Stochastic Processes, Quiz 1, 2024 Spring

Solution and Grading

- Duration: 60 minutes
- Closed material, No calculator

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- Write legibly.
- Justification is necessary unless stated otherwise.

1	20
2	10
3	20
Total	50

#1. Consider a random variable X that follows a uniform distribution with parameter 2 and 3. That is, $X \sim U(2,3)$.

(a) State its pdf [5pts]

$$\text{pdf } f(x) = \begin{cases} 0 & \text{if } x < 2 \\ 1 & \text{if } 2 \leq x \leq 3 \\ 0 & \text{if } x > 3 \end{cases}$$

(b) Find its standard deviation. Justification is necessary.[1] [10pts]

- $\mathbb{E}X = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{2} x f(x)\, dx + \int_{2}^{3} x f(x)\, dx + \int_{3}^{\infty} x f(x)\, dx$
  $= 0 + \int_{2}^{3} x\, dx + 0 = 5/2$
- $\mathbb{E}X^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{-\infty}^{2} x^2 f(x)\, dx + \int_{2}^{3} x^2 f(x)\, dx + \int_{3}^{\infty} x^2 f(x)\, dx$
  $= 0 + \int_{2}^{3} x^2\, dx + 0 = \frac{1}{3}(27-8) = 19/3$
- $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 19/3 - (5/2)^2 = 1/12$
- $sd(X) = \sqrt{1/12}$

(c) What is its coefficient of variation of X? [5pts]

- $cv(X) = sd(X)/\mathbb{E}X = \frac{\sqrt{1/12}}{2/5} = \sqrt{3}/15$

Grading scheme:

- (a) No partial points.
- (b) If solution approach is generally correct but have calculation mistake, then 5 pts.
- (c) If correctly stating the cv formula, then 2 pts.

[1] Hint: For a continuous random variable X, $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$. $sd(X) = \sqrt{Var(X)}$

#2. Let X be an exponential distribution with parameter 5, i.e. $X \sim exp(5)$. Evaluate $\mathbb{E}[min(X,3)]$ `[10pts]`

$$\text{pdf } f(x) = \begin{cases} 5e^{-5x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{E}[min(X,3)] = \int_{-\infty}^{\infty} min(x,3) f(x)\, dx$
$= \int_{-\infty}^{0} min(x,3) f(x)\, dx + \int_{0}^{3} min(x,3) f(x)\, dx + \int_{3}^{\infty} min(x,3) f(x)\, dx$
$= 0 + \int_{0}^{3} x f(x)\, dx + \int_{3}^{\infty} 3 f(x)\, dx$

- $\int_0^3 x f(x)\, dx = \int_0^3 x 5e^{-5x}\, dx = 5\int_0^3 xe^{-5x}\, dx$
  $= \left[-xe^{-5x}\right]_0^3 + \int_0^3 e^{-5x}\, dx$
  $= -3e^{-15} - \frac{1}{5}(e^{-15} - 1)$
- $\int_3^{\infty} 3f(x)\, dx = 15\int_3^{\infty} e^{-5x}$
  $= 3e^{-15}$

Therefore, $\int_0^3 x f(x)\, dx + \int_3^{\infty} 3f(x)\, dx = -3e^{-15} - \frac{1}{5}(e^{-15} - 1) + 3e^{-15} = \frac{1}{5}(1 - e^{-15})$
(See L1. p14 for a detailed explanation)

Grade scheme:

- If pdf is correct but have minor calculation mistake, then 5 pts
- If pdf is correct but have major calculation mistake, then 3 pts
- If pdf is wrong but approach for calculating expectation is correct, then 3 pts

#3. Let X be a Poisson distribution with parameter 4, i.e. $X \sim Poi(4)$.

(a) State its pmf `[10pts]`

$$p(x) = \mathbb{P}(X = x) = \frac{4^x e^{-4}}{x!} \, (x = 0, 1, 2, 3, ...)$$

(b) Let $Y = max(X, 3)$. State its pmf. `[10pts]`

$$p(y) = \mathbb{P}(Y = y) = \begin{cases} \sum_{k=0}^{3} \frac{4^k e^{-4}}{k!} = \frac{71}{3} e^{-4}, & \text{if } y = 3 \\ \frac{4^y e^{-4}}{y!}, & \text{if } y > 3 \\ 0, & \text{otherwise} \end{cases}$$

- $p(0) = p(1) = p(2) = 0 \, (\because Y = max(X, 3) \geq 3$ always)
- $p(3)$ includes all cases where $X \leq 3$

Grade scheme:

- (a) No partial points
- (b) If the range of y is correctly divided but has minor mistake, then 5 pts