A1. Math Review

I. Differentiation and integration

Exercise 1

Suppose $f(x) = xe^x$, find $f'(x)$.

Solution:

$$
\begin{aligned}
& f(x) = xe^x \\
& \Longrightarrow f'(x) = (x)'e^x + x(e^x)' \\
& \Longrightarrow f'(x) = e^x + xe^x
\end{aligned}
$$

Exercise 2

Suppose $f(x) = e^{2x}$, find $f'(x)$.

Solution: To use the theorem of composite function, the function $f()$ can be seen as $f(x) = h(g(x))$, where $h(x) = e^x$ and $g(x) = 2x$. It follows $f'(x) = e^{2x} \times 2 = 2e^{2x}$.

Exercise 3

Derive $\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$. *(Hint: Use Theorem 2 above.)*

Solution: From the theorem, we have $(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x)$. Applying antiderivative to both hand sides gives

$$
f(x) \cdot g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx \tag{1}
$$

It follows $\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$.

Exercise 4

Find $\int xe^x dx$,and evaluate $\int_0^1 xe^x dx$. *(Hint: Use Exercise 3 above.)*

Solution: Since $\int f'(x)g(x)dx = f(x) \cdot g(x) - \int f(x)g'(x)dx$, we have

$$\begin{aligned} \int xe^x \ dx &= e^x x - \int e^x \cdot 1 \ dx \\ &= e^x x - e^x + C \end{aligned}$$

$$\begin{aligned} \int_0^1 xe^x \ dx &= [e^x x - e^x + C]_0^1 \\ &= (e^1 \cdot 1 - e^1 + C) - (e^0 \cdot 0 - e^0 + C) \\ &= (0 + C) - (0 - 1 + C) = 1 \end{aligned}$$

II. Numerical methods for finding a root

Exercise 5

The original R code:

```
f <- function(x) {
  return(1+1/x)
}
tol <- 10^(-5)
x_now <- 0.1
repeat{
  x_next <- f(x_now)
  if (abs(x_next-x_now) < tol) {
    break
  }
  x_now <- x_next
  print(x_next)
}
```

```
## [1] 11
## [1] 1.090909
## [1] 1.916667
## [1] 1.521739
## [1] 1.657143
## [1] 1.603448
## [1] 1.623656
## [1] 1.615894
## [1] 1.618852
## [1] 1.617722
## [1] 1.618153
## [1] 1.617988
## [1] 1.618051
## [1] 1.618027
```

The comparable python code:

```
def f(x):
  return(1+1/x)
tol = pow(10,-5)
x_now = 0.1
while True:
    x_next = f(x_now)
    if (abs(x_next-x_now) < tol): break
    x_now = x_next
    print(round(x_next,6))
```

```
## 11.0
## 1.090909
## 1.916667
## 1.521739
## 1.657143
## 1.603448
## 1.623656
## 1.615894
## 1.618852
## 1.617722
## 1.618153
## 1.617988
## 1.618051
## 1.618027
```

III. Matrix algebra

Exercise 6

Solution:

$$\begin{pmatrix} .6 & .4 \end{pmatrix} \begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix} = (0.6 \cdot 0.7 + 0.4 \cdot 0.5 \quad 0.6 \cdot 0.4 + 0.4 \cdot 0.5) = (0.62 \quad 0.38)$$

Exercise 7

What is P^2?

Solution:

$$P^2 = \begin{pmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.74 & 0.36 \\ 0.6 & 0.4 \end{pmatrix}$$

Exercise 8

$$\begin{aligned} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} \begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix} &= \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} \\ \mathbf{v}_1 + \mathbf{v}_2 &= 1 \end{aligned}$$

Solution: It follows

$$0.7\mathbf{v}_1 + 0.5\mathbf{v}_2 = \mathbf{v}_1 \quad (2)$$

$$0.3\mathbf{v}_1 + 0.5\mathbf{v}_2 = \mathbf{v}_2 \quad (3)$$

$$\mathbf{v}_1 + \mathbf{v}_2 = 1 \quad (4)$$

Thus, $\mathbf{v}_1 = \frac{5}{8}$, $\mathbf{v}_2 = \frac{3}{8}$.

Exercise 9

Solve the following system of equations.

$$
\begin{aligned}
x &= y && (5)\\
y &= 0.5z && (6)\\
z &= 0.6 - 0.4x && (7)\\
x + y + z &= 1 && (8)
\end{aligned}
$$

Solution: $x = \frac{1}{4}, y = \frac{1}{4}, z = \frac{1}{2}$

Exercise 10

$$\begin{pmatrix} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} \begin{pmatrix} -2 & 2 & \\ 3 & -5 & 2 \\ & 3 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 = 1$$

Solution:

$$\begin{pmatrix} -2\mathbf{v}_0 + 3\mathbf{v}_1 & 2\mathbf{v}_0 - 5\mathbf{v}_1 + 3\mathbf{v}_2 & 2\mathbf{v}_1 - 3\mathbf{v}_2 \end{pmatrix} = (0\ 0\ 0)$$

$$\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 = 1$$

Thus, $\mathbf{v}_0 = \frac{9}{19}, \mathbf{v}_1 = \frac{6}{19}, \mathbf{v}_2 = \frac{4}{19}$.

Exercise 11

Solve the following system of equations.

$$\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{pmatrix} \begin{pmatrix} .7 & .3 & & \\ .5 & .5 & & \\ & & .6 & .4 \\ & & .3 & .7 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{pmatrix}$$

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = 1$$

$$\mathbf{v}_1 + \mathbf{v}_2 = a$$

<u>*Solution:*</u> It can be noticed that we are to solve the following two independent system of linear equations.

$$\begin{aligned} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} \begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix} &= \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} \\ \mathbf{v}_1 + \mathbf{v}_2 &= a \end{aligned}$$

and

$$\begin{aligned} \begin{pmatrix} \mathbf{v}_3 & \mathbf{v}_4 \end{pmatrix} \begin{pmatrix} .6 & .4 \\ .3 & .7 \end{pmatrix} &= \begin{pmatrix} \mathbf{v}_3 & \mathbf{v}_4 \end{pmatrix} \\ \mathbf{v}_3 + \mathbf{v}_4 &= 1 - a \end{aligned}$$

Solving the above two systems gives $\mathbf{v}_1 = \frac{5}{8}a, \mathbf{v}_2 = \frac{3}{8}a, \mathbf{v}_3 = \frac{3}{7}(1-a), \mathbf{v}_4 = \frac{4}{7}(1-a)$

Exercise 12

Solve following and express $\mathbf{v}_i$ for $i = 0, 1, 2, ...$

$$
\begin{aligned}
\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + ... &= 1 \\
0.02\mathbf{v}_0 + 0.02\mathbf{v}_1 + 0.02\mathbf{v}_2 + ... &= \mathbf{v}_0 \\
0.98\mathbf{v}_0 &= \mathbf{v}_1 \\
0.98\mathbf{v}_1 &= \mathbf{v}_2 \\
0.98\mathbf{v}_2 &= \mathbf{v}_3 \\
... &= ...
\end{aligned}
$$

Solution: Between $\mathbf{v}_i$ and $\mathbf{v}_0$, there is the following relationships:

$$
\begin{aligned}
\mathbf{v}_1 &= 0.98\mathbf{v}_0 \\
\mathbf{v}_2 &= 0.98\mathbf{v}_1 = 0.98^2\mathbf{v}_0 \\
\mathbf{v}_3 &= 0.98\mathbf{v}_2 = 0.98^3\mathbf{v}_0 \\
... &= ... \\
\mathbf{v}_i &= 0.98^i\mathbf{v}_0
\end{aligned}
$$

Using the original first equation, it follows

$$
\begin{aligned}
& \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + ... = \mathbf{v}_0(1 + 0.98 + 0.98^2 + ...) = 1 \\
\implies & \mathbf{v}_0 \left(\frac{1}{1 - 0.98} \right) = 1 \\
\implies & \mathbf{v}_0 = 0.02
\end{aligned}
$$

In short, $\mathbf{v}_i = (0.02)(0.98)^i$ for all i.

IV. Series and others

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"A1_Solution"
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