# Lecture A6. Simulation 3

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1. I. Recap on estimation of $\pi$
2. II. Running estimate approach

# I. Recap on estimation of $\pi$

# Implementation - basic

- Following is the most basic example of Monte Carlo simulation.

```
set.seed(1234) # fix the random seed
MC_N <- 10^3    1000
x <- runif(MC_N)*2-1 # runif() generates U(0,1)
y <- runif(MC_N)*2-1
t <- sqrt(x^2+y^2)
pi_hat <- 4*sum(t<=1)/MC_N
pi_hat
```

```
## [1] 3.188
```

- One caveat with this vectorized programming style is that we may not observe the intermediate results.
- This may become an issue when each MC repetition takes a long time. Often, observing intermediate result is beneficial.

# Implementation - "The first-timer would write"

- It is mentioned that the first timer would write as below.

```
set.seed(1234)
MC_N <- 10^6
count <- 0
for (MC_i in 1:MC_N) {
  x_i <- runif(1)*2-1
  y_i <- runif(1)*2-1
  t_i <- sqrt(x_i^2+y_i^2)
  if (t_i <= 1) count <- count + 1
}
pi_hat <- 4*count/MC_N
```

$\hat{\pi}$

- This note will rewrite the above code in the way that the estimate for $\pi$ is iteratively updated as `MC_i` increases toward `MC_N`.

# II. Running estimate approach

## Development

- Remind that from `A4, p8`,

$$\hat{\pi} = 4 \times \frac{\text{number of } \{t_i \le 1\}}{N} = 4 \times \frac{\sum_{i=1}^{N} I_{\{t_i \le 1\}}}{N}$$

- This is complete estimate after $N$ number of Monte Carlo repetitions are completed.
- Assume that we have currently completed $n (\le N)$ number of Monte Carlo repetition, then using the completed $n$ samples, we can still generate an estimate as follows.

estimate about pi using n sample

$$\hat{\pi}_n = \frac{\sum_{i=1}^{n} 4 \cdot I_{\{t_i \le 1\}}}{n}$$

성공 횟수 × 4

지금까지의 시행 횟수.

- Note that the notation $\hat{\pi}_n$ speaks itself as "estimate of $\pi$ using $n$ samples".
- In other words, the estimate of $\pi$ is updated as $\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3, \ldots, \hat{\pi}_n, \ldots, \hat{\pi}_N$.

1 . . . . N

$\hat{\pi}_1 . \hat{\pi}_2 \cdots \hat{\pi}_N$

- From the expression for $\hat{\pi}_n$, the numerator implies the running sum up to $n$ repetition(s), while the denumerator counts the number of repetition(s) so far.
- Let's notate the individual result of $i$-th experiment as $A_i$, and the running sum by $n$ repetition as $S_n$, as follows.

$$\hat{\pi}_n = \frac{\sum_{i=1}^{n} 4 \cdot I_{\{t_i \le 1\}}}{n} =: \frac{\sum_{i=1}^{n} A_i}{n} =: \frac{S_n}{n}$$

new_est

$\hat{\pi}_n = \frac{S_n}{n}$

$\hat{\pi}_{n-1} = \frac{S_{n-1}}{n-1}$

- Now, we can rewrite the above as a recursive form.

n번째 까지의 실험으로 얻은 정보의 요약

$$\hat{\pi}_n = \frac{(A_1 + A_2 + \cdots + A_{n-1}) + A_n}{n} = \frac{S_{n-1} + A_n}{n}$$

$$= \frac{\frac{n-1}{n-1} \cdot S_{n-1} + A_n}{n} = \frac{(n-1)\hat{\pi}_{n-1} + A_n}{n}$$

$$= \left(\frac{n-1}{n}\right)\hat{\pi}_{n-1} + \left(\frac{1}{n}\right)A_n$$

n-1번째 까지의 실험으로 얻은 정보의 요약.

n번째 실험의 결과.

- The last expression tells us that, the Monte Carlo updating occurs, from $\hat{\pi}_{n-1}$ to $\hat{\pi}_n$, in a way that the old estimate $(\hat{\pi}_{n-1})$ is updated with the feed of new information $A_n$ as a weighted average of the two.

old-est

현재 n이 10. $\frac{9}{10} \cdot$ old est $+ \frac{1}{10}$ new info

1000. $\frac{999}{1000} \cdot$ " $+ \frac{1}{1000}$ "

# Implementation

$$\hat{\pi}_n = \boxed{\left(\frac{n-1}{n}\right)} \pi_{n-1} + \boxed{\left(\frac{1}{n}\right)} A_n$$

new est — old est — new info

- Another possible benefit of this approach is that we can halt this experiment when $\hat{\pi}_{n-1}$ and $\hat{\pi}_n$ are close enough. (within similar vein with *early stopping* in deep learning domain)
- How would you impose an early stopping condition in this experiement?

??

$$\hat{\pi}_n = \left(\frac{n-1}{n}\right)\hat{\pi}_{n-1} + \left(\frac{1}{n}\right)A_n$$

```
set.seed(1234)
beg_time <- Sys.time() # to time
old_est <- 0
n <- 1
MC_N <- 10^6
repeat{
  x_i <- runif(1)*2-1
  y_i <- runif(1)*2-1
  t_i <- sqrt(x_i^2+y_i^2)
  A_n <- 4*(t_i <= 1)
  new_est <- ((n-1)/n)*old_est + (1/n)*A_n
  if (n > MC_N) break
  n <- n+1
  old_est <- new_est
}
print(new_est)
```

```
## [1] 3.138345
```

```
end_time <- Sys.time() # to time
print(end_time-beg_time) # to time
```

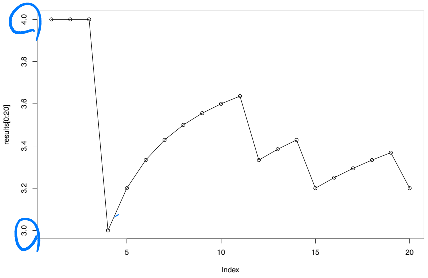
```
## Time difference of 2.953222 secs
```

# Convergence trajectory

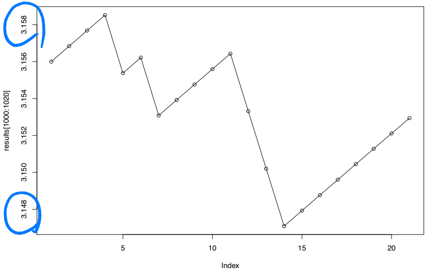
```
set.seed(1234)
beg_time <- Sys.time() # to time
old_est <- 0
n <- 1
MC_N <- 10^6
results <- rep(0, MC_N) # to save
repeat{
  x_i <- runif(1)*2-1
  y_i <- runif(1)*2-1
  t_i <- sqrt(x_i^2+y_i^2)
  A_n <- 4*(t_i <= 1)
  new_est <- ((n-1)/n)*old_est + (1/n)*A_n
  results[n] <- new_est # to save
  if (n > MC_N) break
  n <- n+1
  old_est <- new_est
}
```

1000000

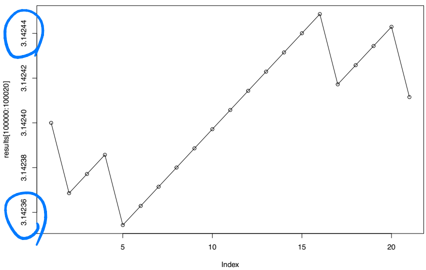
```
plot(results[0:20], type='o')
```



```
plot(results[1000:1020], type='o')
```




```
plot(results[100000:100020], type='o')
```



## Discussion

importance of old est importance of new info

$$\hat{\pi}_n = \left(\frac{n-1}{n}\right)\hat{\pi}_{n-1} + \left(\frac{1}{n}\right)A_n \tag{1}$$

- The above discussion can be generalized into:

$$\hat{\pi}_n = (1-\alpha)\hat{\pi}_{n-1} + \alpha A_n \tag{2}$$

- The previous implementation had $\alpha = \frac{1}{n}$.
  - The quantity $\alpha \in (0, 1)$ implies the importance of the last observation.
  - The quantity $1-\alpha$ implies the importance of the previous estimate.
- Oftentimes, setting $\alpha > \frac{1}{n}$ makes sense. When?

## Discussion

- It can be also written as

$$
\begin{aligned}
\hat{\pi}_n &= \left(\frac{n-1}{n}\right)\hat{\pi}_{n-1} + \left(\frac{1}{n}\right)A_n \\
&= \left[\left(\frac{n-1}{n}\right)\hat{\pi}_{n-1} + \boxed{\left(\frac{1}{n}\right)\hat{\pi}_{n-1}}\right] + \left[\left(\frac{1}{n}\right)A_n - \boxed{\left(\frac{1}{n}\right)\hat{\pi}_{n-1}}\right] \\
&= \hat{\pi}_{n-1} + (1/n)(A_n - \hat{\pi}_{n-1})
\end{aligned}
$$

- Likewise,

$$
\begin{aligned}
\hat{\pi}_n &= (1-\alpha)\hat{\pi}_{n-1} + \alpha A_n \\
&= \hat{\pi}_{n-1} + \alpha(A_n - \hat{\pi}_{n-1})
\end{aligned}
$$

$$\underbrace{\hat{\pi}_n}_{\text{new est}} = \underbrace{\hat{\pi}_{n-1}}_{\text{old est}} + \alpha\underbrace{(\overset{\text{new info}}{A_n} - \overset{\text{old est}}{\hat{\pi}_{n-1}})}_{\text{MC error}} \tag{3}$$

- How would you interpret each term?
  - $\hat{\pi}_n$
  - $\hat{\pi}_{n-1}$
  - $\alpha$ :
  - $(A_n - \hat{\pi}_{n-1})$ : MC error
  - $A_n$ : MC target

$$\hat{\pi}_{n-1} \xrightarrow{\text{이동}} \hat{\pi}_n$$

목적지

$A_n$

"stationary"

## Exercise 1

*Write a python code that produces results in page 10 and 11 using Eq (3). Use the variable names of* `new_est, old_est, alpha, MC_tgt, MC_err` *to demonstrate your understanding.*

```
"If I only had an hour to chop down a tree, I would spend the first 45 minutes sharpening my axe.
- A. Lincoln"
```