School of Computing and Information Systems
COMP30026 Models of Computation
Week 1: Informal Proof

If you finish the exercises early, start on the homework problems!

Homework problems

P1.1 For each case from **??** which has no solution, write a proof of that fact. Try to make your answer as short as possible, but do not sacrifice rigour.

P1.2 Consider a square grid with x columns and y rows. We want to find out how many different paths there are that one can travel along, starting from the bottom left and ending in the top right cell, given that *one can only move up or right.* Let us call that value $f(x, y)$.

(a) Write down a table of the values of $f(x, y)$ when x and y range from 1 to 4. Draw the table as a grid, so that $f(1, 1)$ is in the lower left corner, $f(2, 1)$ is immediately to the right of it, and so on.

(b) Discuss what we mean by "path". How might we represent it mathematically?

(c) Propose and justify two different formulas for calculating $f(x, y)$. What are the benefits and drawbacks of each option? *Hint:* Think dynamic programming!

Solution: One option is to use a recursive approach:

$$f(x, y) = \begin{cases} 1 & \text{if } x = 1 \text{ or } y = 1, \\ f(x-1, y) + f(x, y-1) & \text{otherwise.} \end{cases}$$

This is correct because:

i. There is only one path from $(1, 1)$ to $(x, 1)$. That is, moving right $x - 1$ times.

ii. Similarly, there is only one path from $(1, 1)$ to $(y, 1)$.

iii. To reach (x, y) when $x, y > 1$, you must first reach either $(x-1, y)$ or $(x, y-1)$, and in either case only one move gets you to (x, y).

Note that if you try calculate $f(x, y)$ by recursively expanding this formula, it will take an exponential number of steps. However, almost all of those steps are repeated work. Can you think of a way to optimize the evaluation?

This is a more elegant formula:

$$f(x, y) = \frac{(x+y-2)!}{(x-1)!(y-1)!} = \binom{x+y-2}{x-1}.$$

To explain this, consider that every path to (x, y) consists of exactly $x - 1$ rightward steps and exactly $y - 1$ upward steps. The total number of steps is thus $x + y - 2$.

Now, draw an array of $x + y - 2$ empty slots:

1	2	3	4	5	6	7	8		$x+y-2$
								$\cdots$	

Then, if we fill in $x-1$ of them with R's for (for *right*), and the rest with U's for (for *up*), we will get a path to (x, y). And vice versa: every path corresponds to a unique string of R's and U's. (That is, there is a *bijection* between these paths and such strings.)

So, how many ways can we do this? Well, how many ways are there to choose $x-1$ slots from a set of $x+y-2$ slots?

(d) Prove by induction that the two formulas are equivalent.

P1.3 Put each premise and deduction in your answer to **??** on a separate line of a numbered list.

(a) Next to each premise, write down that it is a premise. Next to each deduction, write down the numbers of the deductions or premises that it follows from.

(b) Split each nontrivial deduction into simpler (possibly still nontrivial) deductions. Repeat the process until the entire proof consists of premises and trivial deductions.

Hint: Your proof can have subproofs. If your original answer proves $\neg P$ by contradiction, convert that into a subproof of $\perp$ whose premise is P. Then you can conclude $\neg P$ from your subproof by *reductio ad absurdum.*

Solution:

(1) Each card is exactly one of the following: red, black, or a joker. (premise)

(2) One of cards 1, 2, or 3 is red. (premise)

(3) One of cards 1, 2, or 3 is black. (premise)

(4) One of cards 1, 2, or 3 is the joker. (premise)

(5) If card 1 is red, then card 3 is the joker. (premise)

(6) If card 1 is black, then card 3 is not the joker. (premise)

(7) If card 2 is red, then card 1 is black. (premise)

(8) If card 2 is black, then card 1 is not black. (premise)

(9) If card 3 is red, then it is the joker. (premise)

(10) If card 3 is black, then it is not the joker. (premise)

(11) Subproof:

 i. Card 3 is red. (assume)
 ii. Card 3 is exactly one of the following: red, black, or a joker. (1)
 iii. Card 3 is not the joker. (P1.3(11)ii, P1.3(11)i)
 iv. Card 3 is the joker. (9, P1.3(11)i)
 v. Contradiction. (P1.3(11)iv, P1.3(11)iii)

(12) Card 3 is not red. (11, proof by contradiction)

(13) Subproof:

 i. Card 2 is red. (assume)
 ii. Card 1 is black. (7, P1.3(13)i)
 iii. Card 3 is not the joker. (6, P1.3(13)ii)
 iv. Card 1 is not the joker. (1, P1.3(13)ii)
 v. Card 2 is not the joker. (1, P1.3(13)i)
 vi. None of cards 1, 2, or 3 are the joker. (P1.3(13)iv, P1.3(13)v, P1.3(13)iii)
 vii. Contradiction. (P1.3(13)vi, 4)

(14) Card 2 is not red. (13, proof by contradiction)

(15) Neither card 3 nor card 2 is red. (12, 14)

(16) Card 1 is red. (15, 2)

(17) Card 3 is the joker. (5, 16)

(18) Card 1 is not black. (16, 1)

(19) Card 3 is not black. (17, 1)

(20) Neither card 1 nor card 3 is black. (18, 19)

(21) Card 2 is black. (3, 20)

(22) Card 1 is red, card 2 is black, and card 3 is the joker. (16, 21, 17)