School of Computing and Information Systems
COMP30026 Models of Computation
Week 5: Translation — Predicate Logic

Exercises

T5.1 Express the following statements in predicate logic, using the one-place predicates D and M, where $D(x)$ means "x is a duck", and $M(x)$ means "x is muddy". Use the constant a as a name for Jemima.

(i) Jemima is a duck.
(ii) If Jemima is a duck, then she is muddy.
(iii) Jemima is a muddy duck.
(iv) Every duck is muddy.
(v) There is a muddy duck.
(vi) Everything is a muddy duck.

T5.2 Let's add a new two-place predicate $L(x, y)$, which means "x lays y", and a one-place predicate $E(z)$ which means "z is an egg". Translate the following statements into predicate logic.

(i) Jemima lays an egg.
(ii) Jemima is muddy and lays an egg.
(iii) Everything lays an egg.
(iv) Every duck lays an egg.
(v) Every muddy duck lays an egg.
(vi) There is a duck who lays every egg.

Continued in P5.1 & P5.2

T5.3 Translate the following predicate logic formulas to natural language, by interpreting $L(x, y)$ as "x loves y". It may help to translate the *inner* formula first. For example, inside the first formula, $\exists y\, L(x, y)$ says "x loves somebody".

(i) $\forall x \exists y\, L(x, y)$
(ii) $\exists y \forall x\, L(x, y)$
(iii) $\forall x \exists y\, L(y, x)$
(iv) $\exists y \forall x\, L(y, x)$
(v) $\forall x \forall y\, L(x, y)$
(vi) $\forall y \forall x\, L(x, y)$

T5.4 What is the effect of quantifier order on a formula? Use your understanding of the formulas in T5.3 to explain which ones are semantic consequences of another. Are any logically equivalent? You do not need to compare every single pair of formulas.

Homework problems

P5.1 Translate the following formulas into predicate logic, using $D(x)$ for "x is a duck", $M(x)$ for "x is muddy", $F(x, y)$ for "x follows y" and $R(x, y)$ for "x is muddier than y". Use the constant a as a name for Jemima, and b for Louise.

(i) Every duck that follows a muddy duck is muddy.
(ii) Any muddy duck is muddier than any duck that isn't muddy.
(iii) Jemima is muddier than any duck that is muddier than Louise.
(iv) There is a duck that is muddier than Louise, but not as muddy as Jemima.

(v) Given any two ducks, one is muddier than the other.

(vi) Given any three ducks, if the first is muddier than the second, and the second is muddier than the third, then the first is muddier than the third.

P5.2 Translate the following formulas into predicate logic, using $D(x)$ for "x is a duck", $M(x)$ for "x is muddy", $E(x)$ for "x is an egg", $L(x, y)$ for "x lays y", and $R(x, y)$ for "x is muddier y". Use the constant a as a name for Jemima, and b for Louise.

(i) Not every duck lays an egg.

(ii) Every duck doesn't lay an egg.

(iii) There is a duck that doesn't lay an egg.

(iv) There isn't a duck who lays every egg.

(v) Louise is not muddier than every duck.

(vi) Louise is not muddier than any duck.

(vii) Every egg has a duck who laid it.

(viii) Not every egg is laid by a muddy duck.

(ix) If every duck is muddy, then no duck lays an egg.

P5.3 For any formula G and variable x, $\neg\forall x\, G \equiv \exists x\, \neg G$, and $\neg\exists x\, G \equiv \forall x\, \neg G$. Interpret the formula $\neg\forall x(D(x) \to \exists y(E(y) \land L(x, y)))$ in natural language, then use these equivalences to "push the negation" through each of the quantifiers and connectives, and re-interpret the result in natural language. Reflect on why these are saying the same thing.

P5.4 In the following formulas, identify which variables are bound to which quantifiers, and which variables are free.

(i) $\forall y(D(x) \land \exists x(E(y) \leftrightarrow L(x, y)))$

(ii) $\exists z(E(z) \land M(y)) \to \forall y(E(z) \land M(y))$

(iii) $\exists x((E(x) \land M(y)) \to \forall y(E(x) \land M(y)))$

(iv) $\forall z((\exists z\, D(z)) \to D(z))$

(v) $\exists u((D(z) \land \forall x(M(x) \to D(x))) \to \forall z(L(x, z)))$

(vi) $\forall x(\forall y(M(x) \to D(x)) \land \exists y(D(y) \land \forall y(L(y, x))))$

P5.5 On the Island of Knights and Knaves, everyone is a knight or knave. Knights always tell the truth. Knaves always lie. You meet three people, let us call them A, B and C. A says to you: "If I am a knight then my two friends here are knaves." Use **propositional** logic to determine whether this statement gives you any real information. What, if anything, can be deduced about A, B, and C?

P5.6 A graph colouring is an assignment of colours to nodes so that no edge in the graph connects two nodes of the same colour. The graph colouring problem asks whether a graph can be coloured using some fixed number of colours. The question is of great interest, because many scheduling problems are graph colouring problems in disguise. The case of three colours is known to be hard (NP-complete).

How can we encode the three-colouring problem in propositional logic—more precisely, in CNF? (One reason we might want to do so is that we can then make use of a SAT solver to determine colourability.) Using propositional variables

- B_i to mean node i is blue,
- G_i to mean node i is green,
- R_i to mean node i is red;
- E_{ij} to mean i and j are different but connected by an edge,

write formulas in CNF for these statements:

(a) Every node (0 to n inclusive) has a colour.
(b) Every node has at most one colour.
(c) No two connected nodes have the same colour.

For a graph with $n + 1$ nodes, what is the total size of these CNF formulas? (The size of a CNF formula is the number of literals it contains.)