Stochastic Processes, Final Exam, 2024 Spring

Solution and Grading

- Duration: 120 minutes
- Closed material, No calculator

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- Write legibly.
- Justification is necessary unless stated otherwise.

1	30
2	40
3	20
Total	90

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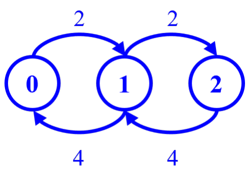
#1. Suppose that you are running a barbershop with a single barber whose service time is exponentially distributed with the mean of 1/4 hour. Currently, the arrival process is given as a Poisson process with the rate of 2 customers per hour. There is only one extra waiting space in the shop. Suppose that the shop operates for 10 hours per day and each customer generates revenue of 20 dollars on average. The barber gets paid 200 dollars per day.

(a) What is the expected waiting time for each customer in the queue? `[10pts]`

(b) What is the expected daily profit of this barber shop? `[10pts]`

Solution:
(a) Let $X(t)$ be the number of customer that are in a barbershop at time t. Then the state space is {0, 1, 2}. The rate diagram is as follows.



Stationary distribution π can be found using the flow balance equations as follows:

$$\begin{aligned} 2\pi_0 &= 4\pi_1 \\ 2\pi_1 &= 4\pi_2 \\ \pi_0 + \pi_1 + \pi_2 &= 1 \end{aligned}$$

Solving this system of linear equations, we have $\pi = (4/7, 2/7, 1/7)$ for a stationary distribution. The expected waiting time in the queue, $\mathbb{E}[W_q]$ is found as

$$\mathbb{E}[W_q] = \frac{L_q}{\lambda_{eff}}$$

where $L_q = 0 \cdot \pi_0 + 0 \cdot \pi_1 + 1 \cdot \pi_2 = 1/7$ and $\lambda_{eff} = 2(1 - \pi_2) = 12/7$. Thus, $\mathbb{E}[W_q] = \frac{(1/7)}{(12/7)} = 1/12$ and the expected waiting time is **1/12 hour or 5 minutes.**

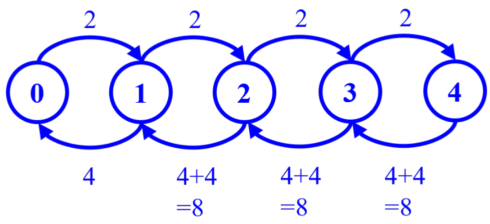
(b) The expected daily profit can be computed as:

Expected profit = (TH) × (Operating hours) × (Avg. revenue per customer) − (Daily wage)

TH is the same as λ_{eff} because the system is stable. Thus, $\mathbb{E}[\text{Profit}] = (12/7) \times 10 \times 20 - 200 =$ **1000/7 dollars**

(c) You are planning to expand this barbershop by hiring another barber and by adding another waiting area. Describe the new service system by presenting the rate matrix <u>or</u> the rate diagram.[1] `[10pts]`

Solution:
(c) By employing a single barber and adding one extra waiting space, the system becomes an M/M/2/2 queueing model. The rate diagram is as follows.

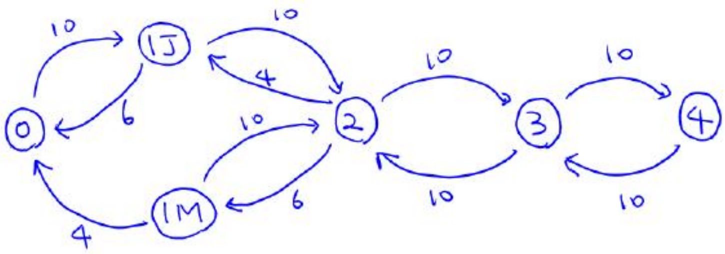


Grading scheme:

- (a)
  - Finding the correct stationary distribution is 5 pts and using Little's law properly is 5 pts.
  - 2.5 pts is deducted for minor mistakes
  - If used Kingman to found W_q, then 5 pts is given
- (b) -2.5 pts deducted for minor mistakes
- (c) -5 pts deduction is applied for each component wrong such as state info or rate
- Minor mistake examples are as follows:
  - Using λ insted of λ_{eff}
  - Not subtracting barber's wage
  - etc..

[1]Present only one. If both are presented, then one with more error will be graded.

#2. Consider the modified version of M/M/2/2 queuing model of the following, where each server has different service rate and the server J is the priority server.[2] The following figure is the rate diagram for the CTMC $X(t)$.[3] (All numbers at diagram are "per minute" rate, i.e. the rate from state 0 to state $1J$ is 10/min)



(a) Present the rate matrix G.`[5pts]`

Solution:

$$G = \begin{array}{c|cccccc} & 0 & 1J & 1M & 2 & 3 & 4 \\ \hline 0 & -10 & 10 & 0 & 0 & 0 & 0 \\ 1J & 6 & -16 & 0 & 10 & 0 & 0 \\ 1M & 4 & 0 & -14 & 10 & 0 & 0 \\ 2 & 0 & 4 & 6 & -20 & 10 & 0 \\ 3 & 0 & 0 & 0 & 10 & -20 & 10 \\ 4 & 0 & 0 & 0 & 0 & 10 & -10 \end{array}$$

Assume now that you have obtained $\pi = \{\pi_0, \pi_{1J}, \pi_{1M}, \pi_2, \pi_3, \pi_4\} = \{0.2, 0.15, 0.05, 0.2, 0.2, 0.2\}$ and use this stationary distribution π to answer following more questions.

(b) What is the long run fraction of time that the server J is busy?`[5pts]`

The server J is working at states 1J, 2, 3, and 4. Thus, the server J is busy with $\pi_{1J} + \pi_2 + \pi_3 + \pi_4 = 1 - \pi_0 - \pi_{1M} = \mathbf{0.75}$

[2]An arrival finding the server J free will begin receiving service by the server J. An arrival finding the server J busy and the server M free will join the server M

[3]For example, $X(t) = 3$ means that there are 3 customers in the system at time t; $X(t) = 1J$ means there is one customer in the middle of the service by server J at time t

(c) What is the throughput (TH)? `[5pts]`

The throughput is the arrival rate multiplied by the probability of the system not being in the state 4. $\text{TH} = \lambda_{\text{eff}} = 10(1 - \pi_4) = 10 \cdot 0.8 = 8$. **Thus, throughput is 8 customers per minute.**

(d) What is the expected number of customers in the system? (L_{sys}) `[5pts]`

$L_{sys} = 0 \cdot \pi_0 + 1 \cdot \pi_{1J} + 1 \cdot \pi_{1J} + 2 \cdot \pi_2 + 3 \cdot \pi_3 + 4 \cdot \pi_4 = 0.15 + 0.05 + 0.4 + 0.6 + 0.8 = 2$. The expected number of customers in the system is 2 customers.

(e) What is the expected total time spent in the system for a customer? (W_{sys}) `[5pts]`

$W_{sys} = \frac{L_{sys}}{\lambda_{eff}} = \frac{L_{sys}}{TH} = \frac{1}{4}$. Thus, the expected total time spent in the system for a customer is 1/4 min.

(f) What is the expected waiting time in the queue for a customer? (W_q) `[5pts]`

$W_q = \frac{L_q}{\lambda_{eff}}$ and $L_q = 0 \cdot \pi_0 + 0 \cdot \pi_{1J} + 0 \cdot \pi_{1J} + 0 \cdot \pi_2 + 1 \cdot \pi_3 + 2 \cdot \pi_4 = 0.2 + 0.4 = 0.6$. Thus, $W_q = \frac{0.6}{8}$ **= 3/40 min.**

(g) (Difficult) What is the probability that a customer is served by server J? (Justification is necessary) [4] `[10pts]`

First, $W_{svc} = W_{sys} - W_q = 7/40$ from (e) and (f). Also, by letting p_J be the probability that a customer is served by server J, $W_{svc} = p_J \cdot \frac{1}{6} + (1 - p_J) \cdot \frac{1}{4}$. Equating these will give p_J equal to 0.9.

Grading scheme:

- (a) 2.5pts is deducted for each error.
- (b) If you give an answer as 0.8, then 2.5 pts
- (c), (d), (e), (f) -2.5 pts is deducted for each error.
- (g) No partial point

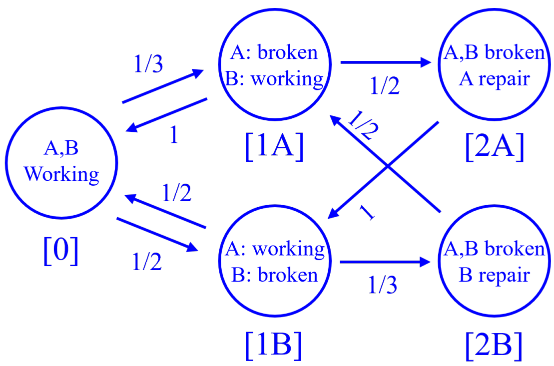
[4] Hint: The correct answer is between 0 and 1. If your answer is not in this range, then it means you did something wrong in the previous questions or on this question. You probably need to use your answer to (f).

#3. Consider a factory with two machines and a repairman. The time until machine A breaks down follows an exponential distribution with mean 3 hours, and the time until machine B breaks down follows an exponential distribution with mean 2 hours. A repairman can work on one broken machine at a time. Repair times for a machine is exponentially distributed. It takes 1 hour on average to fix the machine A and 2 hours on average to fix the machine B. What is the average number of machine under repair? `[20pts]`

The states should be defined as $\{0, 1A, 1B, 2A, 2B\}$, where the "number" indicates the number of broken machines, and the following "alphabet" indicates the machine under repair. Note that the state definion must be made to represent all situations in the most concise manner. Typical mistake in this problem is not to distinguish 2A and 2B, where 2A and 2B are indeed different situation.

- State [0]: A and B are working
- State [1A]: A is broken (and under repair) and B is working
- State [1B]: B is broken (and under repair) and A is working
- State [2A]: Both machines are broken and A is under repair
- State [2B]: Both machines are broken and B is under repair

The diagram is as follows:



Let's consider inflow and outflow of each state.

State	Inflow	Outflow
State 0	$\pi_{1A} + \frac{1}{2}\pi_{1B}$	$\frac{5}{6}\pi_0$
State 1A	$\frac{1}{3}\pi_0 + \frac{1}{2}\pi_{2B}$	$\frac{3}{2}\pi_{1A}$
State 1B	$\frac{1}{2}\pi_0 + \pi_{2A}$	$\frac{5}{6}\pi_{1B}$
State 2A	$\frac{1}{2}\pi_{1A}$	π_{2A}
State 2B	$\frac{1}{3}\pi_{1B}$	$\frac{1}{2}\pi_{2B}$

Using the flow balance relationship above, the stationary distribution is found as $\pi = \{\pi_0, \pi_{1A}, \pi_{1B}, \pi_{2A}, \pi_{2B}\} = \{39/118, 16/118, 33/118, 8/118, 22/118\}$. The number of machine under repair is zero on the state 0. In other states, one machine is under repair. Thus, the average number of machine under repair is $1 - \pi_0 = 1 - 39/118 = 79/118$. This is same as the fraction of the time the repairman is busy.

Grading scheme:

- If the five states are properly identified, then 10 pts were given.
- Think about how you would answer if the problem asked you "avg. number of machine in service" or "avg. number of machine not in service"?

”Examinations are formidable even to the best prepared, for the greatest fool may ask more than the wisest man can answer.” - Charles Caleb Colton.