Unit 8

LOS 1. Solve the Laplacian problem using the radial functions in the separation ansatz

Global problem:

- Problem:

$$
\begin{aligned}
u_{tt} &= \Delta u \\
u_t(x,0) &= \phi(x) \\
u(x,0) &= 0
\end{aligned}
$$

- Translation by fixing at x_0:

$$
\begin{aligned}
\tilde{u}_{tt} &= \Delta \tilde{u} \Leftrightarrow u_{tt} = \Delta u \\
\tilde{u}(x,t) &= u(x+x_0,t) \\
\tilde{\psi}(x) &= \psi(x+x_0)
\end{aligned}
$$

- Solution:

$$
\begin{aligned}
u(0,t) &= t\int_{S^{3-1}} \psi(ty)d\sigma(y) \\
u(0,t) &= \frac{\partial}{\partial t}\left[t\int_{S^{3-1}} \phi(ty)d\sigma(y)\right]
\end{aligned}
$$

$$
\begin{aligned}
u(x_0,t) = \tilde{u}(0,t) &= t\int_{S^{3-1}} \tilde{\psi}(ty)d\sigma(y) \\
&= t\int_{S^{3-1}} \tilde{\psi}(x_0+ty)d\sigma(y)
\end{aligned}
$$

Local problem:

- Problem contained at domain D:

$$
\begin{aligned}
u_{tt} &= \Delta u \\
u|_{\partial D} &= \phi \\
u_t|_{\partial D} &= \psi
\end{aligned}
$$

- The Laplace operator is self-adjoint $\Rightarrow$ it has an ONB of eigenvalues V_n.

- By separation Ansatz:

$$u(t,x) = \sum_n a_n \cos(\sqrt{\lambda_n} t) V_n(x) + \sum_n b_n \sin(\sqrt{\lambda_n} t) V_n(x)$$
$$u(0,x) = \sum_n a_n V_n(x)$$
$$a_n = \frac{(\phi, V_n)}{V_n, V_n}$$

- Issue: we only have information on boundary $u|_{\partial D}$, but we need to evaluate $u(0,x)$, which is inside the boundary.

- Solution:

$$\text{Find harmonic solution to } \Delta\varphi = 0$$
$$\varphi|_{\partial D} = \phi$$

LOS 2. Understand how to solve the Euler and Bessel equations

LOS 3. Understand the importance of Harmonic function in solving PDEs

LOS 4. Solve the Laplace equation with harmonic boundary condition using separation of variables

Sub-problem:

$$\Delta u = -\lambda u$$
$$u(ae^{i\theta}) = f(\theta)$$

When $\lambda = 0$:

- Problem:

$$\Delta u = 0$$
$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0$$

- Separation Ansatz:

$$
\begin{aligned}
&u(r,\theta) = R(r)\Theta(\theta) \\
&\Theta R_{rr} + \frac{\Theta R_r}{r} + \frac{R\Theta_{\theta\theta}}{r^2} = 0 \\
&r^2\frac{R_{rr}}{R} + r\frac{R_r}{R} = -\frac{\Theta_{\theta\theta}}{\Theta} = K \\
&R_{rr} + \frac{R_r}{r} - \frac{KR}{r^2} = 0 && (1) \\
&\Theta_{\theta\theta} = -K\Theta && (2) \\
&\Theta_n = e^{\pm iK\theta} = \begin{cases} \cos n\theta \\ \sin n\theta \end{cases} && K = n^2
\end{aligned}
$$

- Euler differential equation (1):

$$
\begin{aligned}
&R_{rr} + \frac{R_r}{r} - \frac{KR}{r^2} = 0 \\
&R(r) = r^\alpha \\
&\alpha(\alpha-1)r^{\alpha-2} + \frac{\alpha r^{\alpha-1}}{r} - K\frac{r^\alpha}{r^2} = 0 \\
&\alpha(\alpha-1) + \alpha - K = 0 \\
&\alpha^2 - K = 0 \\
&\alpha = \pm\sqrt{K}
\end{aligned}
$$

- Assuming we want a continuous solution (no singularities), choose only positive values:

$$
\begin{aligned}
&a = \sqrt{K} \\
&\therefore R_n(r) = r^n
\end{aligned}
$$

- Going back to initial problem, assume $a = 1$:

$$
\begin{aligned}
u(e^{i\theta}) &= f(\theta) \\
u_n(r,\theta) &= R(r)\Theta(\theta) \\
&= r^n e^{\pm in\theta} \\
u(r,\theta) &= \sum_{n\in\mathbb{Z}} r^{|n|} a_n e^{in\theta}
\end{aligned}
$$

- When $r = 1$:

$$u(1, \theta) = f(\theta) = \sum_n a_n e^{in\theta}$$
$$\therefore a_n = \hat{f}(n)$$

When $\lambda \neq 0$:

- Problem:

$$\Delta u = -\lambda u$$
$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0$$

$$u(r, \theta) = R(r)\Theta(\theta)$$
$$\Theta R_{rr} + \frac{\Theta R_r}{r} + \frac{R\Theta_{\theta\theta}}{r^2} = -\lambda R\Theta$$
$$\Theta R_{rr} + \frac{\Theta R_r}{r} + \lambda R\Theta = -\frac{R\Theta_{\theta\theta}}{r^2}$$
$$\frac{r^2}{R}R_{rr} + r\frac{R_r}{R} + r^2\lambda = -\frac{\Theta_{\theta\theta}}{\Theta} = -K$$
$$R_{rr} + \frac{R_r}{r} + \left(\lambda - \frac{n^2}{r^2}\right)R = 0 \qquad (1)$$

$$\Theta_{\theta\theta} = -K\Theta \qquad (2)$$
$$\Theta_n = e^{\pm iK\theta} = \begin{cases} \cos n\theta \\ \sin n\theta \end{cases} \qquad K = n^2$$

- Bessel equation from (1):

$$Let \rho = \sqrt{\lambda}r \to r = \frac{\rho}{\sqrt{\lambda}}$$
$$R_r = \frac{\partial}{\partial r}R = \frac{\partial R}{\partial \rho}\frac{\partial \rho}{\partial r} = R_p\sqrt{\lambda}$$
$$R_{rr} = \lambda R_{pp}$$
$$\lambda R_{pp} + \sqrt{\lambda}\frac{R_p}{\frac{\rho}{\sqrt{\rho}}} + \left(\lambda - \frac{n^2}{\frac{\rho^2}{\lambda}}\right)R = 0$$
$$R_{pp} + \frac{R_\rho}{\rho} + \left(\lambda - \frac{n^2}{\rho^2}\right)R = 0$$

- Solution to Bessel (Power series):

$$R = \rho^\alpha \sum_{k=0}^{\infty} a_k \rho^k$$

$$[\alpha(\alpha-1) + \alpha - n^2]a_0 = 0 \qquad \to \alpha^2 = n^2$$

$$[(\alpha-1)\alpha + (\alpha-1) - n^2]a_1 = 0$$

$\to$ Let odd coefficient 0, $a_1 = 0$

$$[(\alpha+k)(\alpha+k-1) + (\alpha+k) - n^2]\alpha_{k-2} + a_k = 0$$

$$a_k = -\frac{a_{k-2}}{(\alpha+k)^2 - n^2}$$

- Facts:

$$J_n(p) = \sum_{j=0}^{\infty} (-1)^j \frac{\left(\frac{1}{2}\rho\right)^{n+2j}}{j!(n+j)!}$$

Behaves like $\sqrt{\frac{2}{\pi\rho}} \cos(\rho - \frac{\pi}{4} + \frac{n\pi}{2}) + O(\rho^{-\frac{3}{2}})$

J_n has countably many zeros

- Continuation on solution:

We want to satisfy $u|_{\partial D} = 0$

$\therefore J_n(\rho) = J_n(\sqrt{\lambda}a) = 0$ (zero at the boundary)

We must have

$$\sqrt{\lambda}a \in \{\rho \mid J_n(\rho) = 0\}$$

$$\sqrt{\lambda}a = \{\gamma_{nm} \mid m \in \mathbb{N}\}$$

Assume $a = 1$

$$u(r,\theta) = \sum_{n,m} J_n(\sqrt{\lambda_{nm}}r)(a_n \cos n\theta + b_n \sin n\theta)$$

$$\sqrt{\lambda_{nm}} = \gamma_{nm}$$

$$\lambda_{nm} = \frac{\gamma_{nm}^2}{a^2}$$

We only need J_0 for the solution

- Final answer:

$$\forall n \text{ let } (\gamma_{mn})_{m=1}^{\infty} \text{ the zeros of } J_n(\gamma_{mn}) = 0$$

We need

$$\sqrt{\lambda_{mn}}a = \gamma_{mn}$$
$$\lambda_{mn} = \frac{\gamma_{mn}}{a}^2$$

$$\therefore u(r,\theta) = \sum_{mn} C_{mn} e^{in\theta} J_n(\sqrt{\lambda_{mn}}r)$$

General solution:

- Problem:

$$u_{tt} = \Delta u \qquad \text{on } \mathbb{R}^2$$

- Solution:

$$\begin{aligned} u(r,\theta) &= \phi(r,\theta) \\ &= \sum_{n,m} C_{nm} e^{in\theta} J_n(\sqrt{\lambda_{nm}}r) \end{aligned}$$

$$\begin{aligned} u(r,\theta,t) &= \sum_{n} C_{nm} e^{in\theta} J_0(\sqrt{\lambda_{nm}}r)\sin(\sqrt{\lambda_{nm}}t) \\ &\quad + \sum_{n} D_{nm} e^{in\theta} J_0(\sqrt{\lambda_{nm}}r)\cos(\sqrt{\lambda_{nm}}t) \\ u(r,\theta,0) &= \sum_{n} D_{nm} e^{in\theta} J_0(\sqrt{\lambda_{nm}}r)\cos(\sqrt{\lambda_{nm}}t) \\ u_t(r,\theta,0) &= \sum_{n} C_{nm} e^{in\theta} J_0(\sqrt{\lambda_{nm}}r)\sin(\sqrt{\lambda_{nm}}t)\sqrt{\lambda_{nm}} \\ \text{where } C_{nm} &= \int_0^a \int_0^{2\pi} e^{in\theta} \frac{J_n(\sqrt{\lambda_{nm}}r)}{J_{nm}} \phi(r,\theta) r dr \frac{d\theta}{2\pi} = \frac{(V_{nm},\phi)}{V_{nm}, V_{nm}} \\ J_{nm} &= \int J_n(\sqrt{\lambda_{nm}}r) J_n(\sqrt{\lambda_{nm}}r) dr \end{aligned}$$

- Conclusion: if radius changes, frequency changes

Extension to $\mathbb{R}^3$:

- Problem:

$$u_{tt} = \Delta u \qquad \text{on } \mathbb{R}^3$$

- Separation Ansatz:

$$\begin{aligned}
&u = T(t)V(x) \\
&T''V = T\Delta V \to \frac{T''}{T} = -\lambda = -\gamma^2 \\
&T'' = -\lambda T \\
&\Delta V = -\lambda V
\end{aligned}$$

- Sub-problem:

$$\begin{aligned}
&V(x) = R(r)\alpha(\theta, \varphi) \\
&\Delta V = V_{rr} + \frac{2}{r}V_r + \frac{\Delta_{\theta,\varphi}(u)}{r^2} \\
&\Delta_{\theta,\varphi}(u) = \frac{1}{\sin\theta}V_{\theta\varphi} + \frac{1}{\sin\theta}(\sin\theta V_\theta)\theta
\end{aligned}$$

- Use change of variable:

$$w = \sqrt{r}R(r)$$

LOS 5. Understand the conditions required for the Fourier series expansion

Theorem:

Every function in $L_2\left(D_a\right)$ has a Fourier series decomposition

$$\phi(r,\theta) = v(r,\theta) = \sum_{n,m} C_{nm} e^{in\theta} J_n(\sqrt{\lambda_{nm}} r)$$

For fixed n :

$$m \neq m' \qquad \int J_n(\sqrt{\lambda_{nm}} r) J_n(\sqrt{\lambda_{nm'}} r) r dr = 0$$

Radial function :

$$u(r) = \sum_n C_m J_0(\sqrt{\lambda_{0m}} r)$$