August 20, 2003

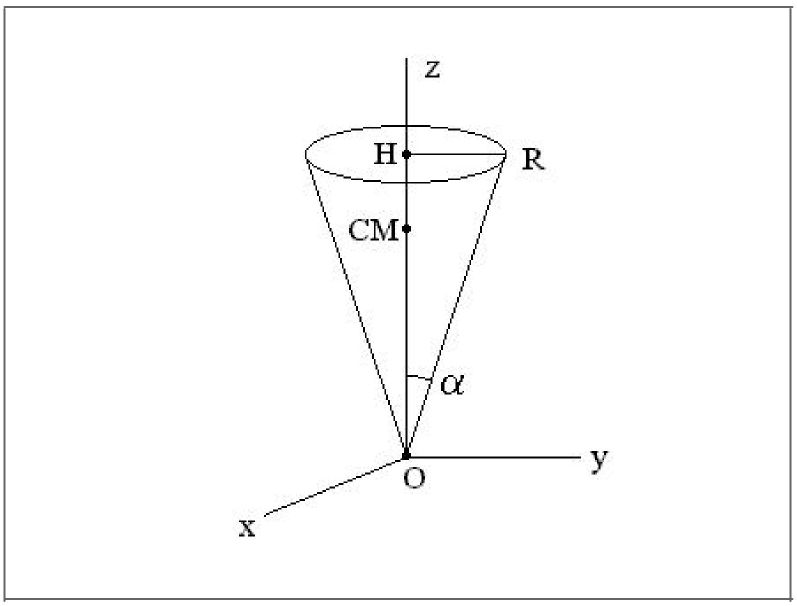
PHYSICS 44 MECHANICS

Homework Assignment VII

SOLUTIONS

Problem 1

(a) Consider a circular cone of height H and base radius $R = H\ \tan\alpha$ with uniform mass density $\rho = 3\,M/(\pi\,HR^2)$.



Show that the non-vanishing components of the inertia tensor $\mathbf{I}$ calculated from the apex O of the cone are

$$I_{xx} \;=\; I_{yy} \;=\; \frac{3}{5}\,M\left(H^2 \,+\, \frac{R^2}{4}\right) \quad \text{and} \quad I_{zz} \;=\; \frac{3}{10}\,M\,R^2$$

(b) Show that the principal moments of inertia calculated in the CM frame (located at a height $h = 3H/4$ on the symmetry axis) are

$$I_1 \;=\; I_2 \;=\; \frac{3}{20}\,M\left(R^2 \,+\, \frac{H^2}{4}\right) \quad \text{and} \quad I_3 \;=\; \frac{3}{10}\,M\,R^2$$

Solution

Throughout this problem we shall use cylindrical coordinates (r, θ, z) so that the infinitesimal volume element is $r\,dr\,d\theta\,dz$, with $0 < \theta < 2\pi$, $0 < r < z\tan\alpha$, and $0 < z < H$. Hence, the position of the center of mass is on the z-axis at a position

$$\begin{aligned} z_{CM} &= \frac{\rho}{M}\int_0^H dz\, z\int_0^{2\pi} d\theta\int_0^{z\tan\alpha} r\,dr \;=\; \frac{\pi\,\rho}{M}\int_0^H dz\, z^3\tan^2\alpha \\ &= \frac{\pi\,\rho}{4M}\,H^4\tan^2\alpha \;=\; \frac{3\,H}{4}, \end{aligned}$$

where $\rho = 3\,M/(\pi\,HR^2)$.

(a) We begin with the component

$$\begin{aligned} I_{xx} &= \rho\int_0^H dz\int_0^{2\pi} d\theta\int_0^{z\tan\alpha} r\,dr\left(z^2 + r^2\sin^2\theta\right) \\ &= \pi\,\rho\int_0^H dz\, z^4\left(\tan^2\alpha + \frac{1}{4}\tan^4\alpha\right) \\ &= \frac{\pi\,\rho}{5}\left(R^2H^3 + \frac{1}{4}R^4H\right) \;=\; \frac{3}{5}\,M\left(H^2 + \frac{R^2}{4}\right). \end{aligned}$$

Next, it is easy to show that

$$I_{yy} = \rho\int_0^H dz\int_0^{2\pi} d\theta\int_0^{z\tan\alpha} r\,dr\left(z^2 + r^2\cos^2\theta\right) \;=\; I_{xx}.$$

Lastly, we have

$$\begin{aligned} I_{zz} &= \rho\int_0^H dz\int_0^{2\pi} d\theta\int_0^{z\tan\alpha} r\,dr\left(r^2\right) \\ &= \frac{\pi\,\rho}{2}\int_0^H dz\, z^4\tan^4\alpha \;=\; \frac{\pi\,\rho}{10}\left(R^4H\right) \;=\; \frac{3}{10}\,MR^2. \end{aligned}$$

(b) Using the Parallel-Axis Theorem, the inertia tensor I_{CM} evaluated at the center of mass is expressed in terms of the inertia tensor I_0 evaluated at the apex – and calculated in Part (a) – as

$$\mathsf{I}_{CM} = \mathsf{I}_0 - M\left(|\mathbf{a}|^2\underline{\mathbf{1}} - \mathbf{a}\,\mathbf{a}\right),$$

where $\mathbf{a} = (3\,H/4)\,\widehat{\mathsf{z}}$ is the vector position of the center of mass as observed at the apex. Here, we find

$$M\left(|\mathbf{a}|^2\underline{\mathbf{1}} - \mathbf{a}\,\mathbf{a}\right) = \frac{9}{16}MH^2\left(\underline{\mathbf{1}} - \widehat{\mathsf{z}}\widehat{\mathsf{z}}\right) = \frac{9}{16}MH^2\left(\widehat{\mathsf{x}}\widehat{\mathsf{x}} + \widehat{\mathsf{y}}\widehat{\mathsf{y}}\right),$$

and thus

$$
\begin{aligned}
I_1 &= I_{xx} - \frac{9}{16} MH^2 = \frac{3}{20} M \left(R^2 + \frac{H^2}{4} \right) \\
I_2 &= I_{yy} - \frac{9}{16} MH^2 = \frac{3}{20} M \left(R^2 + \frac{H^2}{4} \right) \\
I_3 &= I_{zz} = \frac{3}{10} MR^2.
\end{aligned}
$$

Problem 2

(a) The Euler equation for ω_2 for an asymmetric top ($I_1 > I_2 > I_3$), with the two constants of the motion K (the kinetic energy) and L^2 (the square of the magnitude of the angular momentum) related by the initial condition $L^2 = 2\, I_2 K$, is

$$\dot{\omega}_2 = \alpha \left(\varpi^2 - \omega_2^2 \right),$$

where

$$\varpi^2 = \frac{2K}{I_2} \quad \text{and} \quad \alpha = \sqrt{\left(1 - \frac{I_2}{I_1}\right) \left(\frac{I_2}{I_3} - 1\right)}.$$

Using Eqs. (33)-(34) in your Lecture Notes on Rigid Body Motion, show that the solution for $\omega_2(t)$ with the initial condition $\omega_2(0) = 0$ is

$$\omega_2(t) = \varpi \tanh\left(\alpha \varpi\, t\right)$$

Solution

The integral solution begins with

$$\alpha\, t = \int_0^{\omega_2} \frac{d\sigma}{\varpi^2 - \sigma^2} = \frac{1}{\varpi} \int_0^{\arcsin(\omega_2/\varpi)} \sec\theta\, d\theta,$$

where we have made the trigonometric substitution $\sigma = \varpi \sin\theta$ to obtain the second integral. Upon integration, we find

$$\alpha \varpi t = \ln\left(\frac{1 + \omega_2/\varpi}{\sqrt{1 - (\omega_2/\varpi)^2}} \right) = \frac{1}{2} \ln\left(\frac{\varpi + \omega_2}{\varpi - \omega_2} \right).$$

If we now define $\psi = \alpha\varpi t$, then we can solve for ω_2 as

$$\omega_2 = \varpi \left(\frac{e^{2\psi} - 1}{e^{2\psi} + 1} \right) = \varpi \tanh\psi.$$