Stochastic Processes, Quiz 1, 2023 Spring

Solution and Grading

- Duration: 90 minutes
- Closed material, No calculator

- Name: ________________________
- Student ID: ________________________
- E-mail: ______________ `@seoultech.ac.kr`

- Write legibly.
- Justification is necessary unless stated otherwise.

1	10
2	20
3	10
4	30
5	10
6	10
7	10
Total	100

#1. You are considering to sell a certain product. You assessed the potential demand that is 100 items with 1/3 of chance, 200 items with 1/3 of chance, and 300 items with 1/3 of chance. What is the coefficient of variation[1] of the demand? `[10pts]`

- $\mathbb{E}X = \sum_{-\infty}^{\infty} x\mathbb{P}(X = x) = 100 \cdot 1/3 + 200 \cdot 1/3 + 300 \cdot 1/3 = 200$
- $\mathbb{E}X^2 = \sum_{-\infty}^{\infty} x^2\mathbb{P}(X = x) = 100^2 \cdot 1/3 + 200^2 \cdot 1/3 + 300^2 \cdot 1/3 = 140000/3$
- $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 140000/3 - 200^2 = 20000/3$
- $sd(X) = \sqrt{20000/3}$
- $cv(X) = sd(X)/\mathbb{E}X = \frac{\sqrt{20000/3}}{200} = \sqrt{6}/6$

Grading scheme:

- If a very minor mistake in calculating $Var(X)$ or $cv(X)$, then 5 pts.

[1] Hint: $cv(X) = sd(X)/\mathbb{E}X$

#2. Let X be a Poisson random variable with parameter 4, and let $Y = min(X, 3)$.

(a) What is the pmf of Y? (i.e. Specify $\mathbb{P}(Y = i)$ for $i = 0, 1, 2, ...$) `[10pts]`

(b) What is $\mathbb{P}(Y \leq 2 | Y \leq 4)$? `[10pts]`

(a)

- $p(y) = \mathbb{P}(Y = y) = \mathbb{P}(min(X, 3) = y)$ therefore,
- $p(0) = \mathbb{P}(Y = 0) = \mathbb{P}(min(X, 3) = 0) = \mathbb{P}(X = 0) = \frac{4^0 e^{-4}}{0!} = e^{-4}$
- $p(1) = \mathbb{P}(Y = 1) = \mathbb{P}(min(X, 3) = 1) = \mathbb{P}(X = 1) = \frac{4^1 e^{-4}}{1!} = 4e^{-4}$
- $p(2) = \mathbb{P}(Y = 2) = \mathbb{P}(min(X, 3) = 2) = \mathbb{P}(X = 2) = \frac{4^2 e^{-4}}{2!} = 8e^{-4}$
- $$\begin{aligned} p(3) = \mathbb{P}(Y = 3) = \mathbb{P}(min(X, 3) = 3) &= \mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X < 3) \\ &= 1 - (\mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2)) \\ &= 1 - (e^{-4} + 4e^{-4} + 8e^{-4}) \\ &= 1 - 13e^{-4} \end{aligned}$$
- $p(4) = \mathbb{P}(Y = 4) = \mathbb{P}(min(X, 3) = 4) = 0 \qquad (\because min(X, 3) \geq 3 \;\; always)$
  also $p(y) = 0$ for all $y \geq 4$

Therefore the pmf of Y is as follows:

$$p(y) = \begin{cases} e^{-4} & \text{for} \quad y = 0 \\ 4e^{-4} & \text{for} \quad y = 1 \\ 8e^{-4} & \text{for} \quad y = 2 \\ 1 - 13e^{-4} & \text{for} \quad y = 3 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$\begin{aligned} \mathbb{P}(Y \leq 2 | Y \leq 4) &= \frac{\mathbb{P}(Y \leq 2 \cap Y \leq 4)}{\mathbb{P}(Y \leq 4)} = \frac{\mathbb{P}(Y \leq 2)}{\mathbb{P}(Y \leq 4)} \\ &= \frac{\mathbb{P}(Y = 0) + \mathbb{P}(Y = 1) + \mathbb{P}(Y = 2)}{\mathbb{P}(Y = 0) + \mathbb{P}(Y = 1) + \mathbb{P}(Y = 2) + \mathbb{P}(Y = 3) + \mathbb{P}(Y = 4)} \\ &= \frac{e^{-4} + 4e^{-4} + 8e^{-4}}{e^{-4} + 4e^{-4} + 8e^{-4} + 1 - 13e^{-4} + 0} \\ &= 13e^{-4} \end{aligned}$$

Grading scheme:

- (a) If calculation is generally correct but your pmf does not sum up to 1, then 5 pts.
- (b) If conditional probability concepts are correctly applied, (i.e. $\mathbb{P}(Y \leq 2 | Y \leq 4) = \frac{\mathbb{P}(Y \leq 2)}{\mathbb{P}(Y \leq 4)}$), then 5pts. The rest work is 5 pts.

#3. Express S in a number. `[10pts]`

$S = 0.1 + 2 \cdot 0.1^2 + 3 \cdot 0.1^3 + 4 \cdot 0.1^4 + 5 \cdot 0.1^5 + \cdots$

- $S = 10/81$ (See L1.p19)

Grade Scheme

- No partial points unless for a very minor mistake.

#4.

(a) State the definition of the memoryless property. `[10pts]`

(b) State the cdf of random variable that follows exponential distribution with parameter λ. `[10pts]`

(c) Prove that exponential distribution possesses the memoryless property. `[10pts]`

- (a) $\mathbb{P}(X > s + t | X > t) = \mathbb{P}(X > s)$, for all $s, t \geq 0$.
- (b) $F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$
- (c) See L1.p15

Grading scheme:

- (a) No partial points
- (b) If your definition includes $1 - e^{-\lambda x}$ without proper range information, then 5 pts were given.
- (c) 5pts if conditional probability is properly applied, i.e., $\mathbb{P}(X > s + t | X > t) = \frac{\mathbb{P}(X > s+t)}{\mathbb{P}(X > t)}$. 5pts for the rest work.

#5. Suppose $X \sim U(100, 150)$. Evaluate $\mathsf{E}[(120 - X)^+]$. `[10pts]`

- $\mathbb{E}[(120 - X)^+] = 4$ (See L2.p2)

Grade Scheme:

- 5pt for working out until integral expression, i.e. $= \int_{100}^{120} (120 - x) \frac{1}{50} dx$.
- 5pt for the rest work.

#6. Suppose $X \sim exp(3)$. Evaluate $\mathsf{E}[min(X,5)]$. `[10pts]`

- $\mathbb{E}[min(X,5)] = \frac{1}{3}(1-e^{-15})$ (See L2.p4)

Grade Scheme:

- 5pt for working out until integral expression, i.e. $= 3\int_0^5 xe^{-3x}dx + 5\int_5^\infty 3e^{-3x}dx$.
- 5pt for the rest work.

#7. Smith and Jones came to post office together and they are served by two clerks, server A and server B, respectively. Server A has a service time following $exp(5)$ and server B has a service time following $exp(4)$. What is the chance that Smith will be done with the service first? `[10pts]`

- 5/9 (See L1.p17&18)

Grading scheme

- No partial points

(blank)