Unit 5

LOS 1. Compare the solutions for continuous case vs periodic case for a given PDE

Continuous case:

- Problem:

$$\begin{gathered} u_t = u_{xxxxxx} \qquad -\infty < x < \infty \\ u(x,0) = \phi(x) = \int_{\mathbb{R}} e^{i\xi x}\hat{\phi}(k) \\ P(x) = x^6 \end{gathered}$$

- Assume t is fixed:

$$\begin{gathered} \frac{d}{dt}\hat{u}(\xi,t) = (i\xi)^6\hat{u}\xi, t \\ \hat{u}(\xi,t) = e^{-t\xi^6}\hat{u}(\xi,0) \end{gathered}$$

- By Fourier Inversion formula:

$$\begin{gathered} u(x,t) = \int_{\mathbb{R}} e^{i\xi x}\hat{u}(\xi,t)\frac{d\xi}{\sqrt{2\pi}} \\ u(x,t) = \int_{\mathbb{R}} e^{i\xi x}e^{t\xi^6}\hat{u}(\xi,0)\frac{d\xi}{\sqrt{2\pi}} \qquad (1) \end{gathered}$$

- By Fourier expansion:

$$\begin{gathered} f(\xi) = e^{\xi^6} = \int_{\mathbb{R}} e^{i\xi z}\hat{f}(z)\frac{dz}{\sqrt{2\pi}} \\ e^{\xi^6} = \int_{\mathbb{R}} e^{-i\xi z}\hat{f}(z)\frac{dz}{\sqrt{2\pi}} \qquad \text{since } \hat{f} \text{ is even} \\ e^{t\xi^6} = e^{(t^{\frac{1}{6}}\xi)^6} = \int_{\mathbb{R}} e^{-it^{\frac{1}{6}}\xi z}\hat{f}(z)\frac{dz}{\sqrt{2\pi}} \end{gathered}$$

- By change of variable:

$$
\begin{aligned}
y &= t^{\frac{1}{6}} z \\
e^{t\xi^6} &= \int_{\mathbb{R}} e^{-i\xi y} \hat{f}\left(\frac{y}{t^{\frac{1}{6}}}\right) \frac{dy}{t^{\frac{1}{6}}\sqrt{2\pi}} \\
h_t(y) &= \frac{1}{t^{\frac{1}{6}}\sqrt{2\pi}} \hat{f}\left(\frac{y}{t^{\frac{1}{6}}}\right) \\
\therefore e^{t\xi^6} &= \int_{\mathbb{R}} e^{-i\xi y} h_t(y) dy
\end{aligned}
$$

- Substitute back to Equation (1):

$$
\begin{aligned}
u(x,t) &= \int_{\mathbb{R}} e^{i\xi x} e^{t\xi^6} \hat{u}(\xi,0) \frac{d\xi}{\sqrt{2\pi}} \\
u(x,t) &= \int_{\mathbb{R}} e^{i\xi x} \int_{\mathbb{R}} e^{-i\xi y} h_t(y) dy \hat{u}(\xi,0) \frac{d\xi}{\sqrt{2\pi}} \\
u(x,t) &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{i\xi(x-y)} \hat{u}(\xi,0) \frac{d\xi}{\sqrt{2\pi}} \right] h_t(y) dy \\
u(x,t) &= \int_{\mathbb{R}} u(x-y,0) h_t(y) dy \\
u(x,t) &= \int_{\mathbb{R}} \phi(x-y) h_t(y) dy
\end{aligned}
$$

Periodic case:

- Problem:

$$
\begin{aligned}
u_t = u_{xxxxxx} \qquad & -\pi < x < \pi \\
u(x,0) = \phi(x) &= \sum_{k\in\mathbb{Z}} e^{ikx} \hat{\phi}(\xi) \frac{d\xi}{\sqrt{2\pi}} \\
P(x) &= x^6
\end{aligned}
$$

- Assume t is fixed:

$$
\begin{aligned}
\frac{d}{dt}\hat{u}(k,t) &= (ik)^6 \hat{u}k, t \\
\hat{u}(k,t) &= e^{-tk^6} \hat{u}(k,0)
\end{aligned}
$$

- By Fourier Inversion formula:

$$
u(x,t) = \sum_{k\in\mathbb{Z}} e^{-tk^6} \hat{u}(k,0) e^{ikx} \qquad (1)
$$

- Let:

$$H_t(x) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} e^{-tk^6} e^{ikx}$$

$$e^{-tk^6} = \hat{H_t}(x) = \int_{-\pi}^{\pi} e^{-iky} H_t(y)dy$$

- Substitute back to Equation (1):

$$u(x,t) = \sum_{k\in\mathbb{Z}} \left[\int_{-\pi}^{\pi} e^{-iky} H_t(y)dy\right] \hat{u}(k,0)e^{ikx}$$

$$u(x,t) = \int_{-\pi}^{\pi} \left[\sum_{k\in\mathbb{Z}} e^{ik(x-y)}\hat{u}(k,0)\right] H_t(y)dy$$

$$u(x,t) = \int_{-\pi}^{\pi} u(x-y,0)H_t(y)dy$$

$$u(x,t) = \int_{-\pi}^{\pi} \phi(x-y)H_t(y)dy$$

In general:

- Upper half plane (continuous):

$$u(x,t) = \int_{\mathbb{R}} \phi_{ext}(x-y)h_t(y)dy$$

- Periodic:

$$u(x,t) = \int_{-\pi}^{\pi} \phi(x-y)H_t(y)dy$$

Example:

- Problem:

$$u_t = ku_{xx}$$

$$H_t(y) = \frac{c}{\sqrt{kt}} \sum_{j\in\mathbb{Z}} e^{-\frac{|y-j|^2}{4kt}} \qquad -\pi < x < \pi$$

$$h_t(y) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{y^2}{4kt}} \qquad -\infty < x < \infty$$

LOS 2. Analyze the continuity of a solution of a PDE

LOS 3. Describe the maximum principle for heat equation