Stochastic Processes, Quiz 3, 2023 Spring

Solution and Grading

- Duration: 120 minutes
- Closed material, No calculator

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- Write legibly.
- Justification is necessary unless stated otherwise.

1	50
2	25
3	25
Total	100

#1. Fill in the blanks by using the following terms. `[Each 5pts]`

periodic	aperiodic	class	recurrent	transient	absorbing
finite	infinite	irreducible	reducible	reach	communicate

(a)	recurrent
(b)	periodic
(c)	reducible
(d)	aperiodic
(e)	irreducible
(f)	class
(g)	finite
(h)	irreducible
(i)	apreiodic
(j)	recurrent

- An absorbing state is a special case of a [(a)] state.
- A DTMC is said to be [(b)] if there exists a state whose period is higher than one.
- A DTMC is said to be [(c)] if there are more than one class.
- If all diagonal elements of transition matrix in DTMC are positive, then the DTMC has to be [(d)].
- A DTMC is said to be [(e)] if all states communicate.
- A group of states that can reach each other is said to be a [(f)].
- In a DTMC, the limiting probability is same as the unique stationary distribution regardless of initial states if following three conditions of the DTMC are met: [(g)] [(h)] [(i)].
- State i is said to be [(j)] if, starting from state i, the probability of getting back to state i in some future is equal to 1.

Grading scheme:

- 5pts for each problems.
- 2.5pts for answering ”absorbing” in (j).

#2. You either drink coffee or not on any given day. The following is the transition rule:

- If you drank coffee yesterday and today, the chance of you drinking coffee tomorrow is 0.2.
- If you did not drink coffee yesterday but drank coffee today, then the chance of drinking coffee tomorrow is 0.4.
- If you drank coffee yesterday but not today, then the chance of drinking coffee tomorrow is 0.6.
- If you did not drink coffee yesterday and today, then you will drink coffee tomorrow with the probability 0.8.

(a) Suppose you drank coffee yesterday and today, then what is the probability that you will not drink coffee tomorrow but will drink coffee the day after tomorrow? (Answer in number, justification is necessary, and very limited partial points given) `[10pts]`

(b) In the long run, what is the probability that you will be drinking coffee on a given day? (Answer in number, justification is necessary, and very limited partial points will be given) `[15pts]`

Solution:

- C = drink coffee, NC= Not drink coffee.
- (a)

  method 1) Set up transition matrix $\mathbf{P}$ and find quantity from $\mathbf{P}^2$

$$\mathbf{P} = \begin{array}{c} C,C \\ NC,C \\ C,NC \\ NC,NC \end{array} \begin{pmatrix} 0.2 & 0 & 0.8 & 0 \\ 0.4 & 0 & 0.6 & 0 \\ 0 & 0.6 & 0 & 0.4 \\ 0 & 0.8 & 0 & 0.2 \end{pmatrix}$$

  method 2) Apply "path" perspective

$$C,C \xrightarrow{0.8} C,NC \xrightarrow{0.6} NC,C$$

  , thus the probability is 0.48

- (b)

  Step 1) First to calculate stationary distribution, because this MC is finite, aperiodic and irreducible.

$$\begin{pmatrix} \pi_{C,C} & \pi_{NC,C} & \pi_{C,NC} & \pi_{NC,NC} \end{pmatrix} \begin{pmatrix} 0.2 & 0 & 0.8 & 0 \\ 0.4 & 0 & 0.6 & 0 \\ 0 & 0.6 & 0 & 0.4 \\ 0 & 0.8 & 0 & 0.2 \end{pmatrix} = \begin{pmatrix} \pi_{C,C} & \pi_{NC,C} & \pi_{C,NC} & \pi_{NC,NC} \end{pmatrix}$$

$$\begin{aligned}
0.2\pi_{C,C} + 0.4\pi_{NC,C} &= \pi_{C,C} \\
0.6\pi_{C,NC} + 0.8\pi_{NC,NC} &= \pi_{NC,C} \\
0.8\pi_{C,C} + 0.6\pi_{NC,C} &= \pi_{C,NC} \\
0.4\pi_{C,NC} + 0.2\pi_{NC,NC} &= \pi_{NC,NC}
\end{aligned}$$

We can get π as follow:

$$(\pi_{C,C} \quad \pi_{NC,C} \quad \pi_{C,NC} \quad \pi_{NC,NC}) = (1/6 \quad 1/3 \quad 1/3 \quad 1/6)$$

Step 2) Calculate the probability of drinking coffee in a given day by π. Then, you can choose either of true following approach:

- considering the first day $\pi_{C,C} + \pi_{C,NC} = 0.5$
- considering the second day $\pi_{C,C} + \pi_{NC,C} = 0.5$
- considering the both days $(2\pi_{C,C} + \pi_{C,NC} + \pi_{NC,C})/2 = 0.5$

Grading scheme:

- (a)
  - full score is given for correct value.
  - 5pts only if $\mathbf{P}$ is well defiend.
- (b)
  - 3pts know that stationary distribution is necessary and attempt to compute it.
  - 4pts for getting the correct stationary distribution.
  - 8pts for producing correct answers.

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#3. Consider a DTMC with the following transition matrix.

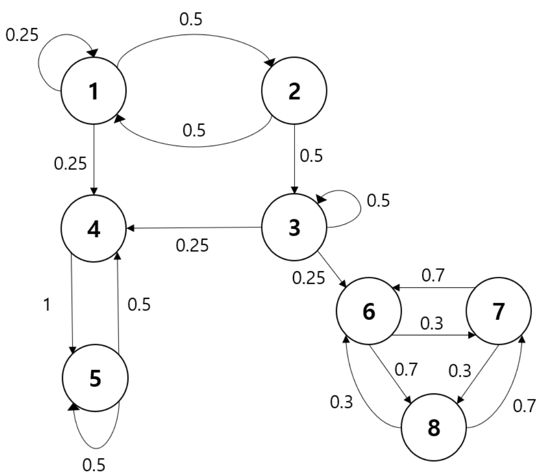
$$\mathbf{P} = \begin{pmatrix} .25 & .5 & & .25 & & & & \\ .5 & & .5 & & & & & \\ & & .5 & .25 & & .25 & & \\ & & & & 1 & & & \\ & & & .5 & .5 & & & \\ & & & & & & .3 & .7 \\ & & & & & .7 & & .3 \\ & & & & & .3 & .7 & \end{pmatrix}$$

(a) How many classes does it have? `[5pts]`

(b) Calculate $\mathbf{P}^{100}$ (Show your work with proper justification.) `[20pts]`

Solution:

- (a) 4 classes.
  Recurrent classes: $\{4,5\}, \{6,7,8\}$. Transient classes: $\{1,2\}, \{3\}$.
- (b)
  Step 1) Draw a transition diagram.



Step 2) Set up p^{100} and identify as many zeros as possible.

$$\mathbf{P}^{100} \approx \lim_{n\to\infty} \mathbf{P}^n = \begin{pmatrix} 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & x & x & 0 & 0 & 0 \\ 0 & 0 & 0 & x & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & x & x \\ 0 & 0 & 0 & 0 & 0 & x & x & x \\ 0 & 0 & 0 & 0 & 0 & x & x & x \end{pmatrix}$$

, where x is non zero number.

Step 3) Identify the "x" in above matrix. The "x" are limiting probability from a recurrent class to the recurrent class itself. Thus, we can find limiting probability by getting stationary distribution of the small Markov chain of recurrent class.

For the first recurrent class of $\{4, 5\}$,

$$\begin{aligned}\pi_4 &= 0.5\pi_5 \\ \pi_5 &= 0.1\pi_4 + 0.5\pi_5 \\ \pi_4 + \pi_5 &= 1\end{aligned}$$

Thus, $(\pi_4, \pi_5) = (1/3, 2/3)$. Similarly, we have $(\pi_6, \pi_7, \pi_8) = (1/3, 1/3, 1/3)$. Now you have identified the "x"s in the previous matrix arriving to

$$\mathbf{P}^{100} \approx \lim_{n\to\infty} \mathbf{P}^n = \begin{pmatrix} 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{pmatrix}$$

Step 4) We shall find the limiting probability going from transient state to recurrent class, i.e. $f_{1,\{4,5\}}, f_{2,\{4,5\}}, f_{3,\{4,5\}}$.
Once you find these three numbers out, you will automatically get $f_{1,\{6,7,8\}}, f_{2,\{6,7,8\}}, f_{3,\{6,7,8\}}$.

$$\mathbf{P}^{100} \approx \lim_{n\to\infty} \mathbf{P}^n = \begin{pmatrix} 0 & 0 & 0 & 1/3 \cdot f_{1,\{4,5\}} & 2/3 \cdot f_{1,\{4,5\}} & 1/3 \cdot f_{1,\{6,7,8\}} & 1/3 \cdot f_{1,\{6,7,8\}} & 1/3 \cdot f_{1,\{6,7,8\}} \\ 0 & 0 & 0 & 1/3 \cdot f_{2,\{4,5\}} & 2/3 \cdot f_{2,\{4,5\}} & 1/3 \cdot f_{2,\{6,7,8\}} & 1/3 \cdot f_{2,\{6,7,8\}} & 1/3 \cdot f_{2,\{6,7,8\}} \\ 0 & 0 & 0 & 1/3 \cdot f_{3,\{4,5\}} & 2/3 \cdot f_{3,\{4,5\}} & 1/3 \cdot f_{3,\{6,7,8\}} & 1/3 \cdot f_{3,\{6,7,8\}} & 1/3 \cdot f_{3,\{6,7,8\}} \\ 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{pmatrix}$$

It remains us to find $f_{1,\{4,5\}}, f_{2,\{4,5\}}, f_{3,\{4,5\}}$ as explained above. So let's begin filling this out by calculating the absorption probabilities into class $\{4, 5\}$

$$f_{1,\{4,5\}} = 0.25 f_{1,\{4,5\}} + 0.5 f_{2,\{4,5\}} + 0.25 \tag{1}$$
$$f_{2,\{4,5\}} = 0.5 f_{1,\{4,5\}} + 0.5 f_{3,\{4,5\}} \tag{2}$$
$$f_{3,\{4,5\}} = 0.5 f_{3,\{4,5\}} + 0.25 \tag{3}$$

Solving by substituting the result of equation (3) (i.e. $f_{3,\{4,5\}} = 0.5$), in the equation (1) and (2) and solve a system of equations of them. we get

$$\begin{aligned} f_{1,\{4,5\}} &= 3/4 \\ f_{2,\{4,5\}} &= 5/8 \\ f_{3,\{4,5\}} &= 1/2 \end{aligned}$$

Obviously, the absorption probabilities into class $\{6, 7, 8\}$ can be calculated directly since for the transient states the sum of all absorption probabilities is 1.

$$\begin{aligned} f_{1,\{6,7,8\}} &= 1 - f_{1,\{4,5\}} = 1/4 \\ f_{2,\{6,7,8\}} &= 1 - f_{2,\{4,5\}} = 3/8 \\ f_{3,\{6,7,8\}} &= 1 - f_{3,\{4,5\}} = 1/2 \end{aligned}$$

Step 5) Deliver solution.

$$\mathbf{P}^{100} \approx \lim_{n\to\infty} \mathbf{P}^n = \begin{pmatrix} 0 & 0 & 0 & 1/4 & 1/2 & 1/12 & 1/12 & 1/12 \\ 0 & 0 & 0 & 5/24 & 5/12 & 1/8 & 1/8 & 1/8 \\ 0 & 0 & 0 & 1/6 & 1/3 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{pmatrix}$$

Grading scheme:

- (a) No partial points.
- (b) 20pts, if you suggest the correct answer but the answer is wrong, the partial points are given as follows:
  - 5pts : $\sim$ Step 2
  - 5pts : Step 3
  - 5pts : Step 4
  - 5pts : Step 5

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Write your name before detaching. Your Name: