Lecture C3. Discrete Time Markov Chain 3

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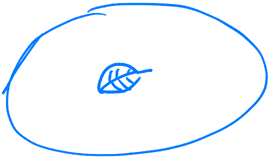
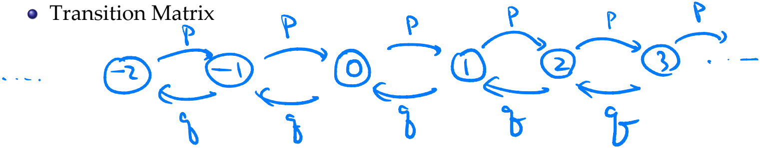
I. Simple Random Walk



Simple Random Walk

- Suppose you toss a coin at each time t and you go up by one if head and down by one if tail. Let the state space $S = \{..., -1, 0, 1, 2...\}$, and S_t is the position after t-th toss of the coin. Suppose the probability of getting head of the coin is p. (and let $q = 1 - p$ for tail)
  - Transition Diagram
  - Transition Matrix



$$\begin{array}{c|ccccccc} & \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\ \hline \vdots & & & & & & & \\ -2 & & 0 & p & 0 & & & \\ -1 & & q & 0 & p & & & \\ 0 & & & q & 0 & p & & \\ 1 & & & & & & & \\ 2 & & & & & & & \\ \vdots & & & & & & & \end{array}$$

tridiagonal matrix

Simple Random Walk

- Symmetric/Asymmetric.

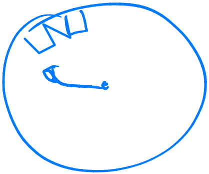
$p=q$

- 1D/2D/3D

- Drunken man, Jumping frog, Casino, Stock Market.

Simple Random Walk - Applications

- Will I ever get back to where I am? (Prob of ever getting back to the original state)
- Do I stand a chance to get to where I want to go?
- How long does it take for a drunken man gets home?
- Can I beat the Casino?
- I have \$50 and I bet \$1 every 30 seconds with $p = 18/38$ on Casino. Can I survive for 30 minutes?
- What is the chance of doubling stock price within 1 year?



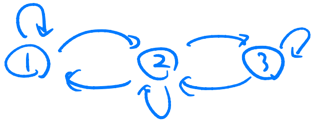
A few definitions (4) - Classifications of state

- A state i is said to be *recurrent* if, starting from i, the probability of getting back to i is 1.
  (There is always a way to get back to state i).
- A state i is said to be *absorbing*, as a special case of recurrent state, if $\mathrm{P}_{ii} = 1$.
  (You can never leave the state i if you get there once).
- A state i is said to be *transient* if, starting from i, the probability of getting back to i is less than 1.
  (It is possible that the process cannot come back to state i)
- Remark: Recurrence and Transience are class property
  - If $i \leftrightarrow j$, then i is recurrent if and only if j is recurrent.
  - If $i \leftrightarrow j$, then i is transient if and only if j is transient.

Exercise 1

For each state of the following DTMC, tell whether the state is i) recurrent, ii) transient, or iii) absorbing.

$$\mathrm{P} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/3 & 2/3 \end{pmatrix}$$



Exercise 2

For each state of the following DTMC, tell whether the state is i) recurrent, ii) transient, or iii) absorbing.

$$\mathrm{P} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 3

For each state of the following DTMC, tell whether the state is i) recurrent, ii) transient, or iii) absorbing.

$$\mathrm{P} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1/2 & 1/6 \\ 0 & 0 & 1 \end{pmatrix}$$

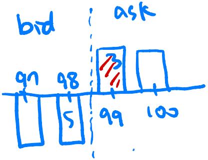
Exercise 4

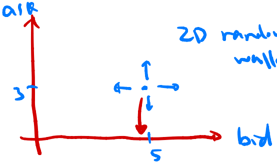
For each state of the following DTMC, tell whether the state is i) recurrent, ii) transient, or iii) absorbing.

$$\mathrm{P} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

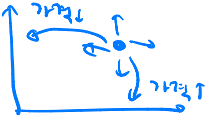
A few remarks

- In a MC with finite states space, not all states can be transient. (i.e., $\exists$ at least one recurrent state)
- A recurrent state is accessible from all states in its class, but is not accessible from recurrent states in other classes.
- A transient state is not accessible from any recurrent state.
- At least one, possibly more, recurrent states are accessible from a given transient state.





2D random walk



midprice = 98.5

double auction

Random Walk with infinite number of states

1. $S = \{..., -1, 0, 1, 2, ...\}$ and $p \neq 0.5$ all states are transient
2. $S = \{..., -1, 0, 1, 2, ...\}$ and $p = 0.5$ (null) recurrent
3. $S = \{0, 1, 2, ...\}$ and $p > 0.5$ transient
4. $S = \{0, 1, 2, ...\}$ and $p = 0.5$ (null) recurrent
5. $S = \{0, 1, 2, ...\}$ and $p < 0.5$ recurrent.

Exercise 5

For $S = \{0, 1, 2, ...\}$ *and* $p = 1/3$ *(#5 in the previous slide), find a stationary distribution using flow balance equation.*

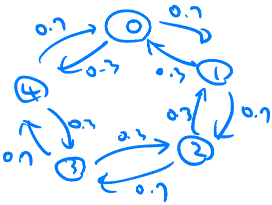
Exercise 6

Does #2 and #4 in the previous slide have stationary distribution? Why or why not?

II. Double stochastic and Chapman-Kolmogrov Equation

Doubly stochastic matrix

- Def. A matrix is said to be *stochastic* if each row sums up to 1.
  - Every legit transition probability matrix in DTMC is *stochastic*.
- Def. A stochastic matrix is said to be *doubly stochastic* if each column sums up to 1 as well.
  - Ex) The first example of periodic matrix.
- Thm. n by n doubly stochastic matrix for finite states DTMC has stationary distribution $\mathrm{v}_i = 1/n$ for every state $i \in S$.
  - Ex) Ring structure DTMC.



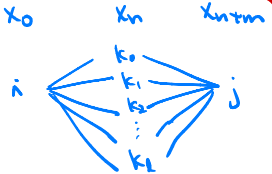
$$\begin{bmatrix} 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 0.3 & 0 & 0.7 & 0 \\ 0 & 0.7 & 0 & 0.3 \\ 0.7 & 0 & 0.3 & 0 \\ 0 & 0.3 & 0 & 0.7 \end{bmatrix}$$

Chapman-Kolmogrov Equation for DTMC

- n-step probability review
- We want to have $(m+n)$-step transition probability (matrix) using m-step and n-step transition matrix.
- $\mathrm{P}_{ij}^{n+m} = \sum_{k \in S} \mathrm{P}_{ik}^{n} \mathrm{P}_{kj}^{m}$
- Perspective of "path".
- pf) $\mathrm{P}_{ij}^{n+m} =$

$$P_{ij}^{n} = \mathbb{P}(X_n = j \mid X_0 = i)$$



$$P_{ij}^{n+m} = \mathbb{P}(X_{n+m} = j \mid X_0 = i)$$

$$= \sum_{\forall k} \mathbb{P}(X_{n+m} = j, X_n = k \mid X_0 = i)$$

$$= \sum_{\forall k} \mathbb{P}(X_{n+m} = j \mid X_n = k, \cancel{X_0 = i}) \times \mathbb{P}(X_n = k \mid X_0 = i)$$

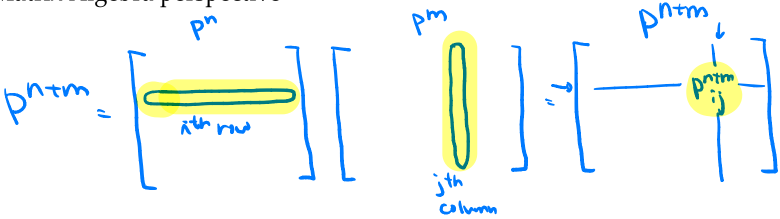
$$= \sum_{\forall k} P_{kj}^{m} P_{ik}^{n} = \sum_{\forall k} P_{ik}^{n} P_{kj}^{m}$$

cf)

$$\mathbb{P}(A \cap B \mid C)$$

$$= \mathbb{P}(A \mid B \cap C)\,\mathbb{P}(B \mid C)$$

- Matrix Algebra perspective



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III. Models beyond Markov property

Motivation

- MC is a powerful tool to analyse the situation when the stochastic evolutions depends only on the *most recent* information.
- But this fact also sounds like the limited applicability of MC to real world, because not all stochastic evolutions should depend only on the most recent information.
- This section challenges the limitation.

Long-term dependency

- Suppose you love coffee. The longer you don't drink coffee, the more you want to drink coffee. Consider the discrete time stochastic process $\{S_t, n \geq 0\}$ has a state space $S = \{C, NC\}$.
  - $S_t = C$ implies you drink coffee on t-th day
  - $S_t = NC$ implies no coffee on t-th day.

- The stochastic evolution of your coffee drinking habit is described following:

> *If you drank coffee yesterday and today, the chance of you drinking coffee tomorrow is 0.2. If you did not drink coffee yesterday but drank coffee today, then the chance of drinking coffee tomorrow is 0.4. If you drank coffee yesterday but not today, then chance of drinking coffee tomorrow is 0.6. If you did not drink coffee yesterday and today, then you will drink coffee tomorrow with probability 0.8.*

(Repeated for your convenience) If you drank coffee yesterday and today, the chance of you drinking coffee tomorrow is 0.2. If you did not drink coffee yesterday but drank coffee today, then the chance of drinking coffee tomorrow is 0.4. If you drank coffee yesterday but not today, then chance of drinking coffee tomorrow is 0.6. If you did not drink coffee yesterday and today, then you will drink coffee tomorrow with probability 0.8.

- Above statement can be expressed mathematically as following. Fill in the blank with decimal number.

$$\mathbb{P}(S_{t+1} = C | S_{t-1} = C, S_t = C) = (\ 0.2 \ )$$
$$\mathbb{P}(S_{t+1} = C | S_{t-1} = NC, S_t = C) = (\ 0.4 \ )$$
$$\mathbb{P}(S_{t+1} = C | S_{t-1} = C, S_t = NC) = (\ 0.6 \ )$$
$$\mathbb{P}(S_{t+1} = C | S_{t-1} = NC, S_t = NC) = (\ 0.8 \ )$$

- Unfortunately, the stochastic process $\{S_t, t \geq 0\}$ is not a DTMC.
- $\{S_t, t \geq 0\}$ is not a DTMC and can be disproved by the following counter-example.

$$\mathbb{P}(S_{t+1} = C | S_t = C, S_{t-1} = C) \neq \mathbb{P}(S_{t+1} = C | S_t = C, S_{t-1} = NC)$$

Remedy

- Letting $Y_t = (S_{t-1}, S_t)$ and consider the discrete time stochastic process $\{Y_t, t \geq 1\}$ with space space $Y = \{(C,C), (NC,C), (C,NC), (NC,NC)\}$.
- Now, this is a DTMC. with following transition probability

$P(Y_{t+1} = (C,C) \mid Y_t = (C,C))$

$$\mathbb{P}(\underline{S_t = C, S_{t+1} = C}|\underline{S_{t-1} = C, S_t = C}) = (\ 0.2\ )$$
$$\mathbb{P}(S_t = C, S_{t+1} = NC|S_{t-1} = C, S_t = C) = (\ 0.8\ )$$
$$\mathbb{P}(S_t = C, S_{t+1} = C|S_{t-1} = NC, S_t = C) = (\qquad)$$
$$\mathbb{P}(S_t = C, S_{t+1} = NC|S_{t-1} = NC, S_t = C) = (\qquad)$$
$$\mathbb{P}(S_t = NC, S_{t+1} = C|S_{t-1} = C, S_t = NC) = (\qquad)$$
$$\mathbb{P}(S_t = NC, S_{t+1} = NC|S_{t-1} = C, S_t = NC) = (\qquad)$$
$$\mathbb{P}(S_t = NC, S_{t+1} = C|S_{t-1} = NC, S_t = NC) = (\qquad)$$
$$\mathbb{P}(S_t = NC, S_{t+1} = NC|S_{t-1} = NC, S_t = NC) = (\qquad)$$

Exercise 7

Express the relevant transition probability matrix for $\{Y_t, n \geq 1\}$.

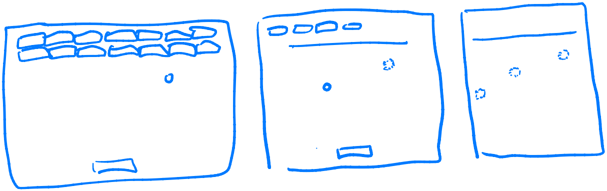
$$\mathrm{P} = \begin{array}{c} (C,C) \\ (NC,C) \\ (C,NC) \\ (NC,NC) \end{array} \begin{pmatrix} .2 & & .8 & \\ .4 & & .6 & \\ & .6 & & .4 \\ & .8 & & .2 \end{pmatrix}$$

Exercise 8

$\mathbb{P}[Y_{t+2} = (NC,C) | Y_t = (C,C)]$=?

Discussion

- This example tells a lot about the modeling principle of stochastic system.
- Ex) The Atari paper written by the deepmind.



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"If I only had an hour to chop down a tree, I would spend the first 45 minutes sharpening my axe.
- A. Lincoln"
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