COMP30026 Models of Computation

Lecture 13: Proving Languages Non-Regular

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The Pumping Lemma for Regular Languages

Lemma

If A is a regular language over Σ, then there is some integer p such that, for all $s \in A$ of length at least p, there exist $x, y, z \in \Sigma^*$ such that $s = xyz$ and

1. $xy^iz \in A$ *for all* $i \geq 0$, *and*
2. $|y| > 0$, *and*
3. $|xy| \leq p$.

A Symbolic View of the Pumping Lemma

The pumping lemma says:

$$A \text{ regular} \Rightarrow \exists p \forall s \in A(|s| \geq p \to \exists x, y, z \in \Sigma^*(s = xyz \wedge \ldots))$$

Equivalently, its contrapositive says:

$$\forall p \exists s \in A(|s| \geq p \wedge \forall x, y, z \in \Sigma^*(s = xyz \to \ldots)) \Rightarrow A \text{ not regular}$$

Quantifier order is critical!!!

Pumping Example 1

Theorem. $B = \{0^n 1^n \mid n \geq 0\}$ is not regular.

Proof. Suppose to the contrary that it is. Let p be the pumping length and let $s = 0^p 1^p$.

Since $s \in B$ by definition and $|s| \geq p$, there exist x, y, z such that $s = xyz$, $xy^i z \in B$ for all $i \geq 0$, $|y| > 0$, $|xy| \leq p$.

In particular, we have $xyyz \in B$.

Now, since $|xy| \leq p$ and xyz starts with p 0s, we have $xy = 0^{|xy|}$ and $z = 0^{p-|xy|}1^p$.

Thus $xyyz = 0^{p+|y|}1^p$. Since $|y| > 0$, we have $p + |y| > p$, so $xyyz \notin B$ by definition.

Contradiction! Hence B is not regular.

Pumping Example 2

Theorem. $C = \{w \mid w \text{ has an equal number of 0s and 1s}\}$ is not regular.

Proof. Assume it is, and let p be the pumping length.

Consider $0^p1^p \in C$ with length greater than p.

By the pumping lemma, $0^p1^p = xyz$ for some x, y, z, with xy^iz in C for all $i \geq 0$, $y \neq \epsilon$, and $|xy| \leq p$. Since $|xy| \leq p$, y consists entirely of 0s.

But then $xyyz \notin C$, a contradiction.

A simpler alternative proof: If C were regular, then also B from before would be regular, since $B = C \cap L(0^*1^*)$ and regular languages are closed under intersection.

Pumping Example 3

Theorem. $D = \{ww \mid w \in \{0,1\}^*\}$ is not regular.

Proof. Assume it is, and let p be the pumping length.

Consider $0^p10^p1 \in D$ with length greater than p.

By the pumping lemma, $0^p10^p1 = xyz$ for some x, y, z, with xy^iz in D for all $i \geq 0$, $y \neq \epsilon$, and $|xy| \leq p$.

Since $|xy| \leq p$, y consists entirely of 0s.

But then $xyyz \notin D$, a contradiction.

Example 4 – Pumping Down

Theorem. $E = \{0^i 1^j \mid i > j\}$ is not regular.

Proof. Assume it is, and let p be the pumping length.

Consider $0^{p+1}1^p \in E$.

By the pumping lemma, $0^{p+1}1^p = xyz$ for some x, y, z, with xy^iz in E for all $i \geq 0$, $y \neq \epsilon$, and $|xy| \leq p$.

Since $|xy| \leq p$, y consists entirely of 0s.

But then $xz \notin E$, a contradiction.