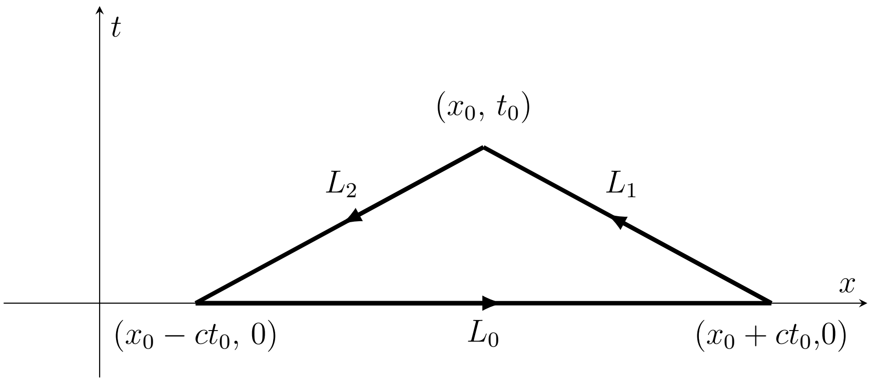
- Evaluate integral of the original equations:

$$u_{tt} - c^2u_{xx} = f(x,t)$$
$$\iint_\Delta (u_{tt} - c^2u_{xx}) = \iint_\Delta f$$

- Green's theorem:

$$\iint_{\Delta} (P_x - Q_t) dx dt = \oint_{\Delta} (P dt + Q dx)$$
$$\iint_{\Delta} (-c^2 u_{xx} - u_{tt}) = \int_{L_0+L_1+L_2} (-c^2 u_x dt + u_t dx)$$

Example, wave equation without reflection:



- On L_0:

$$t = 0$$
$$dt = 0$$
$$u_t(x, 0) = \psi(x)$$
$$\therefore \int_{L_0} = \int_{L_0} (0 - u_t(x, 0) dx) = -\int_{x_0-ct_0}^{x_0+ct_0} \psi(x) dx$$

- On L_1:

$$x + ct = x_0 + ct_0$$
$$dx + cdt = 0 \to dt = \frac{-dx}{c}, dx = -cdt$$
$$-c^2 u_x dt - u_t dx = cu_x dx + cu_t dt = cdu$$
$$\therefore \int_{L_1} = c \int_{L_1} du = c(u(x_0, t_0) - u(x_0 + ct_0, 0)) = cu(x_0, t_0) - c\phi(x_0 + ct_0)$$

- On L_2:

$$x - ct = x_0 - ct_0$$
$$dx - cdt = 0 \to dt = \frac{dx}{c}, dx = cdt$$
$$-c^2 u_x dt - u_t dx = -cu_x dx - cu_t dt = -cdu$$
$$\therefore \int_{L_2} = -c \int_{L_2} du = -c(u(x_0 - ct_0, 0) - u(x_0, t_0)) = -c\phi(x_0 - ct_0) + cu(x_0, t_0)$$

- Solution:

$$\iint_\Delta f = -\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)dx + 2cu(x_0,t_0) - c\phi(x_0+ct_0) - c\phi(x_0-ct_0)$$

$$u(x_0,t_0) = \frac{1}{2c}\iint_\Delta f + \frac{1}{2}[\phi(x_0+ct_0) - \phi(x_0-ct_0)] - \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)dx$$

Solving using operator method:

- By ODE analogy, solution takes the following form:

$$u(t) = S_D(t)v_0 + S_N(t)v_1 + \int_0^t S_N(t-s)f(s)ds$$

- Define source operator, $\mathscr{L}$ such that it solve the Neumann problem:

$$u_{tt} - u_{xx} = f \qquad u(x,0) = 0 \qquad u_t(x,0) = \psi(x)$$

$$\mathscr{L}(t)\psi = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy$$

- Solution for Dirichlet problem:

$$u_{tt} - u_{xx} = f \qquad u(x,0) = \phi(x) \qquad u_t(x,0) = 0$$

$$\frac{\partial}{\partial t}\mathscr{L}(t)\phi = \frac{\partial}{\partial t}\frac{1}{2c}\int_{x-ct}^{x+ct} \phi(y)dy = \frac{1}{2}[\phi(x+ct) - \phi(x-ct)]$$

- Solution for inhomogeneous problem:

$$u_{tt} - u_{xx} = f \qquad u(x,0) = 0 \qquad u_t(x,0) = 0$$

$$u(t) = \int_0^t \left[\frac{1}{2c}\int_{x-(t-s)}^{x+(t-s)} f(y,s)dy\right] ds = \frac{1}{2c}\iint_\Delta f dxdt$$

LOS 4. Learn the Duhamel principle

Using Duhamel's principle we can get the following solution to the wave equation:

$$u_{tt} - c^2u_{xx} = f(x,t)$$
$$u(x,0) = \phi(x)$$
$$u_t(x,0) = \psi(x)$$

$$u(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds$$

Duhamel's principle states that if you can solve the homogeneous equation, you can also solve the inhomogeneous equation:

- Given the following problem:

$$u_{tt} - c^2 u_{xx} = f(x,t)$$
$$u(x,0) = \phi(x)$$
$$u_t(x,0) = \psi(x)$$

- We can solve the following problem to get the inhomogeneous solution:

$$u_{tt} - c^2 u_{xx} = f(x,t)$$
$$u(x,0) = 0$$
$$u_t(x,0) = f(x,t)dt$$

- Where the solution is as follow:

$$u(x,t) = \int \frac{\partial}{\partial t} f(x,t)dt$$

LOS 5. Study well-poseness of wave equation

Conditions for well-poseness:

1. Existence: solution has an explicit formula
2. Uniqueness: solution using different methods are the same
3. Stability: use norms

Wave equation norms:

1. $|u_D(x,t)| \leq \|\phi\|_\infty$ for all x, t
   $\to \sup_x |u_D(x,t)| \leq \|\phi\|_\infty$

2. $|u_N(x,t)| = |\frac{1}{2}\int_{x-ct}^{x+ct} \psi(y)dy| \leq \frac{1}{2}\int_{x-ct}^{x+ct} |\psi(y)|dy \leq t\|\psi\|_\infty$
   $\to \sup_x |u_N(x,t)| \leq t\|\psi\|_\infty$
   $\to \sup_{x,t} |u_N(x,t)| \leq \frac{\|\psi\|_1}{2}$

3. $|u_{source}(x,t)| \leq \frac{1}{2}\int_{x-(t-s)}^{x+(t-s)} dsdy\|g\|_\infty$
   $= \int_0^t (t - sds)\|g\|_\infty = \int_0^t sds\|g\|_\infty \leq \frac{t^2}{2}\|g\|_\infty$
   $\to |u_{source}(x,t)| \leq \frac{t^2}{2}\|g\|_\infty$

LOS 6. Learn how to calculate different norms

Norms:

1. p-norm:

$$\|\phi\|_P = \left(\int_{\mathbb{R}} |\phi(x)|^P dx \right)^{\frac{1}{P}}$$

2. infinity-norm/sup-norm/uniform-norm:

$$\|\phi\|_\infty = \sup_x |\phi(x)|$$
$$\|\phi\| = \max_x |\phi(x)|$$
$$\|\phi\|_T = \max_{x, 0 \le t \le T} |\phi(x,t)|$$

*(sup refers to the smallest upper bound of the set)

3. 1-norm:

$$\|\phi\|_1 = \int_{\mathbb{R}} |\phi(x)| dx$$

4. l^2-norm:

$$\|\phi\|^2 = \int_{\mathbb{R}} |\phi(x)|^2 dx$$
$$\|\phi\| = \sqrt{\int_{\mathbb{R}} |\phi(x)|^2 dx}$$

LOS 7. Study reflection of waves

Lemma:

1. ϕ odd $\implies u_D$ odd, ϕ even $\implies u_D$ even
2. ψ odd $\implies u_N$ odd, ψ even $\implies u_N$ even

Wave-equation homogeneous problem on the half-line:

- Problem:

$$u_{tt} - u_{xx} = 0 \qquad \text{for } 0 < x < \infty$$
$$u(x,0) = \phi(x)$$
$$u_t(x,0) = \psi(x)$$
$$u(0,t) = 0$$

- Use odd extension:

$$\phi_{odd}(x) = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x > 0 \\ 0 & x = 0 \end{cases}$$

$$\psi_{odd}(x) = \begin{cases} \psi(x) & x > 0 \\ -\psi(-x) & x > 0 \\ 0 & x = 0 \end{cases}$$

- Solution:

$$u(x,t) = \frac{\phi_{odd}(x+ct) + \phi_{odd}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(y)dy$$

$$u(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy \qquad \text{if } x > c|t|$$

$$u(x,t) = \frac{\phi(ct+x) - \phi(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(y)dy \qquad \text{if } x < c|t|$$

Wave-equation inhomogeneous problem on the half-line:

- Problem:

$$\begin{aligned} u_{tt} - u_{xx} &= f(x,t) \qquad \text{for } 0 < x < \infty \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \\ u(0,t) &= h(t) \end{aligned}$$

- Solution (reflected once):

$$u(x,t) = u_D(x,t) + u_N(x,t) + h(t - \frac{x}{c}) + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)dyds$$

LOS 8. Find the periodic solutions for the wave equation

Wave-equation homogeneous problem on the finite interval:

- Problem:

$$\begin{aligned} u_{tt} - u_{xx} &= f(x,t) \qquad \text{for } 0 < x < l \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \\ u(0,t) &= h(t) \end{aligned}$$

- Use periodic extension:

$$\phi_{ext}(x) = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x > 0 \\ \text{extended to be of period 2l} \end{cases}$$

$$\psi_{odd}(x) = \begin{cases} \psi(x) & x > 0 \\ -\psi(-x) & x > 0 \\ \text{extended to be of period 2l} \end{cases}$$

- Solution:

$$u(x,t) = \frac{\phi_{ext}(x+ct) + \phi_{ext}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(y)dy$$

Wave-equation inhomogeneous problem on the half-line:

- Problem:

$$u_{tt} - u_{xx} = f(x,t) \qquad \text{for } 0 < x < l$$
$$u(x,0) = \phi(x) \to 0$$
$$u_t(x,0) = \psi(x) \to 0$$
$$u(0,t) = h(t) \qquad u(l,t) = k(t)$$

- Solution:

$$u(x,t) = \sum_{n=0}^{O_{odd}-1} h\left(t - \frac{x}{c} - \frac{2nl}{c}\right) - \sum_{n=0}^{O_{even}-1} h\left(t - \frac{l-x}{c} - \frac{(2n+1)l}{c}\right)$$
$$- \sum_{n=0}^{L_{odd}-1} k\left(t - \frac{x}{c} - \frac{(2n+1)l}{c}\right) + \sum_{n=0}^{L_{even}-1} k\left(t - \frac{l-x}{c} - \frac{2nl}{c}\right)$$

1. O_{odd}: number of odd reflections at $x = 0$
2. O_{even}: number of even reflections at $x = 0$
3. L_{odd}: number of odd reflections at $x = l$
4. L_{even}: number of even reflections at $x = l$

LOS 9. Learn inner product spaces and its properties

Define v as a vector in $\mathbb{C}$ such that:

1. $(v + w) + z = v + (w + z)$
2. $\lambda(v + w) = \lambda v + \lambda w$
3. $(v, w + \lambda z) = (v, w) + \lambda(v, z)$
4. $\overline{(v, w)} = (w, v)$, $b = \alpha + i\beta \rightarrow \bar{b} = \alpha - i\beta$
5. $(v, v) \geq 0$
6. $(v, v) = 0 \iff v = 0$

Inner product definitions:

- For $v = \mathbb{C}$:

$$(v, v) = \bar{v}v \geq 0$$

- For $v = \mathbb{C}^n$:

$$(v, w) = \sum_{j=1}^{n} \bar{v_j} w_j$$
$$(v, v) = \sum_{j=1}^{n} |v_j| = \|v\|^2$$

- For $v = \mathbb{R}^n$:

$$(v, w) = \sum v_j w_j$$
$$(v, v) = v \cdot v \geq 0$$

Examples:

- For $v = C[0, 1]$:

$$(f + \lambda g)(t) = f(t) + \lambda g(t)$$
$$(f, g) = \int_0^1 \overline{f(t)} g(t) dt$$

- For $v = C[0, 2\pi]$:

$$(f, g) = \int_0^{2\pi} \overline{f(t)} g(t) \frac{dt}{2\pi}$$

Cauchy-Swarsch:

- $\|v\| = (v,v)^{\frac{1}{2}}$
- $\|v+w\| \leq \|v\| + \|w\|$
- $v \perp w$ if $(v,w) = 0 \Leftrightarrow \|v+w\| = \|v\| + \|w\|$
- $|(v,w)| \leq \|v\|\|w\|$

LOS 10. Understand the orthogonal and orthonormal families of functions

Orthogonal and orthonormal family:

1. Orthogonal $\Leftrightarrow (v_j, v_{j'}) = 0 \quad \forall j \neq j'$
2. Orthonormal $\Leftrightarrow (v_j, v_{j'}) = 0 \quad \forall j \neq j'$ and $(v_j, v_{j'}) = 1 \quad \forall j = j'$

Let (v_j) $\forall j$ be an orthogonal family:

$$\left\| \sum \lambda_j v_j \right\|^2 = \sum_j |\lambda_j|^2$$

Proposition:

- Let $(v_j)_{j=1}^n$ be orthonormal.
- Let $W = \{\sum_{j=1}^n \lambda_j v_j \ : \ \lambda_j \in \mathbb{C}\}$.
- Let $x \notin W$. Define $P(x) = \sum_{j=1}^n (v_j, x) v_j$
- Then $\inf \|x - y\| = \|x - P_x\|, y \in W$