Stochastic Processes, Quiz 2, 2024 Spring

Solution and Grading

- Duration: 120 minutes
- Closed material, No calculator

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- Write legibly.
- Justification is necessary unless stated otherwise.

1	10
2	10
3	10
4	15
5	20
Total	65

#1. Consider a $M/M/1/1$ queue such as a single doctor's office. Both the inter-arrival time and the service time are exponentially distributed, and the current expected waiting time in the queue for a customer is 80 minutes. You are proposing a strict appointment policy in order to change the inter-arrival time from a random variable with exponential distribution to a constant. This action eliminates all variation in inter-arrival time while keeping other parameters unchanged. What would be the expected waiting time per customer after your suggested policy applied to the system? (Answer in a number and justify it) `[10pts]`

- Kingman's formula: $\mathbb{E}W_q = \mathbb{E}V(\frac{\rho}{1-\rho})(\frac{c_a^2+c_s^2}{2})$
- Old $\mathbb{E}W_q = \mathbb{E}V(\frac{\rho}{1-\rho})(\frac{1^2+1^2}{2}) = 80$mins, since c_a^2 and c_s^2 are 1 because both the inter-arrival time and the service time follow an exponential distribution.
- New $\mathbb{E}W_q = \mathbb{E}V(\frac{\rho}{1-\rho})(\frac{0^2+1^2}{2}) = 40$mins. When the inter-arrival time becomes constant, c_a^2 becomes 0, since the variance of a constant is 0.

$\therefore$ 40 mins.

Grade scheme:

- If correctly stated Kingman's formula, then 2 pts
- If the answer has a minor mistake, then 5 pts

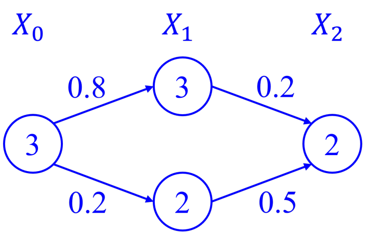
#2. Suppose that we have the following transition matrix for a discrete time Markov chain $\{X_n : n \geq 0,\ n \in \mathbb{N}\}$. Suppose $\mathbb{P}(X_0 = 3) = 1$, then what is $\mathbb{P}(X_2 = 2)$? `[10pts]`

$$\mathbf{P} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 0.7 & ? & 0 \\ 0.5 & ? & 0 \\ 0 & ? & 0.8 \end{pmatrix}$$

The sum of the elements in each row of the transition matrix $\mathbf{P}$ must be 1.

$$\mathbf{P} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.2 & 0.8 \end{pmatrix}$$

Method 1.



$\therefore 0.8 \times 0.2 + 0.2 \times 0.5 = 0.26$

Method 2.

$a_0 = (0 \ \ 0 \ \ 1)$

$$\mathbf{P}^2 = \begin{bmatrix} 0.64 & 0.36 & 0 \\ 0.6 & 0.4 & 0 \\ 0.1 & 0.26 & 0.64 \end{bmatrix}$$

$a_2 = a_0 \cdot \mathbf{P}^2 = (0.1 \ \ 0.26 \ \ 0.64)$

$\therefore \mathbb{P}(X_2 = 2) = 0.26$

Grade scheme:

- If correctly stated transition matrix, then 2 pts
- If the answer has a minor mistake, then 5 pts

#3. In post office, there are two servers, A and B. A's service time follows an exponential distribution with mean 3 minutes and B's service time follows a distribution of $exp(1/5)$. Their service times are independent to each other. Alice and Betty came to post office at noon and services started. (Alice being served by A and Betty being served by B)

(a) What is the chance that Betty will leave the post office before Alice? `[5pts]`

(b) What is the chance that both of them will be still in the service at 12:05 pm? `[5pts]`

$X_A \sim exp(1/3), \;\; X_B \sim exp(1/5)$

- (a) $\quad \mathbb{P}(X_B < X_A) = \frac{\lambda_B}{\lambda_A + \lambda_B} = \frac{3}{8}$
- (b) $\quad \mathbb{P}(X_A > 5, X_B > 5) = \mathbb{P}(X_A > 5) \cdot \mathbb{P}(X_B > 5)$
  $= \Big(1 - (1 - e^{-\frac{1}{3} \cdot 5})\Big) \cdot \Big(1 - (1 - e^{-\frac{1}{5} \cdot 5})\Big)$
  $= e^{-8/3}$

Grade scheme:

- (a) If the answer has a minor calculation mistake, then 3 pts
- (b) If the answer has a minor calculation mistake, then 3 pts

#4. A small bank is staffed by a single server. During a normal business day, the inter-arrival times of customers to the bank follow an exponential distribution with mean 3 minutes. On the other hand, the service time for a customer follows normal distribution with mean 2 minutes and standard deviation of 1 minute. Answer following questions in a number.

It is OK to use previous answer. For example, if you didn't solve problem (a) but know that the answer for (b) is [the answer for (a)] times 5, then you may answer question (b) as "5 × [ans in (a)]"

(a) What is the long-run fraction of times that the server is busy? Is this system stable? `[5pts]`

The long-run fraction $\rho = \frac{1/\mathbb{E}U}{1/\mathbb{E}V} = \frac{2}{3}$. Since $\rho < 1$, this system is stable.

(b) What is the long-run average waiting time of each customer in the queue? `[5pts]`

- Kingman's formula: $\mathbb{E}W_q = \mathbb{E}V(\frac{\rho}{1-\rho})(\frac{c_a^2+c_s^2}{2})$
- Applying Kingman's formula, we find that $\mathbb{E}W_q = 2(\frac{2/3}{1/3})(\frac{1+1/4}{2}) = 2.5$ minutes.
  Here, $\rho = \frac{2}{3}$, which is derived from answer (a). The c_a is 1, since the inter-arrival time follows an exponential distribution. The c_s is 0.5, since the mean and standard deviation of the service time are 2 and 1, respectively.

$\therefore$ 2.5 min.

(c) What is the long-run average number of customers waiting for service? `[5pts]`

- Little's law: $L_q = \lambda W_q$
- Applying Little's law, $L_q = (\frac{1}{3}\text{ customer/min.}) \cdot (2.5\,\text{min.}) = 5/6$ customer.

$\therefore$ 5/6 customer

Grade scheme:

- Common: If the solution approach is perfectly correct but the answer is wrong due to an error in previous answer, it is treated as correct.
- (a)
  If correctly stated ρ, then 1 pts
  If the answer has a minor calculation mistake, then 3 pts
- (b)
  If correctly stated Kingman's formula, then 1 pts
  If the answer has a minor calculation mistake, then 3 pts
- (c)
  If correctly stated Little's law, then 1 pts
  If the answer has a minor calculation mistake, then 3 pts

#5. You are selling lemonade. You have collected the following information.

- The demand is uniformly distributed between 20 gallons and 35 gallons.
- The selling price is 5 dollars per gallon.
- The value of unsold lemonade is 0.5 dollars per gallon.
- Every time you make an order, it costs fixed 25 dollars plus 2 dollars per gallon of lemonade.
- You have 10 gallons of inventory
- Assume that we know $s = 12$ in the (S, s) policy.

(a) What is S? (answer in a number) `[5pts]`

- $c_o = 1.5$, $c_u = 3$, and $F(y) = \frac{y-20}{35-20}$
- We need to find y s.t. $F(y) = \frac{C_u}{C_o+C_u} = \frac{2}{3}$
- $y = 15 \times \frac{2}{3} + 20 = 30$

$\therefore$ S = 30

Now you have figured out (S, s) policy. (The big S is from your answer to (a) and the small s is equal to 12 as given in the question. In case that you have not found an answer to (a), then you may use 27 as big S for the rest of this problem.)

(b) Remind that you have 10 gallons of inventory. What is the optimal order amount? Answer in a number. `[5pts]`

- (S,s) policy is (30,12).
- Since $10 < s = 12$, make an order of 30-10=20 units.

(c) What is the expected profit if you make an optimal order? Answer in a number. (In case that you have not found an answer to (b), then you may use 15 as the optimal order quantity.) `[10pts]`

- $\mathbb{E}[\text{profit}|\text{order } 20] = \mathbb{E}[\text{sales rev}] + \mathbb{E}[\text{salvage rev}] - \mathbb{E}[\text{order cost}]$
- $\mathbb{E}[\text{sales rev}] = 5 \cdot \mathbb{E}[min(D, 30)]$

  $= 5(\int_{20}^{30} y \frac{1}{15} dy + \int_{30}^{35} 30 \cdot \frac{1}{15} dy)$

  $= 5(\frac{1}{30}(30^2 - 20^2) + 2 \times (35 - 30))$

  $= 5(\frac{50}{3} + 10) = \frac{400}{3}$

- $\mathbb{E}[\text{salvage rev}] = 0.5 \cdot \mathbb{E}[(30 - D)^+]$

  $= 0.5(\int_{20}^{30} (30 - y) \frac{1}{15} dy)$

  $= \frac{1}{30} \left[30y - \frac{1}{2}y^2\right]_{20}^{30}$

  $= \frac{1}{30}(900 - 450 - 600 + 200) = \frac{5}{3}$

- $\mathbb{E}[\text{order cost}] = 25 + 2 \times 20 = 65$

$\therefore \mathbb{E}[\text{profit}] = \frac{400}{3} + \frac{5}{3} - 65 = 70$

Grade scheme:

- Common: If the solution approach is perfectly correct but the answer is wrong due to an error in previous answer, it is treated as correct.
- (a) If the answer has a minor calculation mistake, then 3 pts
- (b) No partial points
- (c)
  If correctly stated expected profit, then 2 pts
  If solution approach is generally correct but have major mistake, then 3 pts
  If solution approach is correct but have a minor calculation mistake, then 5 pts