CS 770: Assignment 2

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1 Question 1

Algorithm 1 will explain the implementation of row changing in detail.

Data: $A \in R^n$
Result: L,U,P (LU = PA)
L=eye(n);
P=L;
U=A;
for *k = 1:n* **do**
  ind ←index of the row that its first non-zero number has the largest magnitude;
  if *ind != k* **then**
    switch row k and row ind in U;
    switch row k and row ind in P;
    if $k \geq 2$ **then**
      switch row k and row ind in L;
    end
  end
  L ← usual way to update L;
  U ← usual way to update U;
end

Algorithm 1: LU decomposition with row pivoting

Let

$$A = \begin{bmatrix} 10 & -7 & 0 \\ -3 & 4 & 6 \\ 7 & -2 & 5 \end{bmatrix}$$

By running the algorithm,

$$L = \begin{bmatrix} 1.0000 & 0 & 0 \\ 0.7000 & 1.0000 & 0 \\ -0.3000 & 0.6552 & 1.0000 \end{bmatrix}$$

$$U = \begin{bmatrix} 10.0000 & -7.0000 & 0 \\ 0 & 2.9000 & 5.0000 \\ 0 & 0 & 2.7241 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L * U = P * A = \begin{bmatrix} 10 & -7 & 0 \\ 7 & -2 & 5 \\ -3 & 4 & 6 \end{bmatrix}$$

2 Question 2 [NOT SURE]

2.1 Question 2a

$$M1 = \begin{bmatrix} 1 & 0 & 0 \\ 1000 & 1 & 0 \\ 2000 & 0 & 1 \end{bmatrix}$$

$$M1 * A = A_2 = (1.0e+03) * \begin{bmatrix} 0.000 & 0.002 & 0.003 \\ 0 & 2.004 & 3.005 \\ 0 & 4.001 & 6.006 \end{bmatrix}$$

$$M2 = \begin{bmatrix} 1.000 & 0 & 0 \\ 0 & 1.000 & 0 \\ 0 & -1.997 & 1.000 \end{bmatrix}$$

$$U = M2 * A_2 = A_3 = (1.0e+03) * \begin{bmatrix} 0.000 & 0.002 & 0.003 \\ 0 & 2.004 & 3.005 \\ 0 & -0.001 & 0.005 \end{bmatrix}$$

$$L = (1.0e+03) * \begin{bmatrix} 0.001 & 0 & 0 \\ -1.000 & 0.001 & 0 \\ -2.000 & 0.002 & 0.001 \end{bmatrix}$$

2.2 Question 2b

$$L = \begin{bmatrix} 1.0000 & 0 & 0 \\ 0.5000 & 1.0000 & 0 \\ -0.0005 & 0.6299 & 1.0000 \end{bmatrix}$$

$$U = \begin{bmatrix} -2.0000 & 1..0720 & 5.6430 \\ 0 & 3.1760 & 1.8015 \\ 0 & 0 & 1.8681 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$L * U = \begin{bmatrix} -2.0000 & 1.0720 & 5.6430 \\ -1.0000 & 3.7120 & 4.6230 \\ 0.0010 & 2.0000 & 3.0000 \end{bmatrix}$$

$$P * A = \begin{bmatrix} -2.0000 & 1.0720 & 5.6430 \\ -1.0000 & 3.7120 & 4.6230 \\ 0.0010 & 2.0000 & 3.0000 \end{bmatrix}$$

2.3 Question 2c

Without pivoting,

3 Question 3

Partial Pivoting:
Partial pivoting requires $O(n^2)$ number of number comparisons to determine the largest pivot,

Full Pivoting:
Full pivoting requires $O(n^3)$ number of number comparisons.

4 Question 4

A has an LU factorization $\leftarrow$ each upper left block $A_{1:k,1:k}$ is nonsingular:
If A has an LU factorization, the A can be written as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = LU$$

Where

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

As we can see here, A_{11} is a product of L_{11} and U_{11}, so $\det(A_{11}) = \det(L_{11})\det(U_{11})$, L_{11} and U_{11} are both triangular matrices, the determinant of a triangular matrix is the just the product of its diagonal entries. Since in both L_{11} and U_{11}, their diagonal entries are all non-zero, so $\det(L_{11})\det(U_{11})$ is non-zero, so $\det(A_{11})$ is also non-zero. Therefore, A_{11} is non-singular, so each upper left block $A_{1:k,1:k}$ is non-singular.

Each upper left block $A_{1:k,1:k}$ is nonsingular $\leftarrow$ A has an LU factorization :
This problem can be solved by induction:
Step 1:

When k = 1: $A_{11} = [1][a_{11}]$

Since A_{11} is non-singular, we know that a_{11} is not zero

Step 2:

Assume when k = s, $A_{1:s,1:s} = LU$

Step 3:

When k = s+1:

$$\mathrm{A}_{1:s+1,1:s+1} = \begin{bmatrix} A_{1:s,1:s} & a \\ b & c \end{bmatrix}$$

We can see that when $\mathrm{L}_{new} = \begin{bmatrix} L & 0 \\ bU^{-1} & 1 \end{bmatrix} U_{new} = \begin{bmatrix} L & L^{-1}a \\ 0 & c - bU^{-1}L^{-1}a \end{bmatrix}$

$$A_{1:s+1,1:s+1} = L_{new}U_{new}$$

If we want to show that $L_{new}U_{new}$ is the answer here, we need to show that $U_{22} \neq 0$

As we have seen early,det$(A_{1:s+1,1:s+1})$ = det(L_{new})det(U_{new})

Since both L_{new} and U_{new} are triangular matrices

The determinant of a triangular matrix is just the product of its diagonal entries

Since det$(A_{1:s+1,1:s+1})\neq 0$

Then none of the diagonal entries of L_{new} and U_{new} is 0, so $U_{22} \neq 0$

Proof Done

5 Question 5

5.1 Question 5a

We can write A as

$$A = \begin{bmatrix} a_{11} & a_{12} & ..... & a_{1n} \\ a_{21} & a22 & ..... & a_{2n} \\ .... & & & \\ a_{n1} & a_{n2} & ..... & a_{nn} \end{bmatrix}$$

Let $A^{-1} =$

$$A^{-1} = \begin{bmatrix} b_{11} & b_{12} & ..... & b_{1n} \\ b_{21} & b22 & ..... & b_{2n} \\ .... \\ b_{n1} & b_{n2} & ..... & b_{nn} \end{bmatrix}$$

Such that:

$$AA^{-1} = I$$

To simplify this problem, we can write A^{-1} as

$$A^{-1} = \begin{bmatrix} B_1 & B_2 & ..... & B_n \end{bmatrix}$$

Where

$$B_i = \begin{bmatrix} b_{1i} \\ b_{2i} \\ ..... \\ b_{ni} \end{bmatrix}$$

We can also write I as:

$$I = \begin{bmatrix} I_1 & I_2 & ..... & I_n \end{bmatrix}$$

Where

$$I_i = \begin{bmatrix} I_{1i} \\ I_{2i} \\ ..... \\ I_{ni} \end{bmatrix}$$

To solve A^{-1}, we can do so by solving all the B_is, for each B_i

$$AB_i = I_i$$

This algorithm solves A^{-1} by solving n systems of equations

5.2 Question 5b

To transform the matrix A to its echelon form, we need

$$\frac{n(n-1)}{2}$$

divisions,

$$\frac{2n^3 + 3n^2 - 5n}{6}$$

multiplications and

$$\frac{2n^3 + 3n^2 - 5n}{6}$$

subtractions. Overall, the cost is $O(n^3)$.

Therefore, solving one linear system costs $O(n^3)$, solving n linear systems will cost $O(n^4)$, so it is preferably not be used.

5.3 Question 5c (NOT SURE)

To take the advantage of matrix sparsity, we use LU decomposition for inverting a matrix. In this question, we keep using the notations we denoted in Question 5a.

$$A = LU$$

For each B_i and I_i

$$LUB_i = Ii$$

This method is fast, because matrix I is sparse, forward-substitution and backward-substitution can be calculated efficiently.

6 Question 6 [NOT SURE]

When A is a triangular matrix, the inverse of A will be dense. If A is a large matrix, A^{-1} would have most of its entries being non-zero. Storing such a matrix could be really expensive.

7 Question 7

By LU factorization with pivoting, $LU = PA$, therefore

$$L_1L_2....L_{n-1}U = PA$$

$$U = L'_{n-1}L'_{n-2}....L'_1PA$$

Since PA is just interchanging rows of A, $\|PA\|_{max} = \|A\|_{max}$

For each L'_s

$$
\begin{aligned}
\|L_s^{'}A\|_{max} &= \mathrm{max}_{i,j}|(L_s^{'}A)_{i,j}| \\
&= \mathrm{max}_{i,j}|\textstyle\sum_{a=1}^{n}(\mathrm{L}_s^{'})_{ia}a_{ja}\ | \\
&\leq \mathrm{max}_{i,j}|\textstyle\sum_{a=1}^{n}(\mathrm{L}_s^{'})_{ia}|\mathrm{max}_{i,j}|a_{ij}\ | \\
&= \|L_s^{'}\|_{\infty}\mathrm{max(A)} \\
&\leq 2\mathrm{max(A)}
\end{aligned} \tag{1}
$$

Since there are $n-1$ L, the growth factor ρ has an upper bound of 2^{n-1}

8 Question 8

More text.