Lecture C4. Discrete Time Markov Chain 4

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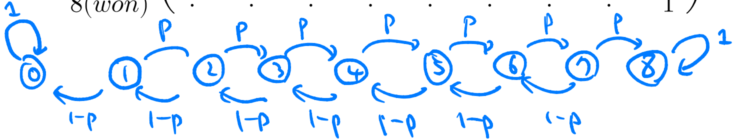
I. Gambler's ruin probability

Gambler's ruin

- Suppose you have $3(=$x$), and bet $1 with winning probability $p = 18/38$ until your wealth becomes 0$(=$a$) or your wealth becomes $8(=$b$). What is the chance of you will leave Casino with $8? (What is the chance that you will reach b before you reach a?)

$$
P = \begin{array}{r}
0(lose) \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8(won)
\end{array}
\begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
.53 & \cdot & .47 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & .53 & \cdot & .47 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & .53 & \cdot & .47 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & .53 & \cdot & .47 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & .53 & \cdot & .47 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & .53 & \cdot & .47 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & .53 & \cdot & .47 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
$$

state
0, 8 : absorbing
1, …, 7 : transient



- Result of $a = 0, b = 8, p = 18/38$

$$
P^{\infty} = \begin{array}{c} 0(lose) \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8(won) \end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
.92 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .08 \\
.82 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .18 \\
.72 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .28 \\
.60 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .40 \\
.48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .52 \\
.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .67 \\
.18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .82 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

- Result of $a = 0, b = 1000, p = 18/38, x = 100.$
- What is the quantity for $\mathrm{P}^{\infty}_{100\$\to win}$?

6.5×10^{-42}

martingale

- Result of $a = 0, b = 1000, p = 19/38, x = 100.$
- What is the quantity for $\mathrm{P}^{\infty}_{100\$\to win}$?

10%

$X_0 = 100$

$X_T = 0$ or 1000

$\mathbb{E}X_0 = \mathbb{E}X_1 = \mathbb{E}X_2 = \cdots = \mathbb{E}X_T = 100$

$\mathbb{E}X_T = \mathbb{P}(win) \times 1000 + \mathbb{P}(lose) \times 0 = 100$

- Result of $a = 0, b = 10 \times 100\$, p = 18/38, x = 1 \times 100$ \$ (bet 100\$ for each)
- What is the quantity for $\mathrm{P}^{\infty}_{1\times 100\$\to win}$?

6%

II. Squash

Skip

Squash

- Racket sports (court number 5 in CRC)
- Rules
  - Two players, three or five games.
  - Only the server scores points.
  - The server, on winning a rally, scores a point
  - The receiver, on winning a rally, becomes the server.
  - The player who scores nine points wins the game

- Rules (cont'd)
  - Suppose A and B are playing for the first set and $8 : \overline{7}$ now.
    (A's score is 8, B's score is 7, and B is serving)
  - Suppose B wins this play so that it becomes $8 : \overline{8}$.
  - Because A got to 8 first, A can decide either
    - i) This set ends at 9
    - ii) This set ends at 10
- Questions
  - Suppose the chance of A winning a play is 0.6, then should A choose i) or ii)?

- Suppose A decides "i) This set ends at 9".
- DTMC
  - Transition diagram and matrix
  - Classification of states
  - What is the chance of A winning this game?

- Suppose A decides "ii) This set ends at 10".
- DTMC

$$
P = \begin{array}{c} lose \\ 8:\overline{8} \\ \overline{8}:8 \\ 8:\overline{9} \\ \overline{8}:9 \\ 9:\overline{8} \\ \overline{9}:8 \\ 9:\overline{9} \\ \overline{9}:9 \\ win \end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
$$

- What is the chance of A winning this game?

- What if the chance of A winning a rally is not 0.6, but for general p?

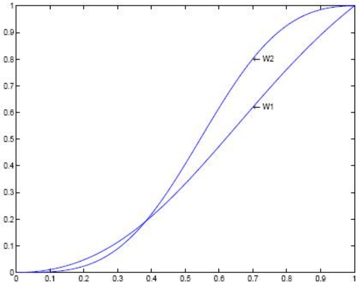


Figure 1: Probability of winning

- optimal decision
  - If $p \leq$, then choose i) ends at 9
  - Otherwise, choose ii) ends at 10
- Upon your decision, you are choosing one DTMC among the two different DTMC.

- Reference
  - Optimal Decision for the Squash Player
  - Jan Vecer, Columbia University, Department of Statistics
  - Journal of Chinese Statistical Association, 2004.
  - www.stat.columbia.edu/∼vecer/squash.ps

III. Tennis

Introduction

- DTMC for tennis game
- Used professional playing records (ATP tour 2011-2015)
- Is Markov chain a valid model for tennis game?
- What are the most important point in tennis?
- How do different court surfaces affect the model?
- How do serving ability of a player affect the model?

Dataset

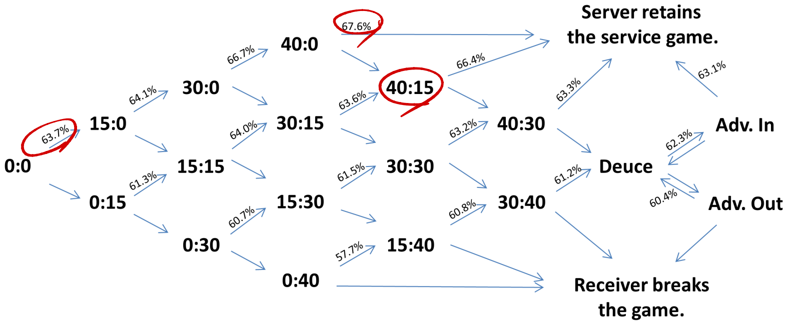
- Men's single matches in ATP tour from 2011 to 2015 are analyzed.
- The dataset includes 10,902 matches, 28,245 sets, 271,856 games, and 1,672,696 points.

Table 1. The structure of the dataset. A record for a single match is presented.

Variable	Value	Note
ID	6493708	
Date	04-Sep-14	
Tournament	Men's US Open	
Player 1	Novak Djokovic	Who serves first
Player 2	Andy Murray	
Winner	1	Player 1 or Player 2
Set 1	RSRRSSRR;SSRRSDRASRRSRR;RSSRRSSS;SRDRR;ARRSRSSS;RRSASS;SSRRSRRR;SRRSSS;RSSRRSSRAS;SRASS;SSRSRS;RSSRSA;S/DR/SR/RR/S	A: ace S: server wins R: receiver wins D: double faults
Set 2	SRSSA;SSSS;ARRRR;SRSSS;SSAS;RRRSR;SRRRR;SRRDSSRR;ASRRSS;SSARS;SSSS;SRSRRSSRSRSS;S/RR/SR/RR/S	
Set 3	SSRSS;SSSS;RSSSS;RRSRR;SRSSRRRSRSSS;SRSSS;RSRSSS;SARRRSRR	
Set 4	SSRDRSSA;SSRSRRSRSS;SSSA;SRDSSRSS;SASDRRAS;SSSRS;SSSS;SSSS;SASRS;RRSRR	
Score	7–6(1) 6–7(1) 6–2 6–4	

DTMC diagram for a regular game

- The point winning probabilities for servers are marked.
- Q. Are they identical?
  - A. Not identical
- Q. Are they path-independent?
  - It depends··· (next page)



Test for path dependency

- Point winning probabilities are path-independent except for `40:15`.
- This singularity is found for all court types (Grass, Hard, and Clay). Why?

63.97
0:15 → 15:15 → 30:15
15:0 → 15:15 → 30:15
63.94

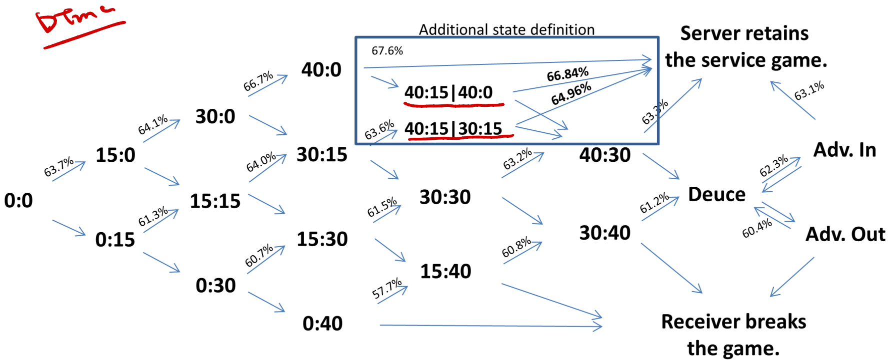
66.84
30:15 → 40:15 → Win
40:0 → 40:15 → Win
64.96

Table 3. Test of path dependency for server's point winning probabilities in regular games.

Current state	Last point won by	Server's winning prob. in the next point (%)	Number of observations	z-statistics
15:15	Server	63.97	60,429	0.12
	Receiver	63.94	62,222	
30:15	Server	63.63	78,443	0.74
	Receiver	63.4	36,960	
15:30	Server	61.86	23,157	1.46
	Receiver	61.29	44,208	
40:15	Server	66.84	73,346	5.33***
	Receiver	64.96	24,010	
30:30	Server	63.15	41,420	−0.53
	Receiver	63.32	42,057	
15:40	Server	61.38	8,672	1.31
	Receiver	60.59	25,945	
40:30	Server	63.35	52,787	0.06
	Receiver	63.33	32,736	
30:40	Server	61.38	21,043	0.55
	Receiver	61.14	30,690	
Deuce	Server	62.37	58,193	0.76
	Receiver	62.16	58,084	

*p < 0.1; **p < 0.05; ***p < 0.01.

Possible remedy to build theoretically valid DTMC.



Additionally defined state prepares a legit Markov chain

Table 5. Comparison of a server's game winning probabilities between the model and the actual data.

	Model	Actual data			Comparison	
State	Server's winning prob. (%)	Server's winning prob. (%)	Number of obs.	Standard deviation (%)	Diff. of prob. (%)	z-statistics
0:0	78.91	78.91	271,856	0.08	0	0.01
0:15	62.12	62.21	98,603	0.15	−0.09	−0.57
15:0	88.46	88.41	173,253	0.08	0.05	0.59
0:30	39.72	40.12	38,174	0.25	−0.4	−1.61
15:15	76.28	76.26	122,651	0.12	0.02	0.18
30:0	95.29	95.17	111,031	0.06	0.12	1.81*
0:40	15.57	16.08	15,017	0.3	−0.51	−1.69*
15:30	55.37	55.23	67,365	0.19	0.14	0.75
30:15	88.06	88.16	115,403	0.1	−0.1	−1.02
40:0	98.9	98.78	74,071	0.04	0.12	2.99***
15:40	26.97	27.11	34,617	0.24	−0.14	−0.59
30:30	73.16	73.13	83,477	0.15	0.03	0.22
40:15|30:15	96.65	96.7	73,346	0.07	−0.05	−0.75
40:15|40:0	96.46	96.23	24,010	0.12	0.23	1.93*
30:40	44.37	44.48	51,733	0.22	−0.11	−0.51
40:30	89.9	89.83	85,523	0.1	0.07	0.67
Deuce	72.45	72.1	116,277	0.13	0.35	2.69***
Adv. Out	43.78	43.37	43,878	0.24	0.41	1.71*
Adv. In	89.83	89.5	72,403	0.11	0.33	2.86***

*p < .1; **p < .05; ***p < 0.01.

Reference

- Sim, M. & Choi, D.* (2019) The winning probability of a game and the importance of points in tennis matches. Research Quarterly for Exercise and Sport. (SSCI & SCIE)

IV. High-frequency financial data

What is OB pressure?

- OB pressure can be seen as a random walk in two-dimensional space.
- The current location of the particle indicates OB state.
- The particle moves to NEWS direction by an OB event.
- Price changes when it touches an axis.

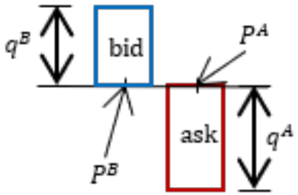


그림 1: Schematic diagram of order book

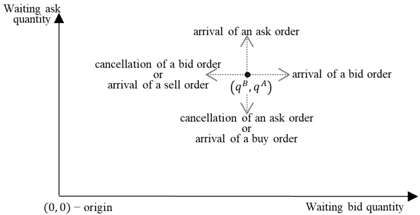


그림 2: Analogy of a 2D random walk

Hidden liquidity

- The presence of HL effectively shifts the both axis.
- That is, HL works as `buffer` of the axis.
- We incorporate the estimated HL and generate OB pressure matrix.

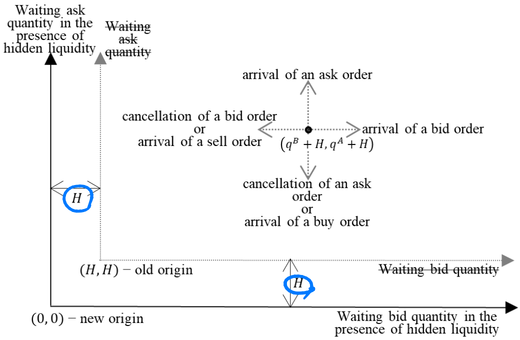


그림 3: HL buffers the axis

Reference

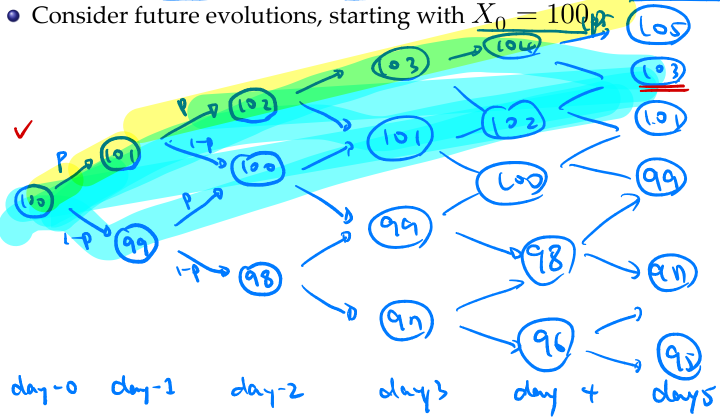
- Sim, M. K., & Deng, S. (2020). Estimation of level-I hidden liquidity using the dynamics of limit order-book. *Physica A: Statistical Mechanics and its Applications*, 540, 122703. (SCIE)

V. Stock price - binomial tree

Stock price - binomial tree

$^nC_r\, p^r (1-p)^{n-r}$

- Let X_n be the closing price of the stock at n-th day.
- Let $p = \mathbb{P}(X_{n+1} = x+1 | X_n = x)$, and $1-p = \mathbb{P}(X_{n+1} = x-1 | X_n = x)$
- Consider future evolutions, starting with $X_0 = 100$.



4 p^5

2 $p^4(1-p)\,{}^5C_4$

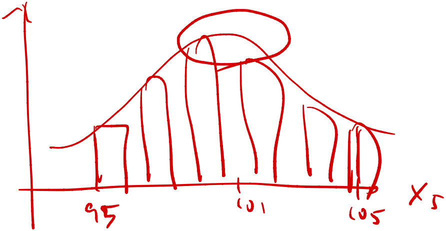
0

0

0

0

- Consider an European call option which matures at day 5 with exercise price 101.
- (If you possess one unit of the call option, then at the day 5, you have a right to buy the stock at 101 dollars.)
- If $X_5 = 103$, then you can buy the stock at 101 and sell at 103. In this case, you earn 2 dollar.
- If $X_5 = 99$, then you still can buy the stock at 101. But you would not do it because you can buy a stock at 99 dollars. (Possessing call option is the "right" not the "obligation")
- i.e., the payoff of a call option is $(X_5 - 101)^+$



- What is the expected payoff for the option, when $p = 0.6$?

Summary

- The behavior of stock price's movement is often believed as random walk.
- The random walk leads to binomial evolution through the discrete time.
- At the far end from the beginning, the binomial expansions converges to normal distribution - In other words, applicable is the famous theorem of "normal approximation of binomial distribution."

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"If I only had an hour to chop down a tree, I would spend the first 45 minutes sharpening my axe. -
A. Lincoln"
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