Stochastic Processes, Mid-term, 2024 Fall

Solution and Grading

- Duration: 80 minutes
- Weight: 30% of final grade
- Closed material, No calculator

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- Write legibly.
- Justification is necessary unless stated otherwise.

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#1. Steel beams arrive at the painting station at a constant rate of one every 100 seconds. An human operator takes an average time of 80 seconds to paint a steel beam and the time follows an exponential distribution. On average, how many painting jobs are waiting for the operator to work on? `[10pts]`

By the Kingman's formula, for a $G/G/1$ queue, the expected waiting time in the queue ($\mathbb{E}[W_q]$)can be found as $\mathbb{E}[W_q] = 80 \cdot \frac{0.8}{1-0.8}\left(\frac{0^2+1^2}{2}\right) = 160$. Since arrival rate ($\lambda$) is 0.01/second, $L_q = \lambda W_q = 1.6$ jobs.

Difficulty: Easy. Partial credit: 5pts for waiting time and remaining 5pts for the number of waiting jobs.

#2. Consider the following DTMC.

$$\mathbf{P} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{pmatrix} 0.6 & 0.4 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0.1 & 0.1 & 0.7 & 0.1 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

a) Fill in all cells by "yes" or "no". `[12pts]`

	recurrent	transient	absorbing
State 1	yes	no	no
State 2	y	n	n
State 3	n	y	n
State 4	n	y	n
State 5	y	n	y

b) How many classes does this DTMC have? `[3pts]`

4 classes

Difficulty: Easy. No partial credit.

#3. Consider a discrete time random walk model with $p < 0.5$ with state space $S = \{0, 1, 2, ...\}$, i.e.

$$\begin{aligned}
&\mathbb{P}(X_{n+1} = i+1 | X_n = i) = p, \text{ for } i = 0, 1, 2, ... \\
&\mathbb{P}(X_{n+1} = i-1 | X_n = i) = 1-p, \text{ for } i = 1, 2, ... \\
&\mathbb{P}(X_{n+1} = 0 | X_n = 0) = 1-p
\end{aligned}$$

Calculate the stationary distribution. `[10pts]`

$\pi_i = \left(\frac{p}{1-p}\right)^i \cdot \frac{1-2p}{1-p}$, for all $i \in \{0, 1, 2, ...\}$. Detailed procedure can be found from lecture note.

Difficulty: Medium. Partial credit: 5pts up to finding relationship between π_i and π_{i+1}.

#4. Consider following transition matrix of DTMC.

$$\mathbf{P} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{pmatrix} 0 & 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0.5 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \end{pmatrix}$$

a) What are the periods of state 1,4, and 6, respectively? Justification is not necessary. `[3pts]`

1, 1, 2

b) What is $\mathbf{P}_{13}^{100}$? `[3pts]`

1/3, because of being double stochastic

c) What is $\mathbf{P}_{15}^{100}$? `[3pts]`

0, because going from recurrent to transient

d) What is $\mathbf{P}_{57}^{100}$? `[3pts]`

0.5, periodic

e) What is $\mathbf{P}_{42}^{100}$? `[3pts]`

$1/5(= 3/5 \cdot 1/3)$

f) Is the stationary distribution unique? Answer yes or no. `[3pts]`

No

g) If your answer to the question f) was "yes", then find the unique stationary distribution. If your answer for question f) was "no", then provide two or more stationary distributions. `[7pts]`

Any two vectors that are in the form of $\alpha(1/3\ 1/3\ 1/3\ 0\ 0\ 0\ 0\ 0) + (1-\alpha)(0\ 0\ 0\ 0\ 1/4\ 1/4\ 1/4\ 1/4)$ where $\alpha \in [0, 1]$.

Difficulty: Medium. No partial credit. If answer for f) was wrong, then score for g) is automatically zero, because you cannot possibly have a unique stationary distribution. If a student answered "yes" to question f, while solving question g, it is expected that the student should realize that the stationary distribution is not unique so that the student would revisit question f - this was the intent of this question.