Lecture A3. Statistics Review

Sim, Min Kyu, Ph.D., mksim@seoultech.ac.kr

SEOULTECH 서울과학기술대학교 데이터사이언스학과

Population and Sample

- A population set (모집단) is the entire group that you want to draw conclusions about.
- A sample set (표본집단) is the subset of population that you have an access to collect data from.
- The size of the sample is always less than the total size of the population.
- It is a researcher's primary concern to draw conclusion on the population set, by studying the behavior from the sample set.

$\hat{m}$ $\hat{v}$ $\hat{Q}$

Population statistics

- Population
  - Suppose that you are interested in Korean male's hand length. Let X be a ~~distribution~~ random var. of population set (entire Korean male's hand length).
  - Let μ be the mean of X and σ^2 be the variance of X.
  - That is, $\mu = \mathbb{E}X$ and $\sigma^2 = \mathbb{E}[(X - \mathbb{E}X)^2]$. (모평균)
  - These *population statistics* are what we are after, specifically, *population mean* and *population variance*. (모분산)
  - Since these are what we aim to estimate, we often call them as $\boxed{\textit{true values}}$, specifically *true mean* and *true variance*.
- Sample
  - In order to estimate μ and σ^2 you collect n samples of Korean male's hand length.
  - Typically, these collected samples are denoted as $X_1, X_2, \ldots, X_n$, or $\{X_i, 1 \le i \le n\}$. sample set.

X에 대한 n번째 관찰값.

$X_1, X_2 \cdots X_n \longrightarrow \mu, \sigma^2.$

Sample statistics

표본 통계량

$\{X_i, 1 \le i \le N\} \rightarrow \boxed{f(X_i)} \approx \mu$ 비슷한것이다.

sample mean　　population mean

- Estimation
  - You want to draw conclusions on the *population mean* (μ) and *population variance* (σ^2) by studying the sample $\{X_i, 1 \le i \le n\}$.
  - From the sample, we compute some value that should be similar to population statistics.
- Sample Mean
  - It is known that $\sum_{i=1}^{n} X_i/n$ is similar value to the population mean. $\bar{X}$ $\leftarrow \mu$
  - This quantity is typically notated as $\overline{X}$, i.e., $\boxed{\overline{X} = \sum_{i=1}^{n} X_i/n.}$
  - This quantity is called as *sample mean* for obvious reason.
  - Sample mean is obtained by taking an arithmetic average of all samples.
- Sample Variance
  - It is known that $\frac{\sum (X_i - \overline{X})^2}{n-1}$ is similar value to the population variance. s^2 σ^2
  - This quantity is typically notated as s^2, i.e. $s^2 = \frac{\sum (X_i - \overline{X})^2}{n-1}$.
  - v This quantity is called as *sample variance* for obvious reason.
  - Sample variance is obtained by 1) summing up squared deviations of all samples and 2) divide it by $n-1$.

- Summary

	Mean	Variance
Population statistics	$\mu = \mathbb{E}X$	$\sigma^2 = \mathbb{E}[(X - \mathbb{E}X)^2]$
Sample statistics	$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$	$s^2 = \frac{\sum (X_i - \overline{X})^2}{n-1}$

$\hat{\mu}$: μ에 대한 estimator (ex) $\overline{X}$)

$\hat{\sigma}^2$: σ^2에 " " (ex) s^2, ...)

Estimation

- Remind that it is mentioned that 'Sample mean is *believed to be a similar value* to the population mean'.
- Like such, we call the process of 'Finding sample statistics that is *believed to be a similar value* to the population statistics.' as **estimation**. 통계적 추정.

  (true statistics)
- For true mean μ, there may be various estimation efforts that aims to find similar value to the μ. We call these *similar value to the true value*, as an **estimator**.
- Again, *estimator* is not a true value, but an estimation effort. To distinguish between the *true value* and *estimator*. Notation of 'hat', or $\hat{\cdot}$ is typically used. For example, $\hat{\mu}$ indicates an estimator for μ, and $\hat{\sigma}^2$ indicates an estimator for σ^2.

- Sample mean serves as *an estimator* for the true mean.
- Sample variance serves as *an estimator* for the true variance.

Desired properties of estimators

10 명 $X_1 \cdots X_{10}$

μ?

A: $\hat{\mu} = \frac{X_1 + \cdots + X_{10}}{10}$

B: $\hat{\mu} = \frac{X_{(5)} + X_{(6)}}{2}$

- Is $\frac{\sum_{i=1}^{n} X_i}{n}$ a good estimator for the true mean? What it means by *good*?
- There are many criteria for *good* estimator such as

  $\mathbb{E}\hat{\mu} = \mu$

  - *unbiased* estimator - Expected value of estimator must be same as true value.
  - *consistent* estimator - As the number of sample increases, the estimator converges to the true value.
  - *maximum-likelihood (ML) estimator* - The probability that the estimator is exactly equal to true value is maximal.
- For mathematical expression, let's notate the true statistics we are after as θ, and the estimator as $\hat{\theta}$. Then,
  - $\hat{\theta}$ is an *unbiased* estimator if $\mathbb{E}\hat{\theta} = \theta$.
  - $\hat{\theta}$ is a *consistent* estimator if $\hat{\theta} \to \theta$ as $n \to \infty$.
  - $\hat{\theta}$ is a *maximum-likelihood (ML)* estimator if $\hat{\theta} = argmax_x \mathbb{P}(\theta = x)$.

$\hat{\theta} = \operatorname{argmax}_x \mathbb{P}(\theta = x)$

- It is known that
  - $\frac{\sum_{i=1}^{n} X_i}{n}$ ← sample mean — is an *unbiased*, *consistent*, and *maximum-likelihood* estimator for the true mean
  - $\frac{\sum (X_i - \overline{X})^2}{n-1}$ ← sample variance — is an *unbiased* and *consistent* estimator for the true variance, but it is not a *maximum-likelihood* estimator.
  - $\frac{\sum (X_i - \overline{X})^2}{n}$ ← is a *consistent* and *maximum-likelihood* estimator for the true variance, but it is not an *unbiased* estimator. In other words, it is *biased* estimator.

$\mu.\ \hat{\mu},\ \overline{X}$

$\sigma^2.\ \hat{\sigma}^2,\ s^2$

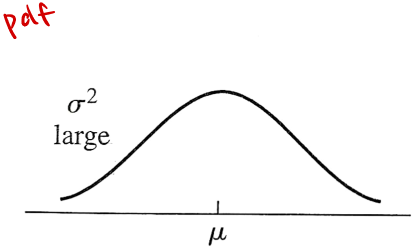
$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$

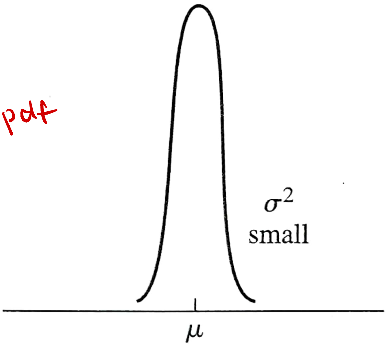
$s^2 = \frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n-1}$

- $\overline{X}$를 $\hat{\mu}$으로 쓰나요..?
- s^2를 $\hat{\sigma}^2$으로 쓰나요?
- $\frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n}$을 $\hat{\sigma}^2$으로 쓰나요?

$\frac{\sum (x_i - \overline{x})^2}{n-1}$ vs $\frac{\sum (x_i - \overline{x})^2}{n}$

- Normal variable $X \sim N(\mu, \sigma^2)$





Central limit theorem (CLT)

as $n \to \infty$

estimator   true value $\frac{\sigma^2}{n} \to 0$

Theorem 1

For a random variable X, whatever the distribution of X is, its sample mean $\overline{X}$ follows a normal distribution as long as the number of samples n is larger than 30. That is

$$\overline{X} \sim N(\mu, \sigma^2/n)$$

- It is intriguing that the population distribution may not be a normal distribution, but the sample mean from the population will always follow a normal distribution as long as the number of sample is larger than 30.
- It is also intriguing that the uncertainty of closeness between the estimator and true value is nicely quantified with the variance σ^2/n.

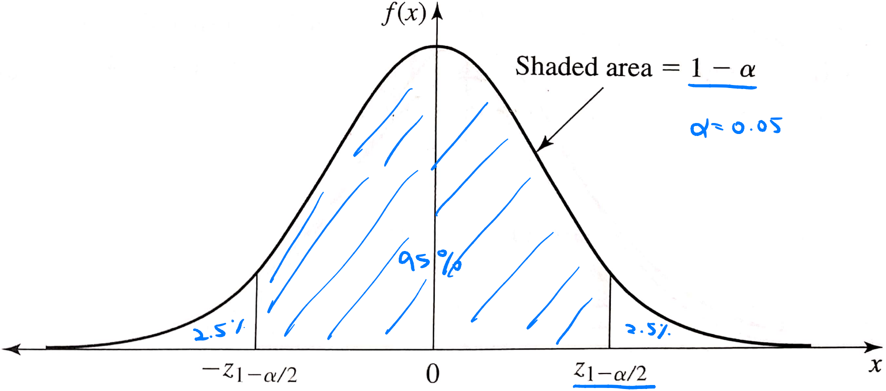
Exercise 1

- *Is $\overline{X}$ an unbiased estimator for μ, why or why not?*
- *Is $\overline{X}$ a consistent estimator for μ, why or why not?*
- *Is $\overline{X}$ a ML estimator for μ, why or why not?*

- ~~Questions~~

Normal variable's quantile

random



$Z \sim N(0,1)$

Confidence interval

$z_{1-0.05/2} = z_{0.975} = 1.96$

- From $\boxed{\overline{X} \sim N(\mu, \sigma^2/n)}$, we can use normal distribution's property to say:

μ? $X_1, X_2, \ldots X_n$ $\bar{X} \sim N(\mu, \sigma^2/n)$

$$\mathbb{P}[\mu - 1.96\frac{\sigma}{\sqrt{n}} \leq \overline{X} \leq \mu + 1.96\frac{\sigma}{\sqrt{n}}] = 0.95$$

① $\bar{X}$에 대한 CI

- Two issues with the above confidence interval.
  1. The above expression is a confidence interval for the estimator ($\overline{X}$) not for the true value (μ).
  2. We do not know the true value σ.

$Z \sim N(0,1)$

$-Z \sim N(0,1)$

- To tackle the first issue, the following effort is made.

$$\overline{X} \sim N(\mu, \sigma^2/n) \quad \Rightarrow \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1) = Z$$

$$\Rightarrow \frac{\mu - \overline{X}}{\sigma/\sqrt{n}} \sim Z$$

$$\Rightarrow \mu \sim N(\overline{X}, \sigma^2/n)$$

② $\mathbb{P}[\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}] = 0.95$

"μ에 대한 CI"

- From the last expression, $\mu \sim N(\overline{X}, \sigma^2/n)$, we still have the second issue of not knowing σ. We must replace σ with s.

$\sigma^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2$

$s^2 = \frac{\sum(x_i - \overline{x})^2}{n-1}$

- In replacing σ with s, it is known that $\boxed{\frac{\mu - \overline{X}}{\sigma/\sqrt{n}} \sim Z}$ becomes

$$\boxed{\frac{\mu - \overline{X}}{s/\sqrt{n}} \sim t_{n-1}}$$

Student t-dist.

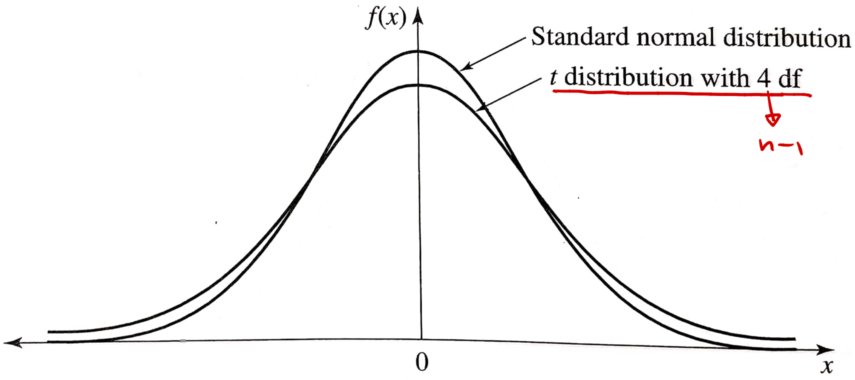
- Now we are ready to state the confidence interval for μ as following.

③

$$\mathbb{P}[\overline{X} - t_{0.975,n-1} \cdot s/\sqrt{n} \leq \mu \leq \overline{X} + t_{0.975,n-1} \cdot s/\sqrt{n}] = 0.95$$

- To get the some sence of what $t_{0.975,n-1}$ might be depending on n,
  - If $n = 30$, $\mathbb{P}[\overline{X} - 2.045 \cdot s/\sqrt{30} \leq \mu \leq \overline{X} + 2.045 \cdot s/\sqrt{30}] = 0.95$
  - If $n = 60$, $\mathbb{P}[\overline{X} - 2.000 \cdot s/\sqrt{60} \leq \mu \leq \overline{X} + 2.000 \cdot s/\sqrt{60}] = 0.95$
  - If $n = 120$, $\mathbb{P}[\overline{X} - 1.980 \cdot s/\sqrt{120} \leq \mu \leq \overline{X} + 1.980 \cdot s/\sqrt{120}] = 0.95$
  - If n is bigger, $\mathbb{P}[\overline{X} - 1.960 \cdot s/\sqrt{n} \leq \mu \leq \overline{X} + 1.960 \cdot s/\sqrt{n}] = 0.95$
- For the most applications in this course, n is so big enough that we are generally just fine using 1.96.

Normal dist. vs t dist.



Exercise 2

You randomly sample 1,600 Korean male and measured their hand length. The sample mean is 20cm and the sample standard deviation is 2cm. What is the 95% confidence interval for Korean male's hand length?

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"Man can learn nothing unless he proceeds from the known to the unknown. - Claude Bernard"
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