Unit 6

LOS 1. Learn how to apply maximum principle for finding a concrete value for two periodic solutions of heat equation

Problems:

$$0 \leq t \leq 1$$
$$0 \leq x \leq 2\pi$$

$$u_t = u_{xx}$$
$$u(0,t) = t^2$$
$$u(2\pi,t) = t^2$$
$$u(x,0) = sin(x)$$

$$w_t = w_{xx}$$
$$w(0,t) = t$$
$$w(2\pi,t) = t$$
$$w(x,0) = \sin(x) + \frac{1}{4}\sin(2x)$$

Find:

$$|u(\pi,1) - w(\pi,1)| \leq ?$$

Method:

- By maximum principle:

$$v = u - w$$
$$\sup_{(x,t)\in\hat{\Omega}} |v(x,t)| \leq \sup_{(x,t)\in\delta\Omega}$$

- Evaluate right hand side of the boundary and let $t = s$:

$$\sup_{0\leq s\leq 1} |u(x,t) - w(x,t)| \leq \sup_{0\leq s\leq 1} |s - s^2|$$

- Find maximum of left hand side:

$$f(s) = s - s^2$$
$$f'(s) = 1 - 2s = 0$$
$$s = \frac{1}{2}$$
$$\therefore \sup_{0 \leq s \leq 1} |s - s^2| = \frac{1}{4}$$

- Test if the above supremum is correct:

$$\sup_x \left| \sin x - (\sin x + \frac{1}{4}\sin 2x) \right| = \frac{1}{4}$$

- Therefore, since $(\pi, 1)$ is an inner point:

$$|u(\pi, 1) - w(\pi, 1)| \leq \frac{1}{4}$$

Is there a solution with $u(0, t) = t^2$ for $0 \leq t \leq \infty$?:

- Short answer: No
- Heat equation is diffusion. Diffusioin decays.
- Proof:

$$u(x,t) = \int_{\mathbb{R}} \phi(y) \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} dy$$
$$u(x,t) = \int_{\mathbb{R}} \phi(y) \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} dy \leq \int_{\mathbb{R}} |\phi(y)| \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} dy$$

$$\text{Assume } \int_{\mathbb{R}} |\phi(y)| dy \leq 1$$
$$e^{-\frac{(x-y)^2}{4kt}} \leq 1$$

$$u(x,t) \leq \int_{\mathbb{R}} |\phi(y)| \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} dy \leq \int_{\mathbb{R}} |\phi(y)| \frac{1}{\sqrt{4\pi kt}} dy \leq \frac{1}{\sqrt{4\pi kt}}$$

- Every solution has to decay uniformly by $\frac{1}{\sqrt{4\pi kt}}$. Therefore, $h(0,t) = t^2$ is not a solution.

LOS 2. Solve a given PDE using separation of variables

Problem:

$$u_t = \mathscr{L}(u) \qquad \hat{\Omega} \leq \mathbb{R}^D$$
$$u(x,0) = \phi(x) \qquad x \in \delta\Omega$$

Ansatz:

$$u(x,t) = T(t)X(x)$$
$$T'X = TL(X)$$
$$\frac{T'}{T} = \frac{L(X)}{X} = \lambda$$
$$T' = e^{\lambda t}T(0)$$
$$L(X) = \lambda X$$

Find an orthonormal basis of $L_2(\hat{\Omega}, m)$ (m is a measure depending on the dimension, for example; for 1-D it will be dx, for 2-D it will be $dxdy$) such that:

$$L(X_k) = \lambda_k X_k$$
$$S(\phi)(x,t) = \sum_k (X_k, \phi) e^{t\lambda k} X_k$$

Example 1-D:

$$\hat{\Omega}[-\pi, \pi]$$
$$X_k = e_k(x) = e^{ikx}$$
$$\mathscr{L}(u) = u_{xx} \to \mathscr{L}(e_k) = -k^2 e_k$$

Example for wave with Dirichlet:

- Problem:

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < l$$
$$u(0,t) = 0 = u(l,t)$$
$$u(x,0) = \phi(x)$$
$$u_t(x,0) = \psi(x)$$

- Separated solution:

$$u(x,t) = X(x)T(t)$$
$$u_{tt} = c^2 u_{xx}$$
$$X(x)T''(t) = c^2 X''T$$
$$-\frac{T''}{c^2 T} = -\frac{X''}{X} = \lambda$$

- Let $\lambda = \beta^2$ where $\beta > 0$:

$$
\begin{gathered}
-\frac{X''}{X} = \beta^2 \\
X'' + \beta^2 X = 0 \\
X(x) = C\cos\beta x + D\sin\beta x
\end{gathered}
$$

$$
\begin{gathered}
-\frac{T''}{c^2 T} = \beta^2 \\
T'' + c^2\beta^2 T = 0 \\
T(t) = A\cos\beta ct + B\sin\beta ct
\end{gathered}
$$

- Impose boundary condition:

$$
\begin{gathered}
u(x,t) = 0 \to u(0,t) = X(0)T(t) = 0 \qquad \to X(0) = 0 \\
X(0) = C \times 1 + D \times 0 = 0 \qquad \to C = 0 \\
u(l,t) = 0 \to u(l,t) = X(l)T(t) = 0 \qquad \to X(l) = 0 \\
X(l) = \cancelto{0}{C}\cos\beta l + D\sin\beta l = D\sin\beta l
\end{gathered}
$$

- To satisfy the above condition:

$$
\begin{gathered}
\beta l = n\pi \qquad n = 1, 2, 3, ... \\
\beta = \frac{n\pi}{l} \\
\lambda_n = \beta^2 = \left(\frac{n\pi}{l}\right)^2 \\
X_n(x) = \sin\left(\frac{n\pi}{l}x\right)
\end{gathered}
$$

- Linear combination of X_n is a solution to X ODE equation. Therefore, the solution to the wave equation is the superposition of all the possible solutions:

$$
\begin{gathered}
u(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi}{l}ct\right) + B_n\sin\left(\frac{n\pi}{l}ct\right)\right]\sin\left(\frac{n\pi}{l}x\right) \\
\phi(x) = \sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi}{l}x\right) \\
\psi(x) = \sum_{n=1}^{\infty}\frac{n\pi}{l}cB_n\sin\left(\frac{n\pi}{l}x\right)
\end{gathered}
$$

Example for diffusion with Dirichlet:

- Problem:

$$u_t = ku_{xx} \qquad 0 < x < l,\ 0 < t < \infty$$
$$u(0,t) = u(l,t) = 0; u(x,0) = \phi(x)$$

- Separation:

$$u(x,t) = X(x)T(t)$$
$$u_t = ku_{xx}$$
$$X(x)T'(t) = kX''(x)T(t)$$
$$XT' = kX''T$$
$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

$$T' = -\lambda kt$$
$$T(t) = Ae^{-\lambda kt}$$

- Let $\lambda = \beta^2$ where $\beta > 0$:

$$\frac{X''}{X} = -\beta^2$$
$$X'' + \beta^2 X = 0$$
$$X(x) = C\cos\beta x + D\sin\beta x$$

$$X(x) = D\sin\beta x$$
$$X(l) = D\sin\beta x = 0$$

- To satisfy the above condition:

$$\beta l = n\pi \qquad n = 1, 2, 3, ...$$
$$\beta = \frac{n\pi}{l}$$
$$\lambda_n = \beta^2 = \left(\frac{n\pi}{l}\right)^2$$
$$X_n(x) = \sin\left(\frac{n\pi}{l}x\right)$$

- Solution:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi}{l}x\right)$$

Solution for diffusion with Neumann:

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \cos\left(\frac{n\pi}{l}x\right)$$
$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{l}x\right)$$

Solution for wave with Neumann:

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right)\right] \cos\left(\frac{n\pi}{l}x\right)$$
$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{l}x\right)$$
$$\psi(x) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} \left(\frac{n\pi}{l}c\right) B_n \cos\left(\frac{n\pi}{l}x\right)$$

LOS 3. Calculate orthonormal basis for Laplace equation

ONB for 2-D periodic:

$$\hat{\Omega}\ [-\pi,\pi]\times[-\pi,\pi]$$
$$\mathscr{L}(u) = u_{xx} + u_{yy} = \Delta u$$
$$e_{kj}(x,y) = e_k(x)e_j(y) = e^{ikx}e^{ijy}$$
$$\mathscr{L}(e_{kj}) = -(k^2+j^2)e_{kj}$$

Conclusion:

- For $[-\pi,\pi]\times[-\pi,\pi]$, product of e_k for multiple domains (e_{kj}) forms an orthonormal basis for the combined domain
- Solution formula for 2-D:
$$S(\phi)(x,y,t) = \sum_k (e_{kj}(x,y),\phi)e^{-t(k^2+j^2)}e_{kj}(x,y)$$
$$\text{Or generally } S(\phi)(x,t) = \sum_k (X_k,\phi)e^{t\lambda k}X_k$$
- Solution formula satisfies the energy estimate inequality:
$$\|u(x,y,t)\|_{L_2} \le \|u(x,y,0)\|_{L_2}$$
$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |u(x,y,t)|\frac{dxdy}{4\pi^2} \le \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |u(x,y,0)|\frac{dxdy}{4\pi^2}$$

- The energy estimate inequality holds because of Parseval identity (L_2 norm is the sum of the Fourier coefficients):

$$\|\phi\|_{L_2}^2 = \sum_k |(\phi, X_k)|^2$$
$$|u(x,y,t)\|_{L_2} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |u(x,y,t)| \frac{dxdy}{4\pi^2}$$
$$\text{Or generally } \|u(x,t)\|_{L_2} = \int_{-\pi}^{\pi} |u(x,t)| dm$$

- If $\lambda_k < 0$:

$$\|u(x,t)\|_{L_2} = \int_{-\pi}^{\pi} \left| \sum_k (X_k, \phi) e^{t\lambda k} X_k \right| dm \leq \int_{-\pi}^{\pi} \left| \sum_k (X_k, \phi) e^{(0)\lambda k} X_k \right| dm$$
$$\|u(x,t)\|_{L_2} \leq \int_{-\pi}^{\pi} \left| \sum_k (X_k, \phi) e^{(0)\lambda k} X_k \right| dm = \|u(x,0)\|_{L_2}$$

Example:

- Problem:

$$\mathscr{L}(u) = u_{xx} + u_{yy} = \Delta u$$
$$\phi(x,y) = \sin x + \cos y$$
$$P(x) = x^2$$
$$\lambda_{kj} = -(k^2 + j^2)$$

- Find Fourier coefficient:

$$\phi(x,y) = \sin x + \cos y = \frac{e^{ix} - e^{-ix}}{2i} e_0(y) + \frac{e^{iy} + e^{-iy}}{2} e_0(x)$$
$$\phi(x,y) = \frac{e_1(x)e_0(y)}{2i} - \frac{e_{-1}(x)e_0(y)}{2i} + \frac{e_0(x)e_1(y)}{2} + \frac{e_0(x)e_{-1}(y)}{2}$$
$$\phi(x,y) = \frac{e_{1,0}}{2i} - \frac{e_{-1,0}}{2i} + \frac{e_{0,1}}{2} + \frac{e_{0,-1}}{2}$$

- Susbstitute ϕ to the solution operator:

$$u(x,y,t) = e^{-t}\frac{e_{1,0}}{2i} - e^{-t}\frac{e_{-1,0}}{2i} + e^{-t}\frac{e_{0,1}}{2} + e^{-t}\frac{e_{0,-1}}{2}$$

Solutions that satisfy $u(x,0) = \sin x$ is infinitely many:

- Previously, it was proven:

$$\lim_{t\to 0} u(x,t) = \phi(x)$$

- Given ϕ on $[0, 2\pi]$ and ϕ bounded on $\mathbb{R}$ and continuous (which is proven above):

$$u(x,t) = \int_{\mathbb{R}} \varphi(x-y) e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{4\pi t}} = \int_{\mathbb{R}} \varphi(z) e^{-\frac{(x-z)^2}{4t}} \frac{dz}{\sqrt{4\pi t}}$$

- Solution is any extension ϕ_{ext} on $\mathbb{R}$

Generically:

- For periodic solution of order 2:

$$u(x,t) \to [0, 2\pi]$$
$$u(x,t) = \sum_k a_k e^{-tk^2} e_k(x)$$
$$a_k = (e_k, \phi)$$
$$(\alpha, \beta) = \sum_k \bar{\alpha}_k \beta_k$$

- By Cauchy–Schwarz:

$$|(\alpha, \beta)| \leq \|\alpha\| \|\beta\| = (\alpha, \alpha)^{\frac{1}{2}} (\beta, \beta)^{\frac{1}{2}}$$
$$|u(x,t)| \leq \left(\sum_k |\bar{\alpha}_k|^2 \right)^{\frac{1}{2}} \left(\sum_k e^{-2tk^2} |e^{ikx}|^2 \right)^{\frac{1}{2}}$$
$$\sum_k |\bar{\alpha}_k|^2 = \|\phi\|_{L_2}$$
$$|e^{ikx}|^2 = 1$$
$$\sum_k e^{-2tk^2} \leq 1 + \int_0^\infty e^{-2tk^2} x$$

- By change of variable:

$$y = \sqrt{8t}x \to dy = \sqrt{8t}dx$$
$$\sum_k e^{-2tk^2} \leq 1 + \int_0^\infty e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}\sqrt{8t}} \leq 1 + \frac{\sqrt{2\pi}}{\sqrt{8t}}$$

- Therefore, roughly:

$$|u(x,t)| \leq C\|\phi\|_{L_2}(1 + t^{-1\frac{1}{4}})$$

- This shows that $u(x,t)$ does not grow indefinitely. This shows a "weak" maximum principle.

LOS 4. Determine the solution operator for the higher order PDE

Problem (Polynomial 2, 6, 10 are good because they are decaying):

$$u_t = u_{xxxxxxxxxx}$$

Find Fourier coefficients:

$$\hat{u}(\xi,t) = (i\xi)^{10}\hat{u}(\xi,0) = -\xi^{10}\hat{u}(\xi,0)$$
$$e^{-\xi^{10}} = \int_{\mathbb{R}} e^{-i\xi y}h(y)\frac{dy}{\sqrt{2\pi}}$$
$$e^{-(t^{\frac{1}{10}}\xi)^{10}} = \int_{\mathbb{R}} e^{-it^{\frac{1}{10}}\xi y}h(y)\frac{dy}{\sqrt{2\pi}}$$

Change of variable:

$$z = t^{\frac{1}{10}}y$$
$$e^{-(t^{\frac{1}{10}}\xi)^{10}} = t^{-\frac{1}{10}}\int_{\mathbb{R}} e^{-i\xi z}h(t^{-\frac{1}{10}}z)\frac{dz}{\sqrt{2\pi}}$$

Solution:

$$u(x,t) = \int_{\mathbb{R}} e^{i\xi x}e^{-t\xi^{10}}\hat{e}(\xi,0)\frac{d\xi}{\sqrt{2\pi}}$$
$$u(x,t) = t^{-\frac{1}{10}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\xi(x-z)}\hat{e}(\xi,0)\frac{d\xi}{\sqrt{2\pi}}h(t^{-\frac{1}{10}}z)\frac{dz}{\sqrt{2\pi}}$$
$$u(x,t) = t^{-\frac{1}{10}}\int_{\mathbb{R}} u(x-z,0)h(t^{-\frac{1}{10}}z)\frac{dz}{\sqrt{2\pi}}$$

If the function h decays fast, values that are far away has little influence to the solution at t.

LOS 5. Understand the theorem on continuous derivatives and pointwise convergence

Properties of diffusion equation:

- Consider the problem: $u_t = u_{xxxx}$
- Property 1: $\lim_{t\to\infty} \int |u(x,t)|^2 dx \neq \infty$. If initial conditions are integrable, L_2 norm is always bounded. In fact, for diffusion, L_2 norm is decreasing.
- Property 2: $\lim_{t\to\infty} u(0,t) = 0$. If initial conditions are integrable, this is always true because h is equals to a constant.

$$u(x,t) = t^{-\frac{1}{4}} \int_{\mathbb{R}} u(x-y,0)h(t^{-\frac{1}{4}}y)\frac{dy}{\sqrt{2\pi}}$$
$$\hat{h}(\xi) = e^{-\xi^4}$$
$$|h(y)| = \left|\int_{\mathbb{R}} e^{-i\xi y}e^{-\xi^4}\frac{d\xi}{\sqrt{2\pi}}\right| \leq \left|\int_{\mathbb{R}} e^{-\xi^4}\frac{d\xi}{\sqrt{2\pi}}\right| = C$$

- Information preservation: for small t, the function h approaches a Dirac delta function and all information is preserved. As t gets larger, the function flattens and information is lost.
- Compared to wave equation: information is preserved for the Dirichlet condition, but not for the Neumann condition.

By separation of variables:

- Problem:

$$u_t = u_{xx}$$
$$u(0,t) = 0$$
$$u_x = (\frac{\pi}{2}, t) = 0$$

- Separation of variables:

$$u(x,t) = X(x)T(t)$$
$$T'X = TX''$$
$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

- Known equalities (f is the superposition of all solutions for X):

$$X(0) = 0 \qquad X'(\frac{\pi}{2}) = 0 \qquad X'' = -\lambda X$$
$$X(x) = \sin \lambda x$$
$$\sqrt{\lambda} = \frac{2k+1}{2} \qquad k \in \mathbb{N}$$
$$\lambda = \frac{(2k+1)^2}{4} \qquad \text{OR} \qquad \lambda = 0$$
$$f \in L_2 \to f(x) = f_0 + \sum a_k \sin(\frac{2k+1}{2}x)$$

- By superposition:

$$u(x,t) = f_0 + \sum_{k=1}^{\infty} a_k e^{-(2k+1)^2 t} \sin(\frac{2k+1}{2}x)$$

- By reflection:

$$u_t = u_{xx} \qquad \tilde{f}[\frac{-\pi}{2}, \frac{3\pi}{2}]$$
$$e_k(x) = e^{2\pi i k(x+\frac{1}{2})}$$
$$\tilde{f} = \sum_k \tilde{f}(k) e_k$$
$$\tilde{f}(k) = -\tilde{f}(-k) \to \sin\left(k + \frac{1}{2}x\right) \text{ and 1 forms an orthogonal system}$$
$$\tilde{f}(\pi - x) = \tilde{f}(x) \to \sin\left(\frac{2k+1}{2}x\right) \text{ is the solution}$$

Theorem A:

- Given f and f' are continuous (C), then we have pointwise convergence:

$$\forall x f(x) = \sum_k \hat{f}(k) e^{ikx}$$
$$S_N(f)(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{ikx}$$
$$\lim_{N \to \infty} S_N(f)(x) = f(x)$$

- To prove, define:

$$K_N(x) = \sum_{k=-N}^{N} e^{ikx}$$
$$f(y) = \sum_{j=-\infty}^{\infty} \hat{f}(j) e^{ijy}$$

- Lemma 1:

$$S_N(f)(x) = \int_{-\pi}^{\pi} K_N(x-y) f(y) \frac{dy}{2\pi} \qquad (1)$$
$$\int_{-\pi}^{\pi} \sum_{k=-N}^{N} e^{ik(x-y)} \sum_{j=-\infty}^{\infty} \hat{f}(j) e^{ijy} \frac{dy}{2\pi} =$$
$$\sum_{k=-N}^{N} \sum_{j=-\infty}^{\infty} e^{ikx} \hat{f}(j) \int_{-\pi}^{\pi} e^{-iky} e^{ijy} \frac{dy}{2\pi} =$$
$$\sum_{k=-N}^{N} \hat{f}(k) e^{ikx}$$

- Convolution trick:

$$S_N(f)(x) = \int_{-\pi}^{\pi} K_N(x-y) f(y) \frac{dy}{2\pi} = \int_{-\pi}^{\pi} K_N(y) f(x-y) \frac{dy}{2\pi}$$

- Lemma 2:

$$K_N(y) = \frac{\sin\left(\frac{2N+1}{2} y\right)}{\sin\left(\frac{y}{2}\right)} \qquad (2)$$
$$\text{Fejer kernel } \rightarrow \sum_{k=-m}^{N} a^k = \sum_{k=-m}^{N} a^k \frac{a-1}{a-1} = \frac{a^{N+1} - a^{-m}}{a-1}$$
$$K_N(y) = \sum_{k=-N}^{N} e^{iky} = \frac{e^{i(N+1)y} - e^{-Ny}}{e^{iy} - 1}$$
$$= \frac{e^{i(N+1)y} - e^{-Ny}}{e^{iy} - 1} \frac{e^{\frac{-iy}{2}}}{e^{\frac{-iy}{2}}}$$
$$= \frac{e^{i(N+\frac{1}{2})y} - e^{-i(N+\frac{1}{2})y}}{e^{\frac{iy}{2}} - e^{\frac{-iy}{2}}} = \frac{2i \sin\left(\frac{2N+1}{2} y\right)}{2i \sin\left(\frac{y}{2}\right)}$$

- Therefore:

$$\begin{aligned} f(x) - S_N(f)(x) &= \int_{-\pi}^{\pi} [f(x) - f(x-y)] K_N(y) dy \\ &= \int_{-\pi}^{\pi} \frac{f(x) - f(x-y)}{\sin\left(\frac{y}{2}\right)} \sin\left(\frac{2N+1}{2} y\right) dy \\ &= \left(g_x(y), \sin\left(\frac{2N+1}{2} y\right)\right) \end{aligned}$$

- Evaluate g_x:

$$g_x(y) = \frac{f(x) - f(x-y)}{\sin\left(\frac{y}{2}\right)}$$

$$|\sin(\frac{y}{2})| \geq \frac{|y|}{\pi}$$

$$\left| \frac{f(x) - f(x-y)}{y} \right| \leq (1+\epsilon)|f'(x)| \qquad y \leq \delta$$

$$\therefore \left| \frac{f(x) - f(x-y)}{\sin\left(\frac{y}{2}\right)} \right| \leq (1+\epsilon)\pi|f'(x)| \qquad y \leq \delta$$

- f is continuous and $\sin(\frac{y}{2}) \neq 0$ and $y \neq 0$ and continuous:

$$g_x \in L_2[-\pi, \pi]$$

- Since g_x is in L_2, then Bessel inequality (below is for orthogonal) holds:

$$\hat{g}(k) = \left(g_x, \sin\left(\frac{2k+1}{2} y\right)\right)$$

$$\sum_k \frac{|\hat{g}(k)|^2}{\left(\sin\left(\frac{2k+1}{2} y\right), \sin\left(\frac{2k+1}{2} y\right)\right)} \leq \|g_x\|_{L_2}^2$$

Generic formula for Bessel, converting from orthogonal to orthonormal: $\sum_k \frac{|f_k, g|^2}{(f_k, f_k)} \leq \|g\|_{L_2}^2$

- For any sum $\sum_k a_k$, if series is convergent, then $lim_{k\to\infty} = 0$. Therefore:

$$\lim_{N\to\infty} \left(g_x(y), \sin\left(\frac{2N+1}{2} y\right)\right) = 0$$

$$\therefore f(x) - S_N(f)(x) = 0$$

Theorem B:

- Applies for pointwise with singularity

- Given f is continuous then:

$$\lim_{N\to\infty} S_N(f)(x) = \tilde{f}(x)$$

- Definition:

$$\tilde{f}(x) = \frac{f(x^+) + f(x^-)}{2}$$

- Proof:

$$\begin{aligned}
\tilde{f}(x) - S_N(f)(x) &= \int_{-\pi}^{\pi} [f(x) - f(x-y)] K_N(y) dy \\
= \int_0^{\pi} \frac{f(x^+) - f(x-y)}{\sin\left(\frac{y}{2}\right)} &\sin\left(\frac{2N+1}{2} y\right) dy \\
+ \int_{-\pi}^{0} \frac{f(x^-) - f(x-y)}{\sin\left(\frac{y}{2}\right)} &\sin\left(\frac{2N+1}{2} y\right) dy \\
g_{x^+} &= \frac{f(x^-) - f(x-y)}{\sin\left(\frac{y}{2}\right)} \\
g_{x^-} &= \frac{f(x^+) - f(x-y)}{\sin\left(\frac{y}{2}\right)} \\
\tilde{f}(x) - S_N(f)(x) &= 0
\end{aligned}$$

LOS 6. Understand the theorem on uniform convergence of partial sums

Theorem:

- Given f is continuous and $f' \in L_2$:

$$\|S_N(f) - f\|_\infty = 0$$

- Proof:

$$f(x) - S_N(f)(x) = \sum_{k>N} \hat{f}(k)e^{ikx}$$
$$f(x) = \sum_k \hat{f}(k)e^{ikx}$$
$$f'(x) = \sum_k ik\hat{f}(k)e^{ikx}$$
$$f'(x) = \sum_k |ik|^2|\hat{f}(k)e^{ikx}|^2 = \|f'\|_{L_2}^2$$

$$f(x) - S_N(f)(x) = \sum_{k>N} \hat{f}(k)e^{ikx}\frac{ik}{ik}$$
$$f(x) - S_N(f)(x) = \sum_{k>N} \left[ik\hat{f}(k)\right]\left[\frac{e^{ikx}}{ik}\right]$$

$$\alpha_k = ik\hat{f}(k) \qquad \beta_k = \frac{e^{ikx}}{ik}$$
$$f(x) - S_N(f)(x) = \sum_k (\bar{\alpha}_k, \beta_k)$$

- By Cauchy–Schwarz:

$$|(\bar{\alpha}, \beta)| \leq \|\bar{\alpha}\|_{L_2}\|\beta\|_{L_2} \leq \|f'\|_{L_2}\left(\sum_{k>N}\left|\frac{e^{ikx^2}}{ik}\right|\right)^{\frac{1}{2}}$$
$$|f(x) - S_N(f)(x)| \leq \sqrt{2}\|f'\|_{L_2}\left(\sum_{k>N}\frac{1}{k^2}\right)^{\frac{1}{2}}$$
$$\left(\sum_{k>N}\frac{1}{k^2}\right)^{\frac{1}{2}} \leq \left(\int_N^\infty \frac{1}{x^2}dx\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{N}}$$
$$\therefore |f(x) - S_N(f)(x)| \leq \sqrt{2}\|f'\|_{L_2}\frac{1}{\sqrt{N}}$$

- Therefore, we have uniform convergence. This only applies when the derivative exists.

LOS 7. Understand the idea of convergence

Uniform convergence:

- Necessary condition: f is continuous and f‘ in L_2

- Then:

$$S_N(f)(x) = \sum_{j=-N}^{N} \hat{f}(j)e^{ijx}$$
$$\lim_{N\to\infty} S_N(f) = f$$
$$\lim_{N\to\infty} \|f - S_N(f)\|_\infty = 0$$

Local (pointwise) convergence:

- Necessary condition: f is piecewise continuous ($f \in L_2$) and f' have left and righ limits at x
- Then

$$\lim_{N\to\infty} S_N(f)(x) = \frac{f(x^+) + f(x^-)}{2}$$

Corollary:

- If f is contiuous and f' is bounded (have left and right limits) then $\lim_{N\to\infty} S_N(f)(x) = f(x)$
- If f' is bounded and have left and right limits, then $f' \in L_2$
- If f and f' are continuous, then we have uniform convergence

Example 1:

- Problem:

$$\lim_{N\to\infty} S_N(f)(0) =?$$
$$f(x) = \begin{cases} 1 & 0 \le x \le \pi \\ 0 & -\pi \le x \le 0 \end{cases}$$

- Find Fourier coefficients:

$$\hat{f}(j) = (e_k, f) = \int_{-\pi}^{\pi} e^{-ikx} f(x) \frac{dx}{2\pi} =$$
$$\int_{0}^{\pi} e^{-ikx}(1) \frac{dx}{2\pi} + \int_{-\pi}^{0} e^{-}ikx(0) \frac{dx}{2\pi} =$$
$$\frac{1}{2\pi} \int_{0}^{\pi} e^{-ikx} dx = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{e^{-ikx}}{-ik}\Big|_0^{\pi} = \frac{(-i)^k - 1}{-ik(2\pi)} & k \neq 0 \end{cases}$$

$$\frac{(-i)^k - 1}{-ik(2\pi)} = 0 \qquad k \text{ even}$$
$$\frac{(-i)^k - 1}{-ik(2\pi)} = \frac{2}{-ik(2\pi)} = \frac{-i}{k\pi} \qquad k \text{ odd}$$

- Therefore:

$$S_N(f)(x) = \frac{1}{2} e^{i(0)x} + \sum_{k=-N,\, k\neq 0,\, k \text{ odd}}^{N} \frac{-i}{k\pi} e^{ikx}$$
$$S_N(f)(x) = \frac{1}{2} + \frac{-i}{\pi} \sum_{k=-N,\, k\neq 0,\, k \text{ odd}}^{N} \frac{e^{ikx}}{k}$$
$$S_N(f)(x) = \frac{1}{2} + \frac{-i}{\pi} \sum_{k=1,\, k \text{ odd}}^{N} \frac{e^{ikx}}{k} + \frac{-i}{\pi} \sum_{k=1,\, k \text{ odd}}^{N} \frac{e^{-ikx}}{-k}$$

- Since N is odd:

$$2m + 1 \leq N$$
$$S_N(f)(x) = \frac{1}{2} + \frac{-i}{\pi} \sum_{l=0}^{m} \frac{e^{i(2l+1)x}}{2l+1} + \frac{-i}{\pi} \sum_{l=0}^{m} \frac{e^{-i(2l+1)x}}{-(2l+1)}$$
$$= \frac{1}{2} + \frac{-i}{\pi} \sum_{l=0}^{m} \left(\frac{e^{i(2l+1)x} - e^{-i(2l+1)x}}{2l+1} \right)$$
$$= \frac{1}{2} + \frac{-i}{\pi} \sum_{l=0}^{m} \frac{2i \sin((2l+1)x)}{2l+1}$$
$$= \frac{1}{2} + \frac{2}{\pi} \sum_{l=0}^{m} \frac{\sin((2l+1)x)}{2l+1}$$

- Try out $x = \frac{\pi}{2}$:

$$\begin{aligned} S_N(f)(x) &= \frac{1}{2} + \frac{2}{\pi}\sum_{l=0}^{m}\frac{\sin((2l+1)\frac{\pi}{2})}{2l+1} \\ &= \frac{1}{2} + \frac{2}{\pi}\sum_{l=0}^{m}\frac{-i^l}{2l+1} \end{aligned}$$

$$\begin{aligned} \sum_{l=0}^{m}\frac{-i^l}{2l+1} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... \\ &= \int_0^1 1 - x^2 + x^4 - x^6 + ... dx \\ &= \int_0^1 \frac{1}{1+x^2}dx \\ &= \arctan 1 - \arctan 0 = \frac{\pi}{4} \end{aligned}$$

$$\therefore S_N(f)(x) = \frac{1}{2} + \frac{2}{\pi}\frac{\pi}{4} = 1$$

Example 2:

- Problem:

$$\lim_{N\to\infty} S_N(f)(0) = ?$$

$$f(x) = \begin{cases} 1 & 0 \le x \le \pi \\ -1 & -\pi \le x \le 0 \end{cases}$$

- If f is odd, Fourier coefficients are also odd:

$$\hat{f}(0) = 0 \qquad\qquad \text{since } f \text{ is odd } \to \hat{f} \text{ is odd}$$

$$\begin{aligned} \hat{f}(j) &= \int_{-\pi} \pi e^{-ijx} f(x)dx \\ &= \int_0 \pi e^{-ijx} f(x)dx + \int_{-\pi} 0 e^{-ijx} f(x)dx \\ &= h(j) + k(j) \end{aligned}$$

- Evaluate:

$$h(j) = \begin{cases} 0 & j \text{ even} \\ \frac{1}{ij\pi} & j \text{ odd} \end{cases}$$
$$k(j) = \begin{cases} 0 & j \text{ even} \\ \frac{1}{-ij\pi} & j \text{ odd} \end{cases}$$
$$\therefore \hat{f}(j) = h(j) - k(j) = \frac{2}{ij\pi}$$
$$S_N(f)(x) = \sum_{j \text{ odd}} \frac{2}{ij\pi} e^{ijx}$$

- Repeat the same method for Example 2.

LOS 8. Solve the heat equation with unusual boundary conditions

Problem:

$$u_t = u_{xx} \qquad 0 \leq x \leq \pi$$
$$u(0, x) = 0$$
$$u(t, 0) = h(t)$$
$$u(0, t) = k(t)$$

Why this does not work:

$$u(x, t) = \sum u_n(t) sin(nx)$$
$$\frac{d}{dt} u_n(t) = -n^2 u_n(t)$$
$$u_n(t) = e^{-tn^2} u_n(0)$$

Since $u(0, x) = 0$:

$$u_n(0) = 0$$
$$\therefore u(x, t) = e^{-tn^2}(0) = 0$$

We only get trivial solution which does not satisfy h and k boundary conditions.
Mistake:

$$\forall t \;\; u(0,t) \in L_2$$
$$u(x,t) = \sum_n u_n(t) \sin(nx)$$
$$\frac{d}{dx} u(x,t) = \sum_n u_n(t)(n \cos nx)$$

$\sum_n \sin x$ is not convergent. $\sum_n n \cos x$ is even worse.