Unit 4

LOS 1. Determine the solution operator for a periodic differential equation

Problem:

$$u_t = \mathscr{L}(u) \qquad -\pi < x < \pi, -\infty < t < \infty$$
$$\mathscr{L}(u) = \sum_{j=0}^{n} a_j \frac{\partial^j u}{\partial x^j}$$
$$u(x,0) = \varphi(x)$$

Use the following notation:

$$u^t(x) = u(x,t)$$

Solution:

$$u(x,t) = \sum_{k\in\mathbb{Z}} e^{tP(ik)}\hat{\varphi}(k)e^{ikx}$$

LOS 2. Learn how to calculate eigen functions for a given PDE

LOS 3. Solve a periodic differential equation using Fourier coefficients

Eigenfunctions:

$$e_k(x) = e^{ikx}$$

1. $\frac{d}{dx}e_k = (ik)e_k$
2. $\left(\frac{d}{dx}\right)^j (e_k) = (ik)^j e_k$
3. $\mathscr{L}(e_k) = P(ik)e_k$

Application to PDE:

- Problem:

$$u_t = \mathscr{L}(u)$$
$$u^0 = \varphi$$

- $\mathscr{L}(\psi_k)$ operator the following orthonormal basis expansion (by rule, every function can be expanded as a generalized Fourier series):

$$\mathscr{L}(\psi_k) = \lambda_k \psi_k$$

- Then, the general solution is:

$$u(x,t) = \sum_{\mathbb{R}} e^{t\lambda_k}(\phi_k, \varphi)\psi_k$$

Example:

- Problem:

$$u_t = \mathscr{L}(u) \qquad -\pi < x < \pi, -\infty < t < \infty$$
$$\mathscr{L}(u) = \sum_{j=0}^{n} a_j \frac{\partial^j u}{\partial x^j}$$

$$\varphi(x) = \begin{cases} 1 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$$

$$\varphi(0) = \frac{1}{2}$$

$$P(x) = x^2 \qquad \text{Polynomial for Diffusion}$$

- Fourier dot product:

$$\hat{\varphi}(k) = \int_{-\pi}^{\pi} e^{-ikx}\varphi(x)\frac{dx}{2\pi}$$

- Solution:

$$2\pi\hat{\varphi}(k) = \int_{-\pi}^{\pi} e^{-ikx}\varphi(x)dx$$
$$2\pi\hat{\varphi}(k) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-ikx}dx$$
$$2\pi\hat{\varphi}(k) = \frac{e^{-ikx}}{-ik}\bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{e^{-ik\frac{\pi}{2}} - e^{ik\frac{\pi}{2}}}{-ik} = \frac{e^{ik\frac{\pi}{2}} - e^{-ik\frac{\pi}{2}}}{ik}$$

- Identities:

  1. For $k = 0$: $e^{ik\frac{\pi}{2}} = e^0 = 0$

2. For $k=1$: $e^{ik\frac{\pi}{2}} = e^{i\frac{\pi}{2}} = i$
3. For $k=2$: $e^{ik\frac{\pi}{2}} = e^{i\pi} = -1$
4. For $k=-1$: $e^{ik\frac{\pi}{2}} = e^{-i\frac{\pi}{2}} = -i$

- Therefore:

$$2\pi\hat{\varphi}(k) = \frac{e^{ik\frac{\pi}{2}} - e^{-ik\frac{\pi}{2}}}{ik} = \begin{cases} 0 & k \text{ even} \\ \frac{2}{k} & k = 4m+1 \\ -\frac{2}{k} & k = 4m+3 \end{cases}$$

- Then, we plug this in into the original solution operator:

$$u(x,t) = \sum_{k\in\mathbb{Z}} e^{tP(ik)}\hat{\varphi}(k)e^{ikx}$$

LOS 4. Solve a periodic differential equation with a source term using Fourier coefficients

Problem:

$$u_t - \mathscr{L}(u) = g \qquad -\pi < x < \pi, -\infty < t < \infty$$
$$u^0 = \varphi(x)$$
$$u(x,t) = u^t$$

Solution:

$$u^t = \int_0^t S(t-s)g(s)ds + S(t)(\varphi)$$
$$S(t)(f) = \sum_{k\in\mathbb{Z}} e^{tP(ik)}\hat{f}(k)e_k$$

Proof for $\varphi = 0$:

- Solution (x is not considered as it is being fixed at a single point):

$$v(x,t) = v^t = \int_0^t S(t-s)g(s)ds$$

- By Liebniz rule of integration:

$$\frac{d}{dt}v = \int_0^t \frac{d}{dt}S(t-s)g(s)ds + S(t-t)g(t)$$

- $S(t-t) = S(0)$ is equal to identity:

$$\frac{d}{dt}v = \int_0^t \frac{d}{dt}S(t-s)g(s)ds + g(t)$$

- Evaluate $\int_0^t \frac{d}{dt} S(t-s) g(s) ds$:

$$\frac{d}{dt} S(t-s) g(s) = \frac{d}{dt} \sum_{k \in \mathbb{Z}} e^{tP(ik)} \hat{g}(k) e_k$$
$$\frac{d}{dt} S(t-s) g(s) = \sum_{k \in \mathbb{Z}} e^{tP(ik)} \hat{g}(k) P(ik) e_k$$

$$P(ik) e_k = \mathscr{L}(e_k) \qquad \to \frac{d}{dt} S(t-s) = \sum_{k \in \mathbb{Z}} e^{tP(ik)} \hat{g}(k) \mathscr{L}(e_k)$$
$$\frac{d}{dt} S(t-s) = \mathscr{L} \left(\sum_{k \in \mathbb{Z}} e^{tP(ik)} \hat{g}(k) e_k \right)$$
$$\frac{d}{dt} S(t-s) = \mathscr{L} S(t-s)$$

- Therefore:

$$\frac{d}{dt} v = \int_0^t \mathscr{L} S(t-s) g(s) ds + g(t)$$
$$\frac{d}{dt} v = \mathscr{L} \int_0^t S(t-s) g(s) ds + g(t)$$
$$\frac{d}{dt} v = \mathscr{L} v + g(t)$$

General application:

- Problem

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$$
$$u_t = \mathscr{L}(P_n(\frac{\partial}{\partial x}))(u) = a_0 u + a_1 u_x + ... + a_n u_{x...x} \qquad \text{x...x} \to \text{n times}$$

- Solution operator:

$$u(x,t) = S(t)(f) = \sum_{k \in \mathbb{Z}} e^{tP(ik)} \hat{f}(k) e^{ikx}$$

- Case I:

$$u_t = u_{xx} - i u_{xxx}$$
$$P(x) = x^2 - i x^3$$
$$e^{tP(ik)} = e^{-tk^2} e^{-tk^3} \to \text{Solution is stable as } e^{tP(ik)} \text{ vanishes as k increases}$$

- Case II:

$$u_t = u_{xx} + iu_{xxx}$$
$$P(x) = x^2 + ix^3$$
$$e^{tP(ik)} = e^{-tk^2}e^{tk^3} \to \text{Solution is not stable as } e^{tk^3} \text{ explodes as k increases}$$

- Case III:

$$u_t = u_{xx} \pm u_{xxx}$$
$$P(x) = x^2 \pm x^3$$
$$e^{tP(ik)} = e^{-tk^2}e^{\pm itk^3} \to \text{Solution is stable as } |e^{-tk^2}e^{\pm itk^3}| = e^{-2tk^2}$$

Example:

- Problem:

$$u_t - u_{xxx} = t\cos(x) \qquad -\pi < x < \pi$$
$$\mathscr{L}(u) = u_{xxx}$$
$$P(x) = x^3$$
$$u(x,0) = 0$$

- Solution operator:

$$u^t = \int_0^t S(t-s)g(s)ds \qquad (1)$$
$$u(x,t) = S(t)(f) = \sum_{k\in\mathbb{Z}} e^{tP(ik)}\hat{f}(k)e^{ikx}$$

- Fourier coefficients:

$$\varphi = \sum a_k e_k$$
$$a_k = \hat{\varphi}(k)$$
$$(e_j, f) = \sum_k a_k(e_j, e_k) = a_j$$

- Therefore:

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$
$$\widehat{\cos}(k) = \begin{cases} \frac{1}{2} & k = 1 \\ \frac{1}{2} & k = -1 \\ 0 & \text{elsewhere} \end{cases}$$

- Continuing on Equation (1):

$$u^t = \int_0^t S(t-s)(s\cos(x))ds$$
$$u^t = \int_0^t S(t-s)(\cos(x))(s)ds$$
$$S(t-s)(\cos(x)) = \sum_{k\in\mathbb{Z}} e^{tP(ik)}\widehat{\cos}(k)e^{ikx} = \frac{1}{2}\left[e^{tP(i)}e_1 + e^{tP(-i)}e_{-1}\right]$$

$$u^t = \int_0^t \frac{1}{2}\left[e^{(t-s)P(i)}e_1 + e^{(t-s)P(-i)}e_{-1}\right] sds$$
$$u^t = \frac{1}{2}e_1\int_0^t e^{(t-s)P(i)}sds + \frac{1}{2}e_{-1}\int_0^t e^{(t-s)P(-i)}sds$$

- Evaluate $\int_0^t e^{(t-s)P(i)}sds$ with integration by parts:

$$\int_0^t e^{(t-s)P(i)}sds = e^{it}\left[\int_0^t e^{-si}sds\right] = e^{it}\left[\frac{e^{-si}s}{-i}\bigg|_0^t - \int_0^t \frac{e^{-si}s}{-i}ds\right]$$
$$= e^{it}\left[-ie^{-ti}t - e^{-ti} + 1\right] = -it + e^{it} - 1$$

- Therefore:

$$u^t = \frac{1}{2}e_1(-it - e^{it} - 1) + \frac{1}{2}e_{-1}(it - e^{-it} + 1)$$
$$u^t = it(\frac{e_1 - e_{-1}}{2}) + (\frac{e_1 - e_{-1}}{2}) + (\frac{e^{it}e_1 - e^{-it}e_{-1}}{2})$$
$$u^t = t\sin(x) + \cos(x) + i\sin(t+x)$$

LOS 5. Solve a diffusion equation using Fourier transformations

Proof that a function can be its own Fourier coefficient:

- Let:

$$f(x) = e^{-\frac{x^2}{2}}$$

- Suppose we want to form the following integral:

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{ax} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \qquad a \in \mathbb{R}$$

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{ax-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$\frac{2ax - x^2}{2} = \frac{-(x^2 - 2ax + a^2 - a^2)}{2} = \frac{-[(x-a)^2 - a^2]}{2} = -\frac{(x-a)^2}{2} + \frac{a^2}{2}$$

$$\therefore \hat{f}(\xi) = \int_{\mathbb{R}} e^{\frac{a^2}{2}} e^{-\frac{(x-a)^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$\hat{f}(\xi) = e^{\frac{a^2}{2}} \int_{\mathbb{R}} e^{-\frac{(x-a)^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

- Change of variable:

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-\frac{(x-a)^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} = 1$$

$$\hat{f}(\xi) = e^{\frac{a^2}{2}}$$

$$a = i\xi$$

$$\therefore \hat{f}(\xi) = e^{-\frac{\xi^2}{2}}$$

Facts:

- Parzeval equality:

$$\|\hat{f}\|_{L_2} = \|f\|_{L_2} \Leftrightarrow \sum_{\mathbb{R}} |f(x)|^2 \frac{dx}{2\pi}$$

- Fourier coefficient for continuous transformation:

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) \frac{dx}{\sqrt{2\pi}}$$

- Fourier inversion (recovery formula):

$$f(x) = \int_{\mathbb{R}} e^{-i\xi x} \hat{f}(\xi) \frac{d\xi}{\sqrt{2\pi}}$$

- Function is its own Fourier coefficient:

  1. Gaussian function: $e^{-\frac{\xi^2}{2}} = \int_{\mathbb{R}} e^{-i\xi x} e^{-\frac{x^2}{4t}} \frac{dx}{\sqrt{2\pi}}$

2. General: $e^{-t|\xi|^2} = \int_{\mathbb{R}^n} e^{-i\xi x} e^{-\frac{|x|^2}{2}} \frac{dx}{\sqrt{4\pi t}^n}$

General solution:

- Problem:

$$u_t = \mathscr{L}_P(u)$$
$$u(x,0) = \varphi(x)$$

- Fourier coefficient:

$$\hat{u}^t(\xi) = \hat{u}(\xi,t) = \int_{\mathbb{R}} e^{-i\xi x} u(x,t) \frac{dx}{2\pi}$$

- Observation 1 (Fourier transformation of u applied by the differential operator is a polynomial times $\hat{u}$ as u itself can be expanded using its orthogonal basis $\hat{u}$):

$$\widehat{\mathscr{L}_P(u)}(\xi) = P(i\xi)\hat{u}(\xi)$$
$$u_t = P(i\xi)\hat{u}(\xi)$$

- Observation 2:

$$\widehat{\left(\frac{d}{dt}u\right)}(\xi) = (i\xi)\hat{u}(\xi)$$

- Derivation:

$$u_t = \mathscr{L}_P(u)$$
$$\widehat{\left(\frac{d}{dt}u\right)}(\xi) = \widehat{\mathscr{L}_P(u)}(\xi)$$
$$\frac{d}{dt}\hat{u}(\xi) = P(i\xi)\hat{u}(\xi) \qquad \to \text{ODE}$$
$$\hat{u}(\xi,t) = e^{tP(i\xi)}\hat{u}(\xi,0)$$
$$\hat{u}(\xi,t) = e^{tP(i\xi)}\hat{\varphi}(\xi)$$

- Using recovery formula:

$$u(x,t) = \int e^{i\xi x} e^{tP(i\xi)} \hat{\varphi}(\xi) \frac{d\xi}{\sqrt{2\pi}}$$

- Properties:
  1. Solution works in L_2 space (the above is an L_2 solution)
  2. For a nice φ (infinitely differetiable and decaying at ∞), solution is unique

Solution for heat equation:

- Problem:

$$u_t = u_{xx}$$
$$u(x,0) = \varphi(x)$$

- Solution:

$$u(x,t) = \int_{\mathbb{R}} e^{i\xi x} e^{tP(i\xi)} \hat{\varphi}(\xi) \frac{d\xi}{\sqrt{2\pi}}$$
$$u(x,t) = \int_{\mathbb{R}} e^{i\xi x} e^{-t\xi^2} \hat{\varphi}(\xi) \frac{d\xi}{\sqrt{2\pi}}$$
$$u(x,t) = \int_{\mathbb{R}} e^{i\xi x} \left(\int_{\mathbb{R}} e^{-i\xi y} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{4\pi t}} \right) \hat{\varphi}(\xi) \frac{d\xi}{\sqrt{2\pi}}$$
$$u(x,t) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{i\xi(x-y)} \hat{\varphi}(\xi) \frac{d\xi}{\sqrt{2\pi}} \right) e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{4\pi t}}$$
$$u(x,t) = \int_{\mathbb{R}} \varphi(x-y) e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{4\pi t}} \qquad \Rightarrow S(t)(\varphi)(x)$$

- Alternative solution expression (by change of variable):

$$u(x,t) = \int_{\mathbb{R}} \varphi(z) e^{-\frac{(x-z)^2}{4t}} \frac{dz}{\sqrt{4\pi t}} \qquad \Rightarrow S(t)(\varphi)(x)$$

Integration by parts $\int_{\mathbb{R}} fg' dx = fg|_{\mathbb{R}} - \int_{\mathbb{R}} f' g dx$