Stochastic Processes, Final Exam, 2023 Spring

Solution and Grading

- Duration: 120 minutes
- Closed material, No calculator

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- Write legibly.
- Justification is necessary unless stated otherwise.
- "Examinations are formidable even to the best prepared, for the greatest fool may ask more than the wisest one can answer" - Charles Caleb Colton

1	10
2	10
3	10
4	10
5	10
6	30
Total	80

#1. Prove that the time to the first arrival in $PP(\lambda)$ follows $exp(\lambda)$. [10pts]

#2. Suppose $X_1 \sim exp(\lambda_1)$, $X_2 \sim exp(\lambda_2)$, and X_1 and X_2 are independent. Prove that $min(X_1, X_2) \sim exp(\lambda_1 + \lambda_2)$. [10pts]

Solution:

- #1. Let T_1 be the time to the first arrival

$$\begin{aligned}
\mathbb{P}(T_1 \leq t) &= 1 - \mathbb{P}(T_1 > t) \\
&= 1 - \mathbb{P}(N(t) - N(0) = 0) \\
&= 1 - \mathbb{P}(Poi(\lambda t) = 0) \\
&= 1 - \frac{(\lambda t)^0 e^{-\lambda t}}{0!} \\
&= 1 - e^{-\lambda t}
\end{aligned}$$

  Thus, $T_1 \sim exp(\lambda)$

- #2.

$$\begin{aligned}
\mathbb{P}(min(X_1, X_2) \leq x) &= 1 - \mathbb{P}(min(X_1, X_2) > x) \\
&= 1 - \mathbb{P}(X_1 > x, X_2 > x) \\
&= 1 - \mathbb{P}(X_1 > x)\mathbb{P}(X_2 > x) \quad (\because \text{independent}) \\
&= 1 - (1 - (1 - e^{-\lambda_1 x}))(1 - (1 - e^{-\lambda_2 x})) \\
&= 1 - (e^{-\lambda_1 x})(e^{-\lambda_2 x}) \\
&= 1 - (e^{-(\lambda_1 + \lambda_2)x})
\end{aligned}$$

  Thus, $min(X_1, X_2) \sim exp(\lambda_1 + \lambda_2)$

Grading scheme:

- (a) No partial points.
- (b) No partial points. 5pts per mistake is deducted only if the mistake is considered a minor mistake.

#3. Suppose each morning a factory posts the number of days worked in a row without injury.[1] Assume that each day is injury free with the probability of 0.98. Assume that whether tomorrow is injury free or not is independent of which of the preceding days were injury free. Let X_n be the number posted on the morning after n full days of work. Let $X_0 = 0$ be the morning the factory first opened.

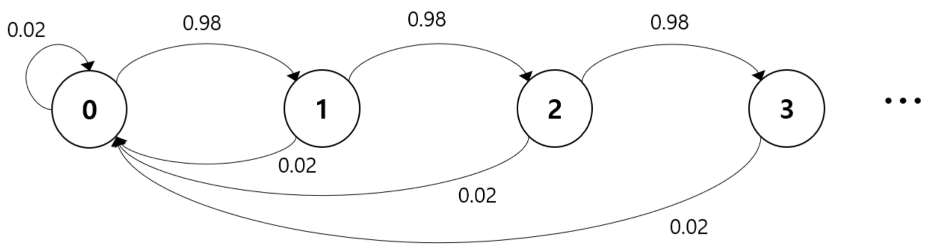
This discrete time stochastic process of X_n can be modelled as a DTMC. Provide either i) *transition matrix* or ii) *transition diagram* so that all the necessary information is ready for calculating its stationary distribution.[2] `[10pts]`

Solution:

i) *transition matrix*

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \end{array} \begin{pmatrix} 0.02 & 0.98 & & & & \\ 0.02 & & 0.98 & & & \\ 0.02 & & & 0.98 & & \\ 0.02 & & & & \dots & \\ \dots & & & & & \dots \\ \dots & & & & & \end{pmatrix}$$

ii) *transition diagram*



Grading scheme:

- No partial points.

[1] i.e. consecutive days without injury

[2] Provide ONLY ONE of those two. If you provide both, the one with more error will be graded. You do NOT need to calculate the stationary distribution

#4. Assume that calls arrive at a customer call center between 9am ($t = 0$) and 2pm ($t = 5$) following a non-homogeneous Poisson process with the rate function of the following:

$$\lambda(t) = \begin{cases} 4 & 0 \le t < 1 \\ 2t + 2 & 1 \le t < 3 \\ 8 & 3 \le t < 5 \end{cases}$$

(a) What is the probability that the call center receives 7^{th} call before 10am ($t = 1$)? `[5pts]`

(b) What is the expected number of arrivals between 11am ($t = 2$) and 1pm ($t = 4$)? `[5pts]`

Solution:

- (a) Let, X is the number of received calls.

$$\begin{aligned} \mathbb{P}\left(N(1) - N(0) \ge 7\right) &= 1 - \mathbb{P}(N(1) - N(0) < 7) \\ &= 1 - \mathbb{P}(X < 7 | X \sim Poi(\int_0^1 4\ dt)) \\ &= 1 - \mathbb{P}\left(X < 7 | X \sim Poi(4)\right) \\ &= 1 - \left(\mathbb{P}\left(Poi(4) = 0\right) + \mathbb{P}\left(Poi(4) = 1\right) + \mathbb{P}\left(Poi(4) = 2\right) + \cdots + \mathbb{P}\left(Poi(4) = 6\right)\right) \\ &= 1 - \left(\frac{4^0 e^{-4}}{0!} + \frac{4^1 e^{-4}}{1!} + \frac{4^2 e^{-4}}{2!} + \cdots \frac{4^6 e^{-4}}{6!}\right) \\ &= 1 - (1 + 4 + 8 + (32/3) + (32/3) + (128/15) + (256/45)) \cdot e^{-4} \\ &= 1 - \frac{437}{9} e^{-4} \end{aligned}$$

- (b) Let, X is the number of arrivals. Then X follows Poisson distribution with parameter $\int_2^4 \lambda(t)\ dt$.

$$\begin{aligned} \int_2^4 \lambda(t)\ dt &= \int_2^3 2t + 2\ dt + \int_3^4 8\ dt \\ &= \left[t^2 + 2t\right]_2^3 + 8(4 - 3) \\ &= (15 - 8) + 8 \\ &= 15 \end{aligned}$$

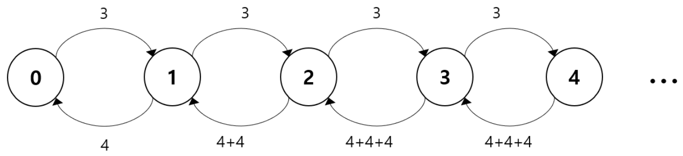
Since $X \sim Poi(15), \mathbb{E}X = 15$.

Grading scheme:

- (a) No partial points. 2.5pts per mistake is deducted only if the mistake is considered a minor mistake.
- (b) No partial points.

#5. Consider a queuing system of $M/M/3/\infty$ with the arrival rate $\lambda = 3/min$ and the service rate $\mu = 4/min$. (All of the three servers have the same service rate) For a stationary distribution $\pi = \{\pi_0, \pi_1, \pi_2, ...\}$, express π_5 in terms of π_0. (Hint: You need to find the constant c such that $\pi_5 = c\pi_0$.) [10pts]

Solution: Let $X(t)$ be the number of customer that are in the system at time t. Then the state space is {0, 1, 2, 3, 4,...}. The rate diagram is as follows.



We can write the following cut equations for the stationary distribution π:

$$
\begin{aligned}
3\pi_0 = 4\pi_1 &\Rightarrow \pi_1 = \frac{3}{4}\pi_0 \\
3\pi_1 = 8\pi_2 &\Rightarrow \pi_2 = \frac{3}{8}\pi_1 = \frac{3}{8}\cdot\frac{3}{4}\pi_0 \\
3\pi_2 = 12\pi_3 &\Rightarrow \pi_3 = \frac{3}{12}\pi_2 = \frac{3}{12}\cdot\frac{3}{8}\cdot\frac{3}{4}\pi_0 \\
3\pi_3 = 12\pi_4 &\Rightarrow \pi_4 = \frac{3}{12}\pi_3 = \left(\frac{3}{12}\right)^2\cdot\frac{3}{8}\cdot\frac{3}{4}\pi_0 \\
3\pi_4 = 12\pi_5 &\Rightarrow \pi_5 = \frac{3}{12}\pi_4 = \left(\frac{3}{12}\right)^3\cdot\frac{3}{8}\cdot\frac{3}{4}\pi_0
\end{aligned}
$$

Thus, $\pi_5 = \left(\frac{3}{12}\right)^3\cdot\frac{3}{8}\cdot\frac{3}{4}\pi_0 = \left(\frac{1}{4}\right)^3\cdot\frac{3}{8}\cdot\frac{3}{4}\pi_0 = \frac{3^2}{2^{11}}\pi_0 = \frac{9}{2048}\pi_0$

Grading scheme:

- No partial points. 5pts per mistake is deducted only if the mistake is considered a minor mistake.

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#6. Suppose that you are running a barbershop with a single barber whose service time is exponentially distributed with the mean of 1/4 hour. Currently, the arrival process is given as a Poisson process with the rate of 2 customers per hour. There is only one extra waiting space in the shop. Suppose that the shop operates for 10 hours per day and each customer generates revenue of 20 dollars on average.

(a) Find a stationary distribution for this system. `[10pts]`

(b) What is the expected waiting time for each customer in the queue? `[10pts]`

(c) What is the expected daily revenue of this barber shop?`[10pts]`

Solution:

- (a) Let $X(t)$ be the number of customer that are in a barbershop at time t. Then the state space is $\{0, 1, 2\}$. The rate diagram is as follows.

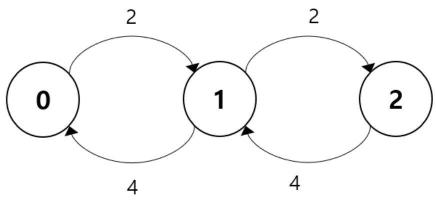

Figure 1: diagram for (a)

We can write the following flow balance equations for the stationary distribution π:

$$\begin{aligned} 2\pi_0 &= 4\pi_1 \\ 2\pi_1 &= 4\pi_2 \\ \pi_0 + \pi_1 + \pi_2 + \pi_2 &= 1 \end{aligned}$$

Solving these equations, we have $\pi = (4/7, 2/7, 1/7)$ for a stationary distribution.

- (b) The expected waiting time in the queue, $\mathbb{E}[W_q]$ is defined as follows.

$$\mathbb{E}[W_q] = \frac{L_q}{\lambda_{eff}}$$

, where L_q and λ_{eff} are obtained by :

  - $L_q = 0 \cdot \pi_0 + 0 \cdot \pi_1 + 1 \cdot \pi_2 = 1/7$
  - $\lambda_{eff} = 2(1 - \pi_2) = 12/7$

Therefore, $\mathbb{E}[W_q] = \frac{(1/7)}{(12/7)} = 1/12$. Thus, the expected waiting time is $1/12$ hour or 5 minutes.

- (c) Let D is the daily revenue in a given day. The expected daily revenue $\mathbb{E}[D]$ can be obtained as follows.

$$\mathbb{E}[D] = (\text{Throughput (TH)}) \times (\text{operating hours per day}) \times (\text{Average revenue per customer})$$

TH is the same as λ_{eff} because the system is stable, i.e. TH $= 12/7$ per hour. It follows

$$\mathbb{E}[D] = (12/7) \times 10 \times 20 = 2400/7 \text{ dollars}$$

Grading scheme:

- (a) No partial points. 5pts per mistake is deducted only if the mistake is considered a minor mistake.

- (b) No partial points, but you can get full credit if you follow the correct process using the answer to the problem (a).

  - 5pts are deducted only if you make a minor mistake when calculating λ_{eff} or L_q.

- (c) You can get full credit if you follow the correct process using the answer to the problem (b).
  - 5pts for having a key understanding that throughput is needed for obtaining expected daily revenue.
  - 5pts for the rest work.

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Write your name before detaching. Your Name: