Solutions

1) a) Write $S_i''(x) = M_i \dfrac{x - x_{i+1}}{\underbrace{x_i - x_{i+1}}_{=-h}} + M_{i+1} \dfrac{x - x_i}{\underbrace{x_{i+1} - x_i}_{=h}}$.

$$\Rightarrow S_i''(x) = \frac{M_i}{h}(x_{i+1} - x) + \frac{M_{i+1}}{h}(x - x_i)$$

Integrate twice

$$S_i(x) = \frac{M_{i+1}}{6h}(x - x_i)^3 + \frac{M_i}{6h}(x_{i+1} - x)^3$$
$$+ C_i(x - x_i) + D_i(x_{i+1} - x)$$

Determine C_i and D_i by Continuity and Interpolation Conditions. Continuity of first derivative gives us the linear system (Tridiagonal)

b) Should be straightforward.

2) $P(x) = 1 + 2x - 3x^2 + x^3$. All bases should give the same result as the interpolating polynomial is unique.

3) $f(x) - P_1(x) = \frac{(x-x_0)(x-x_1)}{2} f''(\xi_x)$, pointwise error

I noticed some students give formula for ξ depends on x, i.e. pointwise error. We are ~~more~~ usually more interested in max. error. To find such ξ, after finding the formula for ξ_x, you can solve the optimization problem.

4) Due to Runge's phenomenon, equidistant interpolation won't work for $f = \frac{1}{1+25x^2}$ $\left(f^{(n+1)}(x) \to \infty \text{ as } n \to \infty \right)$.

$f = |x|$ does not have a continuous derivative at $x=0$. Both equidistant and chebyshev interpolations will fail to capture (or approximate) around zero. Performance of cubic splines depends on how the points are chosen.

$f = \sin(\pi x)$ is a nice function so everything works well.

5) a) Should be trivial.

b) For same number of function evaluations Gauss-Legendre should give a better approximation.

6) As $n \to \infty$;

$$\frac{Q(n) - Q(2n)}{Q(2n) - Q(4n)} = \frac{I - E(n) - I + E(2n)}{I - E(2n) - I + E(4n)}$$

$$I = \int_a^b f(x)dx$$

$$E(n) = \frac{(b-a)^3 f''(\xi_1)}{12n^2} \text{ for some } \xi_1$$

$$= \frac{E(2n) - E(n)}{E(4n) - E(2n)}$$

$$= \frac{f''(\xi_2)/(2n)^2 - f''(\xi_1)/n^2}{f''(\xi_3)/(4n)^2 - f''(\xi_2)/(2n)^2}$$

pick $\xi \in \{\xi_1, \xi_2, \xi_3\}$ s.t. this inequality holds

$$\leq \frac{f''(\xi)\left(\frac{1}{(2n)^2} - \frac{1}{n^2}\right)}{f''(\xi)\left(\frac{1}{(4n)^2} - \frac{1}{(2n)^2}\right)}$$

$$= \frac{-\frac{3}{4n^2}}{-\frac{3}{16n^2}} = 4$$

7) Should be straight forward