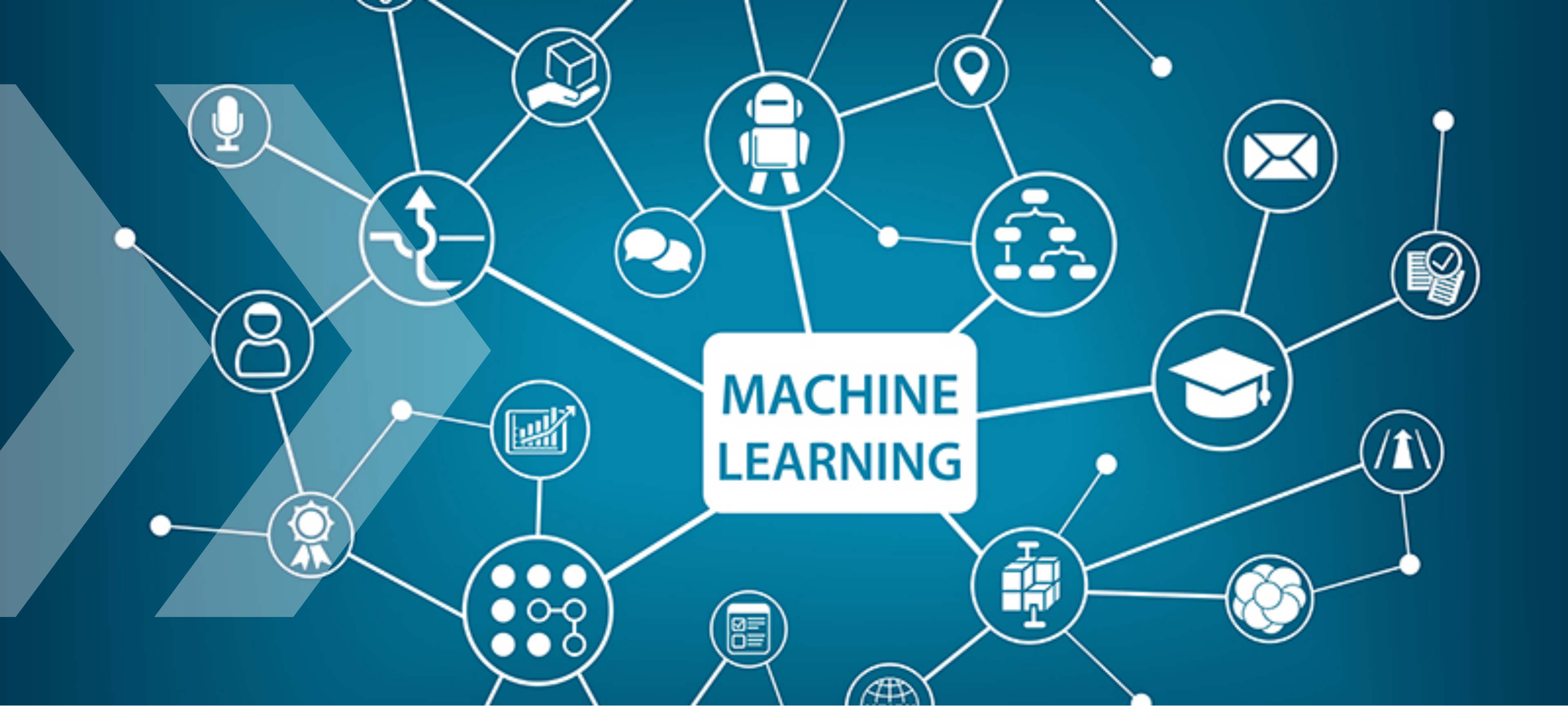


CS489/698: Intro to ML

Lecture 06: Hard-margin SVM

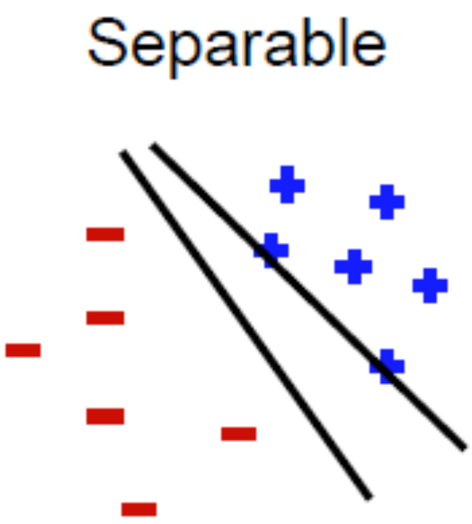
UNIVERSITY OF WATERLOO

Outline

- Maximum Margin
- Lagrangian Dual
- Alternative View

Perceptron revisited

- Two classes: $y = 1$ or $y = -1$



- Assuming linear separable
  - exist **w** and b such that for all i,

$$y_i(\mathbf{w}^\top \mathbf{x}_i + b) > 0$$

- Find any such **w** and b

$$\min_{\mathbf{w},b} \ 0$$

$$\text{s.t. } \forall i, \ y_i(\mathbf{w}^\top \mathbf{x}_i + b) > 0$$

feasibility problem

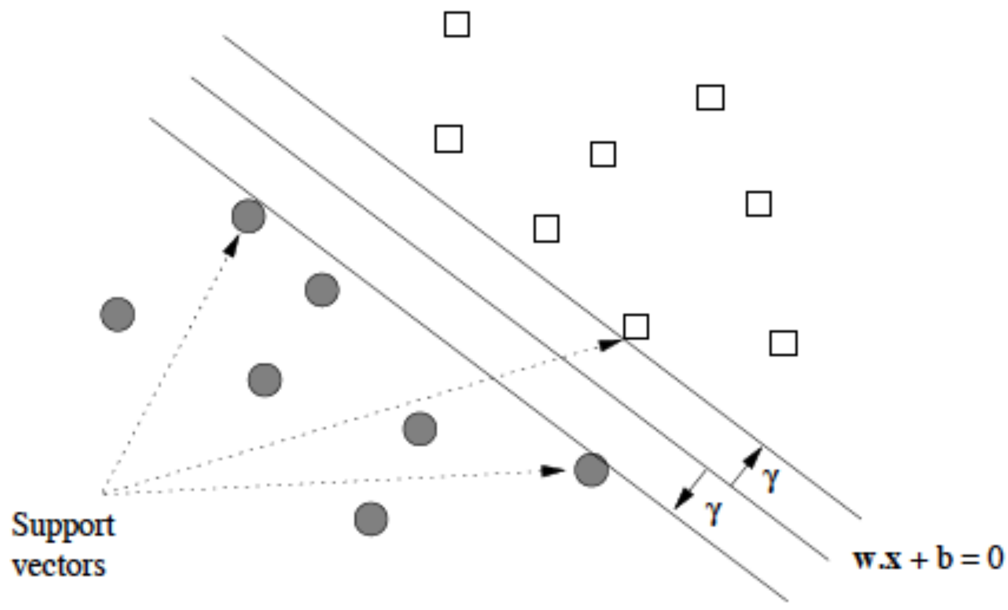
Margin

- Take any linear separating hyperplane H

$$\boxed{\text{for all } i,\ y_i(\mathbf{w}^T\mathbf{x}_i + b) > 0}$$

- Move H until it touches some positive point, H_1
  increase b say

- Move H until it touches some negative point, H_{-1}
  decrease b say



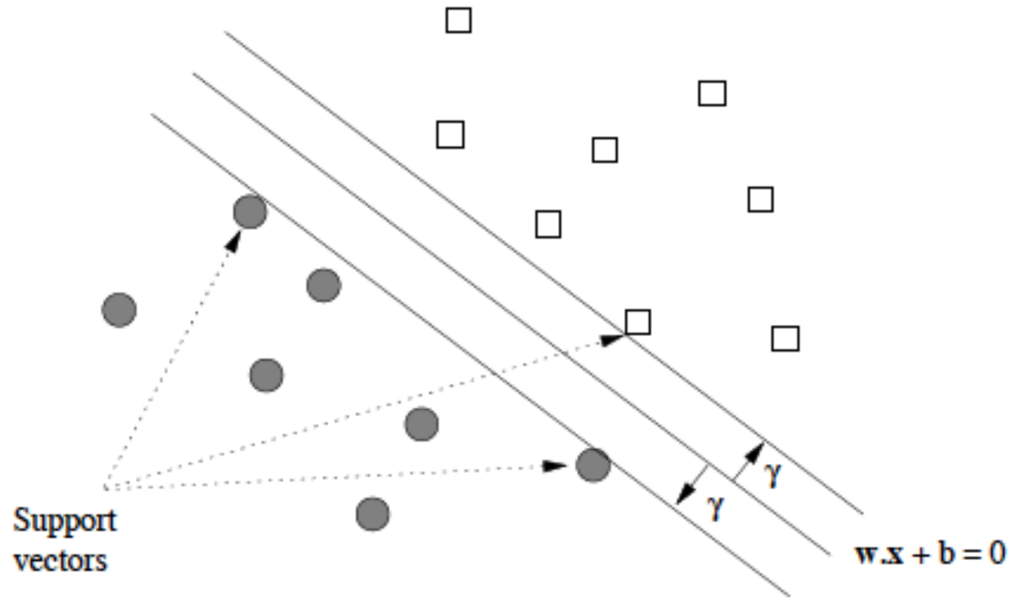
margin $= \mathrm{dist}(H_1,H) \wedge \mathrm{dist}(H_{-1},H)$

Put into formula

$H \quad : \quad \mathbf{w}^T\mathbf{x} + b = 0$

$H_1 : \quad \mathbf{w}^T\mathbf{x} + b = t$

$H_{-1} : \quad \mathbf{w}^T\mathbf{x} + b = -s$



What is the distance between H_1 and H ?

$$\min_{\mathbf{x},\mathbf{z}} \|\mathbf{x}-\mathbf{z}\|_2 \quad \geq \quad \left\| \frac{\mathbf{w}^\top}{\|\mathbf{w}\|_2}(\mathbf{x}-\mathbf{z}) \right\|_2 = \frac{|t|}{\|\mathbf{w}\|_2}$$

$$\text{s.t. } \mathbf{w}^\top \mathbf{x} + b = t$$

$$\mathbf{w}^\top \mathbf{z} + b = 0$$

equality is attained

$$\mathbf{x} = \frac{\mathbf{w}}{\|\mathbf{w}\|_2^2}(t-b)$$

$$\mathbf{z} = \frac{\mathbf{w}}{\|\mathbf{w}\|_2^2}(-b)$$

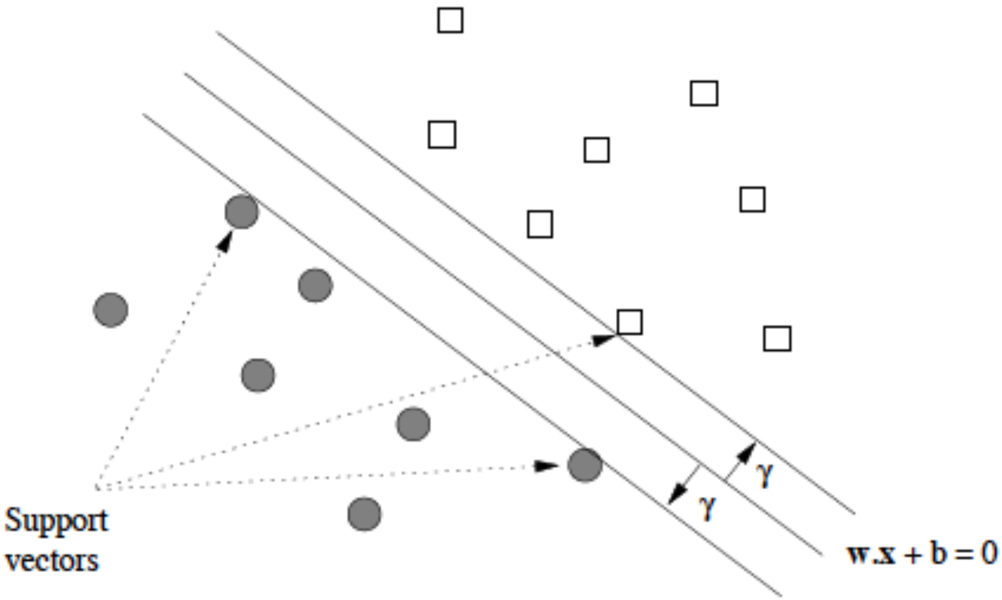
Put into formula

$H \;:\; \mathbf{w}^T\mathbf{x} + b = 0$

$H_1 : \mathbf{w}^T\mathbf{x} + b = t$

$H_{-1} : \mathbf{w}^T\mathbf{x} + b = -t$

What is the distance between H_1 and H ?

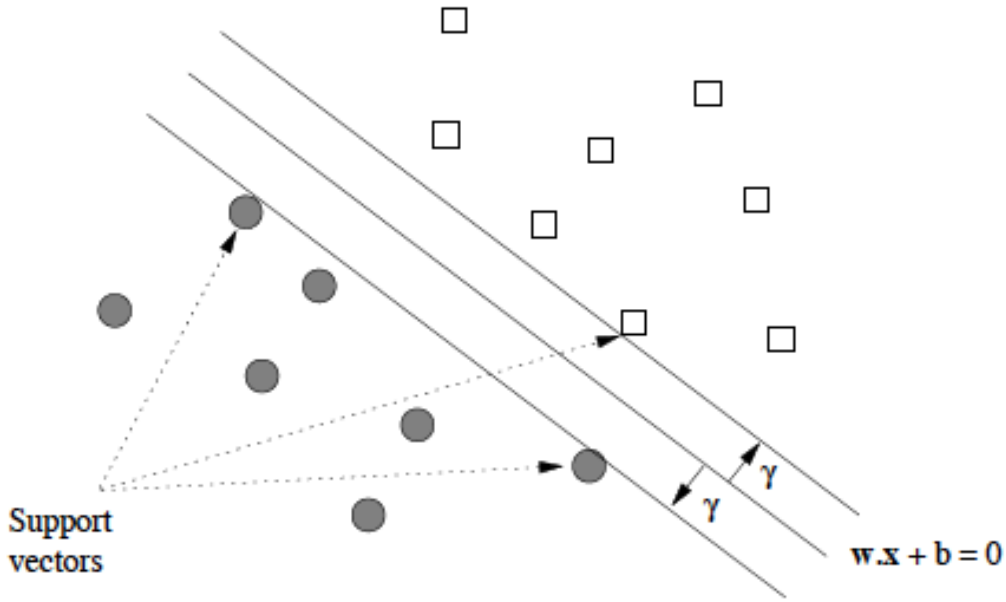


Put into formula

$H\ :\ \mathbf{w}^T\mathbf{x} + b = 0$

$H_1 :\ \mathbf{w}^T\mathbf{x} + b = 1$

$H_{-1} :\ \mathbf{w}^T\mathbf{x} + b = -1$



What is the distance between H_1 and H ?

Maximum Margin

$$\max_{\mathbf{w},b} \ \frac{1}{\|\mathbf{w}\|_2}$$

$$\text{s.t. } \forall i, y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1$$

Important facts.

- For any *f*, $\max_{\mathbf{w}} f(\mathbf{w}) = -\min_{\mathbf{w}} -f(\mathbf{w})$
- For positive *f*, $\max_{\mathbf{w}} \frac{1}{f(\mathbf{w})} = \frac{1}{\min_{\mathbf{w}} f(\mathbf{w})}$
- For s. monotone *g*, $\min_{\mathbf{w}} f(\mathbf{w}) \equiv \min_{\mathbf{w}} g(f(\mathbf{w}))$

Hard-margin Support Vector Machines

$$\max_{\mathbf{w},b} \frac{1}{\|\mathbf{w}\|_2}$$

$$\text{s.t. } \forall i, y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1$$

Linear programming

$$\min_{\mathbf{w},b} \; 0$$

$$\text{s.t. } \forall i, \; y_i(\mathbf{w}^\top \mathbf{x}_i + b) > 0$$

Quadratic programming

margin

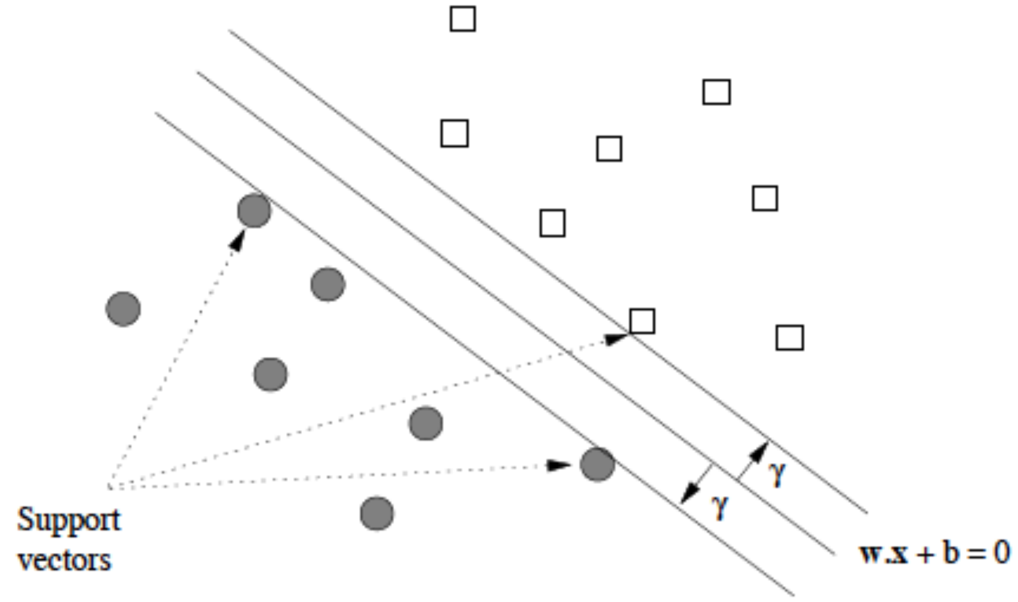
$$\frac{1}{\|\mathbf{w}\|_2}$$

$$\min_{\mathbf{w},b} \frac{1}{2}\|\mathbf{w}\|_2^2$$

$$\text{s.t. } \forall i, y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1$$

Support Vectors

- Those touch the parallel hyperplanes H_1 and H_{-1}
- Usually only a handful
- Entirely determine the hyperplanes!



UNIVERSITY OF WATERLOO

Existence and uniqueness

$$\boxed{\begin{aligned} &\min_{\mathbf{w},b} \ \frac{1}{2}\|\mathbf{w}\|_{2}^{2} \\ &\text{s.t. } \forall i, y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1 \end{aligned}}$$

- Always exists a minimizer **w** and b (if linearly separable)
- The minimizer **w** is unique (strong convexity of $\frac{1}{2}\|\mathbf{w}\|_2^2$)
- The minimizer b is also unique (why?)

UNIVERSITY OF WATERLOO

Outline

- Maximum Margin
- Lagrangian Dual
- Alternative View

Lagrangian

Primal

$$\min_{\mathbf{w},b} \ \frac{1}{2}\|\mathbf{w}\|_2^2$$

$$\text{s.t. } \forall i, y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1$$

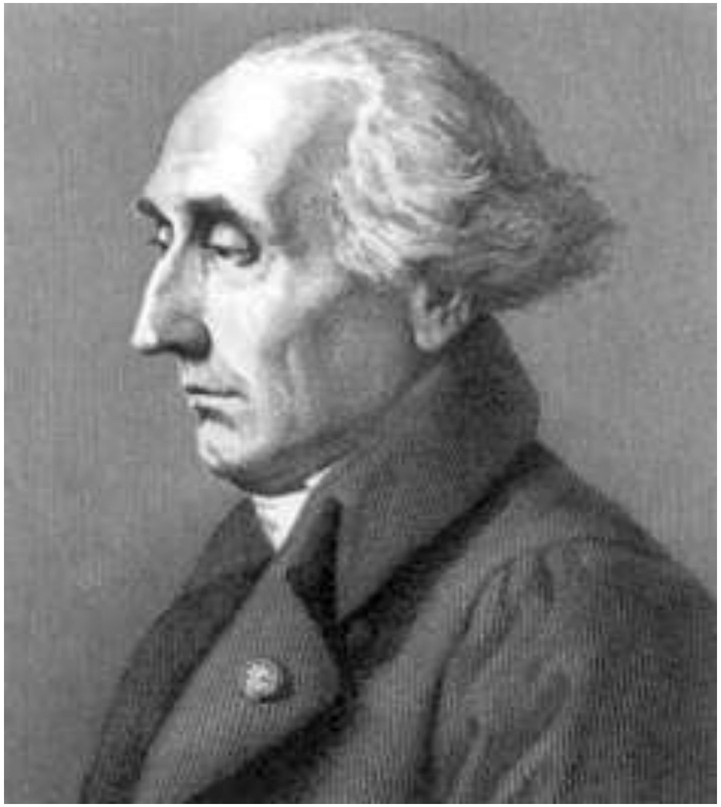
Lagrangian

$$\min_{\mathbf{w},b} \max_{\alpha \geq 0} \frac{1}{2}\|\mathbf{w}\|_2^2 - \sum_i \alpha_i [y_i(\mathbf{w}^\top \mathbf{x}_i + b) - 1]$$

[primal variable]

Lagrangian multiplier
[dual variable]

Joseph-Louis Lagrange (1736-1813)



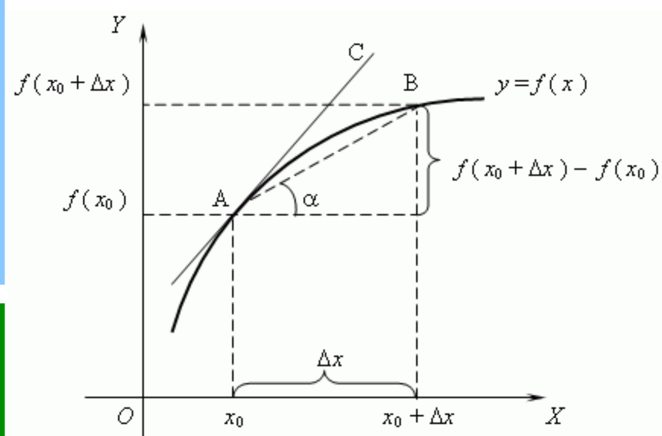
Optimization detour

$$\min_{x} \; f(x)$$

Fermat's Theorem. Necessarily $Df(x) = 0$

(Fréchet) Derivative at x.

$$\lim_{0 \neq \delta \to 0} \frac{|f(x+\delta) - f(x) - Df(x)\delta|}{|\delta|} = 0$$



Example. $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T\mathbf{b} + c$

$Df(\mathbf{x}) = (A+A^T)\mathbf{x} + \mathbf{b}$

UNIVERSITY OF WATERLOO

Deriving the dual

$$\boxed{\min_{\mathbf{w},b} \max_{\alpha \geq 0}} \frac{1}{2}\|\mathbf{w}\|_2^2 - \sum_i \alpha_i [y_i(\mathbf{w}^\top \mathbf{x}_i + b) - 1]$$

$$\boxed{\max_{\alpha \geq 0} \min_{\mathbf{w},b}} \frac{1}{2}\|\mathbf{w}\|_2^2 - \sum_i \alpha_i [y_i(\mathbf{w}^\top \mathbf{x}_i + b) - 1]$$

$$\boxed{\frac{\partial}{\partial b} = \sum_i \alpha_i y_i = 0} \qquad \boxed{\frac{\partial}{\partial \mathbf{w}} = \mathbf{w} - \sum_i \alpha_i y_i \mathbf{x}_i = 0}$$

The dual problem

$$\max_{\alpha \geq 0} \sum_i \alpha_i - \frac{1}{2}\Big\| \sum_i \alpha_i y_i \mathbf{x}_i \Big\|_2^2$$

$$\text{s.t.} \sum_i \alpha_i y_i = 0$$

Only need dot product in the dual !

Dual

$$\min_{\alpha \geq 0} \frac{1}{2} \sum_i \sum_j \alpha_i \alpha_j y_i y_j \boxed{\mathbf{x}_i^\top \mathbf{x}_j} - \sum_k \alpha_k$$

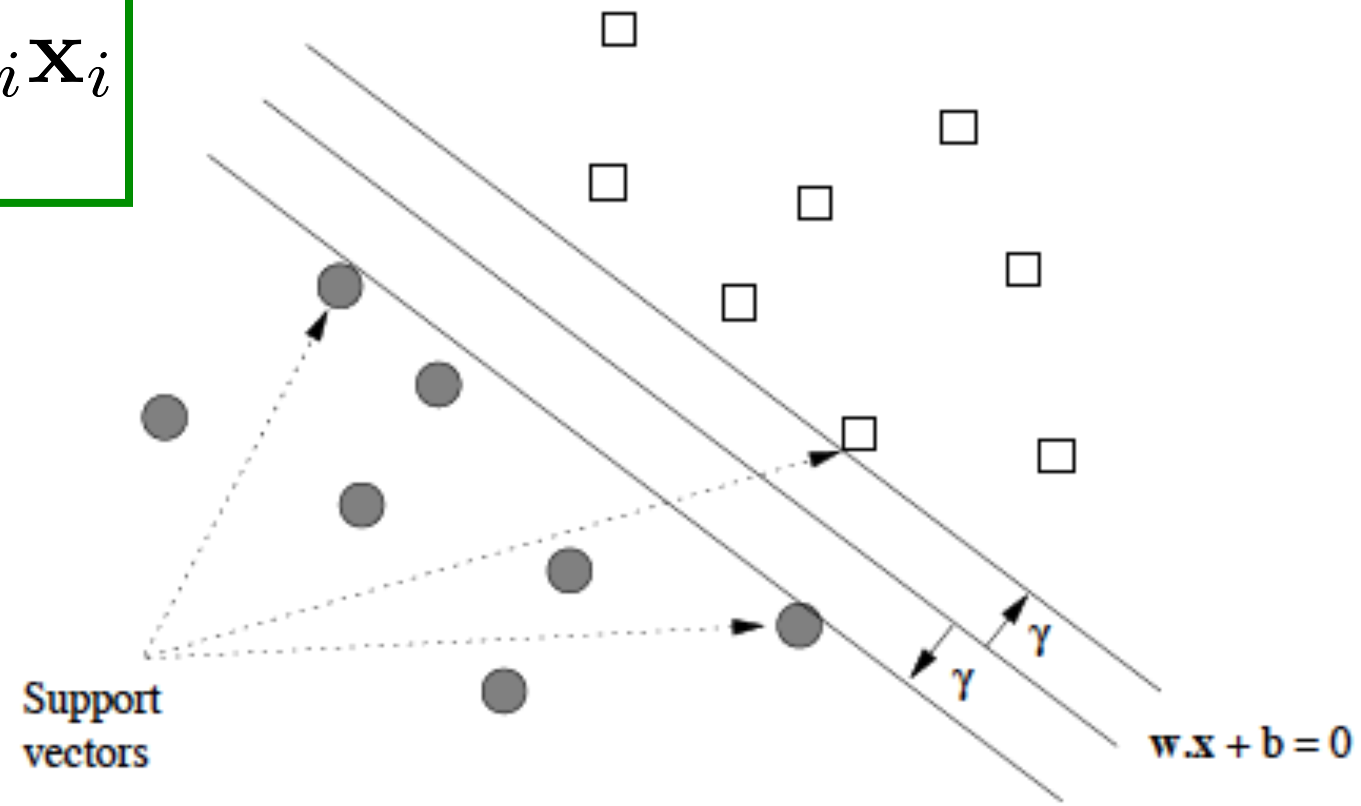
$$\text{s.t.} \sum_i \alpha_i y_i = 0$$

$\mathbf{R}^n$

Support Vectors

$$\boxed{\mathbf{w} = \sum_i \alpha_i y_i \mathbf{x}_i}$$

$$\boxed{\alpha_i > 0}$$



Outline

- Maximum Margin
- Dual
- Alternative View

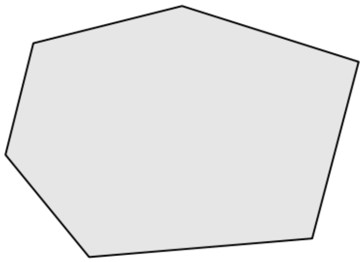
An dual view

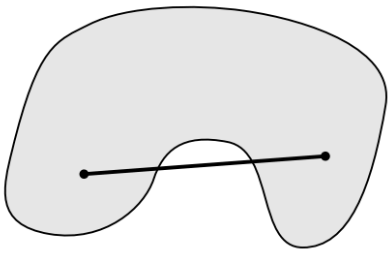
Convex sets and Convex hull

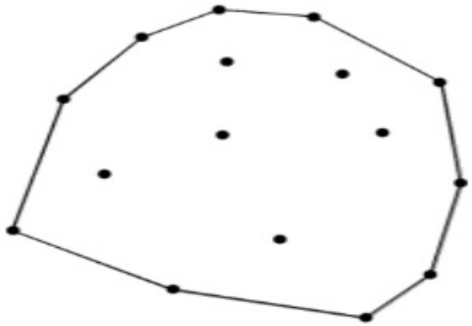
Convex set. A point set $C \in \mathbf{R}^d$ is convex if the line segment [x,y] connecting any two points x and y in C lies entirely in C.

Convex hull. Smallest convex set containing C.

$$\mathrm{ch}(C) := \left\{ \sum_i \alpha_i \mathbf{x}_i : \mathbf{x}_i \in C, \alpha_i \geq 0, \sum_i \alpha_i = 1 \right\}.$$







UNIVERSITY OF WATERLOO

Regularization vs. Constraint

$$\min_W \; \ell(W) + \lambda \cdot r(W)$$

Computationally appealing

Always true

Mild conditions

$$\min_{W : r(W) \leq C} \; \ell(W)$$

Theoretically appealing

can use equality if from top to bottom

UNIVERSITY OF WATERLOO

From regularization to constraint

$$\min_{\alpha \geq 0} \frac{1}{2} \Big\| \sum_i \alpha_i y_i \mathbf{x}_i \Big\|_2^2 - \sum_i \alpha_i$$

$$\text{s.t.} \sum_i \alpha_i y_i = 0$$

$$\Downarrow$$

$$\min_{\alpha \geq 0} \frac{1}{2} \Big\| \sum_i \alpha_i y_i \mathbf{x}_i \Big\|_2^2$$

$$\text{s.t.} \sum_i \alpha_i y_i = 0, \quad \sum_i \alpha_i = C$$

UNIVERSITY OF WATERLOO

Homogeneity

$$\min_{\alpha \geq 0} \frac{1}{2} \Big\| \sum_i \alpha_i y_i \mathbf{x}_i \Big\|_2^2$$

$$\text{s.t.} \sum_i \alpha_i y_i = 0, \ \sum_i \alpha_i = C$$

$$\boxed{\alpha \leftarrow 2\alpha / C}$$

$$\min_{\alpha \geq 0} \frac{1}{2} \Big\| \sum_i \alpha_i y_i \mathbf{x}_i \Big\|_2^2$$

$$\text{s.t.} \sum_i \alpha_i y_i = 0, \ \sum_i \alpha_i = 2$$

Split

$$\min_{\alpha \geq 0} \frac{1}{2} \Big\| \sum_i \alpha_i y_i \mathbf{x}_i \Big\|_2^2$$

$$\text{s.t.} \sum_i \alpha_i y_i = 0, \ \sum_i \alpha_i = 2$$

$$P := \{i : y_i = 1\}$$

$$N := \{i : y_i = -1\}$$

$$\alpha = [\mu; \nu]$$

$$\min_{\mu \geq 0, \nu \geq 0} \frac{1}{2} \Big\| \sum_{i \in P} \mu_i \mathbf{x}_i - \sum_{j \in N} \nu_j \mathbf{x}_j \Big\|_2^2$$

$$\text{s.t.} \sum_i \mu_i = 1, \ \sum_j \nu_j = 1$$

NOW this

Questions?

UNIVERSITY OF WATERLOO