Sample solution for COMP30026 2024 A1

September 2024

Q1

Task A

Interpretations of propositional letters:

- M_A: A is a mimic.
- M_B: B is a mimic.
- M_C: C is a mimic.
- A: A is a knight.
- B: B is a knight.
- C: C is a knight.

Translations of facts:

- $M_A \lor M_B \lor M_C$
- $A \leftrightarrow (C_M \lor C)$
- $B \leftrightarrow \neg(M_A \land \neg C)$
- $C \leftrightarrow (B \to ((M_A \to \neg A) \land (M_B \to \neg B) \land (M_C \to \neg C)))$

Task B

Proposition. *B is the mimic.*

Proof. Suppose to the contrary that C is a knight. Then A is telling the truth, since it is true that either C is the mimic or is a knight. So A is also a knight. Since C is not a knave, it is not the case the both A is the mimic and C is a knave. So B is also telling the truth, and hence must also be a knight. Thus, since C is telling the truth, the mimic must be a knave. But there are no knaves. Contradiction! Therefore C is not a knight, and must hence be a knave.

Suppose to the contrary that the mimic is a knave. Then, if B is a knight, then the mimic is a knave. So C is telling the truth, and is hence a knight. Contradiction! So the mimic is not a knave.

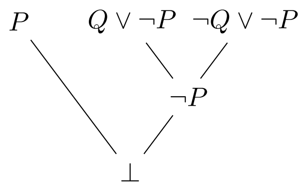
Thus, since C is a knave, C is not the mimic. So it is not true that C is either the mimic or a knight, and so A is lying. Hence A is a knave, and must also not be a mimic. Since neither A nor C are the mimic, and there is a mimic among A, B, and C, the mimic must be B. □

Q2

1. Contingent. True under $\{P \mapsto \mathbf{0}\}$, and false under $\{P \mapsto \mathbf{1}\}$.
2. Unsatisfiable. Conversion to CNF:

$$\begin{aligned} & (P \vee (P \wedge (Q \rightarrow Q))) \wedge (\neg P \vee \neg(\neg Q \rightarrow \neg Q)) \\ \equiv\ & (P \vee (P \wedge (Q \vee \neg Q))) \wedge (\neg P \vee (Q \wedge \neg Q)) \\ \equiv\ & (Q \vee P \vee \neg Q) \wedge (P \vee P) \wedge (Q \vee \neg P) \wedge (\neg Q \vee \neg P) \\ \equiv\ & P \wedge (Q \vee \neg P) \wedge (\neg Q \vee \neg P). \end{aligned}$$

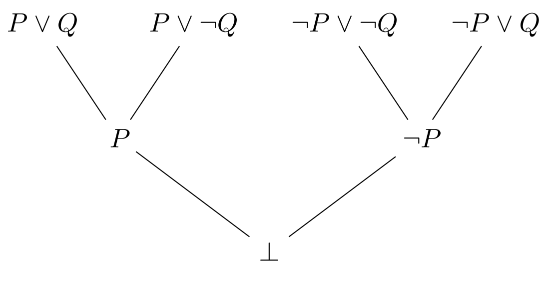
   Refutation:



3. Valid. Convert negation to CNF:

$$\begin{aligned} & \neg((Q \vee P) \rightarrow P) \vee (P \leftrightarrow Q)) \vee (P \wedge \neg Q) \\ \equiv\ & \neg(\neg((Q \vee P) \rightarrow P) \vee ((P \leftrightarrow Q) \vee (P \wedge \neg Q))) \\ \equiv\ & ((Q \vee P) \rightarrow P) \wedge \neg((P \leftrightarrow Q) \vee (P \wedge \neg Q)) \\ \equiv\ & (P \vee \neg(Q \vee P)) \wedge \neg((P \leftrightarrow Q) \vee (P \wedge \neg Q)) \\ \equiv\ & \neg(P \leftrightarrow Q) \wedge (P \vee \neg(Q \vee P)) \wedge \neg(P \wedge \neg Q) \\ \equiv\ & \neg((\neg P \wedge \neg Q) \vee (P \wedge Q)) \wedge (P \vee (\neg Q \wedge \neg P)) \wedge (\neg P \vee Q) \\ \equiv\ & (P \vee Q) \wedge (\neg P \vee \neg Q) \wedge (P \vee \neg Q) \wedge (P \vee \neg P) \wedge (\neg P \vee Q) \\ \equiv\ & (P \vee Q) \wedge (\neg P \vee \neg Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \end{aligned}$$

   Refutation:



4. Contingent. True under $\{P \mapsto \mathbf{0}, Q \mapsto \mathbf{1}, R \mapsto \mathbf{1}\}$, false under $\{P \mapsto \mathbf{0}, Q \mapsto \mathbf{1}, R \mapsto \mathbf{0}\}$.

Q3

Task A

Interpretation:

- fe: iron
- o: oxygen
- $H(x, y)$: x is heavier than y
- $A(x)$: x is an actinide
- $L(x)$: x is a lanthanide
- $R(x)$: x is radioactive
- $I(x, y)$: x is an isotope of y
- pm: promethium

Translation:

1. $H(fe, o)$
2. $\forall x(A(x) \to R(x))$
3. $\exists x(L(x) \land R(x)) \land \exists x(L(x) \land \neg R(x))$
4. $\forall x \forall y((A(x) \land L(y)) \to H(x, y))$
5. $\forall x((A(x) \lor L(x)) \to (H(x, fe) \land H(x, o)))$
6. $\exists x_1 \exists x_2 \exists x_3 \exists y_1 \exists y_2 \exists y_3 (L(x_1) \land L(x_2) \land L(x_3) \land I(y_1, x_1) \land I(y_2, x_2) \land I(y_3, x_3) \land y_1 \neq y_2 \land y_2 \neq y_3 \land y_3 \neq y_1 \land R(y_1) \land R(y_2) \land R(y_3)) \land \forall x(L(x) \to (\neg \exists y(I(y, x) \land \neg R(y)) \leftrightarrow x = pm))$

Task B

Theorem. *Every model of* $\forall x \forall y(P(x, y) \to \neg P(y, x)) \land \forall x \exists y(P(x, y))$ *has at least three elements.*

Proof. Let M be a model of the formula

$$\forall x \forall y(P(x, y) \to \neg P(y, x)) \land \forall x \exists y(P(x, y))$$

with universe U and interpretation function I. Then, by the definition of truth for conjunction, we have both

$$M \vDash \forall x \forall y(P(x, y) \to \neg P(y, x)) \tag{1}$$

and

$$M \vDash \forall x \exists y(P(x, y)). \tag{2}$$

Thus, for all variable assignments v, we have

$$M, v \vDash P(x, y) \to \neg P(y, x), \tag{3}$$

$$M, v \vDash \exists y(P(x, y)). \tag{4}$$

Hence, for all variable assignments v, from (3) and the definitions of truth for implication, negation, and atomic formulas, we have either $(v(x), v(y)) \notin I(P)$ or $(v(y), v(x)) \notin I(P)$. Therefore, for all $s, t \in U$, either $(s, t) \notin U$ or $(t, s) \notin U$.

As an immediate consequence, if $(s,t) \in I(P)$ for some $s,t \in U$, then s and t are distinct. Thus we also have $(s,s) \notin I(P)$ for all $s \in U$.

Now, let v be a variable assignment. From (4), we have $M, v_{y \mapsto a} \vDash P(x,y)$ for some element $a \in U$ by definition. Thus

$$(v(x), v_{y \mapsto a}(y)) = (v(x), a) \in I(P)$$

by definition. Hence $(a, v(x)) \notin I(P)$ by our earlier result, and thus $a \neq v(x)$.

By (4) again, we have $M, v_{x \mapsto a, y \mapsto c} \vDash P(x,y)$ for some $c \in U$. Thus $(a,c) \in I(P)$. However, c cannot be equal to either $v(x)$ or a, since neither $(a, v(x))$ nor (a,a) are in $I(P)$. Hence c, $v(x)$, and a are pairwise distinct elements of U, and thus U has at least 3 elements. $\square$

Q4

Proposition. *A tree consisting of a non-black root with a red left child is not a red-black tree*

Proof. We can express "the root of the tree is not black and its left child is red" with the formula $\neg B(a) \wedge R(l(a))$, whose clausal form is $\{\{\neg B(a)\}, \{R(l(a))\}\}$. It suffices to show that this formula is jointly unsatisfiable with conditions (1) and (2). First, we convert the conditions to clausal form:

1.

$$\begin{aligned}
&\forall x((R(x) \wedge \neg B(x)) \vee (B(x) \wedge \neg R(x))) \\
\equiv\ &(R(x) \wedge \neg B(x)) \vee (B(x) \wedge \neg R(x)) \\
\equiv\ &(R(x) \vee B(x)) \wedge (\neg B(x) \vee \neg R(x)) \\
\equiv\ &\{\{R(x), B(x)\}, \{\neg B(x), \neg R(x)\}\}.
\end{aligned}$$

2.

$$\begin{aligned}
&\forall x(R(x) \rightarrow (\neg R(l(x)) \wedge \neg R(r(x)))) \\
\equiv\ &R(x) \rightarrow (\neg R(l(x)) \wedge \neg R(r(x))) \\
\equiv\ &\neg R(x) \vee (\neg R(l(x)) \wedge \neg R(r(x))) \\
\equiv\ &(\neg R(x) \vee \neg R(l(x))) \wedge (\neg R(x) \vee \neg R(r(x))) \\
\equiv\ &\{\{\neg R(x), \neg R(l(x))\}, \{\neg R(x), \neg R(r(x))\}\}.
\end{aligned}$$

And now we give a resolution refutation of a subset of the union of those sets of clauses:

$$
\begin{tikzcd}[column sep=small]
\neg R(x), \neg R(l(x)) \arrow[ddrr, dash] & & R(x), B(x) \arrow[dr, dash] & & \neg B(a) \arrow[dl, dash] & & R(l(a)) \arrow[dddll, dash] \\
& & & R(a) \arrow[dl, dash] & & & \\
& & \neg R(l(a)) \arrow[drr, dash] & & & & \\
& & & & \perp & &
\end{tikzcd}
$$

Since this subset of the original set of clauses is unsatisfiable, so is the original set, as desired. ☐

Q5

Errors:

1. Rather than checking that ss is the reverse of ss as given by Definition 4(c), the proof jumps to the conclusion that ss is a palindrome, presumably because the expression $a_1 \dots a_n a_n \dots a_1$ looks symmetric.

2. The proof does not consider the case of s being empty.

3. The proof introduces $b_1, \dots, b_{2n}$ and then does not use them.

Corrected:

Theorem. *Let s be a palindrome. Then ss is also a palindrome.*

Proof. If s is empty, then ss is also empty, so $s = ss$ and hence ss is also a palindrome.

Suppose instead that s is nonempty. Then $s = a_1 \dots a_n$ for some symbols $a_1, \dots, a_n$ where n is the length of s. Therefore $ss = a_1 \dots a_n a_1 \dots a_n$ is a string of length $2n$, and hence there exist symbols $b_1, \dots, b_{2n}$ such that $ss = b_1 \dots b_{2n}$, where

$$b_i = \begin{cases} a_i & \text{if } i \leq n, \\ a_{i-n} & \text{otherwise,} \end{cases}$$

for all positive integers $i \leq 2n$.

Now, since s is a palindrome, it is by definition equal to itself under reversal, so $a_i = a_{n-i+1}$ for all positive integers $i \leq n$. Let $i \leq 2n$ be a positive integer. We consider two cases:

1. If $i \leq n$, then $n < 2n - i + 1 \leq 2n$, and so

$$\begin{aligned} b_{2n-i+1} &= a_{(2n-i+1)-n} \\ &= a_{n-i+1} \\ &= a_i \\ &= b_i. \end{aligned}$$

2. Otherwise, we have $i > n$, and so

$$\begin{aligned} b_i &= a_{i-n} \\ &= a_{n-(i-n)+1} \\ &= a_{2n-i+1} \end{aligned}$$

and since $1 \leq 2n - i + 1 \leq n$,

$$= b_{2n-i+1}.$$

So $b_i = b_{2n-i+1}$ in all cases.

Thus $b_i = b_{2n-i+1}$ holds for all positive integers $i \leq 2n$, and so by definition, the string $ss = b_1 \dots b_{2n}$ is equal to its own reverse. Therefore ss is indeed a palindrome. $\square$