

Unit 4

LOS 1. Determine the solution operator for a periodic differential equation

Problem:

$$u_t = \mathcal{L}(u) \quad -\pi < x < \pi, -\infty < t < \infty$$

$$\mathcal{L}(u) = \sum_{j=0}^n a_j \frac{\partial^j u}{\partial x^j}$$

$$u(x, 0) = \varphi(x)$$

Use the following notation:

$$u^t(x) = u(x, t)$$

Solution:

$$u(x, t) = \sum_{k \in \mathbb{Z}} e^{tP(ik)} \hat{\varphi}(k) e^{ikx}$$

LOS 2. Learn how to calculate eigen functions for a given PDE

LOS 3. Solve a periodic differential equation using Fourier coefficients

Eigenfunctions:

$$e_k(x) = e^{ikx}$$

1. $\frac{d}{dx} e_k = (ik) e_k$
2. $\left(\frac{d}{dx}\right)^j (e_k) = (ik)^j e_k$
3. $\mathcal{L}(e_k) = P(ik) e_k$

Application to PDE:

- Problem:

$$\begin{aligned} u_t &= \mathcal{L}(u) \\ u^0 &= \varphi \end{aligned}$$

- $\mathcal{L}(\psi_k)$ operator the following orthonormal basis expansion (by rule, every function can be expanded as a generalized Fourier series):

$$\mathcal{L}(\psi_k) = \lambda_k \psi_k$$

- Then, the general solution is:

$$u(x, t) = \sum_{\mathbb{R}} e^{t\lambda_k}(\phi_k, \varphi) \psi_k$$

Example:

- Problem:

$$u_t = \mathcal{L}(u) \quad -\pi < x < \pi, -\infty < t < \infty$$

$$\mathcal{L}(u) = \sum_{j=0}^n a_j \frac{\partial^j u}{\partial x^j}$$

$$\varphi(x) = \begin{cases} 1 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$$

$$\varphi(0) = \frac{1}{2}$$

$$P(x) = x^2 \quad \text{Polynomial for Diffusion}$$

- Fourier dot product:

$$\hat{\varphi}(k) = \int_{-\pi}^{\pi} e^{-ikx} \varphi(x) \frac{dx}{2\pi}$$

- Solution:

$$2\pi \hat{\varphi}(k) = \int_{-\pi}^{\pi} e^{-ikx} \varphi(x) dx$$

$$2\pi \hat{\varphi}(k) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-ikx} dx$$

$$2\pi \hat{\varphi}(k) = \left. \frac{e^{-ikx}}{-ik} \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{e^{-ik\frac{\pi}{2}} - e^{ik\frac{\pi}{2}}}{-ik} = \frac{e^{ik\frac{\pi}{2}} - e^{-ik\frac{\pi}{2}}}{ik}$$

- Values of e^{ikx} , period 2π :

$$1. \quad e^{i\frac{\pi}{2}} = i$$

2. $e^{i\pi} = -1$
3. $e^{i\frac{3\pi}{2}} = -i$
4. $e^{i2\pi} = 1$

- Therefore:

$$2\pi\hat{\varphi}(k) = \frac{e^{ik\frac{\pi}{2}} - e^{-ik\frac{\pi}{2}}}{ik} = \begin{cases} 0 & k \text{ even} \\ \frac{2}{k} & k = 4m + 1 \\ -\frac{2}{k} & k = 4m + 3 \end{cases}$$

- Then, we plug this in into the original solution operator:

$$u(x, t) = \sum_{k \in \mathbb{Z}} e^{tP(ik)} \hat{\varphi}(k) e^{ikx}$$

LOS 4. Solve a periodic differential equation with a source term using Fourier coefficients

Problem:

$$\begin{aligned} u_t - \mathcal{L}(u) &= g & -\pi < x < \pi, -\infty < t < \infty \\ u^0 &= \varphi(x) \\ u(x, t) &= u^t \end{aligned}$$

Solution:

$$\begin{aligned} u^t &= \int_0^t S(t-s)g(s)ds + S(t)(\varphi) \\ S(t)(f) &= \sum_{k \in \mathbb{Z}} e^{tP(ik)} \hat{f}(k) e_k \end{aligned}$$

Proof for $\varphi = 0$:

- Solution (x is not considered as it is being fixed at a single point):

$$v(x, t) = v^t = \int_0^t S(t-s)g(s)ds$$

- By Liebniz rule of integration:

$$\frac{d}{dt}v = \int_0^t \frac{d}{dt}S(t-s)g(s)ds + S(t-t)g(t)$$

- $S(t-t) = S(0)$ is equal to identity:

$$\frac{d}{dt}v = \int_0^t \frac{d}{dt}S(t-s)g(s)ds + g(t)$$

- Evaluate $\int_0^t \frac{d}{dt} S(t-s)g(s)ds$:

$$\begin{aligned}\frac{d}{dt} S(t-s)g(s) &= \frac{d}{dt} \sum_{k \in \mathbb{Z}} e^{tP(ik)} \hat{g}(k) e_k \\ \frac{d}{dt} S(t-s)g(s) &= \sum_{k \in \mathbb{Z}} e^{tP(ik)} \hat{g}(k) P(ik) e_k\end{aligned}$$

$$\begin{aligned}P(ik)e_k = \mathcal{L}(e_k) &\quad \rightarrow \frac{d}{dt} S(t-s) = \sum_{k \in \mathbb{Z}} e^{tP(ik)} \hat{g}(k) \mathcal{L}(e_k) \\ \frac{d}{dt} S(t-s) &= \mathcal{L} \left(\sum_{k \in \mathbb{Z}} e^{tP(ik)} \hat{g}(k) e_k \right) \\ \frac{d}{dt} S(t-s) &= \mathcal{L} S(t-s)\end{aligned}$$

- Therefore:

$$\begin{aligned}\frac{d}{dt} v &= \int_0^t \mathcal{L} S(t-s)g(s)ds + g(t) \\ \frac{d}{dt} v &= \mathcal{L} \int_0^t S(t-s)g(s)ds + g(t) \\ \frac{d}{dt} v &= \mathcal{L} v + g(t)\end{aligned}$$

General application:

- Problem

$$\begin{aligned}P_n(x) &= a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \\ u_t &= \mathcal{L} \left(P_n \left(\frac{\partial}{\partial x} \right) \right) (u) = a_0 u + a_1 u_x + \dots + a_n u_{x \dots x} \quad \text{x...x} \rightarrow n \text{ times}\end{aligned}$$

- Solution operator:

$$u(x, t) = S(t)(f) = \sum_{k \in \mathbb{Z}} e^{tP(ik)} \hat{f}(k) e^{ikx}$$

- Case I:

$$\begin{aligned}u_t &= u_{xx} - i u_{xxx} \\ P(x) &= x^2 - i x^3 \\ e^{tP(ik)} &= e^{-tk^2} e^{-tk^3} \rightarrow \text{Solution is stable as } e^{tP(ik)} \text{ vanishes as } k \text{ increases}\end{aligned}$$

- Case II:

$$u_t = u_{xx} + iu_{xxx}$$

$$P(x) = x^2 + ix^3$$

$e^{tP(ik)} = e^{-tk^2} e^{tk^3} \rightarrow$ Solution is not stable as e^{tk^3} explodes as k increases

- Case III:

$$u_t = u_{xx} \pm u_{xxx}$$

$$P(x) = x^2 \pm x^3$$

$e^{tP(ik)} = e^{-tk^2} e^{\pm itk^3} \rightarrow$ Solution is stable as $|e^{-tk^2} e^{\pm itk^3}| = e^{-2tk^2}$

Example:

- Problem:

$$u_t - u_{xxx} = t \cos(x) \quad -\pi < x < \pi$$

$$\mathcal{L}(u) = u_{xxx}$$

$$P(x) = x^3$$

$$u(x, 0) = 0$$

- Solution operator:

$$u^t = \int_0^t S(t-s)g(s)ds \quad (1)$$

$$u(x, t) = S(t)(f) = \sum_{k \in \mathbb{Z}} e^{tP(ik)} \hat{f}(k) e^{ikx}$$

- Fourier coefficients:

$$\varphi = \sum a_k e_k$$

$$a_k = \hat{\varphi}(k)$$

$$(e_j, f) = \sum_k a_k (e_j, e_k) = a_j$$

- Therefore:

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\widehat{\cos}(k) = \begin{cases} \frac{1}{2} & k = 1 \\ \frac{1}{2} & k = -1 \\ 0 & \text{elsewhere} \end{cases}$$

- Continuing on Equation (1):

$$\begin{aligned}
 u^t &= \int_0^t S(t-s)(s \cos(x)) ds \\
 u^t &= \int_0^t S(t-s)(\cos(x))(s) ds \\
 S(t-s)(\cos(x)) &= \sum_{k \in \mathbb{Z}} e^{tP(ik)} \widehat{\cos}(k) e^{ikx} = \frac{1}{2} [e^{tP(i)} e_1 + e^{tP(-i)} e_{-1}]
 \end{aligned}$$

$$\begin{aligned}
 u^t &= \int_0^t \frac{1}{2} [e^{(t-s)P(i)} e_1 + e^{(t-s)P(-i)} e_{-1}] s ds \\
 u^t &= \frac{1}{2} e_1 \int_0^t e^{(t-s)P(i)} s ds + \frac{1}{2} e_{-1} \int_0^t e^{(t-s)P(-i)} s ds
 \end{aligned}$$

- Evaluate $\int_0^t e^{(t-s)P(i)} s ds$ with integration by parts:

$$\begin{aligned}
 \int_0^t e^{(t-s)P(i)} s ds &= e^{it} \left[\int_0^t e^{-si} s ds \right] = e^{it} \left[\frac{e^{-si} s}{-i} \Big|_0^t - \int_0^t \frac{e^{-si}}{-i} ds \right] \\
 &= e^{it} [-ie^{-ti} t - e^{-ti} + 1] = -it + e^{it} - 1
 \end{aligned}$$

- Therefore:

$$\begin{aligned}
 u^t &= \frac{1}{2} e_1 (-it - e^{it} - 1) + \frac{1}{2} e_{-1} (it - e^{-it} + 1) \\
 u^t &= it \left(\frac{e_1 - e_{-1}}{2} \right) + \left(\frac{e_1 - e_{-1}}{2} \right) + \left(\frac{e^{it} e_1 - e^{-it} e_{-1}}{2} \right) \\
 u^t &= t \sin(x) + \cos(x) + i \sin(t+x)
 \end{aligned}$$

LOS 5. Solve a diffusion equation using Fourier transformations

Proof that a function can be its own Fourier coefficient:

- Let:

$$f(x) = e^{-\frac{x^2}{2}}$$

- Suppose we want to form the following integral:

$$\begin{aligned}\hat{f}(\xi) &= \int_{\mathbb{R}} e^{ax} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \quad a \in \mathbb{R} \\ \hat{f}(\xi) &= \int_{\mathbb{R}} e^{ax - \frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ \frac{2ax - x^2}{2} &= \frac{-(x^2 - 2ax + a^2 - a^2)}{2} = \frac{-[(x-a)^2 - a^2]}{2} = -\frac{(x-a)^2}{2} + \frac{a^2}{2} \\ \therefore \hat{f}(\xi) &= \int_{\mathbb{R}} e^{\frac{a^2}{2}} e^{-\frac{(x-a)^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ \hat{f}(\xi) &= e^{\frac{a^2}{2}} \int_{\mathbb{R}} e^{-\frac{(x-a)^2}{2}} \frac{dx}{\sqrt{2\pi}}\end{aligned}$$

- Change of variable:

$$\begin{aligned}\hat{f}(\xi) &= \int_{\mathbb{R}} e^{-\frac{(x-a)^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ \hat{f}(\xi) &= \int_{\mathbb{R}} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} = 1 \\ \hat{f}(\xi) &= e^{\frac{a^2}{2}}\end{aligned}$$

$$a = i\xi$$

$$\therefore \hat{f}(\xi) = e^{-\frac{\xi^2}{2}}$$

Facts:

- Parseval equality:

$$\|\hat{f}\|_{L_2} = \|f\|_{L_2} \Leftrightarrow \sum_{\mathbb{R}} |f(x)|^2 \frac{dx}{2\pi}$$

- Fourier coefficient for continuous transformation:

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) \frac{dx}{\sqrt{2\pi}}$$

- Fourier inversion (recovery formula):

$$f(x) = \int_{\mathbb{R}} e^{-i\xi x} \hat{f}(\xi) \frac{d\xi}{\sqrt{2\pi}}$$

- Function is its own Fourier coefficient:

$$1. \text{ Gaussian function: } e^{-\frac{\xi^2}{2}} = \int_{\mathbb{R}} e^{-i\xi x} e^{-\frac{x^2}{4i}} \frac{dx}{\sqrt{2\pi}}$$

2. General: $e^{-t|\xi|^2} = \int_{\mathbb{R}^n} e^{-i\xi x} e^{-\frac{|x|^2}{2}} \frac{dx}{\sqrt{4\pi t^n}}$

General solution:

- Problem:

$$\begin{aligned} u_t &= \mathcal{L}_P(u) \\ u(x, 0) &= \varphi(x) \end{aligned}$$

- Fourier coefficient:

$$\hat{u}^t(\xi) = \hat{u}(\xi, t) = \int_{\mathbb{R}} e^{-i\xi x} u(x, t) \frac{dx}{2\pi}$$

- Observation 1 (Fourier transformation of u applied by the differential operator is a polynomial times \hat{u} as u itself can be expanded using its orthogonal basis \hat{u}):

$$\begin{aligned} \widehat{\mathcal{L}_P(u)}(\xi) &= P(i\xi) \hat{u}(\xi) \\ u_t &= P(i\xi) \hat{u}(\xi) \end{aligned}$$

- Observation 2:

$$\widehat{\left(\frac{d}{dt}u\right)}(\xi) = (i\xi) \hat{u}(\xi)$$

- Derivation:

$$\begin{aligned} u_t &= \mathcal{L}_P(u) \\ \widehat{\left(\frac{d}{dt}u\right)}(\xi) &= \widehat{\mathcal{L}_P(u)}(\xi) \\ \frac{d}{dt} \hat{u}(\xi) &= P(i\xi) \hat{u}(\xi) \quad \rightarrow \text{ODE} \\ \hat{u}(\xi, t) &= e^{tP(i\xi)} \hat{u}(\xi, 0) \\ \hat{u}(\xi, t) &= e^{tP(i\xi)} \hat{\varphi}(\xi) \end{aligned}$$

- Using recovery formula:

$$u(x, t) = \int e^{i\xi x} e^{tP(i\xi)} \hat{\varphi}(\xi) \frac{d\xi}{\sqrt{2\pi}}$$

- Properties:

1. Solution works in L_2 space (the above is an L_2 solution)
2. For a nice φ (infinitely differentiable and decaying at ∞), solution is unique

Solution for heat equation:

- Problem:

$$\begin{aligned} u_t &= u_{xx} \\ u(x, 0) &= \varphi(x) \end{aligned}$$

- Solution:

$$\begin{aligned} u(x, t) &= \int_{\mathbb{R}} e^{i\xi x} e^{tP(i\xi)} \hat{\varphi}(\xi) \frac{d\xi}{\sqrt{2\pi}} \\ u(x, t) &= \int_{\mathbb{R}} e^{i\xi x} e^{-t\xi^2} \hat{\varphi}(\xi) \frac{d\xi}{\sqrt{2\pi}} \\ u(x, t) &= \int_{\mathbb{R}} e^{i\xi x} \left(\int_{\mathbb{R}} e^{-i\xi y} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{4\pi t}} \right) \hat{\varphi}(\xi) \frac{d\xi}{\sqrt{2\pi}} \\ u(x, t) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{i\xi(x-y)} \hat{\varphi}(\xi) \frac{d\xi}{\sqrt{2\pi}} \right) e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{4\pi t}} \\ u(x, t) &= \int_{\mathbb{R}} \varphi(x-y) e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{4\pi t}} \quad \Rightarrow S(t)(\varphi)(x) \end{aligned}$$

- Alternative solution expression (by change of variable):

$$u(x, t) = \int_{\mathbb{R}} \varphi(z) e^{-\frac{(x-z)^2}{4t}} \frac{dz}{\sqrt{4\pi t}} \quad \Rightarrow S(t)(\varphi)(x)$$

Integration by parts $\int_{\mathbb{R}} f g' dx = f g|_{\mathbb{R}} - \int_{\mathbb{R}} f' g dx$