COMP30026 Models of Computation

Lecture 2: Propositional Logic

Mak Nazecic-Andrlon and William Umboh

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Our Goals for the Next Few Lectures

Introduce formal propositional logic.

Use as vehicle for more general logic concepts.

Use for simple mechanised proof.

Pay attention!

Even if you have seen this before, we need the concepts for later.

Propositional = Boolean Logic

Until the mid-19th century, “logic” meant Aristotelian logic.

George Boole took an algebraic view of logic.

Deep connection between logic and arithmetic.

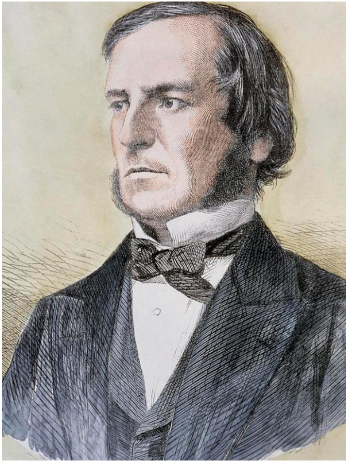

Figure: George Boole, circa 1864

Intro Puzzle

Heidi, Dina and Louise are being questioned by their aunt.

Here is what they say:

> *Heidi: "Dina and Louise had equal share in it; if one is guilty, so is the other."*
> *Dina: "If Heidi is guilty, then so am I."*
> *Louise: "Dina and I are not both guilty."*

Their aunt, knowing that they are honest kids, realises that they cannot tell a lie.

Has she got sufficient information to decide who (if any) are guilty?

Syntax

We shall build propositional formulas from this set of symbols:

$$\underbrace{A, B, C, \ldots, Z}_{\text{prop. letters}}, \underbrace{\neg, \wedge, \vee, \rightarrow, \leftrightarrow}_{\text{connectives}}, (,).$$

Well-formed formulas (wffs) are generated by this grammar:

$$\begin{aligned} \mathit{wff} \coloneqq\ & A \mid B \mid C \mid \ldots \mid Z \\ & \mid \neg \mathit{wff} \\ & \mid (\mathit{wff} \wedge \mathit{wff}) \\ & \mid (\mathit{wff} \vee \mathit{wff}) \\ & \mid (\mathit{wff} \rightarrow \mathit{wff}) \\ & \mid (\mathit{wff} \leftrightarrow \mathit{wff}) \end{aligned}$$

Some Well-Formed Formulas

$$P \tag{1}$$

$$(P \to Q) \tag{2}$$

$$(P \lor \neg P) \tag{3}$$

$$\neg(P \land \neg P) \tag{4}$$

$$(P \leftrightarrow \neg P) \tag{5}$$

$$(((P \to Q) \to P) \to P) \tag{6}$$

Pronunciation Guide

Symbol	Pronunciation
$\neg$	not
$\vee$	or; vee
$\wedge$	and; wedge
$\rightarrow$	if ... then; implies; only if; arrow
$\leftrightarrow$	if and only if; biimplies; double arrow

Warning: This is only pronunciation!

These symbols are not shorthands for English words!

Notational Conveniences

For sake of readability, we will follow these informal rules:

- Drop outermost parentheses.
- Drop inner parentheses in nested uses of $\wedge$ and $\vee$.
  - $P \wedge Q \wedge R$ is short for either of:
    - $((P \wedge Q) \wedge R)$
    - $(P \wedge (Q \wedge R))$
  - Same is true if you replace every "$\wedge$" with "$\vee$".
  - Warning: $P \wedge Q \vee R$ is nonsense.

What is Truth?

What does it mean for $\underset{\text{“}xy\text{”}}{\text{“}P \wedge Q\text{”}}$ to be true?

What about just $\underset{\text{“}x\text{”}}{\text{“}P\text{”}}$?

Is $\underset{\text{“}\max\{x, 1-x\}\text{”}}{\text{“}P \vee \neg P\text{”}}$ true?

Boolean Semantics: Connectives

Definition (Truth function)

A function from truth values to truth values.

Boolean truth values: **t** and **f** (also written **1** and **0**).

Connectives are truth-functional.

Usually presented as a truth table:

A	B	$\neg A$	$A \land B$	$A \lor B$	$A \to B$	$A \leftrightarrow B$
0	**0**	**1**	**0**	**0**	**1**	**1**
0	**1**	**1**	**0**	**1**	**1**	**0**
1	**0**	**0**	**0**	**1**	**0**	**0**
1	**1**	**0**	**1**	**1**	**1**	**1**

Boolean Semantics: Letters

Propositional letters are Boolean variables.

Definition (Truth assignment)

A function from propositional letters to truth values.

Usual notation:

$$v = \{P \mapsto \mathbf{1}, Q \mapsto \mathbf{0}\}.$$

We then have:

$$\begin{aligned} v(P) &= \mathbf{1} \\ v(Q) = v(R) = \cdots = v(Z) &= \mathbf{0}. \end{aligned}$$

Truth of a Formula

Let $v = \{P \mapsto \mathbf{1}, Q \mapsto \mathbf{0}\}$.

Poll: Which of these formulas are true under v?

1. $P \wedge Q$
2. $(P \vee Q) \wedge (P \vee R)$
3. $P \rightarrow Q$
4. $\neg P \rightarrow \neg Q$

Shorthand: "$v \vDash \phi$" means "ϕ is true under v".

Truth Tables for Formulas

P	Q	R	$((P$	$\wedge$	$Q)$	$\vee$	$R)$
0	0	0	0	0	0	**0**	0
0	0	1	0	0	0	**1**	1
0	1	0	0	0	1	**0**	0
0	1	1	0	0	1	**1**	1
1	0	0	1	0	0	**0**	0
1	0	1	1	0	0	**1**	1
1	1	0	1	1	1	**1**	0
1	1	1	1	1	1	**1**	1

Poll

Which of these have the same truth tables?

1. $P \to Q$ and $\neg Q \to \neg P$
2. $(P \to Q) \land (P \to R)$ and $P \to (Q \land R)$
3. $(P \to R) \land (Q \to R)$ and $(P \land Q) \to R$

Hint: $P \to Q$ has the same truth table as $\neg P \lor Q$.

Logical Equivalence

Definition

Formulas are *logically equivalent* iff they have equal truth values under **every** truth assignment.

Shorthand: "$F \equiv G$" means "F is logically equivalent to G".

Material Conditional

Warning: "$\rightarrow$" is weird!

Often read as "implies", but causality is not required!

A	B	$A \rightarrow B$
0	**0**	**1**
0	**1**	**1**
1	**0**	**0**
1	**1**	**1**

1. If volume increases, then pressure falls.
2. If Melbourne is in Queensland, then Brisbane is in Victoria.
3. Melbourne and Brisbane are in different states and if Melbourne is in Queensland then so is Brisbane.

Modus Ponens

$$\frac{\begin{array}{l} P \rightarrow Q \\ P \end{array}}{Q}$$

A rule is sound if every model of the premises is a model of the conclusion.

Challenge: prove that modus ponens is sound.

Exit Puzzle

On the island of Knights and Knaves, everyone is a knight or knave. Knights always tell the truth. Knaves always lie.

Today there is a census on the island!

You are a census taker, going from house to house. Fill in what you know about each of these three houses.

- **In house 1:** We are both knaves.
- **In house 2:** At least one of us is a knave.
- **In house 3:** If I am a knight then so is my wife.

If you like these puzzles, Raymond Smullyan has written lots of books that you will like.