

Unit 1

LOS 1. Differentiate between linear and nonlinear PDEs

Characteristic of linear PDE:

1. $\mathcal{L}(u + v) = \mathcal{L}u + \mathcal{L}v$

2. $\mathcal{L}(cu) = c\mathcal{L}u$

Consequence of linearity:

Solution to homogeneous + solution to inhomogeneous = another solution to inhomogeneous

LOS 2. Distinguish between ODEs and PDEs

In general, ODE involves one dependent variable y which is a function of x . On the other hand, PDE involves one dependent u which is a function multiple independent variables x, t, \dots .

Example of ODE:

$$\frac{\partial y}{\partial x} + y = 0$$

Example of PDE:

$$\begin{aligned}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} &= 0 \\ u_x + u_t &= 0\end{aligned}$$

LOS 3. Identify the degree of a given PDEs

Degree of PDE is equal to the highest degree of derivative in the PDE.

Degree 1: $u_x + u_t = 0$

Degree 2: $u_{xx} + u_t = 0$

Degree 3: $u_{xxx} + u_t = 0$

...

LOS 4. Solve first order PDE using geometric method for constant coefficients

First order PDE with constant coefficients:

$$u_x + cu_y = 0$$

General Solution:

$$u(x, y) = f(cx - y)$$

LOS 5. Solve first order PDE using geometric method for variable coefficients

First order PDE with variable coefficients:

$$u_x + yu_y = 0$$

General Solution:

$$u(x, y) = \phi(ye^x)$$

Example:

$$\begin{aligned}u_x + yu_y &= 0, u(x, 1) = x^2 \\u(x, y) &= \phi(ye^{-x}) \\u(x, 1) &= \phi(e^{-x}) = x^2 \\x &= -\ln t \rightarrow \phi(t) = (-\ln t)^2 \\\therefore u(x, y) &= (-\ln ye^{-x})^2 = (-\ln y + x)^2\end{aligned}$$

LOS 6. Distinguish between parabolic, hyperbolic and elliptic PDEs

In this course, at maximum, we will have second order PDE. General formula of second order PDE:

$$\begin{aligned}a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u &= 0 \\a_{12} &= a_{21} \\u_{xy} &= u_{yx}\end{aligned}$$

Type of PDE depends on determinant $\mathcal{D} = a_{12}^2 - a_{11}a_{22}$:

1. Elliptic: $\mathcal{D} < 0$, all eigenvalues are positive
2. Hyperbolic: $\mathcal{D} > 0$, none of the eigenvalues vanish, one eigenvalue has the opposite sign from the $(n - 1)$ others
3. Parabolic: $\mathcal{D} = 0$, one zero eigenvalues and the rest have the same signs

LOS 7. Identify different types of initial and boundary conditions

Types of initial and boundary conditions:

- (D) Dirichlet: $u(0) = \alpha, u(L) = \beta$
- (I) Initial: $u(0) = \alpha, u'(0) = \beta$
- (N) Neumann: $u_x(0) = \alpha, u_x(L) = \beta$
- (R) Robin: $\beta u_x + \alpha u(0) = 0$, linear combination of other initial or boundary conditions

LOS 8. Analyze a PDE for being well-posed: existence, uniqueness and stability

Well-posedness includes:

- Existence: there exist a solution that can be expressed explicitly, solution provided by boundary conditions satisfy the PDE
- Uniqueness: the solution is not dependent on auxillary variables and the same across the defined range
- Stability: when data are changed very little, the corresponding solution also change very little

LOS 9. Solve wave equation using the Fourier method and the operator method

Wave equation:

$$u_{xx} - u_{tt} = 0$$

Fourier Method:

$$\begin{aligned} u_{\xi, \eta} &= e^{i\xi t} e^{i\eta x} & \xi, \eta &\in \mathbb{R} \\ \mathcal{L} u_{\xi, \eta} &= (-\xi^2 + \eta^2) u_{\xi, \eta} \\ u(x, t) &= \phi(x+t) - \psi(x-t) \\ \phi(x) &= \sum a_k e^{i\xi_k x} & \psi(x) &= \sum b_j e^{i\xi_j x} & a_k, b_j &\in \mathbb{C} \end{aligned}$$

Operator Method:

- Expand the PDE:

$$\begin{aligned}\partial_{tt} - \partial_{xx} &= (\partial_t - \partial_x)(\partial_t + \partial_x) \\ y &= (\partial_t - \partial_x)(\partial_t + \partial_x)\end{aligned}$$

- Assume for solution $u(x, t)$ there is a function v which satisfies the following:

$$\begin{aligned}v \mid v(x, t) &= (\partial_t + \partial_x)u(x, t) \\ (\partial_t - \partial_x)v(x, t) &= 0 \\ v_x - v_t &= 0 \\ v(x, t) &= f(x + t) \\ (\partial_t + \partial_x)u(x, t) &= f(x + t)\end{aligned}$$

- By linearity:

$$\begin{aligned}u &= u^0 + w \\ (\partial_t + \partial_x)u^0 &= f(x + t) \quad (\partial_t + \partial_x)w = 0\end{aligned}$$

- By method of characteristic line:

$$\begin{aligned}(\partial_t + \partial_x)w &= 0 \\ w_t + w_x &= 0 \\ w(x, t) &= g(x - t)\end{aligned}$$

- Substitute $z = x + t$ and assume $h' = f$:

$$\begin{aligned}(\partial_t + \partial_x)u^0 &= f(x + t) \\ \frac{d}{dz}u^0(z) &= f(z) \\ u^0(z) &= h(z) \\ u^0(x + t) &= \frac{h(x + t)}{2} \because (\partial_t + \partial_x)u^0 = \frac{h'(x + t)}{2} + \frac{h'(x + t)}{2} = f(x, t)\end{aligned}$$

- Substitute to the original equation:

$$u(x, t) = \frac{h(x + t)}{2} + g(x - t)$$

LOS 10. Learn the principle of causality

General solution to wave equation:

$$\begin{aligned}u_{xx} - u_{tt} &= 0 \\ u(x, 0) &= \phi(x) \\ u_t(x, 0) &= \psi(x) \\ u(x, t) &= f(x + t) + g(x - t) \quad f, g \in C^2\end{aligned}$$

Example with Dirichlet condition:

- Problem statement:

$$\begin{aligned}u_{xx} - u_{tt} &= 0 \\u(x, 0) &= \phi(x) \\u_t(x, 0) &= 0\end{aligned}$$

- Evaluate the boundary conditions:

$$\begin{aligned}u(x, t) &= f(x + t) + g(x - t) \\u(x, 0) &= f(x) + g(x) = \phi(x) \quad (1) \\u_t(x, 0) &= f'(x + t) - g'(x - t) = f'(x) - g'(x) = 0 \quad (2)\end{aligned}$$

- Integrate (2):

$$\begin{aligned}f'(x) + g'(x) &= 0 \\f(x) &= g(x) + c\end{aligned}$$

- From (1) and assume $c = 0$:

$$\begin{aligned}f(x) + g(x) &= \phi(x) \\f(x) + f(x) - c &= \phi(x) \\2f(x) &= \phi(x) + c \\f(x) &= \phi(x)/2 = g(x) \\\therefore u(x, t) &= \frac{\phi(x + t) + \phi(x - t)}{2}\end{aligned}$$

- From (1) and assume $c = 0$:

$$\begin{aligned}\phi(x) &= u(x, 0) \\\therefore u(x, t) &= \frac{u(x + t, 0) + u(x - t, 0)}{2}\end{aligned}$$

The above equation implies causality. The value of u at point t can be predicted if we know exactly the points at $t = 0$, $u(x + t, 0)$ and $u(x - t, 0)$.

LOS 11. Theorem on solution for wave equation

Example with Neumann condition:

- Problem statement:

$$\begin{aligned}u_{xx} - u_{tt} &= 0 \\u(x, 0) &= 0 \\u_t(x, 0) &= \psi(x)\end{aligned}$$

- Evaluate the boundary conditions:

$$\begin{aligned}u(x, t) &= f(x + t) + g(x - t) \\u(x, 0) &= f(x) + g(x) = 0 \implies f = -g \\u_t(x, 0) &= f'(x + t) - g'(x - t) \rightarrow 2f'(x) = \psi(x) \\f(x) &= \frac{1}{2} \int_{-\infty}^x \psi(y) dy \\\therefore u(x, t) &= \frac{1}{2} \left(\int_{-\infty}^{x+t} \psi(y) dy + \int_{-\infty}^{x-t} \psi(y) dy \right) = \frac{1}{2} \int_{[x+t, x-t]} \psi(y) dy\end{aligned}$$

Therefore, the general solution for the wave equation:

$$\begin{aligned}u_{xx} - u_{tt} &= 0 \\u(x, 0) &= \phi(x) \\u_t(x, 0) &= \psi(x) \\u(x, t) &= \frac{\phi(x + t) + \phi(x - t)}{2} + \frac{1}{2} \int_{[x+t, x-t]} \psi(y) dy\end{aligned}$$

Alternative formulation (S^1 is solution operator for Dirichlet condition and S^2 is solution operator for Neumann condition and \dot{S} is the derivative):

$$\begin{aligned}S^1(\phi)(x, t) &= \frac{\phi(x + t) + \phi(x - t)}{2} \\S^1(\phi)(x, 0) &= \phi(x) = Id \\\dot{S}^1(\phi)(x, 0) &= 0\end{aligned}$$

$$\begin{aligned}S^2(\psi)(x, t) &= \frac{1}{2} \int_{[x+t, x-t]} \psi(y) dy \\S^2(\psi)(x, 0) &= 0 \\\dot{S}^2(\psi)(x, 0) &= \psi(x) = Id\end{aligned}$$

$$\therefore S(\phi, \psi) = S^1(\phi) + S^2(\psi)$$