CS770: Assignment 3

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1 Question 1

1.1 Question 1a

```
function [X,errors] = newtons(f,d,x0,tol,maxiter)
% f is the input function, in our implementation, it's a string
% d is the input derrivative, in our implementation, it's a string
% x0 is the initial guess of the root
% tol is the error tollerance
% maxiter is the maximum iteration

    switch nargin
        case 3
            tol = 10^-4;
            maxiter = 10000;
        case 4
            tol = 10^-4;
    end
    % the default tollerence is set to be 10^-4
    % the default maximum iteration is set to be 10000

    f = inline(f);
    d = inline(d);
    %change the text version of the function and derrivative
    %to numerical version

    X = [];
    X = [X x0];
    errors = [];
    x1 = x0 - f(x0)./d(x0);
    X = [X x1];
    errors = [errors abs(x1-x0)];
```

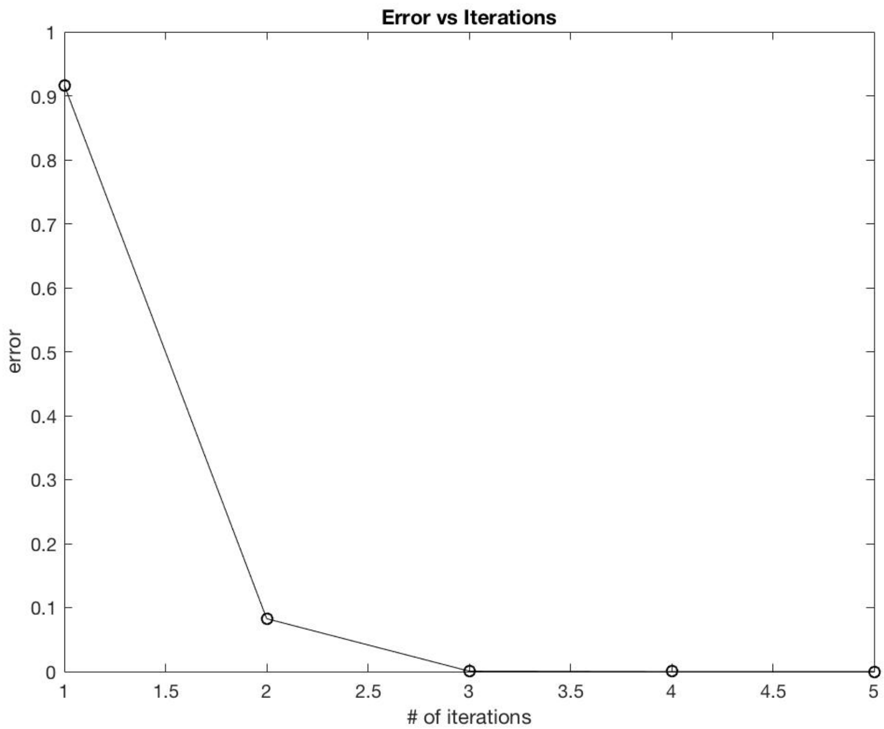
```
k = 1;
while  (errors(k) > tol) && (k <= maxiter)
     x0 = X(k+1);
     tempX = x0- f(x0)./d(x0);
     %calculate the next value
     errors = [errors abs(tempX - X(end))];
     X = [X tempX];
     k = k+1;
end

end
```

1.2 Question 1b

In my Newton's method, I set the tolerance of error to be 10^{-4} and the maximum number of iteration to be 10^4.

- Newton's Method Converges Well:

$$f(x) = x^2 - 10x,\ d(x) = 2x - 10,\ x_0 = 11$$



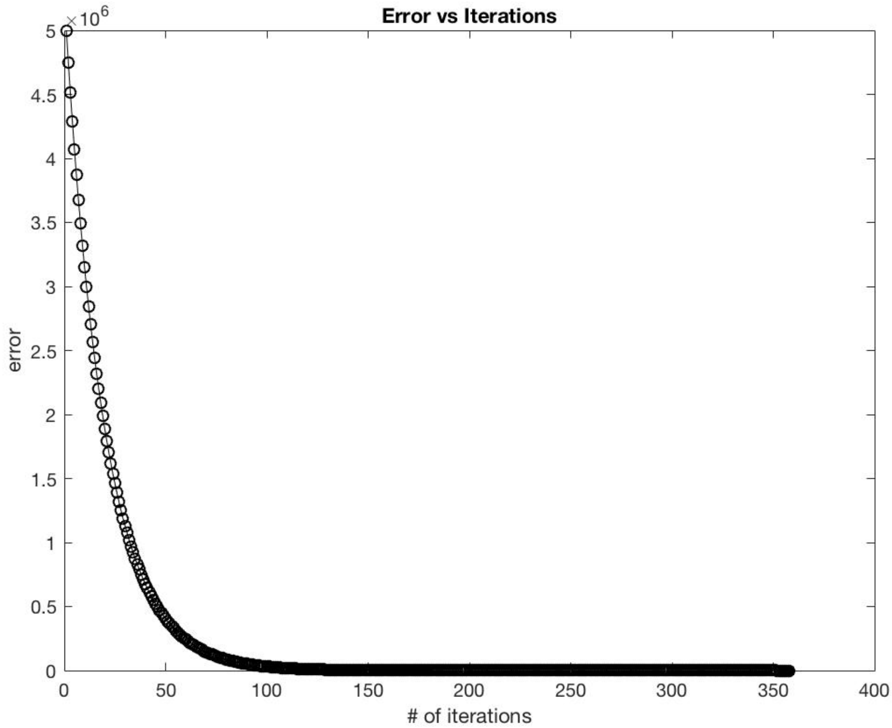
- Newton's Method Doesn't Converges:

$$f(x) = x^2 + 10,\ d(x) = 2x$$

Since $f(x)$ has no roots on its domain, whatever x_0 is chosen, x won't converge.

- Newton's Method Converges Slowly:

$$f(x) = x^{20} - 100,\ d(x) = 20x^{19},\ x_0 = 100000000$$



2 Question 2

2.1 Question 2a

$$f(x) = (x-1)^2 e^x,\ d(x) = 2(x-1)e^x + (x-1)^2 e^x$$

Therefore, when $x \neq 1$, $d(x) \neq 0$, therefore, Newton's iteration is well defined for $x \neq 1$.

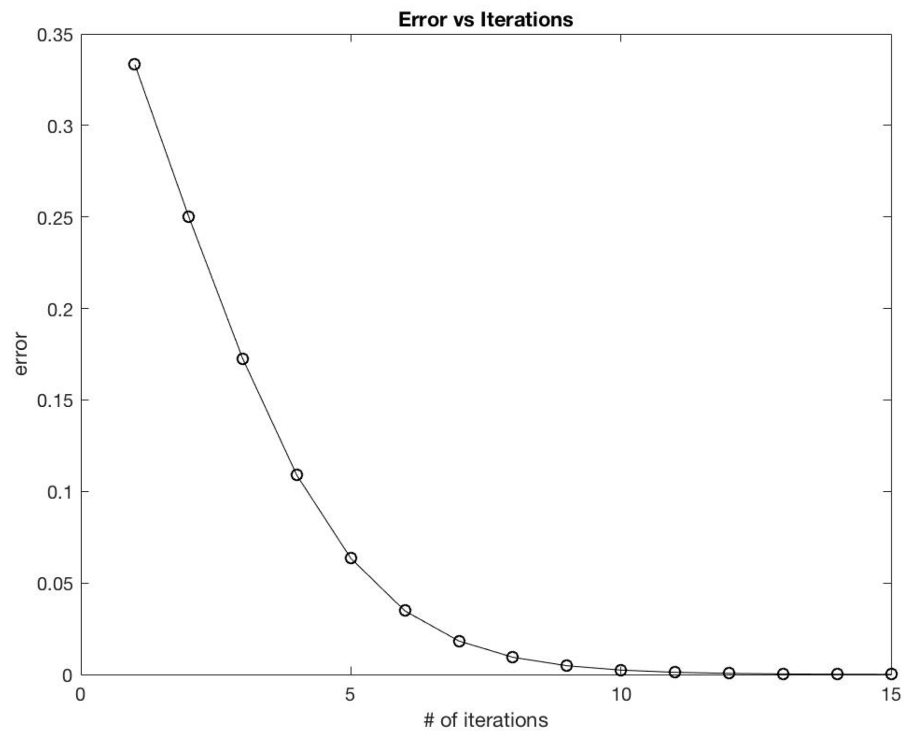
$$f(x^*) = f(x_k) + (x^* - x_k)f'(x_k) + \frac{f''(\epsilon)(x^*-x_k)^2}{2} = 0$$

$$x_{k+1} = x_k - \frac{f(x_k)}{d(x_k)}$$
$$x_{k+1} - x_k = - \frac{f(x_k)}{d(x_k)}$$
$$f(x_k) = -(x_{k+1} - x_k)f'(x_k), \text{ where } f'(x_k) = d(x_k)$$
$$f(x^*) = f(x_k) + (x^* - x_k)f'(x_k) + \frac{f''(\epsilon)(x^* - x_k)^2}{2} = 0 \text{ can be rewritten as}$$
$$f(x^*) = -(x_{k+1} - x_k)f'(x_k) + (x^* - x_k)f'(x_k) + \frac{f''(\epsilon)(x^* - x_k)^2}{2} = 0$$
$$f(x^*) = (x^* - x_{k+1})f'(x_k) + \frac{f''(\epsilon)(x^* - x_k)^2}{2} = 0$$
$$\lim_{k \leftarrow \infty} \frac{|e_{k+1}|}{|e_k|^2} = \frac{f''(x^*)}{2f'(x^*)} = C$$

Therefore, this is a second order method.Beside, since $x_{k+1} = \frac{x_k^2+1}{x_k+1}$, this method will converge slowly when x is big.

2.2 Question 2b

When $x_0 = 2$,



$\frac{f(x_{old})}{d(x_{old})}$ can be written as $1 - \frac{2}{x+1}$, let $s(x) = x - (1 - \frac{2}{x+1})$, then $s(x) = \frac{x^2+1}{x+1}$, then $\frac{\partial}{\partial x}s(x)$=$1 - \frac{2}{(x+1)^2}$, as we can see here, when x is large, $f(x)$ converges slowly, as x decreases, $f(x)$ converges faster.

3 Question 3

3.1 Question 3a

If a polynomial has a multiple root at x^*, then its derivative also has root(s) at x^*. If the multiplicity of root x^* for polynomial $f(x)$ is m, then the multiplicity of root x^* for $f'(x)$ is $m-1$. Therefore, the multiplicity of root x^* for $w(x) = f(x)/f'(x)$ is 1, so $w(x)$ has a simple root at x^*.

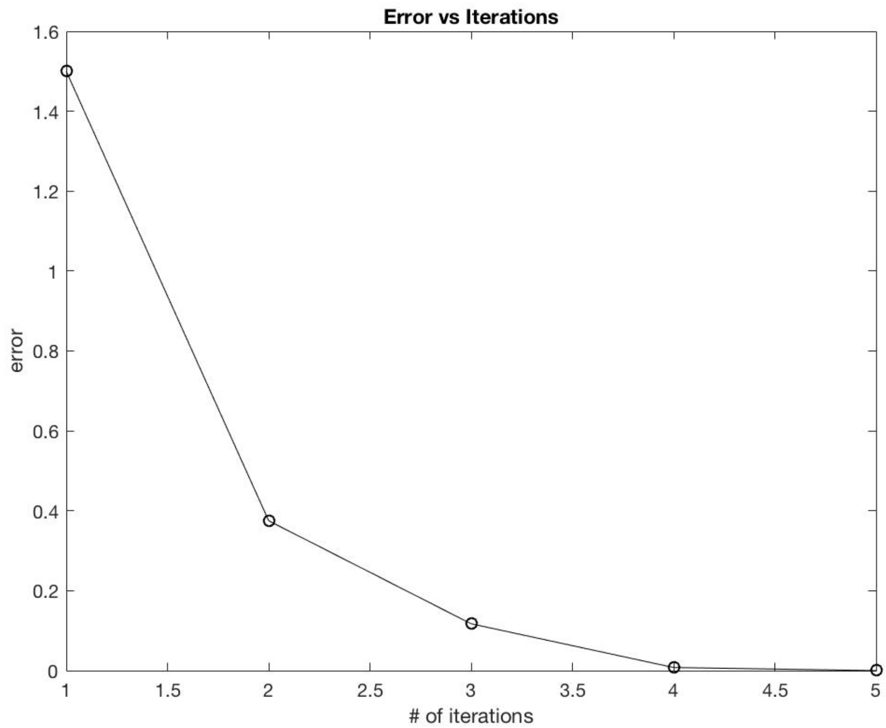
3.2 Question 3b

$$w(x) = f(x)/f'(x),\ w'(x) = \frac{f'(x)f'(x)-f(x)f''(x)}{f'(x)^2}$$

For the update: $x = x_0 - w(x_0)/w'(x_0)$, where x_0 is the value before update

To simplify this equation, $x = \frac{f(x)f'(x)}{f'(x)f'(x)-f(x)f''(x)}$

3.3 Question 3c



Compared to Newton's method, this method converges much faster, however, this method requires the computation of the second derivative which means more calculation is required.

4 Question 4

Proof:

$$x^{k+1} = x^k - f(x^k)\ \frac{f(x^k)}{f(x^k+f(x^k))-f(x^k)}$$

$$\text{Let } F(x^k) = x^k - f(x^k)\ \frac{f(x^k)}{f(x^k+f(x^k))-f(x^k)}, \text{ then } x^{k+1} = F(x^k)$$

$$\text{By Taylor's Series, } f(x^k + f(x^k)) = f(x^k) + f'(x^k)f(x^k) + \frac{f''(\epsilon_k)}{2}f^2(x^k)$$

$$F(x^k) = x^k - \frac{f(x^k)}{f'(x^k)+\frac{f''(\epsilon_k)}{2}f(x^k)}$$

$$e_k = x^* - x_k$$

$$e_{k+1} = x^* - x_{k+1} = F(x^*) - F(x_k) = x^* - x_k + \frac{f(x^k)}{f'(x^k)+\frac{f''(\epsilon_k)}{2}f(x^k)}$$

$$e_{k+1} = \frac{f(x^k)+f'(x^k)(x^*-x_n)+\frac{f''(\epsilon_k)}{2}f(x^k)(x^*-x_k)}{f'(x^k)+\frac{f''(\epsilon_k)}{2}f(x^k)}$$

Since $f(x^*) = 0$, by Taylor's series, we have:

$$0 = f(x_k)+f'(\epsilon_k^*)(x^* - x_k)$$

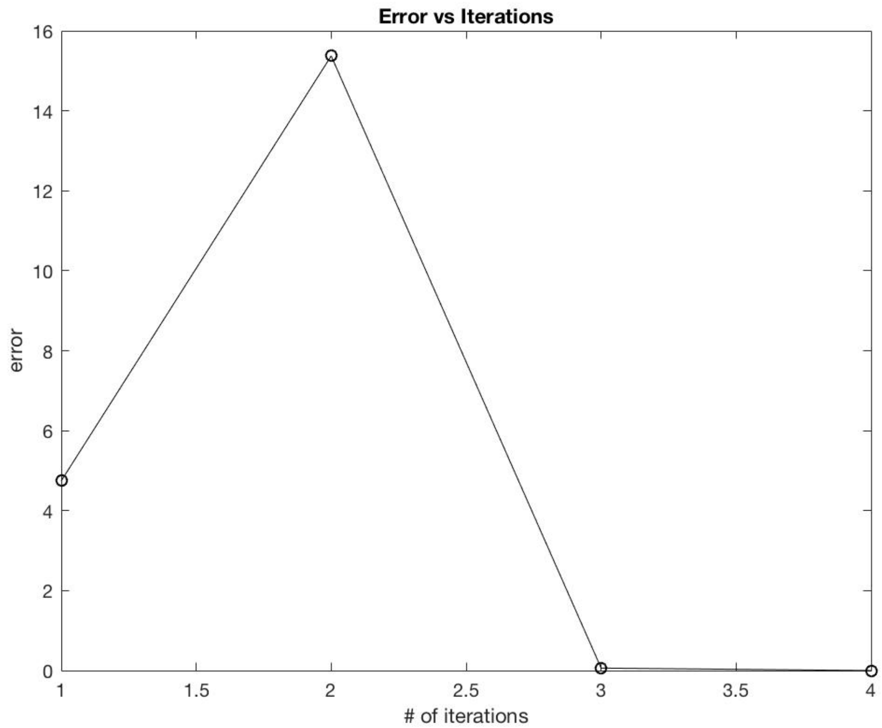
$$\text{Overall, we have } e_{k+1} = -\frac{\frac{f''(\epsilon_k^*)}{2}(x^*-x_k)^2+\frac{f''(\epsilon_k)}{2}f'(\epsilon_k^*)(x^*-x_k)^2}{f'(x^k)+\frac{f''(\epsilon_k)}{2}f(x^k)}$$

We can see $\lim_{x\to\infty} \frac{|e_{k+1}|}{|e_k|^2}$ is a constant

Therefore, Steffensen's method is a second-order method. Proof DONE.

Example:

$x0$ was chosen to be 2, x is converged to 12.566374101689368.



Compared to Secant method, Steffensen's method requires the calculation of $f(x_k + f(x_k))$ while Secant method only evaluates $f(x_k)$ within each iteration. Therefore, Secant is more efficient in terms of the required number of function evaluations.

5 Question 5

For this question, we also set the tolerance to be 10^{-4}, the maximum iteration to be 10^4. And $\alpha_1 = -1$, $\alpha_2 = 2$.

$\bullet \phi(x) = x^2 - 1$
Result: $\phi(x)$ doesn't converge.

$$\phi'(x) = 2x$$

$$\phi'(\alpha_1) = -2,\ \phi'(\alpha_2) = 4$$

$$\mid (\phi'(\alpha_1) \mid = 2 > 1,\ \mid (\phi'(\alpha_2) \mid = 4 > 1$$

Therefore, there is no convergence to α_1 nor α_2 with $\phi(x)$

$\bullet \phi(x) = \sqrt{2+x}$
Result: $\phi(x)$ only converges to α_2.

$$\phi'(x) = \frac{1}{2}(2+x)^{-0.5}$$

$$\phi'(\alpha_1) = \frac{1}{2},\ \phi'(\alpha_2) = \frac{1}{4}$$

$$\mid (\phi'(\alpha_1) \mid = 0.5 < 1,\ \mid (\phi'(\alpha_2) \mid = 0.25 < 1$$

$$\text{However, } \phi(\alpha_1) \neq \alpha_1$$

Therefore, it only converges to α_2 with $\phi(x)$.

$\bullet \phi(x) = -\sqrt{2+x}$
Result: $\phi(x)$ only converges to α_1.

$$\phi'(x) = -\frac{1}{2}(2+x)^{-0.5}$$

$$\phi'(\alpha_1) = -\frac{1}{2},\ \phi'(\alpha_2) = -\frac{1}{4}$$

$$\mid (\phi'(\alpha_1) \mid = 0.5 < 1,\ \mid (\phi'(\alpha_2) \mid = 0.25 < 1$$

$$\text{However, } \phi(\alpha_2) \neq \alpha_2$$

Therefore, it only converges to α_1 with $\phi(x)$.

$\bullet \phi(x) = 1 + \frac{2}{x}$
Result: $\phi(x)$ only converges to α_2.

$$\phi'(x) = \frac{-1}{x^2}$$

$$\phi'(\alpha_1) = -1,\ \phi'(\alpha_2) = \frac{-1}{4}$$

$$\mid (\phi'(\alpha_1) \mid = 1,\ \mid (\phi'(\alpha_2) \mid = \frac{-1}{4} < 1$$

Therefore, it only converges to α_2 with $\phi(x)$.

6 Question 6

Mueller Method READ.