CS 770: Assignment 5

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1 Exercise 1

```
function [C,A] = myDFT(f,X)
% discrete Fourier transform
% Output: C,A
% C contains the DFT coefficients
% A contains the DFT approxmiation
% Input: f,X
% f contains the function values
% X contains the X values which are to be approximated

    n = length(f);
    % n is the number of points

    C=zeros(1,n);
    % initialize the coefficients to 0

    i = sqrt(-1);
    % initialize i

    for k = 0:n-1
        for j = 0:n-1
            C(k+1) = C(k+1)+(1./n)*f(j+1)*exp(-2*pi*k*(j./n)*i);
        end
    end
    % Calculating the DFT coefficients

    N = length(X);
    A = zeros(1,N);

    for i = 1:N
        for j = 1:(n-1)./2
```

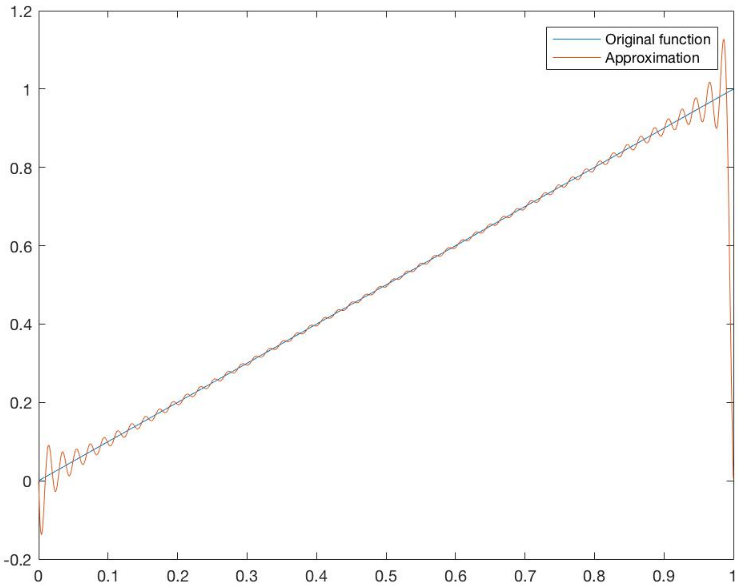
```
            A(i) = A(i) + 2*real(C(j+1))*cos(2*pi*j*X(i))-...
            2*imag(C(j+1))*sin(2*pi*j*X(i));
        end
        A(i) = A(i) + C(1);
    end
    % Calculating the DFT approximations
end
```

Note: this function only works with odd number of sampling points, for question 2, we have only tested it on odd number of sapling points.

2 Exercise 2

2.1 Exercise 2a

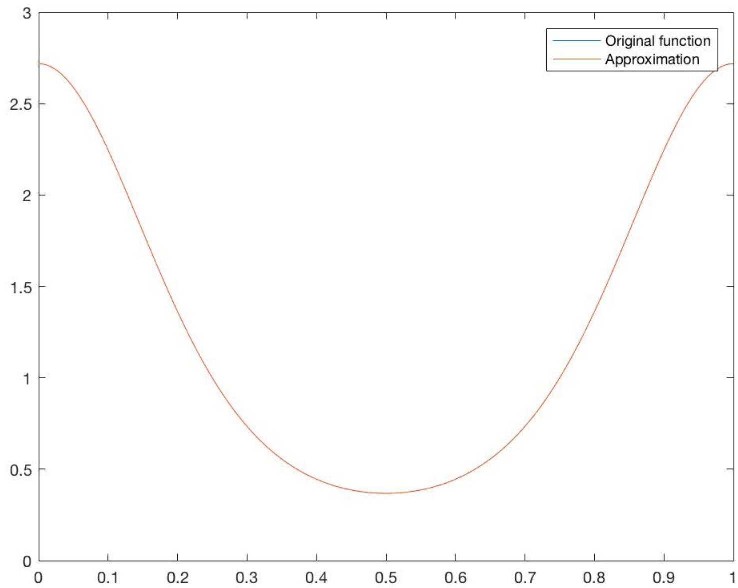
$f(x) = x$, When n = 101



As we can see here, there is oscillation near the discontinuity of the function.

2.2 Exercise 2b

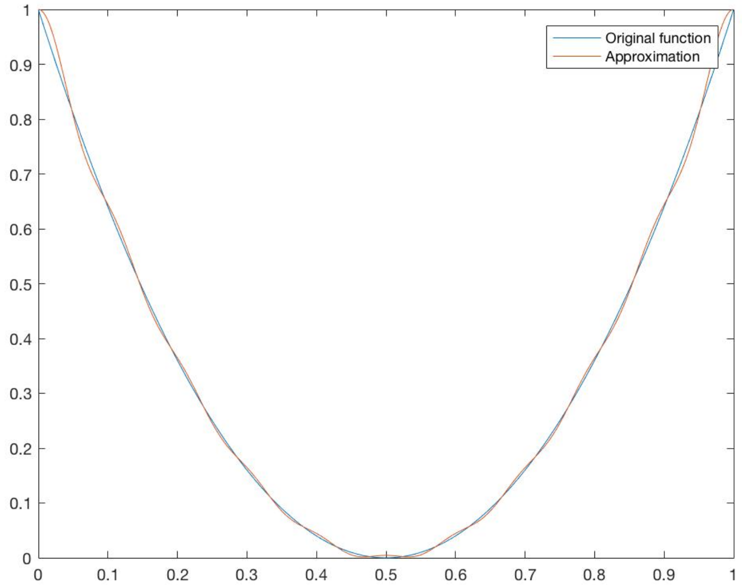
$f(x) = exp(cos(2\pi x))$, When n = 21



We have a good approximation here.

2.3 Exercise 2c

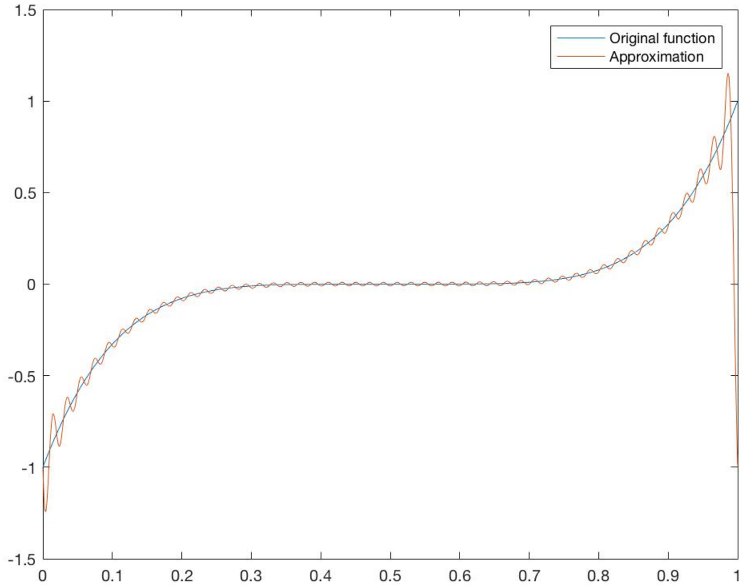
$f(x) = ((x - 0.5)/0.5)^2$, When n = 21



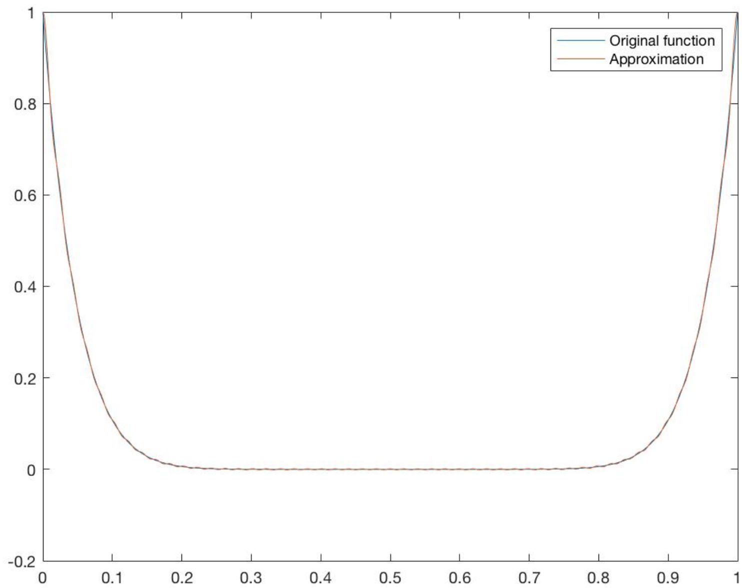
As we can see here, the approximation here is not as good as (b), the reason is that although the function is continuous, however it's not differentiable at boundary points. Therefore

2.4 Exercise 2d

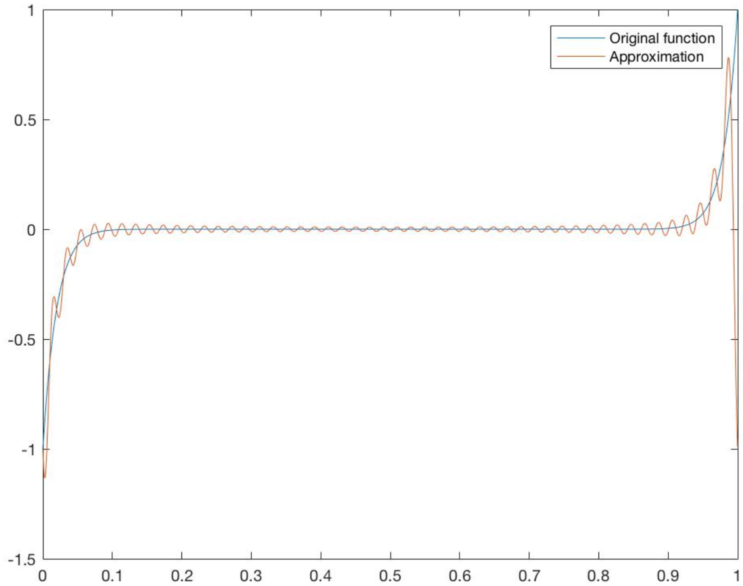
$f(x) = ((x - 0.5)/0.5)^m$, For this question we set n to be 21
When m = 5:



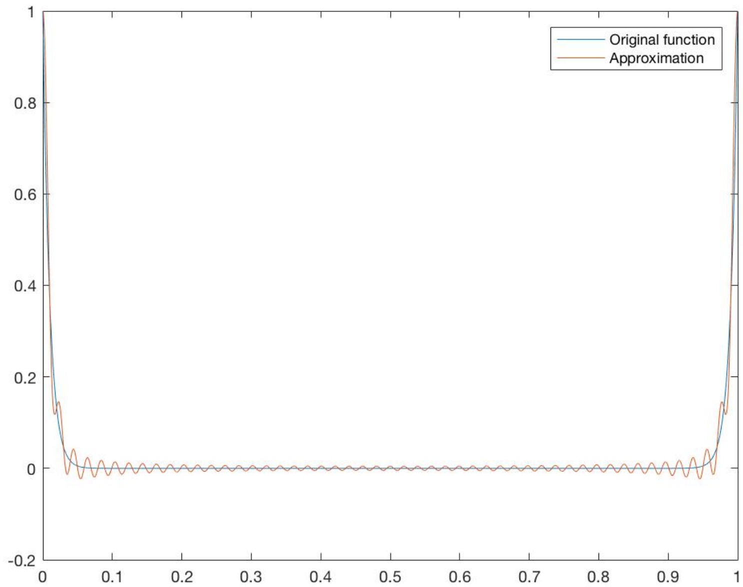
When m = 10:



When m = 25:



When m = 50:



When m is odd, function f is an odd function on the interval, therefore, f is non-differentiable and discontinuous. Compared to an even m, the Fourier approximation is worse for odd functions.

Moreover, when m increases, the function at boundary points become steeper and steeper, the absolute value of the derivatives becomes higher and higher, function becomes less and less differentiable. Therefore, as m increases, the Fourier approximation becomes worse and worse.

3 Exercise 3

For Fourier series,

$$a_k = 2\int_0^1 f(x)cos(2\pi kx)dx,\ b_k = 2\int_0^1 f(x)sin(2\pi kx)dx$$

3.1 Exercise 3a

For $f(x) = (cos(8\pi x))^4$,

$$a_k = 2\int_0^1 (cos(8\pi x))^4 cos(2\pi kx)dx,\ b_k = 2\int_0^1 (cos(8\pi x))^4 sin(2\pi kx)dx$$

$$\text{Since } cos(2x) = 2cos(x)^2 - 1$$

$$\text{Then } cos(8\pi x)^4 = \frac{1}{4}(\frac{cos(32\pi x)}{2} + 2cos(16\pi x) + 1)$$

By the orthogonality property,

$$a_0 = \frac{3}{8},\ a_8 = \frac{1}{2},\ a_{16} = \frac{1}{8},\ \text{All } b_k \text{ is } 0$$

Then the continuous Fourier series is

$$f(x) = \frac{3}{8} + \frac{1}{2}cos(16\pi x) + \frac{1}{8}cos(32\pi x)$$

For discrete Fourier coefficients, when n = 5:

$$a_0 = C_0$$

when n = 11:

$$a_0 = C_0,$$

when n = 21:

$$a_0 = C_0, a_8 = 2 \times Real(C_8)$$

We expect $a_k = 2 \times Real(C_k), b_k = 2 \times Imag(C_k)$ up to n/2, and all other C_ks to have 0 real parts and 0 imaginary parts. However, this is not the case when we don't have enough sampling points. The coefficients of some high frequency terms also show up in the $C_k s$, this is because of aliasing. For example, when n = 11.

$$C_5 = 0.0625$$

-2 $\times$ C_5 equals the coefficient of $cos(32\pi x)$. If we compared between $cos(32\pi x)$ and $cos(10\pi x)$, they give the same values at the sampling points. Similar phenomenons also show up when n equals other values.

3.2 Exercise 3b

For $f(x) = x$,

$$a_k = 2\int_0^1 xcos(2\pi kx)dx,\ b_k = 2\int_0^1 xsin(2\pi kx)dx$$

Let's compute a_k first,

$$\text{Let } u = x,\ \frac{dv}{dx} = cos(2\pi kx)dx$$

Then

$$\frac{du}{dx} = 1,\ v = \frac{1}{2\pi k}sin(2\pi kx)$$

Then

$$\int xcos(2\pi kx)dx = \frac{x}{2\pi k}sin(2\pi kx) \text{ - } \int \frac{1}{2\pi k}sin(2\pi kx)$$

$$\int xcos(2\pi kx)dx = \frac{x}{2\pi k}sin(2\pi kx) + \frac{1}{(2\pi k)^2}cos(2\pi kx)$$

$$a_k = 2\int_0^1 xcos(2\pi kx)dx = 0$$

For b_k

$$\int xsin(2\pi kx)dx = \frac{1}{4\pi^2 k^2}sin(2\pi kx) - \frac{x}{2\pi k}cos(2\pi kx)$$

Then

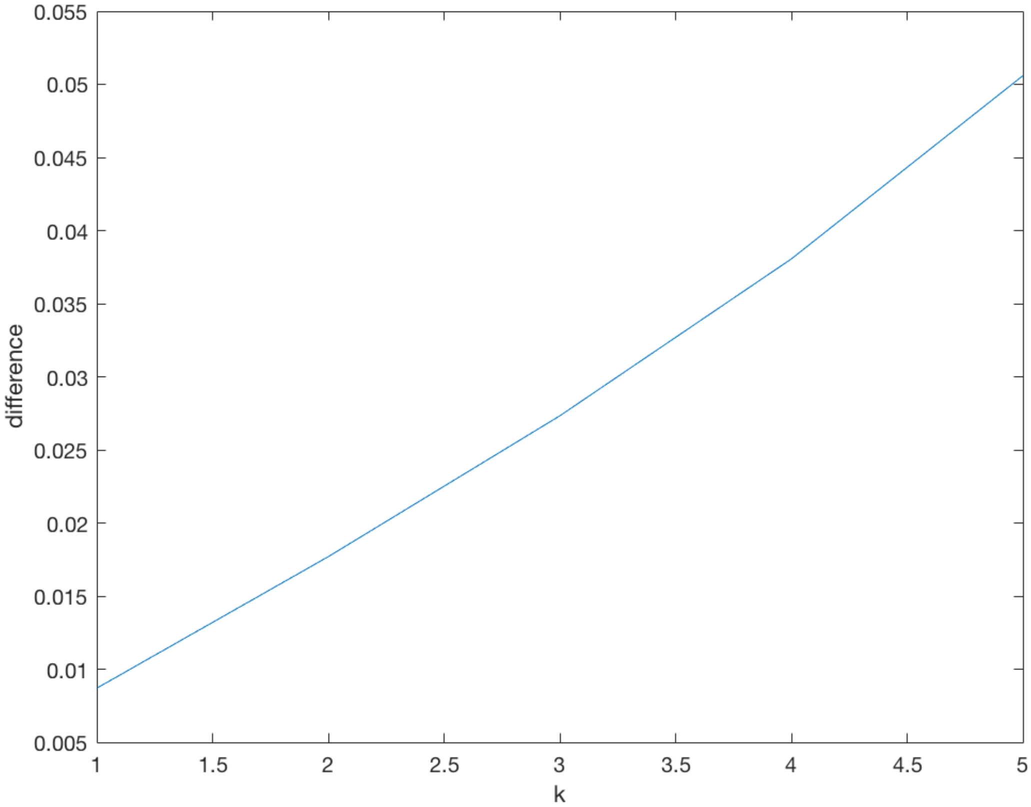
$$b_k = -\frac{1}{\pi k}$$

And a_0 is just $\int_0^1 x = \frac{1}{2}$, then

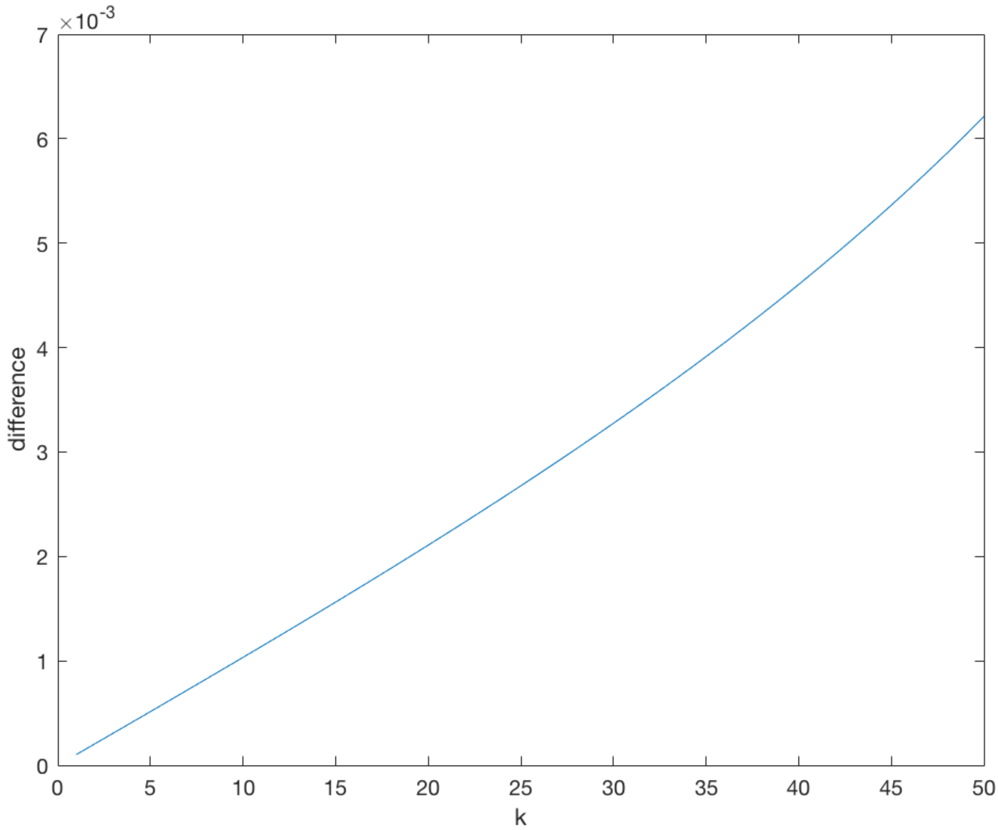
$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty}(-\frac{1}{\pi k})sin(2\pi kx)$$

For discrete Fourier coefficients, similar to question 3a, $a_0 \approx \frac{1}{2}$, and $b_k \approx -2 \times Imag(c_k)$ up to $k = \frac{n}{2}$. When choosing 11 sampling points, the difference between b_k and

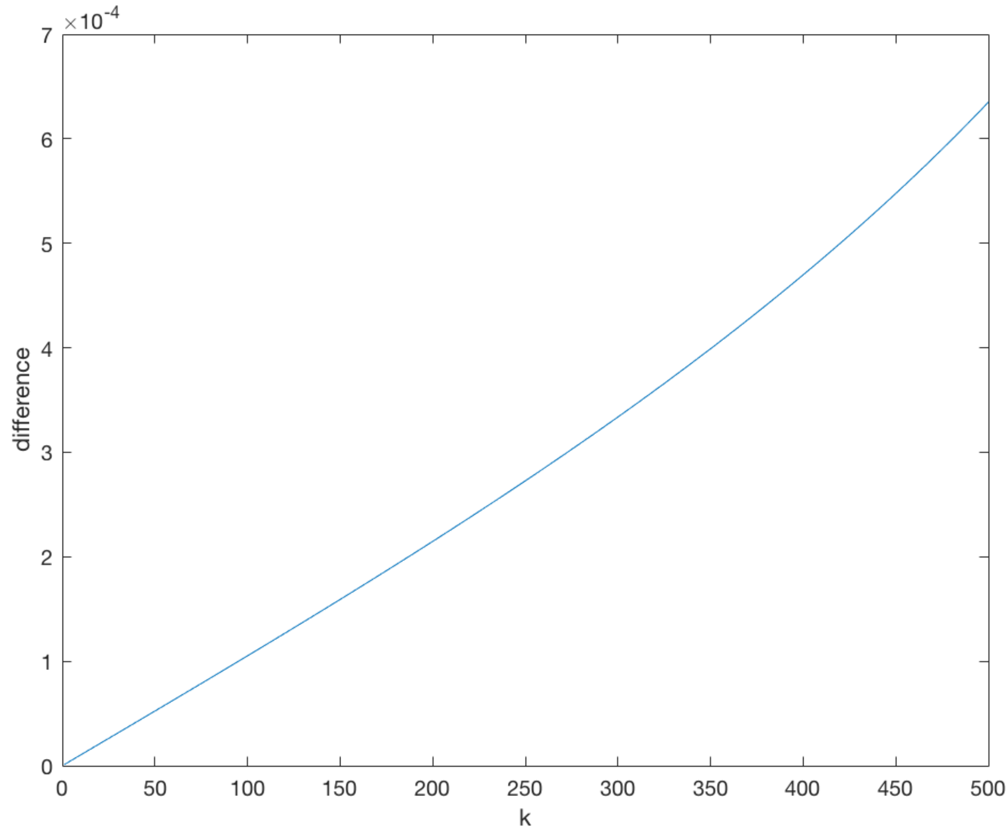
$-2 \times Imag(c_k)$ is



When choosing 101 sampling points, the difference between b_k and $-2 \times Imag(c_k)$ is



When choosing 1001 sampling points, the difference between b_k and $-2 \times Imag(c_k)$ is



As we can see here, the difference decrease really fast as we have more and more sampling points.

4 Exercise 4

Here we introduce a mapping from interval [0,1] to [a,b]

$$t = (b-a)x + a, \text{ where } x \in [0,1], \text{ t} \in [\text{a,b}]$$

Let's call this mapping t(x), and the inverse mapping x(t).

$$x = \frac{t-a}{b-a}$$

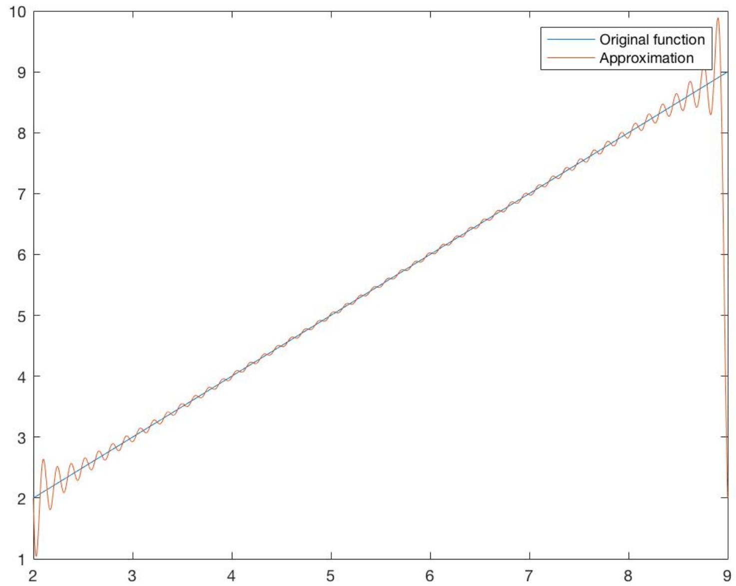
We use the points on [0,1] when calculating the coefficients,

$$\text{Then } C_k = \sum_0^{n-1} f(t_n) e^{-2\pi i k x(t_n)}$$

When approximating the original function, we have

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k cos(2\pi k x(t)) + b_k sin(2\pi k x(t))$$

For example, let f(x) = x, we set the interval to be [2,9], when having 101 sampling the points, the graph is the following:



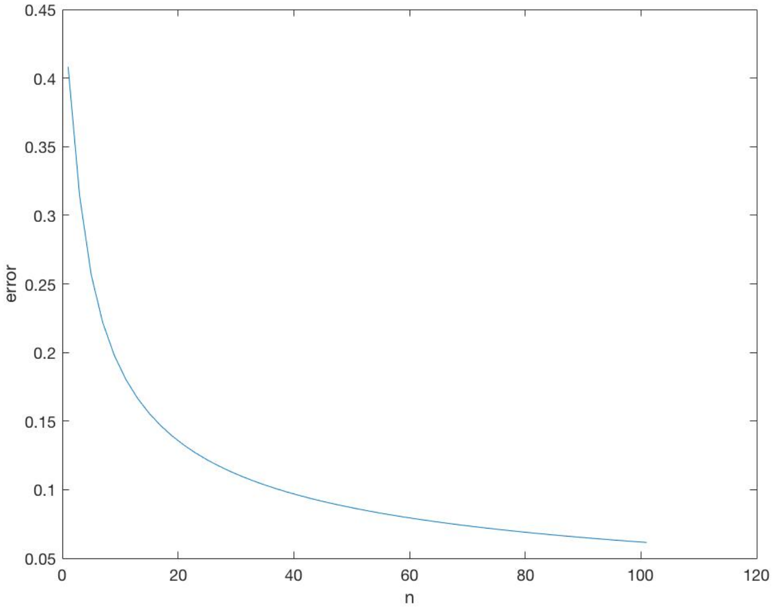
5 Exercise 5

Let's take $f(x) = x^{\frac{3}{2}}$ as an example, this function is 1 time differentiable, the reason is the following, for its second derivative $f''(x) = \frac{3}{4}x^{-\frac{1}{2}}$, it doesn't exist at x=0. Following this pattern, we can find a series of k times differentiable functions. For example, functions that have the form

$$f(x) = Cx^{\frac{1}{2}+k}, \text{ where C is a constant}$$

are k times differentiable.

Take $f(x) = x^{\frac{1}{2}+2}$ which is a 2 times differentiable function, the error is plotted in the following graph:



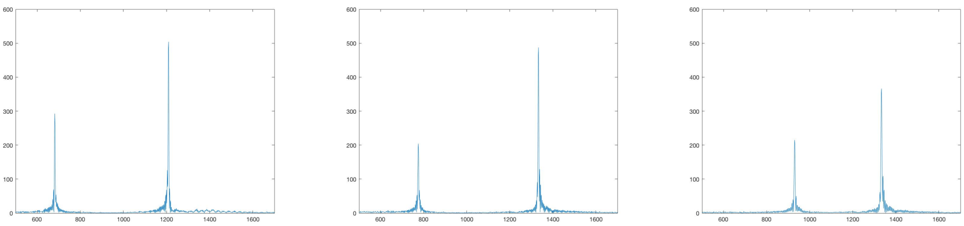
6 Exercise 6

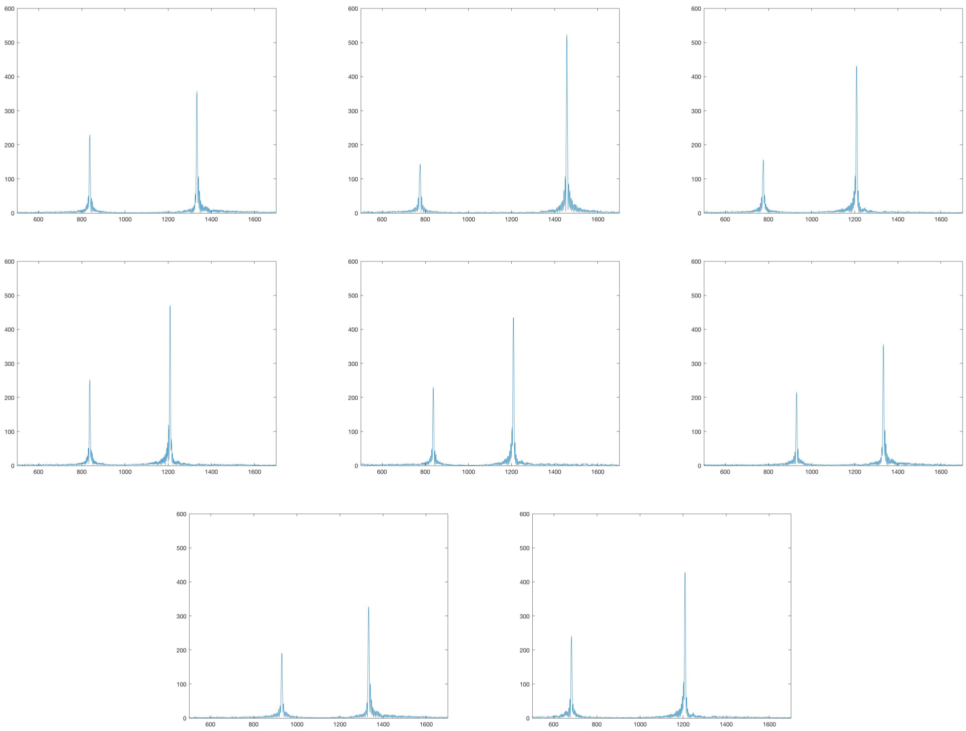
6.1 8.1

The phone number is

$$15086477001$$

The frequency plots of every single digit are the following(Read from left to right and from top to bottom):





6.2 8.2

Dear TA, trust me, my code works, I have tested it on different numbers.
The only below only shows the added part to the function, this piece of code is added to the end of the function.

```
else
    len = length(arg);
    fr = [697 770 852 941];
    fc = [1209 1336 1477];
    Fs = 32768;
    t = 0:1/Fs:0.25;
    for i = 1:1:len
        c = arg(i);
        if c == '-'
            continue
        else
            switch c
                case '*'
                    k = 4;
                    j = 1;
                case '0'
```

```
                k = 4; j = 2;
            case '#'
                k = 4;
                j = 3;
            otherwise
                d = c-'0';
                j = mod(d-1,3)+1;
                k = (d-j)/3+1;
        end
        y1 = sin(2*pi*fr(k)*t);
        y2 = sin(2*pi*fc(j)*t);
        y = (y1 + y2)/2;
        sound(y,Fs)
        pause(0.4)
    end
  end
end
```