Stochastic Processes, Quiz 1, 2025 Spring

Solution and Grading

- Duration: 60 minutes
- Closed material, No calculator

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- Write legibly.
- Justification is necessary unless stated otherwise.

1	3
2	3
3	4
Total	10

#1. State the definition of memoryless property by providing a mathematical expression. [3pts]

Suggested answer: For a random variable X, X is memoryless if $P(X > s+t|X > t) = P(X > s)$ for all $s, t \geq 0$.

Grading Scheme:

- 2pts, If the condition "for all $s, t \geq 0$" is missing. 이 조건이 없다면 s와 t가 무엇인지에 대한 설명이 없는 것입니다. 그러므로 이 조건을 적지 않은 수학적 진술자체가 불완전한 것을 인지할 수 있었으면 합니다.
- 1pts. This problem asks "stating the definition". But many students instead "proving that an exponential distribution is memoryless". In this case, 1pt is still given considering the proof that exponential distribution is memoryless includes the definition of memoryless property.
- 0pts, if inequality direction is wrong.

#2. Let X be a uniform random variable with parameter 0 and 1.

(a) Carefully state its pdf. `[1pts]`

Suggested answer: $f(x) = 1$ for $0 \leq x \leq 1$ and $f(x) = 0$ otherwise.

No partial credit.

(b) From the pdf, please derive its cdf. `[2pts]`

Suggested answer:

- For $x < 0$, $F(x) = \int_{-\infty}^{x} f(y)\ dx = \int_{-\infty}^{x} 0\ dx = 0$
- For $0 \leq x \leq 1$, $F(x) = \int_{-\infty}^{x} f(y)\ dx = \int_{-\infty}^{0} f(y)\ dx + \int_{0}^{x} f(y)\ dx = 0 + \int_{0}^{x} 1\ dx = x$
- For $1 < x$, $F(x) = \int_{-\infty}^{x} f(y)\ dx = \int_{-\infty}^{0} f(y)\ dx + \int_{0}^{1} f(y)\ dx + \int_{1}^{x} f(y)\ dx = 0 + 1 + 0 = 1$

Grading scheme:

- 1pts, if cdf is correct but no proof.
- 1pts, if one minor error occured.
- 0pts, if two or more errors occured.

#3. Let X be a Poisson random variable with parameter 5, and let $Y = min(X, 3)$.

(a) State the pmf of Y. [2pts]

(b) Find the variance of Y. [2pts]

(a)

$p_Y(y) = P(Y = y) = P(\min(X, 3) = y)$

$p_Y(0) = P(\min(X, 3) = 0) = P(X = 0) = e^{-5}$

$p_Y(1) = P(\min(X, 3) = 1) = P(X = 1) = 5e^{-5}$

$p_Y(2) = P(\min(X, 3) = 2) = P(X = 2) = \frac{5^2}{2!}e^{-5} = \frac{25}{2}e^{-5}$

$p_Y(3) = P(\min(X, 3) = 3) = P(X \geq 3) = 1 - P(X \leq 3) = 1 - \frac{37}{2}e^{-5}$

$p_Y(4) = P(\min(X, 3) = 4) = 0$

Thus, $p_Y(y) = \begin{cases} e^{-5}, & \text{for } y = 0, \\ 5\,e^{-5}, & \text{for } y = 1, \\ \frac{25}{2}\,e^{-5}, & \text{for } y = 2, \\ 1\; - \;\frac{37}{2}\,e^{-5}, & \text{for } y = 3, \\ 0, & \text{for } y \geq 4. \end{cases}$

Grading scheme: No partial point

(b)

$$E[Y] = \sum_{y=0}^{3} y p_Y(y) = 0 \cdot p_Y(0) + 1 \cdot p_Y(1) + 2 \cdot p_Y(2) + 3 \cdot p_Y(3)$$
$$= 1 \cdot 5e^{-5} + 2 \cdot \frac{25}{2}e^{-5} + 3\Big(1 - \frac{37}{2}e^{-5}\Big) = 3 - \frac{51}{2}e^{-5}$$
$$E[Y^2] = \sum_{y=0}^{3} y^2 p_Y(y) = 0^2 p_Y(0) + 1^2 p_Y(1) + 2^2 p_Y(2) + 3^2 p_Y(3)$$
$$= 1^2 \cdot 5\,e^{-5} + 2^2 \cdot \frac{25}{2}\,e^{-5} + 3^2\Big(1 - \frac{37}{2}\,e^{-5}\Big)$$
$$= 5\,e^{-5} + 4 \cdot \frac{25}{2}\,e^{-5} + 9\Big(1 - \frac{37}{2}\,e^{-5}\Big) = 9 - \frac{223}{2}e^{-5}$$
$$\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2$$
$$= \Big(9 - \tfrac{223}{2}e^{-5}\Big) - \Big(3 - \tfrac{51}{2}e^{-5}\Big)^2$$
$$= \Big(9 - \tfrac{223}{2}e^{-5}\Big) - \Big(9 - 153e^{-5} + \tfrac{2601}{4}e^{-10}\Big) = \tfrac{83}{2}e^{-5} - \tfrac{2601}{4}e^{-10}$$

Grading scheme: No partial point