CS770: Assignment 4

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1 Question 1

1.1 Question 1a

For cubic spline, let the function $S(x)$ define the spline function
where $a = x_0 < x_1 < x_2 < x_3 < ...... < x_n = b$

$$S(X) = \begin{cases} S_0(x), & x_0 < x < x_1 \\ S_1(x), & x_1 < x < x_2 \\ ...... & \\ S_i(x), & x_i < x < x_{i+1} \\ ...... & \\ S_{n-1}(x), & x_{n-1} < x < x_n \end{cases} \tag{1}$$

Where each $S_i(x)$ has degree 3 in this case.
In cubic spline, $S(x)$ satisfies

$$\begin{cases} S_i(x_i) = S_{i+1}(x_i) \\ S'_i(x_i) = S'_{i+1}(x_i) \\ S''_i(x_i) = S''_{i+1}(x_i) \\ S_i(x_i) = y_i \end{cases}$$

Where each $i = 0, 1, 2, ..., n-2$
By the definition of natural cubic spline, we have two additional constraints,

$$\begin{cases} S''_0(x_0) = 0 \\ S''_{n-1}(x_{n-1}) = 0 \end{cases}$$

1.2 Question 1b

```
function [coeffs] = nSpline(X,y)
%This function returns the coefficients of the natural cubic spline
%X and y are the input points where f(X(i)) = y(i)
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%Each spline function on each interval has degree 3

%Si = a+bx+cx^2+dx^3
%We have n such Si's, where is n = length(X)-1
%coeffs should be [a1;b1;c1;d1;a2;b2;.....;an;bn;cn;dn]
%coeffs is a 4n by 1 vector

    numP = length(X);
    %numP is the number of points
    n =numP - 1;
    %n is the number of spline functions

    K = [];
    %initialize the matrix to be empty
    A = [];
    %A contains all the known values
    %K*coeffs = A

    %we construct the matrix using for loop
    for i = 1:numP
        a = (i-1)*4+1;
        b = a+1;
        c = b+1;
        d = c+1;
        %a,b,c,d are indices for the convinience of calculation
        %a,b,c,d indicate the next polynomial
        %to access the previous polynomial
        %use a-4, b-4, c-4, d-4

        if i==1
            tempK = zeros(1,4*n);
            tempK(1,c) = 2;
            tempK(1,d) = 6*X(i);
            K = [K;tempK];
            A = [A;0];
            tempK = zeros(1,4*n);
            tempK(1,a) = 1;
            tempK(1,b) = X(i);
            tempK(1,c) = X(i)^2;
            tempK(1,d) = X(i)^3;
            K = [K;tempK];
            A = [A;y(i)];
        end
```

```
if i==numP
    tempK = zeros(1,4*n);
    tempK(1,c-4) = 2;
    tempK(1,d-4) = 6*X(i);
    K = [K;tempK];
    A = [A;0];
    tempK = zeros(1,4*n);
    tempK(1,a-4) = 1;
    tempK(1,b-4) = X(i);
    tempK(1,c-4) = X(i)^2;
    tempK(1,d-4) = X(i)^3;
    K = [K;tempK];
    A = [A;y(i)];
end
%these the special end points contraints for natural cubic constraint

if i>1 && i<numP
    tempK = zeros(1,4*n);
    tempK(1,a-4) = 1;
    tempK(1,b-4) = X(i);
    tempK(1,c-4) = X(i)^2;
    tempK(1,d-4) = X(i)^3;
    K = [K;tempK];
    A = [A;y(i)];
    tempK = zeros(1,4*n);
    tempK(1,a) = 1;
    tempK(1,b) = X(i);
    tempK(1,c) = X(i)^2;
    tempK(1,d) = X(i)^3;
    K = [K;tempK];
    A = [A;y(i)];
end
%this is the constraint for S(xi) = yi

if i>1 && i<numP
    tempK = zeros(1,4*n);
    tempK(1,b-4) = 1;
    tempK(1,c-4) = 2*X(i);
    tempK(1,d-4) = 3*X(i)^2;
    tempK(1,b) = -1;
    tempK(1,c) = -2*X(i);
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            tempK(1,d) = -3*X(i)^2;
            K = [K;tempK];
            A = [A;0];
            %this is the constraint for Si'(xi) = Si+1'(xi)

            tempK = zeros(1,4*n);
            tempK(1,c-4) = 2;
            tempK(1,d-4) = 6*X(i);
            tempK(1,c) = -2;
            tempK(1,d) = -6*X(i);
            K = [K;tempK];
            A = [A;0];
            %this is the constraint for Si''(xi) = Si+1''(xi)
        end
    end

    coeffs = K\A;

end
```

This code has been proven to run correctly, the results for Question 4 was obtained using this function.

2 Question 2

For an interpolating polynomial P, it can be expressed as

$$P(x) = a_0p_0(x) + a_1p_1(x) + a_2p_2(x) + ... + a_{n-1}p_{n-1}(x)$$

In this question, we have four give points $(-1,-5)$, $(0,1)$, $(1,1)$, $(2,1)$

- Monomial Basis

For monomial basis, we have

$$p_i(x) = x^{i-1}, \text{ for } i = 1,2,...,n$$

Since we have four points here, the interpolating polynomial should have form of

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

By applying matrix calculation, we get

$$a_0 = 1 \;\; a_1 = 2 \;\; a_2 = -3 \;\; a_3 = 1$$

$$\text{Therefore, } P(x) = 1 + 2x - 3x^2 + x^3$$

- Lagrange Basis

For Lagrange basis, we have

$$p_i(x) = L_i(x) = \prod_{j=1,j\neq i}^{n} \frac{x-x_j}{x_i-x_j}$$

Since we have four points here, the interpolating polynomial should have form of

$$P(x) = a_0p_0(x) + a_1p_1(x) + a_2p_2(x) + a_3p_3(x)$$

Since for Lagrange basis,

$$L_i(x) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Then we have

$$P(x) = \sum_i^n L_i(x) f_i$$

In this case, we have

$$\begin{aligned} P(x) = &-5\frac{x(x-1)(x-2)}{-6} \\ &+1\frac{(x+1)(x-1)(x-2)}{2} \\ &+1\frac{(x+1)x(x-2)}{-2} \\ &+1\frac{(x+1)x(x-1)}{6} \end{aligned} \tag{2}$$

- Newton Basis

For Newton basis, we have

$$p_i(x) = N_i(x) = \prod_{j=1, j\neq i}^n (x - x_j)$$

Our interpolating function should have the form

$$\begin{aligned} P(x) = &a_0 \\ &+ a_1(x - x_0) \\ &+ a_2(x - x_0)(x - x_1) \\ &+ a_3(x - x_0)(x - x_1)(x - x_2) \end{aligned} \tag{3}$$

After calculation we have:

$$a_0 = -5 \quad a_1 = 6 \quad a_2 = -3 \quad a_3 = 1$$

Therefore,

$$\begin{aligned} P(x) = &-5 \\ &+ 6(x+1) \\ &- 3(x+1)x \\ &+ (x+1)x(x-1) \end{aligned} \tag{4}$$

3 Question 3 [NOT DONE]

3.1 Question 3a

For $f(x) = x^3$, we have

$$f(0) = 0, \ f(1) = 1$$

Therefore, $P_n(x) = x$.

Define $\phi(t) = f(t) - P(t) - \frac{f(x)-p(x)}{\omega(x)}\omega(t)$, where $\omega(x) = \prod(x - x_i)$.

In this case, we have $\phi(t) = t^3 - t - \frac{x^3-x}{(x-0)(x-1))}(t^2 - t)$

$\phi'(t) = 3t^2 - 1 - \frac{x^3-x}{(x-0)(x-1))}(2t - 1)$

$\phi''(t) = 6t - 2\frac{x^3-x}{(x-0)(x-1))}$

$\xi = \frac{x+1}{3}$

3.2 Question 3b

For $f(x) = (2x - 1)^4$, we have

$$f(0) = 1,\ f(1) = 1$$

Therefore, $P_n(x) = 1$.

Define $\phi(t) = f(t) - P(t) - \frac{f(x)-p(x)}{\omega(x)}\omega(t)$, where $\omega(x) = \prod(x - x_i)$

In this case, we have $\phi(t) = (2x - 1)^4 - 1 - \frac{(2x-1)^4-1}{(x-0)(x-1))}(t^2 - t)$

$\phi'(t) = 8(2t - 1)^3 - \frac{(2x-1)^4-1}{(x-0)(x-1))}(2t - 1)$

$\phi''(t) = 48(2t - 1)^2 - 2\frac{(2x-1)^4-1}{(x-0)(x-1))}$

ξ

4 Question 4

4.1 • $f(x) = sin(\pi x)$

- Equidistant

- Chebyshev points

4.2 • $f(x) = \frac{1}{1+25x^2}$

- Equidistant

- Chebyshev points

4.3 • $f(x) =\mid x \mid$

- Equidistant

- Chebyshev points

5 Question 5

5.1 Question 5a

Let $x = tan(\theta)$, then

$$\frac{dx}{d\theta} = sec^2\theta$$

Then

$$dx = sec^2\theta d\theta$$

Then

$$\int \frac{4}{1+x^2}dx = \int \frac{4}{1+tan^2(\theta)}sec^2\theta d\theta$$

Since

$$1 + tan^2(\theta) = sec^2(\theta)$$

Then

$$\int \frac{4}{1+x^2}dx = \int 4\ d\theta = 4\ \theta + C, \text{ where } C \text{ is a constant}$$

Since we have set $x = tan(\theta)$, then $\theta = arctan(x)$
Then we have

$$\int \frac{4}{1+x^2}dx = 4arctan(x) + \text{C}$$

Then $\int_0^1 \frac{4}{1+x^2}dx = \pi$

5.2 Question 5b

Let $f(x) = \frac{4}{1+x^2}$
For

6 Question 6

$Q(n) = \frac{h_1}{2}(f(a) + f(b)) + h_1 \sum_{k=1}^{n-1} f(x_k)$, where $h_1 = \frac{b-a}{n}$, $x_k = a + kh_1$

$Q(2n) = \frac{h_2}{2}(f(a) + f(b)) + h_2 \sum_{k=1}^{2n-1} f(x_k)$, where $h_2 = \frac{b-a}{2n}$, $x_k = a + kh_2$

$Q(4n) = \frac{h_4}{2}(f(a) + f(b)) + h_4 \sum_{k=1}^{4n-1} f(x_k)$, where $h_4 = \frac{b-a}{4n}$, $x_k = a + kh_4$

$Q(n) - Q(2n) = \frac{h_1-h_2}{2}(f(a) + f(b)) + h_1 \sum_{k=1}^{n-1} f(x_k) - h_2 \sum_{k=1}^{2n-1} f(x_k)$

$Q(2n) - Q(4n) = \frac{h_2-h_4}{2}(f(a) + f(b)) + h_2 \sum_{k=1}^{2n-1} f(x_k) - h_4 \sum_{k=1}^{4n-1} f(x_k)$

$\frac{Q(n)-Q(2n)}{Q(2n)-Q(4n)} = \frac{\frac{h_2}{2}(f(a)+f(b))+h_1 \sum_{k=1}^{n-1} f(x_k)-h_2 \sum_{k=1}^{2n-1} f(x_k)}{\frac{h_4}{2}(f(a)+f(b))+h_2 \sum_{k=1}^{2n-1} f(x_k)-h_4 \sum_{k=1}^{4n-1} f(x_k)}$

Let $U = h_1 \sum_{k=1}^{n-1} f(x_k) - h_2 \sum_{k=1}^{2n-1} f(x_k)$, $L = h_2 \sum_{k=1}^{2n-1} f(x_k) - h_4 \sum_{k=1}^{4n-1} f(x_k)$

Then $U = 2h_2 f(a + 2h_2) + 2h_2 f(a + 4h_2) .... + 2h_2 f(a + 2(n - 1)h_2) - h_2 f(a + h_2) -$

$h_2f(a+2h_2)....-h_2f(a+(2n-1)h_2)$

$U = h_2f(a+2h_2)+h_2f(a+4h_2)....+h_2f(a+2(n-1)h_2)-h_2f(a+h_2)-h_2f(a+3h_2)....-h_2f(a+(2n-1)h_2)$

When n goes infinity, $f(a+kh_2) = f(a+(k+1)h_2)$, then we have

$U = -h_2f(a+(2n-1)h_2) = -h_2f(b)$

7 Question 7

For Simpson's rule,

$$\int_a^b f(x) \approx \frac{b-a}{6}(f(a)+4f(\frac{a+b}{2})+f(b))$$

Then for composite Simpson's rule, we have:

$$\int_a^b f(x) \approx \frac{h}{3}(f_0+4f_1+2f_2+4f_3+....+2f_{n-2}+4f_{n-1}+f_n)$$

$$\int_a^b f(x) \approx \frac{h}{3}(f(x_0)+2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j})+4\sum_{j=1}^{\frac{n}{2}} f(x_{2j-1})+f(x_n))$$

For the error analysis, let $n = 2M$, then we have

$$E(f) = \int_a^b (f(x)-s(x))dx$$

$$E(f) = \int_a^b \frac{f^{n+1}(\xi)}{(n+1)!}\prod(x-x_i)dx$$

Then

$$E(f) = \frac{f^{n+1}(\xi)}{(n+1)!}\int_a^b \prod(x-x_i)dx, \text{ for some } a < \xi < b$$

For Simpson's Rule, we have:

$$\prod(x-x_i) = (x-a)(x-\frac{a+b}{2})^2(x-b)$$

Therefore, on one subinterval, we have error $O(h^5)$

When having composite Simpson's rule on the entire interval, we have the error being

$$O(h^4)$$

Since we loss one order of accuracy from local to global.