COMP30026 Models of Computation

Lecture 5: Predicate Logic

Mak Nazecic-Andrlon and William Umboh

Semester 2, 2024

Why Predicate Logic?

Express statements about objects more faithfully.

In particular, *finitely* express:

- statements about infinite collections (e.g. the integers);
- transitive verbs (e.g. “loves”) and relative pronouns (e.g. “whose”).

Some Translations

No emus fly: $\forall x(\mathit{Emu}(x) \rightarrow \neg \mathit{Flies}(x))$

There are black swans: $\exists x(\mathit{Black}(x) \wedge \mathit{Swan}(x))$

If all push the cart, the donkey will be happy: $(\forall x\, P(x, c)) \rightarrow H(d)$

If the cart is pushed, the donkey will be happy: $(\exists x\, P(x, c)) \rightarrow H(d)$

Building Blocks

Five new kinds of symbols:

1. constants (e.g. c for "the cart", d for "the donkey")
2. predicates (e.g. P for "__ is pushing __", H for "__ is happy")
   - Identity, $=$, is a special case.
3. quantifiers
   - $\forall$ (pronounced "for all")
   - $\exists$ (pronounced "there exists")
4. variables (e.g. x, y, z)
5. functions (e.g. $+$, $\cdot$, f)

More Translations

Tina found Rover and returned him to Anne:

$$Found(tina, rover) \land Gave(tina, rover, anne)$$

Tina found a dog and gave it to Anne:

$$\exists x(Dog(x) \land Found(tina, x) \land Gave(tina, x, anne))$$

Lea inhabits the house that Jackie built:

$$\exists x(House(x) \land Inhabits(lea, x) \land BuilderOf(jackie, x))$$

Mothers' mothers are grandmothers:

$$\forall x \forall y \forall z((Mother(x, y) \land Mother(y, z)) \rightarrow Grandmother(x, z))$$

Existential Quantification

Existential quantification, $\exists$, is generalised $\vee$.

"Tina found some money and gave it to the Red Cross":

$$\exists x(Money(x) \wedge Found(tina, x) \wedge Gave(tina, x, redcross))$$

means:

$$(Found(tina, \$1) \wedge Gave(tina, \$1, redcross)) \vee$$
$$(Found(tina, \$2) \wedge Gave(tina, \$2, redcross)) \vee$$
$$\vdots$$

Universal Quantification

Universal quantification, $\forall$, is generalised $\wedge$.

"The square of every integer is nonnegative":

$$\forall x(x \in \mathbb{Z} \rightarrow (x \cdot x \geq 0))$$

means:

$$\begin{gathered} 0 \times 0 \geq 0 \wedge \\ 1 \times 1 \geq 0 \wedge \\ \vdots \end{gathered}$$

Quiz: Translate This

Translate "Every Melburnian barracks for a frisbee team".

Use these predicates:

Predicate	Interpretation
$M(x)$	x is a Melburnian
$T(x)$	x is a frisbee team
$B(x, y)$	x barracks for y

Terms

A term represents an individual object.

Terms are not formulas.

Examples:

- $redcross$
- 1
- x
- $f(x + y)$

Atomic Formulas

Atomic formulas represent statements about individual objects.

Examples:

- $\mathit{Flies}(x)$
- $P(x, c)$
- $\mathit{Gave}(\mathit{tina}, x, \mathit{redcross})$
- $2 + 2 = 5$
- $x \in y$

A Notational Convention

A predicate starts with an upper case letter; nothing else does.

- "$parent(rhonda)$" is a term.
  - Likely refers to "the parent of Rhonda".
- "$Parent(rhonda)$" is a formula.
  - Likely represents the proposition "Rhonda is a parent".

Syntax

Well-formed formulas (wffs) are generated by the grammar

$$
\begin{aligned}
\mathit{wff} \rightarrow\ & \mathit{atom} \\
\mid\ & \neg \mathit{wff} \\
\mid\ & (\mathit{wff} \wedge \mathit{wff}) \\
\mid\ & (\mathit{wff} \vee \mathit{wff}) \\
\mid\ & (\mathit{wff} \rightarrow \mathit{wff}) \\
\mid\ & (\mathit{wff} \leftrightarrow \mathit{wff}) \\
\mid\ & \forall \mathit{var}\ \mathit{wff} \\
\mid\ & \exists \mathit{var}\ \mathit{wff}
\end{aligned}
$$

Quantifier Scope

The subformula attached to a quantifier is its scope.

Example: $\forall x \, \underbrace{(P(x) \vee Q(c))}_{\text{scope of } \forall x}$

A quantifier with variable x binds x within its scope.

Example: x is bound in $\forall x \, P(x)$.

If a variable is not bound, it is free.

Example: x is free in $P(x)$.

Poll

Which variables are bound, and which are free?

1. $\forall z(P(x, y, z) \land \forall y(P(f(x), z, y)))$
2. $\forall x \exists y(x < y \land \exists z(y < z))$

Renaming Variables

Bound variables can be renamed unless there is a clash:

$\exists x \forall y (x < y)$ means the same as $\exists x \forall z (x < z)$.

But $\exists x \forall y \ (x \leq y)$ is very different to $\exists x \forall x \ (x \leq x)$.

Renaming free variables changes meaning:

$P(x)$ is different to $P(y)$.

From English to Predicate Logic

A rough guide:

1. Identify nouns, verbs, pronouns, adjectives, relative clauses.
2. Assign:
    - Constant symbols to singular objects (e.g. “Rhonda”).
    - Predicate symbols to verbs and adjectives (e.g. “loves”).
    - Function symbols to relative clauses (e.g. “the parent of”).
    - Variables to indefinite pronouns (e.g. “someone”).
3. Replicate logical structure of sentence.

Example Translations

Let $L(x, y)$ stand for "x loves y".

$L(alex, eva)$	Alex loves Eva
$\forall x\, L(x, eva)$	Everyone loves Eva (incl. Eva)
$\forall x\, (\neg(x = eva) \to L(x, eva))$	Eva is loved by everyone else
$\exists x\, (\neg(x = alex) \land L(x, alex))$	Someone other than Alex loves Alex
$\forall x \exists y\, L(x, y)$	Everybody loves somebody
$\exists y \forall x\, L(x, y)$	Someone is loved by everybody
$\exists x \forall y\, L(x, y)$	Someone loves everybody

Word Order

Consider word order with care:

- “There is something which is not P”: $\exists y \, \neg P(y)$
- “There is not something which is P” (“nothing is P”): $\neg\exists y \, P(y)$
- “All S are not P” vs “not all S are P”: $\forall x \, (S(x) \to \neg P(x))$ vs $\neg\forall x \, (S(x) \to P(x))$

Quantifier Order

Order of different quantifiers is important!!!

$\forall x \exists y\, L(x, y)$ says "everyone has someone they love".

$\exists y \forall x\, L(x, y)$ says "there is someone who is loved by everyone".

But $\forall x \forall y$ is the same as $\forall y \forall x$ and $\exists x \exists y$ is the same as $\exists y \exists x$.

Implicit Quantifiers

Often quantifiers are implicit in English.

Look for nouns (especially plural) without determiners.

"Humans are mortal" means "all humans are mortal":

$$\forall x(Human(x) \rightarrow Mortal(x))$$

"A horse is stronger than a dog" would usually mean:

$$\forall x \forall y((Horse(x) \land Dog(y)) \rightarrow Stronger(x, y))$$

"If a child owns a dog, the child spoils it":

$$\forall x \forall y((Child(x) \land Dog(y) \land Owns(x, y)) \rightarrow Spoils(x, y))$$