Sample solution for COMP30026 2024 A2

October 23, 2024

Q4

Proposition. *The language*

$$L = \{a^n b^m \mid n, m \geq 0\}$$

is context-free.

Proof. Let
$$A = \{a^n b^m \mid n, m \geq 0, 0 \cdot n + 1 \cdot m = 0\}$$
and
$$B = \{a^n b^m \mid n, m \geq 0, 1 \cdot n + 0 \cdot m = 0\}.$$
It is known that A and B are context-free languages.

Now, given any nonnegative integers n and m, we have $0 \cdot n + 1 \cdot m = 0$ if and only if $m = 0$. Thus,

$$\begin{aligned} A &= \{a^n b^m \mid n, m \geq 0, m = 0\} \\ &= \{a^n b^0 \mid n \geq 0\} \\ &= \{a^n \mid n \geq 0\}. \end{aligned}$$

Similarly, given any nonnegative integers n and m, we have $1 \cdot n + 0 \cdot m = 0$ if and only if $n = 0$, and so $B = \{b^m \mid m \geq 0\}$.

Since the context-free languages are closed under concatenation, the language $A \circ B$ is context-free. By the definition of concatenation of languages, we have
$$A \circ B = \{a^n b^m \mid n, m \geq 0\} = L,$$
and thus L is context-free, as desired. □

Q5

Proposition. *The language* $L = \{a^n b^m c^k \mid n, m, k \geq 0, nm = 2k\}$ *is not regular.*

Proof. Suppose to the contrary that L is regular. Let p be its pumping length. Let $s = a^{2p} b c^p$. Then $s \in L$ by definition, and since $|s| = 3p + 1 > p$, by the pumping lemma for regular languages, there exist strings x, y, z such that $s = xyz$, where $|xy| \leq p$, and $|y| > 0$, and $xy^i z \in L$ for all nonnegative integers i. In particular, $xz \in L$.

Since $|xy| \leq p$ and $s = xyz$, the string xy is a substring of the initial block of a's in s. Thus $xz = a^{2p-|y|}bc^p$. However, since $|y| > 0$, we have $(2p-|y|)\cdot 1 < 2p$ and thus $xz \notin L$ by definition. Contradiction! Hence L is not regular. □

Q6

Proposition. *Let* $\Sigma = \{a, b\}$ *and*

$$L = \{w \in \Sigma^* \mid \textit{for all nonempty } s \in \Sigma^*\textit{, the string } sss \textit{ does not occur in } w\}.$$

The language L *is not context-free.*

Proof. Suppose to the contrary that L is context-free. Let p be its pumping length. Then, since L is known to be infinite, there must exist some string $s \in L$ of length at least p. Therefore, by the pumping lemma for context-free languages, there exist strings u, v, x, y, z such that $s = uvxyz$, where $|vy| > 0$ and $uv^ixy^iz \in L$ for all nonnegative integers i. In particular, $s' \in L$ where $s' = uv^3xy^3z$.

Since $|vy| > 0$, either v is nonempty or y is nonempty. If v is nonempty, then since vvv occurs in s', it follows that $s' \notin L$ by definition. If v is instead empty, then y is nonempty, and $s' \notin L$ by definition because yyy occurs in s'.

Thus $s' \notin L$ in all cases. Contradiction! Hence L is not context-free. □