Lecture C1. Discrete Time Markov Chain 1

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I. Motivation

Motivation

- I drink a bottle of soda everyday. I drink either Coke or Pepsi everyday. When I choose what to drink for today, I only consider what I drank yesterday.

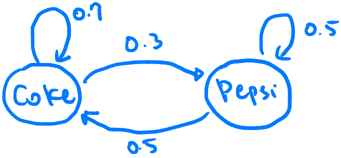
- Specifically,
  - Suppose I drank Coke yesterday, then the chance of drinking Coke again today is 0.7.
  - (What is the chance of drinking Pepsi today then?) 0.3. (1−0.7)
  - Suppose I drank Pepsi yesterday, then the chance of drinking Pepsi again today is 0.5.
  - (What is the chance of drinking Coke today then?) 0.5 (1−0.5)

Representation

State
transition
transition prob
state space
transition diagram / matrix

S_t vs S_{t+1}

- How would you describe this situation in diagram?



- How would you represent this situation to mathematical form?

from \ to	Coke	Pepsi
Coke	0.7	0.3
Pepsi	0.5	0.5

$$\begin{array}{cc} & \begin{array}{cc} \text{Coke} & \text{Pepsi} \end{array} \\ \begin{array}{c} \text{Coke} \\ \text{Pepsi} \end{array} & \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix} \end{array}$$

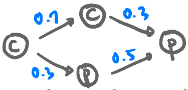
1. square matrix
2. row sum = 1

Some intuitive approaches.

- Suppose I do this for an year. Which brand of soda I will drink more?

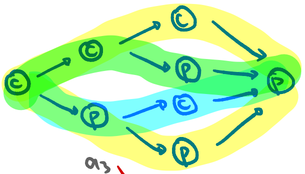
coke

- If I drink Coke today, then what is the chance of drinking Pepsi two days later?



$0.7 \times 0.3 + 0.3 \times 0.5 = 0.36$

- If I drink Coke today, then what is the chance of drinking Pepsi three days later?



a stochastic path

$a_0 \downarrow \qquad a_3$

$$\underset{1\times 2}{[1 \quad 0]}\ \underset{2\times2}{P}\cdot\underset{2\times2}{P}\cdot\underset{2\times2}{P} = \underset{1\times 2}{[\,\_ \quad ?\,]}$$

More questions that we may be interested in answering.

- Given I drink coke today, what is likely my consumption for upcoming 10 days? (Pepsi is \$1 and Coke is \$1.5)
- If I do this for 10 years (3650 days) from now, then how many days I will be drinking Pepsi?
- Suppose Pepsi is \$1 and Coke is \$1.5. How much on average I spend on soda in this month given today is the first day of the month and I drink Pepsi today?
- In answering above question, how much does what I drink today matter?
- Suppose there are a billion customers (who has same type of consuming pattern) like me in the world. You are working for Pepsi and like to boost Pepsi $\rightarrow$ Pepsi probability from 0.5 to 0.6 by marketing promotion. On average, how much additional revenue will be generated by this change for a month?

II. Definitions

- Stochastic process
  - Stochastic means *time* and *randomness* combined.
  - Stochastic process includes *multiple random variables indexed by time*.

- Discrete time stochastic process
  - *Discrete time* stochastic process includes multiple random variables indexed by *discrete time*.
  - For example,
    - $S_0, S_1, S_2, \ldots$, where each implies day-0, day-1, and day-2, $\cdots$
    - $S_t, S_{t+1}, S_{t+2}, \ldots$, where each implies year-t, year-$t+1$, $\cdots$
  - Formally, $\{S_t : t \geq 0, t \in \mathbb{N}\}$

자연수

- Continuous time stochastic process
  - *Continuous time* stochastic process includes multiple random variables indexed by *continuous time*.
  - For example,
    - $\{S_t, t \in [0, \infty)\}$ where each implies daily or yearly evolution of certain quantity.
  - Formally, $\{S_t : t \in \mathbb{R}^+\}$

양의 실수.

- State: value of $S_{\cdot}$.
  - It may be deterministic.
    - Ex) $S_t = c \Leftrightarrow$ I drink coke on day-t, or say, '*The state of* S_t *is* c'.
    - Ex) $S_1 = p \Leftrightarrow$ On day-1, I drink pepsi, or say, '*The state of* S_1 *is pepsi*'.
  - It may be random. (not deterministic)
    - Ex) $\mathbb{P}(S_2 = p) = 0.6 \Leftrightarrow$ The probability that I drink pepsi on day-2 is 0.6.
  - It may be random and often described as a distribution.
    - Ex) $\underline{(\mathbb{P}(S_3 = c), \mathbb{P}(S_3 = p)) = (0.3, 0.7)} \Leftrightarrow$ The probability that I drink coke on day-3 is 0.3 and pepsi is 0.7.

3일날 state에 대한 dist.

- State space: a set of all possible states that $S_{\cdot}$ can take.
  - Ex) A set of all possible kind of sodas that I might drink, i.e. $S = \{c, p\}$.

Markov Property

- Intuitively,
  - The nearest future only depends on the present. Past does not matter.
  - S_{t+1} depends only on the state of S_t.
  - S_{t+1} is function of S_t and some randomness, i.e. $S_{t+1} = f(S_t, \textit{randomness})$.

- A bit rigorously,
  - The future only depends on the recent history that are known.
  - Future is independent of the past, given the present.

- Formally, Markov property holds if

$$\mathbb{P}(S_{t+1} = j | \cancel{S_0 = i_0, S_1 = i_1, \dots,} S_t = i) = P(S_{t+1} = j | S_t = i)$$

- Transitions depend only on the nearest past.
- Transitions depend only on the recent history.

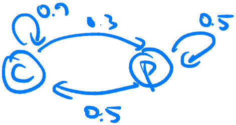
Discrete Time Markov Chain

Definition 1

Discrete Time Markov Chain (DTMC, hereafter) is a *a discrete time stochastic process with Markov Property*.

- To properly describe a DTMC, following components are essential:
  1. State space $S = \{c, p\}$
  2. Transition probability matrix/diagram
  3. Initial distribution

$$\begin{array}{c} c \\ p \end{array} \begin{bmatrix} .7 & .3 \\ .5 & .5 \end{bmatrix}$$



$$a_0 = (\mathbb{P}(S_0 = c), \mathbb{P}(S_0 = p))$$

- Transition probability matrix/diagram.
  - The probability that governs 'transition'.
  - $p_{ij} = \mathbb{P}(S_{t+1} = j | S_t = i) = \mathbb{P}(S_t = j | S_{t-1} = i) = \mathbb{P}(S_1 = j | S_0 = i)$
  - The transition probability matrix P is a collection of p_{ij}, i.e. $\mathbf{P} = [p_{ij}]$.

$$\begin{array}{c c} & \begin{matrix} c & p \end{matrix} \\ \begin{matrix} c \\ p \end{matrix} & \begin{bmatrix} p_{cc} & p_{cp} \\ p_{pc} & p_{pp} \end{bmatrix} \end{array}$$

- Initial distribution
  - The information of where the chain starts at time 0.
  - a_0 := distribution of S_0 in a row vector.
  - Ex) $S_0 = c \Leftrightarrow \mathbb{P}(S_0 = c) = 1, \mathbb{P}(S_0 = p) = 0 \Leftrightarrow a_0 = (1\ 0)$
  - Ex) $\mathbb{P}(S_0 = c) = 0.6, \mathbb{P}(S_0 = p) = 0.4 \Leftrightarrow a_0 = (0.6\ 0.4)$

III. Exercises

Exercise 1

Let's revisit Coke & Pepsi DTMC. Describe the following.

1. *State Space*
2. *Transition Probability Matrix*
3. *Transition Diagram*
4. *Initial Distribution*

Remark 1

$S = \{c, p\}$
$|S|$: # of elements in set S

A few remarks on transition matrix:

1. The size of transition matrix is $|S| \times |S|$, where $|\cdot|$ implies the number of elements in a set.
2. Transition diagram and transition matrix carry exactly same information.
3. A legit transition matrix must have each row summing up to 1.

Exercise 2

$a_0 = [0.6 \quad 0.4]$ $\quad P = \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix}$

Suppose $\mathbb{P}(S_0 = c) = \underline{0.6}$ *and* $\mathbb{P}(S_0 = p) = \underline{0.4}$*, then what is* $\mathbb{P}(S_1 = c) = ?$

1. $\mathbb{P}(S_1 = c) = \mathbb{P}(S_1 = c, S_0 = c) + \mathbb{P}(S_1 = c, S_0 = p)$

$$= \mathbb{P}(S_1 = c \mid S_0 = c)\,\mathbb{P}(S_0 = c) + \mathbb{P}(S_1 = c \mid S_0 = p)\,\mathbb{P}(S_0 = p)$$

$$= 0.7 \times 0.6 + 0.5 \times 0.4$$

$$= 0.62$$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$$\Downarrow$$

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\,\mathbb{P}(B)$$

cf) $\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B)$

2. $\underbrace{[0.6 \quad 0.4]}_{\substack{a_0 \\ \text{dist of time-0}}} \underbrace{\begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix}}_{P} = \underbrace{[0.62 \quad 0.38]}_{\substack{a_1 \\ \text{dist of time-1}}}$

Exercise 3

Suppose $\mathbb{P}(S_0 = c) = 0.6$ *and* $\mathbb{P}(S_0 = p) = 0.4$, *then what is* $\mathbb{P}(S_2 = c) = ?$

$$a_0 \xrightarrow{\times P} a_1 \xrightarrow{\times P} a_2 \cdots \to a_{10} \to \cdots$$

$a_1 = a_0 P$

$a_2 = \underline{\underline{a_1}} P$

$= a_0 P \cdot P$

cf.) $(AB)C = A(BC)$

$\mathbb{P}(S_2 = p) = ?$ $\mathbb{P}(S_2 = p \mid S_0 = c) = ?$

Exercise 4

Suppose $S_0 = c$, *then what is* $\cancel{\mathbb{E}(S_2) =}$? *In other words, what is* $\mathbb{E}(S_2|S_0 = c) = ?$

Transitions in DTMC

Exercise 5

Suppose $S_0 = p$*, then what is* $\mathbb{P}(S_2 = p) = ?$

Exercise 6

What is $\mathbb{P}(S_2 = p | S_0 = p) = ?$

IV. Simulating stochasic paths

Motivation

In this section, we will address the following two questions.

1. Given I drink coke today, what is likely my consumption for upcoming 10 days?
2. What is my expected spending for upcoming 10 days if I drink coke today? (Pepsi is \$1 and Coke is \$1.5)

DTMC Simulator

- For a transition between time t and time $t+1$,
  - A *deterministic transition* can be formulated as
  $$S_{t+1} = f(S_t)$$
  for some function $f(\cdot)$.
  - A *stochastic transition* can be formulated as
  $$S_{t+1} = f(S_t, \boxed{\text{randomness}})$$
  for some function $f(\cdot)$.

- In this light, the soda DTMC's transition can be described as $S_{t+1} = f(S_t,\ u)$, where $u \sim U(0,1)$. (Remind that a simulating a uniform distribution $U(0,1)$ is enough to express possible randomness, based on inverse transformation method in A5.
- Specifically,
  - $f(S_t = c,\ u) = c$ if $u \leq 0.7$, and $= p$ otherwise.
  - $f(S_t = p,\ u) = c$ if $u \leq 0.5$, and $= p$ otherwise.

```r
soda_simul <- function(this_state) {
  u <- runif(1)
  if (this_state == "c") {
    if (u <= 0.7) {
      next_state <- "c"
    }
    else {
      next_state <- "p"
    }
  } else { # this_state=="p"
    if (u <= 0.5) {
      next_state <- "c"
    }
    else {
      next_state <- "p"
    }
  }
  return(next_state)
}
```

- Using the function `soda_simul()`, let's generate 5 possible paths for 10 days.

```r
library(stringr) # for str_sub() and str_count()
for (i in 1:5) {
  path <- "c" # coke today (day-0)
  for (n in 1:9) {
    this_state <- str_sub(path,-1,-1) # last elemen
    next_state <- soda_simul(this_state)
    path <- paste0(path, next_state)
  }
  print(path)
}
```

"c"

```
## [1] "cccccppccp"
## [1] "ccpcppppcc"
## [1] "cppcpppccc"
## [1] "cccccpccpp"
## [1] "ccccccccpp"
```

- To address the second question regarding expected spending, we certainly need more than 5 paths.
- Let's do it with 10,000 Monte-Carlo simulation.
- We need cost evaluating function that calculates cost for each path.

```
cost_eval <- function(path) {
  cost_one_path <-
    str_count(path, pattern = "c")*1.5 +
    str_count(path, pattern = "p")*1
  return(cost_one_path)
}
```

```
MC_N <- 10000
spending_records <- rep(0, MC_N)
for (i in 1:MC_N) {
  path <- "c" # coke today (day-0)
  for (t in 1:9) {
    this_state <- str_sub(path, -1, -1)
    next_state <- soda_simul(this_state)
    path <- paste0(path, next_state)
  }
  spending_records[i] <- cost_eval(path)
}
```

```
mean(spending_records)
```

```
## [1] 13.37395
```

- The above simulation is characterized with
  - Each path has length of 10.
  - The 10,000 number of paths are generated for calculating expected cost.

- In the language of stochastic programming, we prefer to describe it as following
  - In this problem, *time horizon* is 10-days.
  - The MC simulation was conducted with 10,000 *episodes*.
  - Each *episode* is defined as a single full simulation path for *time horizon*.
  - In each *episode*, *a stochastic path* is generated and *total cost for a path* is evaluated.

Exercise. 위의 코드를 python으로.

```
"Faber est suae quisque fortunae - 운명을 만드는 사람은 그 자신이다."
```