Unit 7

LOS 1. Solve the heat equation using Fourier analysis

Problem:

$$
\begin{aligned}
&u_t = u_{xx} && 0 \le x \le \pi \\
&u(0,t) = h(t) \\
&u(\pi,t) = k(t) \\
&u(x,0) = 0
\end{aligned}
$$

Method 1 (Fourier Analysis):

- Assume u is a solution and takes the form of the generic solution for periodic case:

$$
\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n(t) f_n(x) \\
f_n &= \begin{cases} 1 & n = 0 \\ \sin(nx) & n \neq 0 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
u_t(x,t) &= \sum_{n=0}^{\infty} v_n(t) f_n(x) \\
u_{xx}(x,t) &= \sum_{n=0}^{\infty} w_n(t) f_n(x)
\end{aligned}
$$

- All derivatives are square-integrable ($\in L_2$) and therefore have unique Fourier expansion. Assuming solution is smooth ($\in L_2$), the Fourier expansions above converge pointwise

- Evaluate v_n (assume we can interchange differentiation and integration):

$$
\begin{aligned}
&u_t(x,t) = \sum (f_n, u_t) f_n \\
&\therefore v_n(t) = (f_n, u_t) = \int \bar{f}_n u_t dx
\end{aligned}
$$

$$
\begin{aligned}
&v_0 = \int_0^{\pi} u_t(x,t) \frac{dx}{\pi} = \frac{d}{dt} \int_0^{\pi} u(x,t) \frac{dx}{\pi} \\
&v_n = \int_0^{\pi} \sin(nx) u_t(x,t) \frac{2dx}{\pi} = \frac{d}{dt} \int_0^{\pi} \sin(nx) u(x,t) \frac{2dx}{\pi} \\
&\therefore v_n = \frac{d}{dt} u_n = u_n'(t)
\end{aligned}
$$

- Evaluate w_n using integration by parts:

$$\frac{\pi}{2}w_n = \frac{d^2}{dx^2}\int_0^\pi \sin(nx)u(x,t)dx = \int_0^\pi \sin(nx)u_{xx}(x,t)dx$$

$$\begin{aligned}
\frac{\pi}{2}w_n &= \cancelto{0}{\sin(nx)u_x(t)|_0^\pi} - n\int_0^\pi \cos(nx)u_x(x,t)dx \\
&= -n\left[\cos(nx)u_x(t)|_0^\pi\right] - n^2\int_0^\pi \sin(nx)u_x(x,t)dx \\
&= -n[(-1)^n k(t) - h(t)] - n^2\frac{\pi}{2}u_n(t)
\end{aligned}$$

- Establish that $w_n = u_n'(t)$:

$$\begin{aligned}
v_n(t) &= \frac{d}{dx}u_n(t) = u_t = u_n'(t) \\
w_n(t) &= \frac{d^2}{dx^2}u_n(t) = u_x x \\
u_t &= u_{xx} \\
v_n &= w_n \\
u_n'(t) &= w_n
\end{aligned}$$

$$\begin{aligned}
\therefore u_n'(t) &= \frac{2}{\pi}\left(-n[(-1)^n k(t) - h(t)] - n^2\frac{\pi}{2}u_n(t)\right) \\
u_n'(t) &= -\frac{2n}{\pi}[(-1)^n k(t) - h(t)] - n^2 u_n(t) \qquad (1)
\end{aligned}$$

- Solve for (1) using ODE approach:

$$\begin{aligned}
u_n'(t) + n^2 u_n(t) &= -\frac{2n}{\pi}[(-1)^n k(t) - h(t)] \\
g' + n^2 g &= H
\end{aligned}$$

- Solve homogeneous and then use Duhamel's principle:

$$\begin{aligned}
&& g' &= -n^2 g \\
&& S(t)(\phi) &= e^{-tn^2} g(0) = e^{-tn^2}\phi \\
&& g(t) &= \int_0^t S(t-s)H(s)ds + e^{-tn^2}\phi \\
u(x,0) = \phi = 0 && g(t) &= \int_0^t S(t-s)H(s)ds + \cancelto{0}{e^{-tn^2}\phi} \\
&& u_n(t) &= \frac{2n}{\pi}\int_0^t e^{-(t-s)n^2}[h(s) - (-1)^n k(s)]ds
\end{aligned}$$

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n(t) f_n(x) \\
u(x,t) &= u_0(t) f_0(x) + \sum_{n\geq 1}^{\infty} u_n(t)\sin(nx)
\end{aligned}$$

- Evaluate $u_0(t)f_0(x)$:

$$\begin{aligned}
& f_0(x) = 1 \\
& u(x,t) = u_0(t) + \sum_{n\geq 1}^{\infty} u_n(t)\sin(nx) \\
& u(0,t) = u_0(t) + \sum_{n\geq 1}^{\infty} u_n(t)\cancelto{0}{\sin(nx)} \\
& u(0,t) = u_0(t) \to h(t) = u_0(t) \\
& \therefore u(x,t) = h(t) + \sum_{n\geq 1}^{\infty} u_n(t)\sin(nx)
\end{aligned}$$

LOS 2. Solve the heat equation using transformation onto a source equation

Problem:

$$\begin{aligned}
& u_t = u_{xx} && 0 \leq x \leq \pi \\
& u(0,t) = h(t) \\
& u(\pi,t) = k(t) \\
& u(x,0) = 0
\end{aligned}$$

Method 2 (Shifting Data / Transformation onto a source equation):

- Assume a solution w:

$$
\begin{aligned}
&w_t = w_{xx} \\
&w(0,t) = h(t) \\
&w(\pi,t) = k(t) \\
&w(x,t) = \left(1 - \frac{x}{\pi}\right) h(t) + \left(\frac{x}{\pi}\right) k(t)
\end{aligned}
$$

- Assume u is also a solution such that:

$$
\begin{aligned}
&v = u - w \\
&v(0,t) = 0 \\
&v(\pi,t) = 0 \\
&v_{xx} = u_{xx} - w_{xx} \\
&v_t = u_t - w_t \\
&v_{xx} - v_t = u_{xx} - w_{xx} - (u_t - w_t)
\end{aligned}
$$

$$
\begin{aligned}
&w_{xx}(x,t) = 0 \\
&w_t(x,t) = \left(1 - \frac{x}{\pi}\right) h'(t) + \left(\frac{x}{\pi}\right) k'(t) \\
&\therefore v_{xx} - v_t = w_t \\
&v(x,0) = \cancelto{0}{u(x,0)} - w(x,0) = -\left(1 - \frac{x}{\pi}\right) h(0) - \left(\frac{x}{\pi}\right) k(0) = \phi(x)
\end{aligned}
$$

- In general:

$$
\begin{aligned}
&v_t = L(v) \\
&v_t - v_{xx} = g(x,t) \\
&v(x,0) = \phi(x)
\end{aligned}
$$

$$
v(x,t) = \int_0^t S(t-s)g(s)ds + S(t)(\phi)
$$

$$
\int_0^t S(t-s)g(s)ds = \int_0^t \left(\hat{g}_0(x,s) + \sum_{n\geq 1}^{\infty} e^{-(t-s)n^2} \hat{g}_n(x,s)\sin(nx) \right) ds
$$

$$
\begin{aligned}
\hat{g}_n(x,s) &= \frac{2}{\pi}\int_0^\pi \sin(nx) g(x,s)dx \\
\hat{g}_0(x,s) &= \frac{1}{\pi}\int_0^\pi g(x,s)dx
\end{aligned}
$$

$$
\begin{aligned}
S(t)(\phi) &= \tilde{\phi}_0 + \sum_{n\geq 1} e^{-tn^2}\tilde{\phi}_n \sin(nx) \\
\tilde{\phi}_n(x) &= \frac{2}{\pi}\int_0^\pi \sin(nx)\phi(x)dx \\
\tilde{\phi}_0(x) &= \frac{1}{\pi}\int_0^\pi \phi(x)dx
\end{aligned}
$$

- Combining the solutions:

$$
\begin{aligned}
v(x,t) &= \tilde{\phi}_0 + \sum_{n\geq 1} e^{-tn^2}\tilde{\phi}_n \sin(nx) + \sum_{n\geq 1}^{\infty}\int_0^t e^{-(t-s)n^2}\hat{g}_n(x,s)ds\sin(nx) \\
&= \tilde{\phi}_0 + \sum_{n\geq 1} e^{-tn^2}\tilde{\phi}_n \sin(nx) + \sum_{n\geq 1}^{\infty} v_n(t)\sin(nx)
\end{aligned}
$$

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LOS 3. Solve homogeneous heat equation in higher dimensions using Fourier analysis

Problem

$$
\begin{aligned}
&\Delta f = \sum_{k=1}^{n} \frac{d^2}{dx_k^2} f \\
&u_t = \Delta u \\
&u(0, x) = \phi(x)
\end{aligned}
$$

Facts:

- Fourier transform formula:

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i(\xi,x)} f(x) \frac{dx}{\sqrt{2\pi}^n}$$

- Dot product:

$$(\xi, x) = \sum_{j=1}^{n} \xi_j x_j$$

- Lemma 1:

$$
\begin{aligned}
&f(x) = e^{-\frac{|x^2|}{2}} \\
&|x| = (x, x)^{\frac{1}{2}} \\
&\hat{f}(\xi) = e^{-\frac{|\xi^2|}{2}}
\end{aligned}
$$

- Lemma 2 (Parseval):

$$\|f\|_{L_2} = \|\hat{f}\|_{L_2}$$

- Lemma 3 (Fourier inversion - applies when f is sufficiently smooth and decays when approaching infinity):

$$f(x) = \int_{\mathbb{R}} e^{-i(\xi,x)} \hat{f}(\xi) \frac{d\xi}{\sqrt{2\pi}^n}$$

- Lemma 4:

$$
\begin{aligned}
&\widehat{\frac{d}{dx_k} f} = i\xi_k \hat{f}(\xi) \\
&\hat{u}_t(t, \xi) = \widehat{\frac{d}{dt}} u(t, \xi) = \widehat{\frac{d^2}{dx^2}} u(t, \xi) = -|\xi|^2 \hat{u}(t, \xi) \\
&\hat{u}(t, \xi) = e^{-t|\xi|^2} \hat{u}(0, \xi) = e^{-t|\xi|^2} \hat{\phi}(\xi)
\end{aligned}
$$

- Therefore, for u: $\mathbb{R} \times \mathbb{R}^n$ ($\mathbb{R}$ for t space and $\mathbb{R}^n$ for x dimension):

$$\hat{u}(t,\xi) = \int_{\mathbb{R}} e^{-i(\xi,x)} u(t,x) \frac{d\xi}{\sqrt{2\pi}^n}$$

- From Lemma 1:

$$e^{-t|\xi|^2} = e^{-\frac{|\sqrt{2t}\xi|^2}{2}} = \int_{\mathbb{R}^n} e^{i(-\sqrt{2t}\xi,x)} e^{-\frac{|x|^2}{2}} \frac{dx}{\sqrt{2\pi}^n}$$

- By change of variable:

$$\begin{aligned}
&y = \sqrt{2t}x \\
&dy = \sqrt{2t}^n dx \\
&\therefore e^{-t|\xi|^2} = \int_{\mathbb{R}^n} e^{-i(\xi,y)} e^{-\frac{|y|^2}{4t}} \frac{dx}{\sqrt{4\pi t}^n}
\end{aligned}$$

$$\begin{aligned}
u(t,x) &= \int_{\mathbb{R}^n} e^{i(\xi,x)} \hat{u}(t,\xi) \frac{d\xi}{\sqrt{2\pi}^n} \\
&= \int_{\mathbb{R}^n} e^{i(\xi,x)} e^{-t|\xi|^2} \hat{\phi}(\xi) \frac{d\xi}{\sqrt{2\pi}^n} \\
&= \int_{\mathbb{R}^n} e^{i(\xi,x)} \int_{\mathbb{R}^n} e^{-i(\xi,y)} e^{-\frac{|y|^2}{4t}} \frac{dx}{\sqrt{4\pi t}^n} \hat{\phi}(\xi) \frac{d\xi}{\sqrt{2\pi}^n} \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\xi,x)} e^{-i(\xi,y)} \hat{\phi}(\xi) \frac{d\xi}{\sqrt{2\pi}^n} e^{-\frac{|y|^2}{4t}} \frac{dx}{\sqrt{4\pi t}^n} \\
&= \int_{\mathbb{R}^n} \phi(x-y) e^{-\frac{|y|^2}{4t}} \frac{dx}{\sqrt{4\pi t}^n}
\end{aligned}$$

LOS 4. Solve nonhomogeneous heat equation in higher dimensions using Fourier analysis

Problem:

$$\begin{aligned}
u_t - \Delta u &= w \\
u(0,x) &= \phi(x)
\end{aligned}$$

Homogeneous solution:

$$S(t)(\phi)(x) = \int_{\mathbb{R}^n} \phi(x-y) e^{\frac{-|y|^2}{4t}} \frac{dy}{\sqrt{4\pi t}^n}$$

Inhomogeneous solution (using Duhamel):

$$u(t,x) = S(t)(\phi)(x) + \int_0^t S(t-s) w(s) ds$$

LOS 5. Understand the polar coordinates in higher dimensions

Problem:

$$\begin{aligned} u_{tt} &= \Delta u \\ u(0,x) &= \phi(x) \\ u_t(0,t) &= \psi(x) \end{aligned}$$

From LOS 6:

$$\begin{aligned} \hat{u}(t,\xi) &= cos(t|\xi|)\hat{\phi}(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{\psi}(\xi) \\ &= \widehat{u_\phi}(t,\xi) + \widehat{u_\psi}(t,\xi) \end{aligned}$$

Remark:

$$\frac{d}{dt}\hat{u}_\psi(t,\xi) = cos(t|\xi|)\hat{\phi}(\xi)$$

Example in 1-D (shows that we can focus on getting u_ψ only to get the full solution):

$$\begin{aligned} u(t,x) &= \frac{\phi(x+t)+\phi(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \phi(x,s)ds \\ &= u_\phi + u_\psi \end{aligned}$$

$$\frac{d}{dt}u_\psi(t,x) = \frac{\psi(x+t)+\psi(x-t)}{2} \to \text{ same form as Dirichlet solution}$$

Convolution:

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

Lemma 1:

- Claim:

$$\widehat{f * g}(\xi) = \sqrt{2\pi}^n \hat{f}(\xi)\hat{g}(\xi)$$

- Proof:

$$
\begin{aligned}
\widehat{f*g}(\xi) &= \int_{\mathbb{R}^n} e^{-i(\xi,x)} f*g(x) \frac{dx}{\sqrt{2\pi}^n} \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x-y)g(y)dy \frac{dx}{\sqrt{2\pi}^n} \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i[(\xi,x-y)+(\xi,y)]} f(x-y)g(y)dy \frac{dx}{\sqrt{2\pi}^n} \\
&= \sqrt{2\pi}^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(\xi,x-y)} f(x-y) \frac{dy}{\sqrt{2\pi}^n}\ e^{-i(\xi,y)} g(y) \frac{dx}{\sqrt{2\pi}^n}
\end{aligned}
$$

$$
\begin{aligned}
& z = x - y \\
&= \sqrt{2\pi}^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(\xi,z)} f(z) \frac{dz}{\sqrt{2\pi}^n}\ e^{-i(\xi,y)} g(y) \frac{dx}{\sqrt{2\pi}^n} \\
&= \sqrt{2\pi}^n \hat{f}(\xi)\hat{g}(\xi)
\end{aligned}
$$

- Apply the lemma to u_ψ:

$$
u_\psi(t,x) = \sqrt{2\pi}^n \hat{g}^t * \psi
$$

Lemma 2:

- $F : \mathbb{R}^n \to \mathbb{R}$ is radial if $\exists H : \mathbb{R} \to \mathbb{R}$ such that $F(x) = H(|x|)$
- Polar coordinates:

$$
\int_{\mathbb{R}^n} f(x)dx = C_n \int_0^\infty \int_{S^{n-1}} f(ry)d\sigma(y) r^{n-1} dr
$$

- Claim: if F is radial, then $\hat{F}$ is also radial

$$
\begin{aligned}
\hat{F}(\xi) &= \int_{\mathbb{R}^n} e^{-i(\xi,y)} F(x) \frac{dx}{\sqrt{2\pi}^n} \\
&= C_n \int_0^\infty \left[\int_{S^{n-1}} e^{-i(r\xi,y)} d\sigma(y) \right] F(x) r^{n-1} dr \qquad \text{where } x = ry \\
&= C_n \int_0^\infty \left[\int_{S^{n-1}} e^{-i(r\xi,y)} d\sigma(y) \right] H(r) r^{n-1} dr \qquad \text{where } r = |x| \\
V(r|\xi|) &= \int_{S^{n-1}} e^{-i(r\xi,y)} d\sigma(y) \to \text{depends only on } |\xi|\text{, therefore radial}
\end{aligned}
$$

- σ is the probability measure on $S^{n-1} = \{y \mid |y| = 1\}$ which is invariant under rotation (periodic)

- Simple example:

$$F(x) = \begin{cases} 1 & |x| \leq R \\ 0 & |x|\, R \end{cases}$$

$$\begin{aligned} \text{Volume}_n(\text{Ball with radius } R) &= C_n \int_0^R r^{n-1} dr \\ &= \frac{C_n}{n} R^n \end{aligned}$$

- Example for $n = 2$:

$$\begin{aligned} (x, y) &= re^{i\theta} \\ F(x, y) &= H(r) \end{aligned}$$

$$\begin{aligned} S^{n-1} &= \{e^{i\theta} \mid 0 \leq \theta \leq 2\pi\} \\ d\sigma(\theta) &= \frac{d\theta}{2\pi} \end{aligned}$$

$$\begin{aligned} \xi &\in \mathbb{R}^2 \\ \xi &= re^{i\eta} \\ (re^{i\eta}, e^{i\theta}) &= (r, e^{i(\theta-\eta)}) \end{aligned}$$

$$\begin{aligned} \hat{F}(\xi) &= \int_0^{2\pi} e^{-i(r, e^{i(\theta-\eta)})} \frac{d\theta}{2\pi} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \cos(r \cos\theta) d\theta \\ &= V_2(r) \qquad \text{where } r = |\xi| \end{aligned}$$

Lemma 3 (Dilation):

- Claim, for $f : \mathbb{R}^n \to \mathbb{R}$:

$$\begin{aligned} \text{If } f^t(x) &= f(tx) \\ \hat{f}^t(\xi) &= t^{-n} \hat{f}\left(\frac{\xi}{t}\right) \end{aligned}$$

- Proof:

$$\begin{aligned}\hat{f}^t(\xi) &= \int_{\mathbb{R}^n} e^{-i(\xi,x)} f^t(x) \frac{dx}{\sqrt{2\pi}^n} \\ &= \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(tx) \frac{dx}{\sqrt{2\pi}^n}\end{aligned}$$

$$\begin{aligned}&y = tx \\ &dy = t^n dx \\ &= t^{-n} \int_{\mathbb{R}^n} e^{-i(\frac{\xi}{t},y)} f(y) \frac{dy}{\sqrt{2\pi}^n} \\ &= t^{-n} \hat{f}\left(\frac{\xi}{t}\right)\end{aligned}$$

- Apply to $\hat{g}^t$:

$$\hat{g}^t(\xi) = \frac{\sin(t|\xi|)}{|\xi|} = \frac{\sin(t|\xi|)}{|\xi|} \frac{t}{t}$$

$$\begin{aligned}s &= \frac{1}{t} \\ &= \frac{1}{s} \frac{\sin(\frac{|\xi|}{s})}{\frac{|\xi|}{s}} = s^{n-1} s^{-n} \frac{\sin(\frac{|\xi|}{s})}{\frac{|\xi|}{s}}\end{aligned}$$

$$\hat{g}^t(\xi) = t^{1-n} \hat{g}^1\left(\frac{\xi}{t}\right) \qquad \text{where } \hat{g}^1(\xi) = \frac{\sin(|\xi|)}{|\xi|}$$

Apply the two lemmas to the original problem:

$$\begin{aligned}u_\psi(t,0) &= C_n \int_{\mathbb{R}^n} g^t(x-y)\psi(y)dy \\ &= C_n t^{n-1} \int_{\mathbb{R}^n} g^1(\frac{y}{t})\psi(y)dy \\ &= C_n(t) \int_{\mathbb{R}^n} \left[\int_{S^{n-1}} \psi(ry)d\sigma(y)\right] g(t,r) r^{n-1} dr\end{aligned}$$

Conclusion: for any dimension, solution at the space is the overage of all circles, then apply a scalar factor to the radii at every t

LOS 6. Determine solution of wave equation using Fourier transforms

Problem:

$$
\begin{aligned}
u_{tt} &= \Delta u \\
u_t(0,x) &= \phi(x) \\
u_t(0,x) &= \psi(x) \\
\widehat{u_{tt}} &= \widehat{\Delta u} = \widehat{\frac{d^2}{dx^2}u} = -|\xi|^2 \hat{u}(t,\xi)
\end{aligned}
$$

Assume an ODE problem:

$$
\begin{aligned}
f''(t) &= -\alpha f(t) & \alpha > 0 \\
f(t) &= a(0)\cos(\sqrt{a}t) + b(0)\sin(\sqrt{a}t)
\end{aligned}
$$

Translate to the original problem:

$$
\begin{aligned}
\hat{u}(t,\xi) &= \cos(t|\xi|)a(\xi) + \sin(t|\xi|)b(\xi) \\
\hat{u}(0,\xi) &= a(\xi) = \hat{\phi}(\xi) \\
\hat{u}_t(0,\xi) &= |\xi|b(\xi) = \hat{\psi}(\xi) \\
b(\xi) &= \frac{\psi(\xi)}{|\xi|}
\end{aligned}
$$

$$
\begin{aligned}
S_D(t) &= \int_{\mathbb{R}^n} e^{i(\xi,x)} \cos(|\xi|t)\hat{\phi}(\xi)\frac{d\xi}{\sqrt{(2\pi)}^n} \\
S_N(t) &= \int_{\mathbb{R}^n} e^{i(\xi,x)} \sin(|\xi|t)\frac{\hat{\psi}(\xi)}{|\xi|}\frac{d\xi}{\sqrt{(2\pi)}^n}
\end{aligned}
$$

In wave equation, the following equation holds:

$$
\begin{aligned}
S_D(0) &= \mathrm{Id} & S'_D(0) &= 0 \\
S_N(0) &= 0 & S'_N(0) &= \mathrm{Id}
\end{aligned}
$$

Remarks:

$$
\int_0^t \cos(|\xi|s)ds = \frac{sin(|\xi|t)}{|\xi|}
$$

Example for 1-D ($n = 1$):

$$
\begin{aligned}
\cos(|\xi|t) &= \frac{e^{it\xi} - e^{-it\xi}}{2} \\
S_D(t) &= \frac{1}{2}\int_{\mathbb{R}} e^{i\xi x} e^{i\xi t}\hat{\phi}(\xi)\frac{d\xi}{\sqrt{2\pi}} + \frac{1}{2}\int_{\mathbb{R}} e^{i\xi x} e^{-i\xi t}\hat{\phi}(\xi)\frac{d\xi}{\sqrt{2\pi}} \\
&= \frac{1}{2}\left(\phi(x+t) + \phi(x-t)\right) \\
S_N(t) &= \int_{\mathbb{R}} e^{i(\xi,x)} \sin(|\xi|t)\frac{\hat{\psi}(\xi)}{|\xi|}\frac{d\xi}{\sqrt{(2\pi)}} \\
&= \int_{\mathbb{R}} e^{i(\xi,x)} \int_0^t \cos(|\xi|s)ds\ \hat{\psi}(\xi)\frac{d\xi}{\sqrt{(2\pi)}} \\
&= \int_0^t \int_{\mathbb{R}} e^{i(\xi,x)} \cos(|\xi|s)\hat{\psi}(\xi)\frac{d\xi}{\sqrt{(2\pi)}} ds \\
&= \int_0^t S_D(s)ds \\
&= \int_0^t \frac{\psi(x+s) + \psi(x-s)}{2} ds
\end{aligned}
$$

LOS 7. Determine solution of wave equation using polar coordinates

Problem:

$$
\begin{aligned}
u_{tt} &= \Delta u \\
u(0,x) &= \phi(x) \\
u_t(0,x) &= \psi(x)
\end{aligned}
$$

Solution by polar coordinates:

$$
\begin{aligned}
u_\psi(t,x) &= \int g(t,|x-y|)\psi(y)dy \\
u_\psi(t,x) &= \int_0^\infty g(t,r)\int_{S^{n-1}} \psi(ry)d\sigma(y)dr
\end{aligned}
$$

LOS 8. Find the solution of wave equation using spherical means method

By translation:

- Assume u is a solution

- For $x_0 \in \mathbb{R}^2$, the following is also a solution:

$$w(t,x) = u(t, x - x_0)$$

- Hence, we can shift our original wave problem such that:

$$\begin{aligned} &u_{tt} = \Delta u \qquad (1) \\ &u(0,x) = \phi(x) = 0 \\ &u_t(0,x) = \psi(x) \end{aligned}$$

- After solving the above problem, we can shift it back to the original problem

Spherical means:

- Let $u(t,x)$ be a solution:

$$\begin{aligned} &\bar{u}(t,x) = \int_{S^{n-1}} u(t, |x|y) d\sigma(y) \\ &|x| = r \end{aligned}$$

- u above is only dependent on the absolute value of x such that:

$$\begin{aligned} x &\neq x' \\ \text{if } |x| = |x'| &\rightarrow \bar{u}(t,x) = \bar{u}(t,x') \end{aligned}$$

- u over the domain of t is a series of circles with increasing radii as t goes up. As such, the integral is simply an average of all circles over the domain of t.

- Hence:

$$\begin{aligned} \text{Given } g(t,r) &= \int_{S^{n-1}} u(t, ry) d\sigma(y) \\ u_\psi(t,0) &= \int_0^\infty g(t,r)\bar{\psi}(r) dr \end{aligned}$$

Lemma 1:

- If u solves the wave problem (1):

$$\begin{aligned} &\bar{u}_{tt} = \bar{u}_{rr} + \frac{n-1}{r}\bar{u}_r \\ &u(t,0) \text{ only depends on } \bar{u} \end{aligned}$$

- Proof:

$$\begin{aligned}
\bar{u}_{tt}(t,x) &= \int_{S^{n-1}} u_{tt}(t,ry)d\sigma(y) \\
&= \int_{S^{n-1}} (\Delta u)(t,ry)d\sigma(y) \\
&= \Delta \int_{S^{n-1}} u(t,ry)d\sigma(y)
\end{aligned}$$

- Let $G(t,r)$ be a radial function:

$$\begin{aligned}
\frac{\partial}{\partial x_k} G(t,|x|) &= G_r(t,|x|)\frac{\partial}{\partial x_k}|x| \qquad \text{where } |x| = \left(\sum_{j=1}^{n} x_j^2\right)^{\frac{1}{2}} \\
&= G_r(t,|x|)\frac{1}{2}\frac{2x_k}{\sqrt{\sum x_j^2}} \\
&= G_r(t,|x|)\frac{x_k}{|x|}
\end{aligned}$$

$$\frac{\partial^2}{\partial x_k^2} G(t,|x|) = G_{rr}(t,|x|)\frac{x_k^2}{|x|^2} + G_r(t,|x|)\frac{|x| - \frac{x_k^2}{|x|^2}}{|x|^2}$$

$$\begin{aligned}
\therefore \Delta G(t,|x|) &= \sum \frac{\partial^2}{\partial x_k^2} G(t,|x|) \\
&= G_{rr}(t,|x|) + \frac{n}{|x|}G_r(t,|x|) - \frac{1}{|x|}G_r(t,|x|) \\
&= G_{rr}(t,r) + \frac{n-1}{r}G_r(t,r)
\end{aligned}$$

Example for $n = 3$:

- Conversion into polar coordinates:

$$\begin{aligned}
&\bar{u}_{tt} = \bar{u}_{rr} + \frac{2}{r}\bar{u}_r \\
\text{Let } &v(t,r) = r\bar{u}(t,r) \\
&v_{tt} = r\bar{u}_{tt}(t,r) = r[\bar{u}_r r + \frac{2}{r}\bar{u}_r] = r\bar{u}_{rr} + 2\bar{u}_r \\
&v_r = \bar{u}_r + r\bar{u}_r \\
&v_{rr} = \bar{u}_r + \bar{u}_r + r\bar{u}_{rr} = 2\bar{u}_r + r\bar{u}_{rr} = v_{tt}
\end{aligned}$$

- New converted problem:

$$\begin{aligned}
&v_{tt} = v_{rr} \\
&v(0,r) = r\bar{u}(0,r) = r\bar{\phi} = 0 \\
&v_t(0,r) = r\bar{u}_t(r,0) = r\bar{\psi}(r)
\end{aligned}$$

- Solution:

$$\begin{aligned}
v(r,t) &= \frac{1}{2}\int_{r-t}^{r+t} s\bar{\psi}(s)ds \\
\bar{u}(r,t) &= \frac{1}{2r}\int_{r-t}^{r+t} s\bar{\psi}(s)ds
\end{aligned}$$

$$\begin{aligned}
\bar{u}(0,t) &= \lim_{r\to 0}\frac{v(r,t)}{r} \\
&= \frac{\partial v}{\partial r}(r,t)\bigg|_{r=0} \\
&= \frac{1}{2}[(r+t)\bar{\psi}(r+t)+(r-t)\bar{\psi}(r-t)] \\
&= t\bar{\psi}(t)
\end{aligned}$$

$$\therefore u(0,t) = t\int_{S^{n-1}}\psi(ry)d\sigma(y)$$

- Alternative solution:

$$\begin{aligned}
u_\psi(x,t) &= t\int_{S^{3-1}}\psi(ty)d\sigma(y) \\
u_\phi(x,t) &= \frac{\partial}{\partial t}\left[t\int_{S^{3-1}}\phi(ty)d\sigma(y)\right] \\
&= \int_{S^{3-1}}\phi(ty)d\sigma(y) + t\int_{S^{3-1}}\nabla\phi(ty)yd\sigma(y)
\end{aligned}$$

$$\begin{aligned}
\text{Note: } \frac{\partial}{\partial t}\phi(ty_1,ty_2,ty_3) &= \nabla\phi(ty)\cdot y \\
&= \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\phi(ty_j)y_j
\end{aligned}$$

Practical solution for $n=3$:

- Convert into spherical coordinates:

$$\begin{aligned}
w(\theta,\eta) &= (\sin\theta\sin\eta, \sin\theta\cos\eta, \cos\theta) \\
|w(\theta,\eta)|^2 &= 1
\end{aligned}$$

$$\begin{aligned}
u(x,t) &= t\int_{S^{3-1}}\psi(ty)d\sigma(y) \\
&= t\int_0^\pi\int_0^{2\pi}\psi(\sin\theta\sin\eta,\sin\theta\cos\eta,\cos\theta)\frac{d\eta}{2\pi}\frac{d\theta}{\pi}
\end{aligned}$$

- Change of variable:

$$s = \sin\theta$$
$$\frac{ds}{d\theta} = \cos\theta = \sqrt{1-\sin^2\theta}$$
$$d\theta = \frac{ds}{\sqrt{1-\sin^2\theta}} = \frac{ds}{\sqrt{1-s^2}}$$
$$u(x_1, x_2, t) = \frac{t}{\pi}\int_0^{\sin\frac{\pi}{2}=1}\int_0^{2\pi}\phi(ts\sin\eta, ts\cos\eta)\frac{d\eta}{2\pi}\frac{ds}{\sqrt{1-s^2}}$$

- Solution for problem with source:

$$u(x,t) = \int_0^{t-s} S(t-s)g(s)ds$$
$$\text{where } s(t) = t\int_{S^{n-1}}\phi(ty)d\sigma(y)$$

Example for $n = 5$:

$$\bar{u}_{tt} = \bar{u}_{rr} + \frac{4}{r}\bar{u}_r$$

Let $$v(t,r) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)(r^3\bar{u})$$

Lemma $$\frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(r^{2k-1}\phi) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k}(r^{2k}\phi_r)$$

$$\frac{\partial^2}{\partial r^2}v(t,r) = \frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)(r^3\bar{u})$$
$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2(r^4\bar{u}_r)$$
$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{1}{r}(4r^3\bar{u}_r + r^4\bar{u}_{rr})$$
$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)\left[\frac{1}{r}(4\frac{\bar{u}_r}{r} + \bar{u}_{rr})r^3\right]$$
$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)(r^3\bar{u}_{tt}) = v_{tt}$$

Generally:

For $n = 2k-1$
$$v(t,r) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(r^{2k-1}\bar{u})$$

Descent:

- We can solve even numbered dimension using odd numbered dimension problem
- Assume $u(x_1, x_2, t)$ is a solution to the $u_{tt} = \Delta u$
- Consider $\tilde{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t)$ on $\mathbb{R}^3$
- Solve for $\tilde{u}_{tt} = \Delta_{\mathbb{R}^3}\tilde{u}$

LOS 9. Understand the concept of special relativity

Concepts:

- Assume the speed of light $c = 1$
- Light cone $\{(x, v)|\ |x| \leq |v|\}$
- The characteristic surfaces are the only one which propagate singularities
- Light ray $(x, v_0) = x_0 + tv_0|\ |v_0| = 1$
- Characteristic surface is a union of light rays

LOS 10. Solve the Schrödinger equation

Problem:

$$\begin{aligned} &u_t = i\Delta u \\ &u(0, x) = \phi(x) \end{aligned}$$

Solution:

$$\begin{aligned} &\hat{u}_t = -i|\xi|^2\hat{u} \\ &\hat{u}(t, \xi) = e^{-it|\xi|^2}\hat{\phi}(\xi) \\ &u(t, x) = \int e^{i(\xi,x)}e^{-it|\xi|^2}\hat{\phi}\frac{d\xi}{\sqrt{2\pi}^n} \end{aligned}$$

Remarks:

- $\phi \in L_2$
- $\|\hat{u}(t, \xi)\|_{L_2}^2 = \|\hat{\phi}\|_{L_2}^2$ energy is preserved, not decaying
- $\|u(t, \xi)\|_{L_2}^2 = \|\phi\|_{L_2}^2$ solution is radial, therefore convolution applies

Generically:

$$
\begin{aligned}
u(x,t) &= \int \phi(x-y) g\left(\frac{y}{\sqrt{2t}}\right) \frac{dy}{\sqrt{2\pi t}^n} \\
&= \int \phi(x-y) e^{\frac{-|x-y|^2}{4t}} \frac{dy}{\sqrt{2\pi t}^n}
\end{aligned}
$$

LOS 11. Understand the idea of self-adjoint operator for the Laplace operator

Theorem:

- Applies for a compact domain $\bar{D}$
- The Laplace operation with Robin boundary condition is self-adjoint

$$\int_D \Delta u v dx = \int_D u \Delta v dx$$

- Corollary: if we have a self-adjoint operator, then we have discrete eigenvalues

$$\exists \lambda_n \text{ increasing and } V_n \in \text{ONB such that } \Delta(V_n) = -\lambda_n V_n \text{ on } D$$

- Proof:

Let u and v satisfy the boundary conditions

From Robin condition:

$$
\begin{aligned}
au + \frac{\partial u}{\partial n} &= 0 \\
\frac{\partial u}{\partial n} &= -au
\end{aligned}
$$

$$
\begin{aligned}
\int_D (\Delta u)v - (\Delta v)u dx &= \int_{\partial D} \left(\frac{\partial u}{\partial n} v - \frac{\partial v}{\partial n} u\right) dS \\
&= \int_{\partial D} (-auv + avu) dS = 0
\end{aligned}
$$

LOS 12. Understand the divergence theorem and Green's Identity

Requirement:

- Compact, orientable manifold i.e. circle, sphere, cube
- Domain $\Omega \subset \mathbb{R}^2/\mathbb{R}^3$
- Boundary $\partial\Omega$ i.e. perimeter of the circle or the faces of the cube
- Normal vector n on $\partial\Omega$ where $\|n\| = 1$
- Vector field $O_{\text{open}} \supseteq \partial\Omega$

Theorem:

$$\int_\Omega \text{div}(F) dV(x) = \int_{\partial\Omega} F \cdot \mathbf{n}\, dS$$

$$F : O \to \mathbb{R}^n \qquad F = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}$$

V : volume i.e. $dx_1 dx_2$ for circle, $dx_1 dx_2 dx_3$ for cube

$\mathbf{n}$: normal vector

$$\text{div}(F) = \sum_{k=1}^{n} \frac{\partial}{\partial x_k} F_k$$

Application:

$$\begin{aligned} &u_{tt} = \Delta u && \text{(wave)} \\ &u_t = \Delta u && \text{(heat)} \\ &u|_{\partial\Omega} = 0 && (D) \\ &\left.\frac{\partial u}{\partial n}\right|_{\Omega} = 0 && (N) \\ &au + \frac{\partial u}{\partial n} = 0 && (R) \end{aligned}$$

$$\frac{\partial u}{\partial n} = \nabla \cdot \mathbf{n} \qquad \mathbf{n} : \text{ normal vector}$$

Corollary: Green's idendity

u, v two functions defined on Ω and differentiable

$F = (\nabla u)v \qquad H = (\nabla v)u$
F, H are vector fields (gradient times the function)

$$\begin{aligned}
\operatorname{div}(F) &= \sum_{k=1}^{n} \frac{\partial}{\partial k}\left(\left(\frac{\partial}{\partial x_k}u\right)v\right) \\
&= (\Delta u)v + (\nabla u, \nabla v) \\
&= \sum \left(\frac{\partial}{\partial x_k}u\right)v + \sum \frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_k}
\end{aligned}$$

$$\begin{aligned}
\operatorname{div}(F) - \operatorname{div}(H) &= (\Delta u)v + (\nabla u, \nabla v) - (\Delta v)u - (\nabla v, \nabla u) \\
&= (\Delta u)v - (\Delta v)u
\end{aligned}$$

Green's Identity: $\displaystyle\int_\Omega (\Delta u)v - (\Delta v)u dx = \int_{\partial\Omega} \left(\frac{\partial u}{\partial n}v - \frac{\partial v}{\partial n}u\right) dS$

LOS 13. Understand the method of separation of variables for heat and wave equation

Problem (Heat):

$$u_t = \Delta u$$

Separation Ansatz:

$$\begin{aligned}
&u(t,x) = T(t)V(x) \\
&u_t(t,x) = T'(t)V(x) \\
&\Delta u(t,x) = T(t)(\Delta V)(x)
\end{aligned}$$

$$\begin{aligned}
&\frac{T'}{T} = \frac{\Delta V}{V} = \frac{u_t}{u} = \frac{\Delta u}{u} = \frac{\Delta V_n}{V_n} = -\lambda_n \\
&u(x,t) = \sum_n e^{-t\lambda_n} a_n V_n(x) \\
&a_n = \int_D \bar{V}_n \phi(x) dx
\end{aligned}$$

Problem (Wave):

$$
\begin{aligned}
&u_{tt} = \Delta u \\
&\frac{T''}{T} = \frac{\Delta V}{V} = -\lambda_n \\
&T(t) = \cos(\sqrt{\lambda_n} t) + \sin(\sqrt{\lambda_n} t) \\
&u(t,x) = \sum_n a_n \cos(\sqrt{\lambda_n} t) V_n(x) + \sum_n b_n \sin(\sqrt{\lambda_n} t) V_n(x)
\end{aligned}
$$