

Unit 7

LOS 1. Solve the heat equation using Fourier analysis

Problem:

$$\begin{aligned}u_t &= u_{xx} & 0 \leq x \leq \pi \\u(0, t) &= h(t) \\u(\pi, t) &= k(t) \\u(x, 0) &= 0\end{aligned}$$

Method 1 (Fourier Analysis):

- Assume u is a solution and takes the form of the generic solution for periodic case:

$$\begin{aligned}u(x, t) &= \sum_{n=0}^{\infty} u_n(t) f_n(x) \\f_n &= \begin{cases} 1 & n = 0 \\ \sin(nx) & n \neq 0 \end{cases}\end{aligned}$$

$$\begin{aligned}u_t(x, t) &= \sum_{n=0}^{\infty} v_n(t) f_n(x) \\u_{xx}(x, t) &= \sum_{n=0}^{\infty} w_n(t) f_n(x)\end{aligned}$$

- All derivatives are square-integrable ($\in L_2$) and therefore have unique Fourier expansion. Assuming solution is smooth ($\in L_2$), the Fourier expansions above converge pointwise
- Evaluate v_n (assume we can interchange differentiation and integration):

$$\begin{aligned}u_t(x, t) &= \sum (f_n, u_t) f_n \\\therefore v_n(t) &= (f_n, u_t) = \int \bar{f}_n u_t dx\end{aligned}$$

$$\begin{aligned}v_0 &= \int_0^\pi u_t(x, t) \frac{dx}{\pi} = \frac{d}{dt} \int_0^\pi u(x, t) \frac{dx}{\pi} \\v_n &= \int_0^\pi \sin(nx) u_t(x, t) \frac{2dx}{\pi} = \frac{d}{dt} \int_0^\pi \sin(nx) u(x, t) \frac{2dx}{\pi} \\\therefore v_n &= \frac{d}{dt} u_n = u'_n(t)\end{aligned}$$

- Evaluate w_n using integration by parts:

$$\begin{aligned}
\frac{\pi}{2}w_n &= \frac{d^2}{dx^2} \int_0^\pi \sin(nx)u(x,t)dx = \int_0^\pi \sin(nx)u_{xx}(x,t)dx \\
\frac{\pi}{2}w_n &= \cancel{\sin(nx)u_x(t)|_0^\pi}^0 - n \int_0^\pi \cos(nx)u_x(x,t)dx \\
&= -n [\cos(nx)u_x(t)|_0^\pi] - n^2 \int_0^\pi \sin(nx)u(x,t)dx \\
&= -n[(-1)^n k(t) - h(t)] - n^2 \frac{\pi}{2}u_n(t)
\end{aligned}$$

- Establish that $w_n = u'_n(t)$:

$$\begin{aligned}
v_n(t) &= \frac{d}{dx}u_n(t) = u_t = u'_n(t) \\
w_n(t) &= \frac{d^2}{dx^2}u_n(t) = u_{xx} \\
u_t &= u_{xx} \\
v_n &= w_n \\
u'_n(t) &= w_n
\end{aligned}$$

$$\begin{aligned}
\therefore u'_n(t) &= \frac{2}{\pi} \left(-n[(-1)^n k(t) - h(t)] - n^2 \frac{\pi}{2}u_n(t) \right) \\
u'_n(t) &= -\frac{2n}{\pi} [(-1)^n k(t) - h(t)] - n^2 u_n(t) \tag{1}
\end{aligned}$$

- Solve for (1) using ODE approach:

$$\begin{aligned}
u'_n(t) + n^2 u_n(t) &= -\frac{2n}{\pi} [(-1)^n k(t) - h(t)] \\
g' + n^2 g &= H
\end{aligned}$$

- Solve homogeneous and then use Duhamel's principle:

$$\begin{aligned}
 g' &= -n^2 g \\
 S(t)(\phi) &= e^{-tn^2} g(0) = e^{-tn^2} \phi \\
 g(t) &= \int_0^t S(t-s) H(s) ds + e^{-tn^2} \phi \\
 u(x, 0) = \phi = 0 \quad g(t) &= \int_0^t S(t-s) H(s) ds + \cancel{e^{-tn^2} \phi} \xrightarrow{0} 0 \\
 u_n(t) &= \frac{2n}{\pi} \int_0^t e^{-(t-s)n^2} [h(s) - (-1)^n k(s)] ds \\
 u(x, t) &= \sum_{n=0}^{\infty} u_n(t) f_n(x) \\
 u(x, t) &= u_0(t) f_0(x) + \sum_{n \geq 1}^{\infty} u_n(t) \sin(nx)
 \end{aligned}$$

- Evaluate $u_0(t)f_0(x)$:

$$\begin{aligned}
 f_0(x) &= 1 \\
 u(x, t) &= u_0(t) + \sum_{n \geq 1}^{\infty} u_n(t) \sin(nx) \\
 u(0, t) &= u_0(t) + \sum_{n \geq 1}^{\infty} u_n(t) \cancel{\sin(nx)} \xrightarrow{0} 0 \\
 u(0, t) &= u_0(t) \rightarrow h(t) = u_0(t) \\
 \therefore u(x, t) &= h(t) + \sum_{n \geq 1}^{\infty} u_n(t) \sin(nx)
 \end{aligned}$$

LOS 2. Solve the heat equation using transformation onto a source equation

Problem:

$$\begin{aligned}
 u_t &= u_{xx} & 0 \leq x \leq \pi \\
 u(0, t) &= h(t) \\
 u(\pi, t) &= k(t) \\
 u(x, 0) &= 0
 \end{aligned}$$

Method 2 (Shifting Data / Transformation onto a source equation):

- Assume a solution w :

$$w_t = w_{xx}$$

$$w(0, t) = h(t)$$

$$w(\pi, t) = k(t)$$

$$w(x, t) = \left(1 - \frac{x}{\pi}\right) h(t) + \left(\frac{x}{\pi}\right) k(t)$$

- Assume u is also a solution such that:

$$v = u - w$$

$$v(0, t) = 0$$

$$v(\pi, t) = 0$$

$$v_{xx} = u_{xx} - w_{xx}$$

$$v_t = u_t - w_t$$

$$v_{xx} - v_t = u_{xx} - w_{xx} - (u_t - w_t)$$

$$w_{xx}(x, t) = 0$$

$$w_t(x, t) = \left(1 - \frac{x}{\pi}\right) h'(t) + \left(\frac{x}{\pi}\right) k'(t)$$

$$\therefore v_{xx} - v_t = w_t$$

$$v(x, 0) = \cancel{u(x, 0)} \overset{0}{=} w(x, 0) = -\left(1 - \frac{x}{\pi}\right) h(0) - \left(\frac{x}{\pi}\right) k(0) = \phi(x)$$

- In general:

$$v_t = L(v)$$

$$v_t - v_{xx} = g(x, t)$$

$$v(x, 0) = \phi(x)$$

$$v(x, t) = \int_0^t S(t-s)g(s)ds + S(t)(\phi)$$

$$\int_0^t S(t-s)g(s)ds = \int_0^t \left(\hat{g}_0(x, s) + \sum_{n \geq 1} e^{-(t-s)n^2} \hat{g}_n(x, s) \sin(nx) \right) ds$$

$$\hat{g}_n(x, s) = \frac{2}{\pi} \int_0^\pi \sin(nx) g(x, s) dx$$

$$\hat{g}_0(x, s) = \frac{1}{\pi} \int_0^\pi g(x, s) dx$$

$$S(t)(\phi) = \tilde{\phi}_0 + \sum_{n \geq 1} e^{-tn^2} \tilde{\phi}_n \sin(nx)$$

$$\tilde{\phi}_n(x) = \frac{2}{\pi} \int_0^\pi \sin(nx) \phi(x) dx$$

$$\tilde{\phi}_0(x) = \frac{1}{\pi} \int_0^\pi \phi(x) dx$$

- Combining the solutions:

$$v(x, t) = \tilde{\phi}_0 + \sum_{n \geq 1} e^{-tn^2} \tilde{\phi}_n \sin(nx) + \sum_{n \geq 1} \int_0^t e^{-(t-s)n^2} \hat{g}_n(x, s) ds \sin(nx)$$

$$= \tilde{\phi}_0 + \sum_{n \geq 1} e^{-tn^2} \tilde{\phi}_n \sin(nx) + \sum_{n \geq 1} v_n(t) \sin(nx)$$

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LOS 3. Solve homogeneous heat equation in higher dimensions using Fourier analysis

Problem

$$\Delta f = \sum_{k=1}^n \frac{d^2}{dx_k^2} f$$

$$u_t = \Delta u$$

$$u(0, x) = \phi(x)$$

Facts:

- Fourier transform formula:

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i(\xi, x)} f(x) \frac{dx}{\sqrt{2\pi}^n}$$

- Dot product:

$$(\xi, x) = \sum_{j=1}^n \xi_j x_j$$

- Lemma 1:

$$f(x) = e^{-\frac{|x|^2}{2}}$$

$$|x| = (x, x)^{\frac{1}{2}}$$

$$\hat{f}(\xi) = e^{-\frac{|\xi|^2}{2}}$$

- Lemma 2 (Parseval):

$$\|f\|_{L_2} = \|\hat{f}\|_{L_2}$$

- Lemma 3 (Fourier inversion - applies when f is sufficiently smooth and decays when approaching infinity):

$$f(x) = \int_{\mathbb{R}} e^{-i(\xi, x)} \hat{f}(\xi) \frac{d\xi}{\sqrt{2\pi}^n}$$

- Lemma 4:

$$\widehat{\frac{d}{dx_k} f} = i\xi_k \hat{f}(\xi)$$

$$\hat{u}_t(t, \xi) = \widehat{\frac{d}{dt} u(t, \xi)} = \widehat{\frac{d^2}{dx^2} u(t, \xi)} = -|\xi|^2 \hat{u}(t, \xi)$$

$$\hat{u}(t, \xi) = e^{-t|\xi|^2} \hat{u}(0, \xi) = e^{-t|\xi|^2} \hat{\phi}(\xi)$$

- Therefore, for $u: \mathbb{R} \times \mathbb{R}^n$ (\mathbb{R} for t space and \mathbb{R}^n for x dimension):

$$\hat{u}(t, \xi) = \int_{\mathbb{R}^n} e^{-i(\xi, x)} u(t, x) \frac{dx}{\sqrt{2\pi}^n}$$

- From Lemma 1:

$$e^{-t|\xi|^2} = e^{-\frac{|\sqrt{2t}\xi|^2}{2}} = \int_{\mathbb{R}^n} e^{i(-\sqrt{2t}\xi, x)} e^{-\frac{|x|^2}{2}} \frac{dx}{\sqrt{2\pi}^n}$$

- By change of variable:

$$\begin{aligned} y &= \sqrt{2t}x \\ dy &= \sqrt{2t}^n dx \\ \therefore e^{-t|\xi|^2} &= \int_{\mathbb{R}^n} e^{-i(\xi, y)} e^{-\frac{|y|^2}{4t}} \frac{dy}{\sqrt{4\pi t}^n} \end{aligned}$$

$$\begin{aligned} u(t, x) &= \int_{\mathbb{R}^n} e^{i(\xi, x)} \hat{u}(t, \xi) \frac{d\xi}{\sqrt{2\pi}^n} \\ &= \int_{\mathbb{R}^n} e^{i(\xi, x)} e^{-t|\xi|^2} \hat{\phi}(\xi) \frac{d\xi}{\sqrt{2\pi}^n} \\ &= \int_{\mathbb{R}^n} e^{i(\xi, x)} \int_{\mathbb{R}^n} e^{-i(\xi, y)} e^{-\frac{|y|^2}{4t}} \frac{dy}{\sqrt{4\pi t}^n} \hat{\phi}(\xi) \frac{d\xi}{\sqrt{2\pi}^n} \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\xi, x)} e^{-i(\xi, y)} \hat{\phi}(\xi) \frac{d\xi}{\sqrt{2\pi}^n} e^{-\frac{|y|^2}{4t}} \frac{dy}{\sqrt{4\pi t}^n} \\ &= \int_{\mathbb{R}^n} \phi(x - y) e^{-\frac{|y|^2}{4t}} \frac{dy}{\sqrt{4\pi t}^n} \end{aligned}$$

LOS 4. Solve nonhomogeneous heat equation in higher dimensions using Fourier analysis

Problem:

$$\begin{aligned} u_t - \Delta u &= w \\ u(0, x) &= \phi(x) \end{aligned}$$

Homogeneous solution:

$$S(t)(\phi)(x) = \int_{\mathbb{R}^n} \phi(x - y) e^{-\frac{|y|^2}{4t}} \frac{dy}{\sqrt{4\pi t}^n}$$

Inhomogeneous solution (using Duhamel):

$$u(t, x) = S(t)(\phi)(x) + \int_0^t S(t - s)w(s)ds$$

LOS 5. Understand the polar coordinates in higher dimensions

Problem:

$$\begin{aligned}u_{tt} &= \Delta u \\ u(0, x) &= \phi(x) \\ u_t(0, t) &= \psi(x)\end{aligned}$$

From LOS 6:

$$\begin{aligned}\hat{u}(t, \xi) &= \cos(t|\xi|)\hat{\phi}(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{\psi}(\xi) \\ &= \widehat{u_\phi}(t, \xi) + \widehat{u_\psi}(t, \xi)\end{aligned}$$

Remark:

$$\frac{d}{dt}\hat{u}_\psi(t, \xi) = \cos(t|\xi|)\hat{\phi}(\xi)$$

Example in 1-D (shows that we can focus on getting u_ψ only to get the full solution):

$$\begin{aligned}u(t, x) &= \frac{\phi(x+t) + \phi(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \phi(x, s) ds \\ &= u_\phi + u_\psi\end{aligned}$$

$$\frac{d}{dt}u_\psi(t, x) = \frac{\psi(x+t) + \psi(x-t)}{2} \rightarrow \text{same form as Dirichlet solution}$$

Convolution:

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

Lemma 1:

- Claim:

$$\widehat{f * g}(\xi) = \sqrt{2\pi}^n \hat{f}(\xi) \hat{g}(\xi)$$

- Proof:

$$\begin{aligned}
\widehat{f * g}(\xi) &= \int_{\mathbb{R}^n} e^{-i(\xi, x)} f * g(x) \frac{dx}{\sqrt{2\pi}^n} \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i(\xi, x)} f(x - y) g(y) dy \frac{dx}{\sqrt{2\pi}^n} \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i[(\xi, x - y) + (\xi, y)]} f(x - y) g(y) dy \frac{dx}{\sqrt{2\pi}^n} \\
&= \sqrt{2\pi}^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i(\xi, x - y)} f(x - y) \frac{dy}{\sqrt{2\pi}^n} e^{-i(\xi, y)} g(y) \frac{dx}{\sqrt{2\pi}^n}
\end{aligned}$$

$$\begin{aligned}
z &= x - y \\
&= \sqrt{2\pi}^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i(\xi, z)} f(z) \frac{dz}{\sqrt{2\pi}^n} e^{-i(\xi, y)} g(y) \frac{dx}{\sqrt{2\pi}^n} \\
&= \sqrt{2\pi}^n \hat{f}(\xi) \hat{g}(\xi)
\end{aligned}$$

- Apply the lemma to u_ψ :

$$u_\psi(t, x) = \sqrt{2\pi}^n \hat{g}^t * \psi$$

Lemma 2:

- $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is radial if $\exists H : \mathbb{R} \rightarrow \mathbb{R}$ such that $F(x) = H(|x|)$
- Polar coordinates:

$$\int_{\mathbb{R}^n} f(x) dx = C_n \int_0^\infty \int_{S^{n-1}} f(ry) d\sigma(y) r^{n-1} dr$$

- Claim: if F is radial, then \hat{F} is also radial

$$\begin{aligned}
\hat{F}(\xi) &= \int_{\mathbb{R}^n} e^{-i(\xi, y)} F(x) \frac{dx}{\sqrt{2\pi}^n} \\
&= C_n \int_0^\infty \left[\int_{S^{n-1}} e^{-i(r\xi, y)} d\sigma(y) \right] F(x) r^{n-1} dr \quad \text{where } x = ry \\
&= C_n \int_0^\infty \left[\int_{S^{n-1}} e^{-i(r\xi, y)} d\sigma(y) \right] H(r) r^{n-1} dr \quad \text{where } r = |x| \\
V(r|\xi|) &= \int_{S^{n-1}} e^{-i(r\xi, y)} d\sigma(y) \rightarrow \text{depends only on } |\xi|, \text{ therefore radial}
\end{aligned}$$

- σ is the probability measure on $S^{n-1} = \{y \mid |y| = 1\}$ which is invariant under rotation (periodic)

- Simple example:

$$F(x) = \begin{cases} 1 & |x| \leq R \\ 0 & |x| > R \end{cases}$$

$$\begin{aligned} \text{Volume}_n(\text{Ball with radius } R) &= C_n \int_0^R r^{n-1} dr \\ &= \frac{C_n}{n} R^n \end{aligned}$$

- Example for $n = 2$:

$$\begin{aligned} (x, y) &= r e^{i\theta} \\ F(x, y) &= H(r) \end{aligned}$$

$$\begin{aligned} S^{n-1} &= \{e^{i\theta} \mid 0 \leq \theta \leq 2\pi\} \\ d\sigma(\theta) &= \frac{d\theta}{2\pi} \end{aligned}$$

$$\begin{aligned} \xi &\in \mathbb{R}^2 \\ \xi &= r e^{i\eta} \\ (r e^{i\eta}, e^{i\theta}) &= (r, e^{i(\theta-\eta)}) \end{aligned}$$

$$\begin{aligned} \hat{F}(\xi) &= \int_0^{2\pi} e^{-i(r, e^{i(\theta-\eta)})} \frac{d\theta}{2\pi} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \cos(r \cos \theta) d\theta \\ &= V_2(r) \quad \text{where } r = |\xi| \end{aligned}$$

Lemma 3 (Dilation):

- Claim, for $f : \mathbb{R}^n \rightarrow \mathbb{R}$:

$$\begin{aligned} \text{If } f^t(x) &= f(tx) \\ \hat{f}^t(\xi) &= t^{-n} \hat{f}\left(\frac{\xi}{t}\right) \end{aligned}$$

- Proof:

$$\begin{aligned}\hat{f}^t(\xi) &= \int_{\mathbb{R}^n} e^{-i(\xi, x)} f^t(x) \frac{dx}{\sqrt{2\pi}^n} \\ &= \int_{\mathbb{R}^n} e^{-i(\xi, x)} f(tx) \frac{dx}{\sqrt{2\pi}^n}\end{aligned}$$

$$\begin{aligned}y &= tx \\ dy &= t^n dx \\ &= t^{-n} \int_{\mathbb{R}^n} e^{-i(\frac{\xi}{t}, y)} f(y) \frac{dy}{\sqrt{2\pi}^n} \\ &= t^{-n} \hat{f}\left(\frac{\xi}{t}\right)\end{aligned}$$

- Apply to \hat{g}^t :

$$\hat{g}^t(\xi) = \frac{\sin(t|\xi|)}{|\xi|} = \frac{\sin(t|\xi|)}{|\xi|} \frac{t}{t}$$

$$\begin{aligned}s &= \frac{1}{t} \\ &= \frac{1}{s} \frac{\sin(\frac{|\xi|}{s})}{\frac{|\xi|}{s}} = s^{n-1} s^{-n} \frac{\sin(\frac{|\xi|}{s})}{\frac{|\xi|}{s}}\end{aligned}$$

$$\hat{g}^t(\xi) = t^{1-n} \hat{g}^1\left(\frac{\xi}{t}\right) \quad \text{where } \hat{g}^1(\xi) = \frac{\sin(|\xi|)}{|\xi|}$$

Apply the two lemmas to the original problem:

$$\begin{aligned}u_\psi(t, 0) &= C_n \int_{\mathbb{R}^n} g^t(x - y) \psi(y) dy \\ &= C_n t^{n-1} \int_{\mathbb{R}^n} g^1\left(\frac{y}{t}\right) \psi(y) dy \\ &= C_n(t) \int_{\mathbb{R}^n} \left[\int_{S^{n-1}} \psi(ry) d\sigma(y) \right] g(t, r) r^{n-1} dr\end{aligned}$$

Conclusion: for any dimension, solution at the space is the overage of all circles, then apply a scalar factor to the radii at every t

LOS 6. Determine solution of wave equation using Fourier transforms

Problem:

$$\begin{aligned}u_{tt} &= \Delta u \\u_t(0, x) &= \phi(x) \\u(0, x) &= \psi(x) \\\widehat{u_{tt}} &= \widehat{\Delta u} = \frac{\widehat{d^2}}{dx^2} u = -|\xi|^2 \hat{u}(t, \xi)\end{aligned}$$

Assume an ODE problem:

$$\begin{aligned}f''(t) &= -\alpha f(t) & \alpha > 0 \\f(t) &= a(0) \cos(\sqrt{\alpha}t) + b(0) \sin(\sqrt{\alpha}t)\end{aligned}$$

Translate to the original problem:

$$\begin{aligned}\hat{u}(t, \xi) &= \cos(t|\xi|)a(\xi) + \sin(t|\xi|)b(\xi) \\\hat{u}(0, \xi) &= a(\xi) = \hat{\phi}(\xi) \\\hat{u}_t(0, \xi) &= |\xi|b(\xi) = \hat{\psi}(\xi) \\b(\xi) &= \frac{\hat{\psi}(\xi)}{|\xi|}\end{aligned}$$

$$\begin{aligned}S_D(t) &= \int_{\mathbb{R}^n} e^{i(\xi, x)} \cos(|\xi|t) \hat{\phi}(\xi) \frac{d\xi}{\sqrt{(2\pi)^n}} \\S_N(t) &= \int_{\mathbb{R}^n} e^{i(\xi, x)} \sin(|\xi|t) \frac{\hat{\psi}(\xi)}{|\xi|} \frac{d\xi}{\sqrt{(2\pi)^n}}\end{aligned}$$

In wave equation, the following equation holds:

$$\begin{aligned}S_D(0) &= \text{Id} & S'_D(0) &= 0 \\S_N(0) &= 0 & S'_N(0) &= \text{Id}\end{aligned}$$

Remarks:

$$\int_0^t \cos(|\xi|s) ds = \frac{\sin(|\xi|t)}{|\xi|}$$

Example for 1-D ($n = 1$):

$$\begin{aligned}
\cos(|\xi|t) &= \frac{e^{it\xi} - e^{-it\xi}}{2} \\
S_D(t) &= \frac{1}{2} \int_{\mathbb{R}} e^{i\xi x} e^{i\xi t} \hat{\phi}(\xi) \frac{d\xi}{\sqrt{2\pi}} + \frac{1}{2} \int_{\mathbb{R}} e^{i\xi x} e^{-i\xi t} \hat{\phi}(\xi) \frac{d\xi}{\sqrt{2\pi}} \\
&= \frac{1}{2} (\phi(x+t) + \phi(x-t)) \\
S_N(t) &= \int_{\mathbb{R}} e^{i(\xi, x)} \sin(|\xi|t) \frac{\hat{\psi}(\xi)}{|\xi|} \frac{d\xi}{\sqrt{(2\pi)}} \\
&= \int_{\mathbb{R}} e^{i(\xi, x)} \int_0^t \cos(|\xi|s) ds \hat{\psi}(\xi) \frac{d\xi}{\sqrt{(2\pi)}} \\
&= \int_0^t \int_{\mathbb{R}} e^{i(\xi, x)} \cos(|\xi|s) \hat{\psi}(\xi) \frac{d\xi}{\sqrt{(2\pi)}} ds \\
&= \int_0^t S_D(s) ds \\
&= \int_0^t \frac{\psi(x+s) + \psi(x-s)}{2} ds
\end{aligned}$$

LOS 7. Determine solution of wave equation using polar coordinates

Problem:

$$\begin{aligned}
u_{tt} &= \Delta u \\
u(0, x) &= \phi(x) \\
u_t(0, x) &= \psi(x)
\end{aligned}$$

Solution by polar coordinates:

$$\begin{aligned}
u_{\psi}(t, x) &= \int g(t, |x-y|) \psi(y) dy \\
u_{\psi}(t, x) &= \int_0^{\infty} g(t, r) \int_{S^{n-1}} \psi(ry) d\sigma(y) dr
\end{aligned}$$

LOS 8. Find the solution of wave equation using spherical means method

By translation:

- Assume u is a solution

- For $x_0 \in \mathbb{R}^2$, the following is also a solution:

$$w(t, x) = u(t, x - x_0)$$

- Hence, we can shift our original wave problem such that:

$$\begin{aligned} u_{tt} &= \Delta u \\ u(0, x) &= \phi(x) = 0 \\ u_t(0, x) &= \psi(x) \end{aligned} \tag{1}$$

- After solving the above problem, we can shift it back to the original problem

Spherical means:

- Let $u(t, x)$ be a solution:

$$\begin{aligned} \bar{u}(t, x) &= \int_{S^{n-1}} u(t, |x|y) d\sigma(y) \\ |x| &= r \end{aligned}$$

- u above is only dependent on the absolute value of x such that:

$$\begin{aligned} x &\neq x' \\ \text{if } |x| &= |x'| \rightarrow \bar{u}(t, x) = \bar{u}(t, x') \end{aligned}$$

- u over the domain of t is a series of circles with increasing radii as t goes up. As such, the integral is simply an average of all circles over the domain of t .
- Hence:

$$\begin{aligned} \text{Given } g(t, r) &= \int_{S^{n-1}} u(t, ry) d\sigma(y) \\ u_\psi(t, 0) &= \int_0^\infty g(t, r) \bar{\psi}(r) dr \end{aligned}$$

Lemma 1:

- If u solves the wave problem (1):

$$\begin{aligned} \bar{u}_{tt} &= \bar{u}_{rr} + \frac{n-1}{r} \bar{u}_r \\ u(t, 0) &\text{ only depends on } \bar{u} \end{aligned}$$

- Proof:

$$\begin{aligned}
\bar{u}_{tt}(t, x) &= \int_{S^{n-1}} u_{tt}(t, ry) d\sigma(y) \\
&= \int_{S^{n-1}} (\Delta u)(t, ry) d\sigma(y) \\
&= \Delta \int_{S^{n-1}} u(t, ry) d\sigma(y)
\end{aligned}$$

- Let $G(t, r)$ be a radial function:

$$\begin{aligned}
\frac{\partial}{\partial x_k} G(t, |x|) &= G_r(t, |x|) \frac{\partial}{\partial x_k} |x| \quad \text{where } |x| = \left(\sum_{j=1}^n x_j^2 \right)^{\frac{1}{2}} \\
&= G_r(t, |x|) \frac{1}{2} \frac{2x_k}{\sqrt{\sum x_j^2}} \\
&= G_r(t, |x|) \frac{x_k}{|x|}
\end{aligned}$$

$$\frac{\partial^2}{\partial x_k^2} G(t, |x|) = G_{rr}(t, |x|) \frac{x_k^2}{|x|^2} + G_r(t, |x|) \frac{|x| - \frac{x_k^2}{|x|}}{|x|^2}$$

$$\begin{aligned}
\therefore \Delta G(t, |x|) &= \sum \frac{\partial^2}{\partial x_k^2} G(t, |x|) \\
&= G_{rr}(t, |x|) + \frac{n}{|x|} G_r(t, |x|) - \frac{1}{|x|} G_r(t, |x|) \\
&= G_{rr}(t, r) + \frac{n-1}{r} G_r(t, r)
\end{aligned}$$

Example for $n = 3$:

- Conversion into polar coordinates:

$$\bar{u}_{tt} = \bar{u}_{rr} + \frac{2}{r} \bar{u}_r$$

$$\text{Let } v(t, r) = r\bar{u}(t, r)$$

$$v_{tt} = r\bar{u}_{tt}(t, r) = r\left[\bar{u}_r r + \frac{2}{r} \bar{u}_r\right] = r\bar{u}_{rr} + 2\bar{u}_r$$

$$v_r = \bar{u}_r + r\bar{u}_r$$

$$v_{rr} = \bar{u}_r + \bar{u}_r + r\bar{u}_{rr} = 2\bar{u}_r + r\bar{u}_{rr} = v_{tt}$$

- New converted problem:

$$v_{tt} = v_{rr}$$

$$v(0, r) = r\bar{u}(0, r) = r\bar{\phi} = 0$$

$$v_t(0, r) = r\bar{u}_t(r, 0) = r\bar{\psi}(r)$$

- Solution:

$$v(r, t) = \frac{1}{2} \int_{r-t}^{r+t} s \bar{\psi}(s) ds$$

$$\bar{u}(r, t) = \frac{1}{2r} \int_{r-t}^{r+t} s \bar{\psi}(s) ds$$

$$\begin{aligned} \bar{u}(0, t) &= \lim_{r \rightarrow 0} \frac{v(r, t)}{r} \\ &= \left. \frac{\partial v}{\partial r}(r, t) \right|_{r=0} \\ &= \frac{1}{2} [(r+t) \bar{\psi}(r+t) + (r-t) \bar{\psi}(r-t)] \\ &= t \bar{\psi}(t) \end{aligned}$$

$$\therefore u(0, t) = t \int_{S^{n-1}} \psi(ry) d\sigma(y)$$

Example for $n = 5$:

$$\begin{aligned} \bar{u}_{tt} &= \bar{u}_{rr} + \frac{4}{r} \bar{u}_r \\ \text{Let } v(t, r) &= \left(\frac{1}{r} \frac{\partial}{\partial r} \right) (r^3 \bar{u}) \\ \text{Lemma } \frac{\partial^2}{\partial r^2} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{k-1} (r^{2k-1} \phi) &= \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^k (r^{2k} \phi_r) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2}{\partial r^2} v(t, r) &= \frac{\partial^2}{\partial r^2} \left(\frac{1}{r} \frac{\partial}{\partial r} \right) (r^3 \bar{u}) \\ &= \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^2 (r^4 \bar{u}_r) \\ &= \left(\frac{1}{r} \frac{\partial}{\partial r} \right) \frac{1}{r} (4r^3 \bar{u}_r + r^4 \bar{u}_{rr}) \\ &= \left(\frac{1}{r} \frac{\partial}{\partial r} \right) \left[\frac{1}{r} \left(4 \frac{\bar{u}_r}{r} + \bar{u}_{rr} \right) r^3 \right] \\ &= \left(\frac{1}{r} \frac{\partial}{\partial r} \right) (r^3 \bar{u}_{tt}) = v_{tt} \end{aligned}$$

Generally:

For $n = 2k - 1$

$$v(t, r) = \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{k-1} (r^{2k-1} \bar{u})$$

LOS 9. Understand the concept of special relativity

LOS 10. Solve the Schrödinger equation

LOS 11. Understand the idea of self-adjoint operator for the Laplace operator

LOS 12. Understand the divergence theorem and Green's Identity

LOS 13. Understand the method of separation of variables for heat and wave equation