Question 1

1. Let X_1, X_2 and X_3 be random variables taking values in $(-\infty, +\infty)$ and let $Y_1 = X_1X_2$, $Y_2 = X_2X_3$, $Y_3 = X_3X_1$, $Z = X_1X_2X_3$ and $W = X_1 + X_2 + X_3$.

a. Is it generally true that $\mathcal{F}_{Y_1} \subset \mathcal{F}_Z$ and/or $\mathcal{F}_Z \subset \mathcal{F}_{Y_1}$?

Solution.

No, neither inclusion holds in general.

- $\mathcal{F}_{Y_1} \subset \mathcal{F}_Z$ can fail: take $X_3 \equiv 0$ and let (X_1, X_2) be nontrivial (e.g. i.i.d. Rademacher). Then $Z \equiv 0$, so $\mathcal{F}_Z$ is trivial, while $Y_1 = X_1X_2$ is nontrivial, hence $\mathcal{F}_{Y_1}$ is nontrivial, so $\mathcal{F}_{Y_1} \not\subset \mathcal{F}_Z$.
- $\mathcal{F}_Z \subset \mathcal{F}_{Y_1}$ can fail: take $X_1 \equiv X_2 \equiv 1$ and let X_3 be nontrivial. Then $Y_1 \equiv 1$ so $\mathcal{F}_{Y_1}$ is trivial, whereas $Z = X_3$ is nontrivial, so $\mathcal{F}_Z \not\subset \mathcal{F}_{Y_1}$.

That inclusions may hold in special cases follows from the "function–of" rule above, but there is no general inclusion.

□

b. Is it generally true that $\mathcal{F}_{Y_1,Y_2,Y_3} \subset \mathcal{F}_{X_1,X_2,X_3}$ and/or $\mathcal{F}_{X_1,X_2,X_3} \subset \mathcal{F}_{Y_1,Y_2,Y_3}$?

Solution.

Always:

$\mathcal{F}_{Y_1,Y_2,Y_3} \subset \mathcal{F}_{X_1,X_2,X_3}$, because $(Y_1, Y_2, Y_3) = (X_1X_2,\ X_2X_3,\ X_3X_1)$ is a measurable (indeed continuous) function of (X_1, X_2, X_3); thus the information in (X_1, X_2, X_3) suffices to compute (Y_1, Y_2, Y_3).

In general not:

$\mathcal{F}_{X_1,X_2,X_3} \subset \mathcal{F}_{Y_1,Y_2,Y_3}$.

Example: let X_1, X_2, X_3 be independent Rademacher (each ±1 w.p. $1/2$). Then (x_1, x_2, x_3) and $(-x_1, -x_2, -x_3)$ produce the same (Y_1, Y_2, Y_3), so (Y_1, Y_2, Y_3) does not determine (say) the sign of X_1. Hence X_1 cannot be a measurable function of (Y_1, Y_2, Y_3), and therefore $\mathcal{F}_{X_1,X_2,X_3} \not\subset \mathcal{F}_{Y_1,Y_2,Y_3}$.

□

c. Assume that for every ω, $X_1(\omega), X_2(\omega)$ and $X_3(\omega)$ are different from zero. Is it generally true that $\mathcal{F}_{Y_1,Y_2,Z} \subset \mathcal{F}_{X_1,X_2,X_3}$ and/or $\mathcal{F}_{X_1,X_2,X_3} \subset \mathcal{F}_{Y_1,Y_2,Z}$?

Solution.

Equality holds:

$\mathcal{F}_{Y_1,Y_2,Z} = \mathcal{F}_{X_1,X_2,X_3}$.

First, as before, (Y_1, Y_2, Z) is a measurable function of (X_1, X_2, X_3), so $\mathcal{F}_{Y_1,Y_2,Z} \subset \mathcal{F}_{X_1,X_2,X_3}$. Conversely, using $X_i \neq 0$ we can recover each X_i from (Y_1, Y_2, Z) by continuous operations:

$$X_1 = \frac{Z}{Y_2}, \qquad X_2 = \frac{Y_1Y_2}{Z}, \qquad X_3 = \frac{Z}{Y_1}.$$

Hence (X_1, X_2, X_3) is a measurable function of (Y_1, Y_2, Z), so $\mathcal{F}_{X_1,X_2,X_3} \subset \mathcal{F}_{Y_1,Y_2,Z}$. By two inclusions, the sigma–algebras are equal.

□

d. Is it generally true that $\mathcal{F}_{W,Z} \subset \mathcal{F}_{X_1,X_2,X_3}$ and/or $\mathcal{F}_{X_1,X_2,X_3} \subset \mathcal{F}_{W,Z}$?

Solution.

Always:

$\mathcal{F}_{W,Z} \subset \mathcal{F}_{X_1,X_2,X_3}$ since $(W,Z) = (X_1+X_2+X_3,\ X_1X_2X_3)$ is a measurable function of (X_1,X_2,X_3).

In general not:

$\mathcal{F}_{X_1,X_2,X_3} \subset \mathcal{F}_{W,Z}$.

Counterexample: let $\Omega = \{\omega_1,\omega_2\}$, each with probability $1/2$, and set

$$(X_1,X_2,X_3)(\omega_1) = (1,1,0), \qquad (X_1,X_2,X_3)(\omega_2) = (1,0,1).$$

Then $W(\omega_1) = W(\omega_2) = 2$ and $Z(\omega_1) = Z(\omega_2) = 0$, so (W,Z) is constant and $\mathcal{F}_{W,Z}$ is trivial, while X_2 is not constant. Therefore X_2 is not measurable w.r.t. $\mathcal{F}_{W,Z}$ and $\mathcal{F}_{X_1,X_2,X_3} \not\subset \mathcal{F}_{W,Z}$.

□

e. Is it generally true that $\mathcal{F}_{X_1,W,Z} \subset \mathcal{F}_{X_1,X_2,X_3}$ and/or $\mathcal{F}_{X_1,X_2,X_3} \subset \mathcal{F}_{X_1,W,Z}$?

Solution.

Always:

$\mathcal{F}_{X_1,W,Z} \subset \mathcal{F}_{X_1,X_2,X_3}$ because (X_1,W,Z) is a measurable function of (X_1,X_2,X_3).

In general not:

$\mathcal{F}_{X_1,X_2,X_3} \subset \mathcal{F}_{X_1,W,Z}$.

Use the same two–point example as in (d): both ω_1 and ω_2 have

$$(X_1,W,Z) = (1,2,0),$$

yet $X_2(\omega_1) = 1 \neq 0 = X_2(\omega_2)$. Thus X_2 is not a function of (X_1,W,Z), so $\mathcal{F}_{X_1,X_2,X_3} \not\subset \mathcal{F}_{X_1,W,Z}$.

□

Question 2

a. Simulate in Matlab or in Python 1000 random variables with uniform distribution on $[0,1]$: $x_1,\ldots,x_{1000}$.

Solution.

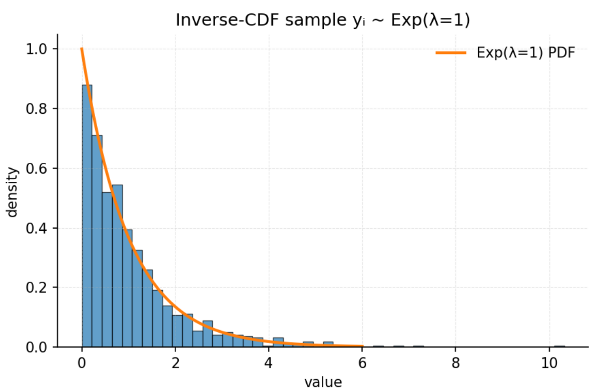
```
import numpy as np

SAMPLE_SIZE: int = 1000
RNG_SEED: int = 40313

rng = np.random.default_rng(RNG_SEED)

# Techically, the value for 1 is excluded as it could pose some issues when
    calculating the inverse at point 2 (the function excludes it
    automatically)
x = rng.uniform(low=0.0, high=1.0, size=SAMPLE_SIZE)
```

□

b. Find $y_i = F^{-1}(x_i)$ where F is the distribution function of the exponential distribution with parameter $\lambda = 1$.

Solution.

```
import numpy as np
import matplotlib.pyplot as plt

def inv_cdf_exp_rate1(u: np.ndarray, allow_inf: bool = False) -> np.ndarray:
    return -np.log1p(-u)

def plot_exp_hist(data: np.ndarray, title: str) -> None:
    fig = plt.figure(figsize=(6.0, 4.0), dpi=150)
    ax = plt.gca()

    # Histogram
    ax.hist(
        data,
        bins="fd",
        density=True,
        alpha=0.7,
        edgecolor="black",
        linewidth=0.6,
    )

    # Theoretical Exp(1) PDF
    x_max = max(6.0, float(np.percentile(data, 99.5)))
    grid = np.linspace(0.0, x_max, 600)
    pdf = np.exp(-grid)
    ax.plot(grid, pdf, linewidth=2.0, label="Exp(\lambda=1) PDF")

    # Labels & cosmetics
    ax.set_title(title)
    ax.set_xlabel("value")
    ax.set_ylabel("density")
    ax.grid(True, alpha=0.3, linestyle="--", linewidth=0.5)
    for spine in ("top", "right"):
        ax.spines[spine].set_visible(False)
    ax.legend(frameon=False)
    fig.tight_layout()
    plt.show()

# Transform and visualize
y = inv_cdf_exp_rate1(x, allow_inf=False)
plot_exp_hist(y, title="Inverse-CDF sample y_i ~ Exp(\lambda=1)")
```

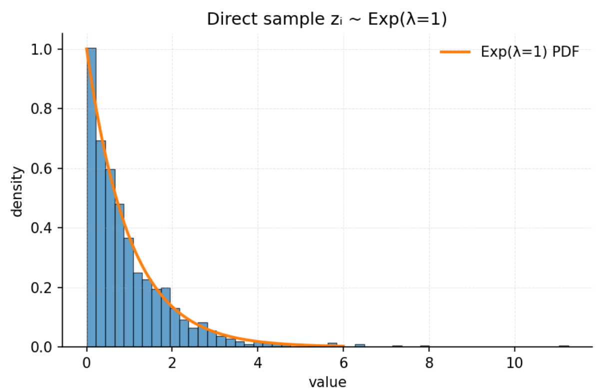


□

c. Simulate 1000 random variables $z_1, \ldots, z_{1000}$ with exponential distribution with parameter $\lambda = 1$.

Solution.

```
import numpy as np
import matplotlib.pyplot as plt

SAMPLE_SIZE: int = 1000
RNG_SEED_2: int = 39  # Different seed to ensure different draws

rng2 = np.random.default_rng(RNG_SEED_2)

z = rng2.exponential(scale=1.0, size=SAMPLE_SIZE)

plot_exp_hist(z, title="Direct sample z_i ~ Exp(\lambda=1)")
```



□

d. Compare the histogram of the y_i with the histogram of the z_i.

Solution.

```
y_f = y[np.isfinite(y)]
z_f = z[np.isfinite(z)]

# 1) Kolmogorov-Smirnov two-sample test
ks = stats.ks_2samp(y_f, z_f, alternative="two-sided", method="auto")

# 2) Cramer-von Mises two-sample test
cvm = stats.cramervonmises_2samp(y_f, z_f)

# 3) Wasserstein-1 distance
w1 = stats.wasserstein_distance(y_f, z_f)

print(f"KS test:                  D = {ks.statistic:.4f}, p = {ks.pvalue:.4g
    }")
print(f"Cramer-von Mises test:    T = {cvm.statistic:.4f}, p = {cvm.pvalue
    :.4g}")
print(f"Wasserstein-1 distance:  W-1 = {w1:.4f}")
```

Across 1,000 draws, the two-sample tests show no evidence that the samples differ: KS $D = 0.047$ ($p = 0.219$) and Cramer–von Mises $T = 0.313$ ($p = 0.124$) both fail to reject the null that y_i (inverse-CDF) and z_i (direct) come from the same distribution. The Wasserstein distance $W_1 = 0.071$ is small in the natural units of the variable (for Exp(1), $\sigma = 1$), i.e., about 7% of one standard deviation—well within Monte Carlo variability.

Overall, the results are consistent with both methods producing i.i.d. Exp(1) samples, confirming the correctness of the inverse-transform step. □