Unit 5

LOS 1. Compare the solutions for continuous case vs periodic case for a given PDE

Continuous case:

- Problem:

$$u_t = u_{xxxxxx} \qquad -\infty < x < \infty$$
$$u(x,0) = \phi(x) = \int_{\mathbb{R}} e^{i\xi x}\hat{\phi}(k)$$
$$P(x) = x^6$$

- Assume t is fixed:

$$\frac{d}{dt}\hat{u}(\xi,t) = (i\xi)^6 \hat{u}\xi, t$$
$$\hat{u}(\xi,t) = e^{-t\xi^6}\hat{u}(\xi,0)$$

- By Fourier Inversion formula:

$$u(x,t) = \int_{\mathbb{R}} e^{i\xi x}\hat{u}(\xi,t)\frac{d\xi}{\sqrt{2\pi}}$$
$$u(x,t) = \int_{\mathbb{R}} e^{i\xi x}e^{t\xi^6}\hat{u}(\xi,0)\frac{d\xi}{\sqrt{2\pi}} \qquad (1)$$

- By Fourier expansion:

$$f(\xi) = e^{\xi^6} = \int_{\mathbb{R}} e^{i\xi z}\hat{f}(z)\frac{dz}{\sqrt{2\pi}}$$
$$e^{\xi^6} = \int_{\mathbb{R}} e^{-i\xi z}\hat{f}(z)\frac{dz}{\sqrt{2\pi}} \qquad \text{since } \hat{f} \text{ is even}$$
$$e^{t\xi^6} = e^{(t^{\frac{1}{6}}\xi)^6} = \int_{\mathbb{R}} e^{-it^{\frac{1}{6}}\xi z}\hat{f}(z)\frac{dz}{\sqrt{2\pi}}$$

- By change of variable:

$$
\begin{aligned}
y &= t^{\frac{1}{6}} z \\
e^{t\xi^6} &= \int_{\mathbb{R}} e^{-i\xi y} \hat{f}\left(\frac{y}{t^{\frac{1}{6}}}\right) \frac{dy}{t^{\frac{1}{6}}\sqrt{2\pi}} \\
h_t(y) &= \frac{1}{t^{\frac{1}{6}}\sqrt{2\pi}} \hat{f}\left(\frac{y}{t^{\frac{1}{6}}}\right) \\
\therefore e^{t\xi^6} &= \int_{\mathbb{R}} e^{-i\xi y} h_t(y) dy
\end{aligned}
$$

- Substitute back to Equation (1):

$$
\begin{aligned}
u(x,t) &= \int_{\mathbb{R}} e^{i\xi x} e^{t\xi^6} \hat{u}(\xi,0) \frac{d\xi}{\sqrt{2\pi}} \\
u(x,t) &= \int_{\mathbb{R}} e^{i\xi x} \int_{\mathbb{R}} e^{-i\xi y} h_t(y) dy \hat{u}(\xi,0) \frac{d\xi}{\sqrt{2\pi}} \\
u(x,t) &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{i\xi(x-y)} \hat{u}(\xi,0) \frac{d\xi}{\sqrt{2\pi}} \right] h_t(y) dy \\
u(x,t) &= \int_{\mathbb{R}} u(x-y,0) h_t(y) dy \\
u(x,t) &= \int_{\mathbb{R}} \phi(x-y) h_t(y) dy
\end{aligned}
$$

Periodic case:

- Problem:

$$
\begin{gathered}
u_t = u_{xxxxxx} \qquad -\pi < x < \pi \\
u(x,0) = \phi(x) = \sum_{k\in\mathbb{Z}} e^{ikx} \hat{\phi}(\xi) \frac{d\xi}{\sqrt{2\pi}} \\
P(x) = x^6
\end{gathered}
$$

- Assume t is fixed:

$$
\begin{gathered}
\frac{d}{dt} \hat{u}(k,t) = (ik)^6 \hat{u}k, t \\
\hat{u}(k,t) = e^{-tk^6} \hat{u}(k,0)
\end{gathered}
$$

- By Fourier Inversion formula:

$$
u(x,t) = \sum_{k\in\mathbb{Z}} e^{-tk^6} \hat{u}(k,0) e^{ikx} \qquad (1)
$$

- Let:

$$H_t(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{-tk^6}e^{ikx}$$
$$e^{-tk^6} = \hat{H_t(x)} = \int_{-\pi}^{\pi} e^{-iky}H_t(y)dy$$

- Substitute back to Equation (1):

$$u(x,t) = \sum_{k\in\mathbb{Z}}\left[\int_{-\pi}^{\pi} e^{-iky}H_t(y)dy\right]\hat{u}(k,0)e^{ikx}$$
$$u(x,t) = \int_{-\pi}^{\pi}\left[\sum_{k\in\mathbb{Z}} e^{ik(x-y)}\hat{u}(k,0)\right]H_t(y)dy$$
$$u(x,t) = \int_{-\pi}^{\pi} u(x-y,0)H_t(y)dy$$
$$u(x,t) = \int_{-\pi}^{\pi} \phi(x-y)H_t(y)dy$$

In general:

- Upper half plane (continuous):

$$u(x,t) = \int_{\mathbb{R}} \phi_{ext}(x-y)h_t(y)dy = \int_{\mathbb{R}} \phi_{ext}(y)h_t(x-y)dy$$

- Periodic:

$$u(x,t) = \int_{-\pi}^{\pi} \phi(x-y)H_t(y)dy = \int_{-\pi}^{\pi} \phi(y)H_t(x-y)dy$$

Example:

- Problem:

$$u_t = ku_{xx}$$
$$H_t(y) = \frac{c}{\sqrt{kt}}\sum_{j\in\mathbb{Z}} e^{-\frac{|y-j|^2}{4kt}} \qquad -\pi < x < \pi$$
$$h_t(y) = \frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}} \qquad -\infty < x < \infty$$

LOS 2. Analyze the continuity of a solution of a PDE

Solution formula:

$$u_t = ku_{xx} \qquad -\infty < x < \infty$$

$$u(x,t) = \int_{\mathbb{R}} \phi(x-y)\frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy = \int_{\mathbb{R}} \phi(y)\frac{1}{\sqrt{4\pi kt}}e^{-\frac{(x-y)^2}{4kt}}dy$$

If ϕ is continuous and bounded:

$$\lim_{t\to 0} u(x,t) = \phi(x)$$

$$\sup_z |\phi(z)| \leq K$$

Proof:

- Let $x \in \mathbb{R}$ since ϕ is continuous at x $\exists\delta > 0$ $\forall y$ where $|y-x| < \delta \Rightarrow$ $|\phi(x) - \phi(y)| < \epsilon$ where $\epsilon > 0$
- Evaluate the difference:

$$\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy = 1$$

$$u(x,t) - \phi(x) = \int_{\mathbb{R}} \phi(x-y)\frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy - \int_{\mathbb{R}} \phi(x)\frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy$$

$$u(x,t) - \phi(x) = \int_{\mathbb{R}} [\phi(x-y) - \phi(x)]\frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy$$

$$u(x,t) - \phi(x) = \int_{|y|<\delta} [\phi(x-y) - \phi(x)]\frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy + \int_{|y|>\delta} [\phi(x-y) - \phi(x)]\frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy$$

$$|u(x,t) - \phi(x)| \leq \int_{|y|<\delta} [\epsilon]\frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy + \int_{|y|>\delta} [K+K]\frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy$$

- Change of variable:

$$z^2 = \frac{y^2}{4t} \Rightarrow z = \frac{y}{\sqrt{2t}} \Rightarrow dz = \frac{dy}{\sqrt{2t}}$$

$$|u(x,t) - \phi(x)| \leq \epsilon \int_{|y|<\delta} \frac{1}{\sqrt{4\pi kt}}e^{-\frac{y^2}{4kt}}dy + 2K\int_{|z|>\frac{\delta}{\sqrt{2t}}} e^{-\frac{z^2}{2}}\frac{dy}{\sqrt{2\pi}}$$

$$|u(x,t) - \phi(x)| \leq \epsilon + 2Ke^{-\frac{\delta}{t}}$$

$$|u(x,t) - \phi(x)| \leq \epsilon + \epsilon \qquad 2Ke^{-\frac{\delta}{t}} \to \epsilon \text{ when } t \text{ is large}$$

$$\therefore |u(x,t) - \phi(x)| \leq 2\epsilon$$

LOS 3. Describe the maximum principle for heat equation

Theorem:

$$u_t = ku_{xx} \qquad 0 < x < l$$

$$\sup_{(x,t)\in\hat{\Omega}} u(x,t) = \sup_{(x,t)\in\delta\Omega} u(x,t) \qquad \text{where } \delta\Omega \text{ is the boundary enclosing } \hat{\Omega}$$

Proof:

- Introduce a perturbation term where $\epsilon > 0$:

$$v(x,t) = u(x,t) + \epsilon x^2$$

- If (x_0, t_0) is an interior point:

$$v_t(x_0,t_0) = 0 \qquad \text{First derivative at maximum is 0}$$
$$v_x x(x_0,t_0) \leq 0 \qquad \text{Second derivative at maximum is negative or 0}$$
$$\therefore \underbrace{v_t - v_{xx}}_{\geq 0} = \underbrace{u_t - u_{xx}}_{=0} - \underbrace{2k\epsilon}_{>0}$$
$$\underbrace{v_t - v_{xx}}_{\geq 0} = \underbrace{u_t - u_{xx} - 2k\epsilon}_{<0}$$

- LHS and RHS contradicts. Therefore, maximum for v cannot be an interior point (1).
- Prove for u:

$$\sup_{(x,t)\in\hat{\Omega}} u(x,t) \leq \sup_{(x,t)\in\hat{\Omega}} v(x,t) \qquad \text{Since } \epsilon > 0$$
$$\sup_{(x,t)\in\hat{\Omega}} u(x,t) \leq \sup_{(x,t)\in\delta\Omega} v(x,t) \qquad \text{By proof in (1)}$$
$$\sup_{(x,t)\in\hat{\Omega}} u(x,t) \leq \sup_{(x,t)\in\delta\Omega} u(x,t) + \sup_{(x,t)\in\delta\Omega} \epsilon x^2$$
$$\sup_{(x,t)\in\hat{\Omega}} u(x,t) \leq \sup_{(x,t)\in\delta\Omega} u(x,t) + \epsilon l^2$$
$$\sup_{(x,t)\in\hat{\Omega}} u(x,t) \leq \sup_{(x,t)\in\delta\Omega} u(x,t) \qquad \text{When } \epsilon \to 0$$

- Maximum principle: maximum exists in the boundary.

Application:

- Theorem: Heat equation has a unique solution

$$
\begin{gathered}
u_t - k u_{xx} = f \qquad k > 0 \\
u(x,0) = \phi(x) \\
u(0,t) = h(t) \\
u(l,t) = k(t)
\end{gathered}
$$

- Proof: Assume u^1 and u^2 are solution to the heat equation:

$$
\begin{gathered}
w = u^1 - u^2 \\
w_t = k w_{xx} \\
w(x,0) = \phi^1(x) - \phi^2(x) = 0 \\
w(0,t) = h^1(x) - h^2(x) = 0 \\
w(l,t) = k^1(x) - k^2(x) = 0 \\
\max_{(x,t)} w(x,t) \le 0 \qquad \text{By maximum principle} \\
\max_{(x,t)} -w(x,t) \le 0 \qquad \text{By maximum principle} \\
\therefore w \equiv 0 \Rightarrow u^1 \text{ and } u^2 \text{ are equal}
\end{gathered}
$$