Stochastic Processes, Final, 2024 Fall

Solution and Grading

- Duration: 80 minutes
- Weight: 35% of final grade
- Closed material, No calculator

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- Write legibly.
- Justification is necessary unless stated otherwise.

#1. The arrival process to an emergency room follows NHPP(Non-homogenous Poisson Process) with the rate function $\lambda(t)$ of following.

$$\begin{aligned} \lambda(t) &= 1/hr, \quad \text{for } 0 < t \leq 6 \\ \lambda(t) &= 2/hr, \quad \text{for } 6 < t \leq 14 \\ \lambda(t) &= \frac{1}{5}(t-4)/hr, \quad \text{for } 14 < t \leq 24 \end{aligned}$$

where $t = 0$ implies the midnight, $t = 1$ implies 1AM, and so on.

(a) Given a patient's arrival, suppose that the patient is male with 60% of chance, and female with 40% of chance. What is the expected number of female patients during 4AM to 8AM? [5pts]

(b) What is the probability that there are exactly 5 customers arrivals between 11AM to 5PM? [5pts]

(a) First to find the expected number of total patients, which is $\int_4^8 \lambda(t)\, dt = 6$. Then, the expected number of female patients simply $6 \cdot 0.4 = 2.4$ by thinning principle.

(b) First to identify $\int_{11}^{17} \lambda(t)\, dt = \int_{11}^{14} 2\, dt + \int_{14}^{17} \frac{1}{5}(t-4)\, dt = 12.9$. Thus the probability is $\frac{12.9^5 \cdot e^{-12.9}}{5!}$

Difficulty: Easy-Medium

#2. Consider one-server service system of the following.

- Interarrival times follows an exponential distribution with mean of 0.5 minutes.
- Service time follows an exponential distribution with mean of 20 seconds.
- The service system can have maximum of four customers at the same time. (One in the service and other three in the waiting space).
- Assume that the stationary distribution is found as $\pi = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_4) = (\frac{3}{18}, \frac{6}{18}, \frac{4}{18}, \frac{3}{18}, \frac{2}{18})$.

Answer the following questions. `[Each 2pts]`

(i) What is the long run fraction of time the system is empty?

(ii) What is the long run fraction of time the server is busy?

(iii) What is the probability that a customer is not accepted to system?

(iv) What is the expected # of customer in the system?

(v) What is the expected # of customer in the queue?

(vi) What is expected total time spent in the system for a customer? (Waiting + Service time)

(vii) What is expected waiting time in a queue for a customer?

(viii) What is TH(throughput)?

(i) $\pi_0 = 3/18 = 1/6$

(ii) $1 - \pi_0 = 15/18 = 5/6$

(iii) $\pi_4 = 2/18 = 1/9$

(iv) $L_{sys} = 1\pi_1 + 2\pi_2 + 3\pi_3 + 4\pi_4 = 31/18$

(v) $L_q = 1\pi_2 + 2\pi_3 + 3\pi_4 = 16/18$

(vi) $L_{sys} = \lambda_{eff} W_{sys} \Rightarrow 31/18\,\text{person} = 2(1 - 2/18)\,\text{person/minute} \cdot W_{sys} \Rightarrow W_{sys} = 31/32\,\text{minutes}$

(vii) $L_q = \lambda_{eff} W_q \Rightarrow 16/18\,\text{person} = 2(1 - 2/18)\,\text{person/minute} \cdot W_{sys} \Rightarrow W_{sys} = 16/32\,\text{minutes}$

(viii) $\lambda_{eff} = 2(1 - 2/18) = 16/9$ person/minute

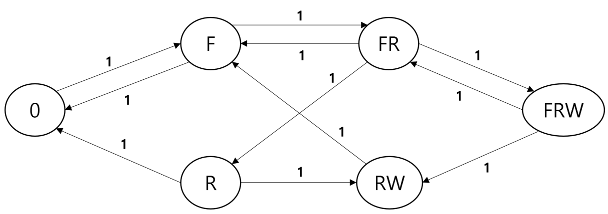
Difficulty: Medium

#3. Consider a very busy, very small gas station in an urban area with the following facts.

- The gas station has two fuel pumps in series, meaning there is a front pump and a rear pump.
- When a car arrives and both pumps are empty, it begins refueling at the front pump.
- When a car arrives and only the front pump is occupied, it begins refueling at the rear pump.
- When a car arrives and the rear pump is occupied, the vehicle waits in the waiting space. (Even if the front pump is empty, it cannot move directly to the front pump.)
- When the waiting car finds the both pumps empty, it immediately moves to the front pump. If finding only the rear pump empty, it immediately moves to the rear pump. If finding only the front pump empty, it continues to wait.
- When a car arrives at the gas station and there is another car waiting, the car leaves without receiving service.
- A car that has finished refueling can leave the gas station immediately. (A car can leave from the rear pump even if another vehicle is refueling at the front pump.)
- On average, one vehicle arrives at the gas station per minute, and it takes an average of one minute to refuel. Both the interarrival time and service time follow an exponential distribution.

How many vehicles, on average, does this gas station provide the service per hour? `[9pts]`

The state space can be defined as $S = \{0, F, R, FR, RW, FRW\}$. Note that state of FW is not possible. The rate diagram can be drawn as follows:



With some calculations, one can find the stationary distribution $\pi = \{\pi_0, \pi_F, \pi_R, \pi_{FR}, \pi_{RW}, \pi_{FRW}\} = \{6/17, 5/17, 1/17, 2/17, 2/17, 1/17\}$. Since a new customer cannot enter the system when the state is RW and FRW, the gas station provides the service to $\lambda_{eff} = \lambda(1 - \pi_{RW} - \pi_{FRW}) = 60 \cdot 14/17 = 840/17 \approx 49.4$ cars per hour. This queuing system loses more than 1/6 of total customers.

Partial credit: 3 pts for identifying all six states correctly. 3 pts for presenting a transition diagram or a transition matrix accurately. 3 pts to work out a stationary distribution and present the final number.

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Just for fun, the following is the answer by chatGPT as of 2024 Aug. Have some pride outsmarting chatGPT from this course!

To answer the question, we need to model the gas station scenario using queuing theory. Specifically, we are dealing with an $M/M/2/2$ queuing system with a special rule that cars wait for the rear pump to be free but cannot use the front pump directly if the rear pump is occupied.

Let's define the system states and use the steady-state probabilities to determine the average service rate.

Parameters:

- Arrival rate (λ): 1 car per minute.
- Service rate (μ): 1 car per minute per pump.

State Definitions:

- State 0: No cars at the gas station.
- State 1: One car at the front pump.
- State 2: One car at the rear pump.
- State 3: Two cars, one at each pump.
- State 4: One car waiting in the queue (since the rear pump is occupied).

State Transition Rates:

- From State 0 to State 1: λ
- From State 1 to State 2: λ
- From State 2 to State 3: λ
- From State 3 to State 4: λ
- Service completion from each pump: μ

Using these states, we can write the balance equations for the steady-state probabilities (P_i) of each state:

1. $P_0 \cdot \lambda = P_1 \cdot \mu$
2. $P_1 \cdot \lambda = P_2 \cdot \mu$
3. $P_2 \cdot \lambda = P_3 \cdot \mu$
4. $P_3 \cdot \lambda = P_4 \cdot \mu$
5. $P_0 + P_1 + P_2 + P_3 + P_4 = 1$

Solving these equations:

$$P_1 = P_0 \cdot \frac{\lambda}{\mu} = P_0 \cdot 1 = P_0$$
$$P_2 = P_1 \cdot \frac{\lambda}{\mu} = P_0 \cdot 1 = P_0$$
$$P_3 = P_2 \cdot \frac{\lambda}{\mu} = P_0 \cdot 1 = P_0$$
$$P_4 = P_3 \cdot \frac{\lambda}{\mu} = P_0 \cdot 1 = P_0$$

Summing all probabilities:

$$P_0 + P_0 + P_0 + P_0 + P_0 = 1$$
$$5P_0 = 1$$
$$P_0 = \frac{1}{5}$$

Therefore:
$$P_1 = P_2 = P_3 = P_4 = \frac{1}{5}$$

Service Rate Calculation:

The average number of cars being serviced per minute is the sum of the cars being serviced in each state where servicing occurs (States 1, 2, and 3):

$$\text{Average service rate per minute} = P_1 \cdot \mu + P_2 \cdot \mu + 2 \cdot P_3 \cdot \mu$$
$$= \frac{1}{5} \cdot 1 + \frac{1}{5} \cdot 1 + 2 \cdot \frac{1}{5} \cdot 1$$
$$= \frac{1}{5} + \frac{1}{5} + \frac{2}{5}$$
$$= \frac{4}{5} \text{ cars per minute}$$

To convert this to cars per hour:
$$\frac{4}{5} \text{ cars per minute} \times 60 \text{ minutes per hour} = 48 \text{ cars per hour}$$

Thus, on average, the gas station provides service to 48 vehicles per hour.

Difficulty: Hard

”If you decide you don’t have to get A’s, you can learn an enormous amount in college” - Isidor Isaac Rabi