Stochastic Processes, Quiz 3, 2024 Spring

Solution and Grading

- Duration: 120 minutes
- Closed material, No calculator

- Name: ____________________
- Student ID: ____________________
- E-mail: ______________ @seoultech.ac.kr

- Write legibly.
- Justification is necessary unless stated otherwise.

1	15
2	15
3	15
4	15
5	10
Total	70

#1. This problem concerns whether the following statement is true or false.

> If a DTMC has a finite number of states and is irreducible, then its stationary distribution is unique.

(a) Circle one of the following: `[5pts]` **(True/False)**

(b) If you answered "true", then explain why it is true and provide an example. If you answered "false", then provide a counterexample and explain it. `[10pts]`

Solution:

- a) True
- b) A finite-state irreducible Markov chain has a unique stationary distribution. We can think about two perspectives: aperodicity and periodicity.

  - Aperiodic: An aperiodic Markov chain will converge to a unique stationary distribution since there are no cyclic behaviors that prevents stabilization. See L7. p22 for theorem.

  - Periodic: Even periodic Markov chains with a finite state space and irreducibility have a unique stationary distribution.

  Example: Consider the following transition matrix for a periodic Markov chain with period 2:

  $$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  To find the stationary distribution, we solve,

  $$\begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix}$$

  which results in the equations: $\pi_2 = \pi_1, \pi_1 = \pi_2$.
  Since the sum of stationary distribution probabilities must be 1, $\pi_1 = \pi_2 = \frac{1}{2}$.
  This shows that even a periodic Markov chain, when it is finite and irreducible, has a unique stationary distribution.

  $\therefore$ This statement is true.

Grade Scheme:

- (a) No partial points
- (b) No partial points

#2. This problem concerns whether the following statement is true or false.

If a DTMC has a finite number of states and is aperiodic, then its stationary distribution is unique.

(a) Circle one of the following: `[5pts]` **(True/False)**

(b) If you answered "true", then explain why it is true and provide an example. If you answered "false", then provide a counterexample and explain it. `[10pts]`

Solution:

- a) False
- b) To demonstrate this, we need a counterexample of a DTMC that has a finite number of states, is aperiodic, but does not have a unique stationary distribution.

  Consider the following transition probability matrix P:

  $$P = \begin{pmatrix} 0.7 & 0.3 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0.3 & 0.7 \end{pmatrix}$$

  Matrix P has **finite number of states**, and **aperiodic** since it has two sub-Markov chains with state space: {1,2} and {3,4}, and each sub-chains have period of 1 (gcd of return time is 1).

  Since the matrix P is not irreducible, its stationary distribution is not unique. It has an infinite number of stationary distributions. We can generalize all the stationary distributions as a linear combination of the two such as: $p(5/8, 3/8, 0, 0) + (1 - p)(0, 0, 3/7, 4/7)$, where $0 \leq p \leq 1$.

  Therefore, even if DTMC has a finite number of states and is aperiodic, when DTMC is reducible, it does not have unique stationary distribution.

Grade Scheme:

- (a) No partial points
- (b) You must provide a correct counterexample. No partial points

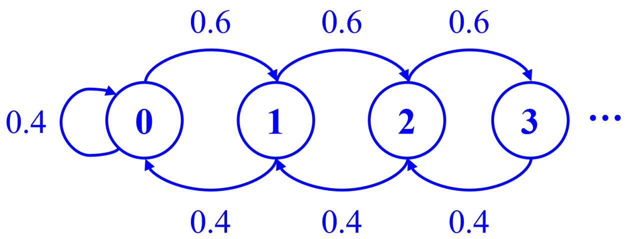
#3. This problem concerns whether the following statement is true or false.

If a DTMC is aperiodic and irreducible, then there exists a stationary distribution.

(a) Circle one of the following: `[5pts]` **(True/False)**

(b) If you answered "true", then explain why it is true and provide an example. If you answered "false", then provide a counterexample and explain it. `[10pts]`

- a) False
- b) Assume the random walk situation with $S = \{0, 1, 2, 3, ...\}$.



This DTMC is aperidoic, irreducible and has infinite state space.

State	Inflow	Outflow	Relationship
State 0:	$0.4\pi_0 + 0.4\pi_1$	$0.6\pi_0 + 0.4\pi_0$	$\pi_1 = 3/2\pi_0$
State 1:	$0.4\pi_1 + 0.4\pi_2$	$0.6\pi_1 + 0.4\pi_1$	$\pi_2 = 3/2\pi_1$
State 2:	$0.4\pi_2 + 0.4\pi_3$	$0.6\pi_2 + 0.4\pi_2$	$\pi_3 = 3/2\pi_2$
State 3:	$0.4\pi_3 + 0.4\pi_4$	$0.6\pi_3 + 0.4\pi_3$	$\pi_4 = 3/2\pi_3$

The value of π_i is greater than 1, so it does not converge, and thus π_0 cannot be defined. Therefore, the stationary distribution cannot be defined.

Grade Scheme:

- (a) No partial points
- (b) You must provide a correct counterexample. No partial points

#4. For a DTMC with the following transition matrix, calculate $\mathbf{P}^{100}$ `[15pts]`

$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 1/6 & 1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution:

Method 1.

$$\mathbf{P^2} = \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{P^3} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore,

$$\mathbf{P^{100}} = \begin{pmatrix} 0 & 1/6 & 1/2 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Method 2.

For the class {2,3},

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{P^2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{P^{100}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

When $x_0 = 1$, the probability that $x_{100} = 2$ is 1/6. This is because if $x_1 = 3$, the states alternate between 2 and 3 (i.e.,$x_0 = 1, x_1 = 3, x_2 = 2, x_3 = 3, ...$).
Simillary, When $x_0 = 1$, $x_{100} = 3$ with probability 1/2.

Therefore,

$$\mathbf{P^{100}} = \begin{pmatrix} 0 & 1/6 & 1/2 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Grade Scheme:

- If your first row is incorrect, then -5pts
- If your second and third rows are incorrect, then -5pts
- If your matrix is not stochastic, then another -5pts

#5. You either drink coffee or not on any given day. The following is the transition rule:

- If you drank coffee yesterday and today, the chance of you drinking coffee tomorrow is 0.2.
- If you did not drink coffee yesterday but drank coffee today, then the chance of drinking coffee tomorrow is 0.4.
- If you drank coffee yesterday but not today, then the chance of drinking coffee tomorrow is 0.6.
- If you did not drink coffee yesterday and today, then you will drink coffee tomorrow with the probability 0.8.

Consider a stochastic process $\{X_n, n \in \{0, 1, 2, ...\}\}$, where $X_n = C$ implies drinking coffee on the n-th day and $X_n = NC$ implies not drinking coffee on the n-th day.

We know that the stochastic process $\{X_n\}$ is not a DTMC. However, one can use $\{X_n\}$ to construct a stochastic process that is a DTMC. Please do so (construct a DTMC using $\{X_n\}$) and present a transition matrix of the constructed DTMC. Make sure to mark the state index in the transition matrix. `[10pts]`

Solution:
Let $Y_n := (X_{n-1}, X_n)$ and consider the discrete time stochastic process $\{Y_n, n \geq 1\}$ with space space $S_Y = \{(C, C), (NC, C), (C, NC), (NC, NC)\}$.

$$\begin{aligned}
\mathbb{P}(X_n = C, X_{n+1} = C | X_{n-1} = C, X_n = C) = (0.2) \\
\mathbb{P}(X_n = C, X_{n+1} = NC | X_{n-1} = C, X_n = C) = (0.8) \\
\mathbb{P}(X_n = C, X_{n+1} = C | X_{n-1} = NC, X_n = C) = (0.4) \\
\mathbb{P}(X_n = C, X_{n+1} = NC | X_{n-1} = NC, X_n = C) = (0.6) \\
\mathbb{P}(X_n = NC, X_{n+1} = C | X_{n-1} = C, X_n = NC) = (0.6) \\
\mathbb{P}(X_n = NC, X_{n+1} = NC | X_{n-1} = C, X_n = NC) = (0.4) \\
\mathbb{P}(X_n = NC, X_{n+1} = C | X_{n-1} = NC, X_n = NC) = (0.8) \\
\mathbb{P}(X_n = NC, X_{n+1} = NC | X_{n-1} = NC, X_n = NC) = (0.2)
\end{aligned}$$

-Transition matrix:

$$\mathbf{P} = \begin{array}{c|cccc}
 & (C,C) & (NC,C) & (C,NC) & (NC,NC) \\
\hline
(C,C) & 0.2 & 0 & 0.8 & 0 \\
(NC,C) & 0.4 & 0 & 0.6 & 0 \\
(C,NC) & 0 & 0.6 & 0 & 0.4 \\
(NC,NC) & 0 & 0.8 & 0 & 0.2
\end{array}$$

Grade Scheme:

- Construction is 5pts and transition matrix is 5pts
- By construction, any reader of your work should be able to see that you are proposing a DTMC of $Y_n = (X_{n-1}, X_n)$.
- By transition matrix, the index must properly marked and the numbers must be correct. Also, it should satisfy the property of DTMC transition matrix (a square matrix having each rowsum equal to 1).