Question 1

1. Let X_n be a Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

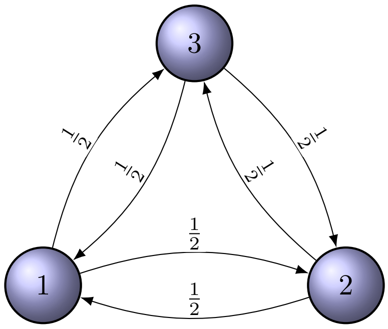


Figure 1-1: Visual repreesntation of the Markov chain

a. Find the 2-step transition probabilities.

Solution.

By definition, $\mathbf{P}^{(2)} = \mathbf{P}^2$. For $i = j$,

$$p_{ii}^{(2)} = \sum_k p_{ik}p_{ki} = p_{ij}p_{ji} + p_{i\ell}p_{\ell i} = \tfrac{1}{2} \cdot \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{2},$$

where $\{j, \ell\} = S \setminus \{i\}$. For $i \neq j$,

$$p_{ij}^{(2)} = \sum_k p_{ik}p_{kj} = p_{ii}p_{ij} + p_{ij}p_{jj} + p_{i\ell}p_{\ell j} = 0 + 0 + \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}.$$

Hence

$$\mathbf{P}^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

□

b. Is the chain periodic?

Solution.

The period of state i is $\gcd\{n \geq 1 : p_{ii}^{(n)} > 0\}$ (Ch. 4.5.1). Since $p_{11}^{(2)} > 0$ and also $p_{11}^{(3)} > 0$ (e.g., $1 \to 2 \to 3 \to 1$ has positive probability), the set of return times contains 2 and 3, so $\gcd(2, 3) = 1$. Thus state 1 has period 1; by irreducibility (which we discuss more in detail in the next subquestion) all states share the same period, so the chain is *aperiodic*.

□

c. Is the chain irreducible?

Solution.

Yes. From each state we can reach any other state in one step (all off-diagonal entries of $\mathbf{P}$ are positive), so all states communicate; by the definition the chain is irreducible.

□

d. Find the stationary distribution.

Solution.

A stationary distribution π satisfies $\pi P = \pi$ and $\sum_i \pi_i = 1$. Here P is doubly stochastic (row and column sums are 1), hence the uniform vector is stationary:

$$\pi = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right).$$

We can check this directly by observing that:

$$\pi \mathbf{P} = \pi \tag{1}$$

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \times \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \tag{2}$$

□

e. If the chain starts in 1, is the chain stationary?

Solution.

No. "Stationary" for a time-homogeneous Markov chain means the initial law is a stationary distribution (so all marginal laws are constant over time). Starting in 1 means the initial distribution is $\delta_1 \neq \pi$, hence the process is *not* stationary.

□

f. Find $\lim_{n\to\infty} \mathbf{P}(X_n = 1)$ if the chain starts in 1.

Solution.

For finite, irreducible, aperiodic Markov chains, $\mathbf{P}^n(i, \cdot) \to \pi(\cdot)$ as $n \to \infty$ (convergence to the unique stationary distribution). Thus

$$\lim_{n\to\infty} \mathbf{P}(X_n = 1 \mid X_0 = 1) = \pi_1 = \tfrac{1}{3}.$$

□

g. Would the answer to point f. change if the initial probabilities were different?

Solution.

No. By the same convergence result, for *any* initial distribution α, we have $\alpha \mathbf{P}^n \to \pi$; in particular $\mathbf{P}(X_n = 1) \to \pi_1 = 1/3$ regardless of the starting law.

□

Question 2

a. Simulate in Matlab or in Python 1000 steps $(x_1, \dots, x_{1000})$ of a Markov chain with state space $S = \{1, 2, 3\}$, initial state 2 and transition matrix

$$\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

Solution.

```python
import numpy as np

rng = np.random.default_rng(40313)

P = np.array([[0.0, 0.5, 0.5],
              [0.5, 0.0, 0.5],
              [0.5, 0.5, 0.0]], dtype=float)

n_steps = 1000
x = np.empty(n_steps, dtype=int)
x[0] = 2  # initial state

states = np.array([1, 2, 3], dtype=int)

for t in range(1, n_steps):
    i = x[t-1] - 1                 # row index for current state
    x[t] = rng.choice(states, p=P[i])

print("First 10 states:", x[:10])
```

```
First 10 states:  [2 3 2 3 1 2 3 2 1 2]
```

□

b. Find the 20-steps transition matrix.

Solution.

```python
import numpy as np

P20 = np.linalg.matrix_power(P, 20)

np.set_printoptions(precision=8, suppress=True)
print("P^20 =\n", P20)
```

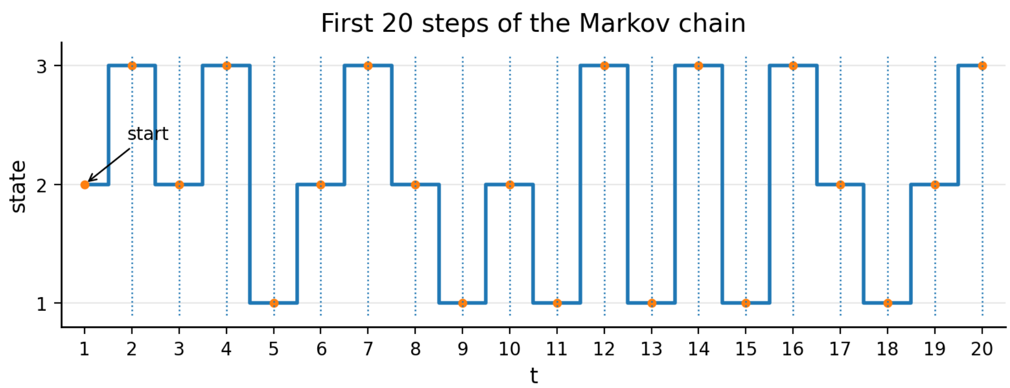
□

c. Draw $x_1, \dots, x_{20}$.

Solution.

```python
import numpy as np
import matplotlib.pyplot as plt

# ---- typographic + layout tweaks ----
plt.rcParams.update({
    "figure.figsize": (8, 3.2),    # wide, compact height
    "figure.dpi": 200,             # high-res
    "axes.titlesize": 14,
    "axes.labelsize": 12,
    "xtick.labelsize": 10,
    "ytick.labelsize": 10,
})
```

```

t = np.arange(1, 21)
y = x[:20]

fig, ax = plt.subplots()

# Main step line + discrete markers
ax.step(t, y, where="mid", linewidth=2)
ax.plot(t, y, linestyle="none", marker="o", markersize=4)

# Vertical dotted lines
jumps = np.where(np.diff(y) != 0)[0] + 1  # indices of change, in 1..19
if jumps.size:
    ax.vlines(t[jumps], ymin=0.9, ymax=3.1, linestyles=":", linewidth=0.9)

# Axes cosmetics
ax.set_xlim(0.5, 20.5)
ax.set_ylim(0.8, 3.2)
ax.set_yticks([1, 2, 3])
ax.set_xticks(np.arange(1, 21))
ax.set_xlabel("t")
ax.set_ylabel("state")
ax.set_title("First 20 steps of the Markov chain")

# Cleaner frame + ticks
for side in ("top", "right"):
    ax.spines[side].set_visible(False)
ax.spines["left"].set_linewidth(1)
ax.spines["bottom"].set_linewidth(1)
ax.tick_params(axis="both", which="both", direction="out", length=4, width=0.8)

# Subtle horizontal gridlines for readability
ax.grid(True, axis="y", alpha=0.3)

# Annotate the starting point
ax.annotate("start",
            xy=(t[0], y[0]),
            xytext=(t[0] + 0.9, y[0] + 0.35),
            arrowprops=dict(arrowstyle="->", lw=0.9),
            ha="left", va="bottom")

fig.tight_layout()

plt.show()
```



□

d. Find the distribution of the last 100 observations.

Solution.

```
import numpy as np

# Assumes x from part (a)
last_100 = x[-100:]
counts = np.bincount(last_100, minlength=4)[1:]  # ignore index 0
proportions = counts / counts.sum()

for s, c, p in zip([1, 2, 3], counts, proportions):
    print(f"State {s}: count = {c:3d}, proportion = {p:.4f}")
```

State	Count	Proportion
State 1	35	0.3500
State 2	31	0.3100
State 3	34	0.3400

Table 2-1: Counts and Proportions for Different States

□