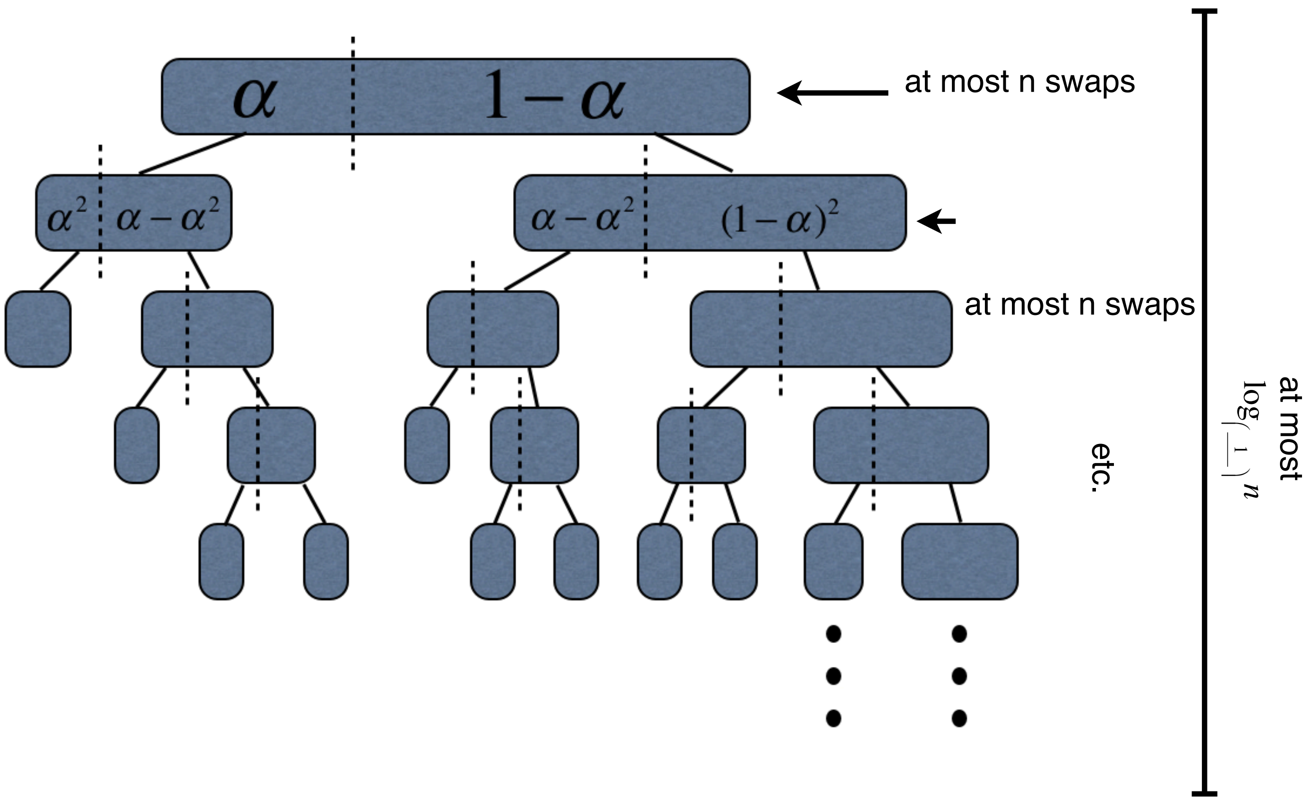
Question 2

Part A

Each left division is α of the size of the array.
Each right division is $1-\alpha$ of the size of the array
Each level will have at most n swaps.
The maximum number of levels will be the amount of $1-\alpha$ splits, therefore the maximum height is $\log_{\left(\frac{1}{1-\alpha}\right)} n$.

Because each level will have at most n swaps, and there will be at most $\log_{\left(\frac{1}{1-\alpha}\right)} n$, the upper bound of quick sort with α partition where $0<\alpha<\frac{1}{2}$ is:

$$n\log_{\left(\frac{1}{1-\alpha}\right)} n$$



Total: at most $n\log_{\left(\frac{1}{1-\alpha}\right)} n$

Part B

Induction Proof for the upper-bound of quicksort with constant alpha

Consider the base case:

$$n = 2$$

The actual run time is:

$$T(2) = 1$$

Theoretical upper-bound of the run time is:

$$2\log_{(\frac{1}{1-\alpha})} 2$$

By comparing the actual and theoretical upper-bound run time:

$$\begin{aligned}
&\mathit{actual} < \mathit{theoretical} \\
&1 < 2\log_{\left(\frac{1}{1-\alpha}\right)} 2 \\
&\frac{1}{2} < \log_{\left(\frac{1}{1-\alpha}\right)} 2 \\
&\left(\frac{1}{2}\right)^{\left(\frac{1}{1-\alpha}\right)} < 2 \\
&\because 0 < \alpha < \frac{1}{2} \\
&\therefore \frac{1}{1-\alpha} > 1 \\
&\therefore \left(\frac{1}{2}\right)^{\left(\frac{1}{1-\alpha}\right)} < 1 < 2 \\
&\mathit{QED}
\end{aligned}$$

Therefore base case is true.

By induction hypothesis, assume the upper-bound

$k\log_{\left(\frac{1}{1-\alpha}\right)} k$ is true for: $0 < k \leq n$

Now consider $n = k + 1$

In order to prove true, consider the worst case run time, which is the first partition splitting only one element:

$$\begin{aligned} &\quad actual: \\ T(k+1) &= T(1) + T((k+1)-1) + k \\ &= k + 1 + T(k) \\ &= k + 1 + k\log_{\left(\frac{1}{1-\alpha}\right)} k \end{aligned}$$

$$\begin{aligned} &theoretical: \\ &\qquad (k+1)\log_{\left(\frac{1}{1-\alpha}\right)}(k+1) \end{aligned}$$

And by comparing the actual with the theoretical upper-bound run-time:

$$k+1+k\left(\log_{\left(\frac{1}{1-\alpha}\right)} k\right) < (k+1)\left(\log_{\left(\frac{1}{1-\alpha}\right)}(k+1)\right)$$

$$\because k+1+k\left(\log_{\left(\frac{1}{1-\alpha}\right)} k\right) < k+1+k\left(\log_{\left(\frac{1}{1-\alpha}\right)} k\right) + \left(\log_{\left(\frac{1}{1-a}\right)} k\right)$$

$$\therefore k+1+k\left(\log_{\left(\frac{1}{1-\alpha}\right)} k\right) + \left(\log_{\left(\frac{1}{1-a}\right)} k\right) < (k+1)\left(\log_{\left(\frac{1}{1-\alpha}\right)}(k+1)\right)$$

$$\begin{aligned} k+1+k\left(\log_{\left(\frac{1}{1-\alpha}\right)} k\right) + \left(\log_{\left(\frac{1}{1-a}\right)} k\right) &< (k+1)\left(\log_{\left(\frac{1}{1-a}\right)}(k+1)\right) \\ (k+1)+(k+1)\left(\log_{\left(\frac{1}{1-\alpha}\right)} k\right) &< (k+1)\left(\log_{\left(\frac{1}{1-a}\right)}(k+1)\right) \\ (k+1)\left(1+\left(\log_{\left(\frac{1}{1-\alpha}\right)} k\right)\right) &< (k+1))\left(\log_{\left(\frac{1}{1-\alpha}\right)}(k+1)\right) \\ 1+\left(\log_{\left(\frac{1}{1-\alpha}\right)} k\right) &< \log_{\left(\frac{1}{1-\alpha}\right)}(k+1) \\ 1 &< \log_{\left(\frac{1}{1-\alpha}\right)}\left(\frac{k+1}{k}\right) \\ 1^{\left(\frac{1}{1-\alpha}\right)} &< \frac{k+1}{k} \\ 1 &< \frac{k+1}{k} \\ &QED \end{aligned}$$

Therefore the, by induction, the upper bound run-time is:

$$n\log_{\left(\frac{1}{1-\alpha}\right)} n$$