Exercise 11 Sthavishtha Bhopalam Rajakumar December 26, 2017

1 Introduction

The objective of the current exercise is to study two-dimensional electrostatic equation (of the form of a Poisson's equation) using Finite Differences, wherein the discretized equations are solved using Gauss-Siedel method and Jacobi method. For this purpose, a point charge is placed at a specific position in a hollow conducting box.

The following tasks are sequentially performed in this work:

1. The Centrally Differenced electrostatic equation is solved using Jacobi relaxation method
2. Similarly, Gauss Siedel relaxation method is used for the same purpose and later compared to the aforementioned Jacobi relaxation method.
3. Further, the work is extended to include multiple point charges inside the hollow conducting box, and also considering the cases when the charges are placed away from the center of the box.
4. The contours of the electric potential are plotted in each task described above.

From a coding perspective, a parallelized numerical code employing shared memory interface (OpenMP) was utilized for running all the simulations.

2 Algorithm Description

A uniform grid size ($N \times N$) of the hollow box has been taken to be 100×100 in the current work. The electric potential inside the box can be mathematically described by the Poisson's equation :

$$\frac{\partial^2 Q(x,y)}{\partial x^2} + \frac{\partial^2 Q(x,y)}{\partial y^2} = -\rho(x,y) \tag{1}$$

where $\rho(x,y) = \delta(x - x_o)\delta(y - y_o)$ with δ being the Dirac Delta function, and (x_o, y_o) being the position of the point charge. The boundary conditions used in the current work include equating $\phi = 0$ at the edges of the box. Thus, the resulting $(N-2) \times (N-2)$ unknowns have to be solved. Since the geometry is simple, a uniform grid spacing along the horizontal and vertical directions have been chosen, for simplicity. Additionally, ρ near the grid points is chosen to be $\frac{1}{\delta x \delta x}$.

After discretization, the Jacobi method can be implemented to solve the resulting system of equations of this form:

$$\phi_{i,j}^{n+1} = \frac{1}{4}[\phi_{i+1,j}^{n} + \phi_{i-1,j}^{n} + \phi_{i,j+1}^{n} + \phi_{i,j-1}^{n}] + \frac{\Delta x^2}{4}\rho_{i,j} \tag{2}$$

with n and $n+1$ denoting the old and new iteration steps respectively. This can be easily parallelized due to the lack of data dependencies.
On the other hand, if Gauss-Siedel relaxation method has to be implemented to solve, the resulting system of equations appear to be of this form:

$$\phi_{i,j}^{n+1} = \frac{1}{4}[\phi_{i+1,j}^{n} + \phi_{i-1,j}^{n+1} + \phi_{i,j+1}^{n} + \phi_{i,j-1}^{n+1}] + \frac{\Delta x^2}{4}\rho_{i,j} \tag{3}$$

This numerical scheme imposes a difficulty to be parallelized due to the inherent data dependences. Hence, a red-black method of updating the variables [1] was implemented for this purpose. In this case, the variables at red and black points are independent of each other, making it easier to be parallelized with two separate `omp for` loops.

[1] `http://stellar.cct.lsu.edu/2011/11/hpx-vs-openmp-gauss-seidel-method/`

3 Results and Discussions

3.1 Comparison of Jacobi and Gauss-Siedel relaxation methods

For comparing both the relaxation schemes (Gauss-Siedel and Jacobi), a graph has been plotted to determine the relative convergence rate and number of iterations it may take for both these methods to yield numerical solutions within a specific tolerance ($1e-05$ in the current case).
From Figure 1, it can be seen that the Gauss-Siedel relaxation method possesses a much higher convergence rate requiring fewer number of iterations than its counterpart Jacobi relaxation method. Additionally, Gauss-Siedel method also offers reduced memory requirements [1]. Hence, the latter method was chosen for the subsequent simulations.

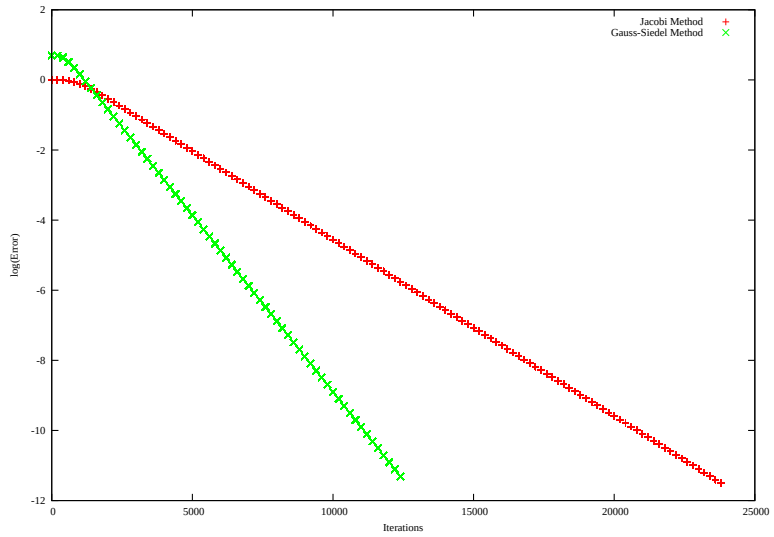


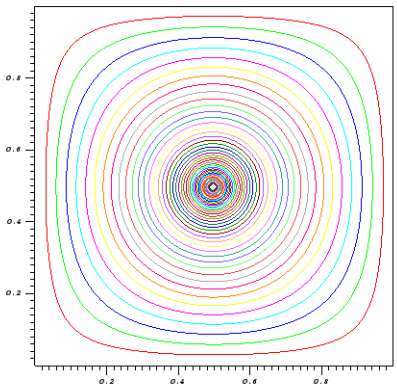
Figure 1: Jacobi Method vs Gauss-Siedel Relaxation Method

3.2 Placing the point charges at different positions

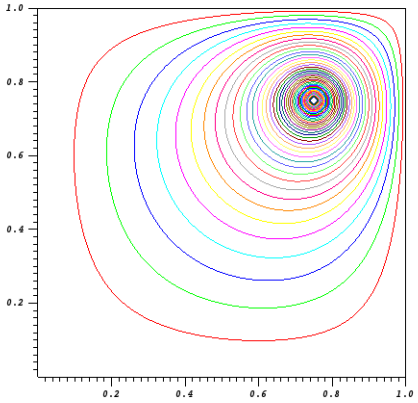
Contour plots were captured for each case as the point charges were placed at various positions in the hollow box as seen in Figures 2, 3, 4. It can be seen that the contour lines get accumulated at those concentrated points where the charged points are placed. For this purpose, single, two and three point charges have been considered in the current study, as seen in these Figures.

References

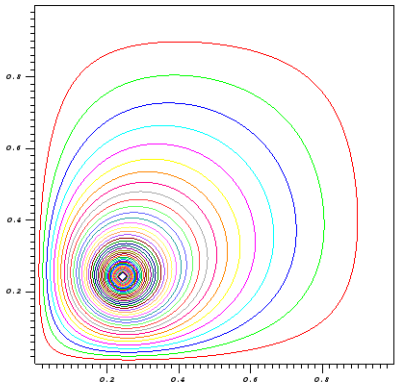
[1] Hans Hermann. Lecture notes on introduction to computational physics. Institute for Building Materials, ETH Zürich, 2017.



(a) $((0.5, 0.5))$

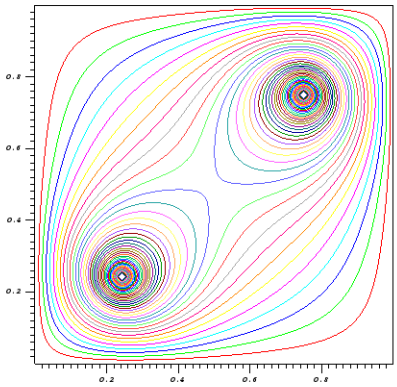


(b) $(0.75, 0.75)$

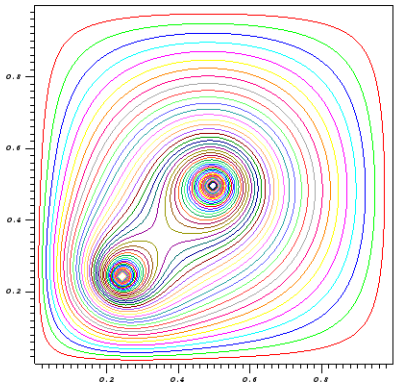


(c) $(0.25, 0.25)$

Figure 2: Potential contours for single point charges at respective positions



(a) $(0.25, 0.25)$ and $(0.75, 0.75)$



(b) $(0.25, 0.25)$ and $(0.5, 0.5)$

Figure 3: Potential contours for two point charges placed at respective positions

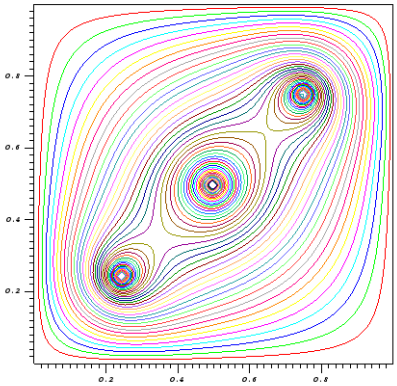


Figure 4: Potential contours for three point charges placed at $(0.25, 0.25)$, $(0.5, 0.5)$ and $(0.75, 0.75)$