School of Computing and Information Systems
COMP30026 Models of Computation
Week 1: Informal Proof

If you finish the exercises early, start on the homework problems!

Exercises

T1.1 (Post's Correspondence Problem) Let's play a game. We have a finite set of "dominoes" such as

$$\left\{ \boxed{\frac{\texttt{b}}{\texttt{ca}}}, \boxed{\frac{\texttt{a}}{\texttt{ab}}}, \boxed{\frac{\texttt{ca}}{\texttt{a}}}, \boxed{\frac{\texttt{abc}}{\texttt{c}}} \right\}$$

Each domino has a string written in the upper half, and one in the lower half. The goal is to find a sequence of dominoes which, when laid side by side, spell out identical non-empty strings across the top and the bottom (or alternatively, determine that no solution is possible). You do not have to use every domino, and any domino can be used multiple times.

For example, the set given above has a solution, namely

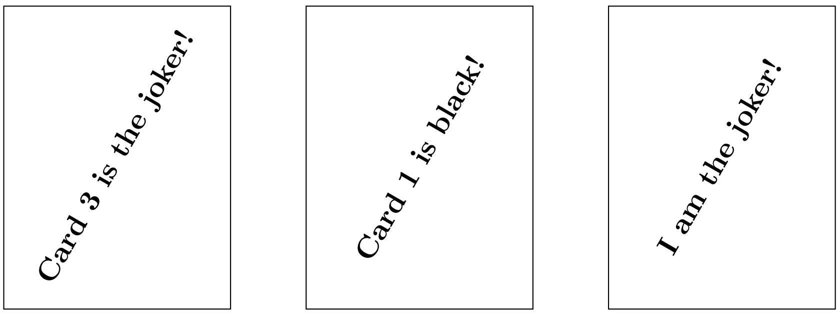
$$\boxed{\frac{\texttt{a}}{\texttt{ab}}}\boxed{\frac{\texttt{b}}{\texttt{ca}}}\boxed{\frac{\texttt{ca}}{\texttt{a}}}\boxed{\frac{\texttt{a}}{\texttt{ab}}}\boxed{\frac{\texttt{abc}}{\texttt{c}}}$$

(a) Can you find a solution to $\left\{ \boxed{\frac{\texttt{baa}}{\texttt{abaaa}}}, \boxed{\frac{\texttt{aaa}}{\texttt{aa}}} \right\}$?

(b) How about $\left\{ \boxed{\frac{\texttt{a}}{\texttt{cb}}}, \boxed{\frac{\texttt{bc}}{\texttt{ba}}}, \boxed{\frac{\texttt{c}}{\texttt{aa}}}, \boxed{\frac{\texttt{abc}}{\texttt{c}}} \right\}$?

(c) How about $\left\{ \boxed{\frac{\texttt{ab}}{\texttt{aba}}}, \boxed{\frac{\texttt{bba}}{\texttt{aa}}}, \boxed{\frac{\texttt{aba}}{\texttt{bab}}} \right\}$?

T1.2 Three playing cards lie face down on a table. One is red, one is black, and one is the joker (which is neither red nor black). Each card has a sentence on its back:



Card 1 Card 2 Card 3

The red card has a true sentence written on its back and the black card has a false sentence.

(a) Determine which card is red, which is black, and which is the joker.

(b) What does it mean for your answer to be correct? How can you check the correctness of your answer? Discuss.

(c) Are there multiple answers? If so, give another answer and check its correctness. Otherwise, give an informal proof (in prose) of the uniqueness of your answer.

Homework problems

P1.1 For each case from T1.1 which has no solution, write a proof of that fact. Try to make your answer as short as possible, but do not sacrifice rigour.

P1.2 Consider a square grid with x columns and y rows. We want to find out how many different paths there are that one can travel along, starting from the bottom left and ending in the top right cell, given that *one can only move up or right.* Let us call that value $f(x, y)$.

(a) Write down a table of the values of $f(x, y)$ when x and y range from 1 to 4. Draw the table as a grid, so that $f(1, 1)$ is in the lower left corner, $f(2, 1)$ is immediately to the right of it, and so on.

(b) Discuss what we mean by "path". How might we represent it mathematically?

(c) Propose and justify two different formulas for calculating $f(x, y)$. What are the benefits and drawbacks of each option? *Hint:* Think dynamic programming!

(d) Prove by induction that the two formulas are equivalent.

P1.3 Put each premise and deduction in your answer to T1.2c on a separate line of a numbered list.

(a) Next to each premise, write down that it is a premise. Next to each deduction, write down the numbers of the deductions or premises that it follows from.

(b) Split each nontrivial deduction into simpler (possibly still nontrivial) deductions. Repeat the process until the entire proof consists of premises and trivial deductions.

Hint: Your proof can have subproofs. If your original answer proves $\neg P$ by contradiction, convert that into a subproof of $\perp$ whose premise is P. Then you can conclude $\neg P$ from your subproof by *reductio ad absurdum.*