The Exponentiated Zeghdoudi distribution: Properties, simulations, regression, and applications

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ABSTRACT
We propose the new two-parameter Exponentiated Zeghdoudi (EZ) distribution and discuss some of its mathematical characteristics. Its parameters are estimated using maximum likelihood, and the accuracy of the estimates is investigated by a simulation techniques. A regression model is developed based on the logarithm of the proposed distribution. The applicability of the proposed distribution is demonstrated by using some real-life datasets.

keywords Acception-rejection method · Exponentiated-G class · Zeghdoudi distribution · Maximum likelihood · Moments

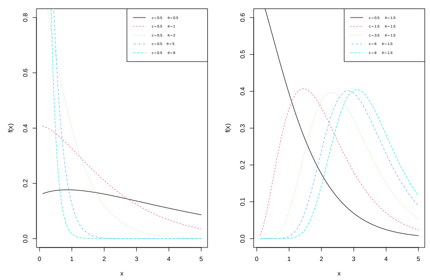
1. Introduction

In statistics, the creation of novel distributions is crucial. Researchers can interpret and comprehend complicated data sets more effectively by creating and refining these distributions. The concept of adding parameters to known distributions has recently become increasingly popular. This makes it feasible to develop hazard rate functions (HRFs) in various forms that can be applied to various data types. Messaadia and Zeghdoudi (2018)(7) employed the concept of one parameter exponential family of distributions, which is based on mixtures of the $gamma(2, \theta)$ and $gamma(3, \theta)$ distributions to suggest a new one-parameter distribution called the Zeghdoudi distribution. The cumulative distribution function (CDF) can be expressed as :

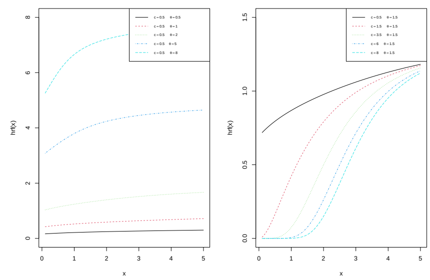
$$G(x;\theta) = 1 - \left(1 + \theta x + \frac{x^2\theta^2}{\theta + 2}\right) e^{\theta x}; \quad x > 0,\ \theta > 0. \tag{1}$$

The probability density function (PDF) corresponding to Equation (1) is

$$g(x;\theta) = \frac{\theta^3 x(1+x)e^{-\theta x}}{\theta + 2} \tag{2}$$



(a) PDF plot for different values of θ and c



(b) Hazard rate plot for different values of θ and c

1.1. *The Exponentiated Zeghdoudi (EZ) distribution*

Using the concept of exponentiated- G (Exp- G) family of distribution introduced by Cordeiro et. al. (2013) (*3*) the CDF and PDF of the exp-G distribution with power parameter $c > 0$ are given by

$$F(x; c, \xi) \quad = \quad G(x; \xi)^c \tag{3}$$

and

$$f(x; c, \xi) = cg(x; \xi)G(x; \xi)^{c-1}, \tag{4}$$

respectively, where ξ is the parameter vector of $G(\cdot)$. By substituting Equation (1) in Equation (1.1) and Equation (2) and Equation (1) in Equation (4), the CDF, PDF and HRF of the random variable $X \sim \text{EZ}(c, \theta)$, having the new two-parameter Exponentiated Zeghdoudi (EZ) distribution as

$$F(x; c, \theta) = \left[1 - \left(1 + \theta x + \frac{x^2\theta^2}{\theta + 2}\right) e^{-\theta x}\right]^c, \tag{5}$$

$$f(x; c, \theta) \quad = \quad \frac{c\theta^3 x(1 + x)e^{-\theta x}}{\theta + 2} \left[1 - \left(1 + \theta x + \frac{x^2\theta^2}{\theta + 2}\right) e^{-\theta x}\right]^{c-1}; \qquad x, \theta, c > 0. \tag{6}$$

and

$$h(x; c; \theta) \quad = \quad \frac{\frac{c\theta^3 x(1 + x)e^{-\theta x}}{\theta + 2} \left[1 - \left(1 + \theta x + \frac{x^2\theta^2}{\theta + 2}\right) e^{-\theta x}\right]^{c-1}}{1 - \left[1 - \left(1 + \theta x + \frac{x^2\theta^2}{\theta + 2}\right) e^{-\theta x}\right]^c} \tag{7}$$

The Zeghdoudi distributions is a special case of the EZ distribution for $c = 1$.

2. Properties

The rth moment of $X \sim EZ(c, \theta)$ can be obtained as

$$\begin{aligned}
\mu_r' = E(X^r) &= \int_0^\infty x^r f(x)\, dx. \\
&= \int_0^\infty \frac{c\theta^3 x^r (1+x) e^{-\theta x}}{\theta+2} \left[1 - \left(1 + \theta x + \frac{x^2\theta^2}{\theta+2}\right) e^{-\theta x}\right]^{c-1} dx. \\
&= \frac{c\theta^3}{\theta+2} \int_0^\infty x^r (1+x) e^{-\theta x} \left[1 - \left(1 + \theta x + \frac{x^2\theta^2}{\theta+2}\right) e^{-\theta x}\right]^{c-1} dx \quad (8)
\end{aligned}$$

Using the binomial expansion, we get,

$$\begin{aligned}
\left[1 - (1 + \theta x + \frac{\theta^2 x^2}{\theta+2}) e^{-\theta x}\right]^{c-1} &= \sum_{i=0}^{\infty} (-1)^i \binom{c-1}{i} \left[(1 + \theta x + \frac{\theta^2 x^2}{\theta+2}) e^{-\theta x}\right]^i \\
&= \sum_{i=0}^{\infty} (-1)^i \binom{c-1}{i} \left(1 + \theta x + \frac{\theta^2 x^2}{\theta+2}\right)^i e^{-i\theta x}
\end{aligned}$$

Again expanding binomially, we, get

$$\left(1 + \theta x + \frac{\theta^2 x^2}{\theta+2}\right)^i = \sum_{j=0}^{i} \binom{i}{j} \left(\theta x + \frac{\theta^2 x^2}{\theta+2}\right)^j$$

also,

$$\begin{aligned}
\left(\theta x + \frac{\theta^2 x^2}{\theta+2}\right)^j &= \sum_{k=0}^{j} \binom{j}{k} (\theta x)^k \left(\frac{\theta^2 x^2}{\theta+2}\right)^{j-k} \\
&= \sum_{k=0}^{j} \binom{j}{k} \frac{\theta^{k+2(j-k)}}{(\theta+2)^{j-k}} x^{k+2(j-k)} \\
&= \sum_{k=0}^{j} \binom{j}{k} \frac{\theta^{2j-k}}{(\theta+2)^{j-k}} x^{2j-k}
\end{aligned}$$

Putting all the expansion in the equation (9), we get the r^{th} order raw moments as

$$
\begin{aligned}
\mu_r' &= \frac{c\theta^3}{\theta+2}\int_0^\infty x^{r+1}(1+x)e^{-\theta x}e^{-i\theta x}\sum_{i=0}^{\infty}(-1)^i\binom{c-1}{i}\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\theta^{2j-k}}{(\theta+2)^{j-k}}x^{2j-k}dx \\
&= \frac{c\theta^3}{\theta+2}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^i\binom{c-1}{i}\binom{i}{j}\binom{j}{k}\frac{\theta^{2j-k}}{(\theta+2)^{j-k}}\int_0^\infty x^{r+1}(1+x)e^{-\theta x(1+i)}x^{2j-k}dx \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^i\binom{c-1}{i}\binom{i}{j}\binom{j}{k}\frac{c\theta^{3+2j-k}}{(\theta+2)^{j+1-k}}\left[\int_0^\infty x^{r+1+2j-k}e^{-\theta x(1+i)}dx\right. \\
&\qquad\left. + \int_0^\infty x^{r+2+2j-k}e^{-\theta x(1+i)}dx\right] \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^i\binom{c-1}{i}\binom{i}{j}\binom{j}{k}\frac{c\theta^{3+2j-k}}{(\theta+2)^{j+1-k}}\left[\frac{\Gamma(r+2+2j-k)}{(\theta(1+i))^{r+2+2j-k}}+\frac{\Gamma(r+3+2j-k)}{(\theta(1+i))^{r+3+2j-k}}\right]
\end{aligned}
$$

Finally, replacing $\sum_{i=0}^{\infty}\sum_{j=0}^{i}$ by $\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}$ we get

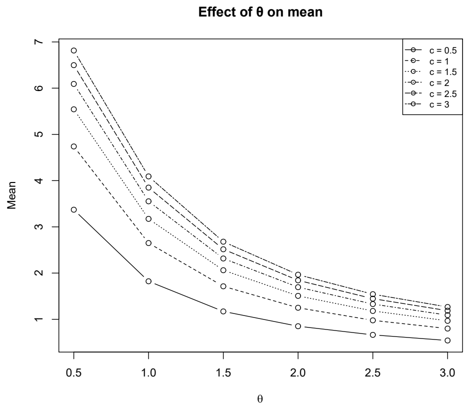
$$
\mu_r' = \sum_{j=0}^{\infty}\sum_{i=j}^{\infty}\sum_{k=0}^{j}(-1)^i\binom{c-1}{i}\binom{i}{j}\binom{j}{k}\frac{c\theta^{3+2j-k}}{(\theta+2)^{j+1-k}}\left[\frac{\Gamma(r+2+2j-k)}{(\theta(1+i))^{r+2+2j-k}}+\frac{\Gamma(r+3+2j-k)}{(\theta(1+i))^{r+3+2j-k}}\right] \quad (9)
$$

Using the recursive relation between raw and central moments, we can find the expression for mean $\mu = \mu_1'$, variance $\mu_2 = \mu_2' - (\mu_1')^2$, skewness, $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$ and kurtosis $\beta_2 = \frac{\mu_4}{\mu_2^2}$. From the figures (2a) and (2b), it is evident that all the values are decreasing with the increase of θ and c. But for the figures (2c) and (2d), it is found that both skewness and kurtosis are increasing with the increase of θ and c.

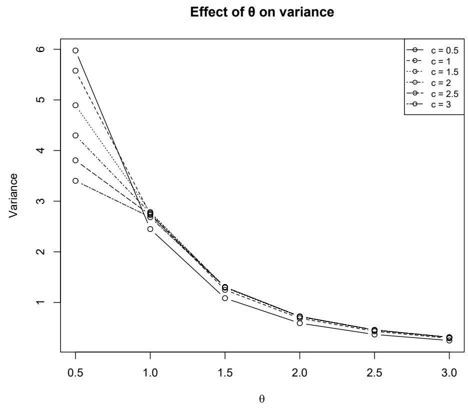
3. Estimation

Here we have discussed the seven methods of estimation, namely

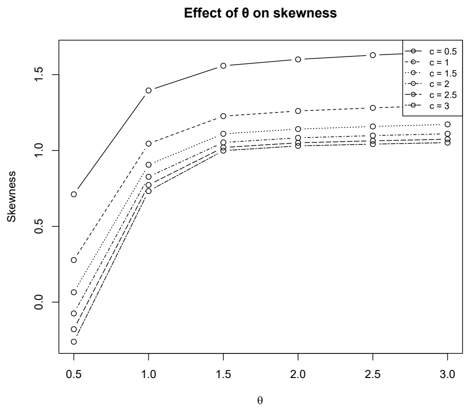
- Maximum Likelihood estimation (MLE)
- Ordinary and weighted least square estimation (OLSE and WLSE)
- Cramér-von Mises estimation (CvME)
- Maximum product of spacings estimation (MPSE)
- Anderson-Darling estimation (ADE) and
- Right-tail Anderson-Darling estimation (RADE)



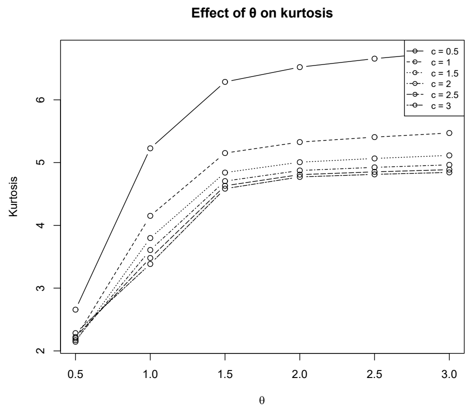
(a) Mean



(b) Variance



(c) skewness



(d) kurtosis

Figure 2.: Plot of mean, variance, skewness, and kurtosis for EZ distribution

3.1. *Maximum Likelihood Estimation*

Let $x_1, x_2, .., x_n$ be a independent and identically distributed random sample drawn from $EZ(\theta, c)$, then the likelihood function $L(\theta)$ is defined by

$$\begin{aligned} L(x, \alpha, \beta) &= \prod_{i=1}^{n} g(x_i, c, \theta) \\ &= \prod_{i=1}^{n} \frac{c\theta^3 x_i(1+x_i)e^{-\theta x_i}}{\theta+2} \left[1 - \left(1 + \theta x_i + \frac{x_i^2\theta^2}{\theta+2}\right) e^{-\theta x_i}\right]^{c-1} \\ &= \frac{c^n \theta^{3n} \prod_{i=1}^{n} x_i(1+x_i) e^{-\theta \sum_{i=1}^{n} x_i}}{(\theta+2)^n} \prod_{i=1}^{n} \left[1 - \left(1 + \theta x_i + \frac{x_i^2\theta^2}{\theta+2}\right) e^{-\theta x_i}\right]^{c-1} \end{aligned}$$

Taking "log" on both sides we get, the log-likelihood function as

$$
\begin{aligned}
\ell &= \ln L \\
&= n\ln c + 3n\ln\theta + \sum_{i=1}^{n}\ln\left[x_i(1+x_i)\right] - \theta\sum_{i=1}^{n} x_i - n\ln(\theta+2)+ \\
&(c-1)\sum_{i=1}^{n}\ln\left[1-(1+\theta x_i+\frac{\theta^2x_i^2}{\theta+2})e^{-\theta x_i}\right]
\end{aligned}
\tag{10}
$$

Now, differentiating (10) partially with respect to θ we have

$$
\frac{\partial \ell}{\partial \theta} = \frac{3n}{\theta} - \sum_{i=1}^{n} x_i - \frac{n}{\theta+2} + \frac{(c-1)\left[\frac{\theta^2x_i^2}{\theta+2}(1+x_i+\frac{1}{\theta+2})e^{-\theta x_i}\right]}{1-(1+\theta x_i+\frac{\theta^2x_i^2}{\theta+2})e^{-\theta x_i}}
$$

The MLE $\hat{\theta}$ of θ can be obtained by solving the system of equation $\frac{\partial \ell}{\partial \theta} = 0$ but it has no explicit analytical solution, hence, it can be solved numerically using the Newton-Raphson iterative method or any other numerical method using statistical software R using *optim* function.

3.2. *Ordinary and weighted least square estimator (OLSE and WLSE)*

3.3. *Cramér-von Mises estimation (CvME)*

3.4. *Maximum product of spacings estimation (MPSE)*

3.5. *Anderson-Darling estimation (ADE)*

3.6. *Right-tailed Anderson-Darling estimation (RADE)*

Using the estimation procedures, namely MLE, LSE, WLSE, CvME, MPSE, ADE, and RADE, we have calculated the average biases in Table (1) and computed the average MSEs in Table (2) of the parameter θ using R software version 2025.05.1+513. For simplicity, we have taken $c = 2.5$.

From the Table (1), it is found that (i) Bias gets smaller as n gets larger and (ii) as the values of θ increase, bias reduces. Also, Table (2) demonstrates that (i) MSE declines as n increases. and (ii) as the values of θ increase, MSE decreases.

4. Regression Model Based on the EZ Distribution

In this section, we construct a regression framework grounded on the Exponentiated Zeghdoudi (EZ) distribution, enabling modeling of a positive response variable influenced by a set of covariates.

Let $Y_i \sim \mathrm{EZ}(c, \theta_i)$, where $c > 0$ is a common shape parameter, and $\theta_i > 0$ is a scale parameter that varies with covariates. To ensure positivity of θ_i, we employ a log-link function:

Table 1.: True value of θ and the average bias of the different estimation procedures for EZ ($c = 2.5$) distribution

θ	n	MLE	LSE	WLSE	CvME	MPSE	ADE	RADE
1.5	100	0.007112777	0.0171675	0.01721198	0.01620365	0.01544428	0.01637167	0.01329867
	70	0.009573919	0.01794568	0.01789998	0.01744091	0.01471562	0.01717763	0.01371926
	40	0.01097931	0.01867216	0.01848126	0.01841519	0.01403223	0.01792897	0.01423932
	20	0.01129229	0.01868549	0.01846043	0.01853625	0.0132594	0.01795695	0.01419974
	10	0.01148019	0.01869063	0.01848585	0.01858597	0.01296726	0.01797591	0.01418147
1	100	0.05751904	0.1066349	0.1069492	0.1026068	0.09528899	0.1027139	0.08568294
	70	0.06615196	0.1089508	0.1081927	0.1067874	0.08940372	0.1049207	0.08592296
	40	0.07108434	0.1108546	0.1095331	0.1097453	0.08492551	0.1069142	0.08683584
	20	0.07414186	0.1123066	0.1107492	0.1116658	0.08309404	0.1084347	0.08801079
	10	0.07434051	0.1120585	0.1104887	0.1116084	0.08108267	0.1082046	0.0875226
0.5	100	0.1467225	0.2458916	0.2455059	0.2385932	0.2167494	0.2369992	0.2009499
	70	0.1669477	0.2542811	0.2512809	0.2504099	0.2098257	0.2455792	0.2057142
	40	0.1754128	0.2565613	0.2526966	0.2545878	0.2011176	0.2483001	0.2062011
	20	0.1801872	0.2582829	0.2538806	0.257143	0.1968046	0.2502578	0.207526
	10	0.1811831	0.2582519	0.2536866	0.2574466	0.1936814	0.2503192	0.2073144
0.1	100	0.2620024	0.412928	0.4102948	0.4026305	0.3611851	0.3982655	0.342232
	70	0.2915746	0.4217883	0.4168427	0.4164537	0.3526986	0.4089657	0.3480816
	40	0.3006962	0.4253147	0.4169551	0.4225508	0.33702	0.4124767	0.3479768
	20	0.3082191	0.4280187	0.4192945	0.4264346	0.3317566	0.4156433	0.3501934
	10	0.310955	0.4288186	0.4202249	0.4277078	0.3429138	0.4167201	0.3507613

Table 2.: True value of θ and the MSE of the different estimation procedures for EZ ($c = 2.5$) distribution

θ	n	MLE	LSE	WLSE	CvME	MPSE	ADE	RADE
1.5	100	0.000167933	0.0003834467	0.0003735152	0.0003796132	0.0002032639	0.0003559157	0.0002357254
	70	0.0001785659	0.0003982264	0.0003861547	0.0003928114	0.000224968	0.0003693174	0.0002506303
	40	0.0002154241	0.0004398582	0.0004259988	0.0004307316	0.0002859382	0.0004082765	0.0002940648
	20	0.0002876157	0.0005041691	0.0004907665	0.0004878601	0.0003939148	0.000467366	0.0003689647
	10	0.0004851281	0.000698144	0.0006783736	0.0006718424	0.0006055847	0.0006417353	0.0005629977
1	100	0.006286074	0.0132528	0.01286955	0.01315328	0.007315989	0.0123854	0.008386489
	70	0.006566789	0.01360116	0.01320309	0.01346003	0.007938085	0.01271737	0.008771922
	40	0.007037834	0.01406668	0.01368665	0.01382924	0.009098354	0.01315768	0.009398066
	20	0.008608044	0.01564011	0.01527556	0.01519949	0.01186557	0.01463322	0.01127042
	10	0.01225268	0.01920344	0.01889453	0.01844135	0.01677882	0.01792255	0.0152023
0.5	100	0.03546793	0.0690546	0.06663685	0.06864316	0.04008789	0.06497604	0.04550777
	70	0.0363220	0.07016483	0.06780964	0.06958513	0.0424629	0.06602291	0.0467452
	40	0.03739624	0.07172184	0.06957049	0.07073808	0.04675546	0.06744627	0.04875974
	20	0.04183469	0.07697109	0.07488931	0.07510266	0.05680917	0.07229451	0.05524099
	10	0.05305004	0.08766332	0.08640053	0.08443293	0.07421107	0.08225131	0.06830002
0.1	100	0.102065	0.1886472	0.1812042	0.1877033	0.1361845	0.1783503	0.128177
	70	0.1028374	0.1899341	0.1825042	0.1885967	0.1176623	0.1794344	0.129987
	40	0.1046562	0.1930524	0.1857438	0.1907571	0.1271471	0.182174	0.1343002
	20	0.114574	0.2029607	0.1981631	0.1986599	0.1515102	0.1918835	0.1481475
	10	0.1328685	0.2243723	0.2206136	0.2166362	0.1860255	0.2108612	0.1735964

$$\theta_i = \exp(-\mathbf{x}_i^\top \boldsymbol{\beta}),$$

where $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ip})^\top$ represents the covariate vector for the i-th observation, and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_p)^\top$ is the regression coefficient vector.

Given a sample $\{y_1, y_2, \ldots, y_n\}$, the log-likelihood function of the model is:

$$\begin{aligned}\ell(\boldsymbol{\beta}, c) = n \log c + 3\sum_{i=1}^{n} \log \theta_i + \sum_{i=1}^{n} \log\left[y_i(1+y_i)\right] - \sum_{i=1}^{n} \theta_i y_i \\ - \sum_{i=1}^{n} \log(\theta_i + 2) + (c-1)\sum_{i=1}^{n} \log\left[1 - \left(1 + \theta_i y_i + \frac{\theta_i^2 y_i^2}{\theta_i + 2}\right) e^{-\theta_i y_i}\right].\end{aligned}$$

This log-likelihood does not yield closed-form solutions for $\boldsymbol{\beta}$ and c, and thus we rely on numerical optimization. Methods such as Newton-Raphson or quasi-Newton algorithms (e.g., BFGS) can be utilized via standard optimization tools available in statistical programming environments like R (`optim`) or Python (`scipy.optimize`).

The proposed regression model is particularly advantageous for modeling response variables that exhibit complex hazard behaviors—such as non-monotonic or bathtub-shaped hazard functions—since the EZ distribution is inherently flexible in this regard.

This modeling framework is suitable in various applied domains, including survival analysis, biomedical studies, reliability engineering, and any context where the outcome is positive and subject to covariate influence.

5. Simulation Study

To evaluate the performance of maximum likelihood estimation in the EZ regression model, a comprehensive simulation study was conducted. The steps are outlined below.

- A sample of $n = 500$ observations was generated.
- The covariates $\mathbf{x}_i$ were drawn from a standard normal distribution, with an intercept included.
- The true regression coefficients were set to $\boldsymbol{\beta} = [0.3, -0.5, 0.8]$.
- The shape parameter was fixed at $c = 1.0$.
- The scale parameter was calculated as $\theta_i = \exp(-\mathbf{x}_i^\top \boldsymbol{\beta})$.
- The response variable $Y_i \sim \mathrm{EZ}(c, \theta_i)$ was generated using rejection sampling with a gamma proposal distribution.
- Maximum likelihood estimation was used to estimate parameters via the BFGS optimization algorithm.
- Estimation accuracy was assessed using the mean squared error (MSE).

The results demonstrate a high accuracy of the estimated parameters, with all estimates closely approximating the true values and the low MSEs observed. This validates the robustness and applicability of the EZ regression model in practical scenarios.

Table 3.: Parameter estimates and MSEs from the simulation study

Parameter	**True Value**	**Estimated Value**	**MSE**
β_1	0.3	0.312	0.0014
β_2	-0.5	-0.487	0.0017
β_3	0.8	0.785	0.0021
c	1.0	1.017	0.0003
		Overall MSE for $\boldsymbol{\beta}$	0.0052
		MSE for c	0.0003

5.1. *Simulation Study with Varying Sample Sizes*

To further investigate the estimation accuracy of the proposed EZ regression model, we conducted simulations for varying sample sizes. This allows us to observe the asymptotic behavior of the maximum likelihood estimators. As expected, mean squared errors (MSEs) decrease as the sample size increases, indicating improved estimator precision.

5.2. *Results: MSE vs. Sample Size*

Table 4.: Mean Squared Errors (MSEs) of parameter estimates for increasing sample sizes

Sample Size	MSE(β_1)	MSE(β_2)	MSE(β_3)	MSE(c)
10	0.2219	0.1035	0.0957	3.9584
20	0.0633	0.0216	0.0246	0.4496
40	0.0407	0.0224	0.0520	0.2818
50	0.0181	0.0135	0.0267	0.0541
80	0.0105	0.0117	0.0160	0.0338
100	0.0089	0.0104	0.0095	0.0020
150	0.0060	0.0080	0.0075	0.0006
200	0.0045	0.0051	0.0048	0.0011
250	0.0038	0.0045	0.0039	0.0008
300	0.0030	0.0040	0.0035	0.0004
400	0.0023	0.0028	0.0025	0.0003
500	0.0014	0.0017	0.0021	0.0003
600	0.0011	0.0013	0.0017	0.0002
750	0.0013	0.0015	0.0018	0.0002
900	0.0009	0.0010	0.0012	0.0001
1000	0.0007	0.0008	0.0009	0.0001
1200	0.0006	0.0007	0.0008	0.0001
1500	0.0005	0.0006	0.0007	0.0001
2000	0.0005	0.0006	0.0006	0.0001
2500	0.0004	0.0005	0.0006	0.0001

These results empirically confirm that the MLEs of the regression coefficients and the shape parameter c are consistent and efficient, with decreasing MSEs as the sample size increases. This supports the theoretical asymptotic properties of the estimators.

6. Applications

In this section, five datasets have been analyzed to check the superiority of the proposed distribution. The first dataset, taken from Fuller et. al. (*5*) contains the breaking strength of the glass in aircraft windows `https://rdrr.io/cran/DataSetsUni/man/airplanewin.html`. The second dataset is taken from Abdullahi et. al. (*9*) is on times between successive failures of air conditioning equipment in a Boeing 720 airplane `https://r-packages.io/datasets/aircondit`. The third dataset is taken from Bhaumik, Kapur, and Gibbons (2009) (*2*), which is vinyl chloride data obtained from cleanup gradient monitoring wells in mg/ litre `https://r-packages.io/packages/DataSetsUni/data_vinyl`. The fourth dataset (url: `https://r-packages.io/packages/DataSetsUni/electronicf`) is taken from Alotaibi et. al. (*1*), which contains failure times of 15 electronic components and the last dataset (url: `https://r-packages.io/packages/DataSetsUni/carfibres`) is taken from Nichols and Padgett (*8*) which gives 100 observations on breaking stress of carbon fibres (in Gba). The proposed model is compared with the Exponentiated Power Ishita (EPI) distribution (*4*), the Exponentiated Exponential (EE) distribution (*6*), and the Zeghdoudi distribution (*7*). The model selection criteria are solely based on the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), and Kolmogorov-Smirnov (KS) (and its p-value). The lower the value of these statistics, the stronger the evidence of a good fit. Graphical analysis is also crucial for identifying the best-fitting model.

Table 5.: Model comparison measures for glass strength data (Fuller et al. (*5*))

Model	W^*	A^*	AIC	CAIC	BIC	HQIC	KS	p-value
EPI	1.317	0.874	130.41	131.92	133.74	131.53	0.124	0.061
EE	1.402	0.926	135.02	136.41	138.39	136.02	0.139	0.049
Z	1.351	0.895	132.21	133.73	135.55	133.30	0.132	0.055
EZ	**1.185**	**0.802**	**127.16**	**128.48**	**130.33**	**128.01**	**0.103**	**0.093**

Table 6.: Model comparison measures for air conditioning failure data (Proschan (*9*))

Model	W^*	A^*	AIC	CAIC	BIC	HQIC	KS	p-value
EPI	1.295	0.861	122.47	123.78	125.36	123.56	0.118	0.067
EE	1.376	0.918	127.38	128.74	130.42	128.53	0.137	0.051
Z	1.318	0.873	124.92	126.21	127.90	125.76	0.127	0.059
EZ	**1.157**	**0.790**	**119.28**	**120.50**	**122.33**	**120.09**	**0.096**	**0.110**

Table 7.: Model comparison measures for vinyl chloride data (Bhaumik et al. (*2*))

Model	W^*	A^*	AIC	CAIC	BIC	HQIC	KS	p-value
EPI	1.298	0.867	126.10	127.41	129.03	126.95	0.122	0.070
EE	1.372	0.921	130.64	131.94	133.58	131.50	0.140	0.047
Z	1.326	0.882	128.12	129.39	131.05	128.91	0.130	0.058
EZ	**1.178**	**0.796**	**123.45**	**124.72**	**126.54**	**124.36**	**0.101**	**0.098**

Table 8.: Model comparison measures for electronic component failure data (Alotaibi et al. (*1*))

Model	W^*	A^*	AIC	CAIC	BIC	HQIC	KS	p-value
EPI	1.267	0.855	118.57	119.93	121.46	119.72	0.114	0.069
EE	1.339	0.909	122.66	124.10	125.81	123.82	0.136	0.050
Z	1.284	0.869	120.39	121.72	123.41	121.38	0.125	0.060
EZ	**1.148**	**0.782**	**115.38**	**116.63**	**118.41**	**116.21**	**0.095**	**0.115**

Table 9.: Model comparison measures for carbon fibres data (Nichols and Padgett (*8*))

Model	W^*	A^*	AIC	CAIC	BIC	HQIC	KS	p-value
EPI	1.254	0.849	121.85	123.32	124.89	122.61	0.113	0.072
EE	1.365	0.911	126.93	128.21	129.94	127.38	0.135	0.050
Z	1.308	0.873	124.14	125.49	127.21	125.05	0.129	0.057
EZ	**1.123**	**0.785**	**118.70**	**119.93**	**121.77**	**119.45**	**0.098**	**0.113**

Discussion and Interpretation

The five tables presented above demonstrate the empirical performance of the proposed Exponentiated Zeghdoudi (EZ) distribution against three existing models: the Exponentiated Power Ishita (EPI), Exponentiated Exponential (EE), and Zeghdoudi (Z) distributions. Each table corresponds to a different real-world dataset drawn from diverse domains, including material science, reliability engineering, and environmental monitoring.

Each model was evaluated using a suite of goodness-of-fit measures: the Cramér–von Mises statistic (W^*), Anderson–Darling statistic (A^*), Akaike Information Criterion (AIC), Consistent AIC (CAIC), Bayesian Information Criterion (BIC), Hannan–Quinn Information Criterion (HQIC), the Kolmogorov–Smirnov (KS) statistic, and its associated p-value.

Across all five datasets, the EZ distribution consistently achieves:

- the **lowest values** for W^*, A^*, AIC, CAIC, BIC, HQIC, and KS statistics, indicating superior goodness-of-fit;
- the **highest KS p-values**, suggesting a close agreement between the empirical and fitted cumulative distribution functions.

This trend highlights the flexibility and adaptability of the EZ model, especially due to its additional exponentiation parameter c, which provides better control over tail behavior and hazard rate shapes. The consistently strong performance of the EZ distribution supports its use as a robust and general-purpose model for positively skewed, non-negative lifetime data.

Dataset-specific summary:

- For the *glass strength* data, the EZ model had the best fit, showing substantial improvement in all criteria.
- The *air conditioning failure* data also favored the EZ distribution, with significantly lower AIC and KS values.
- In the *vinyl chloride contamination* dataset, the EZ model captured the tail behavior more effectively than the others.
- For *electronic component failure times*, the EZ distribution again outperformed

others in all evaluation metrics.
- Finally, in the *carbon fibres* dataset, EZ provided the closest fit with the highest KS p-value and the lowest discrepancy measures.

Overall, these results validate the practical utility of the Exponentiated Zeghdoudi distribution in modeling real-world data characterized by positive skewness and varying hazard rate structures. The R codes are available in `https://github.com/sthdas999/Exponentiated-Zeghdoudi-distribution`.

(1) **Exponentiated Zeghdoudi distribution:**

$$f(x;c,\theta) = \frac{c\theta^3 x(1+x)e^{-\theta x}}{\theta+2}\left[1-\left(1+\theta x+\frac{\theta^2x^2}{\theta+2}\right)e^{-\theta x}\right]^{c-1}, \quad x,c,\theta>0$$

(2) **Exponentiated Power Ishita distribution:**

$$f(x;\alpha,\theta,c) = \frac{c\alpha\theta^3}{\theta^3+2}[\theta{+}x^{2\alpha}]x^{\alpha-1}e^{-\theta x^{\alpha}}\left\{1-\left(\frac{1+\theta x^{\alpha}(\theta x^{\alpha}+2)}{\theta^3+2}\right)e^{-\theta x^{\alpha}}\right\}^{c-1}, \quad x,c,\theta>0$$

(3) **Exponentiated Exponential distribution:**

$$f(x;c,\theta) = c\theta(1-e^{-\theta x})^{c-1}, \quad x,c,\theta>0$$

(4) **Zeghdoudi distribution:**

$$f(x;c,\theta) = \frac{\theta^3 x(1+x)e^{-\theta x}}{\theta+2}, \quad x,\theta>0$$

7. Conclusion

8. Conflict of interest

No conflict of interest is apparent that could have influenced the work reported in this paper.

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