Homework 2

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STAT 520

September 8, 2018

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An experiment was conducted in order to quantify the effect of cloud injection of silver iodide on average rainfall from those clouds. We have a balanced and completely randomized design with 2 groups ($n_1 = n_2 = 26$). The treatment group of clouds received silver iodide injection, while the control group did not. Our response variable is rainfall volume (units/acre-feet), which can take only positive values. We define the response variable as follows.

For cloud $i \in \{1, ..., 26\}$ in group $j \in \{1, 2\}$ (where $j = 1$ is the control), we define $Y_{ij} \sim Gamma\{\alpha_j, \beta_j\}$, where the sample space Ω_y is $[0, \infty)$. For our first model, we consider no shared gamma parameters between groups, so we are concerned with $\theta = (\alpha_1, \beta_1, \alpha_2, \beta_2)$, with parameter space $\Theta : \{\alpha_j > 0, \beta_j > 0\}$. We can write the distribution for Y_{ij} in exponential dispersion family form as:

$$f(y_{ij}|\theta) = exp\{\phi_j\{y_{ij}\theta_j - b(\theta_j)\} + c(yij, \phi_j)\} \tag{1}$$

where

$\phi_j = \alpha_j, \qquad \theta_j = \frac{-\beta_j}{\alpha_j},$
$b(\theta_j) = log\left(\frac{-1}{\phi_j\theta_j}\right), \qquad c(y, \phi_j) = -log(\Gamma(\alpha_j)) - log(y_{ij})$

We consider a model to compare means of the two groups, with the mean of group j given by $\mu_j = -\frac{1}{\theta_j}$ and variance of group j given by $V_j = \frac{\phi_j}{\theta_j^2}$. Additionally, we consider a reduced model with a shared shape parameter across groups, $\alpha_1 = \alpha_2$, which is equivalent to $\phi_1 = \phi_2$ in exponential dispersion family form given by (1). We estimate the parameters via maximum likelihood by maximizing the the sum of the log-likelihoods of the entire sample ($n = 52$) over the parameter space Θ. The R code for the entire analysis is on an attached page.

The actual sample means and variances from the data are given in Table 1. Parameter estimates and associated moments are given in Table 2. We see that for both the reduced and full models provide estimates that are quite close to the sample data. Another initial observation is that although the means of the two groups are quite different, both have large variances among the observations and thus our we have a large amount of uncertainty with a (relatively) small sample size.

Table 1

Sample Estimates		
Estimate	**Seeded (Treatment)**	**Unseeded (Control)**
Mean	441.9731	164.5731
Variance	438342.75	77525.8

Table 2 - Maximum Likelihood Estimates

Full Model		
Log-likelihood $= -301.3529$		
Estimate	**Seeded (Treatment)**	**Unseeded (Control)**
θ	-0.00225	-0.00608
ϕ	0.4580	0.4408
Mean θ^{-1}	443.9047	164.6723
Variance $(\phi\theta^2)^{-1}$	431290.097	60321.95
Reduced Model		
Log-likelihood $= -301.3721$		
Estimate	**Seeded (Treatment)**	**Unseeded (Control)**
θ	-0.00226	-0.00607
ϕ	0.4495	
Mean θ^{-1}	441.9555	164.5843
Variance $(\phi\theta^2)^{-1}$	434501.825	60257.511

In order to compare the fit of the reduced and full models, we conduct a likelihood ratio test. Using the likelihoods from the models, we calculate the likelihood ratio test statistic as $\Lambda = -2(l_{red} - l_{full}) = .0384$. For a χ^2_1 distribution, the p-value for Λ is .8446. Thus at a 5% significance level, we fail to reject the null hypothesis that the reduced model (with shared α parameter) provides a better fit for the data. For the rest of our analysis, we consider only the reduced model with shared shape parameter.

From the results in Table 2, we analyze the question of interest. In order to compare group means, we estimate the ratio of the treatment group mean to the control group mean. Using the Delta method and the information matrix of the maximum likelihood estimates, we get an estimate of the ratio as $\hat{\frac{\mu_2}{\mu_1}} = 2.6853$, with a standard error of 0.9487. A 95% confidence interval for the ratio is estimated as $(0.8258, 4.5447)$.

CONCLUSION: If the two groups had the same mean, their ratio would be equal to 1. Because 1 is inside our estimated 95% confidence interval, we fail to reject the null hypothesis of different rainfall volume (in acre-feet) between groups, at a 5% significance level. As a result, we conclude that there is no statistically significant evidence that seeding clouds with silver iodide has an effect on the average rainfall of the cloud. Because we are considering a reduced model with a shared scale parameter, this conclusion is equivalent to saying that the rate parameters are not different between groups.

Finally, note that we used maximum likelihood instead of moments estimation for the dispersion parameter in order to conduct likelihood ratio tests and confidence intervals. One should note that, especially for relatively small n, our estimates of ϕ in this case are biased. In this case, we see in both models that our estimated variances based on model parameters are slightly smaller than the actual sample variances.

2 Part 2

2.1

Let Y have a gamma distribution with probability density function

$$f_y(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} exp(-\beta y), y > 0.$$

and let $W = Y^{-1}$.

a)

$$\begin{aligned}
f_w(w) &= \frac{\beta^\alpha}{\Gamma(\alpha)} w^{1-\alpha} exp(-\frac{\beta}{w}) \\
&= exp\left\{(-\alpha - 1)log(w) - \beta(\frac{1}{w}) + \alpha log(\beta) - log(\Gamma(\alpha))\right\} \\
&= exp\left\{\theta_1 log(w) + \theta_2(\frac{1}{w}) - \theta_1 log(-\theta_2) - log(\Gamma(-\theta_1) - log(w)\right\}
\end{aligned}$$

for $w > 0$ where $\theta_1 = -\alpha$ and $\theta_2 = -\beta$. Then the pdf is in exponential family form, and note that $\Theta = (\theta_1 < 0, \theta_2 < 0)$

b)

We can use the sufficient statistics from part (a) to find the expectations:

$$\begin{aligned}
E\left[\frac{1}{w}\right] &= \frac{\partial B(\theta)}{\partial \theta_2} \\
&= \frac{-\theta_1}{\theta_2} \\
&\boxed{= \frac{-\alpha}{\beta}}
\end{aligned}$$

and

$$\begin{aligned}
E\left[log(w)\right] &= \frac{\partial B(\theta)}{\partial \theta_1} \\
&= -log(\theta_2) + \frac{-1}{\Gamma'(-\theta_1)} \\
&\boxed{= log(\beta) - \frac{1}{\Gamma'(\alpha)}}
\end{aligned}$$

c)
We use the MGF for W:

$$\begin{aligned}
M_t(u) &= \frac{exp\{B(\theta + u)\}}{exp\{B(\theta)\}} \\
&= E[exp(u'T)]
\end{aligned}$$

Next we note that $exp\{log(w)\} = w$, so we can simply insert $u = (1, 0)$ into the MGF to find $E[W]$:

$$\begin{aligned}
M_t(1) &= \frac{exp\{B(\theta + (1,0))\}}{exp\{B(\theta)\}} \\
&= exp\{(\theta_1 + 1)log(-\theta_2) - log(\Gamma(-\theta_1 + 1)) - (\theta_1)log(-\theta_2) + log(\Gamma(-\theta_1))\} \\
&= exp\{log(-\theta_2) - log(-\theta_1 - 1)\} \\
&\boxed{= \frac{\beta}{\alpha - 1}}
\end{aligned}$$

d)
First, note that $\mu_w = \frac{\beta}{\alpha-1} = \frac{\theta_2}{\theta_1+1}$ and $V(w) = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} = \frac{\theta_2^2}{(-\theta_1-1)^2(-\theta_1-2)} = \mu^2\phi$, where $\phi = \frac{-1}{(\theta_1+2)}$. Then we can solve for θ_1 and θ_2 to get $\theta_1 = \frac{-1}{\phi} - 2$ and $\theta_2 = \mu(\frac{-1}{\phi} - 1)$. We can then input these expressions into the function from (a) to get:

$$\begin{aligned} f_w(w) = exp\Big\{&(\frac{-1}{\phi} - 2)log(w) + \mu(\frac{-1}{\phi} - 1)(\frac{1}{w}) \\ &- (\frac{-1}{\phi} - 2)log(-\mu(\frac{-1}{\phi} - 1)) - log(\Gamma(-(\frac{-1}{\phi} - 2)) \\ &-log(w)\} \end{aligned}$$

and now we have reparameterized the pdf with the same sufficient statistics as before, and now $B(\mu, \phi) = (\frac{-1}{\phi} - 2)log(-\mu(\frac{-1}{\phi} - 1)) + log(\Gamma(\frac{1}{\phi} + 2))$, with $\alpha > 1$.

e)

No. From slide 53 in Chapter 2, we see that a necessary condition is that a sufficient statistic for one of the canonical parameters is W (or a linear function of W). From our results in (a) and (d), we see that this is not the case for the inverse gamma distribution, so we cannot coerce it into exponential dispersion family form.

2.2

Let Y have inverse gamma distribution with pdf

$$f_y(y) = \left[\frac{\tau}{2\pi y^3}\right]^{.5} exp\left\{-\frac{\tau(y-\eta)^2}{2\eta^2 y}\right\}$$

a)
We can write the pdf in exponential dispersion form as

$$\begin{aligned} f_y(y) &= exp\left\{\frac{-\tau}{2}(\frac{y}{\eta^2} - \frac{2}{\eta}) + .5log(-2(\frac{-\tau}{2})) - 1.5logy - .5log(2\pi) - (\frac{\tau}{2})(\frac{1}{y})\right\} \\ &= exp\left\{\phi(y\theta - 2\sqrt{\theta}) + (\frac{\phi}{y}) + .5log(-2\phi) - 1.5logy - .5log(2\pi)\right\} \end{aligned}$$

where $\phi = \frac{-\tau}{2} < 0$ and $\theta = \frac{1}{\eta^2} > 0$, and $y > 0$.

b)
In order to express the mean and variance in terms of τ and η, we take derivatives:

$$\begin{aligned} E[Y] &= b'(\theta) \\ &= \frac{2}{2\sqrt{\theta}} \\ &= \frac{1}{\sqrt{\eta^{-2}}} \\ &\boxed{= \eta} \end{aligned}$$

and

$$
\begin{aligned}
Var[Y] &= \phi^{-1} b''(\theta) \\
&= \phi^{-1} \frac{-1}{2\theta^{1.5}} \\
&= \frac{-2}{\tau} \frac{-1}{2\eta^{-3}} \\
&\boxed{= \frac{\eta^3}{\tau}}
\end{aligned}
$$

c)
We can write the (log)likelihood function as

$$
\begin{aligned}
L(\theta|\mathbf{y}) &= \prod exp\left\{\phi(y_i\theta - 2\sqrt{\theta}) + (\frac{\phi}{y_i}) + .5log(-2\phi) - 1.5log(y_i) - .5log(2\pi)\right\} \\
&= exp\left\{\phi\theta \sum y_i - 2n\phi\sqrt{\theta} + \phi \sum \frac{1}{y} + .5nlog(-2\phi)\right. \\
&- 1.5 \sum log(y_i) - .5nlog(2\pi)\} \\
logL(\theta|\mathbf{y}) &= \phi\theta \sum y_i - 2n\sqrt{\theta} + \phi \sum \frac{1}{y} + .5nlog(-2\phi) - 1.5 \sum log(y_i) - .5nlog(2\pi)
\end{aligned}
$$

We find the MLE of θ by taking the derivative and setting it equal to 0:

$$
\begin{aligned}
\frac{\partial l}{\partial \theta} &= \phi \sum y_i - \frac{n\phi}{\sqrt{\theta}} \\
0 &= \phi \sum y_i - \frac{n\phi}{\sqrt{\theta}} \\
\hat{\theta}_{MLE} &= \left(\frac{n}{\sum y_i}\right)^2 \\
\hat{\eta}_{MLE} &= \frac{1}{\sqrt{\hat{\theta}_{MLE}}} \\
&\boxed{\hat{\eta}_{MLE} = \bar{Y}}
\end{aligned}
$$

bonus
(I think it's false, and I think the gamma distribution would be an example, but didn't have time to work it out.)