Unit 5

LOS 1. Compare the solutions for continuous case vs periodic case for a given PDE

Continuous case:

- Problem:

$$u_t = u_{xxxxxx} \qquad -\infty < x < \infty$$

$$u(x,0) = \phi(x) = \int_{\mathbb{R}} e^{i\xi x}\hat{\phi}(k)$$

$$P(x) = x^6$$

- Assume t is fixed:

$$\frac{d}{dt}\hat{u}(\xi,t) = (i\xi)^6\hat{u}(\xi,t)$$

$$\hat{u}(\xi,t) = e^{-t\xi^6}\hat{u}(\xi,0)$$

- By Fourier Inversion formula:

$$u(x,t) = \int_{\mathbb{R}} e^{i\xi x}\hat{u}(\xi,t)\frac{d\xi}{\sqrt{2\pi}}$$

$$u(x,t) = \int_{\mathbb{R}} e^{i\xi x}e^{t\xi^6}\hat{u}(\xi,0)\frac{d\xi}{\sqrt{2\pi}} \qquad (1)$$

- By Fourier expansion:

$$f(\xi) = e^{\xi^6} = \int_{\mathbb{R}} e^{i\xi z}\hat{f}(z)\frac{dz}{\sqrt{2\pi}}$$

$$e^{\xi^6} = \int_{\mathbb{R}} e^{-i\xi z}\hat{f}(z)\frac{dz}{\sqrt{2\pi}} \qquad \text{since } \hat{f} \text{ is even}$$

$$e^{t\xi^6} = e^{(t^{\frac{1}{6}}\xi)^6} = \int_{\mathbb{R}} e^{-it^{\frac{1}{6}}\xi z}\hat{f}(z)\frac{dz}{\sqrt{2\pi}}$$

- By change of variable:

$$y = t^{\frac{1}{6}}z$$

$$e^{t\xi^6} = \int_{\mathbb{R}} e^{-i\xi y} \hat{f}\left(\frac{y}{t^{\frac{1}{6}}}\right) \frac{dy}{t^{\frac{1}{6}}\sqrt{2\pi}}$$

$$h_t(y) = \frac{1}{t^{\frac{1}{6}}\sqrt{2\pi}} \hat{f}\left(\frac{y}{t^{\frac{1}{6}}}\right)$$

$$\therefore e^{t\xi^6} = \int_{\mathbb{R}} e^{-i\xi y} h_t(y) dy$$

- Substitute back to Equation (1):

$$u(x,t) = \int_{\mathbb{R}} e^{i\xi x} e^{t\xi^6} \hat{u}(\xi,0) \frac{d\xi}{\sqrt{2\pi}}$$

$$u(x,t) = \int_{\mathbb{R}} e^{i\xi x} \int_{\mathbb{R}} e^{-i\xi y} h_t(y) dy \; \hat{u}(\xi,0) \frac{d\xi}{\sqrt{2\pi}}$$

$$u(x,t) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{i\xi(x-y)} \hat{u}(\xi,0) \frac{d\xi}{\sqrt{2\pi}}\right] h_t(y) dy$$

$$u(x,t) = \int_{\mathbb{R}} u(x-y,0) h_t(y) dy$$

$$u(x,t) = \int_{\mathbb{R}} \phi(x-y) h_t(y) dy$$

Periodic case:

- Problem:

$$u_t = u_{xxxxxx} \qquad -\pi < x < \pi$$

$$u(x,0) = \phi(x) = \sum_{k\in\mathbb{Z}} e^{ikx} \hat{\phi}(\xi)$$

$$P(x) = x^6$$

- Assume t is fixed:

$$\frac{d}{dt}\hat{u}(k,t) = (ik)^6 \hat{u}k,t$$

$$\hat{u}(k,t) = e^{-tk^6}\hat{u}(k,0)$$

- By Fourier Inversion formula:

$$u(x,t) = \sum_{k\in\mathbb{Z}} e^{-tk^6}\hat{u}(k,0)e^{ikx} \qquad (1)$$

- Let:

$$H_t(x) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{-tk^6} e^{ikx}$$

$$e^{-tk^6} = \hat{H_t}(x) = \int_{-\pi}^{\pi} e^{-iky} H_t(y) dy$$

- Substitute back to Equation (1):

$$u(x,t) = \sum_{k \in \mathbb{Z}} \left[\int_{-\pi}^{\pi} e^{-iky} H_t(y) dy \right] \hat{u}(k,0) e^{ikx}$$

$$u(x,t) = \int_{-\pi}^{\pi} \left[\sum_{k \in \mathbb{Z}} e^{ik(x-y)} \hat{u}(k,0) \right] H_t(y) dy$$

$$u(x,t) = \int_{-\pi}^{\pi} u(x-y,0) H_t(y) dy$$

$$u(x,t) = \int_{-\pi}^{\pi} \phi(x-y) H_t(y) dy$$

In general:

- Upper half plane (continuous):

$$u(x,t) = \int_{\mathbb{R}} \phi_{ext}(x-y) h_t(y) dy = \int_{\mathbb{R}} \phi_{ext}(y) h_t(x-y) dy$$

- Periodic:

$$u(x,t) = \int_{-\pi}^{\pi} \phi(x-y) H_t(y) dy = \int_{-\pi}^{\pi} \phi(y) H_t(x-y) dy$$

Example:

- Problem:

$$u_t = k u_{xx}$$

$$H_t(y) = \frac{c}{\sqrt{kt}} \sum_{j \in \mathbb{Z}} e^{-\frac{|y-j|^2}{4kt}} \qquad -\pi < x < \pi$$

$$h_t(y) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{y^2}{4kt}} \qquad -\infty < x < \infty$$

LOS 2. Analyze the continuity of a solution of a PDE

Solution formula:

$$u_t = ku_{xx} \qquad -\infty < x < \infty$$

$$u(x,t) = \int_{\mathbb{R}} \phi(x-y)\frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy = \int_{\mathbb{R}} \phi(y)\frac{1}{\sqrt{4\pi kt}}e^{-\frac{(x-y)^2}{4kt}}dy$$

If ϕ is continuous and bounded:

$$\lim_{t\to 0} u(x,t) = \phi(x)$$

$$\sup_z |\phi(z)| \le K$$

Proof:

- Let $x \in \mathbb{R}$ since ϕ is continuous at x $\exists \delta > 0$ $\forall y$ where $|y - x| < \delta \Rightarrow |\phi(x) - \phi(y)| < \epsilon$ where $\epsilon > 0$

- Evaluate the difference:

$$\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy = 1$$

$$u(x,t) - \phi(x) = \int_{\mathbb{R}} \phi(x-y)\frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy - \int_{\mathbb{R}} \phi(x)\frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy$$

$$u(x,t) - \phi(x) = \int_{\mathbb{R}} [\phi(x-y) - \phi(x)]\frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy$$

$$u(x,t) - \phi(x) = \int_{|y|<\delta} [\phi(x-y) - \phi(x)]\frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy$$

$$+ \int_{|y|>\delta} [\phi(x-y) - \phi(x)]\frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy$$

$$|u(x,t) - \phi(x)| \le \int_{|y|<\delta} [\epsilon]\frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy$$

$$+ \int_{|y|>\delta} [K + K]\frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy$$

- Change of variable:

$$z^2 = \frac{y^2}{4t} \Rightarrow z = \frac{y}{\sqrt{2t}} \Rightarrow dz = \frac{dy}{\sqrt{2t}}$$

$$|u(x,t) - \phi(x)| \le \epsilon \int_{|y|<\delta} \frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy + 2K\int_{|z|>\frac{\delta}{\sqrt{2t}}} e^{-\frac{z^2}{2}}\frac{dy}{\sqrt{2\pi}}$$

$$|u(x,t) - \phi(x)| \le \epsilon + 2Ke^{-\frac{\delta}{t}}$$

$$|u(x,t) - \phi(x)| \le \epsilon + \epsilon \qquad 2Ke^{-\frac{\delta}{t}} \to \epsilon \text{ when } t \text{ is large}$$

$$\therefore |u(x,t) - \phi(x)| \le 2\epsilon$$

LOS 3. Describe the maximum principle for heat equation

Theorem:

$$u_t = k u_{xx} \qquad 0 < x < l$$

$$\sup_{(x,t)\in\hat\Omega} u(x,t) = \sup_{(x,t)\in\delta\Omega} u(x,t) \qquad \text{where } \delta\Omega \text{ is the boundary enclosing } \hat\Omega$$

Proof:

- Introduce a perturbation term where $\epsilon > 0$:

$$v(x,t) = u(x,t) + \epsilon x^2$$

- If (x_0, t_0) is an interior point:

$$v_t(x_0, t_0) = 0 \qquad \text{First derivative at maximum is 0}$$

$$v_x x(x_0, t_0) \leq 0 \qquad \text{Second derivative at maximum is negative or 0}$$

$$\therefore \underbrace{v_t - v_{xx}}_{\geq 0} = \underbrace{u_t - u_{xx}}_{=0} - \underbrace{2k\epsilon}_{>0}$$

$$\underbrace{v_t - v_{xx}}_{\geq 0} = \underbrace{u_t - u_{xx} - 2k\epsilon}_{<0}$$

- LHS and RHS contradicts. Therefore, maximum for v cannot be an interior point (1).

- Prove for u:

$$\sup_{(x,t)\in\hat\Omega} u(x,t) \leq \sup_{(x,t)\in\hat\Omega} v(x,t) \qquad \text{Since } \epsilon > 0$$

$$\sup_{(x,t)\in\hat\Omega} u(x,t) \leq \sup_{(x,t)\in\delta\Omega} v(x,t) \qquad \text{By proof in (1)}$$

$$\sup_{(x,t)\in\hat\Omega} u(x,t) \leq \sup_{(x,t)\in\delta\Omega} u(x,t) + \sup_{(x,t)\in\delta\Omega} \epsilon x^2$$

$$\sup_{(x,t)\in\hat\Omega} u(x,t) \leq \sup_{(x,t)\in\delta\Omega} u(x,t) + \epsilon l^2$$

$$\sup_{(x,t)\in\hat\Omega} u(x,t) \leq \sup_{(x,t)\in\delta\Omega} u(x,t) \qquad \text{When } \epsilon \to 0$$

- Maximum principle: maximum exists in the boundary.

Application:

- Theorem: Heat equation has a unique solution

$$u_t - ku_{xx} = f \qquad k > 0$$
$$u(x, 0) = \phi(x)$$
$$u(0, t) = h(t)$$
$$u(l, t) = k(t)$$

- Proof: Assume u^1 and u^2 are solution to the heat equation:

$$w = u^1 - u^2$$
$$w_t = kw_{xx}$$
$$w(x, 0) = \phi^1(x) - \phi^2(x) = 0$$
$$w(0, t) = h^1(x) - h^2(x) = 0$$
$$w(l, t) = k^1(x) - k^2(x) = 0$$
$$\max_{(x,t)} w(x, t) \leq 0 \qquad \text{By maximum principle}$$
$$\max_{(x,t)} -w(x, t) \leq 0 \qquad \text{By maximum principle}$$
$$\therefore w \equiv 0 \Rightarrow u^1 \text{ and } u^2 \text{ are equal}$$