COMP30026 Models of Computation

6: Predicate Logic: Semantics

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Meaning of a Formula

True or false?

$$\forall x \forall z(x < z \rightarrow \exists y(x < y \wedge y < z))$$

Depends on how we interpret it.

1. False if discussing $\mathbb{Z}$ and $<$ is “less than”.
2. True if discussing $\mathbb{Z}$ and $<$ is “less than or equal”.
3. True if discussing $\mathbb{R}$ and $<$ is “less than”.
4. Unconditionally true if universe is $\{0\}$.

Universe of Discourse

The set of things we are talking about.

Quantifiers range over the universe of discourse.

Analogy: Who is "everyone" in "everyone loves Eva"?

Interpretation Functions

Non-logical symbols: predicate, function and constant symbols.

Interpretation function: maps every non-logical symbol to its interpretation.

Constant symbol $\mapsto$ object in U.

Predicate symbol $\mapsto$ relation on U.

Function symbol $\mapsto$ function from U to U.

Models

Model: universe of discourse + interpretation function.

$\mathbb{Z}$ with usual $+$ is a model of $\forall x \exists y (x + y = 1)$.

$\mathbb{N}$ with usual $+$ is not a model of $\forall x \exists y (x + y = 1)$.

Free Variables

Truth of $P(x)$ depends on choice of x.

Variable assignment: a function $v : vars \to U$.

It assigns values to variables.

Example

Consider a model M with universe $U = \{red, blue\}$ and interpretation function I such that $I(P) = \{red\}$. Let $v : var \to U$ be a variable assignment. Then,

- $P(x)$ is true in M under v if $v(x) = red$.
- $P(x)$ is false in M under v if $v(x) = blue$.

Truth of Formulas Depends on Their Parts

The truth of a closed formula depends only on the model.

The only reason why we are interested in formulas with free variables (and hence in variable assignments) is that we want to define the truth of a formula compositionally, as done on the next slide.

Useful notation:

$$v_{x \mapsto d}(y) = \begin{cases} d & \text{if } y = x \\ v(y) & \text{otherwise} \end{cases}$$

Read this as "the map v, updated to map x to d."

Defining Truth in Predicate Logic

In a model M (with universe U and interpretation function I), under a variable assignment v:

- **Atomic formulas:** $P(t_1, \ldots, t_n)$ is true iff $(v(t_1), \ldots, v(t_n)) \in I(P)$.
- **Existential quantifier:** $\exists x\, F$ is true iff there exists at least one $d \in U$ such that F is true in M under $v_{x \mapsto d}$.
- **Universal quantifier:** $\forall x\, F$ is true iff $\neg \exists x\, \neg F$ is true.

Definitions for connectives are same as propositional logic.

Quantifier Order

$\forall x \exists y$ is not the same as $\exists y \forall x$.

The former says each x has a y that satisfies $P(x, y)$; the latter says there's an individual y that satisfies $P(x, y)$ for every x.

$\forall x \forall y$ is the same as $\forall y \forall x$.

$\exists x \exists y$ is the same as $\exists y \exists x$.

Quantified Formulas as a Two-Person Game

Say I make a claim and you try to disprove it. You get to supply values for the universally quantified variables.

- If I claim $\forall x \exists y\ P(x, y)$, then you can challenge me by choosing an x and asking me to find the y that satisfies $P(x, y)$, but I get to know the x you chose.
- If I claim $\exists y \forall x\ P(x, y)$, then you can challenge me by asking me to provide the y, and then you just have to find an x that does not satisfy $P(x, y)$, knowing the y that I chose.
- If I claim $\exists x \exists y\ P(x, y)$, then I have to find both x and y, so it doesn't matter what order they appear.
- If I claim $\forall y \forall x\ P(x, y)$, then you get to pick both x and y, so again their order does not matter.

Rules of Passage for the Quantifiers

We cannot always move subformulas through quantifiers.

Consider $\exists x(P(x) \land Q(x))$.

However, for all formulas F and G:

$$\begin{aligned}
\exists x(\neg F) &\equiv \neg \forall x\, F \\
\forall x(\neg F) &\equiv \neg \exists x\, F \\
\exists x(F \lor G) &\equiv (\exists x\, F) \lor (\exists x\, G) \\
\forall x(F \land G) &\equiv (\forall x\, F) \land (\forall x\, G)
\end{aligned}$$

From this, you can prove

$$\exists x(F \rightarrow G) \equiv (\forall x\, F) \rightarrow (\exists x\, G)$$

More Rules of Passage for Quantifiers

If G is a formula with no free occurrences of x, then we also get

$$\begin{aligned}
\exists x\ G &\equiv G \\
\forall x\ G &\equiv G \\
\exists x(F \land G) &\equiv (\exists x\ F) \land G \\
\forall x(F \lor G) &\equiv (\forall x\ F) \lor G \\
\forall x(F \rightarrow G) &\equiv (\exists x\ F) \rightarrow G \\
\forall x(G \rightarrow F) &\equiv G \rightarrow (\forall x\ F)
\end{aligned}$$

no matter what F is. In particular F may have free occurrences of x.

Satisfication and Consequence

We write $M, v \models F$ to mean "F is true in model M under variable assignment v".

F is true in M iff we have $M, v \models F$ for all variable assignments v. We write $M \models F$.

F semantically entails G iff every model of F is a model of G. We write $F \models G$.

F is logically equivalent to G iff $F \models G$ and $G \models F$. We write $F \equiv G$.

Summarising: Satisfiability and Validity

A closed formula F is

- satisfiable iff $M \models F$ for some model M;
- valid iff $M \models F$ for every model M;
- unsatisfiable iff $M \not\models F$ for every model M;
- non-valid iff $M \not\models F$ for some model M.
- contingent iff it is satisfiable but not valid.

As in the propositional case, we have

- F is valid iff $\neg F$ is unsatisfiable;
- F is non-valid iff $\neg F$ is satisfiable.

Example of Non-Validity

Consider the formula

$$(\forall y \exists x \; P(x, y)) \rightarrow (\exists x \forall y \; P(x, y))$$

It is not valid.

For example, consider the model with universe $U = \mathbb{Z}$, and the predicate P meaning “less than”.

Or, let $U = \{0, 1\}$ and let P mean “equals”.

The formula is satisfiable, as it is true, for example, in the model where $U = \{0, 1\}$ and P means “less than or equal”.

Example of Validity

$F = (\exists y \forall x\ P(x, y)) \rightarrow (\forall x \exists y\ P(x, y))$ is valid.

If we negate F (and rewrite it) we get

$$(\exists y \forall x\ P(x, y)) \wedge (\exists x \forall y\ \neg P(x, y))$$

The right conjunct is made true only if there is some $a \in U$ for which $(a, b) \notin I(P)$ for all $b \in U$.

But the left conjunct requires that $(a, b) \in I(P)$ for at least some b.

Since F's negation is unsatisfiable, F is valid.

Another Example of Validity

Consider

$$F = (\forall x\ P(x)) \rightarrow P(t)$$

F is valid no matter what the term t is.

To see this, again it is easiest to consider

$$\neg F = (\forall x\ P(x)) \land \neg P(t)$$

The term t denotes some element of the universe U, so $\neg F$ cannot be satisfied.

Today's Take-Home Puzzle

Deckard is a blade runner—her job is to identify replicants who look exactly like humans but who have actually been created in the laboratories of Tyrell Corp.

Deckard interviews suspects. The problem is that some replicants are programmed to always speak the truth, while others are programmed to always lie. Unfortunately the same goes for the humans that Deckard deals with: some are always truthful, and the rest always lie.

One day, a suspect makes a simple statement that allows Deckard to conclude the suspect is a lying replicant.

What statement would do that?

Next Lecture: Clausal form for first-order predicate logic.