COMP30026 Models of Computation

Lecture 3: Consequence and Satisfaction

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Recap: Models

"$\models$" is short for "is a model of" or "satisfies".

$v \models F$ iff F is true under v.

Examples:

$$\{P \mapsto \mathbf{1}, Q \mapsto \mathbf{0}\} \models P \vee Q$$
$$\{P \mapsto \mathbf{0}\} \models \neg P$$

Non-examples:

$$\{P \mapsto \mathbf{1}, Q \mapsto \mathbf{0}\} \not\models P \rightarrow Q$$
$$\{P \mapsto \mathbf{1}\} \not\models \neg P$$

Recap: Equivalence

"$\equiv$" is short for "is logically equivalent to".

$A \equiv B$ iff A and B always have equal truth values.

Examples:

$$P \rightarrow Q \equiv \neg P \vee Q$$
$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

Non-examples:

$$P \rightarrow Q \not\equiv R \rightarrow S$$
$$P \wedge Q \not\equiv P \vee Q$$

Semantic Consequence

Definition

G is a *semantic consequence* of F if and only if every model of F is a model of G.

For short, we write "$F \models G$".

"$\models$" is pronounced "(semantically) entails".

Note: $F \equiv G$ iff $F \models G$ and $G \models F$.

Consequence and Implication

Let F and G be formulas.

Theorem

$F \models G$ *if and only if* $\models F \rightarrow G$.

As an immediate corollary:

Corollary

$F \equiv G$ *if and only if* $\models F \leftrightarrow G$.

Poll

Of the following formulas, which allow us to conclude $P \rightarrow Q$?

1. P
2. $\neg P$
3. Q
4. $P \rightarrow (Q \wedge R)$
5. $(P \vee R) \rightarrow Q$
6. $\neg P \vee Q$
7. $\neg Q \rightarrow \neg P$
8. $P \rightarrow (Q \vee R)$
9. $(P \rightarrow Q) \vee R$

Which of the formulas are consequences of $P \rightarrow Q$?

Tautology

Tautology: a formula of propositional logic which is always true.

$(\neg P \wedge Q) \rightarrow (P \rightarrow R)$ is a tautology:

P	Q	R	(	$\neg$	P	$\wedge$	Q	)	$\rightarrow$	(	P	$\rightarrow$	R	)
1	1	1		0	1	0	1		**1**		1	1	1	
1	1	0		0	1	0	1		**1**		1	0	0	
1	0	1		0	1	0	0		**1**		1	1	1	
1	0	0		0	1	0	0		**1**		1	0	0	
0	1	1		1	0	1	1		**1**		0	1	1	
0	1	0		1	0	1	1		**1**		0	1	0	
0	0	1		1	0	0	0		**1**		0	1	1	
0	0	0		1	0	0	0		**1**		0	1	0	

Contradiction

Contradiction: A formula of propositional logic which is always false.

$P \land Q \land (\neg Q \leftrightarrow (\neg P \lor Q))$ is a contradiction.

No need for whole truth table.

If either P or Q are false, $\land$ makes whole formula false.

When P and Q are both true:

P	Q	(	P	$\land$	Q	)	$\land$	(	$\neg$	Q	$\leftrightarrow$	(	$\neg$	P	$\lor$	Q	))
1	1		1	1	1		**0**		0	1	0		0	1	1	1	

Negating a contradiction gives a tautology and vice versa.

Tautologies Are Valid

Consider: "If Bolsonaro is sane, then Bolsonaro is sane."

It is true *regardless* of what "Bolsonaro" or "sane" mean.

Valid: Always true.

Non-valid: Sometimes false.

$\models F$ is short for "F is valid".

Contradictions Are Unsatisfiable

Consider: "Santa Claus is good and Santa Claus is not good."

It is false *regardless* of what "Santa Claus" or "good" mean.

Unsatisfiable: Never true.

Satisfiable: Sometimes true.

Most Statements Are Contingent

Consider: "It is currently raining."

It is true if and only if it is currently raining.

Contingent: Sometimes true, sometimes false.

Poll

Classify the following formulas as valid, contingent, or unsatisfiable:

1. P
2. $P \leftrightarrow \neg P$
3. $P \rightarrow (\neg Q \vee P)$
4. $\neg Q \vee \neg(P \wedge \neg Q)$

Substitution

Replacing all uses of a propositional letter with a given formula.

Example

Consider "$P \rightarrow P$".

Substitute P with "$(Q \wedge R)$".

Result: "$(Q \wedge R) \rightarrow (Q \wedge R)$".

Substitution preserves validity!

Poll

Does substitution preserve unsatisfiability?

Yes!

Negation of contradiction is a tautology.

Poll

Does substitution preserve satisfiability?

No — a counterexample is easy:

Take P (which is clearly satisfiable).

Then substitute P by $Q \land \neg Q$.

Substitution Preserves Logical Equivalence

Denote by $F[A := B]$ the result of substituting A with B in F.

Example: $(P \to P)[P := Q]$ is $(Q \to Q)$.

Theorem

Let F, G, H be formulas and P be any propositional letter.

If $F \equiv G$, then $F[P := H] \equiv G[P := H]$.

Interchange of Equivalents

If $F \equiv G$, then F can be freely replaced with G.

Theorem

Let H' be the result of replacing an instance of F with G in H.

If $F \equiv G$, then $H \equiv H'$.

Result is equivalent: all semantic properties preserved.

Rewrite formulas algebraically!

Some Equivalences

Absorption:
$$P \wedge P \equiv P$$
$$P \vee P \equiv P$$

Commutativity:
$$P \wedge Q \equiv Q \wedge P$$
$$P \vee Q \equiv Q \vee P$$

Associativity:
$$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$$
$$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$$

Distributivity:
$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$
$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$

More Equivalences

$\leftrightarrow$ is also commutative and associative.

Double negation:	$P \equiv \neg\neg P$
De Morgan:	$\neg(P \land Q) \equiv \neg P \lor \neg Q$ $\neg(P \lor Q) \equiv \neg P \land \neg Q$
Implication:	$P \to Q \equiv \neg P \lor Q$
Contraposition:	$\neg P \to \neg Q \equiv Q \to P$ $P \to \neg Q \equiv Q \to \neg P$ $\neg P \to Q \equiv \neg Q \to P$
Biimplication:	$P \leftrightarrow Q \equiv (P \land Q) \lor (\neg P \land \neg Q)$

Quiz

Find an alternative way of writing biimplication—as a conjunction.

Last Equivalences

Let $\bot$ be any unsatisfiable formula and let $\top$ be any valid formula.

Duality:	$\neg\top \equiv \bot$ $\neg\bot \equiv \top$
Negation from absurdity:	$P \to \bot \equiv \neg P$
Identity:	$P \vee \bot \equiv P$ $P \wedge \top \equiv P$
Dominance:	$P \wedge \bot \equiv \bot$ $P \vee \top \equiv \top$
Contradiction:	$P \wedge \neg P \equiv \bot$
Excluded middle:	$P \vee \neg P \equiv \top$

Poll

Which of these claims hold?

1. $P \rightarrow Q \equiv (Q \leftrightarrow (P \vee Q))$
2. $(P \rightarrow Q) \wedge (P \rightarrow R) \equiv P \rightarrow (Q \wedge R)$
3. $(P \rightarrow R) \wedge (Q \rightarrow R) \models (P \wedge Q) \rightarrow R$

Exit Puzzle

Vivian and her partner invited four other couples. When everyone arrived, some of the people shook hands with some of the others. Of course, nobody shook hands with their partner, or themselves, and nobody shook hands with the same person twice.

Afterwards, Vivian asked everyone how many times they shook somebody's hand. She received different answers from all nine!

How many times did Vivian's partner shake hands?

Post your answers to the discussion board!