Unit 7

LOS 1. Solve the heat equation using Fourier analysis

Problem:

$$
\begin{aligned}
&u_t = u_{xx} && 0 \le x \le \pi \\
&u(0,t) = h(t) \\
&u(\pi,t) = k(t) \\
&u(x,0) = 0
\end{aligned}
$$

Method 1 (Fourier Analysis):

- Assume u is a solution and takes the form of the generic solution for periodic case:

$$
u(x,t) = \sum_{n=0}^{\infty} u_n(t) f_n(x)
$$

$$
f_n = \begin{cases} 1 & n = 0 \\ \sin(nx) & n \neq 0 \end{cases}
$$

$$
u_t(x,t) = \sum_{n=0}^{\infty} v_n(t) f_n(x)
$$

$$
u_{xx}(x,t) = \sum_{n=0}^{\infty} w_n(t) f_n(x)
$$

- All derivatives are square-integrable ($\in L_2$) and therefore have unique Fourier expansion. Assuming solution is smooth ($\in L_2$), the Fourier expansions above converge pointwise

- Evaluate v_n (assume we can interchange differentiation and integration):

$$
u_t(x,t) = \sum (f_n, u_t) f_n
$$

$$
\therefore v_n(t) = (f_n, u_t) = \int \bar{f}_n u_t dx
$$

$$
v_0 = \int_0^{\pi} u_t(x,t) \frac{dx}{\pi} = \frac{d}{dt} \int_0^{\pi} u(x,t) \frac{dx}{\pi}
$$

$$
v_n = \int_0^{\pi} \sin(nx) u_t(x,t) \frac{2dx}{\pi} = \frac{d}{dt} \int_0^{\pi} \sin(nx) u(x,t) \frac{2dx}{\pi}
$$

$$
\therefore v_n = \frac{d}{dt} u_n = u_n'(t)
$$

- Evaluate w_n using integration by parts:

$$
\begin{aligned}
\frac{\pi}{2}w_n &= \frac{d^2}{dx^2}\int_0^\pi \sin(nx)u(x,t)dx = \int_0^\pi \sin(nx)u_{xx}(x,t)dx \\
\frac{\pi}{2}w_n &= \cancelto{0}{\sin(nx)u_x(t)|_0^\pi} - n\int_0^\pi \cos(nx)u_x(x,t)dx \\
&= -n\left[\cos(nx)u_x(t)|_0^\pi\right] - n^2\int_0^\pi \sin(nx)u_x(x,t)dx \\
&= -n[(-1)^n k(t) - h(t)] - n^2\frac{\pi}{2}u_n(t)
\end{aligned}
$$

- Establish that $w_n = u_n'(t)$:

$$
\begin{aligned}
&v_n(t) = \frac{d}{dx}u_n(t) = u_t = u_n'(t) \\
&w_n(t) = \frac{d^2}{dx^2}u_n(t) = u_x x \\
&u_t = u_{xx} \\
&v_n = w_n \\
&u_n'(t) = w_n
\end{aligned}
$$

$$
\begin{aligned}
&\therefore u_n'(t) = \frac{2}{\pi}\left(-n[(-1)^n k(t) - h(t)] - n^2\frac{\pi}{2}u_n(t)\right) \\
&u_n'(t) = -\frac{2n}{\pi}[(-1)^n k(t) - h(t)] - n^2 u_n(t) \qquad (1)
\end{aligned}
$$

- Solve for (1) using ODE approach:

$$
\begin{aligned}
&u_n'(t) + n^2 u_n(t) = -\frac{2n}{\pi}[(-1)^n k(t) - h(t)] \\
&g' + n^2 g = H
\end{aligned}
$$

- Solve homogeneous and then use Duhamel's principle:

$$
\begin{aligned}
& g' = -n^2 g \\
& S(g)(t) = e^{-tn^2} g(0) \\
& S(\phi)(t) = \sum \hat{\phi}_n f_n = u(x,0) \\
u(x,0) = \phi = 0 \qquad & g(t) = \int_0^t S(t-s)H(s)ds + \cancelto{0}{S(\phi)(t)} \\
& u_n(t) = \frac{2n}{\pi} \int_0^t e^{-(t-s)n^2} [h(s) - (-1)^n k(s)] ds
\end{aligned}
$$

$$
\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n(t) f_n(x) \\
u(x,t) &= u_0(t) f_0(x) + \sum_{n \geq 1}^{\infty} u_n(t) \sin(nx)
\end{aligned}
$$

LOS 2. **Solve the heat equation using transformation onto a source equation**

LOS 3. **Solve homogeneous heat equation in higher dimensions using Fourier analysis**

LOS 4. **Solve nonhomogeneous heat equation in higher dimensions using Fourier analysis**

LOS 5. **Understand the polar coordinates in higher dimensions**

LOS 6. **Determine solution of wave equation using Fourier transforms**

LOS 7. **Determine solution of wave equation using polar coordinates**

LOS 8. **Find the solution of wave equation using spherical means method**

LOS 9. **Understand the concept of special relativity**

LOS 10. **Solve the Schrödinger equation**

LOS 11. **Understand the idea of self-adjoint operator for the Laplace operator**

LOS 12. **Understand the divergence theorem and Green's Identity**

LOS 13. **Understand the method of separation of variables for heat and wave equation**