Stochastic Processes, Mid-term #2, 2025 Spring

Solution and Grading

- Duration: 90 minutes
- Closed material, No calculator.

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- Write legibly.
- Justification is necessary unless stated otherwise.

1	10
2	10
3	10
4	10
Total	40

#1. The following statement is false. Provide a counterexample and explain why the statement is false. `[10pts]`

> If a DTMC has a finite number of states and is aperiodic, then its stationary distribution is unique.

Suggested Solution:

Even if DTMC has finite states and they are all aperiodic, its stationay distribution is not unique if there are two or more recurrent clasess. Regarding the given statement, counterexamples can be $P = \begin{pmatrix} 0.3 & 0.7 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}$ or $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or anything with two or more recurrent classes. To show the stationary distribution is not unique, a students is expected to present two or more stationary distributions. That is, present two or more π that satisfy $\pi = \pi P$. For example with the second matrix above, $\pi = (1\ 0)$ and $\pi = (0.5\ 0.5)$ satisfy $\pi = \pi P$, so the stationary distribution is not unique.

Grading Scheme:

- If the argument is right track, but presented counter example is not a legit transition matrix (row sum must be equal to 1), then 5pts.
- Otherwise, no partial point.

#2. Consider the following DTMC.

$$P = \begin{pmatrix} 0.3 & 0.7 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 & 0.75 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \end{pmatrix}$$

(a) What is the number of classes? `[5pts]`

(b) For each state, identify its period. `[5pts]`

(a) Three. $\{1, 2\}, \{3, 5\}, \{4\}$.

(b) $d(1) = d(2) = 1$, $d(3) = d(5) = 2$, $d(4) = 1$

Grading Scheme:

- (a) No partial point.
- (b) Each state counts 1 pt.

#3. Prove that the time to the first arrival in $PP(\lambda)$ follows $exp(\lambda)$.`[10pts]`

Let T_1 be the time to the first arrival. Then,

$$\mathbb{P}(T_1 \leq t) = \mathbb{P}(N(t) - N(0) > 0) \tag{1}$$

In the above equality, the LHS event $\{T_1 \leq t\}$ implies that *the first arrival occurs before time t* and the RHS event $\{N(t) - N(0) > 0\}$ implies that *the number of arrivals between time* 0 *and time t is more than 0*. These two events are equivalent, only that the LHS describes in *time domain* and the RHS describes in counting domain. The proof continues as:

$$\begin{aligned}
\mathbb{P}(T_1 \leq t) &= \mathbb{P}(N(t) - N(0) > 0) \\
&= 1 - \mathbb{P}(N(t) - N(0) = 0) \\
&= 1 - \mathbb{P}(Poi(\lambda t) = 0) \\
&= 1 - \frac{(\lambda t)^0 e^{-lambdat}}{0!} \\
&= 1 - e^{-lambdat}
\end{aligned}$$

Therefore, T_1 follows an exponential distribution with parameter λ.■

Grading Scheme:

- This proof contains the following three key steps
  1. T_1 above must be clearly defined.
  2. Explicit translation between time domain and counting domain must be included.
  3. The inequality direction and probability calculation must be correct.
- If only one part among the three key steps is incorrect, then 5pts are given.

#4. Assume that calls arrive at a customer call center between 9am ($t = 0$) and 2pm ($t = 5$) following a non-homogeneous Poisson process with the rate function of the following:

$$\lambda(t) = \begin{cases} 4 & 0 \le t < 1 \\ 2t + 2 & 1 \le t < 3 \\ 8 & 3 \le t < 5 \end{cases}$$

What is the probability that the call center receives exactly 6 calls during 9am and 11am?`[10pts]`

$$\begin{aligned}
\mathbb{P}(N(2) - N(0) = 6) &= \mathbb{P}\left(X = 6 | X \sim Poi\left(\int_0^2 \lambda(t)dt\right)\right) \\
&= \mathbb{P}\left(X = 6 | X \sim Poi\left(\int_0^1 \lambda(t)dt + \int_1^2 \lambda(t)dt\right)\right) \\
&= \mathbb{P}\left(X = 6 | X \sim Poi\left(\int_0^1 4\,dt + \int_1^2 2t + 2\,dt\right)\right) \\
&= \mathbb{P}\left(X = 6 | X \sim Poi(4 + 5)\right) \\
&= \frac{e^{-9} \cdot 9^6}{6!}
\end{aligned}$$

Grading Scheme:

- For the part of identifying the parameter of the Poisson distribution (=9), 5pts.
- For the rest of work, 5pts.

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