Stochastic Processes, Final Exam, 2025 Spring

Solution and Grading

- Duration: 120 minutes
- Closed material, No calculator.

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- Write legibly.
- Justification is necessary unless stated otherwise.

1	16
2	20
3	20
4	14
Total	70

#1. Consider one-server service system of the following.

- Interarrival times follows an exponential distribution with mean of 0.5 minutes.
- Service time follows an exponential distribution with mean of 20 seconds.
- The service system can have maximum of four customers at the same time. (One in the service and other three in the waiting space).
- Assume that the stationary distribution is found as $\pi = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_4) = (\frac{3}{18}, \frac{6}{18}, \frac{4}{18}, \frac{3}{18}, \frac{2}{18})$.

Answer the following questions. `[Each 2pts]`

(i) What is the long run fraction of time the system is empty?

(ii) What is the long run fraction of time the server is busy?

(iii) What is the probability that a customer is not accepted to system?

(iv) What is the expected # of customer in the system?

(v) What is the expected # of customer in the queue?

(vi) What is expected total time spent in the system for a customer? (Waiting + Service time)

(vii) What is expected waiting time in a queue for a customer?

(viii) What is TH(throughput)?

(i) $\pi_0 = 3/18 = 1/6$

(ii) $1 - \pi_0 = 15/18 = 5/6$

(iii) $\pi_4 = 2/18 = 1/9$

(iv) $L_{sys} = 1\pi_1 + 2\pi_2 + 3\pi_3 + 4\pi_4 = 31/18$

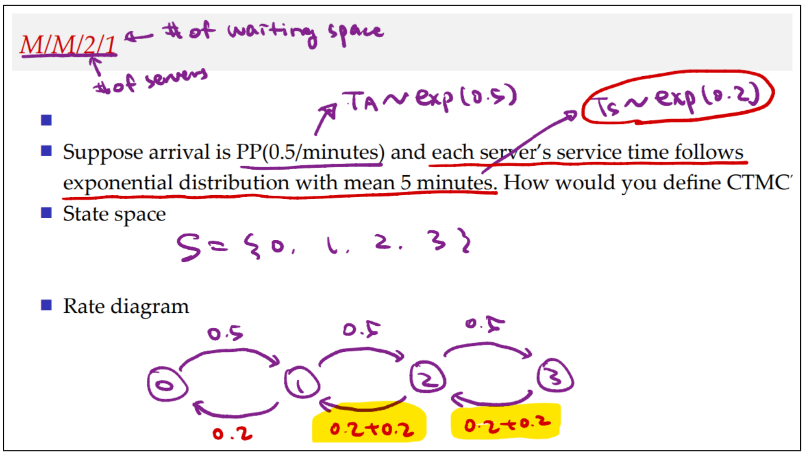
(v) $L_q = 1\pi_2 + 2\pi_3 + 3\pi_4 = 16/18$

(vi) $L_{sys} = \lambda_{eff} W_{sys} \Rightarrow 31/18\,\text{person} = 2(1 - 2/18)\,\text{person/minute} \cdot W_{sys} \Rightarrow W_{sys} = 31/32\,\text{minutes}$

(vii) $L_q = \lambda_{eff} W_q \Rightarrow 16/18\,\text{person} = 2(1 - 2/18)\,\text{person/minute} \cdot W_{sys} \Rightarrow W_{sys} = 16/32\,\text{minutes}$

(viii) $\lambda_{eff} = 2(1 - 2/18) = 16/9$ person/minute

#2. Consider the following lecture note content. To define the $M/M/2/1$ queue as a CTMC, the state is defined as the number of customers in the system, yielding the state space $S = \{0, 1, 2, 3\}$. A corresponding rate diagram has also been constructed.



(a) In the rate diagram, the highlighted portions are labeled as "0.2 + 0.2". This indicates that when there are 2 or 3 customers in the system, both servers are busy, and thus the rate at which the number of customers decreases is the sum of the service rates of the two servers. Formally state the relevant theorem that justifies this modeling approach. (Hint: The theorem is related to the exponential distribution.) `[10pts]`

(b) Prove the theorem you stated above. `[10pts]`

(a) If $X_1 \sim \exp(\lambda_1)$ and $X_2 \sim \exp(\lambda_2)$ and they are independent, then $\min(X_1, X_2) \sim \exp(\lambda_1 + \lambda_2)$.
Grading criteria:

- The **random variables** (e.g., X_1, X_2) and their **distribution parameters** (e.g., λ_1, λ_2) must be correctly stated and clearly distinguished. If failed to do so, then 0pt.
- The theorem must be written in a clear **"if–then"** structure. If failed to do so, then -5pts.

(b)

$$\begin{aligned}
\mathbb{P}\big(\min(X_1, X_2) \le x\big) &= 1 - \mathbb{P}\big(\min(X_1, X_2) > x\big) \\
&= 1 - \mathbb{P}(X_1 > x,\ X_2 > x) \\
&= 1 - \mathbb{P}(X_1 > x)\,\mathbb{P}(X_2 > x) \quad \text{(by independence of } X_1, X_2\text{)} \\
&= 1 - \big(1 - \mathbb{P}(X_1 \le x)\big)\big(1 - \mathbb{P}(X_2 \le x)\big) \\
&= 1 - \big(1 - (1 - e^{-\lambda_1 x})\big)\big(1 - (1 - e^{-\lambda_2 x})\big) \\
&= 1 - \big(e^{-\lambda_1 x}\big)\big(e^{-\lambda_2 x}\big) \\
&= 1 - e^{-\lambda_1 x} e^{-\lambda_2 x} \\
&= 1 - e^{-(\lambda_1 + \lambda_2) x}
\end{aligned}$$

Thus, $\min(X_1, X_2) \sim \exp(\lambda_1 + \lambda_2)$.

Grading Criteria

- No partial credit.
- The proof must be based on the **cumulative distribution function (CDF)**. Proofs using other arguments (e.g., expectation) will not receive any credit.

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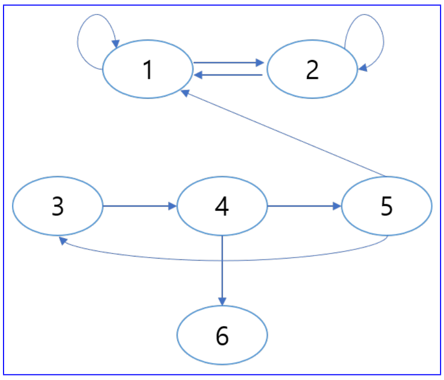
#3. Consider the following transition matrix for a DTMC.

$$P = \begin{pmatrix} .4 & .6 & 0 & 0 & 0 & 0 \\ .7 & .3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & .5 \\ .6 & 0 & .4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) What is the number of classes? For each class, identify its period and identify whether the class is recurrent, transient, or absorbing. [10pts]



- 3 classes
- $\{1,2\}$: period $= 1$, recurrent
- $\{3,4,5\}$: period $= 3$, transient
- $\{6\}$: period $= 1$, recurrent/absorbing

Grading scheme

- number of classes: 2pts
- period: 4pts
- R/T/A: 4pts
- In each category above, each error results in -2pts.

(b) Find P^{100}. `[10pts]`

1) find R $\to$ R part, 2) identify zeros, and 3) insert f-notation as follows.

$$P = \begin{pmatrix} 7/13 & 6/13 & 0 & 0 & 0 & 0 \\ 7/13 & 6/13 & 0 & 0 & 0 & 0 \\ 7/13 f_{3,\{1,2\}} & 6/13 f_{3,\{1,2\}} & 0 & 0 & 0 & f_{3,6} \\ 7/13 f_{4,\{1,2\}} & 6/13 f_{4,\{1,2\}} & 0 & 0 & 0 & f_{4,6} \\ 7/13 f_{5,\{1,2\}} & 6/13 f_{5,\{1,2\}} & 0 & 0 & 0 & f_{5,6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

4) Set up equations for f as follows:

$$\begin{aligned} f_{3,6} &= f_{4,6} \\ f_{4,6} &= 0.5 f_{5,6} + 0.5 \\ f_{5,6} &= 0.4 f_{4,6} \end{aligned}$$

It gives $f_{3,6} = 5/8$, $f_{4,6} = 5/8$, and $f_{5,6} = 1/4$.

Grading scheme

- Part for transitioning from recurrent to recurrent: 2pts
- Part for transitioning from transient to recurrent: 6pts
- Rest of work: 4pts
- In each category above, each error results in -2pts.

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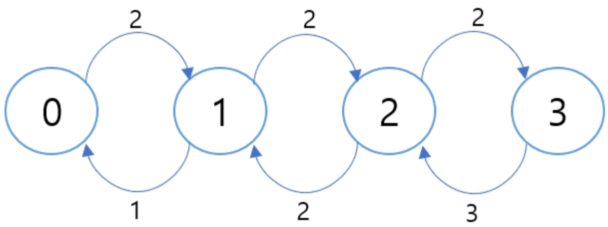
#4. A dormitory is planning to install a set of shared washing machines for its residents. The system operates under the following conditions.

- The interarrival times of residents follow an exponential distribution, with an average arrival rate of 2 residents per hour.
- The washing time for each machine also follows an exponential distribution, with each machine completing 1 wash per hour on average.
- If all machines are in use, an arriving resident does not wait and immediately leaves.

The dormitory director wants to ensure that a resident can use a washing machine with at least 90% probability upon arrival. To meet this requirement, how many machines need to be installed? `[14pts]`

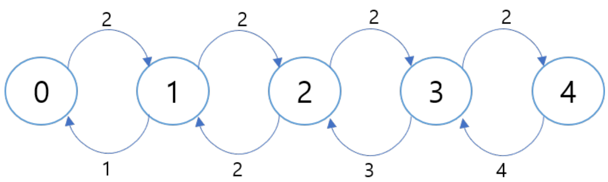
Let X_t be the number of customers at time t, then X_t is a CTMC. Note that if two machines exist, then the demand is not covered. Thus, students are expected to start off the case of three machines.

In three machines case, the rate diagram is as follows.



In this case, the stationary distribution is $\pi = \{\pi_0, \pi_1, \pi_2, \pi_3\} = \{3/19, 6/19, 6/19, 4/19\}$. Thus, a resident will find an available washing machine with probability $1 - \pi_3 = 15/19 \approx 0.789$.

In four machines case, the rate diagram is as follows.



In this case, the stationary distribution is $\pi = \{\pi_0, \pi_1, \pi_2, \pi_3\, \pi_4\} = \{3/21, 6/21, 6/21, 4/21, 2/21\}$. Thus, a resident will find an available washing machine with probability $1 - \pi_4 = 19/21 \approx 0.905$.

Thus, four machines are needed to be installed to meet the 90% criteria.

Grading scheme: The quality of the incorrect answers varied too much that I had to grade by the following criteria.

- If $1 - \pi_4 = 19/21$ is correctly identified for $k = 4$ and this is explained clearly, then full credit.
- If a minor error occured and otherwise a good solution, then 9pts are given.
- If correct methods of CTMC are demonstrated but far from reaching to the correct solutions (e.g. wrong rate diagram), then 4pts are given.

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