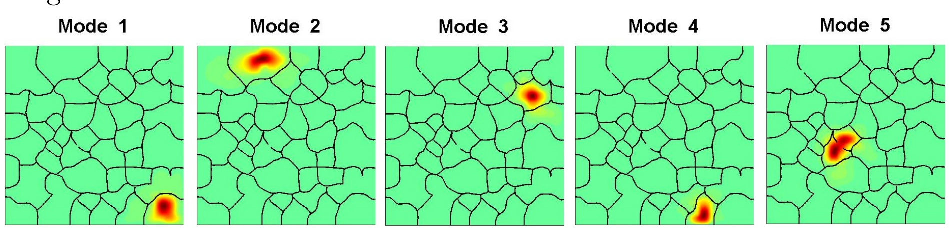
Project Description
Localization of Elliptic PDE and Geometric Measure Theory
Stephen Lewis

---

1. Background and Previous Work

1.1. **Localization of Elliptic PDEs.** The phenomenon of Localization occurs in a vibrating system (acoustic, optic, quantum, etc...) when all of its low energy resonances are highly concentrated. That is, the resonances decay exponentially outside of some small region inside of the domain. Mathematically, this corresponds to the Localization of the low energy eigenfunctions to a corresponding elliptic partial differential operator (PDO). It is often the case that an elliptic PDO without too much symmetry will demonstrate Localization. However, despite being a widely used tool in engineering (including sound insulation, LED design, and avionics), the mathematical connection between a vibrating system and its Localization are not understood.

In a ground-breaking paper [FM12], M. Filoche (Materials Science, École Polytechnique) and the Sponsoring Scientist S. Mayboroda displayed experimentally that Localization regions correspond to a complex network of hypersurfaces which partition the region Ω into weakly coupled subregions Ω_i. They call this network the effective valley network. Further, their theory predicts the eventual delocalization of the eigenfunctions for high enough eigenvalues. They consider, for example, the time-independent Schrödinger operator $L = -\Delta + V$, where V is a random potential function. Pictured below is the resulting effective valley network and the first several eigenfunctions.



A figure from [FM12] showing the effective valley network
and its comparison to the first five eigenmodes

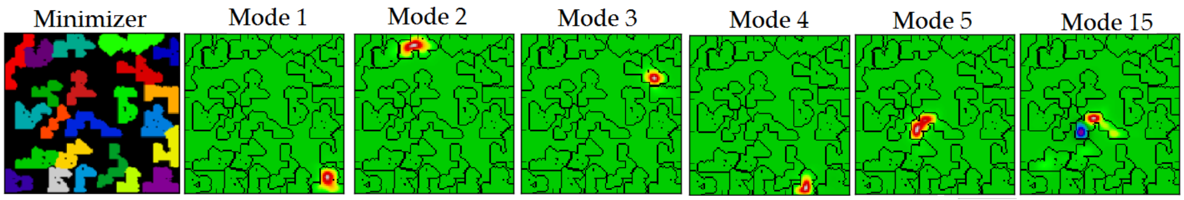
In February 2013, the PI had the opportunity to visit S. Mayboroda at the University of Minnesota to begin work on a new project in Localization. In their manuscript (in progress) [DFJM13], G. David, M. Filoche, D. Jerison, and S. Mayboroda propose a new way to realize the Localization regions by reducing it to a variational problem. The authors define a functional $\tilde{J}$ which takes as inputs disjoint regions $\mathbf{\Omega} = (\Omega_0, \ldots, \Omega_k)$ and functions $\mathbf{u} = (u_0, \ldots, u_k)$, where u_k is supported on $\overline{\Omega}_i$. Let $\mathbf{V} = (\mathcal{L}^n(\Omega_0), \ldots, \mathcal{L}^n(\Omega_k))$ be the vector of volumes. The functional is defined by $\tilde{J}(\mathbf{u}, \mathbf{\Omega}) = J(\mathbf{u}) + G(\mathbf{V})$, where J is an energy with Euler-Lagrange equation L and G is a convex function. For the minimal collection $\mathbf{\Omega}_{\min}$ of $\tilde{J}$, the authors propose that the regions Ω_i correspond to the Localization regions.

For any given collection $\mathbf{\Omega}$, any collection $\mathbf{u}$ of functions which minimize J will the solution to $Lu = 1$ in Ω_i with zero boundary conditions. Thus, if one is willing to solve a large number of PDE, on can consider $\tilde{J}$ as a function of only the regions $\mathbf{\Omega}$.

It was here that the PI began his project. Up to this point, there had been no examples showing if and how the regions Ω_i corresponded to the low energy eigenfunctions. The PI began writing a numerics package in Python which employs a numerical optimization scheme to find approximate minima of the functional $\tilde{J}$. He continued his work with G. David during Spring 2013 as a visiting student at the University of Orsay.

The PI has been actively working on the development of a general and efficient numerics package since his return from Orsay. The diagram below shows the results of early versions of the PI's program. The collection below $\mathbf{\Omega}_{\min}$ compares extremely well to the low energy eigenfunctions.

A minimal collection $\mathbf{\Omega}_{\min}$ and a comparison to its low energy eigenfunctions.



The black region is not in any region Ω_i; we do not ever expect to see a significant portion of an eigenfunction there. The 23 colored regions are the Ω_i. One can visually observe the correspondence between the regions and the eigenfunctions.

1.2. **Nonsmooth Analysis and Geometric Measure Theory (GMT).** Nonsmooth structures are at the heart of many studies in GMT. Smooth (properly embedded C^1) structures can be described in a few ways, including (a) by smooth parametrizations from open subsets of $\mathbb{R}^n$ or (b) by being "well approximated" by their tangent planes in an appropriate sense. By relaxing the assumptions of smooth charts, one can obtain a definition for rectifiable sets (which are sets contained in a countable number of Lipschitz images of $\mathbb{R}^n$). By relaxing the definition of being "well approximated" by tangent planes, one gets the theory of Local Set Approximation (LSA). Rectifiable sets have been studied in great detail [Mat99]. Many beautiful papers exist which employ LSA in a variety of ways [Rei60, Tay76, DDT98, MV90, DKT01, LN08, PTT09].

In [Rei60], the author proves that closed sets in $\mathbb{R}^n$ which are ϵ-approximated by m-planes at all points and all small enough scales (locally uniformly) admit tame bi-Hölder parametrizations by planes. In [DDT98], the authors generalize this result to sets which are well approximated by the minimal cones of [Tay76]. In [DKT01], the authors consider the class of $(n-1)$-asymptotically optimally doubling measures. They show that with an a priori flatness condition on the support of μ, Σ, they achieve local Reifenberg parametrizations of Σ by m-planes. In the case of Hölder-$(n-1)$-asymptotically optimally doubling measures on $\mathbb{R}^n$, they show that with the appropriate a priori flatness condition, Σ admits local $C^{1,\beta}$ parametrizations by planes. In [PTT09], these results are generalized to arbitrary codimension for m-asymptotically optimally doubling measures on $\mathbb{R}^n$. In [Lew13], the PI began extending the scope of [DKT01] to the nonflat points of Σ. He showed that if Σ was the support of an $(n-1)$-asymptotically optimally doubling measure, then it is well-approximated by planes and the Kowalski-Preiss (KP) Cone, $\{x : x_4^2 = x_1^2 + x_2^2 + x_3^2\}$. Moreover, in the case $n = 4$, the PI constructs a $C^{1,\beta}$ parametrization about the nonflat points of Σ by the KP cone. To the PI's knowledge, this is the first construction of a $C^{1,\beta}$ parametrization by a singular set without a minimality condition (see [Tay76]).

However, up to now, there has been little work providing a general framework for LSA. M. Bager (Stony Brook University) and the PI began creating such a framework and studying its theory motivated initially by one goal: to adapt a powerful theorem of GMT to LSA. The powerful theorem is Preiss' Connectedness at Infinity of the Cone of Tangent Measures, which was a key ingredient in several papers of the previous paragraph [DKT01, Lew13, PTT09]. A tangent measure of a Radon measure μ is a limit of blow-up measures, and the collection of all tangent measures to μ at x is denoted $\mathrm{Tan}(\mu, x)$. Preiss' result states a connectedness property for $\mathrm{Tan}(\mu, x)$, which allows one to often classify precisely the collection of tangent measures. Badger and the PI wanted to adapt Preiss' idea to the cone of tangent sets to a set A at x.

A *local approximation class* $\mathcal{S}$ is a collection of nice model sets used to approximate some set A (for example, the collection of planes or the minimal cones of Taylor [Tay76]). The *approximability of a set A by $\mathcal{S}$ in $B(x,r)$* is the quantity $\Theta_A^{\mathcal{S}}(x,r)$, which roughly speaking measures how well the candidate sets of $\mathcal{S}$ approximate A in a Hausdorff distance sense. A *tangent set* to A at x is a limit of blow-ups $(A-x)/r_i$ for some sequence $r_i \to 0$. The collection of tangent sets to A at x is denoted $\operatorname{Tan}(A,x)$. A local approximation class $\mathcal{T}$ is said to be *separated at infinity* from $\mathcal{S}$ if $\inf_{S\in\mathcal{S}} \limsup_{r\to\infty} \Theta_S^{\mathcal{T}}(0,r) > 0$.

Theorem 1 (Connectedness at Infinity of the Tangent Sets). *Let $A \subseteq \mathbb{R}^n$, $x \in A$, $\mathcal{T}$ and $\mathcal{S}$ be local approximation classes. If $\mathcal{T}$ is separated at infinity from $\mathcal{S}$ and $\operatorname{Tan}(A,x) \subseteq \mathcal{S} \cup \mathcal{T}$, then $\operatorname{Tan}(A,x) \subseteq \mathcal{S}$ or $\operatorname{Tan}(A,x) \subseteq \mathcal{T}$.*

This theorem provides a general framework for a decomposition method of LSA employed in [DDT98, DKT01, PTT09, Bad13, Lew13]. In each of these papers, the authors decompose sets well approximated by some $\mathcal{S}$ into flat and strictly nonflat points. In [DDT98] the authors decompose sets A which are ϵ-approximated by the minimal cones of [Tay76] into three components; the flat points, and two types of nonflat points.

2. Proposed Work

The PI and Mayboroda will work closely on numerical and theoretical aspects of Localization. It is true that the PI's primary training has been in generally theoretical areas of analysis rather than numerical computations. However, the PI has experience switching between different areas of math. As an undergraduate and through his first year of graduate school, the PI carried out research in combinatorial algebra with a computational bend ([LT11, AL12]). However, as the PI continued in his Real Analysis class taught by his current advisor, he found himself becoming increasingly fascinated by the subject and would stay after lectures with Toro to discuss further the ideas presented. Since that switch, the PI has a paper on GMT and nonsmooth analysis submitted for publication [Lew13] and one in progress with Badger on Local Set Approximation [BL13]. Having made such a switch, the PI is confident with his entrance to Localization.

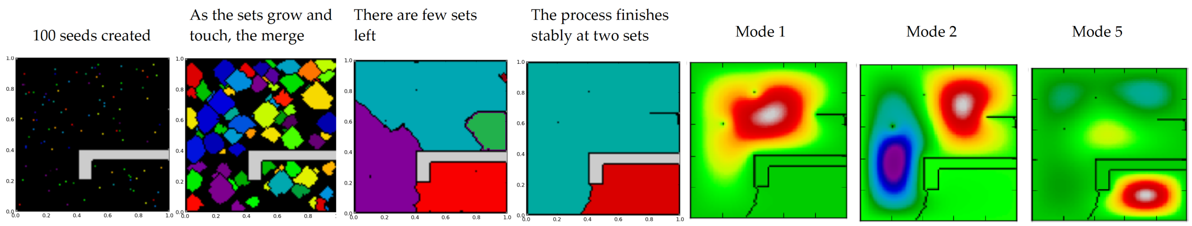
The PI will also continue his study of nonsmooth analysis, LSA, and GMT. In Section 2.2, the PI gives a basic layout of subjects he will pursue in these fields. However, it is the purpose of this application to focus on Localization, due to the PI's shared interest with Mayboroda.

2.1. Localization of Elliptic PDEs.

The PI proposes to pursue the advancement of numerics for Localization and to develop the theory and understanding of its underlying mechanisms. The numerical aspects will be especially well tuned to Anderson and weak Localization. The PI will also develop a highly modular, open-source, and production quality Python software package for use by his collaborators and other interested parties. The PI and Mayboroda will use this software to test theoretical questions numerically, and use the understanding we gain to provide rigorous mathematical examinations. The methods to study these types of questions precisely come from GMT and the study of blow-ups, in which the PI has experience [Lew13, BL13].

Let L be an elliptic operator on a domain Ω and $\mathbf{\Omega}_{\min}$ be a minimizer of the functional $\tilde{J}(\mathbf{\Omega})$. There currently is no understanding of the connection between L, Ω, and the number of Localization regions. Predicting the number of Localization regions has applications to many areas of engineering (acoustic, quantum, etc...), where it is desirable to design an object (piano, charged plate) with a certain number of principal resonances. The PI will address this question through the study of $\mathbf{\Omega}_{\min}$. This study will include both numerical components by studying the problem through examples and test data and theoretical components by providing a deeper understanding of the mechanisms at hand.

The PI has started some early tests in the numerics that result by varying the cardinality of $\mathbf{\Omega}$. The PI has implemented a method which initializes with very large number of small seed sets, and while evolving them, allowing adjacent regions to merge if preferable. The figure below, is the result of an early experiment in this method, where 100 tiny seed sets are planted and then grow, evolve, merge, and eventually stabilize at two sets. The eigenmodes are localized in the predicted regions.



An evolution with merging to a minimal $\mathbf{\Omega}_{\min}$ and a comparison to low energy eigenfunctions.

We recall now that $\mathbf{\Omega} = (\Omega_0, \ldots, \Omega_k)$ is a collection of regions, $\mathbf{u} = (u_0, \ldots, u_k)$ is a collection of functions such that u_i supported on $\overline{\Omega}_i$, and $\mathbf{V} = (\mathcal{L}^n(\Omega_0), \ldots, \mathcal{L}^n(\Omega_k))$ is the volume vector. Further, we recall that we can write $\tilde{J}(\mathbf{u}, \mathbf{\Omega}) = J(\mathbf{u}) + G(\mathbf{V})$, where J is an energy for which L is the Euler-Lagrange equation and G is some convex function of the volumes. However, the minimizer $\mathbf{\Omega}_{\min}$ will be affected by different choices of G. The PI will investigate how sensitive the cardinality $\mathbf{\Omega}_{\min}$ and the regions Ω_i are to changes in G through a combination of numerical examples and pursuing its theory.

In practice, choosing G to have a linear term and a square term with positive coefficients, $G(x_0, \ldots, x_k) = \sum_{i=1}^{k} ax_i + bx_i^2$, seems very effective. It is known that the linear term assures that a minimum will be achieved which has a finite number of sets [DFJM13]. The square term causes there to be a preference for multiple small sets rather than one large set (two sets of measure $1/4$ contribute less than one set of measure $1/2$). The exact balance here is interesting from a numerical perspective and crucial for applications. For example, one would like to know how to construct a region Ω and PDO to guarantee there are some predetermined number of regions.

If one assumes that there is a correct number of Localization regions, then only certain functions G can accurately describe Localization in any given situations. The PI offers another perspective, which is that G acts as a gauge function on the structure of Localization regions, and it is even likely that different applications would prefer different gauge functions. For G as in the previous paragraph, one can increase or decrease the amount of black space (by adjusting a) as well as increase or decrease the number of regions (by adjusting b). In general, collections $\mathbf{\Omega}$ obtained by smaller a will have higher eigenfunctions avoiding the black space while not predicting as accurately the lower ones. Collections $\mathbf{\Omega}$ obtained with a smaller b will have more eigenfunctions which do not cross their boundaries while hugging the lower energy eigenfunctions less accurately (see figure 2 and eigenmode 5 vs 15).

The PI will also study Anderson Localization through these frameworks. Most studied in Quantum Mechanics, Anderson Localization is a continuous variant of Localization (sometimes called weak Localization), where the eigenfunctions decay exponentially outside of some small region. The PI is in agreement with Mayboroda that Anderson Localization could be related to a limit or perhaps family of weak Localizations. The PI expects the exact dependence of $\mathbf{\Omega}_{\min}$ on a and b, the linear and square coefficients, to be relatively stable. Hence the PI suspects that Anderson Localization can be explained as a one or two parameter family of regions. The PI will run numerical experiments to find minimal collections $\mathbf{\Omega}_{\min}$ as a function oft a and b to try and predict decay, and compare it to the efficacy of similar methods from [FM12] using the effective valley network (see fig 4), where the authors demonstrate an exponential decay rule based on crossing boundaries.

The PI will study the Localization of higher order divergence operators as well, for which even monotonicity formulae are rare [MM13]. The PI and Mayboroda will search for monotonicity formulae and generalize the techniques used to study elliptic Localization to higher order differential operators. To begin, they will study variations of second order monotonicity formulae and use numerical methods to develop reasonable conjectures, which will ultimately confirmed mathematically. Further, all of the previous mechanisms of Localization can be considered for higher order divergene operators. Even the simplest example of $L = -\Delta^2$, the biLaplacian shows interesting behavior on asymmetric regions (with clamped boundary conditions) [FM12].

The PI will also distribute and maintain a software package for use by his collaborators and other interested researchers. The NSF Mathematical Sciences Research Postdoctoral Fellowship will provide PI increased opportunity to maintain and improve the package to the needs and new ideas of his collaborators, as well as educate them about the framework work of the package and how to include their own contributions. Decreasing runtime and improving accuracy will be the focus of maintenance. Building new features and smoothly incorporating new contributions will be the focus of improvement.

2.2. **Nonsmooth Analysis and GMT.** The PI's work with Badger has applications to many areas of math. For example, the study of the support Σ of an m-asymptotically optimally doubling measure μ. This implies that the flat points of Σ are open and dense. This is due to *detectability*, a property developed in [BL13]. If $\mathcal{S}$ and $\mathcal{T}$ are approximation classes such that $\mathcal{T} \subseteq \mathcal{S}$, we say $\mathcal{T}$ is *detectable* in $\mathcal{S}$ if for all $S \in \mathcal{S}$, the points of S are uniformly flat at small scales or uniformly nonflat all scales. Badger and the PI proved that if $\mathcal{T}$ is detectable in $\mathcal{S}$ and A is a closed set well approximated by $\mathcal{S}$, then $\mathcal{T}$ is detectable in A in the appropriate sense.

In each of [Rei60, Tay76, DDT98, DKT01, PTT09, Lew13], the authors consider a closed set A which is well approximated by some nice local approximation class. In each paper the authors construct local parametrizations (topological, C^α, $C^{1,\alpha}$, C^∞) of A by some element in $\mathcal{S}$. All of these papers used detectability of flat points. Moreover, [DDT98] uses a stratified notion of detectability with multiple types of nonflat points. The PI will investigate which assumptions on a local approximation class and which approximating schemes allow one to construct parametrizations in general. Detectability is sure to be a useful tool.

Badger and the PI will also investigate the applications of LSA to sets A which are well approximated by zero-sets of harmonic polynomials started in [Bad13]. This application will involves the study of pseudotangent sets, a generalization of tangent sets put forth in [BL13] and modeled after the pseudotangent measures of [KT99] and detectability. Badger and the PI seek to obtain a stratified set of Minkowski dimension bounds based on a decomposition of A into several types of nonflat points.

3. The University of Minnesota

An NSF Mathematical Sciences Research Postdoctoral Fellowship with S. Mayboroda at the University of Minnesota will provide an excellent opportunity for mathematical and professional growth. Mayboroda and the PI already have a working relationship with a shared interest in Localization and Geometric Measure Theory. Mayboroda also has an active research team working on Localization where the PI will fit in smoothly. Prof. Doug Arnold is at UMN and works very broadly in numerical PDE, including numerical Localization. The Institute for Math and its Applications (IMA) is also at UMN, and the PI will benefit from their programs and the source of potential collaborators.

References

[AL12] M. Aguiar and S. et al Lewis, *Supercharacters, symmetric functions in noncommuting variables, and related Hopf algebras*, Advances in Math (2012).

[Bad11] M. Badger, *Harmonic polynomials and tangent measures of harmonic measure*, Rev. Mat. Iberoam. **27** (2011), no. 3, 841–870.

[Bad13] M. Badger, *Flat points in zero sets of harmonic polynomials and harmonic measure from two sides*, J. London Math. Soc. **87** (2013), no. 1, 111–137.

[BL13] M. Badger and S. Lewis, *Local set approximation: Mattila-Vuorinen type sets, Reifenberg type sets, and Pseudotangent sets* (2013), in preparation.

[DDT98] G. David, T. DePauw, and T. Toro, *Generalizations of Reifenberg's theorem in* $\mathbb{R}^3$, Geom. Func. Anal. (1998).

[DFJM13] G. David, M. Filoche, D. Jerison, and S. Mayboroda, *A free boundary problem for the Localization of eigenfunctions* (2013), in preparation.

[DKT01] G. David, C. Kenig, and T. Toro, *Asymptotically optimally doubling measures and Reifenberg flat sets with vanishing constant*, Comm. Pure Appl. Math. **54** (2001), 385-449.

[DS97] G. David and S. Semmes, *Fractured Fractals and Broken Dreams: Self-Similar Geometry Through Metric and Measure*, Oxford Lecture Series in Mathematics and its Applications, vol. 7, The Claredon Press, Oxford University Press, New York, 1997.

[FM12] M. Filoche and S. Mayboroda, *Universal mechanism for Anderson and weak Localization*, PNAS **109** (2012), no. 37.

[KT99] C.E. Kenig and T. Toro, *Free boundary regularity for harmonic measures and Poisson kernels*, Ann. Math. **150** (1999), 369–454.

[LN08] J. L. Lewis and K. Nyström, *Boundary behavior of p-harmonic functions in domains beyond Lipschitz domains*, Adv. Calc. Var. **1** (2008), no. 2, 133-170.

[Lew13] S. Lewis, *Singular points of Hölder asymptotically optimally doubling measures* (2013), preprint, available at `arXiv:1301.1993`.

[LT11] S. Lewis and N. Thiem, *Nonzero coefficients in restriction and tensor products of supercharacters of* $U_n(q)$, Advances in Math (2011).

[Mat99] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability*, Cambridge University Press, 1999.

[MM13] S. Mayboroda and V. Maz'ya, *Regularity of solutions to the polyharmonic equation in general domains*, Inventiones Mathematicae (2013), to appear.

[MV90] P. Mattila and M. Vuorinen, *Linear approximation property, Minkowski dimension, and quasiconformal spheres*, J. London Math. Soc. (2) **42** (1990), 249–266.

[Pre87] D. Preiss, *Geometry of measures in* $\mathbb{R}^n$*: distribution, rectifiability, and densities*, Ann. of Math. **125** (1987), 537–643.

[PTT09] D. Preiss, X. Tolsa, and T. Toro, *On the smoothness of Hölder doubling measures*, Calculus of Variations and PDE's **35** (2009), 339-363.

[Rei60] E. Reifenberg, *Solution of the Plateau problem for m-dimensional surface of varying topological type*, Acta Math. **104** (1960), 1–92.

[Tay76] J. Taylor, *The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces*, Annals Math. **103** (1976), 489-539.