The Interstellar Medium

From a low frequency, and sometimes extragalactic, perspective

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Contents

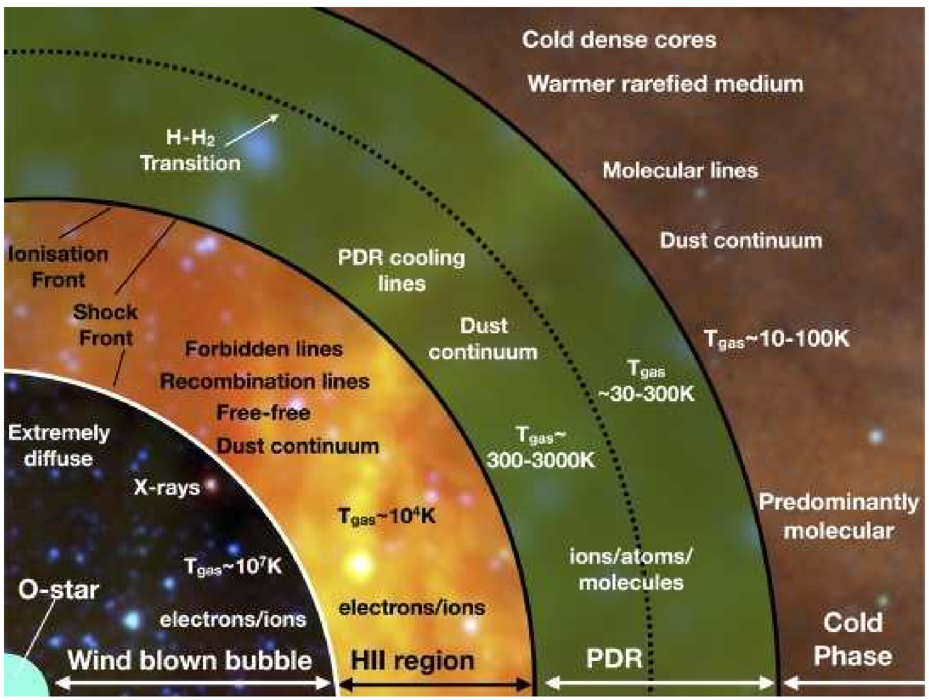
1 Overview of the ISM

The *interstellar medium* (ISM) is a multi-component, multi-phase medium. Its main constituent is hydrogen gas, but all components produce distinctive emission and absorption spectral signatures.

Not all phases are in pressure equilibrium. Interstellar dust, whilst comprising $\lesssim$ 0.1% of the ISM, plays an exceptionally important role in the physics and chemistry of the ISM, star formation, and our interpretation of spectra from astrophysical sources.

Interstellar gas densities range from $\sim 10^4$ atoms m^{-3} to $\sim 10^9$ m^{-3}, with $\sim 10^6$ m^{-3} being typical. Compare this with the best vacuum ever attained in laboratories on Earth, $\sim 10^{10}$ atoms m^{-3}.

Gas component	Phase	T [K]	n [cm^{-3}]	M [M$_\odot$]
neutral	molecular	$\sim 10-100$	10^3	$\sim 10^9$
	cold	~ 150	10	1.5×10^9
	warm	8000	0.2	1.5×10^9
ionised	diffuse	8000	0.03	$\sim 10^9$
	H II	$\sim 10^4$	$1-10^4$	5×10^7
	coronal	5×10^5	6	$\sim 10^8$



Credit:Synthetic observations of star formation and the interstellar medium

Dust particle densities are only $\sim$ 1000 per km^3 (i.e. $\sim 10^{-6}$ m^{-3}). Interstellar gas contains 90% hydrogen 9% helium and 1% heavier elements, which is similar to the overall abundances found in the Sun and other nearby stars.

The physical structure of the ISM depends on the chemical composition of the gas, because the cooling of the gas is dominated by spectral line emission and therefore the abundance of that particular species.

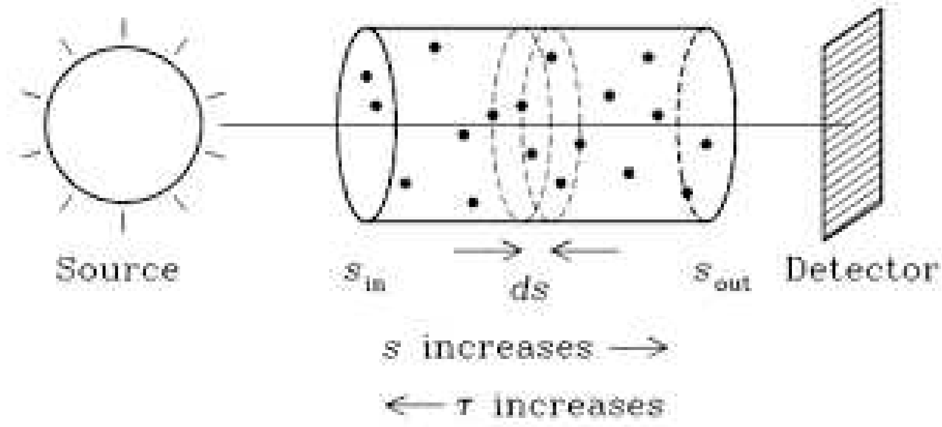
The ISM is rich in chemistry with $\gtrsim$ 200 molecular species having been detected, some very complex.

2 Radiative Transfer

2.1 Radiative transfer along a line-of-sight

2.1.1 The equation of radiative transfer

Consider a beam of radiation from a distant source, passing through the ISM. The change in opacity as the radiation passes through gas of thickness ds is given by



$$dI_\nu = (\underbrace{\epsilon_\nu}_{\text{added}} - \underbrace{\kappa_\nu I_\nu}_{\text{subtracted}})ds,$$

where ϵ_ν and κ_ν are the *emission* and *absorption coefficients*[1], respectively.[2] Note that the change in opacity due to absorption is a fraction of the initial intensity.

Rewriting as $dI_\nu = (\epsilon_\nu - \kappa_\nu I_\nu)ds$, gives the fundamental *equation of radiative transfer*,

$$\boxed{\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + \epsilon_\nu.}$$

[1] κ, as α is reserved for the spectral index, as well as the recombination coefficient.

[2] Note that the ν subscript denotes the specific value at a particular frequency.

2.1.2 Optical depth

In terms of absorption only, the change in opacity through ds is

$$dI_\nu = -\kappa I_\nu ds \Rightarrow \int \frac{dI_\nu}{I_\nu} = -\int_0^s \kappa ds \Rightarrow \ln\left(\frac{I_\nu}{I_\nu(0)}\right) = -\kappa s.$$

where $I_\nu(0)$ is the incident intensity and assuming uniform density and opacity over s.

A useful concept is the *optical depth*, τ, through a slab of the ISM, which is defined via

$$\ln\left(\frac{I_\nu}{I_\nu(0)}\right) = -\tau_\nu, \text{ giving } \frac{I_\nu}{I_\nu(0)} = e^{-\tau_\nu}.$$

Since $I = I(0)e^{-\tau}$, the amount of light transmitted through the slab drops exponentially.

2.1.3 Solutions to the equation of radiative transfer

In the case of absorption only [$\epsilon_\nu = 0$], we have seen that

$$I_\nu = I_\nu(0)e^{-\tau_\nu}.$$

In general, if we divide the equation of radiative transfer by the absorption coefficient, κ_ν,

$$\frac{dI_\nu}{\kappa_\nu ds} = -I_\nu + \frac{\epsilon_\nu}{\kappa_\nu},$$

we see that the denominator on the LHS is the optical depth, $\tau_\nu = \int \kappa_\nu ds$. That is,

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + \frac{\epsilon_\nu}{\kappa_\nu}.$$

Defining the ratio of emission to absorption as the *source function*, the intensity of radiation at a given point within the medium,

$$S_\nu \equiv \frac{\epsilon_\nu}{\kappa_\nu},$$

gives

$$\boxed{\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu.}$$

To solve this first order differential equation, we multiply through by an integrating factor, e^τ, giving

$$e^{\tau_\nu}\frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = S_\nu e^{\tau_\nu}.$$

To simplify this, we note that

$$\frac{d}{d\tau_\nu}[I_\nu e^{\tau_\nu}] = e^{\tau_\nu}\frac{dI_\nu}{d\tau_\nu} + I_\nu \frac{de^{\tau_\nu}}{d\tau_\nu} = e^{\tau_\nu}\frac{dI}{d\tau_\nu} + I_\nu e^{\tau_\nu}$$

So we can rewrite the above as

$$\frac{d}{d\tau_\nu}\left[I_\nu e^{\tau_\nu}\right] = S_\nu e^{\tau_\nu}.$$

$$\Rightarrow \left[Ie^{\tau}\right]_{\tau_1}^{\tau_2} = \int_{\tau_1}^{\tau_2} Se^{\tau}\,d\tau, \text{ where we drop the } \nu \text{ subscripts for clarity.}$$

$$\Rightarrow Ie^{\tau_2} - Ie^{\tau_1} = \int_{\tau_1}^{\tau_2} Se^{\tau}\,d\tau \Rightarrow Ie^{\tau_2} = Ie^{\tau_1} + \int_{\tau_1}^{\tau_2} Se^{\tau}\,d\tau$$

and multiplying though by $e^{-\tau_2}$ gives

$$I = Ie^{\tau_1}.e^{-\tau_2} + \int_{\tau_1}^{\tau_2} Se^{\tau}e^{-\tau_2}\,d\tau = Ie^{-(\tau_2-\tau_1)} + \int_{\tau_1}^{\tau_2} Se^{-(\tau_2-\tau)}\,d\tau.$$

When $\tau_1 = 0$, $I = I_0$,

$$I = I_0 e^{-\tau_2} + \int_0^{\tau_2} Se^{-(\tau_2-\tau)}\,d\tau.$$

For a constant source function, S, this becomes

$$I = I_0 e^{-\tau_2} + S\int_0^{\tau_2} e^{-(\tau_2-\tau)}\,d\tau.$$

$$\int_0^{\tau_2} e^{-(\tau_2-\tau)}\,d\tau = \left[e^{-(\tau_2-\tau)}\right]_0^{\tau_2} = 1 - e^{-\tau_2}$$

$$\Rightarrow I = I_0 e^{-\tau_2} + S\left(1 - e^{-\tau_2}\right)$$

or

$$\boxed{I_\nu = I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right)}$$

In the case of a candle flame, the soot particles cast shadows, attenuating the incident beam, but also add intensity with a source function that depends on soot temperature. The shadow vanishes when thermal equilibrium is reached, $T_{\text{soot}} = T_{\text{light source}}$.

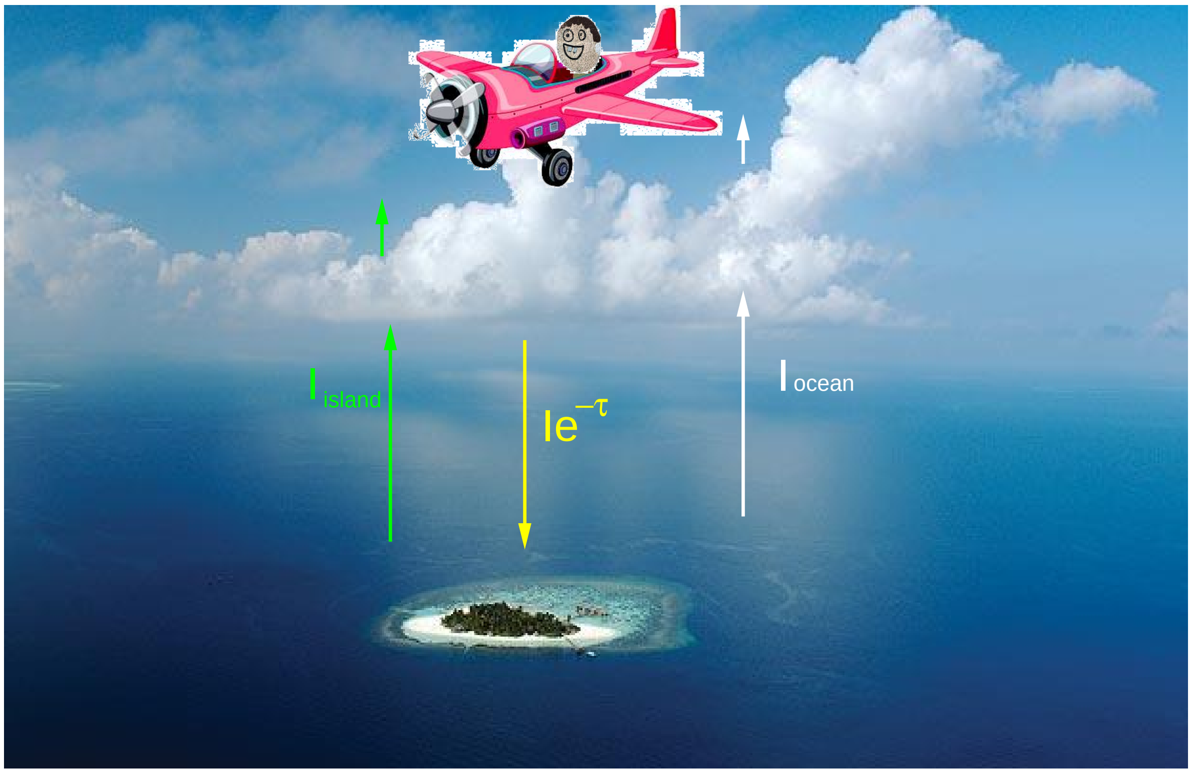
Looking at the extreme values of τ:

- Optically thin ($\tau_\nu \to 0$): $I_\nu \to I_\nu(0)$
- Optically thick ($\tau_\nu \to \infty$): $I_\nu \to S_\nu$

So at large optical depths the specific intensity of the radiation field approaches the source function (the balance between emission and absorption), meaning that the radiation field comes into thermal equilibrium with the medium.
In the top part of the image to the right the air is optically thin to visible light (you could point to the Sun), while in the bottom part the air is optically thick.

Example

You are flying over the ocean, which reflects 10% of the sunlight coming from straight above you, and pass directly over a sandy island, which reflects 30% of the sunlight.



If there is a cloud of opacity $\tau = 0.2$ blocking the sight-line:

(a) How much light is reflected back from the cloud?

If I is incident and $Ie^{-\tau}$ transmitted, then $I - Ie^{-\tau} = I(1 - e^{-\tau})$ is reflected. For $\tau = 0.2$, $I_{\text{reflected}} = 0.182I$.

(b) Can either the island or ocean be seen through the cloud?

For intensity I, $Ie^{-\tau}$ is transmitted through the cloud, so light reflected, from e.g. the ocean is $I_{\text{ocean}} = f_{\text{ocean}} Ie^{-\tau}$, where f is the reflected fraction.

Upon passing through the cloud again on the way up this becomes

$$I_{\text{ocean}} = f_{\text{ocean}} I e^{-0.4\tau}.$$

So, $I_{\text{ocean}} = 0.1 I e^{-0.4} = 0.067 < I_{\text{reflected}}$ and $I_{\text{island}} = 0.3 I e^{-0.4} = 0.201 I > I_{\text{reflected}}$. Therefore, only the island is visible through the cloud.

(c) If the absorption coefficient of seawater is $\kappa_\nu = 0.02\ \text{m}^{-1}$ at visible wavelengths[3], how deep into the water does 1% of the incident intensity remain?

$$I = I_0 e^{-\tau} \Rightarrow \tau = -\ln\left(\frac{I}{I_0}\right), \text{ where } \tau = \int \kappa ds = \kappa s$$

for a constant density[4].

That is,

$$s = \frac{1}{\kappa}\ln\left(\frac{I_0}{I}\right) = 50\ln(100) = 230\ \text{m}.$$

[3]From New Value of "Pure" Seawater Absorption Coefficient

[4]Unlike a gas, liquid is incomprehensible, giving $P = \rho g s$).

2.1.4 The mean free path of a photon

The probability that a photon gets absorbed is $e^{-\tau_\nu}$, so the mean optical depth at which a photon gets absorbed can be determined from the mean of a function

$$\langle x \rangle \equiv \frac{\int_0^\infty x f(x) dx}{\int_0^\infty f(x) dx} \Rightarrow \langle \tau_\nu \rangle = \frac{\int_0^\infty e^{-\tau_\nu} \tau_\nu d\tau_\nu}{\int_0^\infty e^{-\tau_\nu} d\tau_\nu} = \frac{\int_0^\infty e^{-\tau_\nu} \tau_\nu d\tau_\nu}{1}$$

For

$$\int_0^\infty e^{-\tau} \tau d\tau, \text{ let } u = e^{-\tau} \Rightarrow \langle \tau \rangle = \int u \tau d\tau \text{ and } \frac{du}{d\tau} = -e^{-\tau} = -u \Rightarrow \langle \tau \rangle = -\int u\tau \frac{du}{u}$$

Since $u = e^{-\tau}$, $\tau = -\ln u$

$$\Rightarrow \langle \tau \rangle = \int \ln u du = [u \ln u - u] = \left[-e^{-\tau}.\tau - e^{-\tau}\right]_0^\infty = \left[-e^{-\tau}(\tau + 1)\right]_0^\infty = 0 + 1$$

As defined in Sect. 2.1.2, $\tau = \kappa s$, where κ is the absorption coefficient and s the distance through the medium. Therefore the mean distance a photon travels before absorption is given by

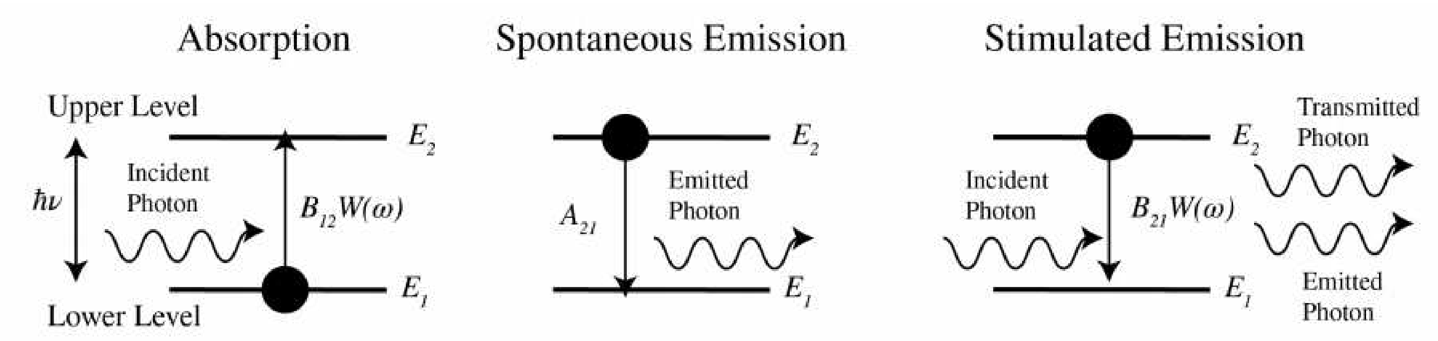
$$\kappa_\nu \langle s_\nu \rangle = \langle \tau_\nu \rangle = 1 \Rightarrow \langle s_\nu \rangle = \frac{1}{\kappa_\nu}.$$

The absorption coefficient [m^{-1}] is the product of the volume density of the absorbing particles, n [m^{-3}], and their cross section, σ [m^2]. Therefore the *mean free path of a photon* is

$$\langle s_\nu \rangle = \frac{1}{n\sigma_\nu}.$$

2.1.5 The equation of radiative transfer for spectral lines

To derive the equation for spectra lines, we require the Einstein coefficients, which describe the probability of absorption or emission of a photon by an atom or molecule:



- *Spontaneous Emission* (A_{21} [s^{-1}]): The probability per unit time for an electron to spontaneously transition from a higher energy level to a lower energy level emitting a photon.

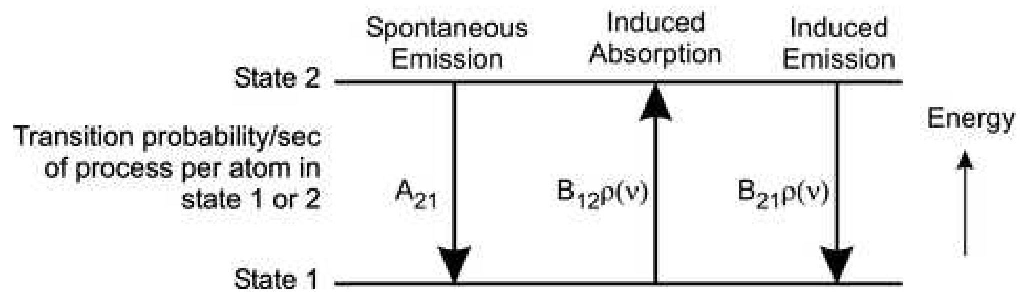
- *Induced Absorption* (B_{12} [J^{-1} m^3 s^{-2}]): The probability per unit time for an electron to absorb a photon and transition from a lower energy level to a higher energy level.

- *Stimulated Emission* (B_{21} [J^{-1} m^3 s^{-2}]): The probability per unit time for an electron to be stimulated by an incident photon to transition from a higher energy level to a lower energy level, emitting another photon that is in phase with the stimulating photon.[5]

[5] The fundamental principle behind the laser.

For a two level stationary system in equilibrium (when emission = absorption) the rate equation is

$$n_2 A_{21} + n_2 B_{21} u_\nu = n_1 B_{12} u_\nu,$$



where n_1 & n_2 are the lower and upper level populations, respectively, and $u_\nu = 4\pi I_\nu / c$ [J m^{-3}] the energy density.

Specific intensity, I_ν, is in units of W m^{-2} Hz^{-1} sr^{-1} and so the net change in energy (energy in–energy out) is related to a change in intensity via

$$dE = dE_{\rm em} + dE_{\rm stim} - dE_{\rm abs} = dI_\nu d\Omega d\mathcal{A} d\nu dt, \qquad (1)$$

where Ω is the solid angle, $\mathcal{A}$ the area of the source, ν the frequency and t time (cf. $L = 4\pi r^2 I$).

Each of coefficients contribute energy $h\nu$ over the full sold angle 4π. Multiplying this by the number of particles and the probability (coefficient) and integrating over the volume (energy density), frequency (specific intensity) and time (intensity gives power), we get, e.g. for spontaneous emission

$$dE_{\rm em} = \frac{h\nu}{4\pi} n_2 A_{21} dV d\Omega \phi(\nu) d\nu dt,$$

where n_2 is the population of the upper level and $\phi(\nu)$ [Hz^{-1}] describes the line profile[6]. To generalise for all coefficients we note that,

$$dE = \frac{h\nu}{4\pi}\underbrace{[n\,\text{Einstein coeff.}]}_{\text{i.e.}\, n_2 A_{21}\ \text{above}}\phi(\nu)d\Omega dV d\nu dt. \tag{2}$$

In terms of the coefficients, the net energy transfer (energy in–energy out) is given by the rate equation

$$n_2 A_{21} + n_2 B_{21} u_\nu - n_1 B_{12} u_\nu, \text{ with } u = \frac{4\pi I_\nu}{c},$$

and inserting each coefficient for Equ. 2 into Equ. 1 gives

$$dI_\nu d\mathcal{A}\cancel{d\Omega d\nu dt} = \frac{h\nu}{4\pi}\left[n_2 A_{21} + n_2 B_{21}\frac{4\pi I_\nu}{c} - n_1 B_{12}\frac{4\pi I_\nu}{c}\right]\phi(\nu)dV\cancel{d\Omega d\nu dt},$$

where $dV = d\mathcal{A}ds$.

$$\Rightarrow \frac{dI_\nu}{ds} = -\frac{h\nu}{c}\left[(n_1 B_{12} - n_2 B_{21})I_\nu + \frac{h\nu}{4\pi}n_2 A_{21}\right]\phi(\nu).$$

If we compare this to equation of radiative transfer of the continuum (Sect. 2.1.1),

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + \epsilon_\nu \Rightarrow \kappa_\nu = \frac{h\nu}{c}(n_1 B_{12} - n_2 B_{21})\phi(\nu) \text{ and } \epsilon_\nu = \frac{h\nu}{4\pi}n_2 A_{21}\phi(\nu)$$

[6]See, e.g. Stellar Atmospheres II

and since $g_1 B_{12} = g_2 B_{21}$, where g_1 and g_2 are the statistical weights of the lower and upper levels, respectively, the absorption coefficient becomes

$$\kappa_\nu = \frac{h\nu}{c} n_1 B_{12} \left(1 - \frac{g_1 n_2}{g_2 n_1}\right) \phi(\nu).$$

In thermodynamic equilibrium, the Boltzmann distribution gives the *excitation temperature*, $T_{\rm ex}$,

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-h\nu/kT_{\rm ex}} \Rightarrow \kappa_\nu = \frac{h\nu}{c} n_1 B_{12} \left(1 - e^{-h\nu/kT_{\rm ex}}\right) \phi(\nu),$$

and since

$$A_{21} = \frac{8\pi h\nu^3}{c^3} B_{21} \text{ with } g_1 B_{12} = g_2 B_{21},$$

the absorption coefficient for a spectral line becomes

$$\boxed{\kappa_\nu = \frac{c^2}{8\pi\nu^2} \frac{g_2}{g_1} n_1 A_{21} \left(1 - e^{-h\nu/kT_{\rm ex}}\right) \phi(\nu).}$$

2.2 The brightness temperature

The *brightness temperature* is a useful diagnostic for temperature measurement if the astronomical source is a blackbody and we are in the *Rayleigh-Jeans* regime (where $h\nu \ll kT$).

This is defined irrespective of the origin of the radiation and is not necessarily the same as the temperature of the emitting body if not thermal and where $h\nu \sim kT$ and above.

However, when the emitted radiation from a surface equals that of a blackbody, the temperature of the blackbody is defined as the brightness temperature.

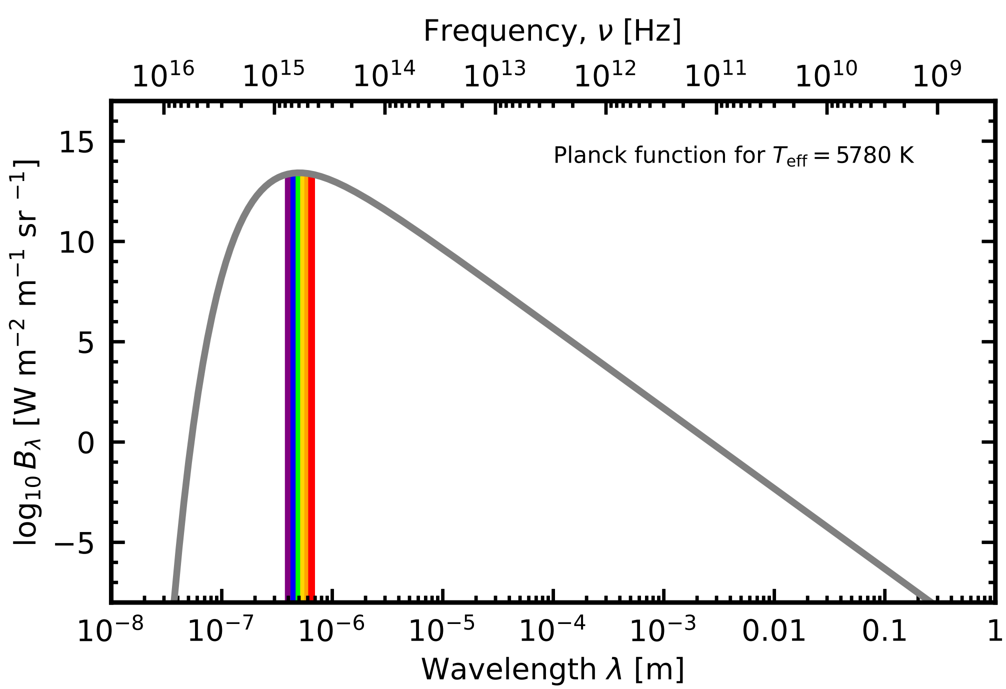
The *specific brightness* is the brightness per unit frequency or wavelength, given by the *Planck function*

$$B_\lambda = \frac{2hc^2}{\lambda^5 \left(\exp\left\{\frac{hc}{\lambda kT}\right\} - 1\right)}$$

in wavelength or

$$B_\nu = \frac{2h\nu^3}{c^2 \left(\exp\left\{\frac{h\nu}{kT}\right\} - 1\right)}$$

in frequency.



At long wavelengths ($h\nu \ll kT$) the Rayleigh-Jeans approximation gives

$$B_\nu \approx \frac{2\nu^2}{c^2}kT, \text{ for } \nu \ll \frac{kT}{h},$$

and thus the *brightness temperature*

$$T_\mathrm{B} \equiv \frac{c^2}{2k\nu^2}B_\nu.$$

Since, in the Rayleigh-Jeans limit, $T_\mathrm{B} \propto B_\nu = I_\nu/\pi$, we can write $I_\nu = I_\nu(0)e^{-\tau_\nu} + S_\nu\,(1 - e^{-\tau_\nu})$ [Sect. 2.1.3] as

$$T_\mathrm{B} = T_\mathrm{B}(0)e^{-\tau_\nu} + T\left(1 - e^{-\tau_\nu}\right).$$

At large optical depths, $T_\mathrm{B} \to T$, i.e. the brightness temperature of the radiation approaches the temperature of the source.

3 Molecular Gas

Although atomic gas contains most of the mass of the ISM, molecular gas is the densest component and is a tracer of cold [$T \sim 10$ K] and dense [$n \gtrsim 10^3$ cm^{-3}] environments – the conditions required for star formation. Most of what we know about ISM molecules comes from spectroscopy. Molecules can have three types of transitions: *electronic, vibrational* and *rotational*. Electronic transitions usually occur in the far-UV, vibrational transitions in the infrared and rotational transitions in the microwave band.

The most abundant molecule is molecular hydrogen, H_2, which is homo-nuclear, thus having no dipole moment and so radiates only through forbidden transitions with very low transition rates at low (molecular gas friendly) temperatures.

Thus, although important at high temperatures [$\sim 10^3$ K], emitting in the near-infrared (λ = 2.2 μm), at low temperatures ($\sim$ 10 K) the presence of H_2 is inferred from the next most abundant molecule carbon monoxide (CO).[a]

[a]Unless otherwise stated, CO refers to the most common isotopomer, $^{12}C^{16}O$.

Being located in the mm-band, the rotational transitions of CO can readily be detected by ground-based telescopes and it is easily excited at low temperatures. making it readily detectable in star-forming regions throughout the Galaxy and up to redshifts $z \gtrsim 6$ (look-back times of $\gtrsim 12.7$ Gyr).

The molecule has a dissociation energy of 11.2 eV, so that

$$E = \frac{hc}{\lambda} = 11.2 \times 1.602 \times 10^{-19} = 1.79 \times 10^{-18} \text{ J} .$$

$$\Rightarrow \lambda = 1.11 \times 10^{-7} \text{ m} = 111 \text{ nm},$$

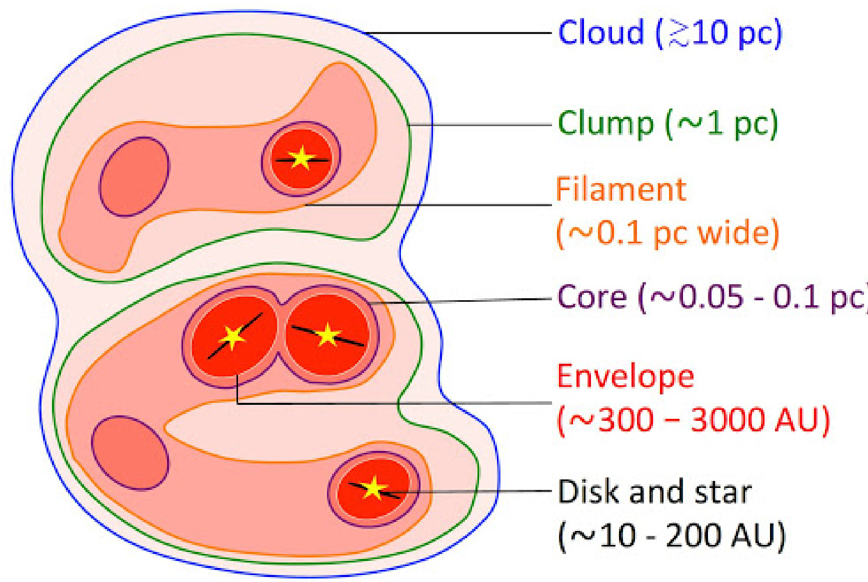
Therefore, a UV photon is required so split this into carbon and oxygen.

The Swedish–ESO Submillimetre Telescope in Chile

Looking at the black body spectrum of the Sun, above, it is seen that the intensity at $\lambda \sim 10^{-7}$ m is $\lesssim 10^6$ magnitudes lower that at visible wavelengths and so most starlight cannot dissociate CO, leaving this the most abundant molecule. Also, given that CO is much heavier than H_2, the spaces between the energy levels are much closer allowing excitation, and thus the detection of spectral lines, at low temperatures.

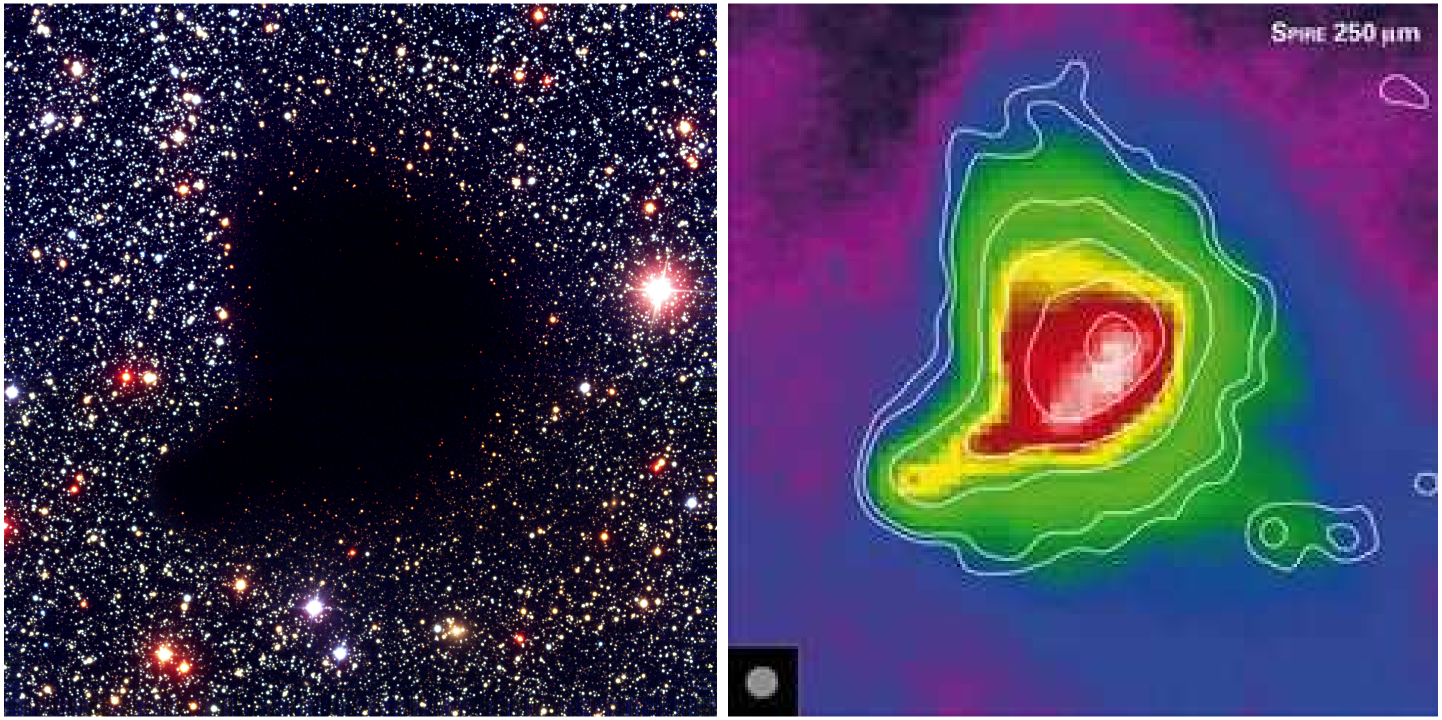
Many other molecules have been detected in the ISM, perhaps up to glycene H_2NCH_2COOH (see here for the latest), whose various transitions provide an excellent probe of the physical and chemical condition of interstellar gas.[7]

A constituent of the ISM are molecular clouds, which are always observed around star-forming regions. New stars condense out of these dense clouds through gravitational collapse triggered by an instability (e.g. Jeans instability for self-gravitation) or shock waves (e.g. from an exploding supernova). *Giant molecular clouds* (GMCs) can have diameters of up to 200 pc and masses up to 10^6 $M_\odot$, with an average density of $\sim 100 - 10^3$ particles cm^{-3}, compared to the 1 cm^{-3} densities near the Sun.



At $T \sim 10$ K and $n(H_2) \gtrsim 10^3$ cm^{-3}, dense cores within GMCs cannot be detected in visible light, and so are primarily observed through the rotational transitions of molecules.

[7] Could we make beer in space?



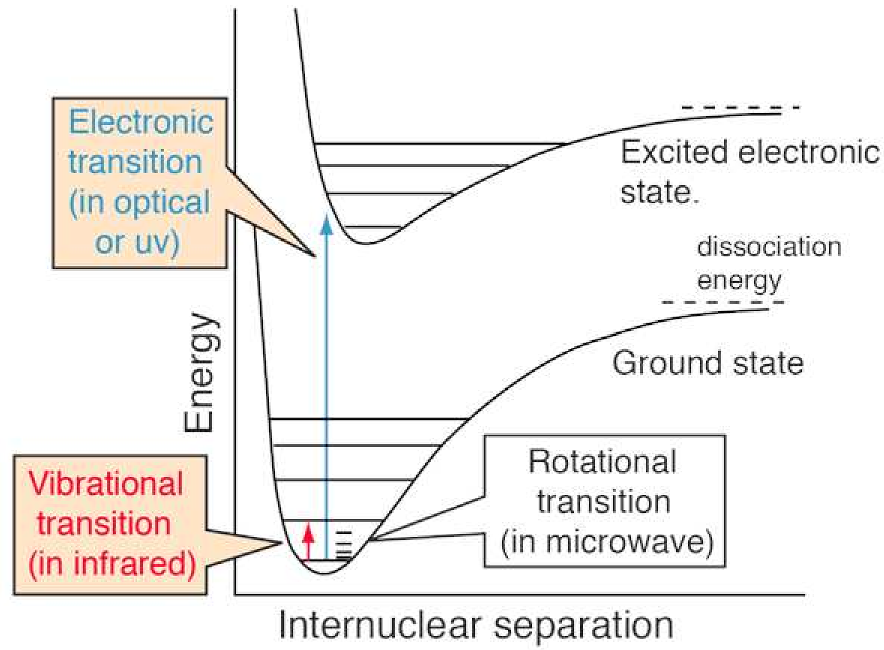
The plots show an optical image of Barnard 68 – a molecular cloud about 500 light-years distant, with a size of about 0.25 light-years, i.e. definitely not "*6-10 billion light-years away, there is a void in space 1 billion light-years across*" So it would subtend 9° of the sky?

"*It is completely empty of both normal and dark matter. It emits not detectable radiation of any kind.*" The 250 μm (FIR) emission[8] in the right image would beg to differ.

[8] From cold dust at $T \sim 10$ K. See Sect. 6.2

3.1 Vibrational and electronic transitions

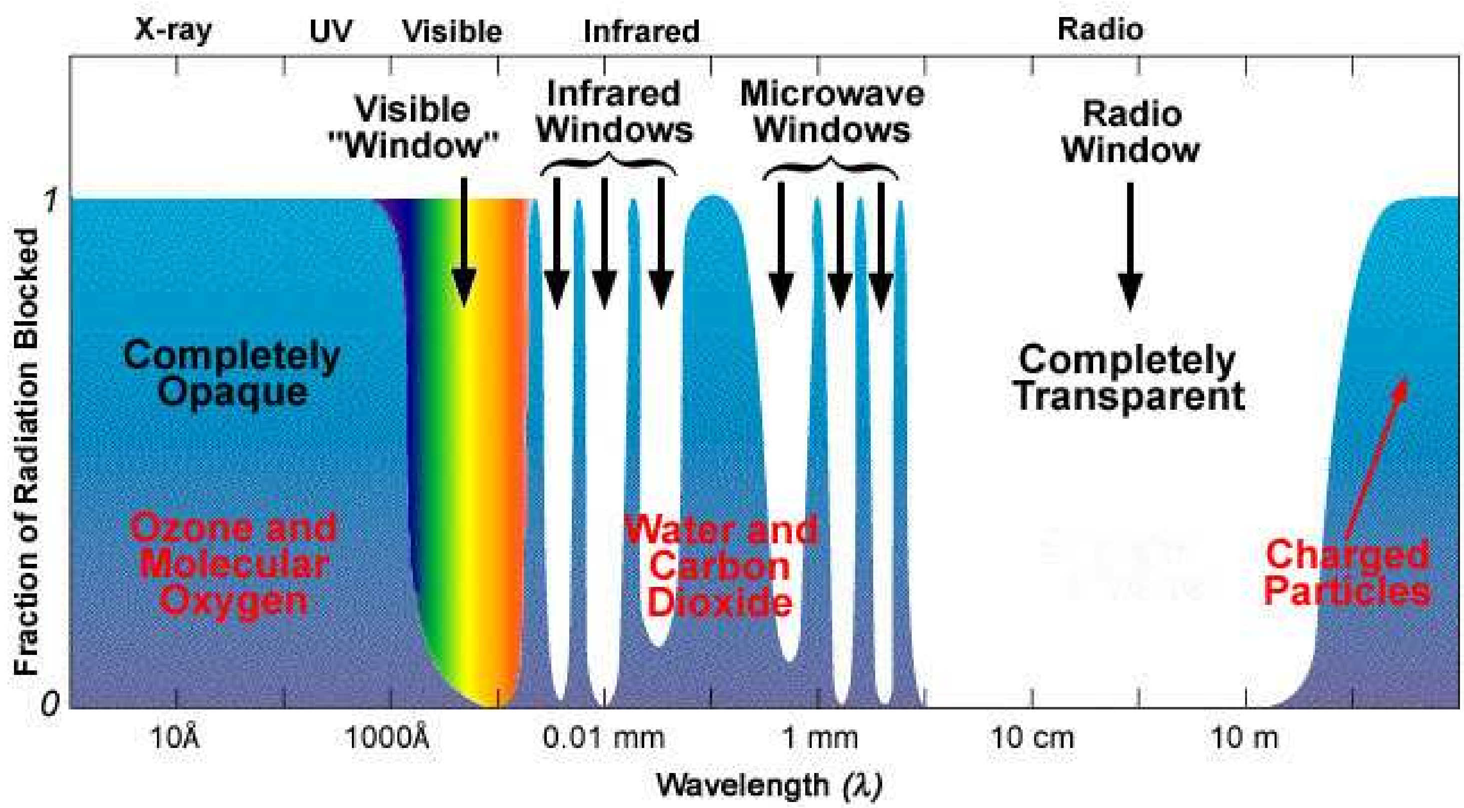
Vibrational transitions of molecules occur between energy levels that result from the quantisation of possible modes of vibration. For diatomic molecules, these are stretching modes corresponding to variations in inter-atomic distances. Vibrational transitions are the mechanism by which greenhouse gases, CO_2, CH_4, NO, O_3, H_2O..., absorb and re-emit infrared radiation back towards the Earth.



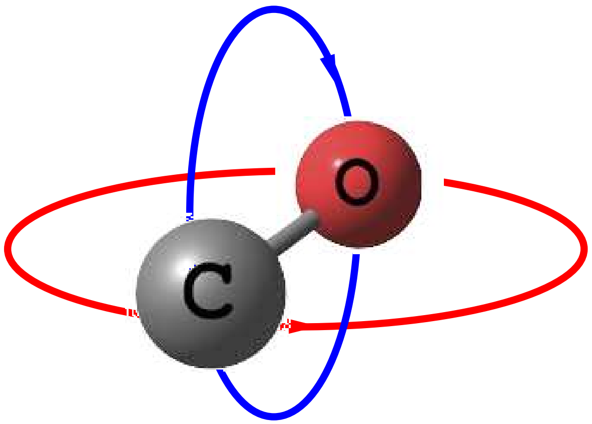
Electronic transitions take place when electrons in a molecule are excited from one energy level to a higher energy level. For CO, these occur in the ultra-violet and so can only be observed from the ground at redshifts of $z \gtrsim 2$ (LBT $\gtrsim$ 10 Gyr), where they are redshifted into the atmospheric window in the optical band.

3.2 Rotational transitions

Observationally, rotational transitions are usually the most important, with the lowest three rotational transitions of CO being at 115, 230 and 345 GHz (λ = 3, 1.3 and 0.9 mm), thus being detectable with ground-based (preferably at high and dry locations) mm-wave band telescopes.



The simplest transitions to consider are those in diatomic and linear molecules, which have two rotational degrees of freedom. Non-linear, polyatomic molecules, whilst detected in the ISM, have many more rotational degrees of freedom and thus are more complex to study.



The rotational energy of a linear molecule is

$$E_{\rm rot} = \frac{1}{2} I\omega^2 = \frac{|\mathbf{J}^2|}{2I},$$

where I is the moment of inertia and $\mathbf{J}$ is the angular momentum. For a diatomic molecule with inter-atomic distance R, $I = m_r R^2$.

The solution to Schrödinger's equation for this system gives the quantisation of rotational energy as

$$E_{\rm rot} = \frac{\hbar^2}{2I} J(J+1), \text{ where } J = 0, 1, 2, 3, \ldots \text{ and } \hbar = \frac{h}{2\pi}.$$

Like electronic transitions, only transitions of $\triangle J = \pm 1$ are permitted. For example, in the case of emission[9], dropping $J \rightarrow J - 1$

$$E_2 = \frac{\hbar^2}{2I} J(J+1) \text{ and } E_1 = \frac{\hbar^2}{2I}(J-1)J,$$

$$\Rightarrow \triangle E = [J(J+1) - J(J-1)]\frac{\hbar^2}{2I} = \frac{\hbar^2 J}{I},$$

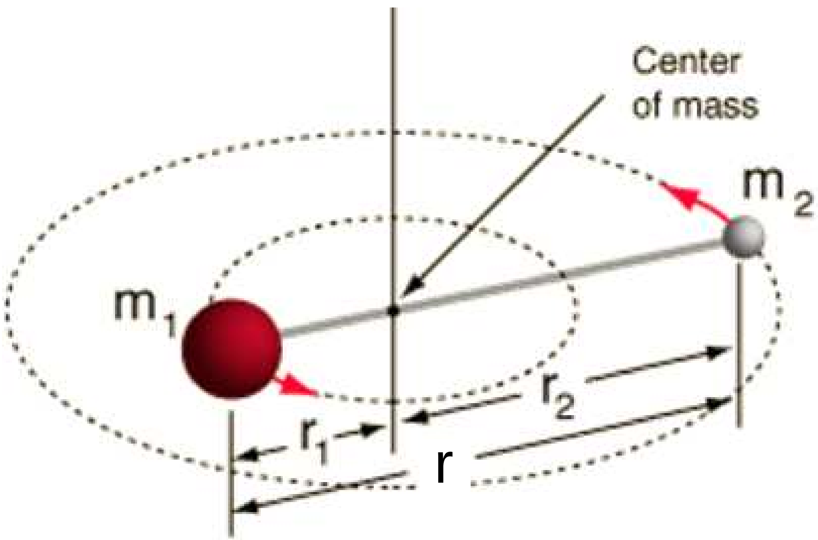
with the frequency of the emitted photon being

$$\nu = \frac{\triangle E}{h} = \frac{hJ}{4\pi^2 I}.$$

[9]Absorption is $J \rightarrow J + 1$.

The moment of inertia is given by $I = m_r r^2$, so that

$$\nu = \frac{\triangle E}{h} = \frac{hJ}{4\pi^2 m_r r^2}.$$



The reduced mass of the system is

$$m_r = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}} = \frac{1}{\frac{m_2+m_1}{m_1 m_2}} = \frac{m_1 m_2}{m_2 + m_1},$$

where $m_r = \text{amu}\left(\frac{12\times16}{12+16}\right) = 1.14 \times 10^{-26}$ kg. $r = 1.13$ Å and $\triangle\nu = 1.153 \times 10^{11}$ Hz gives a ladder where CO rotational transitions increase in steps of ≈ 115.3 GHz

That is, CO $J = 1 \to 0$ at 115.271208 GHz, $J = 2 \to 1$ at 230.542416 GHz, $J = 3 \to 2$ at 345.795990 GHz (see here for higher transitions).

The above assumes a perfectly rigid rotor, where in reality centrifugal stretching makes the molecule slightly elastic, lengthening the bond as the molecule spins faster. This results in the observed frequencies being slightly lower than those predicted.

Adding a correction, from perturbation theory, gives the energy as

$$E_{\rm rot} = \frac{\hbar^2}{2I} J(J+1) - hD[J(J+1)]^2,$$

where D is the *centrifugal distortion constant*.

In terms of *rotational constant*, $B_e \equiv \hbar/(4\pi I)$,

$$\nu = 2B_e(J+1) - 4D(J+1)^3$$

where $D \sim 10^{-5}\,B_e$.

3.3 Molecular gas as a diagnostic

3.3.1 Density

Due to its small dipole moment (μ = 0.122 Debye[10]), CO is primarily excited by collisions with H_2 and has a low critical density compared to other molecules which provide tracers of higher density gas (e.g. HCN with $\mu = 2.98 Debye$).

For an electric dipole transition

$$A_{21} = \frac{64\pi^4}{3hc^3}\nu_{21}^3\mu^2,$$

where μ is the electric dipole moment, and

$$A_{21} = \frac{8\pi h\nu^3}{c^3}B_{21}, \text{ with } g_1 B_{12} = g_2 B_{21}.$$

[10] 1 D ≡ 1 Cm

Including excitation and de-excitation by collision, the rate equation, $n_2 A_{21} + n_2 B_{21} u_\nu = n_1 B_{12} u_\nu$, (Sect. 2.1.5) becomes

$$n_2(A_{21} + B_{21} u_\nu + C_{21}) = n_1(B_{12} u_\nu + C_{12}).$$

Since the main collision partner is H_2, the collision rate is

$$C_{12} = n(\mathrm{H}_2)\left\langle \sigma_{12} v \right\rangle,$$

where σ_{12} is the collision cross-section and v the speed (the mean from a velocity distribution[11]).

In thermodynamic equilibrium, the collisional excitations and de-excitations are in balance, i.e. $n_1 C_{12} = n_2 C_{21}$.

The relative population of the levels defines the *excitation temperature*, via the Boltzmann distribution,

$$\frac{n_{J+1}}{n_J} \equiv \frac{n_u}{n_l} = \frac{g_{J+1}}{g_J} \exp\left(-\frac{h\nu_{ul}}{kT_{\mathrm{ex}}}\right),$$

where n_J is the number density of molecules in the J level and $g_J = 2J + 1$ the statistical weight of the J state.

The excitation due to collisions gives the *kinetic temperature*, defined via,

$$E_{\mathrm{kin}} = \frac{1}{2} m v^2 = \frac{3}{2} k T_{\mathrm{kin}},$$

[11] e.g. A Maxwell–Boltzmann distribution.

and so

$$\frac{C_{12}}{C_{21}} = \frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-h\nu_{21}/kT_{\rm kin}}.$$

That is

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-h\nu_{21}/kT_{\rm kin}} = \frac{B_{12} u_{\nu_{21}} + C_{12}}{A_{21} + B_{21} u_{\nu_{21}} + C_{21}},$$

from the rate equation.

Considering the limiting cases:

1. In the radiative limit (excitation mainly by radiation and collisions unimportant):

$$\frac{g_2}{g_1} e^{-h\nu_{21}/kT_{\rm kin}} = \frac{B_{12} u_{\nu_{21}}}{A_{21} + B_{21} u_{\nu_{21}}},$$

$$\frac{B_{12} u_{\nu_{21}}}{A_{21} + B_{21} u_{\nu_{21}}} = \frac{(g_2/g_1) B_{21} u_{\nu_{21}}}{\frac{8\pi h \nu_{21}^3}{c^3} B_{21} + B_{21} u_{\nu_{21}}},$$

   and the energy density from a background black body[12] is

$$u_\nu = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/kT_{\rm BG}} - 1}$$

   So

$$\frac{\cancel{g_2}}{\cancel{g_1}} e^{-h\nu/kT_{\rm kin}} = \frac{\cancel{g_2}}{\cancel{g_1}} \frac{\cancel{B_{21} u_\nu}}{\cancel{B_{21} u_\nu}(e^{h\nu/kT_{\rm BG}} - 1) + \cancel{B_{21} u_\nu}}$$

[12]The *cosmic microwave background* ($T_{\rm CMB}$ = 2.725 K at z = 0) plus some other (dust) contribution

$$\Rightarrow \frac{1}{e^{h\nu/kT_{\rm kin}}} = \frac{1}{e^{h\nu/kT_{\rm BG}} - 1 + 1}$$

$$\boxed{T_{\rm kin} \rightarrow T_{\rm BG}}$$

2. In the collisional limit:

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-h\nu/kT_{\rm ex}} = \frac{C_{12}}{C_{21}} = \frac{g_2}{g_1} e^{-h\nu/kT_{\rm kin}}$$

$$\boxed{T_{\rm ex} \rightarrow T_{\rm kin}}$$

> When collisions dominate $T_{\rm ex} \rightarrow T_{\rm kin}$ (local thermodynamic equilibrium, LTE).
>
> When radiation dominates $T_{\rm ex} \rightarrow T_{\rm BG}$ (radiative equilibrium).

We can define a critical density as the H_2 density where the collisional de-excitation is similar to the spontaneous radiative rate,

$$n_{\rm crit} \equiv \frac{A_{21}}{\langle \sigma_{12} v \rangle},$$

where the collisional cross-section is typically $\langle \sigma_{12} v \rangle \lesssim 1 \times 10^{-10}$ cm^{-3} s^{-1}.

Molecule	Transition	ν [GHz]	A_{21} [s^{-1}]	$E_2 \equiv h\nu/k$ [K]	$\langle \sigma_{12} v \rangle$ [cm^{-3} s^{-1}]	$n_{\rm crit}$ [cm^{-3}]
CO	$1 \rightarrow 0$	115.271	7.203×10^{-8}	5.53	2.4×10^{-11}	3×10^{3}
	$2 \rightarrow 1$	230.538	6.910×10^{-7}	1.60	6.9×10^{-11}	1×10^{4}
CS	$1 \rightarrow 0$	48.991	1.749×10^{-6}	2.40	1.4×10^{-11}	1×10^{5}
	$2 \rightarrow 1$	97.981	1.679×10^{-5}	7.10	1.1×10^{-10}	7×10^{5}
HCO^+	$1 \rightarrow 0$	89.189	4.187×10^{-5}	4.28	3.2×10^{-10}	1.5×10^{5}
	$3 \rightarrow 2$	267.558	1.453×10^{-3}	25.86	4.8×10^{-10}	3×10^{6}
HCN	$1 \rightarrow 0$	88.631	2.497×10^{-5}	4.25	6.2×10^{-12}	4×10^{6}
	$3 \rightarrow 2$	265.886	8.356×10^{-4}	25.52	8.4×10^{-11}	1×10^{7}
HNC	$1 \rightarrow 0$	90.664	2.690×10^{-5}	4.35	5.7×10^{-12}	4×10^{6}
	$3 \rightarrow 2$	271.981	9.336×10^{-4}	26.11	9.3×10^{-11}	1×10^{7}

From the Leiden Atomic and Molecular Database

Below the critical density, the excitation temperature will be close to the radiation temperature and the emission line will not be visible (sub-thermal excitation). Well above the critical density, the excitation temperature is close to the kinetic temperature of the gas (that is, in LTE)[13]. So species such as HCN will only be detected when $n(H_2) \sim 10^6$ cm^{-3}, thus being tracers of dense cores in

[13]Sometimes described as the line being *thermalised*.

the GMC where star formation may occur, whereas CO is an ubiquitous molecular gas tracer due to its easy excitation.

3.3.2 Temperature

CO is provides a crucial thermometer of the ISM. The lower rotational transitions are very optically thick ($\tau_\nu \gg 1$), meaning the radiation is strongly absorbed and re-emitted from a layer where the temperature is close to the radiation temperature (Sect. 2.2). The relative strengths of the spectral lines can give the excitation temperature of the CO, which will be equal to the kinetic temperature of H_2 if in thermodynamic equilibrium.

Within the molecular cloud, remote from heating sources such as stars (see Sect. 5.1), heating by cosmic rays and collisions with warm dust grains is balanced by molecular line emission cooling the gas[14], maintaining $T \sim 10$ K. In the presence of star formation, temperatures can reach a few hundred K.

As seen above, when collisions are dominant $T_{\rm ex} \approx T_{\rm kin}$ and so for this "thermalised" gas

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-h\nu/kT_{\rm ex}} \Rightarrow \frac{n_J}{g_J} \propto e^{-E_J/kT_{\rm kin}} = Ce^{-E_J/kT_{\rm kin}},$$

[14]CO is a primary coolant of the ISM.

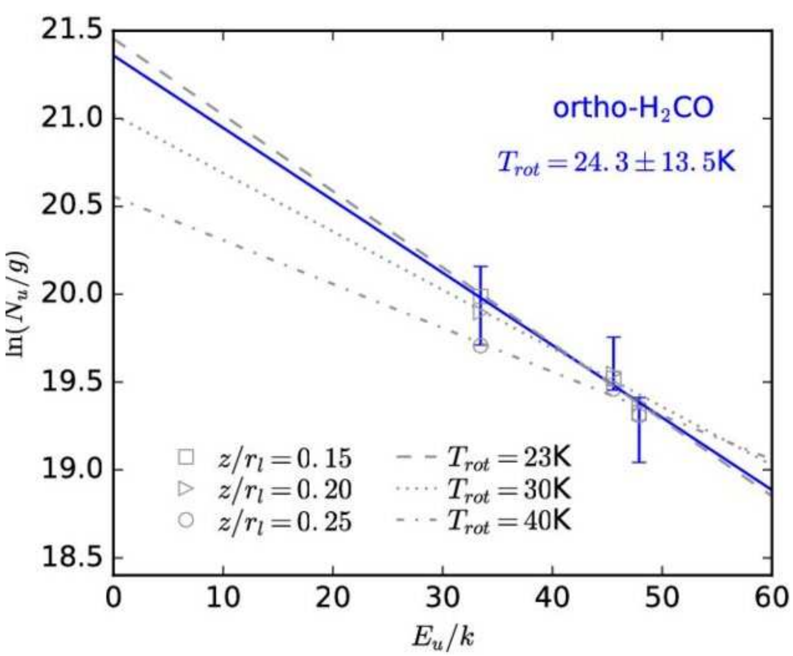
where E_J is the energy of the upper level[15], obtained via

$$\nu = \frac{1}{h}\left(E_J - E_{J-1}\right),$$

and C is a constant. This gives

$$\ln\left(\frac{n_J}{g_J}\right) = -\frac{E_J}{kT} + C \Rightarrow \underbrace{\ln\left(\frac{N_J}{g_J}\right) = -\frac{E_J}{kT} + C,}_{y=mx+c}$$

where $N_J = n_J s$ is the column density (Sect. 4.2) of the transition and C is now another constant.



So by converting the observed integrated line intensities to column densities (which is telescope dependent, Sect. 4.4), the gradient of such an *excitation/rotation diagram* yields the temperature of the gas.

From H_2CO Ortho-to-para Ratio in the Protoplanetary Disk HD 163296

[15]Compiled in the Leiden Atomic and Molecular Database

4 Atomic Gas

4.1 The H I 21-cm transition

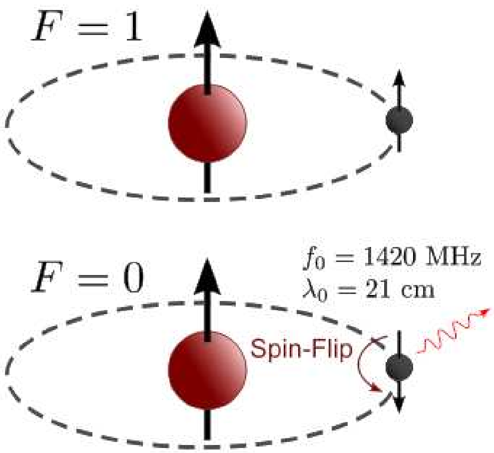
Hydrogen is the most common element in the Universe (75%) and the 21-cm transition allows us to measure the temperature and density of the ISM, as well as galactic dynamics.

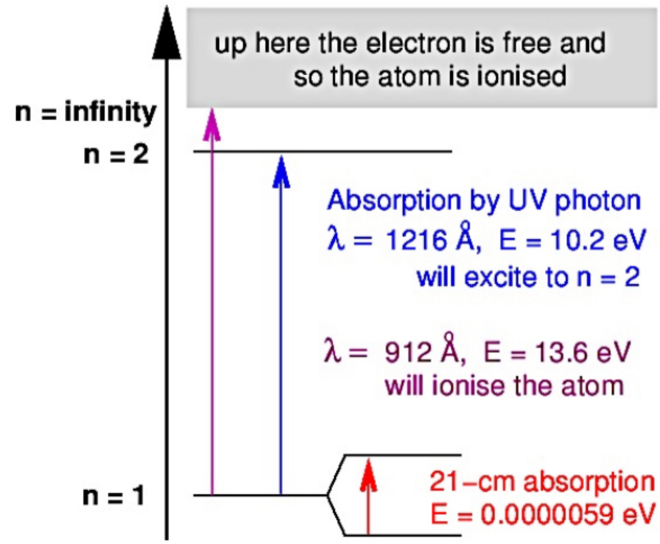
- In 1940, *Oort* suggested emission lines in radio part of spectrum and in 1944 *van de Hulst* predicted that neutral hydrogen (H I) should produce radiation of frequency ν = 1420 MHz (λ = 21.1 cm) from spin-flip of the electron.

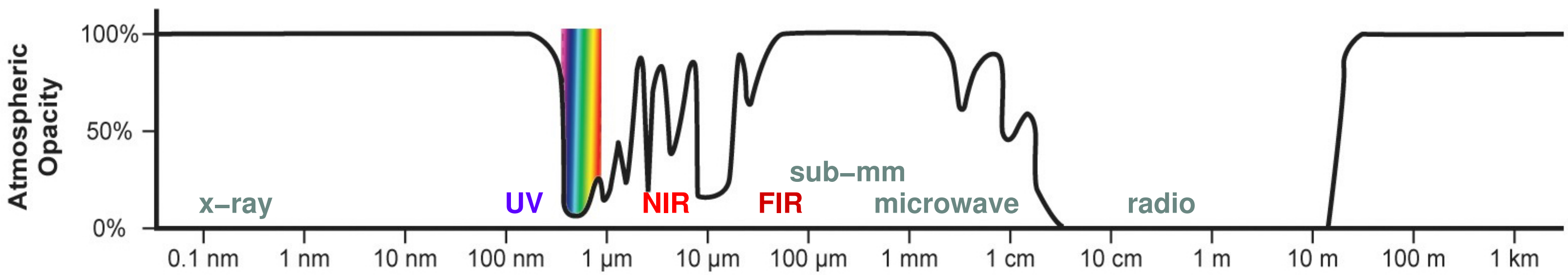
- The 21-cm transition was detected in 1951 by *Ewen & Purcell*, despite being "forbidden" – the Einstein coefficient of spontaneous emission ($A_{21} = 2.87 \times 10^{-15}$ s^{-1}) gives a lifetime of $\sim 10^7$ years, cf. just $\sim 10^{-8}$ s for Lyman-α emission ($n = 2 \rightarrow 1$, λ = 1216 Å).

- A GMC can contain $\sim 10^6 \mathrm{M}_\odot$ of H I and a galaxy $\sim 10^9 \mathrm{M}_\odot$, which is $\sim 10^{63} - 10^{66}$ atoms, making this readily detectable and leading to the birth of spectral line radio astronomy in 1951.

The 21-cm transition results from hyperfine splitting of the ground energy level. The atom is in a lower energy state when the spins of the electron and proton are anti-parallel than when they are parallel, corresponding to the total angular momentum quantum numbers $F = 0$ and $F = 1$, respectively.







While Lyman-α *absorption* yields the total *column density* of H I, with a rest-wavelength of λ = 1216 Å this is restricted to redshifts $z \gtrsim 1.7$ with ground-based optical telescopes. Therefore, at lower redshifts (the last 10 Gyr or 70% the history of the Universe), this is obtained from 21-cm *emission*.

4.2 Column density

The column density, N, of any material along a specified line-of-sight is defined as the number of atoms in a column with unit cross-section: $N = \int n ds$, where n is the number density of atoms and s is the path length along the sight line.

In order to quantify this, we return to the *optical depth* of a line, which is given by the opacity integrated over the line-of-sight, $\tau_\nu = \int \kappa_\nu ds$, where the *absorption coefficient* is (Sect. 2.1.5)

$$\kappa_\nu = \frac{c^2}{8\pi\nu^2}\frac{g_2}{g_1}n_1 A_{21}\left(1 - e^{-h\nu/kT_{\rm ex}}\right)\phi(\nu).$$

That is

$$\tau_\nu = \int \kappa_\nu ds = \int \frac{c^2}{8\pi\nu^2}\frac{g_{F=0}}{g_{F=1}}n_{F=0}A_{F=1\to 0}\left(1 - e^{-h\nu/kT_{\rm spin}}\right)\phi(\nu)ds, \qquad (3)$$

where $g_{F=0}$ and $g_{F=1}$ are the statistical weights of the two states ($g_F = 2F + 1$).

Since $\nu \approx 1.4$ GHz, $h\nu/k \approx 0.07$ K, which is $\ll T_{\rm spin}$, we can use the Rayleigh-Jeans approximation (Sect. 2.2).[16]

Taking the first term of the Taylor expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ..., \text{ then } 1 - e^x \approx 1 - (1 + x) = -x, \text{ where } x = -\frac{h\nu}{kT}.$$

[16]The minimum possible temperature is $T_{CMB} = 2.725$ K at $z = 0$ and we expect $T_{\rm spin} \gtrsim 100$ K.

$$\Rightarrow \tau_\nu \approx \int \frac{c^2}{8\pi\nu^{\cancel{2}}} \frac{g_{F=0}}{g_{F=1}} n_{F=0} A_{F=1\to 0} \frac{h\cancel{\nu}}{kT_{\rm spin}} \phi(\nu) ds. \tag{4}$$

The statistical weights give $(2+1)/(0+1) = 3$ and, assuming that the gas is sufficiently cold so that only the ground state is populated[17], then the total gas density is given by the sum of the two hyperfine levels, i.e.

$$n_{\rm H\,I} = n_{F=0} + n_{F=1} = n_{F=0}\left(1 + \frac{n_{F=1}}{n_{F=0}}\right).$$

The relative populations of two states, again, depends upon the excitation temperature, where it is termed the *spin temperature*

$$\frac{n_{F=1}}{n_{F=0}} = \frac{g_{F=1}}{g_{F=0}} e^{-h\nu/kT_{\rm spin}},$$

under LTE, $T_{\rm spin} = T_{\rm kin}$ (from collisional excitation).

Using the Rayleigh-Jeans approximation,

$$e^{-h\nu/kT_{\rm spin}} \approx 1, \text{ giving } \frac{n_{F=1}}{n_{F=0}} \approx 3 \text{ and } n_{\rm H\,I} \approx n_{F=0}(1+3) = 4n_{F=0}.$$

Substituting this into Equ. 4 and given that $N_{\rm H\,I} = \int n_{\rm H\,I} ds$, for a constant $n_{\rm H\,I}$,

$$\tau_\nu \approx N_{\rm H\,I} \frac{3c^2}{32\pi\nu} A_{F=1\to 0} \frac{h}{kT_{\rm spin}} \frac{1}{\triangle\nu},$$

[17] From $E \sim kT = hc/\lambda$, collisional excitation of the Lyman-α transition requires temperatures of $\sim 10^5$ K.

where ϕ has been replaced by $1/\triangle\nu$ [to account for the units] since Doppler broadening[18] will dominate the line broadening.

Rearranging,

$$N_{\mathrm{H\,I}} \approx \frac{32\pi\nu}{3hc^3 A_{F=1\to 0}} k T_{\mathrm{spin}} \tau_\nu \triangle\nu.$$

We can express the profile in terms of *radial velocity*, via $\triangle v/c \approx \triangle\nu/\nu$, and integrating over the whole profile,

$$N_{\mathrm{H\,I}} \approx \frac{32}{3}\pi \frac{\nu^2 k}{hc^3 A_{F=1\to 0}} \int T_{\mathrm{spin}} \tau_\nu dv. \tag{5}$$

Plugging in the constants gives

$$N_{\mathrm{H\,I}} \approx 1.82 \times 10^{18} \int T_{\mathrm{spin}} \tau_\nu dv \text{ atoms cm}^{-2}\text{, where } dv \text{ is in km s}^{-1}, \tag{6}$$

which is the formula always used by spectral line radio astronomers.[19]

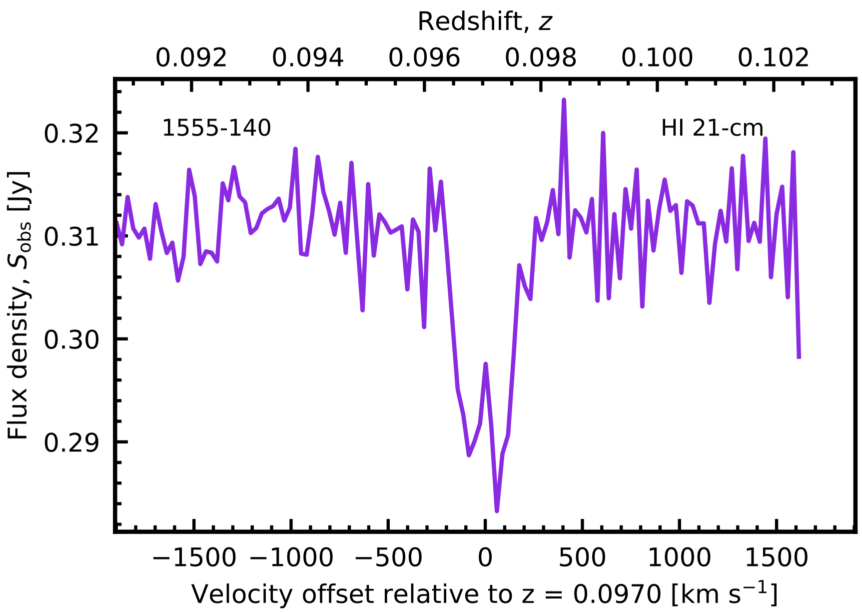
The brightness temperature (Sect. 2.2), a quantity which is measurable from the telescope, is given by

$$T_{\mathrm{B}} = T_{\mathrm{spin}} \left[1 - e^{-\tau_\nu}\right], \tag{7}$$

where τ_ν is the optical depth. Although applicable to emission, this is easiest quantified in terms of absorption.

[18] Of widths kHz – MHz, due to large scale motion.

[19] Usually without knowing where it comes from.



In the absorption spectrum shown, the background continuum flux is S = 0.312 Jy and the peak depth of the profile is at $\triangle S$ = 0.028 Jy relative to this. The optical depth is therefore given by

$$\tau_{peak} = -\ln\left(1 - \frac{\triangle S}{S}\right) = 0.094.$$

That is, at its peak the gas absorbs about 9% of the background 1.4 GHz flux.

We can consider the line *optically thin* if $\triangle S/S \lesssim 0.3$, where

$$\tau_\nu = -\ln\left(1 - \frac{\triangle S}{S}\right) \approx \frac{\triangle S}{S}.$$

Thus, in the optically thin regime, Equ. 7 becomes (cf. Sect. 2.2)

$$T_{\rm B} = T_{\rm spin}\left[1 - e^{-\tau_\nu}\right] \approx T_{\rm spin}\tau_\nu$$

which gives Equ. 6 as

$$\boxed{N_{\rm H\,I} \approx 1.823 \times 10^{18} \int T_{\rm B}\, dv, \text{ in cm}^{-2},}$$

where the velocity integrated brightness temperature can be readily determined from the source flux and telescope parameters.

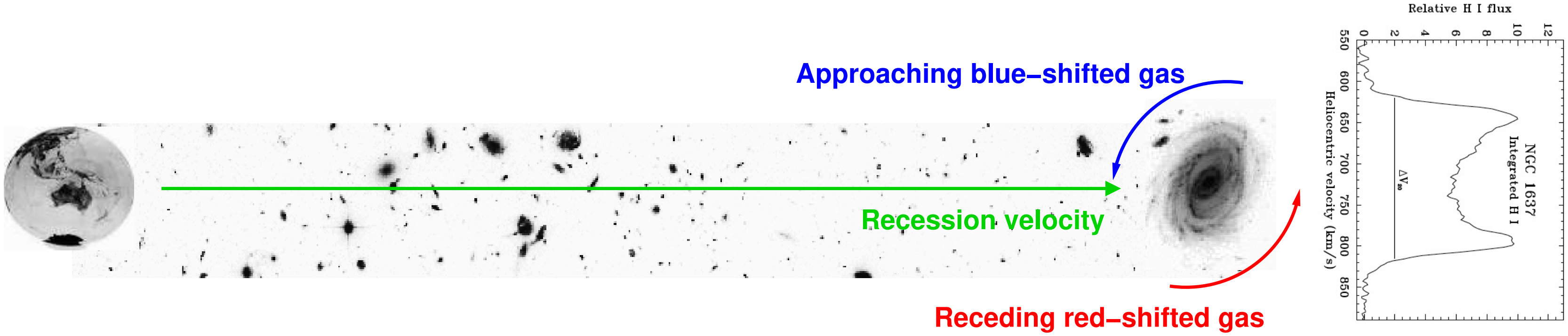
4.3 Line broadening

Being a quantum leap, the absorption line is intrinsically infinitesimally narrow in frequency. There are, however, several broadening mechanisms at play, which allow the line to be resolved by finite resolution spectrometers:

1. *Natural broadening:* Due to the Heisenberg uncertainty of the energy states. This is a very small effect in atomic spectra.

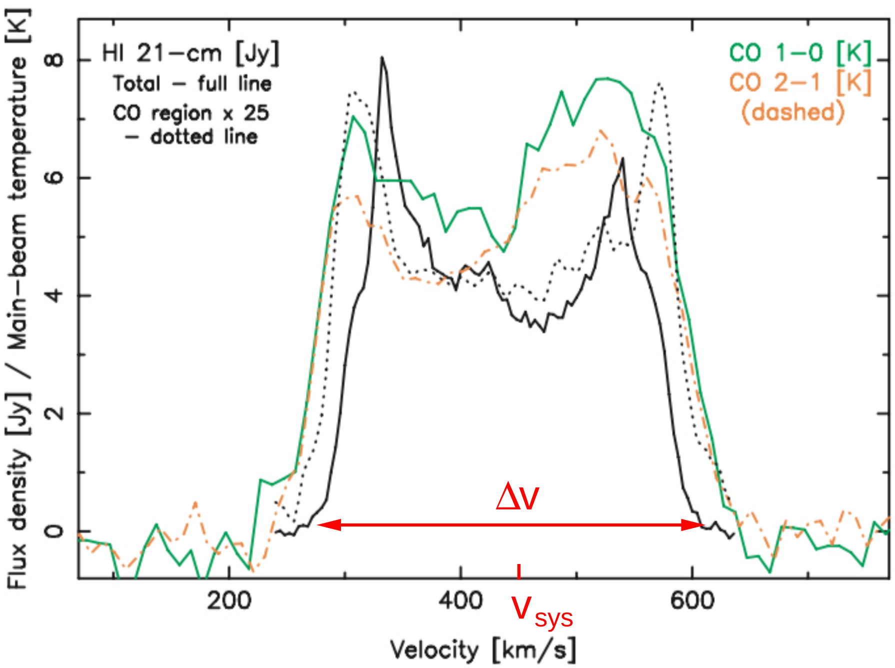
2. *Thermal (Doppler) broadening:* Due to the velocities of the atoms in the gas, where those approaching will emit blue-shifted photons and those receding red-shifted photons. If in LTE the profiles are Gaussian in shape, although turbulence, outflows, etc. can change this. Important in Galactic ISM spectra.

3. *Rotational (Doppler) broadening:* Due to large scale motion – the rotation speed of a galaxy is typically $\Delta v \sim 500$ km s^{-1}. From $\frac{\Delta v}{c} = \frac{\Delta \nu}{\nu_0}$ with $\nu_0 = 1420$ MHz, this gives a line of width of $\Delta\nu \approx 2$ MHz.



4.4 Emission diagnostics

The following plot shows the H I 21-cm emission line from the Circinus galaxy in units of Janskys[20]. Also shown is the molecular (CO) emission in Kelvin. From this we can get:



1. The *systemic velocity*, $v_{\rm sys}$, of the galaxy, which, with the appropriate cosmological model, can give redshift and distance. Here H I is observed at 1418.326 MHz, which gives a redshift of

$$z = \frac{1420.406}{1418.326} - 1 = 0.00147,$$

$\Rightarrow v_r \approx zc$ (non-relativistic) $= 440$ km s^{-1}.

For $z \ll 1$, the distance can be obtained from the radial velocity, via $D = v_r/H_0$ where the Hubble "constant" is (currently) $H_0 = 67.8$ km s^{-1} Mpc^{-1}, giving $D = 6.5$ Mpc. However, this assumes the Hubble flow and Circinus is "too local", with the adopted distance being 4.2 Mpc.[21]

2. The *column density:* The telescope measures the flux density of the source, which is given by

[20] 1 Jy = 1×10^{-26} W m^{-2} Hz^{-1} sr^{-1}.
[21] A Large New Galaxy in Circinus

the brightness integrated over the solid angle of the sky, i.e.

$$S_\nu = \iint_{\text{source}} B_\nu d\Omega = \frac{2\nu^2 k}{c^2} \iint_{\text{source}} T_\text{B} d\Omega,$$

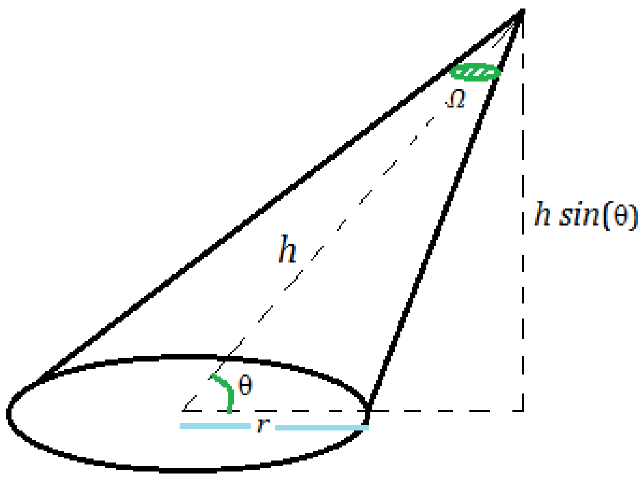
in the Rayleigh-Jeans limit.

The solid angle is defined as

$$\Omega \equiv \frac{A}{r^2},$$

which over the full surface area from inside a sphere is

$$\Omega = \frac{4\pi r^2}{r^2} = 4\pi \text{ steradian (dimensionless)}$$



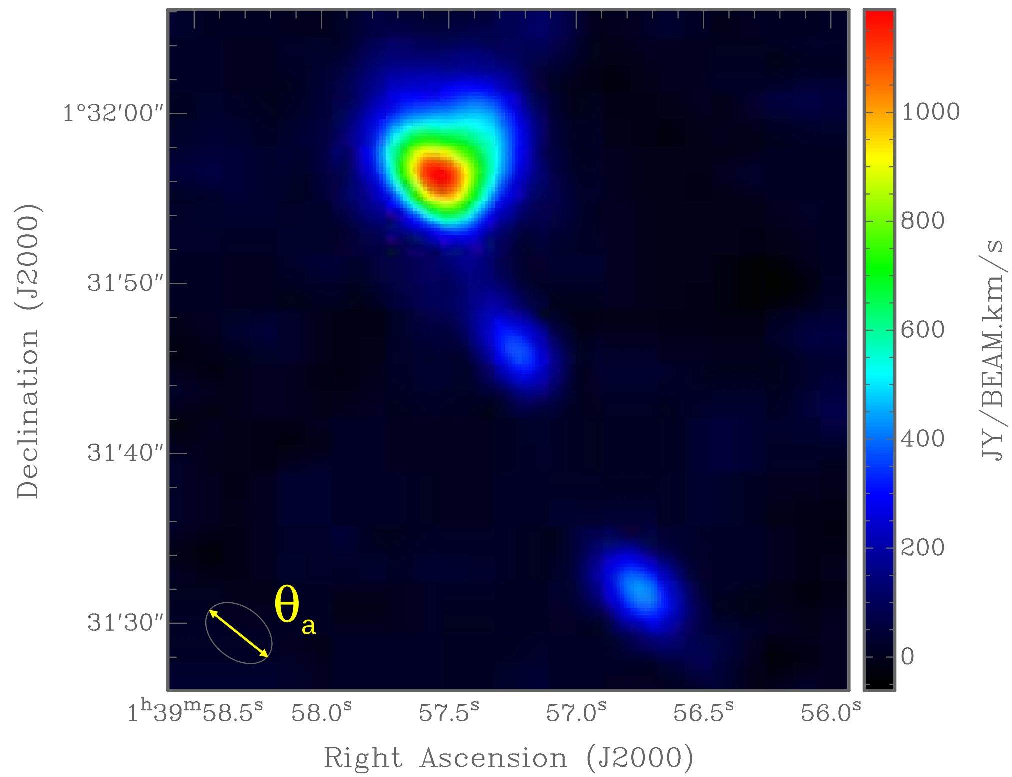
If a source of constant brightness temperature fills the telescope beam, the flux density is

$$\boxed{S_\nu = \frac{2\nu^2}{c^2} k T_\text{B} \Omega.}$$

The area of an ellipse is $A = \pi ab$, where a and b are the major and minor axes. In terms of solid angle formed by the telescope beam and the major and minor axes of the beam (in radians), this is

$$\Omega_{\text{beam}} = \pi \frac{\theta_a}{2}\frac{\theta_b}{2} = \frac{\pi\theta_a\theta_b}{4},$$

since the telescope beam-width is usually given in terms of the diameter.



For a Gaussian beam, the solid angle is therefore

$$\Omega_{\text{beam}} = \frac{\pi\theta_a\theta_b}{4\ln 2} = 1.133\,\theta_a\theta_b,$$

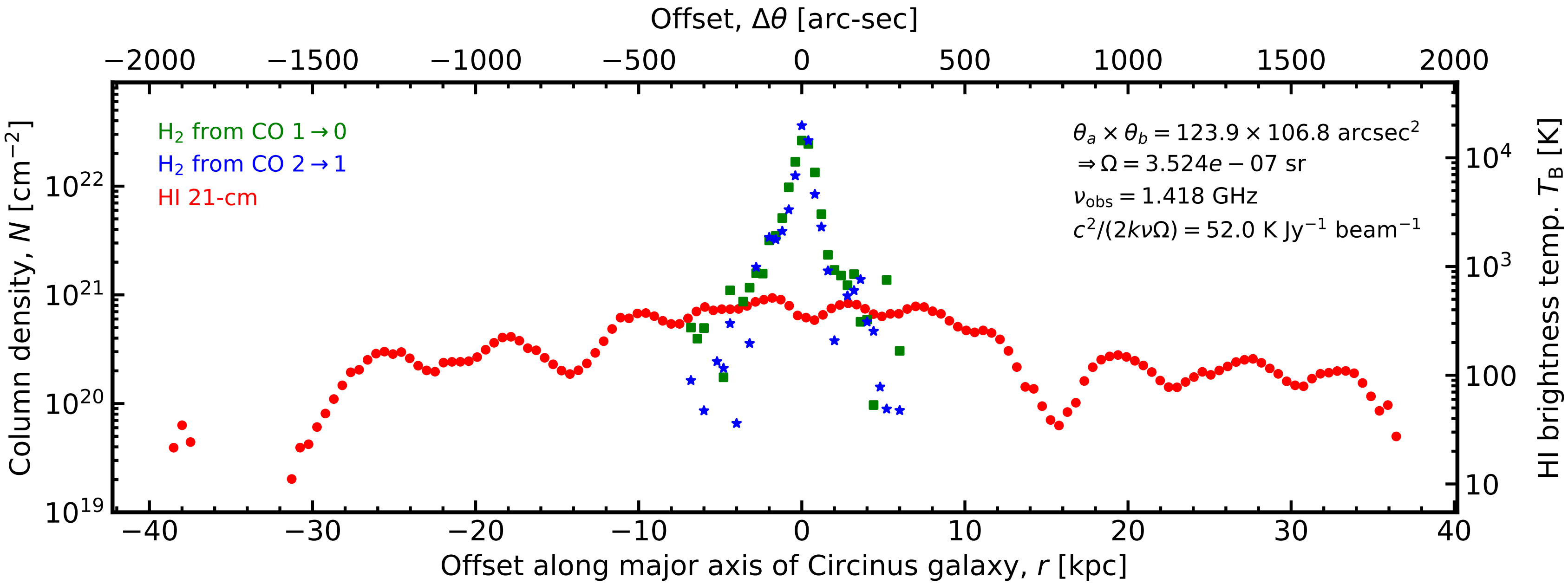
where θ_a and θ_b are the major and minor axes.

So

$$S_\nu = \frac{2\nu^2}{c^2} k T_{\rm B} \Omega = \frac{2\pi}{4\ln 2}\left(\frac{\nu}{c}\right)^2 k T_B\,\theta_a\theta_b \Rightarrow T_B = \frac{4\ln 2}{2\pi}\left(\frac{c}{\nu}\right)^2 \frac{S_\nu}{k\theta_a\theta_b}.$$

Flux densities are usually expressed in Janskys, beam sizes in arc-seconds[22] and, expressing the observed frequency in GHz, gives

$$T_B = 1.222 \times 10^6 \frac{\nu^2}{\theta_a\theta_b} S_\nu.$$



[22] $1'' = 1/3600$ degrees and 1 arc-minute $1' = 1/60$ degrees.

Therefore, in the optically thin regime (where $T_{\rm B} \approx T\tau_\nu$)[23],

$$N_{\rm H\,I} \approx 1.82 \times 10^{18} \int T_{\rm spin}\tau_\nu dv \approx 2.23 \times 10^{24} \frac{\nu^2}{\theta_a \theta_b} \int S_\nu \, dv \text{ in cm}^{-2}.$$

E.g. the maximum flux of Circinus gives a maximum brightness temperature $T_{\rm B}$ = 583 K for $\theta_a \times \theta_b$ = 124" $\times$ 107", giving $N_{\rm H\,I}$ = 1.1×10^{21} cm^{-2} (shown in the column density profile above). As seen in the profile, at higher column densities (e.g. in the centre of the galaxy), the hydrogen becomes more and more molecular.[24]

3. The *total atomic gas mass:* This (and all other texts I could find) state that "it is a straightforward exercise to derive"

$$M_{\rm H\,I} = 2.36 \times 10^5 S_{\rm int} D^2,$$

where $S_{\rm int} = \int S_\nu dv$ [Jy km s^{-1}] and D is the distance to the source in Mpc. Determining the gas mass this way is common practice, although even the original paper[25] is not clear on this. I suspect that this means the common *"it can be shown that"* = *"someone did it once, but I have no idea how"*. So here's my attempt:

$$N_{\rm H\,I} \approx 2.23 \times 10^{28} \frac{\nu^2}{\theta_a \theta_b} S_{\rm int} \text{ [m}^{-2}\text{, where } \nu \text{ is in GHz and } \theta \text{ in arc-seconds]}$$

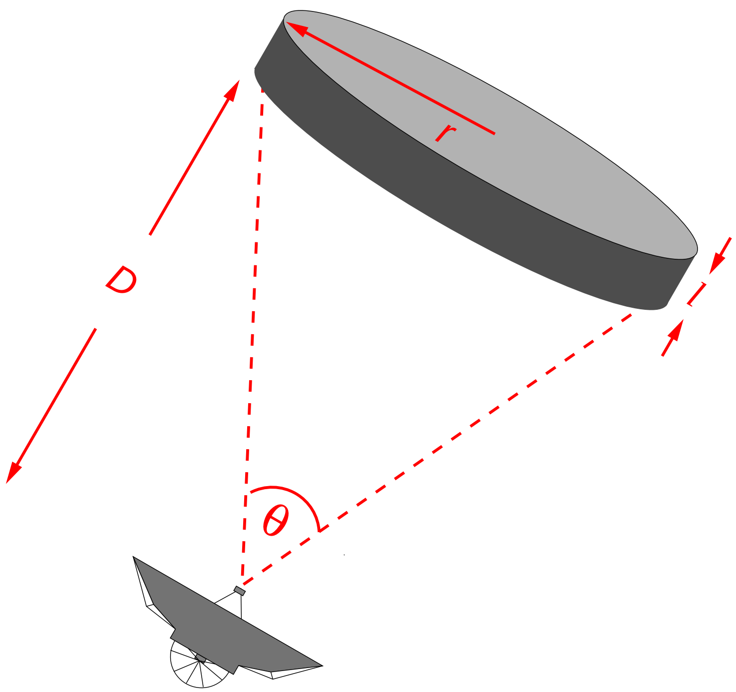
[23] Which is usual for H I 21-cm, although CO is optically thick.

[24] A Physical Upper Limit on the H I Column Density of Gas Clouds

[25] The neutral hydrogen content of late-type spiral galaxies

Assuming a close to circular beam so ($\theta_a \approx \theta_b = \theta = 206304''$ per radian) and noting that in the small angle approximation $\theta \approx 2r/D$, in radians, gives

$$N_{\mathrm{H\,I}} \approx \frac{2.23 \times 10^{28}}{206304^2}\left(\frac{\nu D}{2r}\right)^2 S_{\mathrm{int}} \approx 1.31 \times 10^{17}\left(\frac{\nu D}{r}\right)^2 S_{\mathrm{int}}.$$



Recall that $N_{\mathrm{H\,I}} = \int n_{\mathrm{H\,I}} ds \Rightarrow N_{\mathrm{H\,I}} = \overline{n_{\mathrm{H\,I}}}\ell$, and the mass is related to the density, ρ, via

$$M = \int \rho dV \sim m\overline{n_{\mathrm{H\,I}}}\ell \sim mN_{\mathrm{H\,I}}r^2,$$

where m is the mass of the hydrogen atom.

Substituting in the column density above, gives

$$M \approx 1.29 \times 10^{-10}(\nu D)^2 S_{\mathrm{int}}$$

Inserting $\nu = 1.42$ GHz and converting D from m to Mpc,

$$M = 4.41 \times 10^{-10} \times (1 \times 10^6 \times 3.086 \times 10^{16} D)^2 S_{\mathrm{int}} = 4.20 \times 10^{35} D^2 S_{\mathrm{int}}$$

$$\boxed{M \approx 2.11 \times 10^5 D^2 S_{\rm int} \text{ in solar masses,}}$$

cf. $M \approx 2.36 \times 10^5 D^2 S_{\rm int}$, above.[26] E.g. for Circinus $S_{\rm int}$ = 1330 Jy km s^{-1} $\Rightarrow$ $M \approx 5 \times 10^9$ M$_\odot$.

4. The *velocity dispersion*, $\triangle v$, gives the rotation velocity of the galaxy ($\triangle v \approx \pm 200$ km s^{-1} here), giving the total dynamical mass, $M_{\rm dyn}$, which we now discuss in detail.

4.5 Dark matter

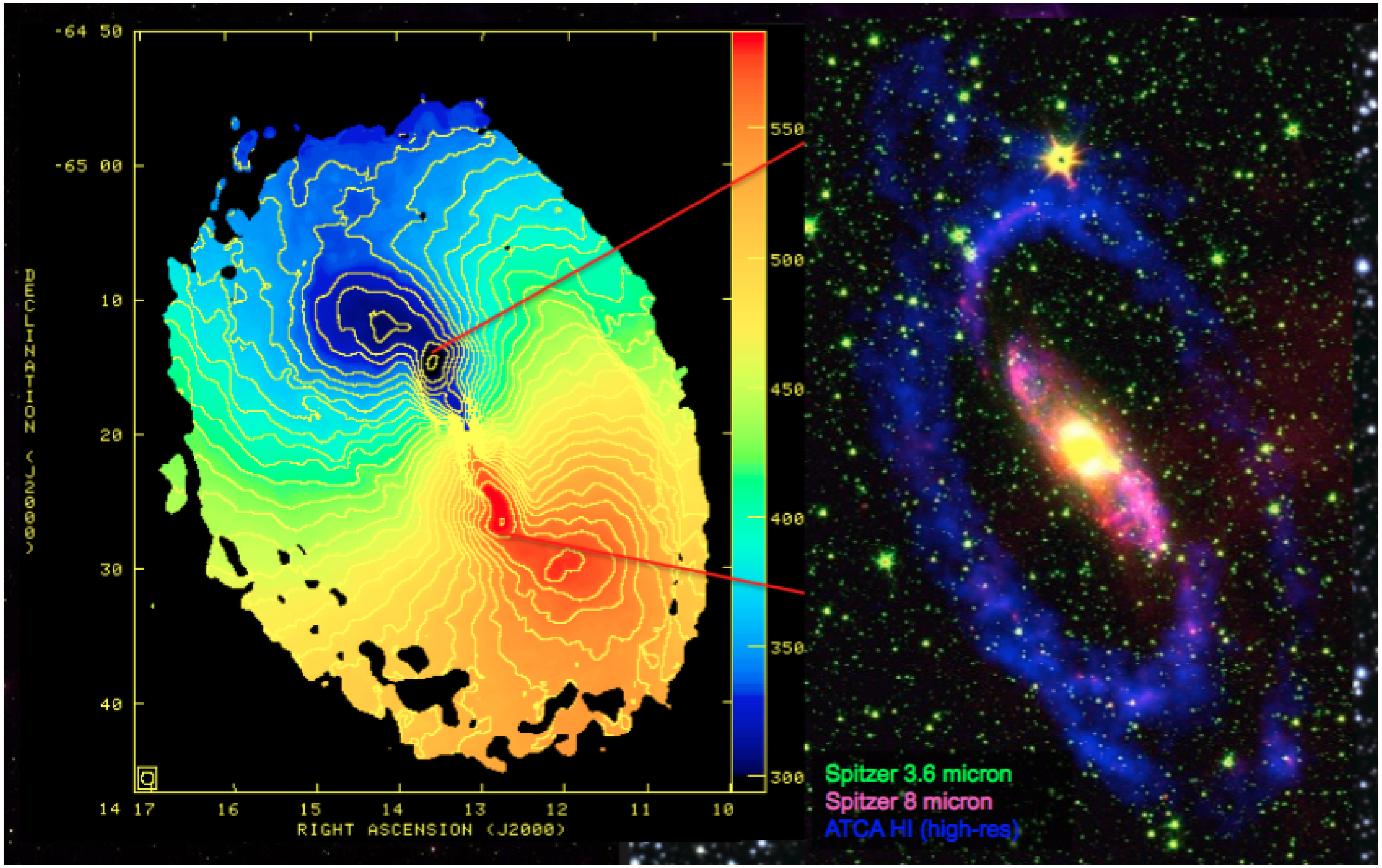
By equating of the Newtonian gravitational and centripetal forces, the total mass enclosed with a radius R is $M_{\rm dyn} = v^2 R/G$, which is generally much larger than all of the EM emitting material can account for [UV, visible, IR, mm, radio].[27]

Below, the panel on the left shows the first moment [velocity dispersion] map of H I emission from the Circinus galaxy. At a distance of 4.2 Mpc, the $\approx 1^\circ$ subtended corresponds to a projected diameter of $d \approx 4.2 \times 10^3 \tan(1) \approx 70$ kpc, which de-projects[28] to ≈ 160 kpc. Comparing this to the near-infrared emission in the right panel, the detection limit of the H I extends much further than the $r \lesssim 10$ kpc of starlight.

[26] Note that if we set $\Omega_{\rm beam} = \theta^2$ (neglecting the $4\ln 2/\pi$ factor), we get $M \approx 2.35 \times 10^5 D^2 S_{\rm int}$. Given the assumptions, it's probably best to stick with $M \sim 2 \times 10^5 D^2 S_{\rm int}$.

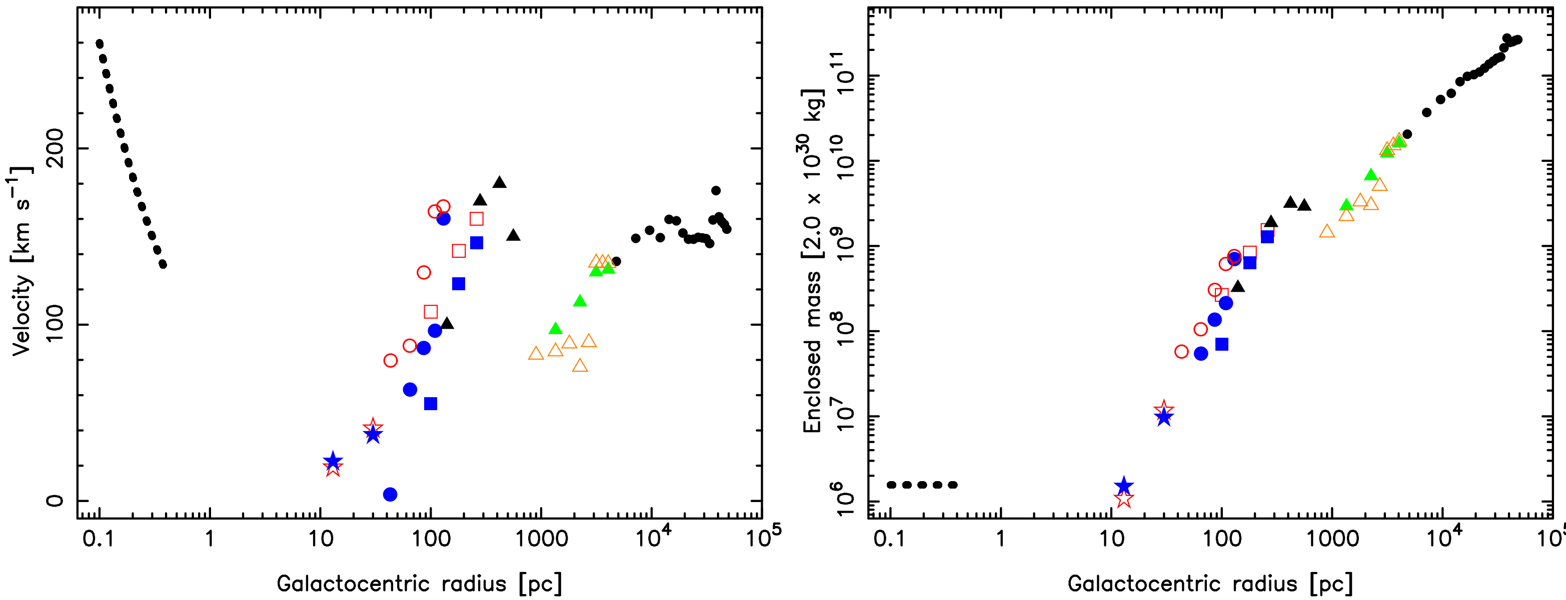
[27] See here and Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions.

[28] Circinus has an inclination of $i = 64^\circ$.



The following plots show the rotational velocities of the various tracers (left panel) and the subsequent dynamical masses (right panel) versus the galactocentric radius:

- The black dotted curve shows the *Keplerian rotation* within the central $\sim$ 1 pc. This is characteristic of orbits around a massive compact object, just as planets around a star. The



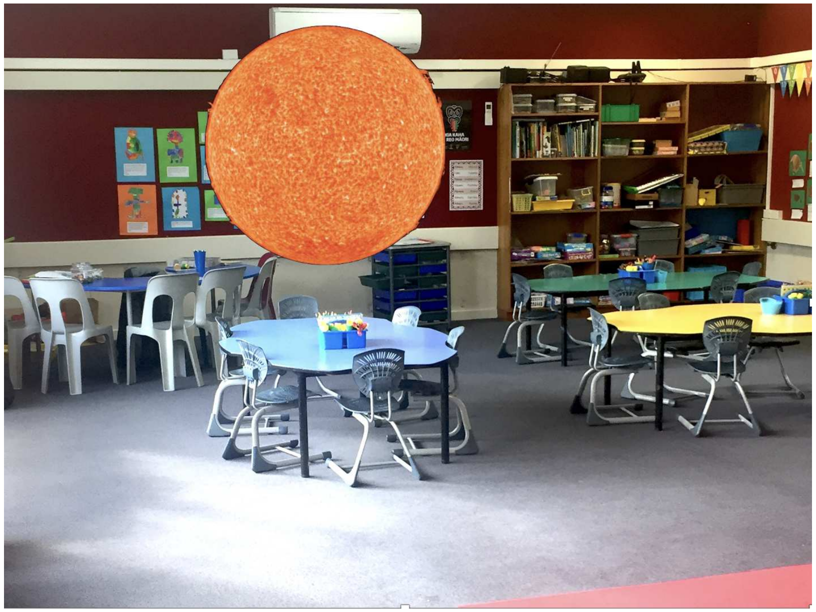
dynamical mass within this region gives $M = 2 \times 10^6$ M$_\odot$ for the super-massive black hole.

- The circles show H_2 data and the filled blue (approaching) and unfilled red (receding) symbols show stellar data, which is too faint to be detected beyond radii of a few hundred pc.
- The triangles show CO ($J = 1 \rightarrow 0$ & $J = 2 \rightarrow 1$) emission, exhibiting the presence of molecular gas out to radii of $\sim$ 5 kpc.
- The black circles show the H I emission which exhibits a *flat rotation curve*.[29] This rotation curve is typical of galaxies and shows that the dynamical mass exceeds 3×10^{11} M$_\odot$ out to the sensitivity of the observations.

[29] It may not look it, but the radius is on a log scale and so the speed is $\triangle v \lesssim \pm 10$ km s^{-1}over $\approx$ 60 kpc.

A flat rotation curve is characteristic of a solid body. However, a galaxy should rotate as a fluid, with the velocity decreasing steeply with radius (as per the Keplerian rotation). From galaxy rotation curves $\sim$ 80% of the mass does not emit detectable electromagnetic radiation.

Let's do a quick comparison: Looking at stars, which are very dense distributions of gas[30], in our neighbourhood, the Sun has radius $R_\odot = 700\,000$ km and next nearest star, Proxima Centauri, is 1.31 pc distant, which is $1.3 \times 3.086 \times 10^{13} = 4.0 \times 10^{13}$ km. That is, $5.7 \times 10^7 R_\odot$.



Or, as I explained to these guys, if the Sun is the size of a desk, Proxima Centauri is 37 000 km away! That is close to the distance of a geostationary orbit (Earth has a radius of 6371 km).

[30]The Sun has a density of $\rho \approx 150\,000$ kg m^{-3} at its core.

Compare this to air at STP ($\rho = 1.22$ kg m^{-3})

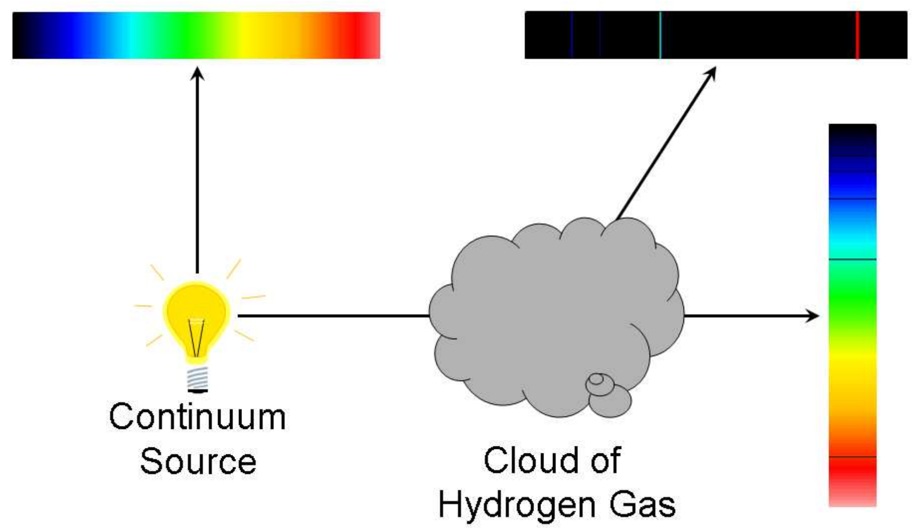
$$\Rightarrow \frac{1.22}{28 \times \text{amu}} \approx 2.6 \times 10^{25} \text{ molecules m}^{-3},$$

assuming mostly N_2 (well it is 80% and O_2 is pretty close in mass).

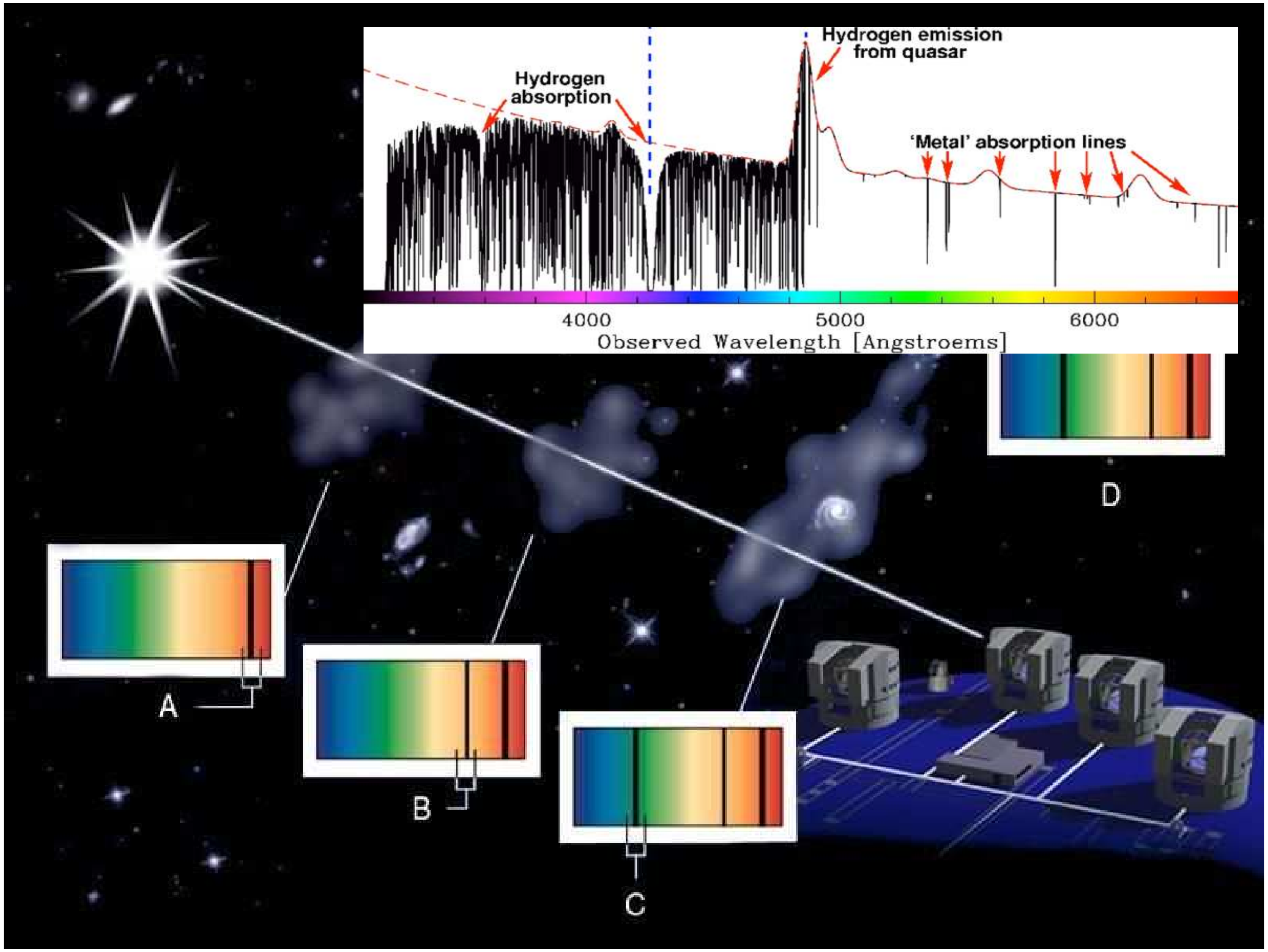
This means that the average air molecule has a volume of 3.8×10^{-26} m^3 all to itself. This gives a linear distance of $(3.8 \times 10^{-26})^{1/3} = 3.4 \times 10^{-9}$ m. The N_2 molecule has a diameter of 3 Å, so the average separation is $3.4 \times 10^{-9}/1.5 \times 10^{-10} \sim 20$ molecular radii (over a million times closer spaced than stars – so galaxies should behave as very rarefied gasses).

4.6 Absorption diagnostics

In addition to emission from warm, diffuse gas, H I can also be detected through absorption by cool, dense gas which is illuminated by a background continuum source. To be absorbed the background source must produce a radio continuum, which covers 1.4 GHz in the rest-frame of the absorber.



A good example is a quasar [31] which can be detected in absorption through Galactic GMCs as well as more distant galaxies.



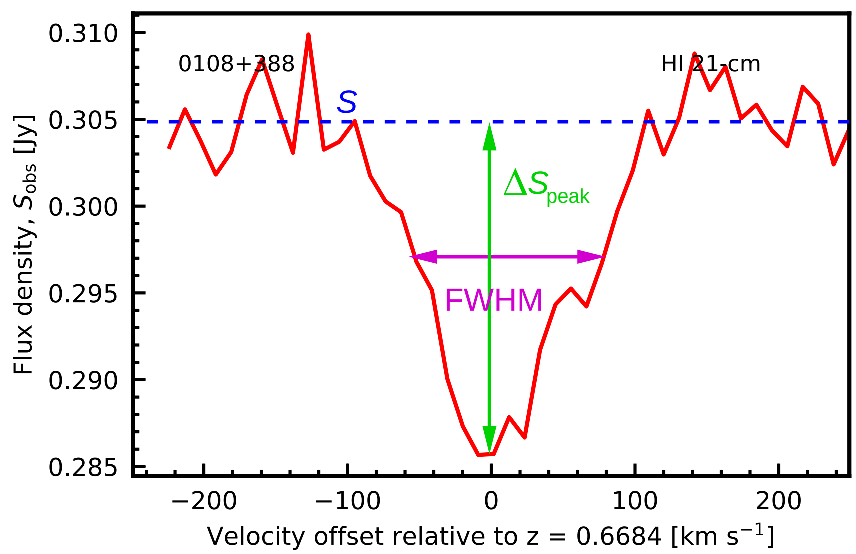
In absorption, the H I column density (obtained either from 21-cm emission, Equ. 6, or Lyman-α

[31] A radio-loud source, as opposed to a *Quasi-Stellar Object* [QSO].

absorption), is related to the velocity integrated optical depth of the line via

$$N_{\rm H\,I} = 1.82 \times 10^{18}\, T_{\rm spin} \int \tau dv,$$

where $T_{\rm spin}$ is the spin temperature and $\int \tau dv$ is the velocity integrated optical depth. However, in the case of absorption, the *observed* optical depth, $\tau_{\rm obs}$, may be not be equal to the intrinsic optical depth, which are related via



$$\tau = -\ln\left(1 - \frac{\tau_{\rm obs}}{f}\right),$$

where $0 \leq f \leq 1$ is the fraction of background flux intercepted by the absorbing gas, the *covering factor*. In the optically thin regime, however, $\tau_{\rm obs} = \triangle S/S_{\rm obs} \lesssim 0.3$, and

$$\tau \approx \frac{\tau_{\rm obs}}{f} \Rightarrow N_{\rm H\,I} = 1.82 \times 10^{18} \frac{T_{\rm spin}}{f} \int \tau_{\rm obs} dv. \qquad (8)$$

So, by assuming $f = 1$, we can obtain the "spin temperature" of the gas via

$$\frac{T_{\rm spin}}{f} = \frac{N_{\rm H\,I}}{1.82 \times 10^{18} \int \tau_\nu dv} \approx \frac{N_{\rm H\,I} S}{1.93 \times 10^{18} \triangle S_{\rm peak} \triangle v},$$ for a Gaussian profile,

where $\triangle v$ is a measure of the line width, usually the full-width half maximum (FWHM).

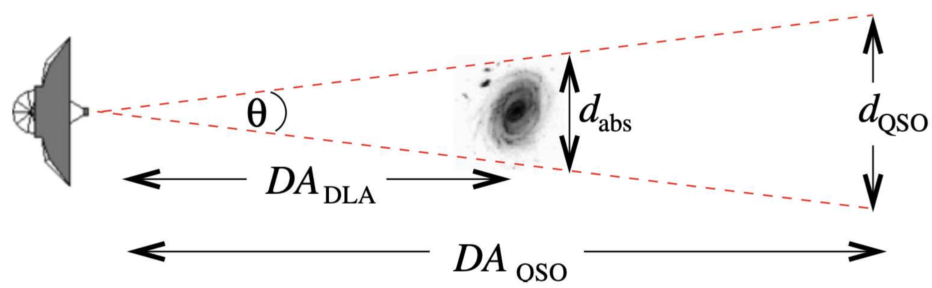
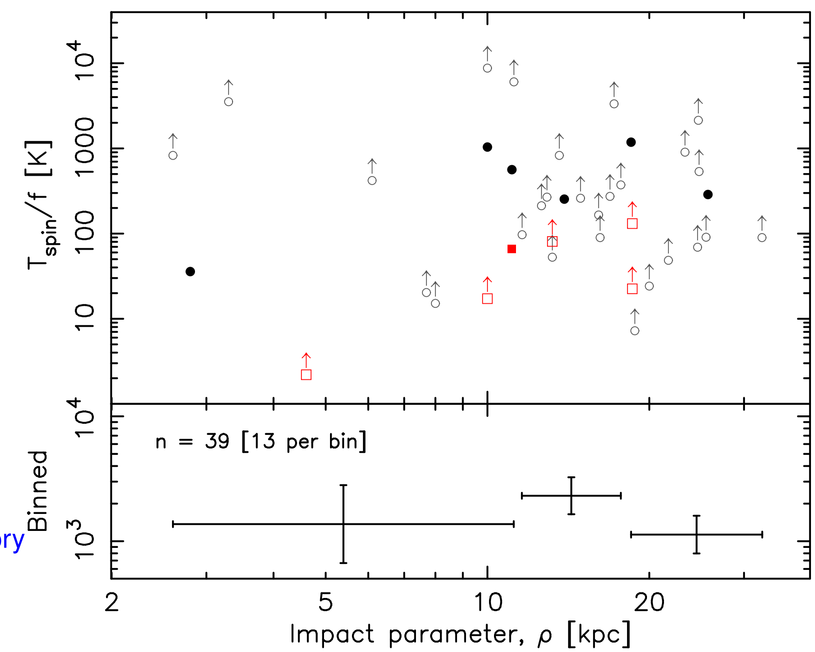


Figure 3. Absorber and quasar cross-sections with respect to the angular diameter distances. If the angle required to fully subtend d_{abs} is less than that to subtend d_{QSO} then $f < 1$.

The Geometry Effects of an Expanding Universe on the Detection of Cool Neutral Gas

However, this assumption is not justified and indeed f may be systematically lower when observing absorption at high redshift. Thus, if you read T_{spin} in an absorption study, it is actually the T_{spin}/f degeneracy.

Another common, but poorly justified, practice is to use Equ. 8 to obtain the column density from $\tau_{obs}\triangle v$, by *assuming* the spin temperature (usually T_{spin} = 100 K, which is of course actually T_{spin}/f = 100 K), when the spin temperature can range from T_{CMB} to $\gtrsim 10^4$ K.[a]



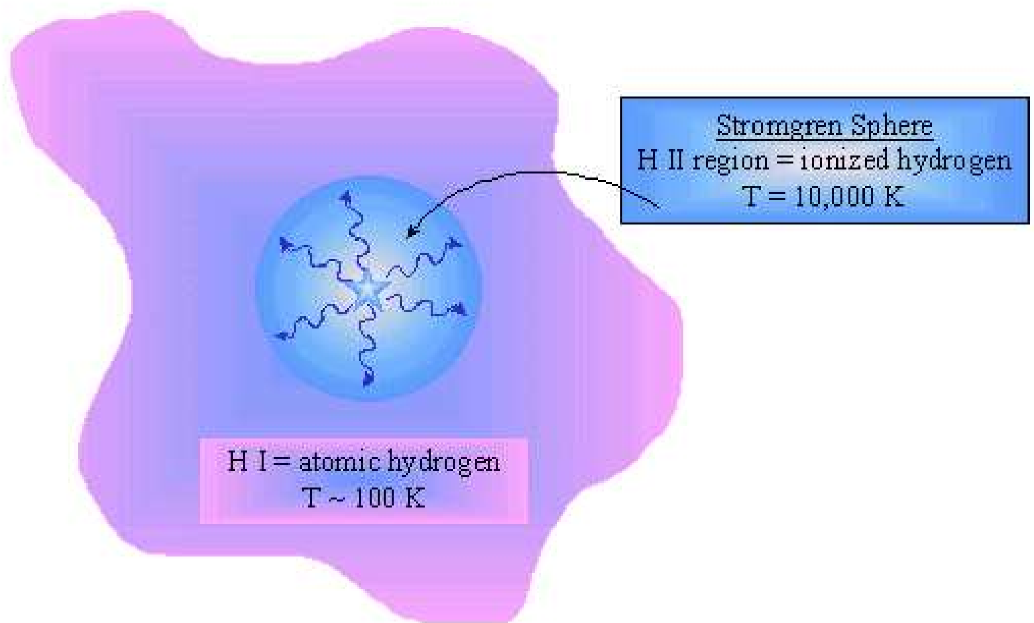
[a]See The evolution of cold neutral gas and the star formation history and High spin temperatures at large impact parameters

5 Ionised Gas

5.1 H II regions

Unlike the Sun (Sect. 2.2), the hottest (most massive) stars (type O/B) emit significant amounts of ionising ($E \geq 13.6$ eV) radiation. As a result, these stars are surrounded by region of ionised hydrogen (a so-called H II region). Because the ionising UV photons are promptly absorbed by neutral hydrogen, the transition between H I and H II is rather abrupt.

Inside the H II region, H^+ ions (protons) and electrons readily recombine but are quickly ionised again by the UV photons. The H II region grows until the rate of recombinations balances the rate of photoionsation. This is photoionsation equilibrium. The gas settles into LTE with a temperature $T \sim 10^4$ K, determined by the balance between heating (due to photo-absorption) and cooling (due to recombination emission).



Ideally, the H II region forms a sphere around the star, known as the Strömgren sphere.

5.2 The Strömgren sphere

Let $Q_\star$ be the number of ionising ($\lambda \leq 912$ Å) photons emitted per second by a source (star/active galactic nucleus). The equilibrium between the photoionsation and the recombination of protons and electrons is given by

$$Q_\star \equiv \int_{\nu_{\rm ion}}^{\infty} \frac{L_\nu}{h\nu}\, d\nu = 4\pi \int_0^{r_{\rm str}} n_{\rm p}\, n_{\rm e}\, \alpha_A\, r^2\, dr, \tag{9}$$

where L_ν is the specific luminosity at frequency ν and h is the Planck constant, giving the number of ionising photons per second.[32] On the right hand side, $r_{\rm str}$ is the extent of the ionisation (the Strömgren radius) and $n_{\rm p}$ and $n_{\rm e}$ are the proton and electron densities, respectively. The radiative recombination rate coefficient of hydrogen, $\alpha_{\rm A}$, is temperature dependent, given by[33]

$$\alpha_{\rm A} = A/\left[\sqrt{\frac{T}{T_0}}\left(1+\sqrt{\frac{T}{T_0}}\right)^{1-B}\left(1+\sqrt{\frac{T}{T_1}}\right)^{1+B}\right],$$

where $A = 8.318 \times 10^{-11}$ cm^3 sec^{-1} , $B = 0.7472$, $T_0 = 2.965$ K and $T_1 = 7.001 \times 10^5$ K.

[32] Osterbrock, 1989, *Astrophysics of Gaseous Nebulae and Active Galactic Nuclei* University Science Books, Mill Valley, California

[33] http://amdpp.phys.strath.ac.uk/tamoc/DATA/RR/

5.2.1 Star surrounded by gas of constant density

For a neutral plasma (i.e. $n_\mathrm{p} = n_\mathrm{e} \equiv n$), the RHS of Equ. 9 is

$$4\pi \int_0^{r_\mathrm{str}} n_\mathrm{p}\, n_\mathrm{e}\, \alpha_A\, r^2\, dr = 4\pi\alpha_\mathrm{A} \int_0^{r_\mathrm{str}} n^2\, r^2\, dr$$

and for a constant density

$$4\pi\alpha_\mathrm{A} n^2 \int_0^{r_\mathrm{str}} r^2\, dr = \frac{4}{3}\pi\alpha_\mathrm{A} n^2 r_\mathrm{str}^3.$$

Equ. 9 therefore becomes,

$$Q_\star = \frac{4}{3}\pi\alpha_\mathrm{A} n^2 r_\mathrm{str}^3$$

giving the Strömgren radius as

$$\boxed{r_\mathrm{str} = \left(\frac{3Q_\star}{4\pi n^2 \alpha_\mathrm{A}}\right)^{1/3}.}$$

To determine the ionising photon rate, $Q_\star$, the specific brightness of a black body is given by

$$B_\nu = \frac{2h\nu^3}{c^2(e^{h\nu/kT} - 1)},$$

and so the specific intensity is $I_\nu = \pi B_\nu$ and the power radiated is

$$L_\nu = \int_{\text{surface}} IdA = 4\pi R^2 I_\nu = 4\pi^2 R^2 B_\nu$$

for a spherical object such as a star.

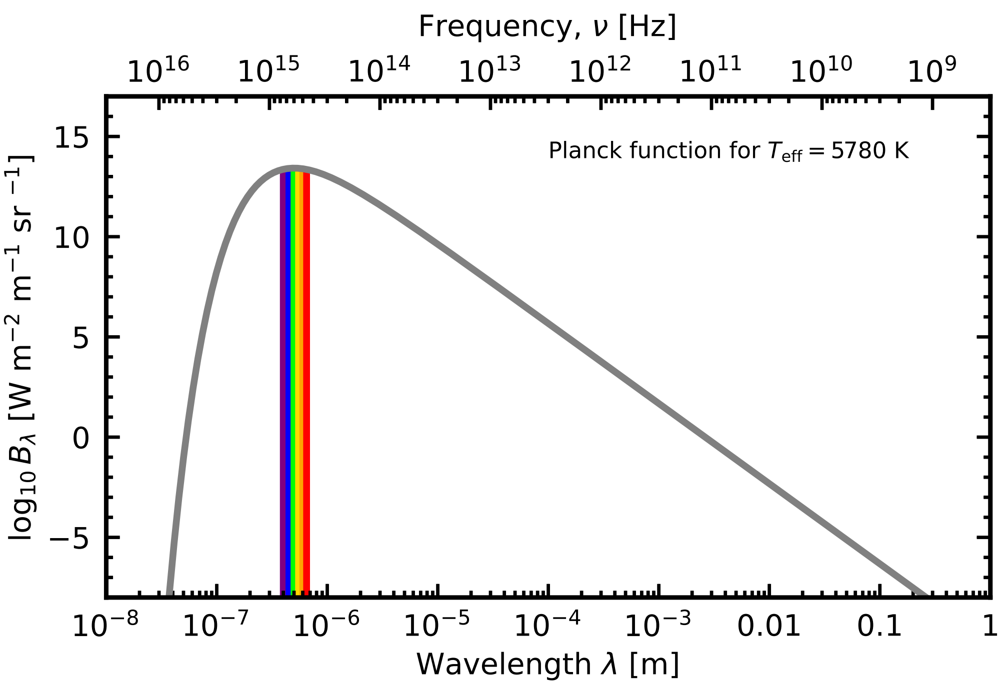
From the LHS of Equ. 9,

$$Q_\star \equiv \int_{\nu_{\text{ion}}}^{\infty} \frac{L_\nu}{h\nu}\, d\nu = 4\pi\, R_\star^2 \int_{\nu_{\text{ion}}}^{\infty} \frac{\pi B_\nu}{h\nu}\, d\nu$$

for a star of temperature T and radius $R_\star$.

$$\Rightarrow Q_\star = 4\pi^2 R_\star^2 \int_{\nu_{\text{ion}}}^{\infty} \frac{2\cancel{h}\nu^3}{c^2 \cancel{h}\nu (e^{h\nu/kT} - 1)}$$

$$= 8\left(\frac{\pi\, R_\star}{c}\right)^2 \int_{\nu_{\text{ion}}}^{\infty} \frac{\nu^2 d\nu}{(e^{h\nu/kT} - 1)}.$$



Given that $h\nu \gg kT$, we can simply this via the *Wien approximation*, $e^{h\nu/kT} - 1 \approx e^{h\nu/kT}$,

$$\Rightarrow Q_\star \approx 8\left(\frac{\pi\, R_\star}{c}\right)^2 \int_{\nu_{\text{ion}}}^{\infty} \frac{\nu^2 d\nu}{e^{h\nu/kT}} = 8\left(\frac{\pi\, R_\star}{c}\right)^2 \left[\left(\frac{kT}{h}\right)^3 e^{-\frac{h\nu}{kT}} \left\{\left(-\frac{h\nu}{kT}\right)^2 + 2\left(\frac{h\nu}{kT}\right) - 2\right\}\right]_{\nu_{\text{ion}}}^{\infty},$$

where the dominant term is ν^2 since $[\nu \gg kT/h]$, giving

$$Q_\star \approx 8 \left(\frac{\pi R_\star \nu_{\rm ion}}{c} \right)^2 \frac{kT}{h} e^{-h\nu_{\rm ion}/kT}, \tag{10}$$

where $h\nu_{\rm ion}$ = 13.6 eV, the ionisation potential of hydrogen.

From Equ. 10, it is seen that the radius of the star is required to obtain the ionising photon rate. This can be estimated by comparing the bolometric intensity, $I = \int_0^\infty I_\nu d\nu$, with the bolometric luminosity obtained from the main sequence for a star of that surface temperature.

E.g. for a type OB star of $T \sim 35\,000$ K, $Q_\star \approx 2 \times 10^{48}$ s^{-1}.

Embedding such a star in interstellar gas, with a density $n = 1$ cm^{-3} at a temperature of $T \sim 10^4$ K ($\alpha_A = 4.19 \times 10^{-13}$ cm^3 s^{-1}), gives

$$r_{\rm str} \sim \left(\frac{3 \times 10^{48}}{4\pi \times (1)^2 \times 10^{-13}} \right)^{1/3}$$

$$\approx 8.2 \times 10^{19} \text{ cm} \approx 30 \text{ pc}.$$

$T_\star$ [K]	20 000	35 000	50 000
$L_{\rm total}$ [$L_\odot$]	4200	2×10^5	2×10^6
Radius [$R_\odot$]	5.4	11	17
Mass [$M_\odot$]	11	31	60
$t_{\rm MS}$ [yr]	3×10^7	2×10^6	4×10^5
$Q_\star$ [s^{-1}]	2.7×10^{46}	1.9×10^{48}	3.4×10^{49}
$N_{\rm stars}$	7.9×10^8	7.0×10^7	1.5×10^7
SFR [$M_\odot$ yr^{-1}]	300	1100	2500

Notes. $L_\star$ is the stellar luminosity estimated from the main sequence, followed by the radius required to equate this to the bolometric intensity obtained from the Planck function. The stellar mass is estimated from the mass–luminosity relation, $L_\star/L_\odot = (M_\star/M_\odot)^{3.5}$, and the main sequence lifetime, $t_{\rm MS}$, from $t_{\rm MS} \propto 1/M^{2.5}$. $N_{\rm stars}$ is the number of stars of this mass according to the Salpeter initial mass function, normalised to a total stellar mass of 3×10^{12} $M_\odot$, followed by the star formation required to maintain this (see main text).

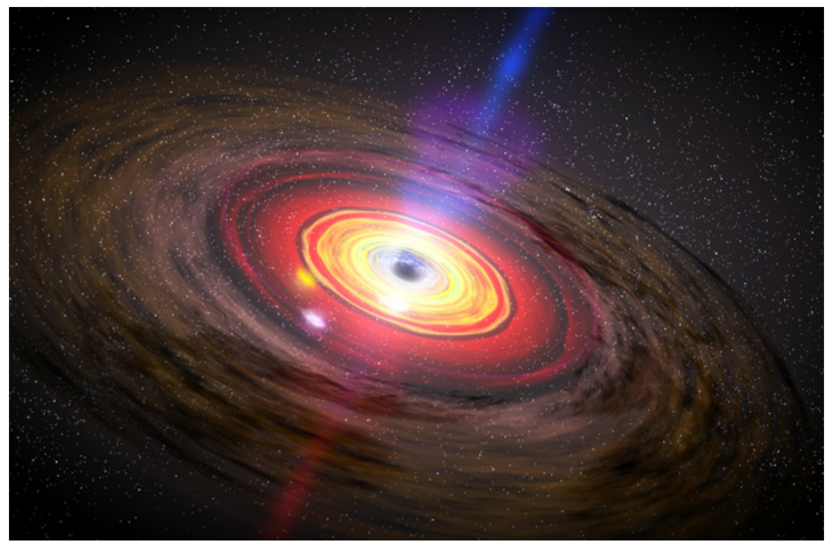
Ultraviolet photo-ionisation in far-infrared selected sources

That is, if there was such a star within 100 light-years of us, we'd be getting fried within its H II region. Just as well these don't live very long (as you should know by now).

5.2.2 Ionising source surrounded by gas of non-constant density

Not all ionising sources are, however, stars. Quasars, located within the nuclei of some galaxies (active galactic nuclei, AGN), also ionise the surrounding gas, suppressing star formation.

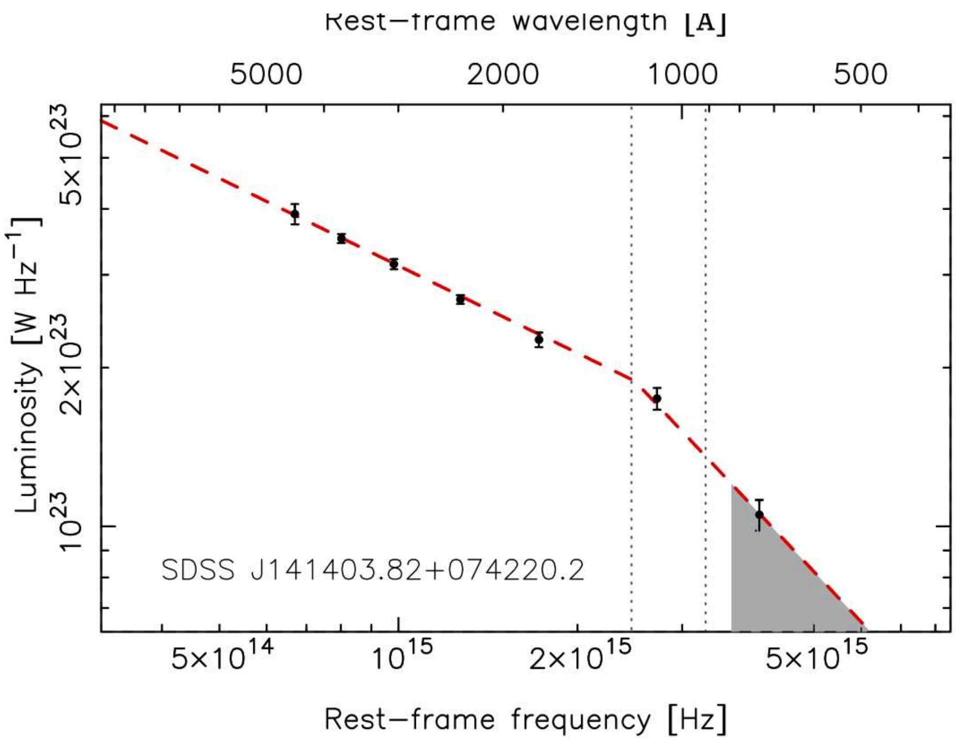
Given that the optical-UV emission from a quasar, or in this case a *quasi-stellar object* (QSO), arises from accretion of material onto the central super-massive black hole, the spectral energy distribution (SED) comprises multiple black bodies of different temperatures resulting in a power-law in this part of the SED.



A power-law is a linear fit in log-log space, the gradient of which gives the spectral index, α, of the SED. Specifically, the luminosity of the source at a given frequency is $L_\nu \propto \nu^\alpha$. So for a power law

$$\log_{10} L_\nu = \alpha \log_{10} \nu + \mathcal{C} \Rightarrow L_\nu = 10^{\mathcal{C}} \nu^\alpha,$$

where $\mathcal{C}$ is the intercept in log space.



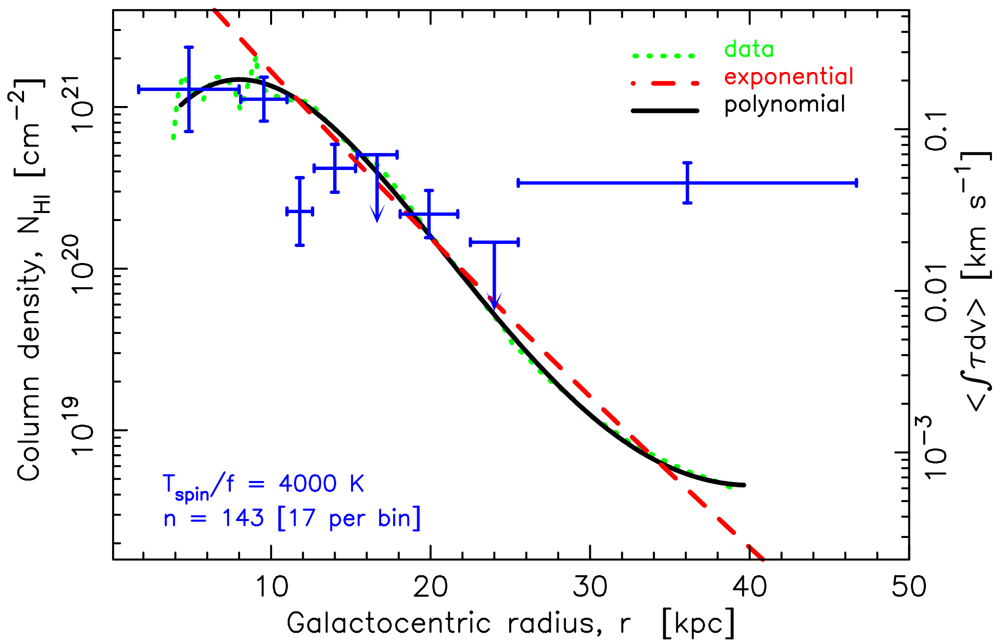
Therefore, the ionising photon rate is given by

$$Q \equiv \int_{\nu_{\text{ion}}}^{\infty} \frac{L_\nu}{h\nu}\, d\nu = \frac{10^{\mathcal{C}}}{h} \int_{\nu_{\text{ion}}}^{\infty} \nu^{\alpha-1}\, d\nu = \frac{10^{\mathcal{C}}}{\alpha h} \left[\nu^\alpha\right]_{\nu_{\text{ion}}}^{\infty} = \frac{-10^{\mathcal{C}}}{\alpha h} \nu_{\text{ion}}^\alpha, \text{ where } \alpha < 0.$$

E.g. if $L_\nu = 2 \times 10^{23}$ W Hz^{-1} at $\nu_{\text{ion}} = 3.29 \times 10^{15}$ Hz and $\alpha = -2$, then $\mathcal{C} = 54.33$ and

$$Q = \frac{-10^{54.33}}{-2 \times 6.63 \times 10^{-34}} (3.29 \times 10^{15})^{-2} = 1.5 \times 10^{56} \text{ s}^{-1}$$

Furthermore, over these radii a constant density is not a good approximation: Spiral galaxies exhibit an exponential gas density profile of the form $n = n_0\, e^{-r/R}$, where n_0 is the gas density at $r = 0$ and R is a scale-length describing the rate of decay of this with radius.



In this case, the RHS of Equ. 9 becomes

$$4\pi \int_0^{r_{\rm str}} n_{\rm p}\, n_{\rm e}\, \alpha_A\, r^2\, dr$$

$$= 4\pi\, \alpha_A\, n_0^2 \int_0^{r_{\rm str}} e^{-2r/R}\, r^2\, dr$$

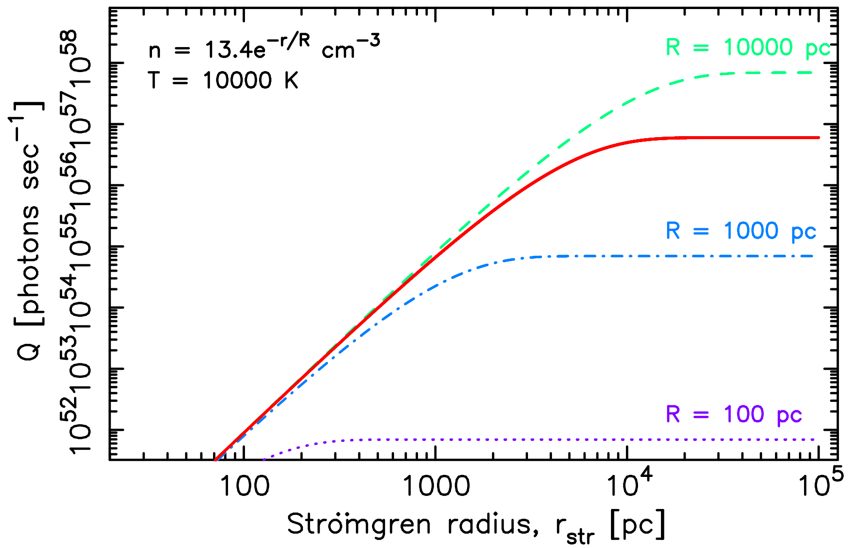
$$= \pi\, \alpha_A\, n_0^2 \left[R^3 - R\, e^{-2r_{\rm str}/R} \left(2\, r_{\rm str}^2 + 2\, r_{\rm str} R + R^2 \right) \right].$$

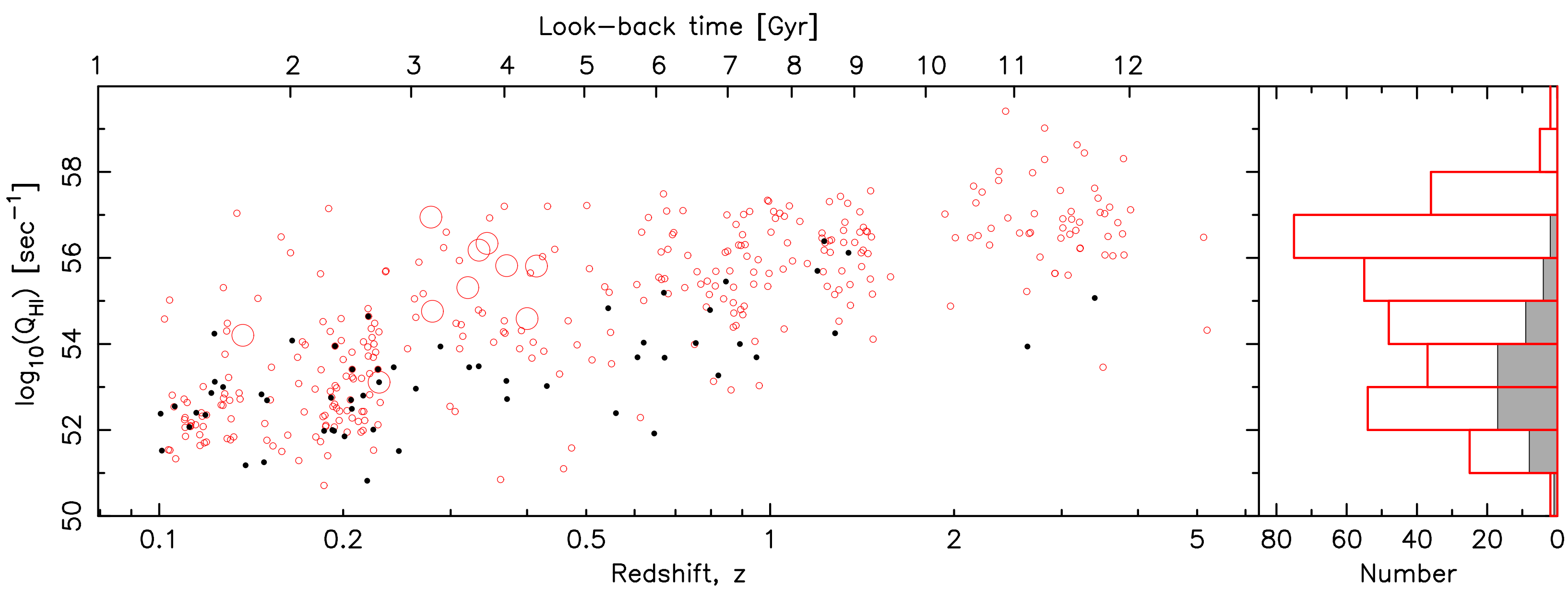
Unlike the constant density distribution, this becomes independent of r at sufficiently large radii, i.e. $Q \to \pi\, \alpha_A\, n_0^2\, R^3$ and $r_{\rm str} \to \infty$.

Conversely, for a given scale-length, R, there *always* exists a "ceiling luminosity" ($Q \times h$) for which *all* of gas is ionised. That is, *a finite ionising rate gives an infinite Strömgren radius.*[34]

For the gas distribution of the Milky Way (n_0 = 13.4 cm^{-3} and R = 2.9 kpc), a large spiral galaxy, this is Q = 6.0 × 10^{56} s^{-1} at T = 10 000 K, close to the limit where H I absorption has ever been detected (Q = 2.5 × 10^{56} s^{-1}).[a]

[a] Ionisation of the Atomic Gas in Redshifted Radio Sources



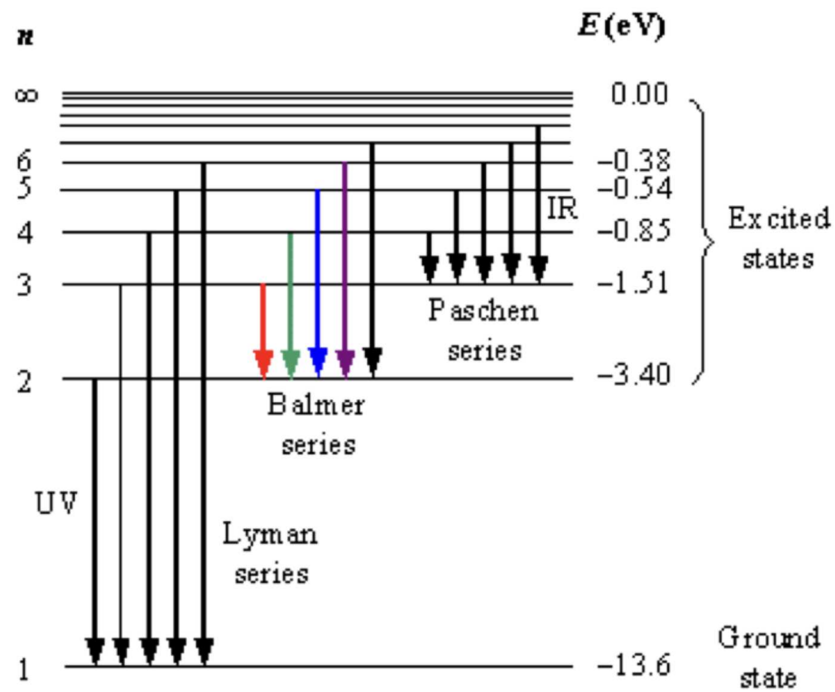


[34] Complete Ionisation of the Neutral Gas: Why There Are So Few Detections of 21-cm Hydrogen in High Redshift Radio Galaxies and Quasars

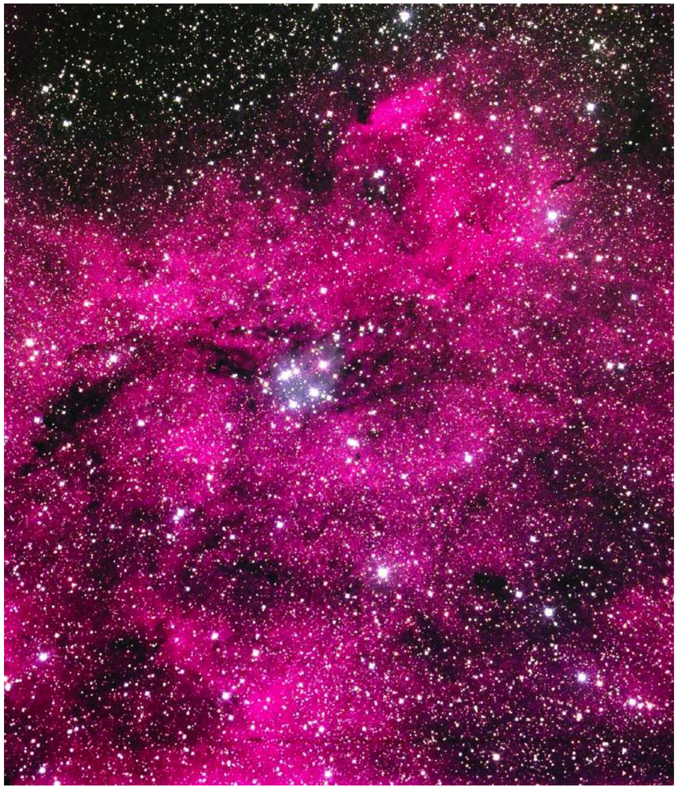
5.3 Tracing ionised gas

5.3.1 Spectral lines

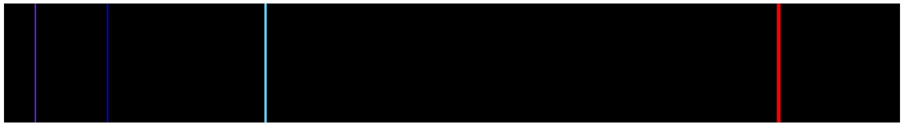
The presence of ionised gas can be inferred from the recombination lines of hydrogen. The strongest is the Lyman-α $n = 2 \to 1$ transition – although at $n = 2$ the atom is still neutral, the level is at 75% that required for ionisation. Furthermore, the lifetime in this state is only $\sim 10^{-8}$ s and so Lyman-α emission is detected where ionised gas is being recombined (cf. Lyman-α absorption which traces the neutral hydrogen, Sect. 4.6).



However, with a rest frequency of $\lambda = 1216$ Å, the $n = 2 \to 1$ transition occurs in the ultra-violet band and can only be detected by ground-based telescopes at redshifts of $z \gtrsim 1.7$. Therefore, apart from space-based observations (HST), Lyman-α, in absorption, is used primarily in the study of neutral hydrogen in the very distant Universe, e.g. in damped Lyman-α absorbers (gas-rich galaxies) and the Lyman-α forest (more diffuse gas located between us and distant QSOs).



During recombination an electron undergoing the $n = 3 \rightarrow 2$ Balmer transition emits so called Hα emission, one of the strongest emission lines in galaxy spectra. Although a tracer of ionised hydrogen (via recombination), the drop in energy is relatively small corresponding to a wavelength of $\lambda = 656$ nm, giving the deep red colour observed in Galactic emission spectra and in images of other galaxies.

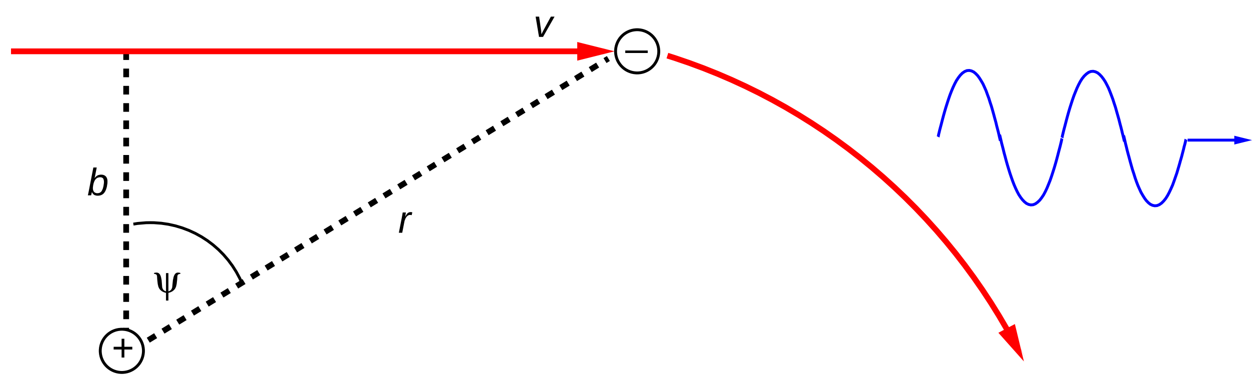


The four visible hydrogen emission spectrum lines in the Balmer series.

5.3.2 Continuum emission

The ionised gas itself cannot emit spectral lines but does emit continuum radiation in the radio band due to thermal bremsstrahlung.[35] In an H II region this is known as *free-free emission*, since it is produced by electrons accelerated by ions without being captured. We will discuss this in context of ionised interstellar gas.

[35] Thermal radiation is dependent on the temperature of the emitter, cf, non-thermal such as electrons trapped in magnetic fields.



A fast electron passing a heavy ion is deflected to a parabolic trajectory, although the weak interaction[36] means we can approximate this as a straight line, giving the impact parameter $b = r\cos\psi$ (closest approach).

We can equate the parallel and perpendicular components of the acceleration with the electrostatic force from a charge *Ze* on the electron [Coulomb's Law]

$$m_e a_{\parallel} = \frac{Ze^2}{4\pi\epsilon_0 r^2}\sin\psi = \frac{Ze^2 \sin\psi\cos^2\psi}{4\pi\epsilon_0 b^2},$$
$$m_e a_{\perp} = \frac{Ze^2}{4\pi\epsilon_0 r^2}\cos\psi = \frac{Ze^2 \cos^3\psi}{4\pi\epsilon_0 b^2},$$

Free accelerated charges radiate according to *Larmor's formula*

$$P = \frac{2}{3}\frac{e^2 a_{\perp}^2}{4\pi\epsilon_0 c^3}.$$

[36] $E_{\text{photon}} = h\nu \sim 10^{-24}$ J $\ll E_k$, resulting in a deviation of $\psi \ll 1$ radian.

Only the component of the electric field perpendicular to the line-of-sight contributes to the electromagnetic radiation at large distances, giving the instantaneous power emitted as

$$P = \frac{2e^2}{3(4\pi\epsilon_0)^3 c^3} \frac{Z^2 e^4}{m_e^2} \left(\frac{\cos^3\psi}{b^2} \right)^2 ,$$

where the total energy emitted by the pulse is $W = \int_{-\infty}^{\infty} P dt$.

Given $E_{\text{photon}} \ll E_k$, we make the approximation of near constant velocity for the electron, giving the displacement along the direction of travel as

$$x = b \tan\psi \Rightarrow \frac{dx}{d\psi} = \frac{b}{\cos^2\psi}$$

and

$$v \equiv \frac{dx}{dt} = \frac{dx}{d\psi}\frac{d\psi}{dt} = \frac{b}{\cos^2\psi}\frac{d\psi}{dt} \Rightarrow dt = \frac{b}{v}\frac{d\psi}{\cos^2\psi}.$$

Therefore,

$$W = \frac{2Z^2 e^6}{3(4\pi\epsilon_0)^3 c^3 m_e^2 b^4} \int_{-\infty}^{\infty} \cos^6\psi dt = \frac{2Z^2 e^6}{3(4\pi\epsilon_0)^3 c^3 m_e^2 b^3 v} \int_{-\pi/2}^{\pi/2} \cos^4\psi d\psi$$

$$\Rightarrow W = \frac{Z^2 e^6}{3(4\pi\epsilon_0)^3 c^3 m_e^2 b^3 v} \int_0^{\pi/2} \cos^4 \psi d\psi,$$

and $\int_0^{\pi/2} \cos^4 \psi d\psi = 3\pi/16$, giving

$$W = \frac{\pi Z^2 e^6}{16(4\pi\epsilon_0)^3 c^3 m_e^2} \left(\frac{1}{b^3 v}\right),$$

for the energy radiated by a single electron moving with velocity v and passing within b of an ion of charge Ze.

While $r \approx b$, the "period" of the electron path around the ion is $\approx 2\pi b/v$, giving the maximum frequency as $\nu_{max} \approx v/(2\pi b)$ and so we can define the average energy per unit frequency as

$$W_\nu \approx \frac{W}{\nu_{max}} = \left(\frac{\pi Z^2 e^6}{16(4\pi\epsilon_0)^3 c^3 m_e^2 b^3 v}\right)\left(\frac{2\pi b}{v}\right) = \frac{\pi^2}{8} \frac{Z^2 e^6}{(4\pi\epsilon_0)^3 c^3 m_e^2} \left(\frac{1}{bv}\right)^2.$$

At $T = 10^4$ K, the rms speed of an electron is 6.7×10^5 m s^{-1}, giving $\nu < \nu_{max} \approx 10^5/b$, which, for a typical $b \sim 10^{-9}$ m, gives $\nu_{max} \sim 10^{14}$ Hz. This is in the near-infrared and so $\nu_{max} \gg \nu$ at radio frequencies ($\sim 10^9$ Hz).

Conversely, we can obtain the maximum impact parameter which will emit significant power at frequency ν from

$$\boxed{b_{max} \approx \frac{v}{2\pi\nu},}$$

which, for the $\nu \sim 1$ GHz typically observed, gives $b_{max} \sim 10^{-4}$ m.

For the minimum impact parameter, the maximum possible momentum transfer is double the impulse,

$$m_e \triangle v = \int_{-\infty}^{\infty} F dt = \frac{Ze^2}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\cos^3\psi}{b^2} dt, \text{ for the perpendicular component.}$$

Again, using

$$dt = \frac{b}{v} \frac{d\psi}{\cos^2\psi} \Rightarrow m_e v = \frac{Ze^2}{4\pi\epsilon_0 bv} \int_{-\pi/2}^{\pi/2} \cos\psi d\psi = \frac{Ze^2}{2\pi\epsilon_0 bv},$$

giving

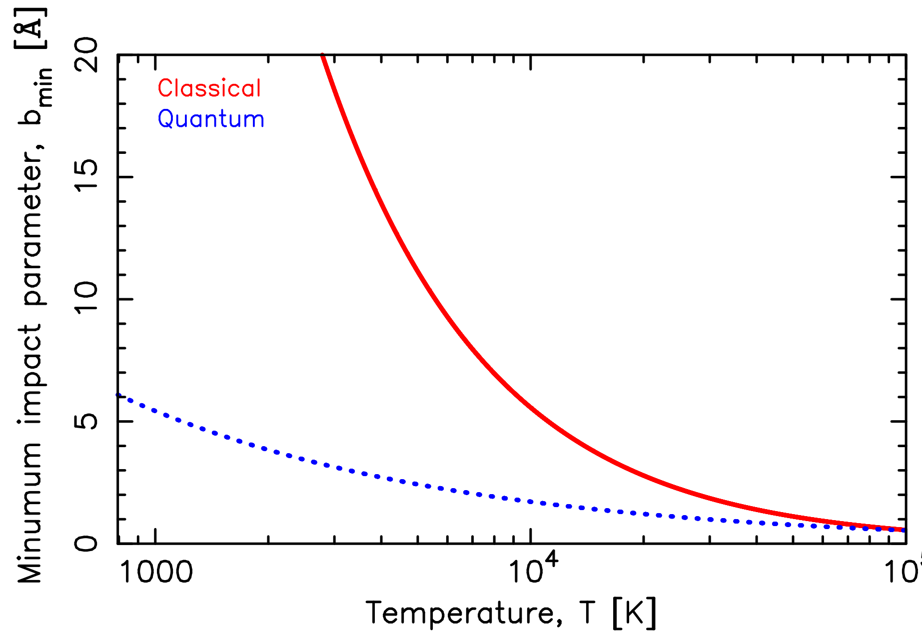
$$\boxed{b_{min} \approx \frac{Ze^2}{4\pi\epsilon_0 m_e v^2}.}$$

E.g. $b_{min} \approx 5.6 \times 10^{-10}$ m for ionised hydrogen at $T = 10^4$ K.

Note, however, that there is a limit dictated by the Heisenberg uncertainty ($\triangle x \approx \hbar/\triangle p$) giving

$$b_{min} = \frac{\hbar}{m_e v},$$

which is significantly smaller than the classical limit, although they converge at $T \sim 10^5$ K.



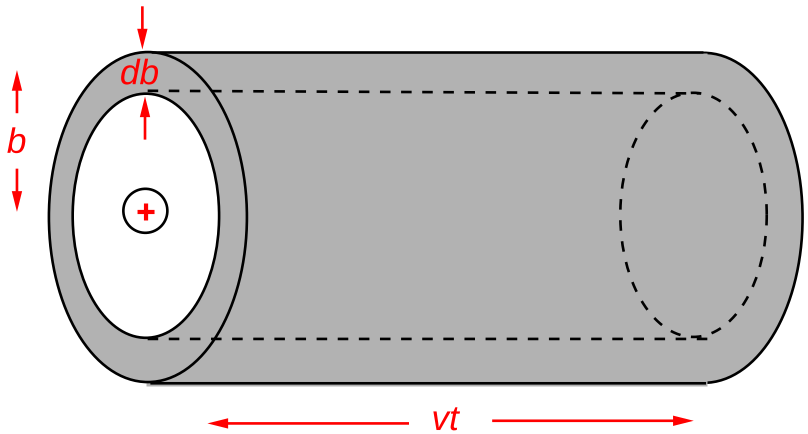
The fact that the opacity depends upon the frequency gives an H II region a distinctive spectrum.

Recall that the optical depth is given by $\tau_\nu = \int \kappa_\nu ds$, where κ_ν is the absorption coefficient. In local thermodynamic equilibrium, this is given by the ratio of the emission coefficient and the Planck function

$$\kappa_\nu = \frac{\epsilon_\nu}{B_\nu} \approx \frac{\epsilon_\nu c^2}{2kT\nu^2}, \text{ in the Rayleigh-Jeans limit.} \tag{11}$$

For N electrons in the volume $\pi b^2 vt$, the number of electron-ion encounters per unit volume, n, per unit time between impact parameters $b + db$ of speed $v + dv$ is

$$ndvdb = n_e n_i 2\pi\, bdb\, vf(v)dv,$$



where n_e and n_i are the electron and ion volume densities, respectively, and $f(v)$ is the normalised [$\int f(v)dv = 1$] speed distribution of the electrons.

For emission coefficient ϵ_ν [J s^{-1} m^{-3} sr^{-1}], the spectral power at frequency ν emitted isotropically [J s^{-1} m^{-3}] is

$$4\pi\epsilon_\nu = \int_{b=0}^{\infty}\int_{v=0}^{\infty} W_\nu n dv db = \int_{b=0}^{\infty}\int_{v=0}^{\infty} \frac{\pi^2}{8}\frac{Z^2 e^6}{(4\pi\epsilon_0)^3 c^3 m_e^2}\left(\frac{1}{bv}\right)^2 n_e n_i \, 2\pi\, b db\, v f(v) dv,$$

$$\Rightarrow \epsilon_\nu = \frac{\pi}{16}\frac{Z^2 e^6 n_e^2}{(4\pi\epsilon_0)^3 c^3 m_e^2}\int_{v=0}^{\infty}\frac{f(v)}{v}dv\int_{b=0}^{\infty}\frac{db}{b}, \text{ for a neutral plasma } [n_e = n_i]. \tag{12}$$

For the first integral, we use the Maxwell–Boltzmann distribution for the velocity

$$f(v) = \frac{4}{\sqrt{\pi}}\left(\frac{m_e}{2kT}\right)^{3/2} v^2 \exp\left\{-\frac{m_e v^2}{2kT}\right\}.$$

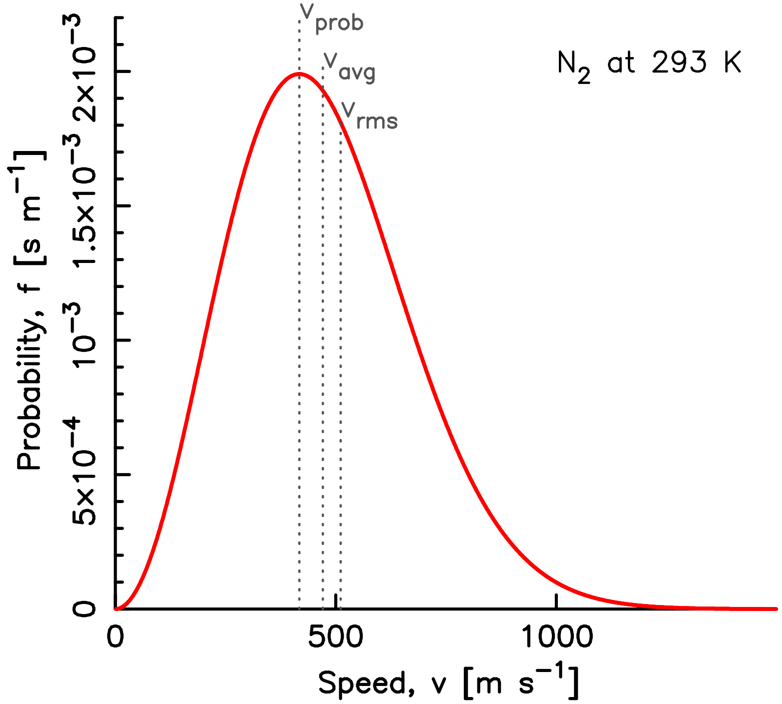
Hence,

$$\int_{v=0}^{\infty} \frac{f(v)}{v} dv = \frac{4}{\sqrt{\pi}} \left(\frac{m_e}{2kT}\right)^{3/2} \int_{v=0}^{\infty} v \exp\left\{-\frac{m_e v^2}{2kT}\right\} dv,$$

$$= -\frac{4}{\sqrt{\pi}} \left(\frac{m_e}{2kT}\right)^{3/2} \frac{kT}{m_e} \left[exp\left\{-\frac{m_e v^2}{2kT}\right\}\right]_{v=0}^{\infty}$$

which gives

$$\int_{v=0}^{\infty} \frac{f(v)}{v} dv = \left(\frac{2m_e}{\pi kT}\right)^{1/2}.$$



The other integral diverges logarithmically

$$\int_{b=0}^{\infty} \frac{db}{b} = [\ln b]_{b=0}^{\infty},$$

and so, in order to impose physical limits on the impact parameter, we use the values of b_{min} and b_{max}, above, giving

$$\int_{b_{min}}^{b_{max}} \frac{db}{b} = \ln\left(\frac{b_{max}}{b_{min}}\right).$$

Inserting these into Equ. 12, gives the *free-free emission coefficient*

$$\boxed{\epsilon_\nu = \frac{\pi}{16} \frac{Z^2 e^6 n_e^2}{(4\pi\epsilon_0)^3 c^3 m_e^2} \left(\frac{2m_e}{\pi k T}\right)^{1/2} \ln\left(\frac{b_{max}}{b_{min}}\right).}$$

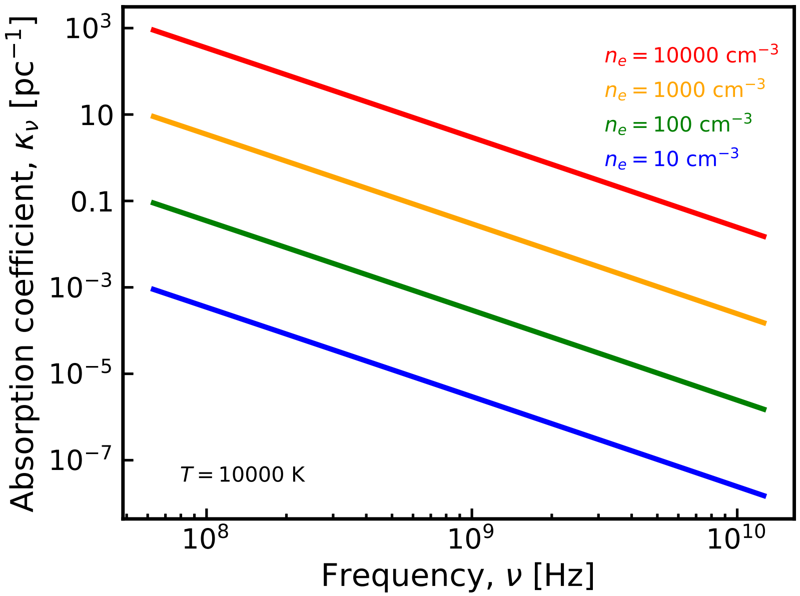
Inserting this into Equ. 11 gives the *absorption coefficient*

$$\boxed{\kappa_\nu = \frac{\sqrt{2\pi}}{32} \frac{Z^2 e^6}{(4\pi\epsilon_0)^3} \frac{n_e^2}{c(m_e k T)^{3/2} \nu^2} \ln\left(\frac{b_{max}}{b_{min}}\right).}$$

For hydrogen ($Z = 1$),

$$\kappa_\nu = 7.192 \times 10^{-14} \left(\frac{n_e}{\nu}\right)^2 \frac{1}{T^{3/2}} \ln\left(\frac{3kT}{\nu h}\right)$$

using the non-classical b_{min}, which then gives

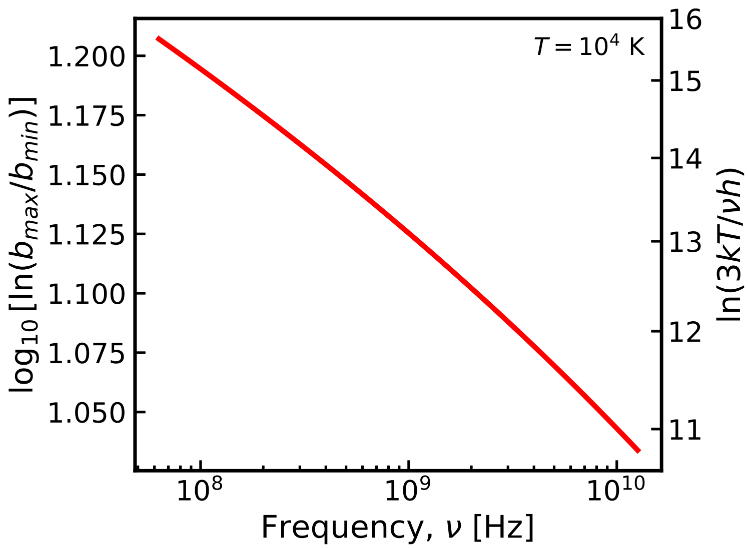


$$\kappa_\nu = 2.219 \times 10^{15} \left(\frac{n_e}{\nu}\right)^2 \frac{1}{T^{3/2}} \ln\left(\frac{3kT}{\nu h}\right) [\mathrm{pc}^{-1}],$$

where n_e is in cm^{-3}.

The opacity, $\tau_\nu = \int \kappa_\nu ds$ (Sect. 2.1.2), for a constant density is

$$\tau_\nu = 2.219 \times 10^{15} \left(\frac{n_e}{\nu}\right)^2 \frac{1}{T^{3/2}} \ln\left(\frac{3kT}{\nu h}\right) s.$$



The plot of $\ln(b_{max}/b_{min})$ has a gradient of ≈ -0.08 in log space, i.e.

$$\ln\left(\frac{b_{max}}{b_{min}}\right) \propto \nu^{\approx -0.08},$$

giving at $T = 10\,000$ K

$$\tau_\nu \approx 7.56 \times 10^{16} \frac{n_e^2}{T^{3/2}\nu^{2.08}} s.$$

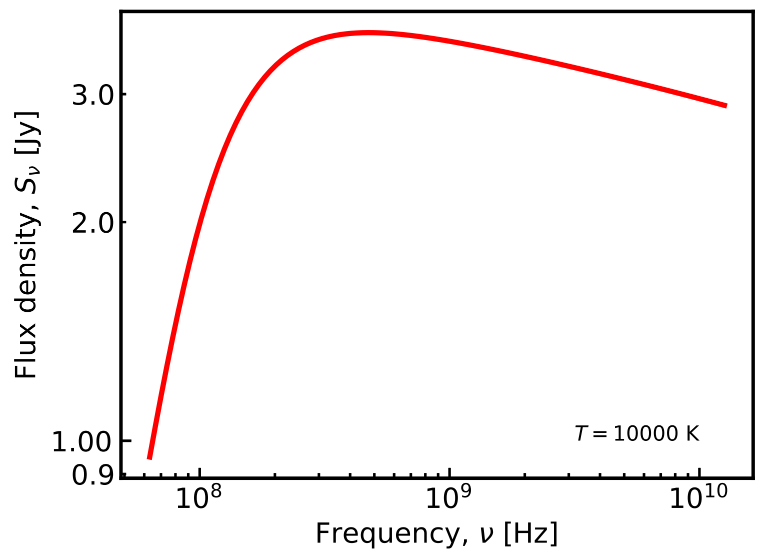
Scaling the brightness of the emission by the optical depth, gives the *observed flux*

$$S_\nu \propto (1 - e^{-\tau_\nu}) B_\nu,$$

where $B_\nu = 2kT\nu^2/c^2$.

This gives the characteristic spectrum with a spectral break (turnover, where $\tau_\nu \approx 1$) in the radio-band.

At frequencies below the turnover ($\nu < \nu_{\rm TO}$), the spectral index $\alpha \approx 2$ and the plasma is opaque ($\tau_\nu \gtrsim 1$) with the spectrum of a black body ($B_\nu \propto \nu^2$).
Above the turnover ($\nu > \nu_{\rm TO}$), $\alpha \approx 2 - 2.08 \approx -0.1$ the plasma is transparent ($\tau_\nu \lesssim 1$) and the spectrum departs from the Planck function becoming nearly flat.



For example, the above spectrum, of a 1 kpc distant H II region, exhibits a turnover at $\nu_{\rm TO}$ = 440 MHz, where S_ν = 3.67 Jy and has a flux density of 2.0 Jy at 0.1 GHz.

The flux from a source at distance d is $S_\nu = L_\nu / 4\pi d^2$, where the luminosity of the source is

$$L_\nu = \int_{\rm surface} I_\nu dA = 4\pi R^2 I_\nu = 4\pi^2 R^2 B_\nu$$

for a spherical emission region of radius R.

If the distance to the source is much larger than this ($d \gg R$), we can treat this is a point source giving

$$S_\nu \propto (1 - e^{-\tau_\nu})B_\nu = \frac{L_\nu}{4\pi d^2} = \frac{\cancel{4}\pi^{\cancel{2}} R^2 B_\nu}{\cancel{4}\pi d^2}$$

$$\Rightarrow S_\nu = (1 - e^{-\tau_\nu})\pi \left(\frac{R}{d}\right)^2 B_\nu.$$

In the optically thick part of the spectrum ($\tau_\nu \gg 1$), $1 - e^{-\tau_\nu} \approx 1$, and in the Rayleigh-Jeans limit, $B_\nu \approx 2(\nu/c)^2 kT$,

$$S_\nu \approx 2\pi kT \left(\frac{R\nu}{dc}\right)^2,$$

$$\Rightarrow R = d\frac{c}{\nu}\sqrt{\frac{S_\nu}{2kT}}.$$

Assuming $T = 10\,000$ K,

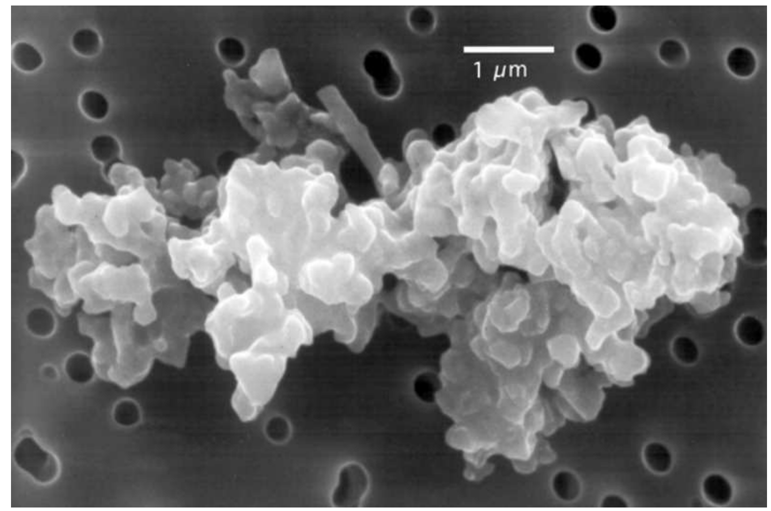
$$R = d \times \frac{3.0 \times 10^8}{0.1 \times 10^9}\sqrt{\frac{2.0 \times 10^{-26}}{2 \times 1.38 \times 10^{-23} \times 10^4}} = 8.1 \times 10^{-4} d.$$

So for a 1000 pc distant source, the radius of the Strömgren sphere is $R \approx 0.8$ pc.

6 Dust

6.1 Interstellar Dust

Dust grains, with sizes ranging from nanometres to micrometres, are intimately mixed with the interstellar gas. Some infra-red data suggests silicates, graphite and even iron may be present in the dust grains, mixed with water ice, methane and ammonia. This implies that their composition may be similar to that of cometary nuclei. The partially-polarised light produced by dust suggests that the grains are elongated in shape.



Dust grains are formed in the atmospheres of evolved stars (red giants, super giants) as well as in novae and supernovae. Dust emission is also found to correlate well with the spatial distribution of hydrogen. Despite representing a tiny fraction of the interstellar mass, dust plays several important roles in the ISM:

1. *Interstellar extinction* – dust absorbs and scatters starlight; because scattering is wavelength dependent and is most efficient towards shorter wavelengths, it causes interstellar reddening.

2. *Infrared emission* – the radiative energy absorbed by dust grains heats them up and is then re-radiated in the mid/far-IR.

3. *ISM chemistry* – atoms and molecules adhere to dust grains, forming a surface mantle on which chemical reactions can take place; heating then releases the mantle, ejecting new compounds, elements and even electrons into the ISM.

Interstellar dust particles have low densities but their sizes, $\sim 10^{-7}$ m, are comparable to the wavelength of visible light, which they therefore scatter. This is stronger for shorter wavelength light (ultra-violet), see here.

6.2 Dust emission

6.2.1 Temperature

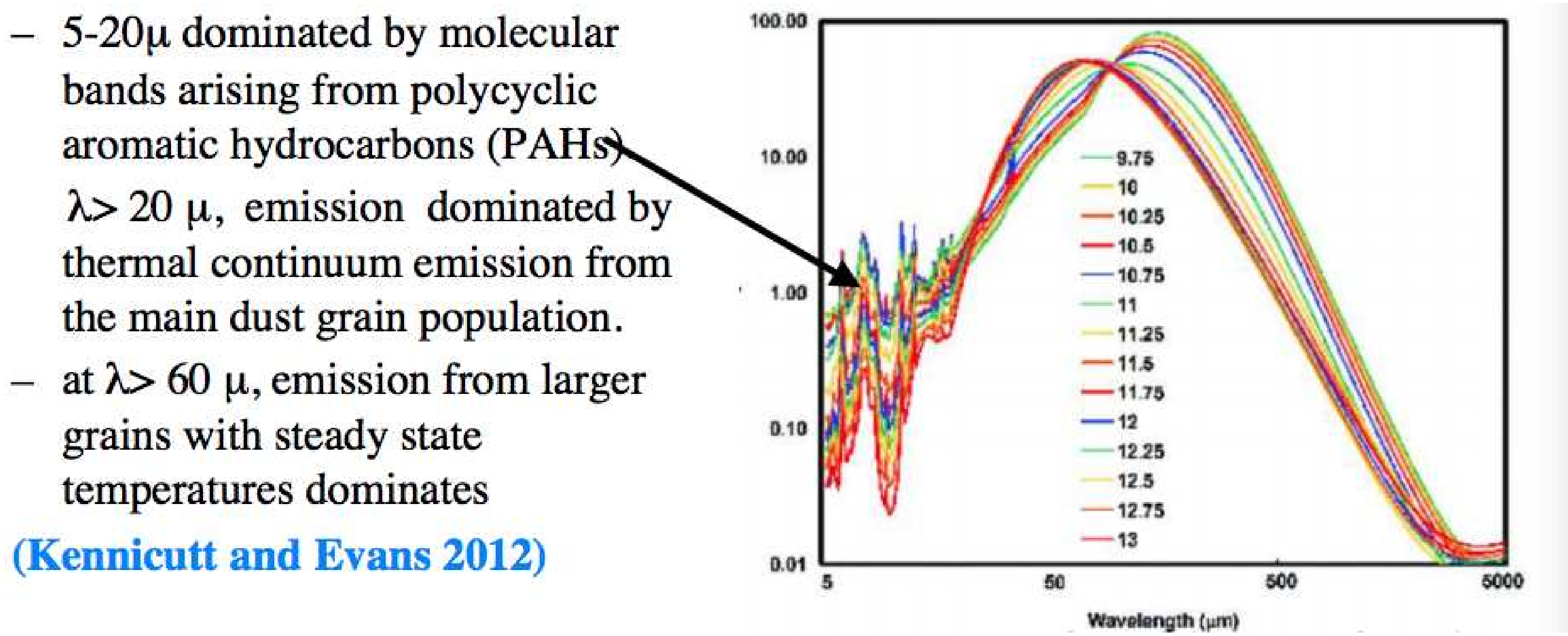
The radiative energy absorbed by dust grains heats them up to temperatures of $T_{\rm dust} \sim 20 - 200$ K. The grains then re-radiate this energy as thermal (greybody) emission of the form

$$S_\nu \propto \frac{\nu^{3+\beta}}{e^{h\nu/kT_{\rm dust}} - 1},$$

in the optically thin regime[37], where S_ν is the flux density and $\beta \approx 1 - 1.5$ is the spectral emissivity index.

[37] Far-infrared spectral energy distribution fitting for galaxies near and far

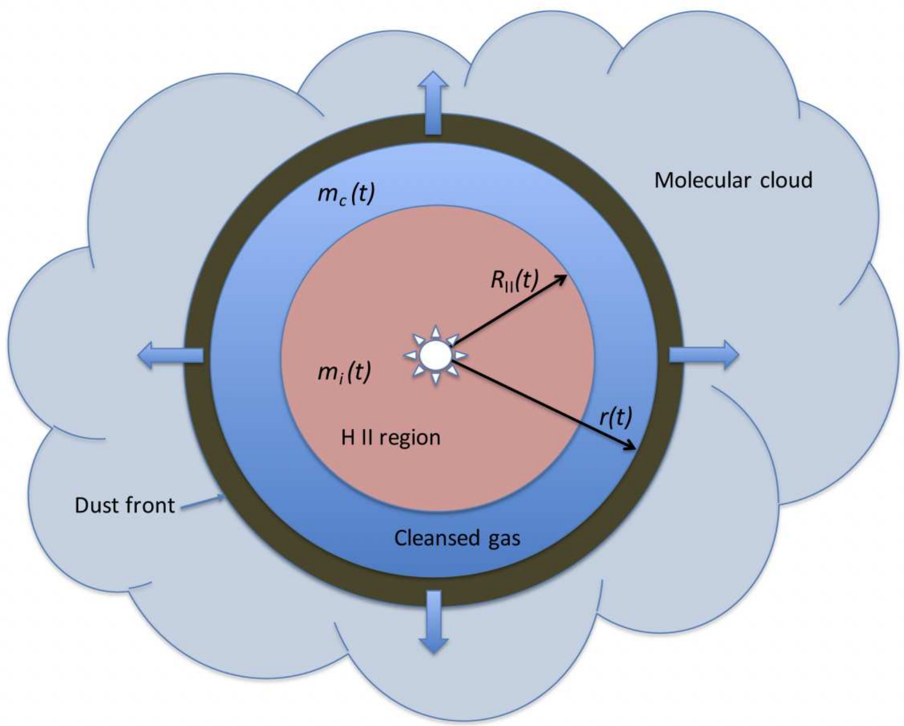
This emission is mostly in the IR (1 – 100 μm) and in practice dust grains heated by starlight gain a wide range of temperatures, so the net emission spectrum is a combination of many black bodies.



From here, referring to Star Formation in the Milky Way and Nearby Galaxies

6.2.2 Location

We can use the temperature to estimate the distance of the dust from the star which heats it.



From Dust cleansing of star-forming gas

At distance d, the side of the grain facing the radiation source absorbs

$$P_{\text{in}} = (1 - a) I A_{\text{CSA}} = \frac{(1 - a) L A_{\text{CSA}}}{4\pi d^2},$$

where A_{CSA} is the cross-sectional area of the grain and a its albedo (the fraction of radiation reflected).

Assuming black body emission ($\beta = 0$), the power emitted by the grain is given by *Stefan's Law*, $P_{\text{out}} = A_{\text{surf}} \sigma \epsilon T_{\text{dust}}^4$, where A_{surf} is its surface area, $\sigma = \frac{2\pi k}{15 c^2 h^3} = 5.6704 \times 10^{-8}$ W m^{-2} K^{-4}, the Stefan–Boltzmann constant and ϵ the emissivity.[a]

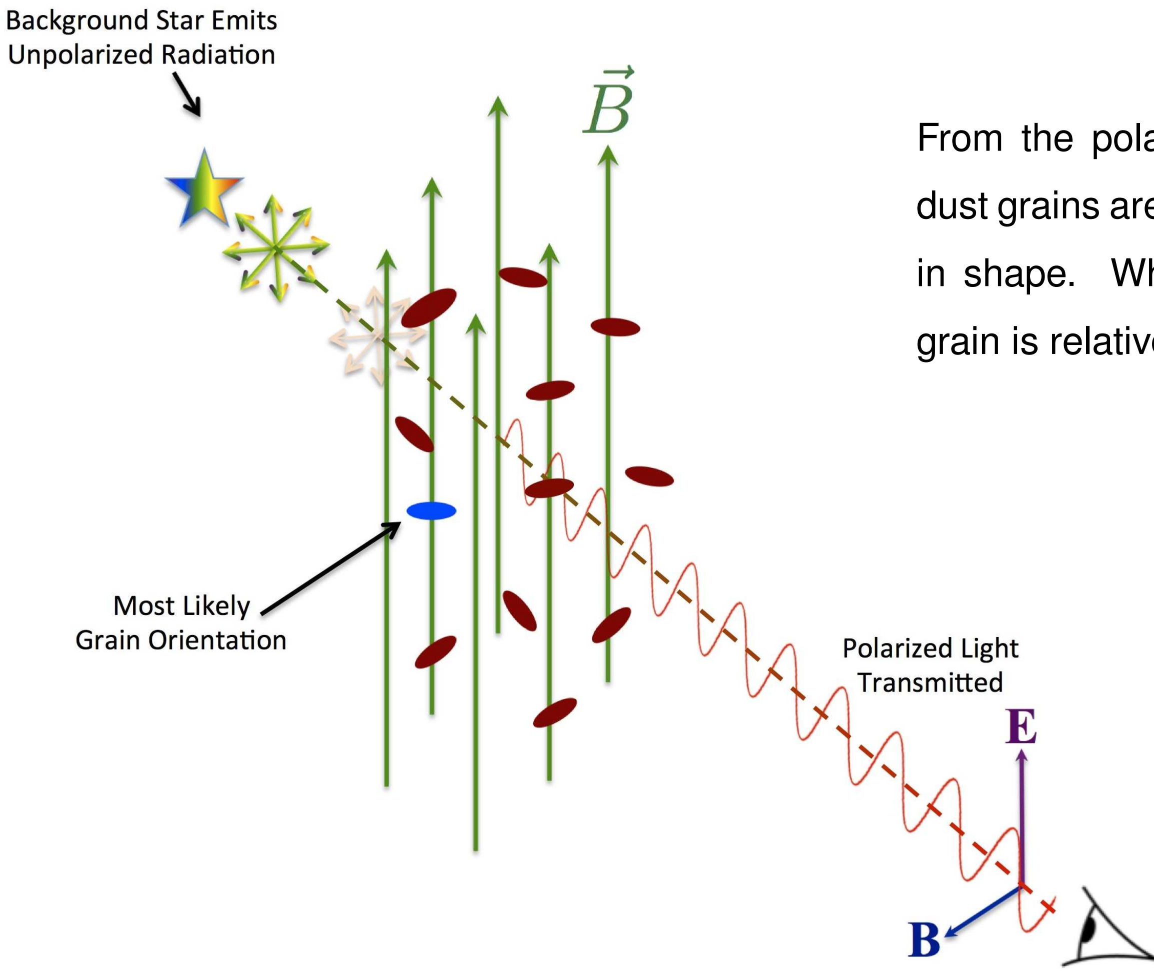
[a] How efficiently a surface emits absorbed radiation.

In thermodynamic equilibrium, the power absorbed by a grain is equal to the power emitted, $P_{\text{in}} = P_{\text{out}}$. That is,

$$\frac{(1 - a) L A_{\text{CSA}}}{4\pi d^2} = A_{\text{surf}} \sigma \epsilon T_{\text{dust}}^4,$$

giving the distance as

$$d = \frac{1}{T_{\text{dust}}^2}\sqrt{\frac{(1-a)}{4\pi\sigma\epsilon}\left(\frac{A_{\text{CSA}}}{A_{\text{surf}}}\right)L}.$$

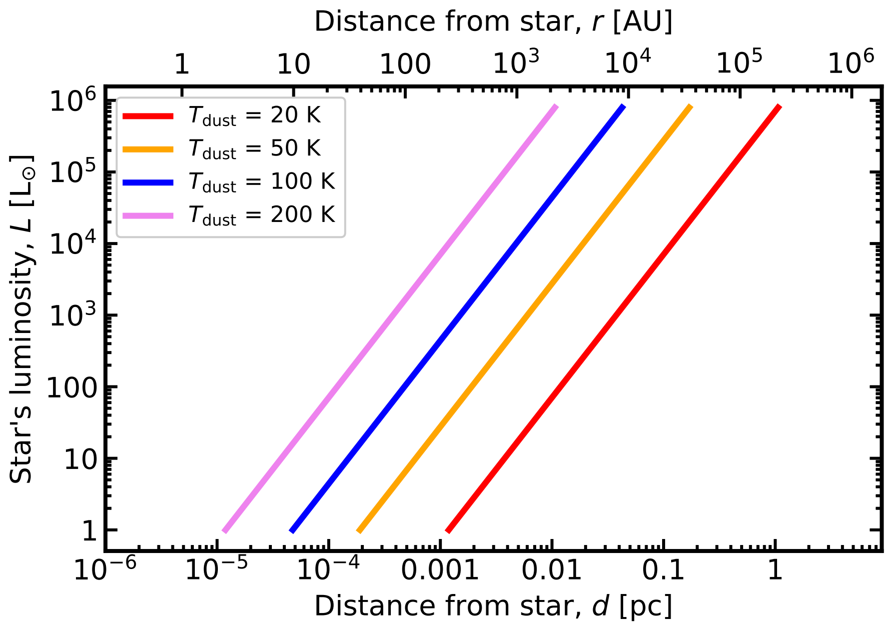


From the polarisation of the radiation, dust grains are believed to be elongated in shape. Whatever the shape, if the grain is relatively flat, $A_{\text{surf}} \approx 2A_{\text{CSA}}$.

From Dust Grain Alignment in the Interstellar Medium

The albedo is wavelength dependent, but generally low $a \lesssim 0.25$ in the ultra-violet. For $\epsilon \sim 1$, this gives

$$d \approx \frac{1}{T_{\rm dust}^2}\sqrt{\frac{L}{10\pi\sigma}}.$$



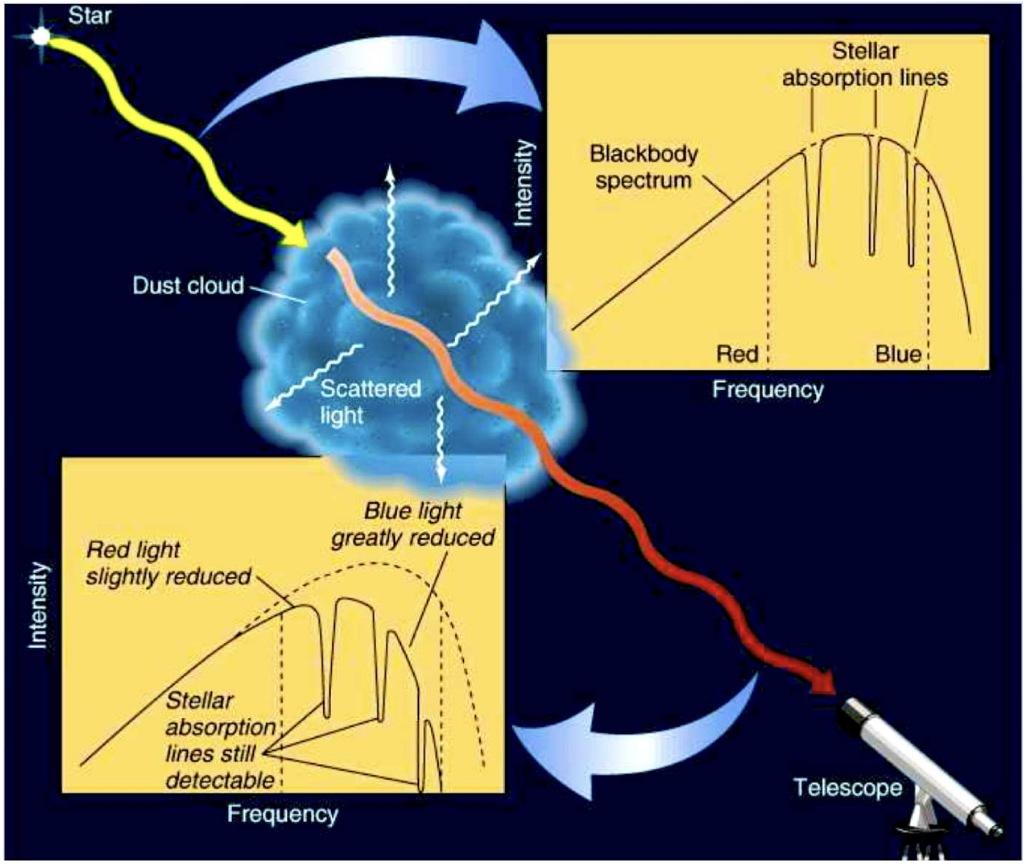
Using the above equation, the plot shows the distance of the dust from the stars of different luminosities and the resulting temperatures. Note that for non-zero grain width, $A_{\rm surf} > 2A_{\rm CSA}$, or the grain face not being normal to the radiation, would allow the grain to radiate more heat decreasing d.

Note that $\epsilon < 1$ would have the effect of increasing d. Furthermore, we have assumed blackbody emission from the dust ($\beta = 0$).[38]

Nevertheless, we would expect dust in our solar system to have a temperature of $T_{\rm dust} \sim 200$ K at a distance of $d \sim 2$ AU, or around the orbit of the asteroid belt.

[38] A greybody version of Stefan's Law could use this as a starting point.

6.3 Extinction



Dust grains dim the light from stars by absorption and scattering, which lead to obscuration and reddening, respectively. This combined effect is referred to as interstellar extinction and is quantitatively represented by the parameter A_V, which is defined as the difference between the observed magnitude, m, and the magnitude, m_0, that would be observed in the absence of dust in a particular waveband (usually the V band):

$$\boxed{A_V \equiv (m - m_0)_V}$$

6.3.1 Obscuration

The *apparent magnitude* of a source is a measure of its observed brightness in a particular waveband and is defined on a logarithmic scale:

$$m_V = -2.5 \log_{10} F_V + C,$$

where F_V is the flux in the V-band and C is a calibration constant.

Thus, the extinction is

$$A_V = -2.5 \log_{10}\left(\frac{F}{F_0}\right) = -2.5 \log_{10}(e^{-\tau_V}) = 2.5\tau_V \log_{10} e \approx 1.09\,\tau_V.$$

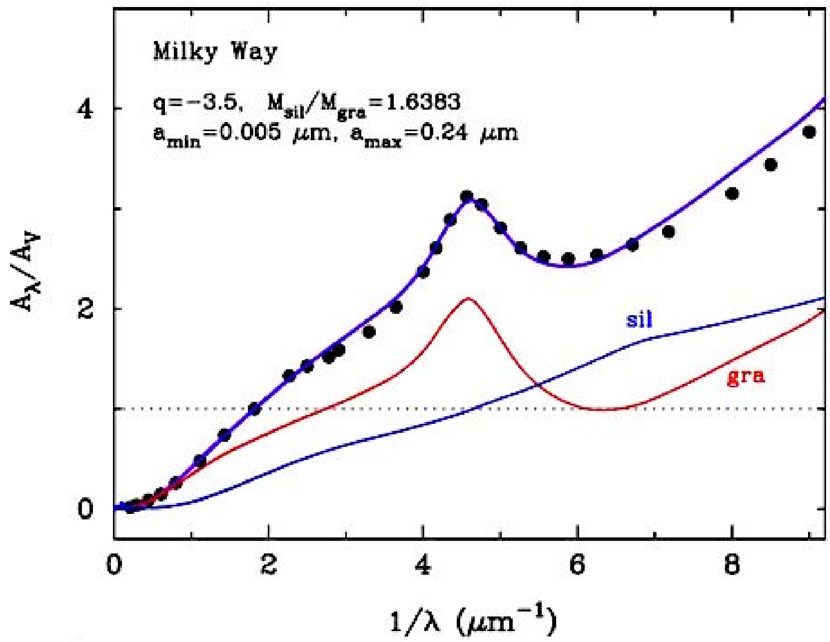
That is, extinction in a given band, measured in magnitudes, is approximately equal to the line-of-sight optical depth in that band.

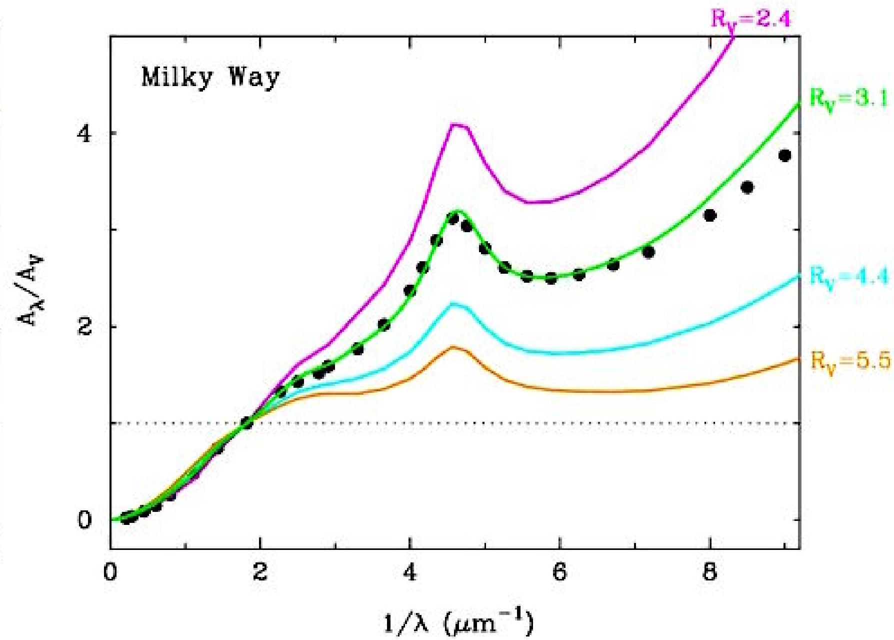
The plots show extinction in the Milky Way, where the circles represent the measured values and the curves show the models,

$$R_V = \frac{A_V}{A_B - A_V},$$

where A_B is the average blue magnitude extinction.[39]

[39] The Relationship between Infrared, Optical, and Ultraviolet Extinction





○ MRN dust model

(Mathis, Rumpl, & Nordsieck 1977)

- **composition :**
  silicate and graphite
- **size distribution :**
  n(a) ∝ a^{-q} with q=3.5,
  0.005 μm ≤ a ≤ 0.25 μm

total-to-selective extinction ratio
$R_V = A_V / (A_B - A_V)$

- **small grains (small Rv)**
  → steep extinction curve
- **large grains (large Rv)**
  → flat extinction curve

The curves are a combination of large, cylindrical grains small graphite grains and poly-aromatic hydrocarbons (PAHs). A large R_V implies larger grains and $R_V = 3.1$ provides the best fit to the data.

6.3.2 Reddening

The *colour index*, or simply colour, is defined as the difference between apparent magnitudes in two different photometric wavebands, usually the B (blue, centred near 4400 Å) and the V (visual, centred near 5500 Å). Thus, the $B - V$ colour is defined as $m_B - m_V$.

This quantity essentially measures the ratio of fluxes in the two wavebands and thus, it is independent of the distance to a source. The amount of reddening suffered by light is measured by the *colour excess*, which is defined as the difference between its measured colour and the intrinsic colour given by its spectral type, i.e.

$$\boxed{E(B-V) = (m_B - m_V) - (m_B - m_V)_0 = A_B - A_V,}$$

where the second equality follows from the definition of A_V given above . Note that because extinction in B is usually greater than in V, $E(B-V)$ is positive.

Empirical measurements have revealed that along a typical line-of-sight in the ISM, $E(B-V)$ is approximately proportional to the column density of hydrogen, according to

$$E(B-V) = \frac{N_{\rm H}}{5.8 \times 10^{25}\ {\rm m}^2}.$$

This indicates that the column density of dust along a typical sight-line is correlated with the column density of hydrogen. i.e. interstellar dust tends to follow the inhomogeneous distribution of interstellar gas. Since the number density of hydrogen the ISM is typically $N_{\rm H} \sim 10^6\ {\rm cm}^{-3}$, then along a line-of-sight of distance d [kpc] , the column density is typically $N_{\rm H} \approx 3 \times 10^{25} d\ {\rm cm}^{-2}$.

Putting this into the above relation gives

$$E(B-V) \approx 0.5d,$$

and empirical studies show that interstellar extinction in the V band follows

$$A_V \approx 1.6d,$$

where again d is in kpc.