- First, I will explain Quiz 1.
- Homework 3 is updated with an example session for Problem 4.
- Please submit papers instead of notebooks for Homework 3.
- Course project deadline is 2013/12/1 23:59:59. Start early!

Solution by Xiao Jia

Homework 2

Problem 1. (10 points) An expression e is *closed* iff $\mathrm{FV}(e) = \emptyset$. Prove: If e_1 is closed, and $e_1 \to^* e_2$, then e_2 is closed. Hint: Define the lemma involving $\to$.

Solution. We first prove the following lemma.
Lemma 1: If e_1 is closed, and $e_1 \to e_2$, then e_2 is closed.

Proof. By induction of the derivation of $e_1 \to e_2$.
Remember, $v ::= \lambda x.e$, i.e. only functions can be values.

1. Case $\dfrac{}{(\lambda x.e)\ v \to e[v/x]}$

   We first need a property of $e[v/x]$.

   Property 1: $\mathrm{FV}(e_1[e/x]) \subseteq (\mathrm{FV}(e_1) - \{x\}) \cup \mathrm{FV}(e)$
   $\subseteq$ may be easier to prove.

   1. Case $x[e/x] = e$
   2. Case $y[e/x] = y$ (if $y \neq x$)
   3. Case $(e_a\ e_b)[e/x] = ((e_a[e/x])\ (e_b[e/x]))$
   4. Case $(\lambda x.e_a)[e/x] = \lambda x.e_a$
   5. Case $(\lambda y.e_a)[e/x] = \lambda y.(e_a[e/x])$ (if $y \neq x$)

   (1) $\mathrm{FV}((\lambda x.e)\ v) = \emptyset$ (by assumption)

   (2) $\mathrm{FV}(\lambda x.e) = \mathrm{FV}(e) - \{x\} = \mathrm{FV}(v) = \emptyset \Rightarrow (\mathrm{FV}(e) - \{x\}) \cup \mathrm{FV}(v) = \emptyset$

   (3) $\mathrm{FV}(e[v/x]) \subseteq \emptyset \Rightarrow \mathrm{FV}(e[v/x]) = \emptyset$

2. Case $\dfrac{e_1 \to e_1'}{e_1\ e_2 \to e_1'\ e_2}$

   (1) $\mathrm{FV}(e_1\ e_2) = \emptyset$ (by assumption)

(2) $\mathrm{FV}(e_1) = \mathrm{FV}(e_2) = \emptyset$

(3) $\mathrm{FV}(e_1') = \emptyset$ (by I.H. and (2))

(4) $\mathrm{FV}(e_1'\ e_2) = \emptyset$

3. Case $\dfrac{e_2 \to e_2'}{v\ e_2 \to v\ e_2'}$

(1) $\mathrm{FV}(v\ e_2) = \emptyset$ (by assumption)

(2) $\mathrm{FV}(v) = \mathrm{FV}(e_2) = \emptyset$

(3) $\mathrm{FV}(e_2') = \emptyset$ (by I.H. and (2))

(4) $\mathrm{FV}(v\ e_2') = \emptyset$

$\square$

Then prove by induction on the derivation of $e_1 \to^* e_2$.

1. Case $\dfrac{}{e_1 \to^* e_1}$ (trivial, because $\mathrm{FV}(e_1) = \emptyset$)

2. Case $\dfrac{e_1 \to e_2 \quad e_2 \to^* e_3}{e_1 \to^* e_3}$

(1) $\mathrm{FV}(e_1) = \emptyset$ (by assumption)

(2) $\mathrm{FV}(e_2) = \emptyset$ (by Lemma 1 and (1))

(3) $\mathrm{FV}(e_3) = \emptyset$ (by I.H. and (2))

$\to^*$ is defined by its properties, i.e. reflexivity and transitivity. As discussed in the solution of Homework 1, it is non-trivial to convince ourselves that we are relying on strictly smaller structures (since we are using structural induction), e.g. $e_2 \to^* e_3$ above. In order to be more precise, we can actually define the multi-step relation by explicitly using the number of steps:

$$\frac{}{e_1 \to^0 e_1} \qquad \frac{e_1 \to^1 e_2 \quad e_2 \to^n e_3}{e_1 \to^{n+1} e_3}$$

where $\to^1$ is the same as $\to$. Then $\to^*$ can be defined as

$$e_1 \to^* e_2 \iff \exists n \in \mathbb{N}. e_1 \to^n e_2$$

$\blacksquare$

Problem 2. (5 points) Prove the substitution lemma: If $\Gamma, x : t' \vdash e : t$, and $\Gamma \vdash v : t'$, then $\Gamma \vdash e[v/x] : t$.

Solution. Prove by induction on the derivation of $\Gamma, x : t' \vdash e : t$.

1. Case $\Gamma, x : t' \vdash y : (\Gamma, x : t')(y)$

Subcase $y = x$

(1) $(\Gamma, x : t')(x) = t'$

(2) $t = t'$ (by (1) and assumption)

(3) $x[v/x] = v$

(4) $\Gamma \vdash v : t'$ (by assumption)

(5) $\Gamma \vdash x[v/x] : t$ (by (2), (3) and (4))

Subcase $y \neq x$

(1) $(\Gamma, x : t')(y) = \Gamma(y)$

(2) $t = \Gamma(y)$ (by (1) and assumption)

(3) $\Gamma \vdash y : t$ (by (2))

(4) $y[v/x] = y$

(5) $\Gamma \vdash y[v/x] : t$ (by (3) and (4))

2. Case $\Gamma, x : t' \vdash e_1\ e_2 : t$

(1) $\exists t_1, t_2$ where $\Gamma, x : t' \vdash e_1 : t_1$ and $\Gamma, x : t' \vdash e_2 : t_2$

(2) $\Gamma \vdash e_1[v/x] : t_1$ (by I.H.)

(3) $\Gamma \vdash e_2[v/x] : t_2$ (by I.H.)

(4) $\Gamma \vdash e_1[v/x]\ e_2[v/x] : t$ (by (2) and (3))

(5) $\Gamma \vdash (e_1\ e_2)[v/x] : t$ (by (4))

3. Case $e = \lambda y.e'$ (can assume $y \neq x$ and $y \notin \mathrm{Dom}(\Gamma)$)

(1) $\exists t_1, t_2$ where $\Gamma, x : t', y : t_1 \vdash e' : t_2$ and $t = t_1 \to t_2$

(2) $\Gamma, y : t_1, x : t' \vdash e' : t_2$ (by Exchange Lemma)

(3) $\Gamma \vdash v : t'$ (by assumption)

(4) $\Gamma, y : t_1 \vdash v : t'$ (by (3) and Weakening Lemma)

(5) $\Gamma, y : t_1 \vdash e'[v/x] : t_2$ (by (2), (4) and I.H.; using $\Gamma, y : t_1$ for Γ)

(6) $\Gamma \vdash \lambda y.e'[v/x] : t_1 \to t_2$

(7) $\Gamma \vdash (\lambda y.e')[v/x] : t_1 \to t_2$

$\blacksquare$

Problem 3. (20 points) Evaluate the following λ expressions using call-by-value and call-by-name. Show the complete steps of evaluation.

(a) $((\lambda x.\ x \times x)\ 5)$

(b) $((\lambda y.((\lambda x.\ x + y + z)\ 3))\ 2)$

(c) $((\lambda v.(\lambda w.w))\ ((\lambda x.x)\ (y\ (\lambda z.z))))$

(d) $((\lambda x.\ x\ x)\ (\lambda y.\ y\ y))$

Solution.

(a) Call-by-name and call-by-value:

$$\begin{aligned} &((\lambda x.\ x \times x)\ 5) \\ \to\ & 5 \times 5 \\ \to\ & 25 \end{aligned}$$

(b) Call-by-name and call-by-value:

$$\begin{aligned} &((\lambda y.((\lambda x.\ x + y + z)\ 3))\ 2) \\ \to\ & ((\lambda x.\ x + 2 + z)\ 3) \\ \to\ & 3 + 2 + z \\ \to\ & 5 + z \end{aligned}$$

(c) Call-by-name:

$$\begin{aligned} &((\lambda v.(\lambda w.w))\ ((\lambda x.x)\ (y\ (\lambda z.z)))) \\ \to\ & \lambda w.w \end{aligned}$$

Call-by-value:

$$\begin{aligned} &((\lambda v.(\lambda w.w))\ ((\lambda x.x)\ (y\ (\lambda z.z)))) \\ \ldots\ & ((\lambda v.(\lambda w.w))\ ((\lambda x.x)\ (y\ (\lambda z.z)))) \\ \ldots\ & \text{No reductions can be applied!} \end{aligned}$$

(d) Call-by-name and call-by-value:

$$\begin{aligned} &((\lambda x.\ x\ x)\ (\lambda y.\ y\ y)) \\ \to\ & ((\lambda y.\ y\ y)\ (\lambda y.\ y\ y)) \\ \to\ & \ldots \\ \ldots\ & \text{(Infinite sequence of reductions!)} \end{aligned}$$

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Problem 4. (25 points) Church encoding is a means of embedding data and operators into the λ calculus, the most familiar form being the Church numerals, a representation of the natural numbers using λ notation. Church numerals $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$, are defined as follows:

$$
\begin{aligned}
\mathbf{0} &= \lambda f.\lambda x.\ x \\
\mathbf{1} &= \lambda f.\lambda x.\ f\ x \\
\mathbf{2} &= \lambda f.\lambda x.\ f\ (f\ x) \\
\mathbf{3} &= \lambda f.\lambda x.\ f\ (f\ (f\ x)) \\
&\ldots \\
\mathbf{n} &= \lambda f.\lambda x.\ f^n\ x \\
&\ldots
\end{aligned}
$$

For each of the following arithmetic operations for Church numerals:

- If the definition is correct, explain why.
- If the definition is wrong, fix it and explain why your fixed one is correct.

(a) The addition function $plus = \lambda n.\lambda m.\lambda f.\lambda x.\ m\ f\ (n\ f\ x)$

(b) The successor function $succ = \lambda n.\lambda f.\lambda x.\ f\ (n\ f\ x)$

(c) The multiplication function $mult = \lambda n.\lambda m.\lambda f.\ m\ (n\ f)$

(d) The exponentiation function $exp = \lambda n.\lambda m.\ n\ m$

(e) The predecessor function $pred = \lambda n.\lambda f.\lambda x.\ n\ (\lambda g.\lambda h.\ h\ (g\ f))\ (\lambda u.x)\ (\lambda u.u)$

Solution.

(a) Correct; $f^{(n+m)}(x) = f^m(f^n(x))$

(b) Correct; $\mathrm{succ}(n) = n + 1$ is β-equivalent to (plus 1).

(c) Correct; $f^{(n\times m)}(x) = (f^n)^m(x)$

(d) Wrong; fix: $exp = \lambda n.\lambda m.\ m\ n$ which is to apply n for m times, i.e. n^m.

Now observe the form of the Church numerals. For $\mathbf{n}$, f is applied for n times.

(e) Correct; $\mathrm{pred}(n) = \begin{cases} 0 & \text{if } n = 0, \\ n-1 & \text{otherwise} \end{cases}$ works by generating an n-fold composition of functions that each apply their argument g to f; the base case discards its copy of f and returns x.

The $\mathrm{pred}(n)$ function is probably the most difficult of the basic operations to understand. To gain insight into the operation of $\mathrm{pred}(n)$ it is instructive to look at the expansion of its body when it is applied to the first few Church numerals.

The subtraction function can be written based on the predecessor function.

$$\text{sub} = \lambda m.\lambda n.\ (n\ \ \text{pred})\ m$$

The zero predicate can be written as:

$$\text{zero?} = \lambda n.\ n\ (\lambda x.F)\ T$$

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Problem 5. (40 points) Write functions in λ calculus and Church numerals to calculate ...

(a) ... the factorial of n.

(b) ... the greatest common divisor of x and y.

(c) ... the n-th Fibonacci number F_n ($F_0 = 0$ and $F_1 = 1$)

(d) ... the i-th prime number p_i ($p_1 = 2$)

Solution. Read the following materials for solutions.
`www.ics.uci.edu/~lopes/teaching/inf212W12/readings/lambda-calculus-handout.pdf`
`jwodder.freeshell.org/lambda.html`
`www.mathstat.dal.ca/~selinger/papers/lambdanotes.pdf` ■