Unit 9

LOS 1. Solve wave equation using different separation techniques

LOS 2. Understand theorem on an orthonormal basis of eigenfunctions

Problem:

$$\begin{aligned} u_{tt} &= \Delta u \\ u|_{\partial D} &= \phi \\ u_t|_{\partial D} &= \psi \end{aligned}$$

Separation 1:

$$\begin{aligned} &u = T(t)V(x) \\ &T''V = T\Delta V \\ &T'' = -\lambda T && \to T(t) = Ce^{-\lambda t} \\ &\Delta V = -\lambda V && (1) \end{aligned}$$

Separation 2:

$$\begin{aligned} \text{From (1) } V(r,\theta,\varphi) &= R(r)P(\theta)Q(\varphi) \\ &= R(r)Y(\theta,\varphi) \end{aligned}$$

$$\begin{aligned} &\Delta V = R_{rr}Y + \frac{2}{r}R_rY + \frac{1}{r^2}\Delta_{\theta,\varphi}Y = -\lambda RY \\ &r^2\frac{R_{rr}}{R} + 2r\frac{R_r}{R} + r^2\lambda = \frac{\Delta Y}{Y} = -\gamma \end{aligned}$$

$$\begin{aligned} &r^2\frac{R_{rr}}{R} + 2r\frac{R_r}{R} + r^2\lambda = -\gamma \\ &R_{rr} + \frac{2}{r}R_r + \left(\lambda - \frac{\gamma}{r^2}\right)R = 0 \qquad (2) \end{aligned}$$

$$\Delta Y = -\gamma Y \qquad (3)$$

Consider $\gamma = 0$:

$$
\begin{aligned}
& R_{rr} + \frac{2}{r} R_r + \lambda R = 0 \\
\text{Change of variable: } & w = \sqrt{r} R(r) \\
& w_r = \sqrt{r} R_r + \frac{1}{2\sqrt{r}} R \\
& w_{rr} = \sqrt{r} R_{rr} + \frac{1}{2\sqrt{r}} R_r + \frac{1}{2\sqrt{r}} R_r - \frac{1}{4} r^{-\frac{3}{2}} R
\end{aligned}
$$

Substitute to (2) to get the Bessel function:

$$
w_{rr} + \frac{1}{r} w_r + \left(\lambda - \frac{\gamma + \frac{1}{2}}{r^2} \right) w = 0
$$

Change of variable: $\rho = \sqrt{\lambda} r$: $\qquad w(r) = J_{\gamma + \frac{1}{2}}(\sqrt{\lambda} r)$

Boundary condition: $\qquad J_{\gamma + \frac{1}{2}}(\sqrt{\lambda} a) = 0$

Since $\gamma = 0$: $\qquad J_{\frac{1}{2}}(\sqrt{\lambda} a) = 0$

Theorem: Let Laplacian on S^2 admits an ONB of eigenfunction Y_{kj} such that:

$$\Delta_{\theta,\varphi} Y_{jm} = -\gamma_{jm} Y_{jm}$$

Then: $Y_{jm} = e^{im\varphi} P_l^{|m|} \cos\theta$ (Legendre polynomials)

Hence, solution:

$$
\begin{aligned}
u(t, r, \theta, \varphi) = & \sum_{j,m,k} \frac{1}{\sqrt{r}} \cos\left(t \sqrt{\lambda(\gamma_{mj})} \right) J_{\gamma_{mj}} \left(r \sqrt{\lambda_k(\gamma_{mj})} \right) Y_{mj} \\
& + \sum_{j,m,k} \frac{1}{\sqrt{r}} \sin\left(t \sqrt{\lambda(\gamma_{mj})} \right) J_{\gamma_{mj}} \left(r \sqrt{\lambda_k(\gamma_{mj})} \right) Y_{mj}
\end{aligned}
$$

$$
\underbrace{\frac{1}{\sqrt{r}} J_{\gamma_{mj}} \left(r \sqrt{\lambda_k(\gamma_{mj})} \right)}_{\text{radial part}} \overbrace{Y_{mj}}^{\text{spherical part}} \quad \text{is the eigenfunction to } \Delta V = -\lambda V
$$

Solution to heat problem:

- Problem:

$$
\begin{aligned}
&u_t = \Delta u \\
&u = T(t)V(x) \\
&T(t) = e^{-\lambda t} \\
&\Delta V = -\lambda V
\end{aligned}
$$

- When $\gamma = 0$:

$$
\begin{aligned}
&J_{\frac{1}{2}}(\sqrt{\lambda_k}a) = 0 \\
&u(x,t) = \sum_k \frac{1}{\sqrt{r}} e^{-t\lambda k} a_k J_{\frac{1}{2}}(\sqrt{\lambda_k}r)
\end{aligned}
$$

LOS 3. Learn theory about Hermite polynomials along with its properties

Properties:

1. Generating function: $e^{xz-\frac{z^2}{2}} = \sum_{n=0}^{\infty} P_n(x)\frac{z_n}{n!}$
2. $P_n(x) = \frac{d}{dz}^n\Big|_0$

$$
\begin{aligned}
&P_0(x) = 1 && P_2(x) = x^2 - 1 \\
&P_1(x) = x && P_3(x) = x^3 - 3x
\end{aligned}
$$

3. Let $d\mu(x) = e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}$, then:

$$
\int_{n,m\in\mathbb{N}} P_n(x)P_m(x)d\mu(x) = n!\delta_{nm}
$$

$$
\text{Proof: } \int_{\mathbb{R}} e^{xz-\frac{z^2}{2}} e^{xw-\frac{w^2}{2}} d\mu(x) = \sum_{nm} \frac{z^n w^m}{n!m!} \int_{\mathbb{R}} P_n(x)P_m(x)d\mu(x)
$$

$$
\text{LHS } e^{-\frac{z^2}{2}} e^{-\frac{w^2}{2}} \int_{\mathbb{R}} e^{x(z+w)} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} =
$$

$$
e^{-\frac{z^2}{2}} e^{-\frac{w^2}{2}} e^{\frac{(z+w)^2}{2}} = e^{zw}
$$

$$
\text{RHS } \sum_{k\in\mathbb{R}} z^k w^k \frac{1}{k!}
$$

$$
\text{For } n \neq m \quad \int_{\mathbb{R}} P_n(x)P_m(x)d\mu(x) = 0
$$

$$
\text{For } n = m \quad \frac{1}{n!}\frac{1}{n!} \int_{\mathbb{R}} P_n(x)P_m(x)d\mu(x) = \frac{1}{n!}
$$

4. Previous proof implies theorem:

$$\hat{P}_j(x) = \frac{P_j(x)}{\sqrt{j!}} \text{ ONB for } L_2(R, u)$$

5. $P_n'(x) = nP_{n-1}(x)$

$$\begin{aligned} \text{Proof: } P'(x) &= \frac{d}{dx} e^{xz - \frac{z^2}{2}} \\ &= z e^{xz - \frac{z^2}{2}} \\ &= \sum_{n=0}^{\infty} P_n(x) \frac{z^{n+1}}{n!} = nP_{n-1}(x) \end{aligned}$$

6. $P_{n+1}(x) = xP_n(x) - nP_{n-1}(x)$ or $P_n(x) = xP_{n-1}(x) - (n-1)P_{n-2}(x)$

$$\begin{aligned} \text{Proof: } \frac{d}{dz} e^{xz - \frac{z^2}{2}} &= (x - z) e^{xz - \frac{z^2}{2}} \\ \text{LHS} &= \sum_{n=0}^{\infty} P_n(x) \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} P_{n+1}(x) \frac{z^n}{n!} \\ \text{RHS} &= x \sum_{n=0}^{\infty} P_n(x) \frac{z^n}{n!} - \sum_{n=0}^{\infty} P_n(x) \frac{z^{n+1}}{n!} \end{aligned}$$

7. P_n are eigenfunctions for:

$$A = -\frac{d^2}{dx^2} + x\frac{d}{dx} = -\Delta + (x, \nabla) \qquad \text{(Ornstein-Uhlenbek)}$$
$$AP_n(x) = nP_n(x)$$

Proof $\psi(x,z) = e^{xz - \frac{z^2}{2}}$

$$\frac{d}{dx}\psi(x,z) = z\psi(x,z)$$
$$\frac{d^2}{dx^2}\psi(x,z) = z^2\psi(x,z)$$

$$\begin{aligned} A &= (-z^2 + zx)\psi(x,z) \\ &= z(x-z)\psi(x,z) \\ &= z\frac{d}{dz}\psi(x,z) \end{aligned}$$

$$\begin{aligned} \text{LHS } &= \sum_{n=0}^{\infty} A(P_n)(x)\frac{z^n}{n!} \\ \text{RHS } &= \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} z(nz^{n-1}) \\ &= \sum_{n=0}^{\infty} nP_n(x)\frac{z^n}{n!} \end{aligned}$$

8. A is self-adjoint and $(1+A)^{-1}$ is compact, therefore satisfies:

$$(f, Ah)_\mu = \int_{\mathbb{R}} f'h' d\mu(x)$$
$$(f, h)_\mu = \int_{\mathbb{R}} \overline{f(x)} h(x) d\mu(x)$$

Proof: $\displaystyle\int_{\mathbb{R}} f'h'e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} =$

$$-\int_{\mathbb{R}} f(h'e^{-\frac{x^2}{2}})'\frac{dx}{\sqrt{2\pi}} =$$
$$-\int_{\mathbb{R}} f(h'' - xh')e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} = (f, Ah)_\mu$$

9. Solution to A:

$$A = -\frac{d^2}{dx^2} + x\frac{d}{dx}$$

$$\begin{aligned}
&AP_n(x) = nP_n(x) \\
&-P_n'' + xP_n' = nP_n \\
&0 = P_n'' - xP_n' + nP_n
\end{aligned}$$

$$\begin{aligned}
&Q_n(x) = P_n(\sqrt{2}x) && \text{By change of variables} \\
&Q'(x) = \sqrt{2}P_n'(\sqrt{2}x) \\
&Q''(x) = 2P_n''(\sqrt{2}x)
\end{aligned}$$

$$\begin{aligned}
&0 = P_n''(\sqrt{2}x) - \sqrt{2}xP_n'(\sqrt{2}x) + nP_n(\sqrt{2}x) \\
&0 = Q_n''(x) - 2xQ_n' + 2nQ_n
\end{aligned}$$

LOS 4. Solve harmonic oscillator PDE using separation ansatz and power series ansatz

Problem:

$$-uu_t = u_{xx} - x^2u$$

Separation:

$$\begin{aligned}
&u(t,x) = T(t)V(x) \\
&-iT'V = V'' - x^2T \\
&-i\frac{T'}{T} = \frac{V'' - x^2V}{V} = -\lambda
\end{aligned}$$

$$\begin{aligned}
&T' = \frac{\lambda}{i}T = -i\lambda T \\
&T(t) = Ce^{-i\lambda t}
\end{aligned}$$

$$\begin{aligned}
&V'' + (\lambda - x^2)V = 0 \\
&w(x) = e^{\frac{x^2}{2}}V(x) && \text{By change of variable} \\
&V = e^{-\frac{x^2}{2}}w \\
&V' = xe^{-\frac{x^2}{2}}w + e^{-\frac{x^2}{2}}w \\
&V'' = x^2e^{-\frac{x^2}{2}}w - 2xe^{-\frac{x^2}{2}}w' + e^{-\frac{x^2}{2}}w'' - e^{-\frac{x^2}{2}}w \\
&\quad = -v + x^2v + 2xe^{-\frac{x^2}{2}}w' + e^{-\frac{x^2}{2}}w''
\end{aligned}$$

$$w'' - 2xw' + (\lambda - 1)w = 0$$

Solution:

$$\begin{aligned}
&w(x) = H_{k_0}(x) = \sum_k a_kx^k \\
&(k+2)(k+1)a_{k+2}x^k - 2ka_kx^k + (\lambda - 1)a_kx^k = 0 && k \in \mathbb{Z}^+ \\
&a_{k+2} = \frac{2k - (\lambda - 1)}{(k+2)(k+1)}a_k
\end{aligned}$$

For terminating power series: $2k_0 = \lambda - 1$

$$\begin{aligned}
&H_0(x) = 1 && \lambda = 1 \\
&H_1(x) = x && \lambda = 3 \\
&H_2(x) = 4x^2 - 1 && \lambda = 5 \\
&\vdots
\end{aligned}$$

Observation:

$$
\begin{aligned}
&H_n(x) = 2^{\frac{n}{2}} Q_n = P_n(\sqrt{2}x) \\
&\lambda = 2n + 1 && \text{(from } 2k_0 = \lambda - 1) \\
&u(t, x) = \sum_n a_n e^{-i(2n+1)t} P_n(\sqrt{2}x) e^{-\frac{x^2}{2}} \\
&u(0, \frac{x}{\sqrt{2}}) = \sum_n a_n P_n(x) e^{-\frac{x^2}{2}}
\end{aligned}
$$

The above solution applies if boundary condition satisfies $u(0, \frac{x}{\sqrt{2}}) e^{-\frac{x^2}{2}} \in L_2(\gamma)$