Unit 3

LOS 1. Understand the relation between Hilbert spaces and convergence

Hilbert space norm is L^2 norm.

LOS 2. Distinguish between L2-norm and uniform norm

Infinity-norm is used to assert stability. If boundary values are uniformly close, their solution should be uniformly close (fot finite time interval).

Infinity-norm:

$$\|\phi\|_\infty = \sup_x |\phi(x)|$$
$$\|\phi\| = \max_x |\phi(x)|$$
$$\|\phi\|_T = \max_{x, 0\leq t\leq T} |\phi(x,t)|$$

L^2-norm:

- Definition:

$$\|\phi\| = \sqrt{\int_{\mathbb{R}} |\phi(x)|^2 dx}$$
$$\|f\| = (f,f)^{\frac{1}{2}}$$

- For $v = C[0, 2\pi]$:

$$(f,g) = \int_0^{2\pi} \overline{f(t)} g(t) \frac{dt}{2\pi}$$
$$\|f\| = (f,f)^{\frac{1}{2}} = \left(\int_0^{2\pi} |f(t)|^2 \frac{dt}{2\pi} \right)^{\frac{1}{2}}$$

LOS 3. Distinguish between uniform convergence and L2-convergence

Relationship between L^2 and uniform norm:

$$\int |f(t)|^2 \frac{dt}{x\pi} \leq \int \max_s |f(s)|^2 \frac{dt}{2\pi} = \|f\|_\infty^2 \int_{-\pi}^{\pi} \frac{dt}{2\pi} \leq \|f\|_\infty^2$$
$$\left(\int |f(t)|^2 \frac{dt}{x\pi}\right)^{\frac{1}{2}} \leq \|f\|_\infty$$
$$\|f\|_2 \leq \|f\|_\infty$$
$$\therefore 0 \leq \|f_n - f\|_2 \leq \|f_n - f\|_\infty$$

If uniform convergence goes to zero, then L^2 goes to zero by squeeze theorem. Therefore, uniform convergence implies L^2 convergence.

LOS 4. Learn the connection of convergence with the theory of orthonormal systems

Definition of an orthonormal system:

$$(x_j, x_k) = \delta_{j,k}$$

$$\delta_{j,k} = \begin{cases} 0 & x \neq j \\ 1 & x = j \end{cases}$$

Assume we are estimating function x which we are estimating with $(x_1, x_2, x_3, ...)$:

$$x = \sum a_j x_j$$
$$\| \sum a_j x_j \|^2 = \sum_j |g_j|^2$$
$$a \perp b \to \|a + b\|^2 = \|a\|^2 + \|b\|^2$$

Lemma:

$$W = \text{span}\{x_1, x_2, x_3, ...\}$$
$$W \in V$$
$$x \notin V$$
$$P_x = \sum_{j=1}^{n} (x_j, x) x_j$$
$$\|x - y\|^2 = \|x - P_x + P_x - y\|^2 = \|x - P_x\|^2 + \|P_x - y\|^2 \geq \|x - P_x\|^2$$

P_x is a projection of x unto W. To minimize $\|x-y\|^2$, we have to minimize $\|x-P_x\|^2+\|P_x-y\|^2$. This upper bound is minimized when $y=P_x$. Therefore:

$$\|x-y\|^2 \geq \|x-P_x\|^2$$
$$\|x-y\| \geq \|x-P_x\|$$
$$\inf \|x-y\| = \|x-P_x\|, y \in W$$

*(inf refers to the greatest lower bound of the set)

To ensure convergence:

$$\lim_{n\to\infty} \|x-\sum_{j=1}^{n} a_j x_j\| = 0$$
$$\lim_{n\to\infty} \|x-P_x\| = 0$$
$$\therefore W_n = \text{span}\{x_1, x_2, x_3, ..., x_n\} \to \lim_{n\to\infty} \|x-P_{W_n}\|$$

Since $\sum_{j=1}^{n} a_j x_j$ must equal P_x to minimize $\|x-\sum_{j=1}^{n} a_j x_j\|$, then:

$$\sum_{j=1}^{n} a_j x_j = \sum_{j=1}^{n} (x_j, x) x_j$$
$$a_j = (x_j, x)$$

The above coefficient is referred as generalized Fourier coefficient. The best approximation is therefore:

$$x = P_{W_n} = \sum_{j=1}^{n} (x_j, x) x_j$$

LOS 5. Study theorems on orthonormal projections and basis

Theorem:

1. Theorem A. $|g|_H$ is Hilbert-borm (the same as L^2 norm):

$$f \in C[-\pi, \pi]$$
$$\lim_{n\to\infty} \|f-P_n(f)\|_H = 0$$

2. Theorem B. All function has a Fourier expansion that converges.

$$f \in C[-\pi, \pi]$$

$$\epsilon > 0 \qquad \exists g_m = \sum_{k=-m}^{m} a_k e_k$$

$$\|f - g_m\|_\infty \leq \epsilon$$

$$\therefore \|f - P_n(f)\|_H = \|f - g_m - P_n(f - g_m)\|_H \leq \|f - gm\|_H \leq \|f - g_m\|_\infty \leq \epsilon$$

Implication of theorems:

- Fourier expansion of any function will converge to f. Finite dimension $P_n(f)$ approximation makes the norm smaller:

$$f = \sum_{k \in \mathbb{Z}} \hat{f}(k) f_k$$

$$\int_{-\pi}^{\pi} |f(t)|^2 \frac{dt}{2\pi} = \sum_{k \in \mathbb{Z}} |f\hat{(k)}|^2$$

$$\|P_n(f)\|^2 = \sum_{k=-n}^{n} |f\hat{(k)}|^2$$

$$\|P_n(f)\|^2 \leq \|f\|^2$$

  Above is the Bessel's inequality. $\|f\|^2$ is the sum over all the space while $\|P_n(f)\|^2$ is the sum over a finite space.

- For function to satisfy $\sum |f\hat{(k)}|^2 < \infty$:

$$L^2\text{-space} = \{f = \lim_n f_n \text{ almost everywhere where}$$
$$\exists f_n \in C[-\pi, \pi],\ \|f_n - fn + 1\| < c2^{-n}\}$$

LOS 6. Understand the importance of orthonormal basis in PDE systems

Using appropriate orthonormal basis, PDE can be solved in an easier manner.

Example of orthonormal basis to solve PDE:

1. Fourier coefficients: $e_k(t) = e^{ikt}$ (a periodic function). General formulation:

$$W_n = \text{span}\{e_k \mid -n \leq k <\leq n\}$$

$$P_{W_n} = \sum_{k=-n}^{n} (e_k, f) e_k = \sum_{k=-n}^{n} \hat{f}_k e_k$$

$$\lim_n P_{W_n}(f) = f$$

*When f_odd, $\hat{f}_odd$.

2. Discrete Fourier coefficients: $\tilde{e}_k(j) = e^{\frac{2\pi ik}{n}j}$. Used to estimate piecewise, locally constant function. Continuous function can be approximated as the linear combination of the piecewise function.

3. Hermite polynomials. $He_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{\frac{-x^2}{2}}$. Used to estimate $\partial\gamma(x) = e^{\frac{-|x|^2}{2}}\partial x$.

4. Sine function. $S_k = sin(kx)$ forms an orthonormal basis in $f \in L^2(0,\pi)$.

Example problem using Fourier coefficients:

- Function to estimate:

$$f = \mathbb{1}_{[\frac{-\pi}{2},\frac{\pi}{2}]}$$

- Estimate:

$$W_1 = \text{span}\{\mathbb{1}, x_1, -x_1\}$$

$$P_{W_1} = \sum_{j=1}^{n}(x_j, x)x_j = (\mathbb{1}, f)\mathbb{1} + (x_1, f)x_1 + (x_2, f)x_2$$

$$x_k = e^{ikt}$$

$$(x_1, f) = \hat{f}(1) = \int_{-\pi}^{\pi} e^{-it} f(t)\frac{dt}{2\pi} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-it}\frac{dt}{2\pi} = \frac{e^{-i\frac{\pi}{2}} - e^{i\frac{\pi}{2}}}{-2i\pi}$$

- e^{ikx} is periodic:

$$e^{i\frac{\pi}{2}} = i \qquad e^{-i\frac{\pi}{2}} = -i$$

$$\frac{e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}}}{-2i\pi} = \frac{-i-i}{-2i\pi} = \frac{1}{\pi}$$

$$(x_1, f) = \frac{1}{\pi}$$

- Using the same formula:

$$(\mathbb{1}, f) = \frac{1}{2} \qquad (x_2, f) = -\frac{1}{pi}$$

- Therefore:

$$P_{W_1} = \frac{1}{2}\mathbb{1} + \frac{1}{\pi}e^{it} - \frac{1}{\pi}e^{-it}$$

General application to PDE:

- Assume there is operator such that:

$$\mathscr{L} = \sum_{j=0}^{n} a_j \frac{\partial^j}{\partial x^j} u$$

- Alternatively (P_n below is a polynomial, not a projection):

$$P_n(x) = \sum_{j=0}^{n} a_j x^j$$
$$\mathscr{L} = P_n(\frac{\partial}{\partial x})$$

- Problem:

$$u_t = \mathscr{L}(u) \qquad u(x,0) = \phi(x)$$

- Assume that the following is true:

$$u(x,t) = \sum_{k\in\mathbb{Z}} \phi(k,t) e_k(x)$$

- Apply the operator:

$$u_t(x,t) = \sum_{k\in\mathbb{Z}} \frac{d}{dt}\phi(k,t) e_k(x)$$
$$\mathscr{L}(u)(x,t) = \sum_{k\in\mathbb{Z}} \phi(k,t)\mathscr{L}(e_k)(x) \qquad (1)$$

- Evaluate $\mathscr{L}(e_k)(x)$:

$$\frac{d}{dt}e^{ikx} = (ik)^j e^{ikx}$$
$$\therefore L(e_k)(x) = P_n(ik)e_k$$

 Where P_n is a polynomial of ik (not the same as P_n defined above)

- Continuing on Equation (1):

$$\mathscr{L}(u)(x,t) = \sum_{k\in\mathbb{Z}} \phi(k,t)P_n(ik)e_k(x) = u_t(x,t)$$
$$\sum_{k\in\mathbb{Z}} \phi(k,t)P_n(ik)e_k(x) = \sum_{k\in\mathbb{Z}} \frac{d}{dt}\phi(k,t)e_k(x)$$

- Fourier coefficients must be the same:

$$\phi(k,t)P_n(ik) = \frac{d}{dt}\phi(k,t)$$
$$\phi(k,t) = C_k e^{tP_n(ik)}$$
$$\therefore u(x,t) = \sum_{k\in\mathbb{Z}} C_k e^{tP_n(ik)} e_k$$

- When $t = 0$:

$$u(x,0) = \phi(x) = \sum_{k \in \mathbb{Z}} \hat{\phi}(k) e_k$$
$$\therefore C_k = \hat{\phi}(k)$$

- Convergence can be evaluated by looking at the $P_n(ik)$ term. For evaluation:

$$\mathscr{L} = \frac{d^2}{dx^2}$$
$$P_n(x) = x^2 P_n(ik) = (ik)^2 = -k^2 \rightarrow \text{ in } u(x,t)\text{, the terms vanish}$$
$$f\mathscr{L} = \frac{d^4}{dx^4}$$
$$P_n(x) = x^4 P_n(ik) = (ik)^4 = k^4 \rightarrow \text{ in } u(x,t)\text{, the terms does not vanish}$$

LOS 7. Apply Fourier series for energy problem

Problem:

$$\mathscr{L} = -\frac{\partial^4}{\partial x^4}$$
$$u_t = \mathscr{L} u = -u_{xxxx}$$
$$u(x,0) = \phi(x)$$

Assume solution is in L^2:

$$u^t = \sum_{k \in \mathbb{Z}} \hat{u}^t(k) e_k$$
$$\frac{d}{dt} u^t = \sum_{k \in \mathbb{Z}} \hat{u}^t(k) \mathscr{L}(e_k)$$

$$\mathscr{L}(e_k) = \mathscr{L}(e^{ik}) = -\frac{\partial^4}{\partial x^4}(e^{ikx})$$
$$\mathscr{L}(e_k) = -k^4 e^{ikx} = -k^4 e_k$$
$$\frac{d}{dt} u^t = \sum_{k \in \mathbb{Z}} -k^4 \hat{u}^t(k) e_k$$

Assume x is constant:

$$\frac{d}{dt} u^t = \sum_{k \in \mathscr{Z}} \frac{d}{dt} \hat{u}^t(k) e_k = \sum_{k \in \mathbb{Z}} -k^4 \hat{u}^t(k) e_k$$

For two functions with Fourier coefficients, their Fourier coefficients have to be equal:

$$\frac{d}{dt}\hat{u}^t(k) = -k^4\hat{u}^t(k)$$

The above problem is an ODE:

$$\hat{u}^t(k) = e^{-tk^4}\hat{u}^0(k) = e^{-tk^4}\hat{\phi}(k)$$
$$\therefore u(x,t) = \sum_{k\in\mathbb{Z}} e^{-tk^4}\hat{\phi}(k)e_k$$

Energy can be measured by the L^2 norm:

$$\|u^t\|_2^2 = \sum \|\hat{u}^t(k)\|^2 = \sum_k \|e^{-tk^4}\hat{\phi}(k)\|^2 \leq \sum_k \|\hat{\phi}(k)\|^2 = \|u^0\|_2^2$$
$$\|u^t\|_2^2 \leq \|u^0\|_2^2$$
$$\|u^t\|_2 \leq \|u^0\|_2$$