Stochastic Processes, Quiz 2, 2023 Spring

Solution and Grading

- Duration: 120 minutes
- Closed material, No calculator

- Name: ____________________
- Student ID: ____________________
- E-mail: ______________ @seoultech.ac.kr

- Write legibly.
- Justification is necessary unless stated otherwise.

1	20
2	20
3	30
4	40
Total	110

#1. There are two servers (John and Paul) in the bank. The service time of John follows an exponential distribution with rate of 1/2 per minute. The service time of Paul follows exponential distribution with mean of 5 minutes. At noon, the three customers A,B,C arrived to post office same time. Immediately, John starts serving A and Paul starts serving B. As soon as either A or B leaves the bank, C will start getting served. The service times of John and Paul are independent.

(a) What is the probability that A will leave the bank before B? `[10pts]`

(b) What is the probability that all of the customers will be in the bank at 12:07 PM? `[10pts]`

(a)

- Let $X_A \sim exp(1/2)$, $X_B \sim exp(1/5)$
  $\mathbb{P}(X_A < X_B) = \frac{\lambda_A}{\lambda_A + \lambda_B} = \frac{1/2}{1/2+1/5} = 5/7$

(b)

- Both A & B are in the system

  $$\begin{aligned}
  &\mathbb{P}(X_A > 7, X_B > 7) \\
  &= \mathbb{P}(X_A > 7)\mathbb{P}(X_B > 7) \quad (\because \text{ independent}) \\
  &= (1 - \mathbb{P}(X_A \leq 7))(1 - \mathbb{P}(X_B \leq 7)) \\
  &= e^{-\frac{1}{2}\times 7} \cdot e^{-\frac{1}{5}\times 7} = e^{-4.9}
  \end{aligned}$$

Grading scheme:

- (a) No partial points.
- (b)
  - 5pts for having a key understanding that both A and B must be in the system at the moment.
  - 5pts for the rest of work.

#2. A bank has only one server. The interarrival times of customers to the bank follow exponential distribution with mean 5 minutes and the service time for a customer follows normal distribution with mean 4 minutes and standard deviation of 1 minute. Answer the following questions in a number.[1]

(a) What is the long-run fraction of times that the server is busy? `[10pts]`

(b) What is the long-run average waiting time of each customer in the queue? `[10pts]`

(a)

- $\rho = \frac{1/\mathbb{E}U}{1/\mathbb{E}V} = \frac{1/5}{1/4} = 0.8 \quad \therefore u = min(1, \rho) = 0.8$

(b)

$$\begin{aligned}\mathbb{E}W &= \mathbb{E}V\left(\frac{\rho}{1-\rho}\right)\left(\frac{c_a^2 + c_b^2}{2}\right)\\ &= 4 \times \frac{0.8}{1-0.8} \times \left(\frac{1^2 + (1/4)^2}{2}\right)\\ &= 8.5 \text{ minutes}\end{aligned}$$

Grading scheme:

- (a) No partial points.
- (b) No partial points. 5pts per mistake is deducted only if the mistake is considered a minor mistake.

[1] It is OK to reference the previous answer. For example, if you didn't solve problem (a) but know that the answer for (b) is [the answer for (a)] times 5, then you may answer question (b) by "5 × [ans in (a)]".

#3. You are selling lemonade. The demand is uniformly distributed between 20 gallons and 35 gallons. The selling price is 5 dollars per gallon; and the value of unsold lemonade is 0.5 dollars per gallon; every time you make an order, it costs fixed 25 dollars plus 2 dollars per gallon of lemonade. You are considering to adopt a (S, s) policy in order to maximize expected daily profit.

(a) What is c_o and c_u? [10pts]

(b) What is the S? [10pts]

(c) Set up a quadratic equation where one of the two solutions to the equation is s. That is, you need to present $as^2 + bs + c = 0$ where a, b, c are numbers.[2][3] [10pts]

[2]If you have not found S from the previous question, you may use $S = 32$.

[3]To ease your calculation time, the following may be helpful: If $D \sim U(20, 35)$ and s is between 20 and 35, then $5\mathbb{E}[min(D, s)] + 0.5\mathbb{E}[(s - D)^+] = 0.18s^2 - 7.3s + 211$.

(a)

- $c_o = 1.5$
- $c_u = 3$

(b)

- $F(y) = \frac{c_u}{c_o + c_u} \Rightarrow \frac{y-20}{35-20} = \frac{3}{1.5+3} \Rightarrow y = 30 (= S)$

(c)

- $\mathbb{E}[\text{profit}|\text{order 0 at s}] = \mathbb{E}[\text{profit}|\text{order 30-s at s}]$
- $\Rightarrow 5\mathbb{E}[min(D,s)] + 0.5\mathbb{E}[(s-D)^+] = 5\mathbb{E}[min(D,30)] + 0.5\mathbb{E}[(30-D)^+] - (25 + 2(30-s))$
- $\Rightarrow 0.18s^2 - 7.3s + 211 = 5 \cdot \frac{80}{3} + 0.5 \cdot \frac{10}{3} - (85 - 2s)$
- $\Rightarrow 0.18s^2 - 7.3s + 211 = 50 + 2s$
- $\Rightarrow 0.18s^2 - 9.3s + 161 = 0$

Grading scheme:

- (a) 5pt for each of them (c_o and c_u).
- (b) You can get full credit if you use a clear process with the results of a problem (a).
- (c)
  - 3pts for the equality using the unknown s, i.e. $\mathbb{E}[\text{profit}|\text{order 0 at s}] = \mathbb{E}[\text{profit}|\text{order 30-s at s}]$.
  - 3pts for setting up the equation, i.e. $5\mathbb{E}[min(D,s)] + 0.5\mathbb{E}[(s-D)^+] = 5\mathbb{E}[min(D,30)] + 0.5\mathbb{E}[(30-D)^+] - (25 + 2(30-s))$.
  - 4pts for the rest work

#4. You are working for the revenue management team at an airline company. Your task is to determine how many flight tickets to sell for the 6 AM flight from GMP to CJU on the upcoming Thursday. The cabin has a capacity of 200, and the retail ticket price is \$150.

After tickets are sold, some customers may not show up for flying[4]. Suppose that if z tickets are sold, then the number of passengers who show up will be between $z-10$ and $z-1$, with equal probability for each number. In other words, there is a 10% chance of each scenario where 1,2,3,...,10 people will not show up[5].

If a passenger shows up and there are no available seats, the airline company must compensate them \$500. The company will not issue a refund for the ticket price and has no further liability after the compensation is provided. Marginal cost of carrying an passenger from GMP to CJU is nominal and can be ignored to be zero.

(a) During the classes, we defined overstock cost as "economic cost due to having overstock", and described the situation as "too much preparation for demand". How would you quantify overstock cost in this example? (answer in a dollar amount) `[10pts]`

(b) During the classes, we defined understock cost as "economic cost due to having understock", and described the situation as "too less preparation for demand". How would you quantify understock cost in this example? (answer in a dollar amount) `[10pts]`

[4] Indeed, it is generally known that about 1%-2% of people who buy flight tickets do not show up as passenger for whatever reason.

[5] For example, if you sell 200 tickets, then the possible number of passengers will be between 190 to 199, with each outcome having an equal probability of 0.1.

(c) Suppose you sold 205 tickets and let X be the number of passenger who show up at airport gate for flying. Then X is distributed between 195 and 204, equally likely. What is the total expected profit, where the profit is defined as sale revenue minus overbooking compensation cost. Express the answer in terms of expectation and evaluate the expectation to present answer in a number. `[10pts]`

(d) Find the optimal number of ticket sale to maximize the expected profit. Show your work. `[10pts]`

(a)

- 150 (stock ¿ demand)

(b)

- 350 (=500-150)

(c)

$$
\begin{aligned}
\mathbb{E}[\text{profit}] &= \mathbb{E}[\text{rev}] - \mathbb{E}[\text{cost}] \\
&= \mathbb{E}[\text{ticket sale}] - \mathbb{E}[\text{overbooking compensation}] \\
&= 205 \times 150 - 500 \cdot \mathbb{E}[(X - 200)^+] \\
&= 30750 - 500 \\
&= 30250
\end{aligned}
$$

(d) First to obtain

$$F(x) \geq \frac{c_u}{c_o + c_u} = 0.7$$

. Desired approach is to consider to find z that matches the above inequality. If selling z tickets, each scenario gives the following pmf for number of passengers that show up.

	z-10	z-9	...	z-4	...	z-1
$\mathbb{P}(D = d)$	0.1	0.1	...	0.1	...	0.1
$\mathbb{P}(D \leq d)$	0.1	0.2	...	0.7	...	1.0

To match $\mathbb{P}(D \leq d) = 0.7$, $z - 4$ must be equal to 200, hence $z = 204$. Coincidentally, selling 203 tickets gives the same expected profit, thus qualified for the full credit answer.

Alternative approach goes as follows: Knowing that Profit(201) ¿ Profit(200) and Profit(210) ¿ Profit(211), a student may calculate all of the 10 scenarios including Profit(201), Profit(202), $\cdots$, and Profit(210). It would conclude the maximum occur at Profit(203) and Profit(204).

Grading scheme:

- (a) No partial points
- (b) If 500, 5pts were given.
- (c) 5pts for setting up for $\mathbb{E}[\text{profit}]$, i.e. $\mathbb{E}[\text{ticket sale}] - \mathbb{E}[\text{overbooking compensation}]$. 5pts for the rest work
- (d) As long as you find answer is 203 or 204, full credit is given. Otherwise, no credit is given.

(blank)

(Detachable)