School of Computing and Information Systems
COMP30026 Models of Computation
Week 5: Predicate Logic — Semantics and Resolution

Exercises

T5.1 For each of the following predicate logic formulas, give a model in which the formula is true, and a model in which the formula is false.

(a) $\forall x \forall y\, P(x,y)$

(b) $\forall x \exists y (P(x,y) \land \neg P(y,x))$

(c) $(\forall x \exists y\, \neg P(x,y)) \land (\forall x \exists y\, P(y,x))$

T5.2 Use resolution to show that the following set of clauses is unsatisfiable:

$$\{Q(f(u), g(x)), P(x)\}, \{\neg P(b)\}, \{\neg Q(z,y), R(z,z)\}, \{\neg R(w, f(u))\}$$

T5.3 Turn $\exists z \forall x\ \exists y \left[\exists u\ \forall v \Big(P(x,y) \lor Q(z,u,v) \Big) \to \forall u\ \Big(\neg R(f(x),u,y) \Big) \right]$ into clausal form.

T5.4 Use the following theorem to prove that $\forall x\, P(x) \models \exists x\, P(x)$ holds. (Recall that the universe of discourse is nonempty by definition.) Does $\exists x\, P(x) \models \forall x\, P(x)$ also hold?

Theorem. Given any model M, variable assignment v, variable x and formula F, if $M, v \models \forall x\, F$, then $M, v \models F$.

Homework problems

P5.1 Consider the following predicates:

- $C(x)$, which stands for "x is a cat"
- $M(x)$, which stands for "x is a mouse"
- $L(x,y)$, which stands for "x likes y"

Express the statement "No mouse likes a cat who likes all mice" as a formula in first-order predicate logic. Once you have, convert that formula into clausal form.

P5.2 For this question use the following predicates:

- $G(x)$ for "x is a green dragon"
- $R(x)$ for "x is a red dragon"
- $H(x)$ for "x is a happy dragon"
- $S(x)$ for "x can spit fire"
- $P(x,y)$ for "x is a parent of y"
- $C(x,y)$ for "x is a child of y"

(a) Express the following statements as formulas in first-order predicate logic:

i. x is a parent of y if and only if y is a child of x.
ii. A dragon is either green or red; not both.
iii. A dragon is green if and only if at least one of its parents is green.
iv. Green dragons can spit fire.
v. A dragon is happy if all of its children can spit fire.

(b) Translate each of the five formulas to clausal form.

(c) Prove, using resolution, that all green dragons are happy.

P5.3 Consider the following formulas:

(a) $\forall x \neg L(x,x)$
(b) $\forall x \exists y\, L(x,y)$
(c) $\forall x \forall z (L(x,z) \rightarrow \exists y (L(x,y) \wedge L(y,z)))$

Give a model which satisfies all three formulas. Can this be done with a finite universe? If so, how many elements does the smallest such universe have?

P5.4 Using equivalences, show that $\neg \forall x \exists y\, (\neg P(x) \wedge P(y))$ is valid. Then convert the formula to clausal form.

P5.5 Prove the theorem given in T5.4.

P5.6 Turn $\neg \forall x\ \exists y \left[\forall z \left(Q(x,z) \wedge P(y) \right) \wedge \forall u \left(\neg Q(u,x) \right) \right]$ into clausal form.

P5.7 Turn $\forall x \forall y \exists z \Big(P(x) \rightarrow \forall y \forall z (Q(y,z)) \Big)$ into a simpler, equivalent formula of the form $\varphi \rightarrow \psi$.

P5.8 For each of the following pairs of terms, determine whether the pair is unifiable. If it is, give the most general unifier. (Don't forget our agreed convention: for constants we use letters from the beginning of the alphabet, here a and b, whereas for variables we use letters from the end of the alphabet.)

(i) $h(f(x), g(y, f(x)), y)$ and $h(f(u), g(v,v), u)$
(ii) $h(f(g(x,y)), y, g(y,y))$ and $h(f(u), g(a,v), u)$
(iii) $h(g(x,x), g(y,z), g(y, f(z)))$ and $h(g(u,v), g(v,u), v)$
(iv) $h(v, g(v), f(u,a))$ and $h(g(x), y, x)$
(v) $h(f(x,x), y, y, x)$ and $h(v, v, f(a,b), a)$

P5.9 Consider the following predicates:

- $E(x,y)$, which stands for "x envies y"
- $F(x,y)$, which stands for "x is more fortunate than y"

(a) Using 'a' for Adam, express, in first-order predicate logic, the sentence "Adam envies everyone more fortunate than him."
(b) Using 'e' for Eve, express, in first-order predicate logic, the sentence "Eve is no more fortunate than any who envy her."
(c) Formalise an argument for the conclusion that "Eve is no more fortunate than Adam." That is, express this statement in first-order predicate logic and show that it is a logical consequence of the other two.

P5.10 Using the unification algorithm, determine whether $Q(f(g(x), y, f(y,z,z)), g(f(a,y,z)))$ and $Q(f(u, g(a), v), u)$ are unifiable. If they are, give a most general unifier. (As usual, we use letters from the end of the alphabet for variables, and letters from the beginning of the alphabet for constants.)

P5.11 Determine whether $P(f(g(x), f(g(x), g(a))), x)$ and $P(f(u, f(v,v)), u)$ are unifiable. If they are, give a most general unifier.

P5.12 The barber paradox is a variant of Russell's paradox: say there is a barber who shaves those and *only* those who do not shave themselves. Does the barber shave themselves? The question has no answer: both "yes" and "no" immediately lead to a contradiction.

Using $B(x)$ to mean "x is a barber" and $S(u, v)$ to mean "u shaves v", translate the premise of the paradox into predicate logic. Then, using resolution (with factoring!), show that it is unsatisfiable.

P5.13 Consider the following unsatisfiable set of clauses:

$$\{\{P(x)\}, \{\neg P(x), \neg Q(y)\}, \{Q(x), \neg R(y)\}, \{R(x), S(a)\}, \{R(b), \neg S(x)\}\}$$

What is the simplest refutation proof, if "simplest" means "the refutation tree has minimal depth"? What is the simplest refutation proof, if "simplest" means "the refutation tree has fewest nodes"?

P5.14 Consider these statements:

S_1: "No politician is honest."
S_2: "Some politicians are not honest."
S_3: "No Australian politician is honest."
S_4: "All honest politicians are Australian."

(a) Using the predicate symbols P and H for being a politician and being honest, respectively, express S_1 and S_2 as formulas of predicate logic F_1 and F_2.

(b) Is $F_1 \rightarrow F_2$ satisfiable?

(c) Is $F_1 \rightarrow F_2$ valid?

(d) Using the predicate symbol A for "is Australian", express S_3 and S_4 in clausal form.

(e) Using resolution, show that S_1 is a logical consequence of S_3 and S_4.

(f) Prove or disprove the statement "S_2 is a logical consequence of S_3 and S_4."

P5.15 Consider a model M with universe $U = \{\text{Jemima}, \text{Thelma}, \text{Louise}\}$ and interpretation function I such that

$$\begin{aligned}
I(a) &= \text{Jemima}, \quad I(b) = \text{Thelma}, \quad I(c) = \text{Louise}, \\
I(F) &= \{(\text{Jemima}, \text{Louise}), (\text{Thelma}, \text{Jemima}), (\text{Thelma}, \text{Thelma}), (\text{Louise}, \text{Thelma})\}, \\
I(M) &= \{\text{Jemima}, \text{Louise}\}.
\end{aligned}$$

Let v be a variable assignment such that $v(x) = \text{Louise}, v(y) = \text{Thelma}, v(z) = \text{Jemima}$.

For each of the following formulas, determine whether it is true or false in the model M under the variable assignment v. In each instance, prove your claim from the formal semantics.

(i) $F(x, a)$

(ii) $\exists y\, F(x, y)$

(iii) $\forall x \exists y\, F(x, y)$

(iv) $\forall y\, F(b, y)$

(v) $\exists x \forall y\, F(x, y)$

P5.16 Show that following "equivalences" are incorrect, by specifying a model which makes one formula true and the other false.

(i) $\exists x(F(x) \land G(x)) \overset{?}{\equiv} \exists x F(x) \land \exists x G(x)$

(ii) $\forall x(F(x) \lor G(x)) \overset{?}{\equiv} \forall x F(x) \lor \forall x G(x)$

(iii) $\forall x \exists y R(x, y) \overset{?}{\equiv} \exists x \forall y R(x, y)$