School of Computing and Information Systems
COMP30026 Models of Computation
Week 4: Translation and Resolution

Homework problems

P4.1 Using resolution, show that the following set of conjunctive clauses is unsatisfiable:

$$\{\{P, R, \neg S\}, \{P, S\}, \{\neg Q\}, \{Q, \neg R, \neg S\}, \{\neg P, Q\}\}.$$

P4.2 Find the reduced CNF of $\neg((\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B))$ and express the result in clausal form. Then determine whether a refutation of the resulting set is possible.

Solution: Let us follow the method given in a lecture, except we do the double-negation elimination aggressively, as soon as opportunity arises:

$\neg((\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B))$	
$\neg(\neg(B \vee \neg A) \vee \neg(B \vee A) \vee B)$	(unfold $\rightarrow$ and eliminate double negation)
$(B \vee \neg A) \wedge (B \vee A) \wedge \neg B$	(de Morgan for outermost neg; elim double neg)

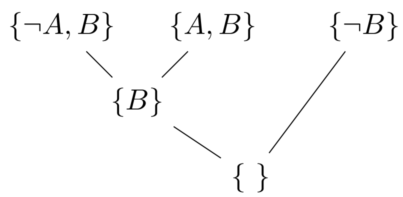
This is RCNF without further reductions.

We could also have used other transformations—sometimes this can shorten the process. For example, we could have rewritten the sub-expression $\neg B \rightarrow \neg A$ as $A \rightarrow B$ (the contraposition principle). You may want to check that this does not change the result.

The resulting formula, written as a set of sets of literals:

$$\{\{\neg A, B\}, \{A, B\}, \{\neg B\}\}$$

We can now construct the refutation:



P4.3 Use resolution to show that each of these formulas is a tautology:

(a) $(P \vee Q) \rightarrow (Q \vee P)$
(b) $(\neg P \rightarrow P) \rightarrow P$
(c) $((P \rightarrow Q) \rightarrow P) \rightarrow P$
(d) $P \leftrightarrow ((P \rightarrow Q) \rightarrow P)$

Solution:

(a) $(P \vee Q) \rightarrow (Q \vee P)$. First negate the formula (why?), to get $\neg((P \vee Q) \rightarrow (Q \vee P))$. Then we can use the usual techniques to convert the negated proposition to RCNF. Here is a useful shortcut, combining $\rightarrow$-elimination with one of de Morgan's laws:

$$\neg(A \rightarrow B) \equiv A \wedge \neg B.$$

So:

$$\begin{array}{ll}
\neg((P \vee Q) \rightarrow (Q \vee P)) & \\
(P \vee Q) \wedge \neg(Q \vee P) & \text{(shortcut)} \\
(P \vee Q) \wedge \neg Q \wedge \neg P & \text{(de Morgan)}
\end{array}$$

The result allows for a straight-forward refutation:

$$\begin{tikzcd}[column sep=small]
\{P, Q\} \arrow[dr, dash] & & \{\neg Q\} \arrow[dl, dash] & & \{\neg P\} \arrow[ddll, dash] \\
& \{P\} \arrow[dr, dash] & & & \\
& & \{\,\} & &
\end{tikzcd}$$

(b) $(\neg P \rightarrow P) \rightarrow P$. Again, first negate the formula, to get $\neg((\neg P \rightarrow P) \rightarrow P)$. Then turn the result into RCNF:

$$\begin{array}{ll}
\neg((\neg P \rightarrow P) \rightarrow P) & \\
(\neg P \rightarrow P) \wedge \neg P & \text{(shortcut from above)} \\
(\neg\neg P \vee P) \wedge \neg P & \text{(unfold } \rightarrow\text{)} \\
(P \vee P) \wedge \neg P & \text{(eliminate double negation)} \\
P \wedge \neg P & (\vee\text{-absorption})
\end{array}$$

The resolution proof is immediate; we will leave it out.

(c) $((P \rightarrow Q) \rightarrow P) \rightarrow P$. Again, negate the formula, to get $\neg(((P \rightarrow Q) \rightarrow P) \rightarrow P)$. Then turn the result into RCNF:

$$\begin{array}{ll}
\neg(((P \rightarrow Q) \rightarrow P) \rightarrow P) & \\
((P \rightarrow Q) \rightarrow P) \wedge \neg P & \text{(shortcut, outermost } \rightarrow\text{)} \\
((\neg P \vee Q) \rightarrow P) \wedge \neg P & \text{(unfold } \rightarrow\text{)} \\
(\neg(\neg P \vee Q) \vee P) \wedge \neg P & \text{(unfold } \rightarrow\text{)} \\
((\neg\neg P \wedge \neg Q) \vee P) \wedge \neg P & \text{(de Morgan)} \\
((P \wedge \neg Q) \vee P) \wedge \neg P & \text{(double negation)} \\
(P \vee P) \wedge (\neg Q \vee P) \wedge \neg P & \text{(distribution)} \\
P \wedge (\neg Q \vee P) \wedge \neg P & \text{(absorption)}
\end{array}$$

Again this gives an immediate refutation: just resolve $\{P\}$ against $\{\neg P\}$.

(d) $P \leftrightarrow ((P \rightarrow Q) \rightarrow P)$. Negating the formula, we get $P \oplus ((P \rightarrow Q) \rightarrow P)$. Let us turn the resulting formula into RCNF:

$$\begin{array}{ll}
P \oplus ((P \rightarrow Q) \rightarrow P) & \\
(P \vee ((P \rightarrow Q) \rightarrow P)) \wedge (\neg P \vee \neg((P \rightarrow Q) \rightarrow P)) & \text{(eliminate } \oplus\text{)} \\
(P \vee ((P \rightarrow Q) \rightarrow P)) \wedge (\neg P \vee ((P \rightarrow Q) \wedge \neg P)) & \text{(shortcut from above)} \\
(P \vee (\neg(\neg P \vee Q) \vee P)) \wedge (\neg P \vee ((\neg P \vee Q) \wedge \neg P)) & (\rightarrow\text{-elimination}) \\
(P \vee (\neg\neg P \wedge \neg Q) \vee P) \wedge (\neg P \vee ((\neg P \vee Q) \wedge \neg P)) & \text{(de Morgan)} \\
(P \vee (P \wedge \neg Q) \vee P) \wedge (\neg P \vee ((\neg P \vee Q) \wedge \neg P)) & \text{(double negation)} \\
P \wedge (P \vee \neg Q) \wedge (\neg P \vee ((\neg P \vee Q) \wedge \neg P)) & (\vee\text{-absorption, distribution}) \\
P \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge \neg P & (\vee\text{-absorption, distribution})
\end{array}$$

Once again, now just resolve $\{P\}$ against $\{\neg P\}$.

P4.4 For each of the following clause sets, write down a propositional formula in CNF to which it corresponds. Which of the resulting formulas are satisfiable? Give models of those that are.

(a) $\{\{A, B\}, \{\neg A, \neg B\}, \{\neg A, B\}\}$

(b) $\{\{A, \neg B\}, \{\neg A\}, \{B\}\}$

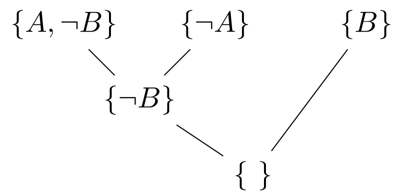
(c) $\{\{A\}, \emptyset\}$

(d) $\{\{A, B\}, \{\neg A, \neg B\}, \{B, C\}, \{\neg B, \neg C\}, \{A, C\}, \{\neg A, \neg C\}\}$

Solution:

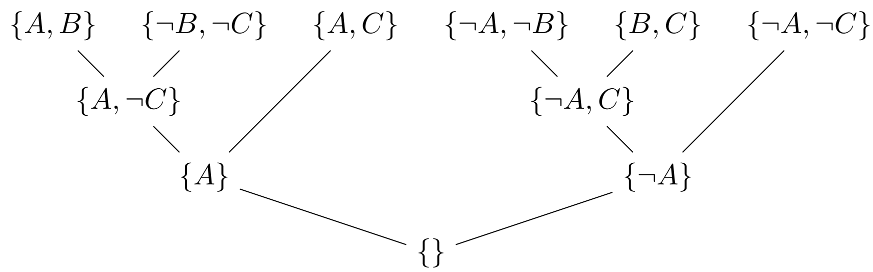
(a) $\{\{A, B\}, \{\neg A, \neg B\}, \{\neg A, B\}\}$ stands for the formula $(A \vee B) \wedge (\neg A \vee \neg B) \wedge (\neg A \vee B)$. This is satisfiable by $\{A \mapsto \mathbf{f}, B \mapsto \mathbf{t}\}$.

(b) $\{\{A, \neg B\}, \{\neg A\}, \{B\}\}$ stands for $(A \vee \neg B) \wedge \neg A \wedge B$. A refutation is easy:



(c) $\{\{A\}, \emptyset\}$ stands for $A \wedge \mathbf{f}$, which is clearly not satisfiable.

(d) We have $\{\{A, B\}, \{\neg A, \neg B\}, \{B, C\}, \{\neg B, \neg C\}, \{A, C\}, \{\neg A, \neg C\}\}$. This set is not satisfiable, as a proof by resolution shows:



P4.5 Consider these assumptions:

(a) If Ann can clear 2 meters, she will be selected.

(b) If Ann trains hard then, if she gets the flu, the selectors will be sympathetic.

(c) If Ann trains hard and does not get the flu, she can clear 2 meters.

(d) If the selectors are sympathetic, Ann will be selected.

Does it follow that Ann will be selected? Does she get selected if she trains hard? Use any of the propositional logic techniques we have discussed, to answer these questions.

Solution: Let us give names to the propositions:

- C: Ann clears 2 meters
- F: Ann gets the flu
- K: The selectors are sympathetic
- S: Ann is selected
- T: Ann trains hard

The four assumptions then become:

(a) $C \rightarrow S$
(b) $T \rightarrow (F \rightarrow K)$
(c) $(T \wedge \neg F) \rightarrow C$
(d) $K \rightarrow S$

It is easy to see that S is not a logical consequence of these, as we can give all five variables the value *false*, and all the assumptions will thereby be true.

To see that $T \rightarrow S$ is a logical consequence of the assumptions, we can negate it, obtaining $T \wedge \neg S$. Then, translating everything to clausal form, we can use resolution to derive an empty clause.

Alternatively, note that $T \rightarrow (F \rightarrow K)$ is equivalent to $(T \wedge F) \rightarrow K$. Since also $K \rightarrow S$, we have $(T \wedge F) \rightarrow S$. Similarly, $(T \wedge \neg F) \rightarrow C$ together with $C \rightarrow S$ gives us $(T \wedge \neg F) \rightarrow S$.

But from $(T \wedge F) \rightarrow S$ and $(T \wedge \neg F) \rightarrow S$ we get $T \rightarrow S$. (You may want to check that by massaging the conjunction of the two formulas.)

P4.6 Consider the following four statements:

(a) The commissioner cannot attend the function unless he resigns and apologises.
(b) The commissioner can attend the function if he resigns and apologises.
(c) The commissioner can attend the function if he resigns.
(d) The commissioner can attend the function only if he apologises.

Identify the basic propositions involved and translate the statements into propositional logic. In particular, what is the translation of a statement of the form "X does not happen unless Y happens"? Identify cases where one of the statements entails some other statement in the list.

Solution: Let us give names to the propositions:

- A: The commissioner apologises
- F: The commissioner can attend the function
- R: The commissioner resigns

The four statements then become

(a) $F \rightarrow (A \wedge R)$
(b) $(R \wedge A) \rightarrow F$
(c) $R \rightarrow F$
(d) $F \rightarrow A$

The first translation may not be obvious. But to say "X does not happen unless Y happens" is the same as saying "it is not possible to have X happen and at the same time Y does not happen." That is, $\neg(X \wedge \neg Y)$, which is equivalent to $X \rightarrow Y$. Note that (a) entails (d) and (c) entails (b).

P4.7 Letting F and G be two different formulas from the set

$$\{(P \wedge Q) \vee R, \quad (P \vee Q) \wedge R, \quad P \wedge (Q \vee R), \quad P \vee (Q \wedge R)\}$$

list all combinations that satisfy $F \models G$.

P4.8 A formula is in *disjunctive* normal form (DNF) iff it is a disjunction of conjunctions of literals. We say a formula is in RDNF (reduced disjunctive normal form) iff it is in DNF and no conjunctive clause has duplicate literals, is a contradiction, or subsumes another conjunctive clause. (If F and G are *conjunctive* clauses, we say F subsumes G iff $F \vee G \equiv F$.)

We claim that the formula

$$(P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge R) \vee (Q \wedge \neg R)$$

is logically equivalent to the simpler

$$\neg P \vee Q$$

with both being in reduced disjunctive normal form (RDNF). Show that the claim is correct.

P4.9 In P4.8 we saw that truth of this formula does not depend on the truth of R:

$$(P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge R) \vee (Q \wedge \neg R)$$

Find a shortest logically equivalent CNF formula which includes R.