School of Computing and Information Systems
COMP30026 Models of Computation
Week 2: Propositional Logic

If you find yourself getting stuck on a particular question in the tutorial, try to move onto other questions until you have a chance to ask your tutor for help.

Homework problems

P2.1 Which of the following pairs of formulas are equivalent?

(a) $\neg P \to Q$ and $P \to \neg Q$
(b) $\neg P \to Q$ and $Q \to \neg P$
(c) $\neg P \to Q$ and $\neg Q \to P$
(d) $(P \to Q) \to P$ and P
(e) $P \to (Q \to R)$ and $Q \to (P \to R)$
(f) $P \to (Q \to R)$ and $(P \to Q) \to R$
(g) $(P \land Q) \to R$ and $P \to (Q \to R)$
(h) $(P \lor Q) \to R$ and $(P \to R) \land (Q \to R)$

Solution:

(a) Not equivalent:

P	Q	$\neg P$	$\to$	Q	P	$\to$	$\neg Q$
0	0	1	0	0	0	1	1
0	1	1	1	1	0	1	0
1	0	0	1	0	1	1	1
1	1	0	1	1	1	0	0
			✗			✗	

We see that the columns for the two implications are different.

(b) Not equivalent:

P	Q	$\neg P$	$\to$	Q	Q	$\to$	$\neg P$
0	0	1	0	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	1	0	0	1	0
1	1	0	1	1	1	0	0
			✗			✗	

(c) Equivalent:

P	Q	$\neg P$	$\to$	Q	$\neg Q$	$\to$	P
0	0	1	0	0	1	0	0
0	1	1	1	1	0	1	0
1	0	0	1	0	1	1	1
1	1	0	1	1	0	1	1
			✓			✓	

(d) Equivalent:

P	Q	$(P$	$\to$	$Q)$	$\to$	P
0	0	0	1	0	0	0
0	1	0	1	1	0	0
1	0	1	0	0	1	1
1	1	1	1	1	1	1
✓					✓	

(e), (f) : the pair in (e) equivalent; but the pair in (f) are not equivalent:

P	Q	R	P	$\to$	$(Q$	$\to$	$R)$	Q	$\to$	$(P$	$\to$	$R)$	$(P$	$\to$	$Q)$	$\to$	R
0	0	0	0	1	0	1	0	0	1	0	1	0	0	1	0	0	0
0	0	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1
0	1	0	0	1	1	0	0	1	1	0	1	0	0	1	1	0	0
0	1	1	0	1	1	1	1	1	1	0	1	1	0	1	1	1	1
1	0	0	1	1	0	1	0	0	1	1	0	0	1	0	0	1	0
1	0	1	1	1	0	1	1	0	1	1	1	1	1	0	0	1	1
1	1	0	1	0	1	0	0	1	0	1	0	0	1	1	1	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
				✓					✓								
				✗												✗	

Note that the formulas in (e) are both equivalent to $(P \land Q) \to R$, which explains why the order of P and Q does not matter here.

(g) Equivalent:

P	Q	R	$(P$	$\land$	$Q)$	$\to$	R	P	$\to$	$(Q$	$\to$	$R)$
0	0	0	0	0	0	1	0	0	1	0	1	0
0	0	1	0	0	0	1	1	0	1	0	1	1
0	1	0	0	0	1	1	0	0	1	1	0	0
0	1	1	0	0	1	1	1	0	1	1	1	1
1	0	0	1	0	0	1	0	1	1	0	1	0
1	0	1	1	0	0	1	1	1	1	0	1	1
1	1	0	1	1	1	0	0	1	0	1	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1
						✓			✓			

(h) Equivalent:

P	Q	R	$(P$	$\lor$	$Q)$	$\to$	R	$(P$	$\to$	$R)$	$\land$	$(Q$	$\to$	$R)$
0	0	0	0	0	0	1	0	0	1	0	1	0	1	0
0	0	1	0	0	0	1	1	0	1	1	1	0	1	1
0	1	0	0	1	1	0	0	0	1	0	0	1	0	0
0	1	1	0	1	1	1	1	0	1	1	1	1	1	1
1	0	0	1	1	0	0	0	1	0	0	0	0	1	0
1	0	1	1	1	0	1	1	1	1	1	1	0	1	1
1	1	0	1	1	1	0	0	1	0	0	0	1	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
						✓					✓			

P2.2 Define your own binary connective $\Box$ by writing out a truth table for $P \Box Q$ (fill in the middle column however you like). Can you write a formula which has the the same truth table as $P \Box Q$ using only the symbols P, Q, $\neg$, $\land$, $\lor$, and $\to$? Repeat the exercise once.

Solution: Here's an example

P	Q	P	$\Box$	Q
0	0	0	1	0
0	1	0	1	1
1	0	1	1	0
1	1	1	0	1

With this truth table for $\Box$, $P \Box Q$ is logically equivalent to $\neg(P \land Q)$.

P2.3 How many distinct truth tables are there involving two fixed propositional letters? In other words, how many meaningfully distinct connectives could we have defined in the previous question?

Solution: There are four rows in the truth table, so that's 2 choices for each row, which amounts to $2 \times 2 \times 2 \times 2 = 16$ possibilities.

P2.4 Find a formula that is equivalent to $P \leftrightarrow (P \wedge Q)$ but shorter.

Solution: $(P \wedge Q) \leftrightarrow P$ is logically equivalent to $P \to Q$. This is easily checked with a truth table, but how can we simplify $(P \wedge Q) \leftrightarrow P$ when we don't know what it is supposed to be equivalent to? Well, we can just try. Let us expand the biimplication and obtain $(P \to (P \wedge Q)) \wedge ((P \wedge Q) \to P)$. Intuitively, the conjunct on the right is just true, and we can check that with a truth table. So we have found that the original formula is equivalent to $P \to (P \wedge Q)$, which isn't any shorter, but still. We can rewrite the result as $(P \to P) \wedge (P \to Q)$, and now it becomes clear that all we need is $P \to Q$.

P2.5 Find a formula that is equivalent to $(\neg P \vee Q) \wedge R$ using only $\to$ and $\neg$ as logical connectives.

Solution: $(\neg P \vee Q) \wedge R$ is logically equivalent to $\neg((P \to Q) \to \neg R)$

P2.6 Consider the formula $P \to \neg P$. Is that a contradiction (is it *unsatisfiable*)? Can a proposition imply its own negation?

Solution: There is no contradiction at all. The formula is true if (and only if) P is false. The point of a conditional formula is to make a claim about the scenario where the premise (P) is true. If the premise of $\to$ is false, the formula is satisfied. For the same reason, $\neg P \to P$ is satisfiable; it is not a contradiction. But $P \leftrightarrow \neg P$ is clearly a contradiction. (If you disagree with any of these statements, draw truth tables.)

P2.7 By negating a satisfiable proposition, can you get a tautology? A satisfiable proposition? A contradiction? Illustrate your affirmative answers.

Solution: If you negate a satisfiable proposition, you can never get a tautology, since at least one truth table row will yield false.

You will get another satisfiable proposition iff the original proposition is not valid. For example, P is satisfiable (but not valid), and indeed $\neg P$ is satisfiable.

Finally, if we have a satisfiable formula which is also valid, its negation will be a contradiction. Example: $P \vee \neg P$.

P2.8 For each of the following propositional formulas, determine whether it is satisfiable, and if it is, whether it is a tautology:

(a) $P \leftrightarrow ((P \to Q) \to P)$
(b) $(P \to \neg Q) \wedge ((P \vee Q) \to P)$

Solution: Let us draw the truth tables.

(a)

P	Q	P	$\leftrightarrow$	$((P$	$\to$	$Q)$	$\to$	$P)$
0	0	0	1	0	1	0	0	0
0	1	0	1	0	1	1	0	0
1	0	1	1	1	0	0	1	1
1	1	1	1	1	1	1	1	1
			$\Uparrow$					

Hence satisfiable, and in fact valid (all 1).

(b)

P	Q	$(P$	$\to$	$\neg$	$Q)$	$\wedge$	$((P$	$\vee$	$Q)$	$\to$	$P)$
0	0	0	1	1	0	1	0	0	0	1	0
0	1	0	1	0	1	0	0	1	1	0	0
1	0	1	1	1	0	1	1	1	0	1	1
1	1	1	0	0	1	0	1	1	1	1	1
						$\Uparrow$					

Hence satisfiable (at least one 1), but not valid (not all 1). The truth table shows the formula is equivalent to $\neg Q$.

P2.9 Complete the following sentences, using the words "satisfiable, valid, non-valid, unsatisfiable".

(a) F is satisfiable iff F is not ______
(b) F is valid iff F is not ______
(c) F is non-valid iff F is not ______
(d) F is unsatisfiable iff F is not ______
(e) F is satisfiable iff $\neg F$ is ______
(f) F is valid iff $\neg F$ is ______
(g) F is non-valid iff $\neg F$ is ______
(h) F is unsatisfiable iff $\neg F$ is ______

Solution:

(a) F is satisfiable iff F is not unsatisfiable
(b) F is valid iff F is not non-valid
(c) F is non-valid iff F is not valid
(d) F is unsatisfiable iff F is not satisfiable
(e) F is satisfiable iff $\neg F$ is non-valid
(f) F is valid iff $\neg F$ is unsatisfiable
(g) F is non-valid iff $\neg F$ is satisfiable
(h) F is unsatisfiable iff $\neg F$ is valid

P2.10 Show that $P \leftrightarrow (Q \leftrightarrow R) \equiv (P \leftrightarrow Q) \leftrightarrow R$. This tells us that we could instead write

$$P \leftrightarrow Q \leftrightarrow R \tag{1}$$

without introducing any ambiguity. Mind you, that may not be such a good idea, because many people (incorrectly) tend to read "$P \leftrightarrow Q \leftrightarrow R$" as

$$P,\ Q, \text{ and } R \text{ all have the same truth value} \tag{2}$$

Show that (1) and (2) are incomparable, that is, neither is a logical consequence of the other.

Solution: Even with three variables the truth table is manageable, so let us construct it.

				(1A)						(1B)			(2)	
P	Q	R	P	$\leftrightarrow$	$(Q$	$\leftrightarrow$	$R)$	$(P$	$\leftrightarrow$	$Q)$	$\leftrightarrow$	R	$P \wedge Q \wedge R$	$\vee$	$\neg P \wedge \neg Q \wedge \neg R$
0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	1
0	0	1	0	1	0	0	1	0	1	0	1	1	0	0	0
0	1	0	0	1	1	0	0	0	0	1	1	0	0	0	0
0	1	1	0	0	1	1	1	0	0	1	0	1	0	0	0
1	0	0	1	1	0	1	0	1	0	0	1	0	0	0	0
1	0	1	1	0	0	0	1	1	0	0	0	1	0	0	0
1	1	0	1	0	1	0	0	1	1	1	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
				✓							✓			✗	

B is a logical consequence of A (we write $A \models B$) iff B is true for any assignment of variables which makes A true. So we see that (1) $\not\models$ (2), because cases like 1 0 0 that make (1) true but (2) false. Similarly, (2) $\not\models$ (1), because the case 0 0 0 makes (2) true but (1) false.

P2.11 Let F and G be propositional formulas. What is the difference between "$F \equiv G$" and "$F \leftrightarrow G$"? Prove that "$F \leftrightarrow G$" is valid iff $F \equiv G$.

Solution: The connective $\leftrightarrow$ is part of the language that we study, namely the language of propositional logic. So $A \leftrightarrow B$ is just a propositional formula.

The symbol $\equiv$ belongs to a *meta-language*. The meta-language is a language which we use when we reason *about* some language. In this case we use $\equiv$ to express whether a certain relation holds between formulas in propositional logic.

More specifically, $F \equiv G$ means that we have both $F \models G$ and $G \models F$. In other words, F and G have the same value for every possible assignment of truth values to their variables. The two formulas are logically equivalent.

On the other hand $F \leftrightarrow G$ is just a propositional formula (assuming F and G are propositional formulas). For some values of the variables involved, $F \leftrightarrow G$ may be false, for other values it may be true. By the definition of validity, $F \leftrightarrow G$ is *valid* iff it is true for *every* assignment of propositional variables in F and G.

We want to show that $F \equiv G$ iff $F \leftrightarrow G$ is valid.

(a) Suppose $F \equiv G$. Then F and G have the same values for each truth assignment to their variables[1]. But that means that, when we construct the truth table for $F \leftrightarrow G$, it will have a t in every row, that is, $F \leftrightarrow G$ is valid.

(b) Suppose $F \leftrightarrow G$ is valid. That means we find a t in each row of the truth table for $F \leftrightarrow G$. But we get a t for $F \leftrightarrow G$ iff the values for F and G agree, that is, either both are f, or both are t. In other words, F and G agree for every truth assignment. Hence $F \equiv G$.

You may think that this relation between validity and biimplication is obvious and should always be expected, and indeed we will see that it carries over to first-order predicate logic. But there are (still useful) logics in which it does not hold.

P2.12 Is $(P \land Q) \leftrightarrow P$ logically equivalent to $(P \lor Q) \leftrightarrow Q$?

Solution: Yes, $(P \land Q) \leftrightarrow P \equiv (P \lor Q) \leftrightarrow Q$, and both of these formulas are logically equivalent to $P \rightarrow Q$

P	Q	$(P$	$\land$	$Q)$	$\leftrightarrow$	P	$(P$	$\lor$	$Q)$	$\leftrightarrow$	Q
0	0	0	0	0	1	0	0	0	0	1	0
0	1	0	0	1	1	0	0	1	1	1	1
1	0	1	0	0	0	1	1	1	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1
					✓					✓	

[1] We should perhaps be more careful here, because F and G can be logically equivalent without F having the exact same set of variables as G—can you see how? So we should say that we consider both of F and G to be functions of the *union* of their sets of variables.